

## MORE ON WEAK DIAMOND

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ABSTRACT. We deal with the combinatorial principle Weak Diamond. We prove that, if it holds for a given cardinal, we can get this principle with more than two colours or some relevant ideal is not too saturated. Then we point out a model theoretic consequence of Weak Diamond.

### 0. BASIC DEFINITIONS

In this section we present basic notations, definitions and results.

The paper was circulated (including the math arXiv) and accepted to the East-West Journal of Math around 2000, but due to some problems between the editors has not appeared. Meanwhile Asperó, Larson and Moore [ALM13] with a related result was done. Weak diamond was introduced in [DS78], lately see [GS21].

- Notation 0.1.*
- (1)  $\kappa, \lambda, \theta, \mu$  will denote cardinal numbers and  $\alpha, \beta, \delta, \varepsilon, \xi, \zeta, \gamma$  will be used to denote ordinals.
  - (2) Sequences of ordinals are denoted by  $\nu, \eta, \rho$  (with possible indexes).
  - (3) The length of a sequence  $\eta$  is  $\ell g(\eta)$ .
  - (4) For a sequence  $\eta$  and  $\ell \leq \ell g(\eta)$ ,  $\eta \upharpoonright \ell$  is the restriction of the sequence  $\eta$  to  $\ell$  (so  $\ell g(\eta \upharpoonright \ell) = \ell$ ). If a sequence  $\nu$  is a proper initial segment of a sequence  $\eta$  then we write  $\nu \triangleleft \eta$  (and  $\nu \trianglelefteq \eta$  has the obvious meaning).
  - (5) For a set  $A$  and an ordinal  $\alpha$ ,  $\alpha_A$  stands for the function on  $A$  which is constantly equal to  $\alpha$ .
  - (6) For a model  $M$ ,  $|M|$  stands for the universe of the model.
  - (7) The cardinality of a set  $X$  is denoted by  $\|X\|$ . The cardinality of the universe of a model  $M$  is denoted by  $\|M\|$ .

**Definition 0.2.** Let  $\lambda$  be a regular uncountable cardinal and  $\theta$  be a cardinal number, possibly finite.

- (1) A  $(\lambda, \theta)$ -colouring is a function  $F : \text{DOM} \rightarrow \theta$ , where  $\text{DOM}$  is either  ${}^{<\lambda}2 = \bigcup_{\alpha < \lambda} {}^\alpha 2$  or  $\bigcup_{\alpha < \lambda} {}^\alpha (\mathcal{H}(\lambda))$ . In the first case we will write  $\text{DOM}_\alpha = \text{DOM}_\alpha(F) = {}^{1+\alpha}2$ , in the second case we let  $\text{DOM}_\alpha = \text{DOM}_\alpha(F) = {}^{1+\alpha}(\mathcal{H}(\lambda))$  (for  $\alpha \leq \lambda$ ).

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If the choice does not matter we shall not mention it; for the main definitions the choice does not matter, see 1.10.

If  $\lambda$  is understood from the context, we may omit it; if  $\theta = 2$  then we may omit it (thus a  $\lambda$ -colouring means a  $(\lambda, 2)$ -colouring and a *colouring* is a  $(\lambda, 2)$ -colouring).

- (2) For a  $(\lambda, \theta)$ -colouring  $F$  and a set  $S \subseteq \lambda$ , we say that a function  $\eta \in {}^S \theta$  is an  $F$ -weak diamond sequence for  $S$  when for every  $f \in \text{DOM}_\lambda$  the set

$$\{\delta \in S : \eta(\delta) = F(f \upharpoonright \delta)\}$$

is stationary.

- (3)  $\text{WdId}_\lambda$  is the collection of all sets  $S \subseteq \lambda$  such that for some colouring  $F$  there is no  $F$ -weak diamond sequence for  $S$ .

*Remark 0.3.* In the definition of  $\text{WdId}_\lambda$  (0.2(3)), the choice of  $\text{DOM}$  (see 0.2(1)) does not matter; see [She98, AP, §1], remember that  $\|\mathcal{H}(\lambda)\| = 2^{<\lambda}$ .

**Theorem 0.4.** [Devlin and Shelah [DS78]; see [She98, AP, §1] too]  
Assume that  $2^\theta = 2^{<\lambda} < 2^\lambda$  (e.g.  $\lambda = \mu^+$ ,  $2^\mu < 2^\lambda$ ). Then for every  $\lambda$ -colouring  $F$  there exists an  $F$ -weak diamond sequence for  $\lambda$ . Moreover,  $\text{WdId}_\lambda$  is a normal ideal on  $\lambda$  (and  $\lambda \notin \text{WdId}_\lambda$ ).

*Remark 0.5.* One could wonder why the weak diamond (and  $\text{WdId}_\lambda$ ) is interesting. Below we list some of the applications, limitations and related problems.

- (1) Weak diamond is really weaker than diamond, but provably (in ZFC) it holds true for some cardinals  $\lambda$ . Note that under GCH,  $\diamond_{\mu^+}$  holds true for each  $\mu > \aleph_0$ , so the only interesting case then is  $\lambda = \aleph_1$ .
- (2) Original interest in this combinatorial principle comes from interest in Whitehead groups:
 

if  $G$  is a strongly  $\lambda$ -free Abelian group and  $\Gamma(G) \notin \text{WdId}_\lambda$   
then  $G$  is not Whitehead.
- (3) A related question was: can we have stationary subsets  $S_1, S_2 \subseteq \omega_1$  such that  $\diamond_{S_1}$  but  $\neg \diamond_{S_2}$ ? (See [She77].)
- (4) Weak diamond has been helpful particularly in problems where we have some uniformity, e.g.:
  - (\*)<sub>1</sub> Assume  $2^\lambda < 2^{\lambda^+}$ . Let  $\psi \in \mathbb{L}_{\lambda^+, \omega}$  be categorical in  $\lambda, \lambda^+$ .  
Then  $(\text{MOD}_\psi, \prec_{\text{Frag}(\psi)})$  has the amalgamation property in  $\lambda$ .
  - (\*)<sub>2</sub> If  $G$  is a group of cardinality  $\lambda > \aleph_0$  then we can find subgroups  $G_i$  of  $G$  (for  $i < \lambda$ ) non-conjugate in pairs (see [She87b]).
- (5) One may wonder if assuming  $\lambda = \mu^+$ ,  $2^\lambda > 2^\mu$  (and e.g.  $\mu$  regular) we may find a regular  $\sigma < \mu$  such that

$$\{\delta < \lambda : \text{cf}(\delta) = \sigma\} \notin \text{WdId}_\lambda(\lambda).$$

Of course, by the “normal ideal” result (see 0.4), it follows that there is such  $\sigma$ , but does  $\sigma$  depends on the present set theory? e.g. does it hold for every regular  $\sigma \neq \text{cf}(\mu)$  below  $\mu$ ?

Unfortunately, this is not the case (see [She85] even for  $\mu = \aleph_1$ ).

- (6) We would like to prove
- (a)  $\text{WdId}_\lambda$  is not  $\lambda^+$ -saturated or
  - (b) a strengthening, e.g. weak diamond for more (than two) colours.
- We will get (a variant of) a local version of the disjunction, where we essentially fix  $F$ . There are two reasons for interest in **(a)**: understanding  $\lambda^+$ -saturated normal ideals (e.g. we get more information on the case  $\text{CH} + “\mathcal{D}_{\omega_1}$  is  $\aleph_2$ -saturated”); see also Zapletal Shelah [SZ99]), and non  $\lambda^+$ -saturation helps in “non-structure theorems in model theory” (see [She83], [She01], [She09a], [She09b]). That is, having  $2^\mu < 2^{\mu^+} < 2^{\mu^{++}}$  and some “bad” (i.e. “non-structure”) properties for models in  $\mu$  we get  $2^{\mu^{++}}$  models in  $\mu^{++}$  when  $\text{WdId}_\lambda$  is not  $\lambda^{++}$ -saturated (and using the local version does not hurt).
- (7) Note that for  $S \notin \text{WdId}_\lambda$  we have a weak diamond sequence  $f \in {}^S 2$  such that the set of “successes” (=equalities) is stationary, but it does not have to be in  $(\text{WdId}_\lambda)^+$ . We would like to start and end in the same place: being positive for the same ideal. Also, in **(b)** above the set of places we guess was stationary, when we start with  $S \in (\text{WdId}_\lambda)^+$ .

Note that it may well be that  $\lambda \in \text{WdId}_\lambda$  (if  $(\exists \theta < \lambda)(2^\theta = 2^\lambda)$  this holds), but some “local” versions may still hold. E.g. in the Easton model, we have  $F$ -weak diamond sequences for all  $F$  which are reasonably definable (see [She98, AP, §1]; define e.g.

$$F(f) = 1 \Leftrightarrow L[X, f] \models \varphi(X, f)$$

for a fixed first order formula  $\varphi$ , where  $X \subseteq \lambda$  depends on  $F$  only). So the case  $\text{WdId}_\lambda = \mathcal{P}(\lambda)$  has some interest.

- (8) Related later works are [She08], [She20].

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## 1. WHEN COLOURINGS ARE ALMOST CONSTANT

In 1.1, 1.2 we now “slice”  $\text{WdId}_\lambda$  by finer approximations meaningful even when  $\text{WdId}_\lambda$  fails.

**Definition 1.1.** Let  $\lambda$  be a regular uncountable cardinal.

- (1) Let  $F$  be a  $(\lambda, \theta)$ -colouring.
- (a) Let  $S \subseteq \lambda$ . We say that a sequence  $\eta \in {}^S \theta$  is coded by  $F$  if there exists  $f \in \text{DOM}_\lambda$  such that

$$\alpha \in S \Leftrightarrow \eta(\alpha) = F(f \upharpoonright (1 + \alpha)).$$

(b) We let

$$\mathfrak{B}(F) \stackrel{\text{def}}{=} \{\eta \in {}^\lambda\theta : \eta \text{ is coded by } F\}.$$

- (2) For a family  $\mathcal{A}$  of subsets of  $\lambda$  let  $\text{ideal}_\lambda(\mathcal{A})$  be the  $\lambda$ -complete normal ideal on  $\lambda$  generated by  $\mathcal{A}$  (i.e. it is the closure of  $\mathcal{A}$  under unions of  $< \lambda$  elements, diagonal unions, containing singletons, and subsets).

[Note that  $\text{ideal}_\lambda(\mathcal{A})$  does not have to be a proper ideal.]

- (3) For a  $\lambda$ -colouring  $F$  (so  $\theta = 2$ ) we define by induction on  $\alpha$ :

$$\text{ID}_0^-(F) = \emptyset, \quad \text{ID}_0(F) = \{S \subseteq \lambda : S \text{ is not stationary}\},$$

for a limit  $\alpha$

$$\text{ID}_\alpha^-(F) = \bigcup_{\beta < \alpha} \text{ID}_\beta(F), \quad \text{ID}_\alpha(F) = \text{ideal}_\lambda\left(\bigcup_{\beta < \alpha} \text{ID}_\beta(F)\right),$$

and<sup>1</sup> for  $\alpha = \beta + 1$

$$\text{ID}_\alpha^-(F) = \{S \subseteq \lambda : \text{for each } S^* \subseteq S \text{ there is } f \in \text{DOM}_\lambda \text{ such that } \{\delta < \lambda : \delta \in S^* \Leftrightarrow F(f \upharpoonright \delta) = 0\} \in \text{ID}_\beta(F)\};$$

$$\text{ID}_\alpha(F) = \text{ideal}_\lambda(\text{ID}_\alpha^-(F)).$$

Finally we let  $\text{ID}^a(F) = \bigcup_\alpha \text{ID}_\alpha(F)$ .

- (4) We say that  $F$  is *rich* when if  $\text{DOM}(F) = \bigcup_{\alpha < \lambda} {}^\alpha\mathcal{H}(\lambda)$ , and for every function  $f \in \text{DOM}_\lambda$  and  $\alpha < \lambda$  and a set  $A \subseteq \alpha$  there is  $f' \in \text{DOM}_\lambda$  such that:

$$(\forall i \in (\lambda \setminus \alpha)(f(1+i) = f'(1+i) \ \& \ F(f \upharpoonright (\alpha+i)) = F(f' \upharpoonright (\alpha+i)))$$

$$\text{and } (\forall j \in \alpha \setminus \{0\})(F(f' \upharpoonright j) = 1 \Leftrightarrow j \in A).$$

**Definition 1.2.** Let  $\lambda$  be a regular uncountable cardinal and let  $F$  be a  $\lambda$ -colouring.

- (1)  $\text{WDmId}_\lambda(F)$  is the family of all sets  $S \subseteq \lambda$  with the property that for every  $S^* \subseteq S$  there is  $f \in \text{DOM}_\lambda$  such that the set

$$\{\delta \in S : \delta \in S^* \Leftrightarrow F(f \upharpoonright \delta) = 1\}$$

is not stationary, (note, the difference with 0.2(3)).

- (2)  $\mathfrak{B}^+(F)$  is the closure of (see 1.1(1)):

$$\mathfrak{B}(F) \cup \{S \subseteq \lambda : S \text{ is not stationary}\}$$

under unions of  $< \lambda$  sets, complement and diagonal unions (here, in  $\mathfrak{B}(F)$ , we identify a subset of  $\lambda$  with its characteristic function).

