# APPLYING SET THEORY
# E88

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Abstract. We prove some results in set theory as applied to general topology and model theory. In particular, we study $\aleph_1$-collectionwise Hausdorff, Chang Conjecture for logics with Malitz-Magidor quantifiers and monadic logic of the real line by odd/even Cantor sets.

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## § 0. Introduction

In §1 we prove a result in general topology saying: if $\diamondsuit^*_{\aleph_1}$ then any normal space is $\aleph_1 -$ CWH (= collectionwise Hausdorff ); done independently of and in parallel to Fleisner and Alan D. Taylor.

In §2 we prove the Chang Conjecture for Magidor-Malitz Quantifiers. A recent work is [HU17].

In §3 we prove the Monadic Theory of the tree ${}^{\omega >}2$ is complicated under a quite weak set theoretic assumption.

Earlier [She75] proved this (i.e. the result on the monadic logic) assuming CH or at least a consequence of it. The present note was circulated in the Spring of 1979 in a collection including others, and see [She85].

Later, Gurevich-Shelah [GS82] proved undecidability in ZFC, with further developments then more in Shelah [She88], still the older proof gives information not covered by them. For more see [BS87], [GKKS02], [GGK04].

The results are old, still in particular, §1 gives a direct proof of the result compared to others and §3 gives a considerably more transparent easier proof of the result of [GS82].

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## § 1. A note in general topology if $\diamondsuit^*_{\aleph_1}$ then any normal space is $\aleph_1 -$ CWH (= collectionwise Hausdorff)

The normal Moore space problem has been a major theme in general topology, see the recent survey Dow-Tall [DT18]. In this connection, Fleissner [Fle74, p.6] proved: ($\mathbf{V} = \mathbf{L}$) every normal first countable (topological) space is CWH (CWH means collectionwise Hausdorff ). He used a strengthening of diamond. The author proved Fleissner strengthening (for $\aleph_1$) does not follow from ZFC+$\diamondsuit^+_{\aleph_1}$ (see [She81, Th.5,pg.31]). Here we prove nevertheless $\diamondsuit^*_{\aleph_1}$ implies every normal first countable space is $\aleph_1 -$ CWH.

The central idea of the proof in §1 is inspired by one key idea in Fleissner [Fle74]. Fleissner implicitly used a stronger combinatorial principle $\diamondsuit_{SS}$. In 1979, the author and independently both Fleissner and Alan D. Taylor all saw (as mentioned in [Tay81], [SS00] that a weaker principle, $\diamondsuit^*_{\omega_1}$, would suffice. Later Smith and Szeptycki [SS00] derive better results. On more recent results on diamond and strong negation see [She10] and references there.

**Convention 1.1.** Below $\delta$ always denotes a limit ordinal ($< \omega_1$).

For transparency, below we refer to the following equivalent form of $\diamondsuit^*_{\omega_1}$.

**Definition 1.2.** Let $\diamondsuit^*_{\aleph_1}$ mean that there exist a sequence $\langle \mathbf{g}_\delta : \delta < \omega_1 \rangle$ where $\mathbf{g}_\delta = \langle \bar{g}^{\delta,k} : k < \omega \rangle$ is of the form $\bar{g}^{\delta,k} = \langle g^{\delta,k}_n : n < \omega \rangle$, where $g^{\delta,k}_n : \delta \to \omega$ has the property that, for any sequence $\bar{g} = \langle g_n : n < \omega \rangle$ with $g_n : \{\delta : \delta < \omega_1\} \to \omega$, there is a club (closed unbounded) set $C \subseteq \omega_1$ such that, for each $\gamma \in C$, there is $k = k(\gamma) \in \omega$ with

$$\bar{g} \restriction \gamma := \langle g_n \restriction \gamma : n < \omega \rangle = \bar{g}^{\gamma,k} = \langle g^{\gamma,k}_n : n < \omega \rangle.$$

**Claim 1.3.** *Assume $\diamondsuit^*_{\aleph_1}$. If $X$ is Hausdorff first countable normal and $|X| = \aleph_1$ then $X$ is* CWH.

