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CANONICAL UNIVERSAL LOCALLY FINITE GROUPS SH1175

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Abstract. We prove that for $\lambda = \beth_\omega$ or just λ strong limit singular of cofinality $\aleph_0$, if there is a universal member in the class $\mathbf{K}^{\rm lf}_\lambda$ of locally finite groups of cardinality λ, then there is a canonical one (parallel to special models for elementary classes, which is the replacement of universal homogeneous ones and saturated ones in cardinals $\lambda = \lambda^{<\lambda}$).

For this we rely on the existence of enough indecomposable such groups, as proved in “Density of indecomposable locally finite groups”. We also more generally deal with the existence of universal members in general classes for such cardinals.

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The author thanks Alice Leonhardt for the beautiful typing. First typed February 18, 2016 as part of [She]. In References [She17a, 0.22=Lz19] means [She17a, 0.22] has label z19 there, L stands for label; so will help if [She17a] will change. Also [She20, Th.3.5=Th.1.5=Lb24] refer to Th.3.5 in the published version, Th. 1.5 in the arXive version, and label b24 in the latex file. The reader should note that the version in my website is usually more updated than the one in the mathematical archive.

§ 0. Introduction

§ 0(A). Background and aims.

Our motivation is investigating the class $\mathbf{K}_{\rm lf}$ of locally finite groups so the reader may consider only this case ignoring the general case; or consider universal classes (see Def. 0.5). We continue [She17a], see history there and on earlier history the book [KW73].
We wonder:

Problem 0.1. 1) Is there a universal $G \in \mathbf{K}^{\rm lf}_\lambda$ (= the class of members of $\mathbf{K}_{\rm lf}$ of cardinality λ), see 0.4(1); e.g. for $\lambda = \beth_\omega$? Or just λ strong limit of cofinality $\aleph_0$ (which is not above a compact cardinal)?
2) May there be a universal $G \in \mathbf{K}^{\rm lf}_\lambda$, when $\lambda < \lambda^{\aleph_0}$, e.g. for $\lambda = \aleph_1 < 2^{\aleph_0}$, i.e. consistently?

Generally on the problem of the existence of a universal model of a class in cardinality λ see the classical Jonsson [Jón56], [Jón60], Morley-Vaught [MV62] and the recent survey [Shear].

Returning to locally finite groups, concerning 0.1(1) recall that by Grossberg-Shelah [GS83], if $\lambda = \lambda^{\aleph_0}$ then there is no universal member of $\mathbf{K}^{\rm lf}_\lambda$. but if λ, a strong limit cardinal of cofinality $\aleph_0$ is above a compact cardinal κ, then there is $G \in \mathbf{K}^{\rm lf}_\lambda$ which is universal. So Problem 0.1 address the open cases.

Let us consider the model theory of locally finite groups. Recall

Definition 0.2. 1) G is a lf (locally finite) group if G is a group and every finitely generated subgroup is finite.
2) G is an exlf (existentially closed lf) group (in [KW73] it is called ulf, universal locally finite group) when G is a locally finite group and for any finite groups $K \subseteq L$ and embedding of K into G, the embedding can be extended to an embedding of L into G.
3) Let $\mathbf{K}_{\rm lf}$ be the class of lf (locally finite) groups (partially ordered by $\subseteq$, being a subgroup) and let $\mathbf{K}_{\rm exlf}$ be the class of existentially closed $G \in \mathbf{K}_{\rm lf}$.

Wehrfritz asked about the categoricity of the class of exlf groups in any $\lambda > \aleph_0$. This was answered by Macintyre-Shelah [MS76] which proved that in every $\lambda > \aleph_0$ there are 2^λ non-isomorphic members of $\mathbf{K}^{\rm exlf}_\lambda$. This was disappointing in some sense: in $\aleph_0$ the class is categorical, so the question was perhaps motivated by the hope that also general structures in the class can be understood to some extent.

The existence of a universal can be considered as a weak positive answer.

A natural and frequent question on a class of structures is the existence of rigid members, i.e. ones with no non-trivial automorphism. Now any exlf group $G \in \mathbf{K}_{\rm exlf}$ has non-trivial automorphisms - the inner automorphisms (recalling it has a trivial center). So the natural question is about complete members where a group is called complete iff it has no non-inner automorphism.