- (3)  $\text{ID}^b \stackrel{\text{def}}{=} \{S \subseteq \lambda : \text{for every } Y \subseteq S \text{ for some } B \in \mathfrak{B}^+(F) \text{ we have } B \cap S = Y\}$

- (4)  $\text{ID}^c(F)$  is the collection of all  $S \subseteq \lambda$  such that for some  $X \in \mathfrak{B}^+(F)$  we have:  $S \subseteq X$  and there is a partition  $X_0, X_1$  of  $X$  such that  $(\alpha) \mathcal{P}(X_\ell) = \{Y \cap X_\ell : Y \in \mathfrak{B}^+(F)\}$  for  $\ell = 0, 1$ , and

<sup>1</sup> Note that  $\text{ID}_\alpha^-(F) \neq \emptyset$  iff  $\alpha > 0$

( $\beta$ ) for  $\ell < 2$  there is no  $Y \in \mathfrak{B}^+(F)$  satisfying

$$Y \setminus X_\ell \in \text{ID}^b(F) \quad \& \quad Y \notin \text{ID}^b(F).$$

**Proposition 1.3.** *Assume  $\lambda$  is a regular uncountable cardinal and  $F$  is a  $\lambda$ -colouring.*

- (1)  $\text{ideal}_\lambda(\mathcal{A})$  is the collection of all diagonal unions  $\bigvee_{\xi < \lambda} A_\xi$  such that  $A_\xi \in \mathcal{A}$  for  $\xi < \lambda$ , when  $\mathcal{A}$  is a family of subsets of  $\lambda$  such that
  - ( $\otimes_{\mathcal{A}}$ ) if  $S_0 \subseteq S_1$  and  $S_1 \in \mathcal{A}$  and  $A \in [\lambda]^{<\lambda}$  then  $S_0 \cup A \in \mathcal{A}$ ,
- (2) The condition ( $\otimes_{\text{ID}_\alpha^-(F)}$ ) (see above) is true for each  $\alpha$ . Consequently, if  $\alpha = \beta + 1$  then  $\text{ID}_\alpha(F) = \{ \bigvee_{i < \lambda} A_i : \langle A_i : i < \lambda \rangle \subseteq \text{ID}_\alpha^-(F) \}$ , and if  $\alpha$  is limit then  $\text{ID}_\alpha(F) = \{ \bigvee_{i < \lambda} A_i : \langle A_i : i < \lambda \rangle \subseteq \bigcup_{\beta < \alpha} \text{ID}_\beta(F) \}$ .
- (3)  $\text{ID}^a(F)$  and  $\text{ID}_\alpha(F)$  are  $\lambda$ -complete normal ideals on  $\lambda$  extending the ideal of non-stationary subsets of  $\lambda$  (but they do not have to be proper). For  $\alpha < \gamma$  we have  $\text{ID}_\alpha(F) \subseteq \text{ID}_\gamma(F)$  and hence  $\text{ID}^a(F) = \text{ID}_\alpha(F)$  for every large enough  $\alpha < (2^\lambda)^+$ .
- (4) Suppose  $\bar{B} = \langle B_\ell : \ell \leq m \rangle$ , where  $B_\ell \subseteq B_{\ell+1}$  (for  $\ell < m$ ) and  $B_m \in \text{ID}^a(F)$ . Then  $\bar{B}$  has an  $F$ -representation, which means that there are a well founded tree  $T \subseteq \omega^{>\lambda}$ , sequences  $\langle B_\eta^\ell : \eta \in T, \ell \leq \ell_\eta \rangle$ , and  $\langle f_\eta^k : \eta \in T, k \leq k_\eta \rangle$  such that  $k_\eta \leq \ell_\eta + 1 \leq m + 1$  and
  - (a)  $B_\emptyset^\ell = B_\emptyset$ ,  $\ell_\emptyset = m$ ,  $B_\eta^\ell \subseteq B_\eta^{\ell+1} \subseteq \lambda$ ,  $f_\eta^\ell \in {}^\lambda 2$ ,
  - (b)  $(\forall \eta \in T \setminus \max(T))(\forall i < \lambda)(\eta \frown \langle i \rangle \in T)$ ,
  - (c) for each  $\eta \in T \setminus \max(T)$  there is  $\alpha_\eta < \lambda$  such that for all  $\ell \leq \ell_\eta$ ,  $\delta \in \lambda \setminus \alpha_\eta$  we have:
    - ( $\oplus$ )  $\delta \in B_\eta^\ell$  iff
 
$$(\exists i < \delta)(\delta \in B_{\eta \frown \langle i \rangle}^\ell) \text{ or}$$

$$F(f_\eta^\ell \upharpoonright \delta) = 1 \ \& \ \neg(\exists i < \delta)(\exists k)(\delta \in B_{\eta \frown \langle i \rangle}^k),$$
  - (d) for each  $\eta \in \max(T)$ ,  $B_\eta$  is a bounded subset of  $\lambda$  with  $\min(B_\eta) > \sup(\{\eta(n) : n < \ell g(\eta)\})$ .
- (5) If for some  $f^* \in {}^\lambda 2$  we have  $(\forall \alpha < \lambda)(F(f^* \upharpoonright \alpha) = 0)$  then in part (4) above we can demand that  $k_\eta = \ell_\eta + 1$ .
- (6) If  $F$  is rich then in part (4) above we can add
  - (e)  $\alpha_\eta = 0$  for  $\eta \in T \setminus \max(T)$  and  $B_\eta = \emptyset$  for  $\eta \in \max(T)$ .
- (7)  $\text{ID}^a(F)$  is the minimal normal ideal  $D$  on  $\lambda$  such that there is no  $S \in D^+$  satisfying
 
$$(\forall S^* \subseteq S)(\exists A \in \mathfrak{B}(F))(S^* \triangle A \in D).$$
- (8) If  $X \in \mathcal{P}(\lambda) \setminus \text{ID}^b(F)$  then there is  $\eta \in {}^\lambda 2$  which is a weak diamond even modulo  $\text{ID}^b(F)$  which mean that:
 

for every  $f \in \text{DOM}(F)$  we have:  $\{\delta \in X : F(f \upharpoonright \delta) = \eta(\delta)\} \neq \emptyset \pmod{\text{ID}^b(F)}$ .

$$(9) \text{ ID}^1(F) = \{S \subseteq \lambda : (\exists X \in \mathfrak{B}^+(F))(S \subseteq X \ \& \ \mathcal{P}(X) \subseteq \mathfrak{B}^+(F))\}.$$

*Proof.* (1) Should be clear.

(2) If  $\alpha = 0$  then  $(\otimes_{\mathcal{A}})$  holds trivially because there is no such  $S_1$ .

If  $\alpha$  is a limit ordinal then the condition holds because for every  $S_1 \in \text{ID}_{\alpha}^-(F)$  there is  $\beta < \alpha$  such that  $S_1 \in \text{ID}_{\beta}^-(F)$  and we can use the induction hyp.

Lastly, if  $\alpha = \beta + 1$  this is easy too.

(3) For the first sentence  $\text{ID}_{\alpha}^-(F)$  is a normal ideal by its definition; this implies  $\text{ID}^{\alpha}(F)$  is a normal ideal by the second sentence. We still have to prove the second sentence.

By induction on  $\gamma < \lambda$  and then by induction on  $\alpha < \gamma$  we show that  $(\forall \gamma < \lambda)(\forall \alpha < \gamma)(\text{ID}_{\alpha}(F) \subseteq \text{ID}_{\gamma}(F))$ . If  $\gamma = 1$  then this follows immediately from definitions; similarly if  $\gamma$  is limit. So suppose now that  $\gamma = \gamma_0 + 1$  and we proceed by induction on  $\alpha \leq \gamma_0$ . There are no problems neither when  $\alpha = 0$  nor when  $\alpha$  is limit. So suppose that  $\alpha = \beta + 1 < \gamma$  (so  $\beta < \gamma_0$ ). By the inductive hypothesis we know that  $\text{ID}_{\beta}(F) \subseteq \text{ID}_{\gamma_0}(F)$ . Let  $A \in \text{ID}_{\beta+1}(F)$ . By (2) there are  $A_{\xi} \in \text{ID}_{\beta+1}^-$  (for  $\xi < \lambda$ ) such that  $A = \nabla_{\xi < \lambda} A_{\xi}$ . Now

look at the definition of  $\text{ID}_{\beta+1}^-(F)$ : since  $\text{ID}_{\beta}(F) \subseteq \text{ID}_{\gamma_0}(F)$  we see that  $A_{\xi} \in \text{ID}_{\gamma_0+1}^-(F)$ . Hence  $A \in \text{ID}_{\gamma}$ .

(4) By induction on  $\alpha$  we show that: if  $\bar{B} = \langle B_{\ell} : \ell \leq m \rangle$ , where  $B_{\ell} \subseteq B_{\ell+1}$  (for  $\ell < m$ ) and  $B_m \in \text{ID}_{\alpha}(F)$  then  $\bar{B}$  has an  $F$ -representation.

CASE 1:  $\alpha = 0$ .

Thus the set  $B_m$  is not stationary and we may pick up a club  $E$  of  $\lambda$  disjoint from  $B_m$ . Let  $E = \{\alpha_{\zeta} : \zeta < \lambda\}$  be the increasing enumeration. Put  $T = \{\langle \rangle\} \cup \{\langle i \rangle : i < \lambda\}$ ,  $\alpha_{\langle \rangle} = 1$ ,  $\ell_{\langle \rangle} = \ell_{\langle i \rangle} = m$ ,  $B_{\langle \rangle}^{\ell} = B_{\ell}$  and  $B_{\langle i \rangle}^{\ell} = B_{\ell} \cap \alpha_{i+1}$ . Now check.

CASE 2:  $\alpha$  is limit.

It follows from (2) that  $B_{\ell} = \nabla_{i < \lambda} B_{\ell,i}$  for some  $B_{\ell,i} \in \bigcup_{\beta < \alpha} \text{ID}_{\beta}(F)$ . Let  $B'_{\ell,i}$

be defined as follows:

$$\begin{aligned} &\text{if } i = (m+1)j + t, \ell < t \leq m \text{ then } B'_{\ell,i} = \emptyset, \\ &\text{if } i = (m+1)j + t, t \leq m, t \leq \ell \text{ then } B'_{\ell,i} = B_{\ell,i}. \end{aligned}$$

Then for each  $i, \ell$  we may find  $\langle B_{\eta}^{i,\ell}, f_{\eta}^{i,\ell'}, \alpha_{\eta}^i : \eta \in T_i, \ell < \ell_{\eta}^{i,1}, \ell' < \ell_{\eta}^{i,2} \rangle$  satisfying clauses (a)–(d) and such that  $\langle B_{\langle \rangle}^{\ell,i,k} : k \leq k_{\eta}^1 \rangle = \langle B'_{\ell,i} : \ell \leq m \rangle$  (by the induction hypothesis). Put

$$\begin{aligned} T &= \{\langle \rangle\} \cup \{\langle i \rangle \frown \eta : \eta \in T_i\}, \\ \ell_{\langle \rangle} &= m, \quad \ell'_{\langle \rangle} = 0, \quad \ell_{\langle i \rangle \frown \eta} = \ell_{\eta}^{i,1}, \quad \ell'_{\langle i \rangle \frown \eta} = \ell_{\eta}^{i,2} \\ B_{\langle \rangle}^{\ell} &= B_{\ell}, \quad B_{\langle i \rangle \frown \eta}^{\ell} = B_{\eta}^{i,\ell}, \quad f_{\langle i \rangle \frown \eta}^{\ell'} = f_{\eta}^{i,\ell'}, \\ \alpha_{\langle \rangle} &= \omega, \quad \alpha_{\langle i \rangle \frown \eta} = \alpha_{\eta}^i. \end{aligned}$$

Checking that  $\langle B_{\eta}^{\ell}, f_{\eta}^{\ell'}, \alpha_{\eta} : \eta \in T, \ell \leq \ell_{\eta}, \ell' \leq \ell'_{\eta} \rangle$  is as required is straightforward.

CASE 3:  $\alpha = \beta + 1$ .

By (2) above and the proof of Case 2 we may assume that  $B_m \in \text{ID}_\alpha^-(F)$ . It follows from the definition of  $\text{ID}_\alpha^-(F)$  that there are  $f_\ell \in \lambda_2$  (for  $\ell \leq m$ ) such that

$$B_\ell^\oplus \stackrel{\text{def}}{=} \{\delta < \lambda : \delta \text{ is limit and } F(f_\ell \upharpoonright \delta) = 0 \Leftrightarrow \delta \in B_\ell\} \in \text{ID}_\beta(F),$$

and hence  $B^\oplus \stackrel{\text{def}}{=} \bigcup_{\ell \leq m} B_\ell^\oplus \in \text{ID}_\beta(F)$ . Therefore  $B_\ell^* \stackrel{\text{def}}{=} B_\ell \cap B^\oplus \in \text{ID}_\beta(F)$ .

Now apply the inductive hypothesis for  $\beta$  and  $\bar{B}^* = \langle B_\ell^* : \ell \leq m \rangle$  to get the sequences  $\langle B_\eta^{\ell,*}, f_\eta^{k,*} : \eta \in T^*, \ell \leq \ell_\eta^*, k \leq k_\eta^* \rangle$  satisfying clauses (a)–(d) and such that  $\langle B_\langle \rangle^{\ell,*} : \ell \leq \ell_\langle \rangle^* \rangle = \langle B_\ell^* : \ell \leq m \rangle$ . Put

$$\begin{aligned} T &= \{\langle \rangle\} \cup \{\langle i \rangle : i < \lambda\} \cup \{\langle 0 \rangle \frown \eta : \eta \in T^*\}, \\ \ell_{\langle 0 \rangle \frown \eta} &= \ell_\eta^*, \quad k_{\langle \rangle} = m + 1, \quad k_{\langle 0 \rangle \frown \eta} = k_\eta, \\ B_{\langle 0 \rangle \frown \eta}^\ell &= B_\eta^{\ell,*}, \quad B_{\langle 0 \rangle \frown \langle i \rangle}^\ell = B_\ell \cap (i + \omega), \\ f_{\langle \rangle}^k &= f_k, \quad f_{\langle 0 \rangle \frown \eta}^k = f_\eta^{k,*}, \\ \alpha_{\langle \rangle} &= \omega, \quad \alpha_{\langle 0 \rangle \frown \eta} = \alpha_\eta^*. \end{aligned}$$

(5) If  $f_\eta^\ell$  is not defined then choose  $f^*$  as it.

(6), (7), (8), (9) Easy too.  $\square$

*Remark 1.4.* Note that it may happen that  $\lambda \in \text{ID}^a(F)$ . However, if  $\eta \in \lambda_2$  is a weak diamond sequence for  $F$  then the set  $\{\gamma < \lambda : \eta(\gamma) = 0\}$  witnesses  $\lambda \notin \text{ID}_1^-(F)$ . And conversely, if  $\lambda \notin \text{ID}_1^-(F)$  and  $S^* \subseteq \lambda$  witnesses it, then the function  $0_{S^*} \cup 1_{\lambda \setminus S^*}$  is a weak diamond sequence for  $F$ .