*Proof.* Let $\langle \mathbf{g}_\delta : \delta < \omega_1 \rangle$ be as in 1.2.

Without loss of generality $X_* = \{\delta : \delta < \omega_1\} \subseteq X$ and $X_*$ is closed discrete in the space $X$. Let $U^\delta_n (n < \omega)$ be a basis of open neighborhoods of $\delta$ (for $\delta < \omega_1$). We shall define by induction on $\alpha < \omega_1$ a limit ordinal $\gamma_\alpha < \omega_1$ and $\langle f_n(\gamma) : n < \omega, \gamma < \gamma_\alpha \rangle$ such that $\gamma_\alpha$ is increasing continuous with $\alpha$ and $\gamma_0 = 0$. For $\alpha = 0$ choose $\gamma_\alpha = \omega; f_n(\gamma) = 0$. For limit $\alpha$ let $\gamma_\alpha$ be $\cup\{\gamma_\beta : \beta < \alpha\}$ For $\alpha = \beta + 1$ if $\gamma_\alpha > \alpha$ then we let $\gamma_\alpha = \gamma_\beta + \omega$ and let $f_n(\gamma) = 0$ for $\gamma \in [\gamma_\beta, \gamma_\alpha)$. Finally assume that $\alpha = \delta^*, \gamma_{\delta^*} = \delta^*$ so $\delta^* \in X_*$.

We have chosen above the functions $\langle g^{\delta^*,k}_n : n < \omega, k < \omega \rangle$ with $g^{\delta^*,k}_n : \delta^* \to \omega$; now for each $n, k < \omega$ let $A^{\delta^*,n,k}_\ell = \cup\{U^\delta_{g^{\delta^*,k}_n(\delta)} : \delta < \delta^*, f_n(\delta) = \ell\}$ (for $n < \omega, \ell < 2$). Call $k < \omega$ <u>good</u> for $\delta^*$ when for infinitely many (pairs) $n, \ell$ we have

$$B^{\delta^*,n,k}_\ell := \mathrm{cl}(A^{\delta^*,n,k}_\ell) \cap (X_* \setminus \delta^*) \neq \emptyset.$$

We let $\gamma_\alpha = \gamma_{\delta^*+1} = \min\{\delta : \delta > \delta^*$ and if $\ell < 2$ and $n, k < \omega$ and $B^{\delta^*,n,k}_\ell \neq \emptyset$ then $(\delta^*, \delta) \cap B^{\delta^*,n,k}_\ell \neq \emptyset\}$.

Now we choose $f_n \restriction [\delta^*, \gamma_\alpha)$ such that for any $k$ good for $\delta^*$, for some $n, \ell, \delta \geq \delta^*$ we have

$$f_n(\delta) = 1 - \ell( \text{ for } \delta \in \text{cl}(A_\ell^{\delta^*,n,k})).$$

Then we complete arbitrarily the $f_n$ so that their domain is $\gamma_\alpha$.

Thus we have defined $f_n(n < \omega)$ with $f_n : \omega_1 \to 2$. For each $n$ the sets $f_n^{-1}\{1\} \cap X_*, f_n^{-1}\{0\} \cap X_*$ forms a partition of $X_*$, both are closed and discrete subsets of $X$. But $X$ is normal. So there are functions $g_n : X_* \to \omega$ for $n < \omega$ so that letting for $\ell = 0, 1$

$$A_\ell^n = \cup\{U_{g_n(\delta)}^\delta : \delta \in X_*, f_n(\delta) = \ell\}$$