Concerning the existence of a complete, locally finite group of cardinality λ: Hickin [Hic78] proved one exists in $\aleph_1$ (and more, e.g. he finds a family of $2^{\aleph_1}$ such groups pairwise far apart, i.e. no uncountable group is embeddable in two of them). Thomas [Tho86] assumed G.C.H. and built one in every successor cardinal (and more, e.g. it has no Abelian or just solvable subgroup of the same cardinality). Related are Giorgetta-Shelah [GS84], Shelah-Zigler [SZ79], which investigate $\mathbf{K}_{G_*}$ getting similar results. Dugas-Göbel [DG93, Th.2] prove that for $\lambda = \lambda^{\aleph_0}$ and $G_0 \in \mathbf{K}^{\rm lf}_{\le\lambda}$ there is a complete $G \in \mathbf{K}^{\rm exlf}_{\lambda^+}$ extending G_0; moreover 2^{λ^+} pairwise non-isomorphic ones. Then Braun-Göbel [BG03] got better results for complete locally finite p-groups.

Now [She17a] show that though the class $\mathbf{K}_{\rm exlf}$ is very "unstable" there is a large enough set of definable types so we can imitate stability theory and have reasonable control in building exlf groups, using quantifier free types. This may be considered a "correction" to the non-structure results discussed above. This was applied to build canonical extension of a locally finite group of the same cardinality and also endo-rigid locally finite groups in a more relaxed way.

Returning to the present work, here we return to the universality problem for $\mu = \beth_\omega$ or just strong limit of cofinality $\aleph_0$. We prove for $\mathbf{K}_{\rm lf}$ and similar classes that if there is a universal model of cardinality μ, then there is something like a special model of cardinality μ, in particular, universal and unique up to isomorphism. This relies on [She20], which proves the existence and even the density of so-called θ-indecomposable (i.e. θ is not a possible cofinality) models in $\mathbf{K}_{\rm lf}$ of various cardinalities continuing Carson-Shelah [CS20] which deal with the class of groups.

Returning to Question 0.1(1), a possible avenue is to try to prove the existence of universal members in μ when $\mu = \Sigma_{n<\omega}\mu_n$ each μ_n measurable $< \mu$, i.e. maybe for some reasonable classes this holds.

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§ 0(B). **Definitions.**

Context 0.3. $\mathbf{K}$ will be one of the following cases:

Case 1: $\mathbf{K} = \mathbf{K}_{\rm lf}$, the class of locally finite groups, so the submodel relation is just being a subgroup,

Case 2: $\mathbf{K}$ is a universal class, see Def 0.5(1) below, the submodel is just a submodel,

Case 3 $\mathbf{K}$ is $\mathfrak{k} = (K_{\mathfrak{k}}, \le_{\mathfrak{k}})$ an a.e.c. with $\mathrm{LST}_{\mathfrak{k}} < \mu$, see [She09, §1]; we shall only comment on it. In particular, in this context, in the definitions, $M \subseteq N$ should be replaced by $M \le_{\mathfrak{k}} N$.

Definition 0.4. 1) We say that $M \in \mathbf{K}_\mu$ is universal (in $\mathbf{K}$ or in $\mathbf{K}_\mu$) when every member of $\mathbf{K}_\mu$ can be embedded into it.
2) We say that $M \in \mathbf{K}$ is universal for $\mathbf{K}_{<\mu}$ when every $M \in \mathbf{K}_{<\mu}$ can be embedded into it; see Def 0.5(4) below.
3) We define similarly "$M \in \mathbf{K}$ is universal for $\mathbf{K}_\mu$" and "$M \in \mathbf{K}$ is universal for $\mathbf{K}_{\le\mu}$".

Definition 0.5. 1) We shall say that $\mathbf{K}$ is a universal class when for some vocabulary $\tau = \tau_{\mathbf{k}}$:

(a) $\mathbf{K}$ is a class of τ-models, closed under isomorphisms,
(b) a τ-model belongs to $\mathbf{K}$ iff every finitely generated sub-model belongs to it,

3) Let $\mathbf{K}_\mu$ be the class of $M \in \mathbf{K}$ of cardinality μ. We define $\mathbf{K}_{<\mu}, \mathbf{K}_{\le\mu}$ naturally.
4) For cardinals $\lambda \le \mu$ let $\mathbf{K}_{\mu,\lambda}$ be the class of pairs (N, M) such that $N \in \mathbf{K}_\mu, M \in \mathbf{K}_\lambda$ and $M \subseteq N$.