**Definition 1.5.** For a  $\lambda$ -colouring  $F$  we define  $\lambda$ -colourings  $F^\oplus$  and  $F^\otimes$  as follows.

- (1) A function  $g \in {}^\gamma(\mathcal{H}(\lambda))$  is called  $F^\oplus$ -standard if there is a tuple  $(T, \bar{f}, \bar{\alpha}, \bar{A})$  (called a witness) such that
  - (i)  $T \subseteq \omega^{>\gamma}$  is a well founded tree (so  $\langle \rangle \in T$ ,  $\nu \triangleleft \eta \in T \Rightarrow \nu \in T$  and  $T$  has no  $\omega$ -branch);
  - (ii)  $\bar{f} = \langle f_\eta^\ell : \eta \in T, \ell \leq k_\eta \rangle$ , where  $f_\eta^\ell \in \text{DOM}(F) \cap {}^\gamma(\mathcal{H}(\lambda))$ ;
  - (iii)  $\bar{\alpha} = \langle \alpha_\eta : \eta \in T \rangle$ , where  $\alpha_\eta < \lambda$ ;
  - (iv)  $\bar{A} = \langle A_\eta^\ell : \eta \in T, \ell \leq \ell_\eta \rangle$ , where  $A_\eta^\ell \subseteq \alpha_\eta$ ;
  - (v)  $g(\beta) = (T \cap \omega^{>\beta}, \langle f_\eta^\ell \upharpoonright \beta : \eta \in T \cap \omega^{>\beta}, \ell < k_\eta \rangle, \langle \alpha_\eta : \eta \in T \cap \omega^{>\beta} \rangle, \langle A_\eta^\ell : \eta \in T \cap \omega^{>\beta}, \ell \leq \ell_\eta \rangle)$  for each  $\beta < \gamma$ .
- (2)  $\text{DOM}(F^\oplus) = \bigcup_{\alpha < \lambda} {}^\alpha(\mathcal{H}(\lambda))$  and for  $g \in {}^\gamma(\mathcal{H}(\lambda))$ :
  - $(\oplus)_\alpha$  if  $\gamma = 0$  then  $F^\oplus(g) = 0$ ,
  - $(\oplus)_\beta$  if  $\gamma > 0$  and  $g$  is not standard then  $F^\oplus(g) = 0$ ,
  - $(\oplus)_\gamma$  if  $\gamma > 0$  and  $g$  is standard as witnessed by  $\langle \bar{T}, \bar{f}, \bar{\alpha}, \bar{A} \rangle$  then  $F^\oplus(g) = \mathbf{t}_{F,g}^0(\langle \rangle)$ , where  $\mathbf{t}_{F,g}^\ell(\eta) \in \{0, 1\}$  (for  $\eta \in T$ ,  $\ell = 0, 1$ ) are defined by downward induction as follows.
    - If  $\eta \in \max(T)$  then  $\mathbf{t}_{F,g}^\ell(\eta) = 1$  iff  $\gamma \in A_\eta$ ,

$$\begin{aligned}
& \text{if } \eta \in T \setminus \max(T), \gamma < \alpha_\eta \text{ then } \mathbf{t}_{F,g}^\ell(\eta) = 1 \text{ iff } \gamma \in A_\eta, \\
& \text{if } \eta \in T \setminus \max(T), \gamma \geq \alpha_\eta \text{ then} \\
\mathbf{t}_{F,g}^1(\eta) = 1 & \quad \text{iff } F(f_\eta) = 1 \text{ or } (\exists i < \gamma)(\mathbf{t}_{F,g}^1(\eta \frown \langle i \rangle) = 1), \\
\mathbf{t}_{F,g}^0(\eta) = 1 & \quad \text{iff } (\exists i < \gamma)(\mathbf{t}_{F,g}^0(\eta \frown \langle i \rangle) = 1) \text{ or} \\
& \quad F(f_\eta) = 1 \ \& \ (\forall i < \gamma)(\mathbf{t}_{F,g}^1(\eta \frown \langle i \rangle) = 0).
\end{aligned}$$

- (3) A function  $g \in {}^\gamma(\mathcal{H}(\lambda))$  is called  $F^\otimes$ -standard if there is a tuple  $(T, \bar{f}, \bar{\ell}, \bar{\alpha}, \bar{A})$  (called a witness) such that
- (i)  $T \subseteq \omega^{>\gamma}$  is a well founded tree;
  - (ii)  $\bar{f} = \langle f_\eta : \eta \in T \rangle$ , where  $f_\eta \in \text{DOM}(F) \cap {}^\gamma(\mathcal{H}(\lambda))$ ;
  - (iii)  $\bar{\ell} = \langle \ell_\eta : \eta \in T \rangle$ , where  $\ell_\eta : {}^3\{0, 1\} \rightarrow \{0, 1\}$ ;
  - (iv)  $\bar{\alpha} = \langle \alpha_\eta : \eta \in T \rangle$ , where  $\alpha_\eta < \lambda$ ;
  - (v)  $\bar{A} = \langle A_\eta : \eta \in T \rangle$ , where  $A_\eta \subseteq \alpha_\eta$ ;
  - (vi)  $g(\beta) = (T \cap \omega^{>\beta}, \langle f_\eta \upharpoonright \beta : \eta \in T \cap \omega^{>\beta} \rangle, \langle \ell_\eta : \eta \in T \cap \omega^{>\beta} \rangle, \langle \alpha_\eta : \eta \in T \cap \omega^{>\beta} \rangle, \langle A_\eta : \eta \in T \cap \omega^{>\beta} \rangle)$  for each  $\beta < \gamma$ .
- (4)  $\text{DOM}(F^\otimes) = \bigcup_{\alpha < \lambda} {}^\alpha(\mathcal{H}(\lambda))$  and for  $g \in {}^\gamma(\mathcal{H}(\lambda))$ :
- $(\otimes)_\alpha$  if  $\gamma = 0$  then  $F^\otimes(g) = 0$ ,
  - $(\otimes)_\beta$  if  $\gamma > 0$  and  $g$  is not  $F^\otimes$ -standard then  $F^\otimes(g) = 0$ ,
  - $(\otimes)_\gamma$  if  $\gamma > 0$  and  $g$  is  $F^\otimes$ -standard as witnessed by  $\langle \bar{T}, \bar{f}, \bar{\ell}, \bar{\alpha}, \bar{A} \rangle$  then  $F^\otimes(g) = \mathbf{t}_{F,g}(\langle \cdot \rangle)$ , where  $\mathbf{t}_{F,g}(\eta) \in \{0, 1\}$  (for  $\eta \in T$ ) are defined by downward induction as follows.

$$\begin{aligned}
& \text{If } \eta \in \max(T) \text{ then } \mathbf{t}_{F,g}(\eta) = 1 \text{ iff } \gamma \in A_\eta, \\
& \text{if } \eta \in T \setminus \max(T), 1 + \gamma < \alpha_\eta \text{ then } \mathbf{t}_{F,g}(\eta) = 1 \text{ iff } \gamma \in A_\eta, \\
& \text{if } \eta \in T \setminus \max(T), 1 + \gamma \geq \alpha_\eta \text{ then}
\end{aligned}$$

$$\mathbf{t}_{F,g}(\eta) = \ell_\eta(F(f_\eta), \max\{\mathbf{t}_{F,g}(\eta \frown \langle \beta \rangle) : \beta < \gamma\}, \min\{\mathbf{t}_{F,g}(\eta \frown \langle \beta \rangle) : \beta < \gamma\}).$$

*Remark 1.6.* On  $F^\oplus, F^\otimes$  see 1.7, 1.12 below.

**Proposition 1.7.** *Let  $F$  be a  $\lambda$ -colouring. Then  $F^\oplus$  is a  $\lambda$ -colouring and*

- (a) *if  $S \in \text{ID}^a(F)$  then  $0_S \cup 1_{\lambda \setminus S} \in \mathfrak{B}(F^\oplus)$  and  $\mathfrak{B}(F) \subseteq \mathfrak{B}(F^\oplus)$ ,*
- (b)  *$\text{ID}^a(F) \subseteq \text{ID}_1(F^\oplus) = \text{ID}_1^-(F^\oplus) = \text{ID}^a(F^\oplus)$ ,*

*Proof.* (a) Check.

(b) The main point is proving  $\text{ID}^a(F) \subseteq \text{ID}_1(F^\oplus)$ .

Suppose that  $B \in \text{ID}^a(F)$ . We are going to show that then  $B \in \text{ID}_1^-(F^\oplus)$ . So suppose that  $B' \subseteq B$ . We want to find  $g \in \text{DOM}_\lambda(F^\oplus)$  such that the set

$$\{\delta < \lambda : \delta \text{ is limit and } F(g \upharpoonright \delta) = 0 \Leftrightarrow \delta \in B'\}$$

is in  $\text{ID}_0(F^\oplus)$  (what just means that it is non-stationary). Since  $B \in \text{ID}^a(F)$  we have  $B' \in \text{ID}^a(F)$ , so by 1.3(4) we may find  $\langle B_\eta^\ell, f_\eta^k, \alpha_\eta : \eta \in T, \ell \leq \ell_\eta, k < k_\eta \rangle$  such that the clauses (a)–(d) of 1.3(4) are satisfied with  $\ell_\emptyset = 0$ ,  $B' = B_\emptyset^0$ . Define  $g$  as follows. For  $\beta < \lambda$  let  $T_\beta = T \cap \omega^{>\beta}$  and

$$g(\beta) = (T_\beta, \langle f_\eta^k : \eta \in T_\beta, k \leq k_\eta \rangle, \langle \alpha_\eta : \eta \in T_\beta \rangle, \langle B_\eta^\ell \cap \alpha_\eta : \ell \leq \ell_\eta, \eta \in T_\beta \rangle).$$

Now look at the demands in 1.5(2) – they are exactly what 1.3(4) guarantees us.  $\square$

**Definition 1.8.** Let  $F_1, F_2$  be  $\lambda$ -colourings (with  $\text{DOM}(F_\ell)$  being either  $\lambda > 2$  or  $\bigcup_{\alpha < \lambda} \alpha(\mathcal{H}(\lambda))$ , see 0.2(1)).

- (1) We say that  $F_1 \leq F_2$  when there is  $h : \text{DOM}(F_1) \rightarrow \text{DOM}(F_2)$  such that:
  - (a)  $\eta \sqsubseteq \nu \Rightarrow h(\eta) \sqsubseteq h(\nu)$ ,
  - (b)  $h(\eta) = \lim_{\alpha < \delta} h(\eta \upharpoonright \alpha)$ , for every  $\eta \in \delta 2$ ,  $\delta$  a limit ordinal,
  - (c)  $(\forall \eta \in \text{DOM}(F_1))(0 < \ell g(\eta) = \ell g(h(\eta)) \Rightarrow F_1(\eta) = F_2(h(\eta)))$ .
- (2) We say that  $F_1 \leq^* F_2$  when there is  $h : \text{DOM}(F_1) \rightarrow \text{DOM}(F_2)$  such that the clauses (a)–(c) above hold but
  - (d) if  $\eta \in \text{DOM}_\lambda(F_1)$  and  $\lim_{\alpha < \lambda} h(\eta \upharpoonright \alpha)$  has length  $< \lambda$  then  $F_1(\eta \upharpoonright \alpha) = 0$  for every large enough  $\alpha$ .

**Proposition 1.9.** (1)  $\leq^*$  and  $\leq$  are transitive relations on  $\lambda$ -colourings, satisfying  $\leq^* \subseteq \leq$ .  
(2)  $\leq$  is  $\lambda^+$ -directed.

**Proposition 1.10.** (1) For every colouring  $F_1 : \bigcup_{\alpha < \lambda} \alpha(\mathcal{H}(\lambda)) \rightarrow 2$  there is a colouring  $F_2 : \lambda > 2 \rightarrow 2$  such that  $F_1 \leq F_2 \leq^* F_1$ .  
(2) For every  $\lambda$ -colouring  $F_2 : \lambda > 2 \rightarrow 2$  there is a  $\lambda$ -colouring  $F_1 : \bigcup_{\alpha < \lambda} \alpha(\mathcal{H}(\lambda)) \rightarrow 2$  such that  $F_2 \leq F_1 \leq^* F_2$ .

*Proof.* 1) Let  $F_1 : \bigcup_{\alpha < \lambda} \alpha(\mathcal{H}(\lambda)) \rightarrow 2$ . Let  $h_0$  be a one-to-one function from  $\mathcal{H}(\lambda)$  to  $\lambda > 2$ , say  $h_0(\eta) = \langle \ell_{\eta, i} : i < \ell g(h_0(\eta)) \rangle$ . Define a function  $h_1 : \mathcal{H}(\lambda) \rightarrow \lambda > 2$  by:

$$\begin{aligned} \ell g(h_1(\eta)) &= 2\ell g(h_0(\eta)) + 2, \\ h_1(\eta)(2i) &= h_0(\eta)(i), \quad h_1(\eta)(2i+1) = 0 \quad \text{for } i < \ell g(h_0(\eta)), \quad \text{and} \\ h_1(\eta)(2\ell g(h_0(\eta))) &= h_1(\eta)(2\ell g(h_0(\eta)) + 1) = 1. \end{aligned}$$

Next, by induction on  $\ell g(\eta)$ , we define a function  $h^+ : \bigcup_{\alpha < \lambda} \alpha(\mathcal{H}(\lambda)) \rightarrow \lambda > 2$  as follows:

$$h^+(\langle \rangle) = \langle \rangle, \quad h^+(\eta \frown \langle x \rangle) = h^+(\eta) \frown h_1(x).$$

and if  $\eta \in \bigcup_{\alpha < \lambda} \alpha(\mathcal{H}(\lambda))$  has length the limit ordinal  $\delta$  then  $h^+(\eta) = \bigcup \{h^+(\eta \upharpoonright \beta) : \beta < \delta\}$

Clearly  $h^+$  is one to one with the right domain and range.