we have

$$A_0^n \cap A_1^n = \emptyset.$$

Let $g_n^+$ be any function from $\omega_1$ to $\omega$ extending $g_n$. For some closed unbounded set $C \subseteq X_*$ we have: $\delta^* \in C \Rightarrow (\exists k)(\langle g_n^+ \restriction \delta^* : n < \omega\rangle = \langle g_n^{\delta^*,k} : n < \omega\rangle)$. Let the first such $k$ be denoted $k(\delta^*)$. Without loss of generality every $\delta^* \in C$ satisfy $\gamma_{\delta^*} = \gamma$ hence if $\delta^* \in C \wedge n < \omega \wedge k < \omega \wedge \ell < 2$ and $B_\ell^{\delta^*,n,k} = \text{cl}(A_\ell^{\delta^*,n,k}) \cap (X_* \backslash \delta^*) \neq \emptyset$ then $\min(B_\ell^{\delta^*,n,k}) < \min(C \backslash \delta^*)$.

For $\delta^* \in C$ now $k(\delta^*)$ cannot be good for $\delta^*$, (by the definition).

Now for at least one $n$ (in fact, for infinitely many $n$-s) we have $\text{cl}(A_\ell^n | \delta^*) \cap (X_* \setminus \delta^*) = \emptyset$ for $\ell \in \{0, 1\}$, let $n(\delta^*)$ be the first such $n$.

Define

$$B_n = \{\delta : \text{ for some } \delta^* \in C \cup \{0\} \text{ we have } \delta^* \leq \delta < \min(C \backslash \delta) \text{ and } n = \max\{n(\delta^*), n(\delta)\}$$

Now $\bigcup_n (g_n \restriction B_n)$ almost exhibits $X_*$ has the right sequence of neighborhoods. Now we can deal with each $B_n$ separately (just choose $\mathscr{U}_n$ by induction on $n$ such that $\mathscr{U}_n$ is open, $\mathscr{U}_n \cap X_* = B_n$ and $\mathscr{U}_n \subseteq X \backslash c\ell(\bigcup_{\ell<n} \mathscr{U}_\ell)$, possible by normality).

By dealing as follows with each interval $[\delta^*, \min(C \backslash (\delta^* + 1))$ for $\delta^* \in C \cup \{0\}$ we have $U_{g_n(\delta)}^\delta (\delta \in B_n)$ as required.

That is, for $\gamma \in C \cup \{0\}$ with $\gamma^+$ its successor in $C$, choose a (countable) family of pairwise disjoint open sets $\mathscr{U}_\gamma(\beta)$ for $\beta \in X_* \wedge \gamma \leq \beta < \gamma^+$, with $\beta \in \mathscr{U}_\gamma(\beta)$, this is possible as in the choice of the $\mathscr{U}_n$'s.

Now for $\beta \in X_*$ we let $W_\beta = \mathscr{U}_{n(\beta)} \cap \mathscr{U}_{\gamma(\beta)}(\beta) \cap \mathscr{U}_{g_{n(\beta)}(\beta)}^\beta$ where:

- $\gamma(\gamma) = \max(C \cap (\beta + 1))$
- $m(\beta) = \max\{n(\delta^*), n((\delta^*)^+) : \delta^* = \max(C \cap \beta) \leq \beta < (\delta^*)^+\}$

Finally $\langle W_\beta : \beta \in X_*\rangle$ is a sequence of pairwise disjoint open sets of $X$ with $\beta \in X_* \Rightarrow \beta \in W_\beta$, so we are done. $\square_{1.3}$

*Remark* 1.4. As in [Fle74] it suffices to assume every point in the space has a neighborhood basis of cardinality $\aleph_1$.

## § 2. Chang Conjecture for Magidor-Malitz Quantifiers

Silver (see [Sil71]) had proved the consistency of Chang conjecture, i.e.