5) Let $(N_1, M_1) \leq_{\mu,\lambda} (N_2, M_2)$ mean that $(N_\ell, M_\ell) \in \mathbf{K}_{\mu,\lambda}$ for $\ell = 1, 2$ and $M_1 \subseteq M_2, N_1 \subseteq N_2$.
6) For $\mu \leq \lambda$ we define $\mathbf{K}_{\mu,<\lambda}$ and $\leq_{\mu,<\lambda}$ similarly.
7) A universal class $\mathbf{K}$ can be considered as the a.e.c. $\mathfrak{k} = (\mathbf{K}, \subseteq)$

Notation 0.6. 1) Let M, N and also G, H, L denote members of $\mathbf{K}$.
2) Let $|M|$ be the universe = set of elements of M and $\|M\|$ its cardinality.
3) Let a, b, c, d denote members of such M,
and $\bar{a}, \bar{b} \ldots$ denote sequences of such element.

Definition 0.7. 1) We say the pair (N, M) is an (χ, μ, κ)-amalgamation base (or amalgamation pair, but may omit χ when $\chi = \mu$, we may even omit μ, κ too) when:

(a) $(N, M) \in \mathbf{K}_{\mu,\kappa}$
(b) if $N_1 = N$ and $M \subseteq N_2 \in \mathbf{K}_\chi$ then N_1, N_2 can be amalgamted over M, this mean that for some N_3, f_1, f_2 we have $M \subseteq N_3 \in \mathbf{K}$ and f_ℓ-embeds N_ℓ into N_3 over M.

2) We say that the pair (N, M) is a universal (μ, λ)-amalgamation base (we may omit μ, λ) when:

(a) $(N, M) \in \mathbf{K}_{\mu,\lambda}$
(b) if $N \subseteq N' \in \mathbf{K}_\mu$ then N' can be embedded into N over M.

3) We may in parts (1),(2) omit μ, κ when $(\mu, \lambda) = (\|N\|, \|M\|)$.

§ 1. Indecomposability

In this section we deal with indecomposability, equivalently $\mathrm{CF}(M)$, see e.g. [ST97]; we have $\mathbf{K}_{\mathrm{lf}}$ in mind but still is meaningful and of interest also for other classes.

Why do we deal with indecomposable members $\mathbf{K}$? When we shall try to understand universal members M of $\mathbf{K}_\mu$ we shall use some θ-indecomposable $N \subseteq M$ of cardinality $< \mu$. How will this help us? The point is that $N \in \mathbf{K}_{<\mu}$ may have too many embeddings into M, but if ($\theta = \mathrm{cf}(\theta) > \mathrm{cf}(\mu)$ and $\alpha < \mu \Rightarrow |\alpha|^{\|N\|} < \mu$) and N is θ-indecomposable and θ is regular uncountable $< \mu$ this is not the case.

We need indecomposable $\mathbf{c} : [\lambda] \to \theta$ in order to build enough θ-indecomposable locally finite groups (as done in [She20]).

Definition 1.1. 1) We say M is θ-decomposable or $\theta \in \mathrm{CF}(M)$ <u>when</u>: θ is regular and if $\langle M_i : i < \theta\rangle$ is $\subseteq$-increasing with union M, <u>then</u> $M = M_i$ for some i.
2) We say M is Θ-indecomposable <u>when</u> it is θ-indecomposable for every $\theta \in \Theta$. We say M is Θ^{orth}-indecomposable <u>when</u> it is θ-indecomposable for every regular $\theta \notin \Theta$.
3) We say G is θ-indecomposable inside G^+ <u>when</u>:

- (a) $\theta = \mathrm{cf}(\theta)$;
- (b) $G \subseteq G^+$;
- (c) if $\langle G_i : i \le \theta\rangle$ is $\subseteq$-increasing continuous and $G_\theta = G^+$ (hence $G \subseteq G_\theta$) <u>then</u> for some $i < \theta$ we have $G \subseteq G_i$.