Finally we define a colouring  $F_2 : \lambda > 2 \rightarrow 2$  by

$$F_2(\nu) = \begin{cases} F_1(\eta) & \text{if } \nu = h^+(\eta), \\ 0 & \text{if } \nu \notin \text{rng}(h^+). \end{cases}$$

It is easy to check that  $F_2$  is as required.  $\square$

**Proposition 1.11.** *Assume that  $F_1, F_2$  are  $\lambda$ -colourings such that  $F_1 \leq F_2$ , or  $F_1 \leq^* F_2$ . Then:*

- (1) *For every  $\eta \in \text{DOM}_\lambda(F)$  there are  $\nu \in \text{DOM}_\lambda(F)$  and a club  $E$  of  $\lambda$  such that*

$$(\forall \delta \in E)(F_1(\eta \upharpoonright \delta) = F_2(\nu \upharpoonright \delta)).$$

- (2)  $\text{ID}_\alpha(F_1) \subseteq \text{ID}_\alpha(F_2)$ ,  $\text{ID}_\alpha^-(F_1) \subseteq \text{ID}_\alpha^-(F_2)$ ; hence  $\text{ID}^a(F_1) \subseteq \text{ID}^a(F_2)$  and  $\mathfrak{B}^+(F_1) \subseteq \mathfrak{B}^+(F_2)$ .  
 (3) *For every colouring  $F$  we have  $\text{ID}^c(F) \subseteq \text{WDmId}_\lambda$*

*Proof.* Straightforward. □

*Conclusion 1.12.* Assume that  $\lambda$  is a regular uncountable cardinal and  $F : \lambda^{>2} \rightarrow 2$  is a  $\lambda$ -colouring. Let

$$F^\otimes : \bigcup_{\alpha < \lambda} \alpha(\mathcal{H}(\lambda)) \rightarrow 2$$

be the colouring defined for  $F$  in Definition 1.5(4). Let  $\iota \in \{a, b\}$ . Then:

- (a)  $F \leq F^\otimes$ .  
 (b)  $\text{ID}^\iota(F^\otimes)$  is a normal ideal on  $\lambda$ .  
 (c)  $\mathfrak{B}(F) \subseteq \mathfrak{B}(F^\otimes)$  and  $\text{ID}^\iota(F) \subseteq \text{ID}^\iota(F^\otimes) = \text{WDmId}_\lambda(F^\otimes)$ .  
 (d)  $F^\otimes$  relates to itself as it relates to  $F$ , i.e. if  $\alpha^* < \lambda^+$ ,  $\langle S_\alpha : \alpha < \alpha^* \rangle$  is increasing continuous modulo  $\text{ID}^\iota(F^\otimes)$ ,  $S_{\alpha+1} = S_\alpha \cup A_\alpha \text{ mod } \text{ID}^\iota(F^\otimes)$ ,  $A_\alpha \in \mathfrak{B}(F^\otimes)$ ,  $\ell_\alpha \in 2$ ,  
 then for some  $f \in {}^\lambda(\mathcal{H}(\lambda))$

$$\{\alpha < \lambda : F(f \upharpoonright \alpha) = 1\} / \mathcal{D}_\lambda$$

is, in  $\mathcal{P}(\lambda) / \mathcal{D}_\lambda$ , the least upper bound of the family  $\{(A_\alpha \setminus S_\alpha) / \mathcal{D}_\lambda : \ell_\alpha = 1\}$  (where  $\mathcal{D}_\lambda$  stands for the club filter).

- (e) The family  $\mathfrak{B}(F^\otimes)$  is closed under complements, unions and intersections of less than  $\lambda$  sets, diagonal unions and diagonal intersections and it includes bounded subsets of  $\lambda$ . Moreover  $\mathfrak{B}^+(F^\otimes) = \mathfrak{B}(F^\otimes)$ .  
 (f) If  $\mathcal{P}(\lambda) / \text{ID}^\iota(F^\otimes)$  is  $\lambda^+$ -saturated then

for every set  $X \subseteq \lambda$  there are sets  $A, B \in \mathfrak{B}(F^\otimes)$  such that:

- ( $\alpha$ )  $A \subseteq X \subseteq B$ ,  
 ( $\beta$ ) for every  $Y \in \mathfrak{B}(F^\otimes)$  one of the following occurs:  
 (i) the sets  $(X \setminus A) \cap Y$ ,  $(X \setminus A) \setminus Y$ ,  $(B \setminus X) \cap Y$ ,  $(B \setminus X) \setminus Y$  are<sup>2</sup> not in  $\text{ID}^\iota(F^\otimes)$ ,  
 (ii)  $Y \cap (B \setminus A) \in \text{ID}^\iota(F^\otimes)$ ,  
 (iii)  $(B \setminus A) \setminus Y \in \text{ID}^\iota(F^\otimes)$ .

In the situation as above we denote  $A = \max_{F^\otimes}(X)$ ,  $B = \min_{F^\otimes}(X)$  (note that these sets are unique only modulo  $\text{ID}^\iota(F^\otimes)$ ). Moreover

- (g) In clause (f), if  $A \subseteq \min_{F^\otimes}(B)$  then  $\min_{F^\otimes}(A) \subseteq \min_{F^\otimes}(B) \text{ mod } \text{ID}^\iota(F^\otimes)$ .

<sup>2</sup> hence none of  $X \setminus A$ ,  $B \setminus A$  includes (modulo  $\text{ID}^\iota(F^\otimes)$ ) a member of  $\mathfrak{B}(F^\otimes) \setminus \text{ID}^\iota(F^\otimes)$

(h) In clause (f), when  $\iota = b$ , if  $X \subseteq \lambda$ ,  $X \notin \text{ID}'(F^\otimes)$  then for some  $Y_1, Y_2 \subseteq X$  which are not in  $\text{ID}'(F^\otimes)$  we have

$$\max_{F^\otimes}(Y_1) = \max_{F^\otimes}(Y_2) = \emptyset \quad \text{and} \quad \min_{F^\otimes}(Y_1) = \min_{F^\otimes}(Y_2) \notin \text{ID}'(F^\otimes).$$

(i) In clause (f),  $\min_{F^\otimes}$  and  $\max_{F^\otimes}$  commute with the union of  $< \lambda$  and the intersection of  $< \lambda$  sets.

*Proof.* Clauses (a) and (b): Should be clear.

Clause (e): Note that as  $\theta = 2$  we identify a sequence  $\eta \in {}^\lambda 2$  with  $\{i < \lambda : \eta(i) = 1\}$ .

**$\mathfrak{B}(F^\otimes)$  is closed under complementation.**

Suppose that  $A \in \mathfrak{B}(F^\otimes)$ . First, assume  $A$  is bounded then let  $g, (T, \bar{f}, \bar{\ell}, \bar{\alpha}, \bar{A})$  be as in 1.5(3) with  $T = \{\langle \rangle\} \cup \{\langle i \rangle : i < \lambda\}$ ,  $A_{\langle \rangle} = \alpha_{\langle \rangle} \setminus A$ ,  $\alpha_{\langle \rangle} > \sup(A)$ ,  $\ell_{\langle \rangle}$  constantly 1. Then  $(\forall \alpha < \lambda)(F^\otimes(g \upharpoonright (1 + \alpha)) = 1 \Leftrightarrow \alpha \in A)$ , so  $F$  codes  $\lambda \setminus A$ .

Second, suppose that  $\sup(A) = \lambda$ . Pick  $g$  such that

$$(\forall \alpha < \lambda)(F^\otimes(g \upharpoonright (1 + \alpha)) = 1 \Leftrightarrow \alpha \in A).$$

By our assumption, for arbitrarily large  $\beta < \lambda$  we have  $F^\otimes(g \upharpoonright \beta) = 1$ , so  $g \upharpoonright \beta$  is

$$(T_\beta, \langle f_\eta^\beta : \eta \in T_\beta \rangle, \langle \alpha_\eta^\beta : \eta \in T_\beta \rangle, \langle \ell_\eta^\beta : \eta \in T_\beta \rangle, \langle A_\eta^\beta : \eta \in T_\beta \rangle)$$

and it is as in 1.5(3). If  $\beta_1 < \beta_2$  then the two values necessarily cohere, in particular  $T_{\beta_1} = T_{\beta_2} \cap {}^{\omega^>}(\beta_1)$ . Consequently there is  $(T, \bar{f}, \bar{\ell}, \bar{\alpha}, \bar{A})$  such that  $T = \bigcup_{\beta < \lambda} T_\beta \subseteq {}^{\omega^>} \lambda$  is closed under initial segments and is well founded

(as  $T_\beta$  increase with  $\beta$  and  $\text{cf}(\lambda) > \aleph_0$ ). Thus we have proved

( $\boxtimes$ ) if  $A \subseteq \lambda$  is unbounded and  $F^\otimes$  coded by  $g$  then there is  $\mathbf{p} = (T, \bar{f}, \bar{\ell}, \bar{\alpha}, \bar{A})$  such that the clauses (i)–(vi) of 1.5(3) hold for  $\gamma = \lambda$  and  $g \upharpoonright \beta = \mathbf{p} \upharpoonright \beta$ .

Now define  $\mathbf{p}'$  like  $\mathbf{p}$  (with the same  $T$  etc) except that  $\ell_{\langle \rangle}^{\mathbf{p}'} = 1 - \ell_{\langle \rangle}^{\mathbf{p}}$  and  $A_{\langle \rangle}^{\mathbf{p}'} = A_{\langle \rangle}^{\mathbf{p}}$ .

**$\mathfrak{B}(F^\otimes)$  contains all bounded subsets of  $\lambda$ .**

By the first part of the arguments above all co-bounded subsets of  $\lambda$  are in  $\mathfrak{B}(F^\otimes)$ , so (by the above) their complements are there too.

**$\mathfrak{B}(F^\otimes)$  is closed under unions of length  $< \lambda$ .**

Let  $B = \bigcup_{i < \alpha} B_i$  where  $\alpha < \lambda$  and  $B_i \in \mathfrak{B}(F^\otimes)$ . Let  $w = \{i < \alpha : \sup(B_i) = \lambda\}$  and for  $i \in w$  let  $B_i$  be represented by  $g_i \in {}^\lambda(\mathcal{H}(\lambda))$  which, by ( $\boxtimes$ ), comes

from  $\mathbf{p}^i = (T^i, \bar{f}^i, \bar{\ell}^i, \bar{\alpha}^i, \bar{A}^i)$ . We may assume that  $w = \beta \leq \alpha$ . Let

$$\begin{aligned} T &= \{\langle \rangle\} \cup \{\langle i \rangle : i < \lambda\} \cup \{\langle i \rangle \frown \eta : \eta \in T^i, i < \beta\}, \\ f_{\langle i \rangle \frown \eta} &= f_{\eta}^i, \text{ etc} \\ \alpha_{\langle \rangle} &\text{ is the first } \gamma \geq \omega \text{ such that } \gamma \geq \alpha \ \& \ (\forall i \in [\beta, \alpha))(B_i \subseteq \gamma), \\ B_{\langle i \rangle} &= \emptyset \quad \text{if } i \geq \beta, \\ A_{\langle \rangle} &= \bigcup_{i < \alpha} B_i \cap \alpha_{\langle \rangle}, \\ \ell_{\langle \rangle}(i_0, i_1, i_2) &= i_1. \end{aligned}$$

Checking is straightforward.

**$\mathfrak{B}(F^\otimes)$  is closed under diagonal unions.**

Let  $B = \bigtriangleup_{i < \lambda} B_i$ , where each  $B_i \in \mathfrak{B}(F^\otimes)$  is represented by  $g_i \in {}^\lambda \mathcal{H}(\lambda)$

which, by  $(\boxtimes)$ , comes from  $\mathbf{p}^i = (T^i, \bar{f}^i, \bar{\ell}^i, \bar{\alpha}^i, \bar{A}^i)$ . Let  $T = \{\langle \rangle\} \cup \{\langle i \rangle \frown \eta : \eta \in T_i, i < \lambda\}$ ,  $f_{\langle i \rangle \frown \eta} = f_{\eta}^i$ , etc,  $\alpha_{\langle \rangle} = \omega$ ,  $B_{\langle \rangle} = B \cap \omega$  and  $\ell_{\langle \rangle}(i_0, i_1, i_2) = i_1$ .

So we have proved the first sentence in clause (e). The second sentence there follows by it and Def. 1.2(2). Note that including the family of non-stationary sets follows by including the family of bounded subsets and being closed under diagonal unions.

Clause (c): We concentrate on the case  $\text{ID}^t = \text{ID}^a$ . First note that  $\mathfrak{B}(F) \subseteq \mathfrak{B}(F^\otimes)$  as  $\mathfrak{B}(F) \subseteq \mathfrak{B}^+(F) \subseteq \mathfrak{B}^+(F^\otimes) = \mathfrak{B}(F^\otimes)$  (the second inclusion by (a) and 1.11, the last equality by (e)). Next note that

$$\text{WdId}_\lambda(F^\otimes) \subseteq \text{ID}_1^-(F^\otimes) \subseteq \text{ID}_1(F^\otimes) \subseteq \text{ID}^t(F^\otimes).$$

Now by induction on  $\alpha$  we are proving that  $\text{ID}_\alpha(F^\otimes) \subseteq \text{WdId}_\lambda(F^\otimes)$ . So suppose that we have arrived to a stage  $\alpha$ .

If  $\alpha = 0$  then we use the fact that every non-stationary subset of  $\lambda$  is in  $\mathfrak{B}(F^\otimes)$  (by (e)).

If  $\alpha$  is limit then, by the induction hypothesis,  $\text{ID}_\alpha^-(F^\otimes) \subseteq \mathfrak{B}(F^\otimes)$  and hence  $\text{ID}_\alpha(F^\otimes) \subseteq \mathfrak{B}(F^\otimes)$  (as  $\mathfrak{B}(F^\otimes)$  is closed under diagonal unions by (e); remember 1.3(3)).