$\oplus$ any model $M$ with universe $\aleph_2$ (and countable signature= vocabulary) $\tau$, has an elementary submodel $N, \|N\| = \aleph_1, |\|N\| \cap \omega_1| = \aleph_0$)

Silver did this by starting with a model $\mathbf{V}$ with $\kappa$ Ramsey (in fact, something weaker suffices), forcing MA and then collapsing $\kappa$ to $\aleph_2$ by $\mathbb{P}^\kappa_{\rm Set} = \{f : {\rm Dom}(f) \subseteq \{\mu : \aleph_1 < \mu < \kappa, \mu \text{ a cardinal}\}$ has cardinality $\leq \aleph_1$, and for some $\alpha < \omega_1$, $(\forall \mu \in {\rm Dom} f)(f(\mu)$ is a function from $\alpha$ to $\mu)\}$. See also Koszmider [Kos05] for a topological application.

We can ask whether this submodel $N$ can inherit more properties from $M$.

**Definition 2.1.** Let us define a (technical variant of) Magidor-Malitz quantifiers.

$M \models (Q^n \bar{x})\varphi(x_1, \ldots, x_n)$ means that there is a set $A \subseteq M, A$ is of cardinality $\|M\|$ such that $(\forall a_1 \ldots a_n \in A)\varphi(a_1 \ldots a_n)$.

The result is that:

**Claim 2.2.** *In $\oplus$ above, we can have $N$ an elementary submodel of $M$ even for the logic $\mathbb{L}(Q^0, Q^1, \ldots)_{n<\omega}$. So e.g. Suslinity of trees is preserved.*

For this we need the following.

**Definition 2.3.** Call a forcing $\mathbb{P}$ <u>suitable</u> <u>when</u> for any sequence $\langle p_i : i < \omega_1\rangle$ of members of $\mathbb{P}$ there is a set $\mathscr{U} \subseteq \omega_1$ of cardinality $\aleph_1$ such that: for any finite $u \subseteq \mathscr{U}$ there is $q \in \mathbb{P}$ such that $\bigwedge\limits_{i \in u} q \geq p_i$.

**Claim 2.4.** *Forcing by suitable forcing preserves satisfaction of sentences of Magidor-Malitz quantifiers for models of power $\aleph_1$.*

*Proof.* See [BJ95, 1.5-13,pg.34]. $\square_{2.4}$

**Claim 2.5.** *There is a suitable forcing $\mathbb{P}, |\mathbb{P}| = 2^{\aleph_1}$, such that in $\mathbf{V}^{\mathbb{P}}$: if $\mathbb{Q}$ is a suitable forcing of power $\aleph_1, \underset{\sim}{M}$ a $\mathbb{Q}$-name of a model of power $\aleph_1$, in a language $L \in \mathbf{V}$, universe $\aleph_1$, <u>then</u> there is a directed $\mathbf{G} \subseteq \mathbb{P}$, which determines $\underset{\sim}{M}$ as $M$ and* such that *for any sentence $\psi$ from the $\mathbb{L}(Q^0, Q^1, \ldots)$ (the variant of Magidor-Malitz logic from Definition 2.1)*

$$\Vdash_{\mathbb{Q}} \text{“}\underset{\sim}{M} \models \psi\text{” implies } M \models \psi.$$

*Proof.* Just iterate the required forcings, with direct limit (i.e. finite support) and remembering it is known that suitability is preserved under iteration, i.e. 2.4.

<u>Proof of Main result 2.2</u>:

Do as Silver, start with $\mathbf{V} \models$ “$\kappa$ Ramsey”, force by $\mathbb{P}$ <u>from Claim 2.5</u>, and then use $\mathbb{P}^\kappa_{\rm Set}$. The rest is as in his proof.

But we have to choose $G$ as in Claim 2.5, and notice that more is reflected to the submodel he uses, (just check the definition carefully) and work a little, and remember that $\aleph_1$-complete forcing preserves satisfaction of sentences in $\mathbb{L}(Q^0, \ldots)$ (and $\mathbb{P}$ is $\aleph_1$-complete). $\square_{2.5}$