4) For $\theta = \mathrm{cf}(\theta) \le \lambda \le \mu$ such that $\theta \notin \Theta_\lambda$ we say $\mathbf{K}$ is (μ, λ, θ)-indecomposable <u>when</u> for every pair $(N, M) \in \mathbf{K}_{\mu,\lambda}$ there is $(N_1, M_1) \in \mathbf{K}_{\mu,\lambda}$ which is $\le_{\mu,\lambda}$-above it and M_1 is θ-indecomposable (really, not just inside N_1). For $\theta = \mathrm{cf}(\theta) < \lambda \le \mu$ we say $\mathbf{K}$ is $(\mu, <\lambda, \theta)$-indecomposable <u>when</u>:
if $\theta = \mathrm{cf}(\theta) \le \lambda_1 < \lambda, \theta \notin \Theta_{\lambda_1}$ then $\mathbf{K}$ is (μ, λ_1, θ)-indecoposable [1]
5) We say $\mathbf{c} : [\lambda]^2 \to S$ is θ-indecomposable <u>when</u>: if $\langle u_i : i < \theta\rangle$ is $\subseteq$-increasing sequence of sets with union λ then $S = \{\mathbf{c}\{\alpha, \beta\} : \alpha \neq \beta \in u_i\}$ for some $i < \theta$;
6) We may replace above the cardinal θ by a set or class Θ of regular cardinals, (as done in 1.1(2)).

A group G may be considered indecomposable as a group or as a semi-group; our default choice is semi-group; but note that for locally finite groups the two interpretations are equivalent.

The following was proved in [She20].

Theorem 1.2. *1) If $\lambda \ge \aleph_1$ and we let $\Theta_\lambda = \{\mathrm{cf}(\lambda)\}$ except that $\Theta_\lambda = \{\mathrm{cf}(\lambda), \partial\} = \{\lambda, \partial\}$ when $(c)_{\lambda,\partial}$ below holds, <u>then</u> clauses (a),(b) holds, where:*

- (a) *some $\mathbf{c} : [\lambda]^2 \to \lambda$ is θ-indecomposable for every $\theta = \mathrm{cf}(\theta) \notin \Theta_\lambda$*
- (b) *for every $G_1 \in \mathbf{K}^{\mathrm{lf}}_{\le\lambda}$ there is an extension $G_2 \in \mathbf{K}^{\mathrm{lf}}_\lambda$ which is $\Theta^{\mathrm{orth}}_\lambda$-indecomposable*
- $(c)_{\lambda,\partial}$ *for some $\mu, \lambda = \mu^+, \mu > \partial = \mathrm{cf}(\mu)$ and $\mu = \sup\{\theta < \mu : \theta$ is a regular Jonsson cardinal$\}$.*

[1] we could have asked less, replacing $= \lambda$ by $< \lambda$. Does not matter so far.

2) If $\mu \geq \lambda \geq \theta = \mathrm{cf}(\theta)$, and $\theta \notin \Theta_\lambda, \lambda > \aleph_1$ <u>then</u> $\mathbf{K}_{\mathrm{lf}}$ is (μ, λ, θ)-indecomposable.
3) If $\mu \geq \lambda$ and $(H_1, G_1) \in K_{\leq \mu, \leq \lambda}$ <u>then</u> we can find a pair $(H_2, G_2) \in K_{\mu, \lambda}$ such that:

(a) *G_2 is $\Theta^{\mathrm{orth}}_\lambda$-indecomposable,*
(b) *if $\mu > \lambda$ then the pair (H_2, G_1) is θ-indecomposale for every regular θ.*
(c) *H_2 is $\Theta^{\mathrm{orth}}_\mu$-indecomposable*

Proof. 1) By [She20, Th.3.5=Th.1.5=Lb24] .
2) The proof will serve also for part (3). Let $(N, M) \in \mathbf{K}_{\mu,\lambda}$ be given. We choose a pair (χ, ∂) of cardinals and $\mathbf{c}$ such that $\lambda \leq \chi \leq \mu$ $\partial = \mathrm{cf}(\partial) \leq \lambda, \partial \neq \theta$ and $\mathbf{c} : [\chi]^2 \to \chi$ is θ-indecomposable; (possible here as $\theta \notin \Theta_\lambda, \lambda > \aleph_1$ even for $\chi = \lambda$).

By induction of $\alpha \leq \partial$ we choose H_α, L_α, but L_α is chosen together with $H_{\alpha+1}$ when α is a successor ordinal, such that:

(a) $(H_\alpha, L_\alpha) \in \mathbf{K}_{\mu,\lambda}$ is increasing continuous with α
(b) $(H_0, L_0) = (N, M)$
(c) if $\alpha = \beta + 1 < \theta$ then and L_β is θ-indecomposable inside L_α

Why can we carry the induction? For $\alpha = 0$ this is trivial; similarly for α a limit ordinal. Lastly by clause (b) of part (1), for $\alpha = \beta + 1 \leq \alpha_*$, recall the proof of [She20, 3.4=1.4=Lb15], pedantically by first applying first [She20, 3.5=1.5(1)=Lb24] and then applying [She20, 3.4=1.4=Lb15].
3) We deal with every regular $\theta \leq \mu$ successively. Fixing θ we can use the proof of part (2). $\square_{1.2}$

Now comes the central definition, what is its role?