So suppose that  $\alpha = \beta + 1$  and  $B \in \text{ID}_\alpha(F^\otimes)$ . Suppose  $B' \subseteq B$  (so  $B' \in \text{ID}_\alpha^-(F^\otimes)$ ). There is  $B'' \in \mathfrak{B}(F)$  such that  $B'' \triangle B' \in \text{ID}_\beta(F)$ . By the first part we know that  $B'' \in \mathfrak{B}(F^\otimes)$  and by the induction hypothesis  $B' \triangle B'' \in \mathfrak{B}(F^\otimes)$ . Consequently  $B' \in \mathfrak{B}(F^\otimes)$ .

Together we have proved that  $\text{ID}^t(F^\otimes) = \text{WdId}_\lambda(F^\otimes)$ . The inclusion  $\text{ID}^t(F) \subseteq \text{ID}^t(F^\otimes)$  is easy.

Clause (d) Easy.

Clause (f)

Let  $\mathcal{D}_1$  be  $\{A \subseteq X : A \in \mathfrak{B}(F^\otimes) \setminus \text{ID}^t(F^\otimes)\}$  and let  $\{A_i : i < i_*\}$  be a maximal sub-family of  $\mathcal{D}_1$  such that  $i < j < i_* \Rightarrow A_i \cap A_j \in \text{ID}^t(F^\otimes)$ . By the assumption of clause (f) necessarily  $i_* < \lambda^+$  so without loss of generality  $i_* \leq \lambda$ . Let  $A$  be  $\cup\{A_i : i < i_*\}$  if  $i_* < \lambda$  and the diagonal union if  $i_* = \lambda$ . Clearly  $A \in \mathfrak{B}^+(F) = \mathfrak{B}(F^\otimes)$ .

Let  $A'$  be chosen similarly replacing  $X$  by  $\lambda \setminus X$  and let  $B = \lambda \setminus A'$

Clearly  $A, B$  are as required.

Clause (g) Easy.

Clauses (h), (i) Easy.  $\square$

**Proposition 1.13.** *Let  $\lambda$  be a regular uncountable cardinal and  $F$  be a  $\lambda$ -colouring.*

- (1) *If  $ID_\alpha(F)$  is  $\lambda^+$ -saturated then for some  $\beta < \lambda^+$  we have  $ID_{\alpha+\beta}(F) = ID^a(F)$ .*
- (2)  $ID_\alpha(F) \subseteq WDMId_\lambda$ , see 1.2(3);
- (3) *If  $ID_\alpha(F)$  is  $\lambda^+$ -saturated and  $\lambda \notin WDMId_\lambda$  then  $WDMId_\lambda = ID_1(F')$  for some  $\lambda$ -colouring  $F'$ .*
- (4)  $ID^b(F), ID^c(F)$  are normal ideals, and  $ID^1(F) \subseteq ID^b(F) \subseteq ID^c(F) \subseteq WDMId_\lambda$ .
- (5)  $ID^1(F^\otimes) = WDMId_\lambda(F^\otimes)$ .
- (6)  $WDMId_\lambda = \cup\{ID^\iota(F) : F \text{ a function form } \lambda^{>2} \text{ to } \} = \cup\{ID_1(F) : F \text{ a function form } \lambda^{>2} \text{ to } \} \cup \{WDMId_\lambda(F) : F \text{ a function form } \lambda^{>2} \text{ to } \}$  for  $\iota = a, b, c$ .

*Proof.* 1) It follows from 1.3(3) that  $ID_\gamma(F)$  increases with  $\gamma$  and  $\beta < \gamma, ID_\beta(F) = ID_{\beta+1}$  implies  $ID_\beta(F) = ID_\gamma$ ; so the assertion should be clear.

2) By the definition (and 1.12(c)).

3) Assume that  $ID_\alpha(F)$  is  $\lambda^+$ -saturated and  $\lambda \notin WDMId_\lambda$ . By induction on  $\beta < \lambda^+$  we try to choose colourings  $F_\beta$  such that

- (a)  $ID(F) \subseteq ID(F_\alpha)$
- (b) if  $\beta < \gamma$  then  $ID^a(F_\beta) \subseteq ID^a(F_\gamma)$ ,
- (c)  $ID^a(F_\beta) \neq ID^a(F_{\beta+1})$ .

So we let  $F_0 = F$ . If  $\beta$  is limit then we use 1.9(2) to choose  $F_\beta$  so that  $(\forall \gamma < \beta)(F_\gamma \leq F_\beta)$ . Finally, if  $\beta = \gamma + 1$  then we let  $F'_\beta = (F_\gamma)^\otimes$  (so  $ID^a(F_\gamma) \subseteq ID_1(F'_\beta) = ID^a(F'_\beta) \subseteq WDMId_\lambda$ ). If  $ID^a(F'_\beta) \neq WDMId_\lambda$  then we choose a set  $A \in WDMId_\lambda \setminus ID^a(F'_\beta)$  and  $F_\beta^*$  witnessing  $A \in WDMId_\lambda$ . We may assume that  $(\forall \alpha \in \lambda \setminus A)(\forall \eta \in {}^\alpha 2)(F_\beta^*(\eta) = 0)$ . Now take a colouring  $F_\beta$  such that  $F'_\beta, F_\beta^* \leq F_\beta$ .

After carrying out the construction choose  $S_\beta^0 \in ID^a(F_{\beta+1}) \setminus ID^a(F_\beta)$  (for  $\beta < \lambda^+$ ) and let  $S_\beta = S_\beta^0 \setminus \bigcap_{\gamma < \beta} S_\gamma^0$ . Then  $\langle S_\beta : \beta < \lambda^+ \rangle$  is a sequence of pairwise disjoint members of  $\mathcal{P}(\lambda) \setminus ID^a(F_0) \subseteq \mathcal{P}(\lambda) \setminus ID_\alpha(F)$ , contradicting our assumptions.

4), 5), 6) Easy too.  $\square$

For the rest of this section we will assume the following

**Hypothesis 1.14.** (1) We assume that

- (a)  $\lambda$  is a regular uncountable cardinal,
- (b)  $F$  is a  $\lambda$ -colouring,
- (c)  $\lambda \notin ID^b(F^\otimes)$ , and

- (d)  $\text{ID}^b(F^\otimes)$  is  $\lambda^+$ -saturated, that is there is no sequence  $\langle A_\alpha : \alpha < \lambda^+ \rangle$  such that for each  $\alpha < \beta < \lambda^+$

$$A_\alpha \notin \text{ID}^b(F^\otimes) \quad \text{and} \quad \|A_\alpha \cap A_\beta\| < \lambda.$$

- (2) For each limit ordinal  $\alpha \in [\lambda, \lambda^+)$  fix an enumeration  $\langle \varepsilon_i^\alpha : i < \lambda \rangle$  of  $\alpha$ .

**Construction 1.15.** Fix a sequence  $\eta \in {}^\lambda 2$  for a moment. We choose a sequence

$$\langle S_\alpha[\eta], A_\alpha[\eta], B_\alpha[\eta], \ell_\alpha[\eta], m_\alpha[\eta], f_\alpha[\eta] : \alpha < \alpha^*[\eta] \rangle$$

as follows. By induction on  $\alpha < \lambda^+$  we try to choose  $S_\alpha[\eta] = S_\alpha$ ,  $A_\alpha[\eta] = A_\alpha$ ,  $B_\alpha[\eta] = B_\alpha$ ,  $\ell_\alpha[\eta] = \ell_\alpha$ ,  $m_\alpha[\eta] = m_\alpha$ ,  $f_\alpha[\eta] = f_\alpha$  such that:

- (a)  $S_\alpha, A_\alpha, B_\alpha \subseteq \lambda$ ,  $\ell_\alpha, m_\alpha \in \{0, 1\}$ ,  $f_\alpha \in {}^\lambda 2$ ,
- (b)  $A_\alpha \notin \text{ID}^b(F^\otimes)$ ,  $A_\alpha \cap S_\alpha = \emptyset$ ,
- (c) if  $\alpha = 0$  then  $S_\alpha = \emptyset$
- (d)  $S_{\alpha+1} = S_\alpha \cup A_\alpha$ ;
- (e) if  $\alpha < \lambda$  is limit then  $S_\alpha = \bigcup_{\beta < \alpha} S_\beta$ ; if  $\alpha \in [\lambda, \lambda^+)$  is limit then  $S_\alpha = \{\gamma < \lambda : (\exists i < \gamma)(\gamma \in S_{\varepsilon_i^\alpha})\}$ ,  $S_0 = \emptyset$ ,
- (f)  $B_\alpha \in \text{ID}^b(F^\otimes)$ ,
- (g) for every  $\delta \in \lambda \setminus (S_\alpha \cup B_\alpha)$

$$\eta(\delta) = m_\alpha \quad \Rightarrow \quad F(f_\alpha \upharpoonright \delta) = \ell_\alpha,$$

- (h)  $A_\alpha = \{\delta \in \lambda \setminus S_\alpha : F(f_\alpha \upharpoonright \delta) = 1 - \ell_\alpha\}$ .

It follows from 1.14 that at some stage  $\alpha^* = \alpha^*[\eta] < \lambda^+$  we get stuck (remember clause (b) above). Still, we may define  $S_{\alpha^*}$  as in clause (c).

**Proposition 1.16.** *Assume 1.14. Then:*

- (1) *There exists  $\eta \in {}^\lambda 2$  such that*

$$\lambda \setminus S_{\alpha^*[\eta]}[\eta] \notin \text{ID}^b(F^\otimes).$$

- (2) *If  $S \in \mathfrak{B}(F^\otimes) \setminus \text{ID}^b(F^\otimes)$  then we can demand  $S \subseteq S_{\alpha^*[\eta]}[\eta]$ .*

*Proof.* Assume not. Then for each  $\eta \in {}^\lambda 2$  the set  $B_{\alpha^*[\eta]} \stackrel{\text{def}}{=} \lambda \setminus S_{\alpha^*[\eta]}$  is in  $\text{ID}^b(F^\otimes)$ . Now,

$$\{\alpha \in B_{\alpha^*[\eta]} : \eta(\alpha) = 1\} \in \text{ID}^b(F^\otimes) \subseteq \mathfrak{B}(F^\otimes)$$

(see 1.7).

**Claim 1.16.1.** *For each  $\alpha$ ,  $S_\alpha \in \mathfrak{B}(F^\otimes)$ .*

*Proof of the claim.* We show it by induction on  $\alpha$ . If  $\alpha = 0$  then  $S_\alpha = \emptyset \in \mathfrak{B}(F^\otimes)$  (see 1.12(c)). If  $\alpha < \lambda$  is a limit ordinal then  $S_\alpha = \bigcup_{\beta < \alpha} S_\beta$  and

by the inductive hypothesis  $S_\beta \in \mathfrak{B}(F^\otimes)$ , so by 1.12(e) we are done (as  $\mathfrak{B}(F^\otimes)$  is closed under unions of  $< \lambda$  elements). If  $\alpha \in [\lambda, \lambda^+)$  is limit then we use the fact that  $\mathfrak{B}(F^\otimes)$  is closed under diagonal unions. If  $\alpha = \beta + 1$

then  $A_\beta \in \mathfrak{B}(F^\otimes)$  or  $\lambda \setminus A_\beta \in \mathfrak{B}(F^\otimes)$  and hence we may conclude that  $A_\beta \in \mathfrak{B}(F^\otimes)$  (remember 1.12(e)). Since  $\mathfrak{B}(F^\otimes)$  is closed under unions of length  $< \lambda$  we are done.  $\square$

**Claim 1.16.2.** For each  $\alpha$ ,  $Y_\alpha \stackrel{\text{def}}{=} \{\beta < \lambda : \eta(\beta) = 1\} \cap S_\alpha \in \mathfrak{B}(F^\otimes)$ .

*Proof of the claim.* We prove it by induction on  $\alpha$ . If  $\alpha = 0$  then  $Y_\alpha = \emptyset$  and there is nothing to do. The case of limit  $\alpha$  is handled like that in the proof of 1.16.1. So suppose that  $\alpha = \beta + 1$ . It suffices to show that the set  $Y_\alpha \cap (S_\alpha \setminus S_\beta)$  is in  $\mathfrak{B}(F^\otimes)$ , which means that  $Y_\alpha \cap A_\alpha$  is there (remember clauses (g) and (h) of 1.15). Note that if  $\delta \in A_\alpha \setminus B_\alpha$  then  $F(f_\alpha \upharpoonright \delta) = 1 - \ell_\alpha \neq \ell_\alpha$  and hence  $\eta(\delta) \neq m_\alpha$  so  $\eta(\delta) = 1 - m_\alpha$ . Consequently  $Y_\alpha \cap (A_\alpha \setminus B_\alpha) \in \{A_\alpha \setminus B_\alpha, \emptyset\}$ . But  $\mathcal{P}(B_\alpha) \subseteq \mathfrak{B}(F^\otimes)$  so together we are done.  $\square$

It follows from 1.16.1, 1.16.2 that

$$\{\beta : \eta(\beta) = 1\} \cap S_{\alpha^*[\eta]} \in \mathfrak{B}(F^\otimes).$$

But  $\lambda \setminus S_{\alpha^*[\eta]} \in \text{ID}^b(F^\otimes)$ , so  $\mathcal{P}(\lambda \setminus S_{\alpha^*[\eta]}) \subseteq \mathfrak{B}(F^\otimes)$  so we get a contradiction.  $\square$

*Conclusion 1.17.* Assume 1.14. Let  $\eta \in \lambda_2$ ,  $X_\ell[\eta] = (\lambda \setminus S_{\alpha^*[\eta]}) \cap \eta^{-1}(\{\ell\})$  (for  $\ell = 0, 1$ ). Then one of the following occurs:

- (A)  $\lambda \setminus S_{\alpha^*[\eta]} \in \text{ID}^b(F^\otimes)$ ,
- (B)  $X_0[\eta], X_1[\eta] \notin \text{ID}^b(F^\otimes)$ , and  $X_0[\eta] \cup X_1[\eta] \in \mathfrak{B}(F^\otimes)$ ,  $X_0[\eta] \cap X_1[\eta] = \emptyset$ , and for every  $f \in \lambda_2$ ,

either the sequence  $\langle F(f \upharpoonright \delta) : \delta \in (\lambda \setminus S_{\alpha^*[\eta]}) \rangle$  is  $\text{ID}^b(F^\otimes)$ -almost constant

or both sequences  $\langle F(f \upharpoonright \delta) : \delta \in X_0[\eta] \rangle$  and  $\langle F(f \upharpoonright \delta) : \delta \in X_1[\eta] \rangle$  are not  $\text{ID}^b(F^\otimes)$ -almost constant.