## § 3. A Remark on the Monadic Theory of Order

In [She75] we prove the undecidability of the monadic theory of (the order) $R$, assuming CH, or the weaker Baire-like hypothesis that $\mathbb{R}$ is not the union of fewer than continuum sets of first category sets. This condition is weaken below to "not (St) at least for $T$ where a closely related theory is the monadic theory $T$ of $M = ({}^{\omega\geq}2, \triangleleft)$ where ${}^{\omega\geq}2$ is the set of sequences of zeros and ones of length $\leq \omega, \triangleleft$ is the (partial) order of being initial segment. $T$ is closely related to Rabin's monadic theory of $({}^{\omega>}2, \triangleleft)$ which he proved decidable [M.O69]. It is still unknown whether we can prove those results in ZFC. We prove here that a statement "not(St)" implies the undecidability of $T$ (and all results on its complexity, see [She75] and the paper of Gurevich on the subject) <u>but</u> it is not clear (at that time) whether (St) is consistent with ZFC.

**Definition 3.1.** A Cantor [set] $C$ is a non-empty subset of ${}^{\omega\geq}2$ with the properties

(a) $C$ is closed under initial segments,
(b) if $\eta$ has length $\omega$ then $\eta \in C \equiv (\forall n)(\eta{\restriction}n \in C)$,
(c) $\eta \in C \cap {}^{\omega>}2$ implies $\eta \frown \langle 0\rangle \in C$ or $\eta \frown \langle 1\rangle \in C$,
(d) for every $\eta \in C \cap {}^{\omega>}2$, there is $\nu \in C \cap {}^{\omega>}2$ such that $\eta \triangleleft \nu$ <u>and</u> $\nu \frown \langle 0\rangle \in C, \nu \frown \langle 1\rangle \in C$.

**Definition 3.2.** 1) For a Cantor $C$, the set of its splitting points is $\mathrm{Sp}(C) = \{\eta \in {}^{\omega>}2 : \eta \frown \langle 0\rangle \in C$ and $\eta \frown \langle 1\rangle \in C\}$.
2) For a set $A \subseteq {}^{\omega>}2, C$ is an $A$-Cantor, if $\mathrm{Sp}(C) \subseteq A$.
3) For a set $S \subseteq \omega, C$ is called an $S$-Cantor, if

$$\mathrm{Sp}(C) \subseteq \bigcup_{n\in S} {}^{n}2.$$

4) An odd Cantor is one that is an $\{2n+1 : n < \omega\}$-Cantor. An even Cantor is one that is an $\{2n : n < \omega\}$-Cantor.

Now the statement we speak about is

**Definition 3.3.** Let (St) mean: the set ${}^{\omega}2$ is the union of less than $2^{\aleph_0}$ Cantors each of them odd or even.

**Problem 3.4.** Is (St) consistent with ZFC?

**Claim 3.5.** *Let $\{C_i : i < \alpha\}$ be a family of odd and even Cantors, ${}^{\omega\geq}2 = \bigcup_{i<\alpha} C_i$. Then $2^{\aleph_0} \leq |\alpha|^+$.*

*Proof.* Let for $\eta, \nu \in {}^{\omega}2, \rho = p(\eta,\nu)$ be defined by $\rho(2n) = \eta(n), \rho(2n+1) = \nu(n)$, and then let $\eta = \mathrm{pr}_1(\rho), \nu = \mathrm{pr}_2(\rho)$.

Now for any even $C$, and $\eta$ there is at most one $\nu$ such that $p(\eta,\nu) \in C$; why? if $\nu_0, \nu_1$ are such $\nu$'s, $\rho_\ell = p(\eta,\nu_\ell)$, then, by the definition of $p(-,-)$, for some $m < \omega, \rho_0{\restriction}m = \rho_1{\restriction}m, \rho_0(m) \neq \rho_1(m)$. If $m = 2n$ then $\rho_\ell(m) = \rho_\ell(2n) = \eta(n)$ so they are equal, contradiction. If $m = 2n+1$, then $(\rho_0(m) \neq \rho_1(m)$ and) $\rho_0{\restriction}m = \rho_1{\restriction}m$ is a splitting point of $C$, however $m$ is odd and $C$ is an even Cantor, a contradiction. So really there is at most one $\nu$, and let $\varrho(\eta, C)$ be the unique $\nu$ such that $p(\eta,\nu) \in C$ if there is one and $\eta$ otherwise.