We like to sort out when there is a universal member of $\mathbf{K}_\mu$ and when there is a canonical universal member. For reasons explained above we concentrate on the case μ is strong limit of cofinality $\aleph_0$, for example $\beth_\omega$. To find out when the answer to those two questions for every universal class $\mathbf{K}$ seem too much to hope. The Def 1.3 accomplish a more modest task: it give a large frame satisfied by a large family of pairs $(\mathbf{K}, \mu)$ for which we shall prove an equivalence. In particular our class $\mathbf{K}_{\mathrm{lf}}$ belongs to this family.

Definition 1.3. We say that $\mathbf{K}$ is μ-nice <u>when</u>:

(a) $\tau_{\mathbf{k}}$ has cardinality $< \mu$,
(b) for every $M \in \mathbf{K}_{<\mu}$ there is $N \in \mathbf{K}_\mu$ extending M,
(c) $\mathbf{K}$ has the JEP (joint embedding property),
(d) $\mathbf{K}$ is $(\mu, <\mu, \mathrm{cf}(\mu))$-indecomposable.

Naturally we like to prove that the pair $(\mathbf{K}_{\mathrm{lf}}, \beth_\omega)$ fall under the frame of Def 1.3. This is the role of 1.4, 1.5. In §3 we point out an additional family. For the main case, μ is a strong limit of cofinality $\aleph_0$.

Claim 1.4. $\mathbf{K}_{\mathrm{lf}}$ *is μ -nice when $\mu > \aleph_1$*

Proof. In Def 1.3 clause (a) is trivial and as $\mathbf{K}_{\mathrm{lf}}$ is closed under products clearly clauses (b),(c) are clear and clause (d), for μ regular is trivial (and is not used) and for μ singular it holds by 1.2(3) or see more 1.5(2) below. $\square_{1.4}$

We give below more than what is strictly needed.

Claim 1.5. *Assume $\mathbf{K} = \mathbf{K}_{\mathrm{lf}}$.*
1) We have $(A) \Rightarrow (B)$ where:

(A) (a) $\mu \geq \aleph_1$
 (b) $\delta_* \leq \mu$ *and* $\lambda_\alpha < \mu$ *for* $\alpha < \delta_*$
 (c) $\lambda_\alpha \geq |\alpha|$ *is non-decreasing,*
 (d) $G_1 \in \mathbf{K}_{\leq \lambda}$
 (e) $G_{1,\alpha} \in \mathbf{K}_{\leq \lambda_\alpha}$ *for* $\alpha < \delta_*$

(B) *There are* $G_2, \bar{G}_2$ *such that:*
 (a) $G_2 \in \mathbf{K}_\lambda$ *extends* G_1
 (b) $\bar{G}_2 = \langle G_{2,\alpha} : \alpha < \delta_* \rangle$ *is increasing,*
 (c) $G_{2,\alpha} \in \mathbf{K}_{\lambda_\alpha}$ *extend* $G_{1,\alpha}$
 (d) G_2 *is* Θ*-indecomposable where* $\Theta = (\Theta_\lambda \cup \{\mathrm{cf}(\delta_*)\})^{\mathrm{orth}}$
 (e) $G_{2,\alpha}$ *is* $\Theta^{\mathrm{orth}}_{\lambda_\alpha}$*-indecomposable (not just inside* H_2 *for every* $\alpha < \delta_*$
 (f) *if* $\mu = \Sigma\{\lambda_\alpha : \alpha < \delta_*\}$ *the* $G_2 = \cup\{G_{2,\alpha} : \alpha < \delta_*\}$

2) If $\mu > \lambda \geq \aleph_1$ *<u>then</u>* $\aleph_0 \in \Theta^{\mathrm{orth}}_{\mathrm{cf}(\mu)} \cap \Theta^{\mathrm{orth}}_\lambda$ *except possibly when* $\mu = \lambda^+, \mathrm{cf}(\lambda) = \aleph_0$.