*Proof.* Assume that the first possibility fails, so  $\lambda \setminus S_{\alpha^*[\eta]} \notin \text{ID}^b(F^\otimes)$ .

Assume  $X_0[\eta] \in \text{ID}^b(F^\otimes)$ . Take any  $f_{\alpha^*[\eta]} \in \lambda_2$  and choose  $\ell_{\alpha^*[\eta]} \in \{0, 1\}$  so that

$$\{\delta \in \lambda \setminus S_{\alpha^*[\eta]} : F(f_{\alpha^*[\eta]} \upharpoonright \delta) = 1 - \ell_{\alpha^*[\eta]}\} \notin \text{ID}^b(F^\otimes).$$

Putting  $m_{\alpha^*[\eta]} = 0$  and  $B_{\alpha^*[\eta]} = X_0[\eta]$  we get a contradiction with the definition of  $\alpha^*[\eta]$ . Similarly one shows that  $X_1[\eta] \notin \text{ID}^b(F^\otimes)$ .

Suppose now that  $f \in \lambda_2$  and the sequence  $\langle F(f \upharpoonright \delta) : \delta \in (\lambda \setminus S_{\alpha^*[\eta]}) \rangle$  is not  $\text{ID}^b(F^\otimes)$ -almost constant but, say, the sequence  $\langle F(f \upharpoonright \delta) : \delta \in X_0[\eta] \rangle$  is  $\text{ID}^b(F^\otimes)$ -almost constant (and let the constant value be  $\ell_{\alpha^*[\eta]}$ ). Let  $m_{\alpha^*[\eta]} = 0$ ,  $B_{\alpha^*[\eta]} = \{\delta \in X_0[\eta] : F(f \upharpoonright \delta) = 1 - \ell_{\alpha^*[\eta]}\}$ . Then  $B_{\alpha^*[\eta]} \in \text{ID}^b(F^\otimes)$  and since necessarily

$$\{\delta \in X_0[\eta] \cup X_1[\eta] : F(f \upharpoonright \delta) = 1 - \ell_{\alpha^*[\eta]}\} \notin \text{ID}^b(F^\otimes),$$

we immediately get a contradiction. Similarly in the symmetric case.  $\square$

*Remark 1.18.* Note that, in 1.17, if  $S \in \mathfrak{B}(F^\otimes) \setminus \text{ID}^b(F^\otimes)$  then there is  $\eta \in {}^\lambda 2$  such that  $\eta^{-1}[\{0\}] \supseteq \lambda \setminus S$  and above  $X_0, X_1 \subseteq S$  and possibility (A) fails.

**Proposition 1.19.** *Assume 1.14.*

- (1) We can find  $S_F^*, S_0^*$  and  $S_1^*$  such that:
  - (a)  $S_F^* \in \mathfrak{B}(F^\otimes)$ ,
  - (b)  $S_F^* = S_0^* \cup S_1^*$ ,  $S_0^* \cap S_1^* = \emptyset$ , and  $S_0^*, S_1^*$  witness  $S_F^* \in \text{ID}^c(F^\otimes)$
  - (c) if  $S_F^* \neq \lambda$  then  $\text{ID}^c(F^\otimes) \upharpoonright \mathcal{P}(\lambda \setminus S_F^*) = \text{WDMId}_\lambda(F^\otimes) \upharpoonright \mathcal{P}(\lambda \setminus S_F^*)$ ,  $\lambda \setminus S_F^* \notin \text{ID}^c(F^\otimes)$ .
  - (d) if  $S_F^* \neq \emptyset$  then  $S_F^* \notin \text{ID}^b(F^\otimes)$  and

$$\{(S_0^* \cap F^\otimes(f)/\text{ID}^b(F^\otimes), S_1^* \cap F^\otimes(f)/\text{ID}^b(F^\otimes)) : f \in \text{DOM}_\lambda\}$$

is an isomorphism from  $\mathcal{P}(S_0^*)/\text{ID}^b(F^\otimes)$  onto  $\mathcal{P}(S_1^*)/\text{ID}^b(F^\otimes)$ .

- (2) If in 1.17,  $S_F^* \subseteq S_{\alpha^*[\eta]} \text{ mod } \text{ID}^b(F)$  then we can add

(\*) for some  $\rho \in X_{12}$  for every  $f \in {}^\lambda 2$  we have

$$\{\delta \in X_1 : F(f \upharpoonright \delta) = \rho(\delta)\} \neq \emptyset \text{ mod } \text{ID}^b(F^\otimes).$$

*Proof.* 1) We try to choose by induction on  $\alpha < \lambda^+$  sets  $S_\alpha, S_{\alpha,0}, S_{\alpha,1}$  such that

- (a)  $S_\alpha \subseteq \lambda$ ,
- (b)  $S_\alpha = S_{\alpha,0} \cup S_{\alpha,1}$ ,  $S_{\alpha,0} \cap S_{\alpha,1} = \emptyset$ ,
- (c) if  $\beta < \alpha$  and  $\ell < 2$  then

$$S_\beta \subseteq S_\alpha \text{ mod } \text{ID}^b(F^\otimes) \quad \text{and} \quad S_{\beta,\ell} \subseteq S_{\alpha,\ell} \text{ mod } \text{ID}^b(F^\otimes),$$

- (d) the sets  $S_{\alpha,0}, S_{\alpha,1}$  witness that  $S_\alpha \in \text{ID}^c(F^\otimes)$  (see 1.2(4)).

At some stage  $\alpha < \lambda^+$  we have to be stuck (as  $\text{ID}^b(F^\otimes)$  is  $\lambda^+$ -saturated) and then  $(S_\alpha, S_{\alpha,0}, S_{\alpha,1})$  can serve as  $(S_F^*, S_0^*, S_1^*)$ .

- 2) By the choice of  $S_F^*$ , for some  $\ell < 2$  we have

$$\mathcal{P}(X_\ell) \neq \{F^\otimes(f) \cap X_\ell : f \in {}^\lambda 2\},$$

so let  $Y \subseteq X_\ell$  be such that  $Y \notin \{F^\otimes(f) \cap X_\ell : f \in {}^\lambda 2\}$ . Let  $\rho = 0_Y \cup 1_{X_\ell \setminus Y}$ . Since without loss of generality  $\ell = 1$ , we are done.  $\square$

*Remark 1.20.* (1) Recall that if  $\lambda \notin \text{WDMId}_\lambda$  then  $S_F^* \neq \lambda$ .

- (2) Recall:  $\text{ID}^a(F^\otimes) = \text{WDMId}_\lambda(F^\otimes)$  is a normal ideal and  $\text{ID}^b(F^\otimes) \subseteq \text{ID}^c(F^\otimes)$  are normal ideals extending it.

## 2. WEAK DIAMOND FOR MORE COLOURS

In this section we deduce a weak diamond for, say, three colours, assuming the weak diamond for two colours and assuming that a certain ideal is saturated.

**Proposition 2.1.** *Assume that  $\lambda$  is a regular uncountable cardinal and  $\mu \leq 2^{<\lambda}$ . Let  $F_i : \lambda^{>2} \rightarrow \{0,1\}$  be  $\lambda$ -colourings for  $i < \mu$ . Then there is a colouring  $F : \lambda^{>2} \rightarrow \{0,1\}$  such that  $F_i \leq F$  for every  $i < \mu$ .*

*Proof.* CASE 1.  $\mu \leq 2^{|\alpha|}$  for some  $\alpha < \lambda$ .

Let  $\rho_i \in {}^\alpha 2$  for  $i < \mu$  be pairwise distinct. For  $\eta \in \lambda^{>2}$  let  $h_i(\eta) = \rho_i \widehat{\ } \eta$ . Define  $F$  by:

$$F(\nu) = \begin{cases} 0 & \text{if } \ell g(\nu) < \alpha, \text{ or } \ell g(\nu) \geq \alpha \\ & \text{but } \nu \upharpoonright \alpha \notin \{\rho_i : i < \mu\}, \\ F_i(\langle \nu(\alpha + \varepsilon) : \varepsilon < \ell g(\nu) - \alpha \rangle) & \text{if } \ell g(\nu) \geq \alpha \text{ and for some } i < \mu, \nu \upharpoonright \alpha = \rho_i. \end{cases}$$

It is easy to see that  $F : \lambda^{>2} \rightarrow \{0,1\}$  and  $h_i$  exemplifies that  $F_i \leq F$ .

CASE 2.  $\mu = \lambda$ .

For  $\eta \in \lambda^{>2}$ ,  $i < \mu$ , we define  $h_i(\eta) \in {}^{i+1+\ell g(\eta)} 2$  as follows: for  $\gamma < i + 1 + \ell g(\eta)$  we let

$$h_i(\eta)(\gamma) = \begin{cases} 0 & \text{if } \gamma < i, \\ 1 & \text{if } \gamma = i, \\ \eta(\gamma - (i + 1)) & \text{otherwise.} \end{cases}$$

Next, for  $\nu \in \lambda^{>2}$  define:

$$F(\nu) = \begin{cases} F_i(\langle \nu(i + 1 + \gamma) : \gamma < \ell g(\nu) - (i + 1) \rangle) & \text{if } i = \min\{j : \nu(j) = 1\} \\ 0 & \text{if there is no such } i. \end{cases}$$

Now check.

CASE 3. Otherwise, for each  $\alpha < \lambda$  choose  $F^\alpha : \lambda^{>2} \rightarrow \{0,1\}$  such that  $(\forall i < 2^{|\alpha|})(F_i \leq F^\alpha)$  (exists by Case 1). Let  $F : \lambda^{>2} \rightarrow \{0,1\}$  be such that  $(\forall \alpha < \lambda)(F^\alpha \leq F)$  (exists by Case 2).

The proposition follows.  $\square$

**Theorem 2.2.** *If (A) then (B) where*

- (A) (a)  $\lambda$  is a regular uncountable cardinal.  
 (b)  $F^{\text{tr}} : \lambda^{>2} \rightarrow 3$ .  
 (c) For  $i < 3$  let  $F_i : \lambda^{>2} \rightarrow \{0,1\}$  be such that

$$F_i(\eta) = \begin{cases} 1 & \text{if } F^{\text{tr}}(\eta) = i, \\ 0 & \text{otherwise,} \end{cases}$$

- (d) Let  $F : \lambda^{>2} \rightarrow \{0,1\}$  be such that  $(\forall i < 3)(F_i \leq F)$ . Moreover  $F'_2 \leq F$  where  $F'_2$  is the function with domain  $\lambda^{<2}$  defined by  $F'_2(\eta) = \min\{1, F(\eta)\}$ . (remember 1.11(3)),

(e)  $\text{ID}^b(F^\otimes)$  is  $\lambda^+$ -saturated, i.e. there is no sequence  $\langle A_\alpha : \alpha < \lambda^+ \rangle$  such that<sup>3</sup>

$$(\forall \alpha < \beta < \lambda^+)(A_\alpha \notin \text{ID}^b(F^\otimes) \quad \& \quad \|A_\alpha \cap A_\beta\| < \lambda).$$

(B) Then there is a weak diamond sequence for  $F^{\text{tr}}$ , even for every  $S \in \mathfrak{B}(F^\otimes) \setminus \text{ID}^c(F^\otimes)$ .

*Proof.* Let  $S_F^*$  be as in 1.19. Since we are assuming  $\lambda \notin \text{ID}^c(F^\otimes)$  necessarily  $\lambda \setminus S_F^* \notin \text{ID}^b(F^\otimes)$ .

[Why? because by 1.20(1)(b) we have  $\text{ID}^c(F^\otimes) = \text{ID}^b(F^\otimes) + S_F^*$ ].

It follows from 1.16 and 1.17 that there are disjoint sets  $X_0, X_1 \subseteq \lambda$  (even disjoint from  $S_F^*$  from 1.19) such that  $X_0, X_1 \notin \text{ID}^b(F^\otimes)$ ,  $X_0 \cup X_1 \in \mathfrak{B}(F^\otimes)$  and for every  $f \in {}^\lambda 2$  we have one of the following:

- (a) the sequence  $\langle F(f \upharpoonright \delta) : \delta \in X_0 \cup X_1 \rangle$  is  $\text{ID}^b(F^\otimes)$ -almost constant, or
- (b) both sequences  $\langle F(f \upharpoonright \delta) : \delta \in X_0 \rangle$  and  $\langle F(f \upharpoonright \delta) : \delta \in X_1 \rangle$  are not  $\text{ID}^b(F^\otimes)$ -almost constant.

It follows from 1.19(2) that we may assume that there is  $\eta \in {}^{X_1} 2$  such that for every  $f \in {}^\lambda 2$  the set

$$\{\delta \in X_1 : F(f \upharpoonright \delta) = \eta(\delta)\}$$

is stationary. Define a function  $\rho \in {}^\lambda 2$  as follows:

$$\rho(\alpha) = \begin{cases} 1 + \eta(\alpha) & \text{if } \alpha \in X_1, \\ 0 & \text{otherwise.} \end{cases}$$

**Claim 2.2.1.**  $\rho$  is a weak diamond sequence for  $F^{\text{tr}}$  even on  $X_0 \cup X_1$ .

*Proof of the claim.* Let  $f \in {}^\lambda 2$  and we shall prove that  $Y_f = \{\delta \in X_0 \cup X_1 : F^{\text{tr}}(\eta \upharpoonright \delta) = \eta(\delta)\} \neq \emptyset \pmod{\text{ID}^b(F^\otimes)}$ .