Similarly if $C$ is odd and $\eta \in {}^\omega 2$, then for at most one $\nu, p(\nu,\eta) \in C$ and let $\varrho(\eta, C)$ be $\nu$ for this $\eta$, and let $\varrho(\eta, C) = \eta$ otherwise. Our definition of the function $\varrho$ does not contradict, because no Cantor is odd and even.

Let for $\eta \in {}^\omega 2, \mathrm{Dp}(\eta) = \{\varrho(\eta, C_i) : i < \alpha\}$. So clearly $\mathrm{Dp}(\eta)$ is a subset of ${}^\omega 2$ of cardinality $\leq |\alpha|$.

Now if $\eta, \nu \in {}^\omega 2$, by hypothesis $\rho = p(\eta,\nu)$ belongs to some $C_i$. If $C_i$ is odd this implies $\nu = \varrho(\eta, C_i) \in \mathrm{Dp}(\eta)$ and if $C_i$ is even this implies $\eta = \varrho(C_i, \nu) \in \mathrm{Dp}(\nu)$.

If $|\alpha|^+ < 2^{\aleph_0}$ we can easily find a counterexample. $\square_{3.5}$

**Claim 3.6.** *Assume* $\neg$(St).

*1) If $S_n \subseteq \omega$ are infinite pairwise almost disjoint (for $n \in \{0,1,2\}$), $C_i (i < \alpha < 2^{\aleph_0})$ are Cantors, each an $S_n$-Cantor for some $n$ (or just an $S_n \cup S_2$-Contor for some $n$), $C$ is a Cantor* such that *for every $\eta \in C \cap {}^{\omega>}2, \ell \in \{0,1\}$, there is $\nu$, such that $\eta \triangleleft \nu \in \mathrm{Sp}(C), \nu \in \bigcup_{k \in S_\ell} {}^k 2$.*

*Then there is $\eta \in C \setminus \bigcup_{i<\alpha} C_i \setminus {}^{\omega>}2$.*

*2) Similarly for $S_n \subseteq {}^{\omega>}2$*

*Proof.* 1) We can find a Cantor $C' \subseteq C$, and $0 = k(0) < k(1) < \ldots < k(n) < \ldots < \omega$ such that:

(∗) if $\eta \in {}^{k(n)}2$, then there are exactly two $\nu \in {}^{k(n+1)}2 \cap C', \eta \triangleleft \nu$, and if they are $\nu_1, \nu_2$ and $m := \min\{m : \nu_1(m) \neq \nu_2(m)\}$ then $m \in S_0 \cup S_1$ but $\notin S_2 \cup (S_0, \cap S_1)$. Moreover $m \in S_0$ iff $n$ is even.

Let $A = \{\eta \restriction k(n) : n < \omega, \eta \in C'\}$, so $A \subseteq C'$. Clearly there is an isomorphism $f$, of the models $({}^{\omega\geq}2, \triangleleft), (C', \triangleleft)$.

Let $C'_i = \{f(\eta) : \eta \in C', \eta \in C_i\}$, it is easy to check that each $C'_i$ is countable, or the union of a countable set and a Cantor which is odd or is even.

We can find odd Cantor $C'_i (\alpha \leq i < \alpha\omega)$ such that all countable sets we mentioned are covered by them. Now by - "not (St)" there is $\eta \in {}^{\omega\geq}2$ such that $\eta \notin \bigcup_{i<\alpha\omega} C'_i$ (as $\alpha\omega < 2^{\aleph_0}$) and $f^{-1}(\eta)$ is the required elements.

2) Similarly. $\square_{3.6}$

Now[1]

**Claim 3.7.** *Assume* $\neg$(St).

*1) The monadic theory $T$ is undecidable.*

*Proof.* Below let $P$ vary on Cantors and not that We can repeat the proof of [She75] with small adaptation (and prove $T$ is undecidable). That is, the change needed is in [She75, 7.4] which has a set-theoretic hypothesis (CH or the Baire-like hypothesis mentioned above), so we repeat it with the needed changes below.

$\square_{3.7}$

**Lemma 3.8.** *Assume not(*St*) and let $J$ be an index-set of cardinality at most $2^{\aleph_0}$,*

[1]We have added 3.7(1) and 3.8 in 2019

*1) Assume the* $D_i (i \in J)$ *countable dense subsets of* ${}^{\omega>}2$ *and* $D = \bigcup_{i \in J} D_i$ *and* $\bar{D} = \langle D_i : i \in J \rangle$[2]. *Then there is* $Q \subseteq {}^{\omega}2 \backslash D, Q = Q[\bar{D}]$ *such that for every Cantor* $P$ *:*

(A) *if* $P \cap D \subseteq D_i (i \in J)$ *and* $D_i$ *is dense in* $P$ *then* $|P \cap Q| < 2^{\aleph_0}$

(B) *if for some* $i \in J$ *the sets* $P \cap D_i, P \setminus D_i$ *are dense in* $P$ *then* $P \cap Q \neq \emptyset$.

*2) For some such* $\bar{D}$ *we can strengthen clause (B) above to*

(B) *if* $P$ *is a Cantor and for every* $i \in J$ *the set* $D_i \cap P$ *is nowhere-dense in* $P$ *then for every , dense subsets* $D_1^*, D_2^*$ *of* $P \cap D$ *we can find* $D_1^\bullet \subseteq D_1^*, D_2^\bullet \subseteq D_2^*$ *satisfying for any* $P$ *we have: is* $P \cap D_1^\bullet, P \cap D_2^\bullet$ *are dense in* $P$ *then* $P \cap Q \neq \emptyset$.

*Proof.* 1) Let $\{P_\alpha : 0 < \alpha < 2^{\aleph_0}\}$ be any enumeration of the Cantor sets. We define $x_\alpha, \alpha < 2^{\aleph_0}$ by induction on $\alpha$.

For $\alpha = 0, x_\alpha \in \mathbb{R}$ is arbitrary.

For any $\alpha > 0$, if $P_\alpha$ does not satisfy the assumptions of (B) then let $x_\alpha = x_0$ and if $P$ satisfies the assumptions of (B) (hence in particular $D$ is dense in $P$) let $x_\alpha \in P_\alpha - \bigcup\{P_\beta : \beta < \alpha, (\exists i \in J)(P_\beta \cap D \subseteq D_i$ and $D$ is dense in $P_\beta)\} = D$.

This is possible; to prove this let $\mathscr{U} = \{\beta < \alpha :$ there is $i \in J$ such that $P_\beta \cap D \subseteq D_i\}$ and for $\beta \in \mathscr{U}$ let $i_\beta \in J$ be such that $P_\beta \subseteq D_{i_\beta}$ Let $i(*) \in J$ be such that $P \cap D_{i(*)}, P \setminus D_{i(*)}$ are dense in $P$. Now we apply 3.6(2), (or 3.6(1) if we restrict the $D_i$-s, does not matter)

So by (St) and the hypothesis $|P_\alpha \cap D| < 2^{\aleph_0}$ there exists such $x_\alpha$.

Now let $Q = \{x_\alpha : \alpha < 2^{\aleph_0}\}$. If $P$ satisfies the assumptions of (A), then $P \in \{P_\alpha : 0 < \alpha < 2^{\aleph_0}\}$. Hence for some $\alpha, P = P_\alpha$, hence $P \cap D \subseteq \{x_\beta : \beta < \alpha\}$, so $|P \cap D| < 2^{\aleph_0}$. If $P = P_\alpha$ satisfies the assumption of (B) then $x_\alpha \in P_\alpha, x_\alpha \in Q$, hence $P_\alpha \cap Q \neq \emptyset$.