If  $\{\alpha \in X_0 : F^{\text{tr}}(f \upharpoonright \alpha) = 2\} \notin \text{ID}^b(F)$  then we are done (remember 1.3(3)). Otherwise (by the definition of  $F_0$ ), we have

$$\{\alpha \in X_0 : F_2(f \upharpoonright \alpha) = 1\} \in \text{ID}^b(F).$$

For  $\ell < 3$ , as  $F_\ell \leq F$  let  $f_\ell \in {}^\lambda 2$  be such that the set  $\{\alpha < \lambda : F_\ell(f \upharpoonright \alpha) = F(f_\ell \upharpoonright \alpha)\}$  contains a club of  $\lambda$  and  $g \in {}^\lambda 2$  such that the set  $\{\alpha < \lambda : F'_2(g \upharpoonright \alpha) = F(f \upharpoonright \alpha)\}$  contains a club of  $\lambda$ , exist by clause (A)(d) of the assumption of the theorem.

We now use  $f_2$ . Then

$$\{\alpha \in X_0 : F(f_2 \upharpoonright \alpha) = 1\} \in \text{ID}^b(F^\otimes),$$

and hence, by the choice of the sets  $X_0, X_1$ , clause (b) there fails hence clause (a) holds, so

$$\{\alpha \in X_1 \cup X_1 : F(f_2 \upharpoonright \alpha) = 1\} \in \text{ID}^b(F^\otimes).$$

<sup>3</sup> As is well known, writing below  $A_\alpha \cap A_\beta \in \text{ID}^b(F^\otimes)$  instead  $\|A_\alpha \cap A_\beta\| < \lambda$  does not change anything.

Consequently,

$$Z = \{\alpha \in X_1 : F^{\text{tr}}(f \upharpoonright \alpha) = 2\} = \{\alpha \in X_1 : F_2(f \upharpoonright \alpha) = 1\} \in \text{ID}^b(F^\otimes).$$

Now we use the choice of  $\eta$ , by it we know that the set

$$Y = \{\delta \in X_1 : F(g \upharpoonright \delta) = \eta(\delta)\}$$

is stationary and even  $\neq \emptyset \pmod{\text{ID}^b(F^\otimes)}$ . Hence for some  $k \in \{0, 1\}$  the set

$$Y_k = \{\delta \in X_1 : F(g \upharpoonright \delta) = k = \eta(\delta)\}$$

is stationary and even  $\neq \emptyset \pmod{\text{ID}^b(F^\otimes)}$ , but  $\{\delta \in X_1 : F(g \upharpoonright \delta) = F'_2(f \upharpoonright \delta)\}$  contains a club. Hence

$$Y_k^* = \{\delta \in X_1 : F(g \upharpoonright \delta) = k = \eta(\delta) \text{ and } F(g \upharpoonright \delta) = F'_2(f \upharpoonright \delta) \text{ and } \delta \notin Z\}$$

is stationary and even  $\neq \emptyset \pmod{\text{ID}^b(F^\otimes)}$ . Finally note that

$$\delta \in Y_k^* \Rightarrow F(f_1 \upharpoonright \delta) = \eta(\delta) = F_1(f \upharpoonright \delta) = k \Rightarrow F^{\text{tr}}(f \upharpoonright \delta) = k.$$

Thus the claim and the theorem are proved.  $\square$

$\square$

**Theorem 2.3.** *Suppose  $F^{\text{tr}}$  is a  $(\lambda, \theta)$ -colouring,  $\theta \leq \lambda$  and  $F_i$  (for  $i < \theta$ ) are given by*

$$F_i(f) = \begin{cases} 1 & \text{if } F(f) = i, \\ 0 & \text{otherwise.} \end{cases}$$

Let  $F : \lambda > 2 \rightarrow 2$  be such that  $(\forall i < \theta)(F_i \leq F)$  and let  $F^\otimes$  be as in 1.5 for  $F$ . Suppose that  $\text{ID}^b(F^\otimes)$  is  $\lambda^+$ -saturated, and  $S_{F^\otimes}^* \neq \lambda$  (equivalently  $\lambda \notin \text{ID}^c(F^\otimes)$ ).

If  $(\otimes)$  then  $(\star)$  where:

$(\otimes)$  there are sets  $Y_i \subseteq \lambda \setminus S_{F^\otimes}^*$  for  $i < \theta$  such that

(a)  $(\forall i < \theta)(Y_i \notin \text{ID}^b(F^\otimes))$ ,

(b) the sets  $Y_i$  are pairwise disjoint or at least

$$(\forall i < j < \theta)(Y_i \cap Y_j \in \text{ID}^b(F^\otimes)),$$

(c)  $\bigcap_{i < \theta} \min_{F^\otimes}(Y_i) \notin \text{ID}^b(F^\otimes)$ , see 1.12(h).

$(\star)$  there is a weak diamond sequence  $\eta \in {}^\lambda \theta$  for  $F^{\text{tr}}$ , i.e.

$(\forall f \in {}^\lambda 2)(\{\delta < \lambda : F^{\text{tr}}(f \upharpoonright \delta) = \eta(\delta)\}$  is stationary and even  $\neq \emptyset \pmod{\text{ID}^b(F^\otimes)}$ );

moreover

$$(\forall f \in {}^\lambda 2)(\{\delta < \lambda : F^{\text{tr}}(f \upharpoonright \delta) = \eta(\delta)\} \notin \text{ID}^b(F^\otimes)).$$

*Proof.* We may assume that the sets  $\langle Y_i : i < \theta \rangle$  are pairwise disjoint (otherwise we use  $Y'_i = Y_i \setminus \bigcup_{j < i} Y_j$ ). Let  $\eta \in {}^\lambda \theta$  be such that  $(\forall i < \theta)(\eta \upharpoonright Y_i = i)$ .

Note that if

$$\{\delta \in Y_i : F^{\text{tr}}(f \upharpoonright \delta) = i\} \in \text{ID}^b(F^\otimes)$$

then we also have

$$\{\delta < \lambda : F^{\text{tr}}(f \upharpoonright \delta) = i\} \in \mathfrak{B}(F^\otimes)$$

(use  $F_i \leq F \leq F^\otimes$ ). Consequently, in this case, we have

$$\{\delta \in \min_{F^\otimes}(Y_i) : F^{\text{tr}}(f \upharpoonright \delta) = i\} \in \text{ID}^b(F^\otimes).$$

If this occurs for every  $i < \theta$  then

$$\{\delta \in \bigcap_{i < \theta} \min_{F^\otimes}(Y_i) : (\exists i < \theta)(F(f \upharpoonright \delta) = i)\} \in \text{ID}^b(F^\otimes),$$

but for each  $\delta$ , for some  $i < \theta$  we have  $F(f \upharpoonright \delta) = i$ , a contradiction.  $\square$

**Proposition 2.4.** *Under the assumptions of 2.3 (so the ideal  $\text{ID}^b(F^\otimes)$  is  $\lambda^+$ -saturated), if  $X \subseteq \lambda \setminus S_{F^\otimes}^*$ ,  $X \notin \text{ID}^b(F^\otimes)$  then there is a partition  $(X_0, X_1)$  of  $X$  (so  $X_0 \cup X_1 = X$ ,  $X_0 \cap X_1 = \emptyset$ ) such that*

$$X_0, X_1 \notin \text{ID}^b(F^\otimes), \quad \text{and} \quad \min_{F^\otimes}(X_0) = \min_{F^\otimes}(X_1) = \min_{F^\otimes}(X).$$

*Proof.* Let

$$\mathcal{A}_{F^\otimes} \stackrel{\text{def}}{=} \{Z \subseteq \lambda : Z \notin \text{ID}^b(F^\otimes) \text{ and there is a partition } (Z_0, Z_1) \text{ of } Z \text{ such that } \min_{F^\otimes}(Z_0) = \min_{F^\otimes}(Z_1) \pmod{\text{ID}^b(F^\otimes)}\}.$$

Note that, by 1.12(h),

$$(*) \quad (\forall Y \in \text{ID}^b(F^\otimes)^+)(\exists Z \in \mathcal{A}_{F^\otimes})(Z \subseteq Y).$$

Let  $X \subseteq \lambda$ ,  $X \notin \text{ID}^b(F^\otimes)$  and let  $\langle Z_\alpha : \alpha < \alpha^* \rangle$  be a maximal sequence such that for each  $\alpha < \alpha^*$ :

$$Z_\alpha \in \mathcal{A}_{F^\otimes}, \quad Z_\alpha \subseteq X, \quad \text{and} \quad (\forall \beta < \alpha)(Z_\alpha \cap Z_\beta \in \text{ID}^b(F^\otimes)).$$

Necessarily  $\alpha^* < \lambda^+$ , so without loss of generality  $\alpha^* \leq \lambda$ ,  $\min(Z_\alpha) > \alpha$  and  $Z_\alpha \cap Z_\beta = \emptyset$  for  $\alpha < \beta < \alpha^*$ . Let  $\langle Z_\alpha^0, Z_\alpha^1 \rangle$  be a partition of  $Z_\alpha$  witnessing  $Z_\alpha \in \mathcal{A}_{F^\otimes}$ . Put

$$Z_0 \stackrel{\text{def}}{=} \bigcup_{\alpha < \alpha^*} Z_\alpha^0 \quad \text{and} \quad Z_1 \stackrel{\text{def}}{=} \bigcup_{\alpha < \alpha^*} Z_\alpha^1.$$

Then  $Z_0 \cap Z_1 = \emptyset$ ,  $Z_0 \cup Z_1 \subseteq X$ . Note that  $\bigcup_{\alpha < \alpha^*} Z_\alpha$  is equal to the diagonal union and, by (\*) above,  $X \setminus \bigcup_{\alpha < \alpha^*} Z_\alpha \in \text{ID}^b(F^\otimes)$ . Consequently we may assume  $Z_0 \cup Z_1 = \bigcup_{\alpha < \alpha^*} Z_\alpha = X$ . Next, since

$$\min_{F^\otimes}(Z_0) \supseteq \min_{F^\otimes}(Z_\alpha^0) \supseteq Z_\alpha^0 \cup Z_\alpha^1 = Z_\alpha,$$

we get

$$\min_{F^\otimes}(Z_0) \supseteq \bigcup_{\alpha < \alpha^*} Z_\alpha = X = Z_0 \cup Z_1,$$

and similarly one shows that  $\min_{F^\otimes}(Z_1) \supseteq X$ . Now we use 1.12(h) to finish the proof.  $\square$

**Proposition 2.5.** *Under the assumptions of 2.3:*

- (1) If  $2^\theta < \lambda$  then there is a sequence  $\langle Y_i : i < \theta \rangle$  as required in 2.3( $\otimes$ )
- (2) Similarly if  $\theta \leq \aleph_0$ .
- (3) In both cases, if  $S \notin \text{ID}^b(F^\otimes)$  then we can demand  $(\forall i < \theta)(Y_i \subseteq S)$ .

*Proof.* 1) By induction on  $\alpha \leq \theta$  we choose sets  $X_\eta \subseteq \lambda$  for  $\eta \in {}^\alpha 2$  such that:

- (i)  $X_\emptyset \notin \text{ID}^b(F^\otimes)$ ,
- (ii) if  $\alpha$  is limit then  $X_\eta = \bigcap_{i < \alpha} X_{\eta \upharpoonright i}$ ,
- (iii) if  $\alpha = \beta + 1$ ,  $\eta \in {}^\beta 2$  and  $X_\eta \in \text{ID}^b(F^\otimes)$  then  $X_{\eta \frown \langle 0 \rangle} = X_\eta$ ,  $X_{\eta \frown \langle 1 \rangle} = \emptyset$ ;  
if  $\alpha = \beta + 1$ ,  $\eta \in {}^\beta 2$  and  $X_\eta \notin \text{ID}^b(F^\otimes)$  then  $(X_{\eta \frown \langle 0 \rangle}, X_{\eta \frown \langle 1 \rangle})$  is a partition of  $X_\eta$  such that  $\min_{F^\otimes}(X_{\eta \frown \langle 0 \rangle}) = \min_{F^\otimes}(X_{\eta \frown \langle 1 \rangle}) = \min_{F^\otimes}(X_\eta)$ .

It follows from 2.4 that we can carry out the construction.

Clearly  $\langle X_\eta : \eta \in {}^\theta 2 \rangle$  is a partition of  $X_\emptyset$ , so (as  $2^\theta < \lambda$  and  $\text{ID}^b(F^\otimes)$  is  $\lambda$ -complete) we can find a sequence  $\eta \in {}^\theta 2$  such that  $X_\eta \notin \text{ID}^b(F^\otimes)$ . Then

$$(\forall \alpha < \theta)(X_{\eta \upharpoonright \alpha} \notin \text{ID}^b(F^\otimes))$$

(as each of these sets includes  $X_\eta$ ). Moreover, for each  $\alpha < \theta$  and for  $\ell = 0, 1$  we have

$$\min_{F^\otimes}(X_{\eta \upharpoonright \alpha \frown \langle \ell \rangle}) \supseteq X_{\eta \upharpoonright \alpha} \supseteq X_\eta.$$

Put  $Y_\alpha = X_{\eta \upharpoonright \alpha \frown \langle 1 - \eta(\alpha) \rangle}$ . Then  $\langle Y_\alpha : \alpha < \theta \rangle$  is a sequence of pairwise disjoint sets (as  $X_{\eta \upharpoonright \alpha \frown \langle 0 \rangle} \cap X_{\eta \upharpoonright \alpha \frown \langle 1 \rangle} = \emptyset$ ) and for every  $\alpha < \theta$

$$Y_\alpha \notin \text{ID}^b(F^\otimes) \quad \text{and} \quad \min_{F^\otimes}(Y_\alpha) \supseteq X_{\eta \upharpoonright \alpha} \supseteq X_\eta.$$

Hence  $\bigcap_{\alpha < \theta} \min_{F^\otimes}(Y_\alpha) \notin \text{ID}^b(F^\otimes)$ . Let  $Z_\alpha = Y_\alpha \cap \min_{F^\otimes}(X_\eta)$ . Note that  $\min_{F^\otimes}(Z_\alpha) = \min_{F^\otimes}(X_\eta)$  (the “ $\leq$ ” is clear; if  $\min_{F^\otimes}(Z_\alpha) < \min_{F^\otimes}(X_\eta)$  then  $\min_{F^\otimes}(X_\eta) \setminus \min_{F^\otimes}(Z_\alpha)$  contradicts the definition of  $\min_{F^\otimes}(Y_\alpha)$ ). Thus the sequence  $\langle Z_\alpha : \alpha < \theta \rangle$  is as required. Moreover

$$\min_{F^\otimes}(Z_\alpha) = \bigcup_{\beta} \min_{F^\otimes}(Z_\beta).$$

2) Let  $X \subseteq \lambda$ ,  $X \notin \text{ID}^b(F^\otimes)$ . By induction on  $n$  we choose sets  $X'_n, X''_n$  such that  $X'_n \cap X''_n = \emptyset$ ,  $X'_n \cup X''_n \supseteq X$ , and

$$\min_{F^\otimes}(X'_n) = \min_{F^\otimes}(X''_n) = \min_{F^\otimes}(X).$$

For  $n = 0$  we use 2.4 for  $X$  to get  $X'_0, X''_0$ . For  $n + 1$  we use 2.4 for  $X''_n$  to get  $X'_{n+1}, X''_{n+1}$ .