2) Similarly.

So we have proved the lemma. $\square_{3.8}$

*Remark* 3.9. We can interpret the monadic theory of $(R, <)$ in $T$, but the converse was not clear at the time, but looking at it again probably we can carry the proof for $\mathbb{R}$.

## References


[BJ95] Tomek Bartoszyński and Haim Judah, *Set theory: On the structure of the real line*, A K Peters, Wellesley, Massachusetts, 1995.

[BS87] Andreas R. Blass and Saharon Shelah, *There may be simple* $P_{\aleph_1}$*- and* $P_{\aleph_2}$*-points and the Rudin-Keisler ordering may be downward directed*, Ann. Pure Appl. Logic **33** (1987), no. 3, 213–243. MR 879489

[DT18] Alan Dow and Franklin D. Tall, *Normality versus paracompactness in locally compact spaces*, Canad. J. Math. **70** (2018), no. 1, 74–96. MR 3744886

[Fle74] William G. Fleissner, *Normal moore spaces in the constructible universe*, Proceedings of the American Mathematical Society **46** (1974), 294–298.

[GGK04] Stefan Geschke, Martin Goldstern, and Menachem Kojman, *Continuous Ramsey theory on Polish spaces and covering the plane by functions*, J. Math. Log. **4** (2004), no. 2, 109–145.


[2] The main case is that the $D_i$-s are pairwise almost disjoint

[GKKS02] S. Geschke, M. Kojman, W. Kubiś, and R. Schipperus, *Convex decompositions in the plane and continuous pair colorings of the irrationals*, Israel J. Math. **131** (2002), 285–317.

[GS82] Yuri Gurevich and Saharon Shelah, *Monadic theory of order and topology in ZFC*, Ann. Math. Logic **23** (1982), no. 2-3, 179–198 (1983). MR 701125

[HU17] Yair Hayut and Spencer Unger, *The strong tree property and weak square*, MLQ Math. Log. Q. **63** (2017), no. 1-2, 150–154.

[Kos05] P. Koszmider, *Projections in weakly compactly generated Banach spaces and Chang's conjecture*, J. Appl. Anal. **11** (2005), no. 2, 187–205.

[M.O69] M.O.Rabin, *Decidability of second order theories and automata on infinite trees*, Trans. A.M.S. **141** (1969), 1–35.

[Shea] Saharon Shelah, *Applying set theory.*

[Sheb] ———, *Dependent dreams: recounting types*, arXiv: 1202.5795.

[She75] ———, *The monadic theory of order*, Ann. of Math. (2) **102** (1975), no. 3, 379–419. MR 0491120

[She81] ———, *On Fleissner's diamond*, Notre Dame J. Formal Logic **22** (1981), no. 1, 29–35. MR 603754

[She85] ———, *Monadic logic and Löwenheim numbers*, Ann. Pure Appl. Logic **28** (1985), no. 2, 203–216. MR 779162

[She88] ———, *Notes on monadic logic. Part A. Monadic theory of the real line*, Israel J. Math. **63** (1988), no. 3, 335–352. MR 969946

[She10] ———, *Diamonds*, Proc. Amer. Math. Soc. **138** (2010), no. 6, 2151–2161, arXiv: 0711.3030. MR 2596054

[Sil71] Jack Silver, *The independence of Kurepa's conjecture and two-cardinal conjectures in model theory*, Axiomatic Set Theory (Proc. Sympos. Pure Math., Vol. XIII, Part I, Univ. California, Los Angeles, Calif., 1967), Amer. Math. Soc., Providence, R.I., 1971, pp. 383–390.

[SS00] Kerry D. Smith and Paul J. Szeptycki, *Paranormal spaces under* $\diamondsuit^*$, Proc. Amer. Math. Soc. **128** (2000), no. 3, 903–908.

[Tay81] Alan D. Taylor, *Diamond principles, ideals and the normal Moore space problem*, Canadian J. Math. **33** (1981), no. 2, 282–296.



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