Finally we let  $Y_n = X''_n$  (note that  $\min_{F^\otimes}(Y_n) = \min_{F^\otimes}(X)$ ).  $\square$

*Conclusion 2.6.* Assume that

- (A)  $\lambda$  is a regular uncountable cardinal,

- (B)  $F$  is a  $(\lambda, \theta)$ -colouring such that  $\lambda \notin \text{ID}^b(F)$  and  $\text{ID}^b(F)$  is  $\lambda^+$ -saturated,
- (C)  $2^\theta < \lambda$  or  $\theta = \aleph_0$ ,
- (D)  $(\exists \mu < \lambda)(2^\mu = 2^{<\lambda} < 2^\lambda)$  or at least  $\lambda \notin \text{WDMId}_\lambda$  or at least  $\lambda \notin \text{ID}^c(F)$ .

Then there is a weak diamond sequence for  $F$ . Moreover, there is  $\eta \in {}^\lambda\theta$  such that for each  $f \in \text{DOM}_\lambda(F)$  we have

$$\{\delta < \lambda : F(f \upharpoonright \delta) = \eta(\delta)\} \notin \text{ID}^b(F).$$

### 3. AN APPLICATION OF WEAK DIAMOND

In this section we present an application of Weak Diamond in model theory. For more on model-theoretic investigations of this kind we refer the reader to [She01] and earlier work [She87a] (and see [She09c]), and to an excellent survey by Makowsky, [Mak85].

**Definition 3.1.** Let  $\mathfrak{K}$  be a collection of models.

- (1) For a cardinal  $\lambda$ ,  $\mathfrak{K}_\lambda$  stands for the collection of all members of  $\mathfrak{K}$  of size  $\lambda$ .
- (2) We say that a partial order  $\leq_{\mathfrak{K}}$  on  $\mathfrak{K}_\lambda$  is  $\lambda$ -nice if
  - ( $\alpha$ )  $\leq_{\mathfrak{K}}$  is a suborder of  $\subseteq$  and it is closed under isomorphisms of models (i.e. if  $M, N \in \mathfrak{K}_\lambda$ ,  $M \leq_{\mathfrak{K}} N$  and  $f : N \rightarrow N' \in \mathfrak{K}_\lambda$  is an isomorphism then  $f[M] \leq_{\mathfrak{K}} N'$ ),
  - ( $\beta$ )  $(\mathfrak{K}_\lambda, \leq_{\mathfrak{K}})$  is  $\lambda$ -closed (i.e. any  $\leq_{\mathfrak{K}}$ -increasing sequence of length  $\leq \lambda$  of elements of  $\mathfrak{K}_\lambda$  has a  $\leq_{\mathfrak{K}}$ -upper bound in  $\mathfrak{K}_\lambda$ ) and
  - ( $\gamma$ ) if  $\bar{M} = \langle M_\alpha : \alpha < \lambda \rangle$  is an  $\leq_{\mathfrak{K}}$ -increasing sequence of elements of  $\mathfrak{K}_\lambda$  then  $\bigcup_{\alpha < \lambda} M_\alpha$  is the  $\leq_{\mathfrak{K}}$ -upper bound to  $\bar{M}$  (so  $\bigcup_{\alpha < \lambda} M_\alpha \in \mathfrak{K}_\lambda$ ).
- (3) Let  $N \in \mathfrak{K}_\lambda$ ,  $A \subseteq |N|$ . We say that *the pair  $(A, N)$  has the w. amalgamation property in  $\mathfrak{K}_\lambda$*  if for every  $N_1, N_2 \in \mathfrak{K}_\lambda$  such that  $N \leq_{\mathfrak{K}} N_1$ ,  $N \leq_{\mathfrak{K}} N_2$  there are  $N^* \in \mathfrak{K}_\lambda$  and  $\leq_{\mathfrak{K}}$ -embeddings  $f_1, f_2$  of  $N_1, N_2$  into  $N^*$ , respectively, such that  $f_1(A) = f_2(A)$ . (In words:  $N_1, N_2$  can be amalgamated over  $(A, N)$  setwise.)
- (4) We say that  $(\mathfrak{K}, \leq_{\mathfrak{K}})$  has *the amalgamation property for  $\lambda$*  if for every  $M_0, M_1, M_2 \in \mathfrak{K}_\lambda$  such that  $M_0 \leq_{\mathfrak{K}} M_1$ ,  $M_0 \leq_{\mathfrak{K}} M_2$  there are  $M \in \mathfrak{K}_\lambda$  and  $\leq_{\mathfrak{K}}$ -embeddings  $f_1, f_2$  of  $M_1, M_2$  into  $M$ , respectively, such that

$$M_0 \leq_{\mathfrak{K}} M \quad \text{and} \quad f_1 \upharpoonright M_0 = f_2 \upharpoonright M_0 = \text{id}_{M_0}.$$

**Theorem 3.2.** *Assume that  $\lambda$  is a regular uncountable cardinal for which the weak diamond holds (i.e.  $\lambda \notin \text{WDMId}_\lambda$ ). Suppose that  $\mathfrak{K}$  is a class of models,  $\mathfrak{K}$  is categorical in  $\lambda$  (i.e. all models from  $\mathfrak{K}_\lambda$  are isomorphic), it is closed under isomorphisms of models, and  $\leq_{\mathfrak{K}}$  is a  $\lambda$ -nice partial order on  $\mathfrak{K}_\lambda$  and  $M \in \mathfrak{K}_\lambda$ . Let  $\bar{A} = \langle A_\alpha : \alpha < \lambda \rangle$  be an increasing continuous*

sequence of subsets of  $|M|$  such that

$$(\forall \alpha < \lambda)(\|A_\alpha\| < \lambda) \quad \text{and} \quad \bigcup_{\alpha < \lambda} A_\alpha = M.$$

Then the set

$$S_M^{\bar{A}} \stackrel{\text{def}}{=} \{\alpha < \lambda : (A_\alpha, M) \text{ does not have the w. amalgamation property}\}$$

is in  $\text{WdMId}_\lambda$ .

*Proof.* Assume that  $S_M^{\bar{A}} \notin \text{WdMId}_\lambda$ .

We can fix a partition  $\langle D_i : i < \lambda \rangle$  of  $\lambda$  to sets each of cardinality  $\lambda$ .

We may assume that  $|M| = \lambda$ . By induction on  $i < \lambda$  we choose pairs  $(B_\eta, N_\eta)$  and sequences  $\langle C_j^\eta : j < \lambda \rangle$  for  $\eta \in {}^i 2$  such that

- (a)  $\|B_\eta\| < \lambda$ ,  $N_\eta \in \mathfrak{K}_\lambda$ ,  $B_\eta \subseteq |N_\eta| \subseteq \cup\{D_j : j < 1 + \ell g(\eta)\}$
- (b)  $\langle C_j^\eta : j < \lambda \rangle$  is increasing continuous,  $\bigcup_{j < \lambda} C_j^\eta = |N_\eta|$ ,  $\|C_j^\eta\| < \lambda$ ,
- (c) if  $\nu \triangleleft \eta$  then  $N_\nu \leq_{\mathfrak{K}} N_\eta$  and  $B_\nu \subseteq B_\eta$ ,
- (d) if  $j_1, j_2 < i$  then  $C_{j_2}^{\eta \upharpoonright j_1} \subseteq B_\eta$ ,
- (e) if the pair  $(B_\eta, N_\eta)$  does not have the w. amalgamation property in  $\mathfrak{K}_\lambda$  then  $N_{\eta \frown \langle 0 \rangle}$ ,  $N_{\eta \frown \langle 1 \rangle}$  witness it (i.e. they cannot be w. amalgmated over  $B_\eta$ ),
- (f) if  $i$  is limit and  $\eta \in {}^i 2$  then  $B_\eta = \bigcup_{j < i} B_{\eta \upharpoonright j}$ ,  $\bigcup_{j < i} N_{\eta \upharpoonright j} \subseteq N_\eta$ .

There are no problems with carrying out the construction (remember that  $\leq_{\mathfrak{K}}$  is a nice partial order. Finally, for  $\eta \in {}^\lambda 2$  we let  $B_\eta = \bigcup_{i < \lambda} B_{\eta \upharpoonright i}$  and  $N_\eta = \bigcup_{i < \lambda} N_{\eta \upharpoonright i}$ . Clearly, by 3.1(2)( $\gamma$ ), we have  $N_\eta \in \mathfrak{K}$  and  $B_\eta \subseteq |N_\eta|$  for each  $\eta \in {}^\lambda 2$ . Moreover,

$$|N_\eta| = \bigcup_{j < \lambda} |N_{\eta \upharpoonright j}| = \bigcup_{j < \lambda} \bigcup_{i < \lambda} C_i^{\eta \upharpoonright j} = \bigcup_{j^* < \lambda} \bigcup_{j_1, j_2 < j^*} C_{j_2}^{\eta \upharpoonright j_1} \subseteq \bigcup_{j^* < \lambda} B_{\eta \upharpoonright j^*} = B_\eta,$$

and thus  $B_\eta = |N_\eta|$ . Since  $\mathfrak{K}$  is categorical in  $\lambda$ , for each  $\eta \in {}^\lambda 2$  there is an isomorphism  $f_\eta : N_\eta \xrightarrow{\text{ontq}} M$ .

Fix  $\eta \in {}^\lambda 2$  for a moment.

Let  $E_\eta = \{\delta < \lambda : f_\eta[B_{\eta \upharpoonright \delta}] = A_\delta = \delta\}$ . Clearly,  $E_\eta$  is a club of  $\lambda$ . Note that if  $\delta \in E_\eta$  then:

- ( $\boxtimes$ )  $\delta \in S_M^{\bar{A}} \Rightarrow (A_\delta, M)$  does not have the w. amalgamation property
- $\Rightarrow (B_{\eta \upharpoonright \delta}, N_\eta)$  fails the w. amalgamation property
- $\Rightarrow (B_{\eta \upharpoonright \delta}, N_{\eta \upharpoonright \delta})$  fails the w. amalgamation property
- $\Rightarrow N_{\eta \upharpoonright \delta \frown \langle 0 \rangle}$ ,  $N_{\eta \upharpoonright \delta \frown \langle 1 \rangle}$  cannot be w. amalgmated over  $(B_{\eta \upharpoonright \delta}, N_{\eta \upharpoonright \delta})$
- $\Rightarrow$  for each  $\nu \in {}^\lambda 2$  such that  $\eta \upharpoonright \delta \frown \langle 1 - \eta(\delta) \rangle \triangleleft \nu$  we have  $f_\nu \upharpoonright B_{\eta \upharpoonright \delta} \neq f_\eta \upharpoonright B_{\eta \upharpoonright \delta}$ .

We define a colouring

$$F : \bigcup_{\alpha < \lambda} \alpha(\mathcal{H}(\lambda)) \longrightarrow \{0, 1\}$$

by letting, for  $f \in \text{DOM}_\alpha$ ,  $\alpha < \lambda$ ,

$$F(f) = 1 \quad \text{iff} \quad (\exists \eta \in {}^\lambda 2)(\eta(\alpha) = 0 \ \& \ (\forall i < \alpha)(f(i) = (\eta(i), f_\eta^{-1}(i)))).$$

We have assumed  $S_M^{\bar{A}} \notin \text{WDMId}_\lambda$ , so there is  $\rho \in {}^\lambda 2$  such that for each  $f \in \text{DOM}_\lambda$  the set

$$S_f = \{\delta \in S_M^{\bar{A}} : \rho(\delta) = F(f \upharpoonright \delta)\}$$

is stationary and even  $\neq \emptyset \pmod{\text{ID}^b(F^\otimes)}$ . Let  $f \in \text{DOM}_\lambda$  be defined by  $f(i) = (\rho(i), f_\rho^{-1}(i))$  (for  $i < \lambda$ ). Note that

$$\text{if } \alpha \in E_\rho, \rho(\alpha) = 0$$

then  $\rho$  is a witness to  $F(f \upharpoonright \alpha) = 1$  and hence  $\alpha \notin S_f$ .

Since  $S_f$  is stationary and even  $\neq \emptyset \pmod{\text{ID}^b(F^\otimes)}$  and  $E_\rho$  is a club of  $\lambda$  we may pick  $\delta \in S_f \cap E_\rho$ . Then  $\rho(\delta) = 1$  and hence  $F(f \upharpoonright \delta) = 1$ , so let  $\eta_\delta \in {}^\lambda 2$  be a witness for it. It follows from the definition of  $F$  that then  $\eta_\delta(\delta) = 0$ , and  $\eta_\delta \upharpoonright \delta = \rho \upharpoonright \delta$ , and  $f_{\eta_\delta}^{-1} \upharpoonright \delta = f_\rho^{-1} \upharpoonright \delta$ . Hence  $f_{\eta_\delta} \upharpoonright B_{\eta_\delta \upharpoonright \delta} = f_\rho \upharpoonright B_{\rho \upharpoonright \delta}$ , so both have range  $A_\delta = \delta$  (and  $\delta \in E_{\eta_\delta} \cap E_\rho \cap S_M^{\bar{A}}$ ). But now we get a contradiction with  $(\boxtimes)$ .  $\square$

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