Proof. 1) By induction of $\alpha \leq \delta_*$ we choose $H_\alpha, \bar{H}_\alpha, L_\alpha$, but L_α is chosen together with $H_{\alpha+1}$ and not chosen for $\alpha = \alpha_*$, such that:

(a) H_α is increasing continuous with α
(b) $H_0 = G_1$ and $\alpha > 0 \Rightarrow H_\alpha \in \mathbf{K}_\lambda$
(c) $(H_\alpha, L_\beta) \in \mathbf{K}_{\lambda,\lambda_\beta}$ when $\alpha = \beta + 1 \leq \alpha_*$
(d) $\bar{H}_\alpha = \langle H_{\alpha,\varepsilon} : \varepsilon < \delta_* \rangle$ such that if $\mu = \Sigma\{\lambda_\varepsilon : \varepsilon < \delta_*\}$ then this sequence is increasing with union H_α and $H_{\alpha,\varepsilon}$ has cardinality λ_ε when $\alpha > 0$ and $\leq \lambda_\varepsilon$ when $\alpha = 0$
(e) $G_{1,\beta}, H_{\beta,\varepsilon}, L_\gamma$ are sub-groups of L_α when $\beta \leq \alpha, \varepsilon \leq \alpha, \gamma < \alpha$
(f) L_β is $\Theta^{\mathrm{orth}}_{\lambda_\beta}$-indecomposable,
(g) G_2 is Θ-indecomposable where $\Theta = (\Theta_\mu \cup \{cf(\delta_*)\})^{\mathrm{orth}}$

Why can we carry the induction? We choose $\bar{H}_\alpha$ just afer H_α was chosen. For $\alpha = 0$ this is trivial (note that L_α is not chosen), similarly for α a limit ordinal. Lastly for $\alpha = \beta + 1 \leq \alpha_*$, Def 1.1(4) 1.2(3) gives the desired conclusion. In details, first choose $L^+_\beta \subseteq H_\beta$ of cardinality at most λ_α satisfying the desired sets (listed in clause (e)). Then apply 1.2(3) to the pair (H_β, L^+_β) to get (H_α, L_α). Lastly let $G_2 \in \mathbf{K}_\lambda$ extend H_{α_*} and satisfies the indecomposablity demand, and letting $G_{2,\alpha} = L_\alpha$ we are done.
2) Easy. $\square_{1.5}$

Claim 1.6. *If* μ *is strong limit singular and* $N \in \mathbf{K}_\mu$ *<u>then</u> the set* $\mathrm{IDC}_{<\mu}(N)$ *has cardinality* $\leq \mu$*, if fact even equal; where, for* $N \in \mathbf{K}_\mu$,

(*) $\mathrm{IDC}_{<\mu}(N) = \{M : M \subseteq N$ *has cardinality* $< \mu$ *and is* $\mathrm{cf}(\mu)-$*indecomposable* $\}$.

Proof. Easy. $\square_{1.6}$

§ 2. Universality

For quite many classes, there are universal members in any (large enough) μ which is strong limit of cofinality $\aleph_0$, see [She17b] which include history. Below we investigate "is there a universal member of $\mathbf{K}^{\rm lf}_\mu$ for such μ". We prove that if there is a universal member, e.g. in $\mathbf{K}^{\rm lf}_\mu$, then there is a canonical one.

What do we mean by "canonical"? This is not a precise definition, but we mean it is unique up to isomorphism, by a natural definition. Examples we have in mind are the algebraic closure of a field, the saturated model of a complete first order theory T in cardinality $\mu^+ = 2^\mu > |T|$ and the special model of a complete first order theory T in a singular strong limit cardinal $\mu > |T|$, see [CK62]. The last one means:

- (*) for such T, μ we say that M is a special model of T of cardinality μ when some $\bar M$ witness is which means:
  - (a) $\bar M = \langle M_i : i < {\rm cf}(\mu)\rangle$
  - (b) M_i is $\prec$-increasing with i
  - (c) each M_i has cardinality $< \mu$
  - (d) $M = \cup\{M_i : i < {\rm cf}(\mu)\}$
  - (e) for every $\lambda < \mu$ for every large enough $i < {\rm cf}(\mu)$ the model M_i is λ^+-saturated.

Considering our main case, $\mathbf{K}_{\rm lf}$, a major difference (between what we prove here e.g. for $\mathbf{K}_{\rm lf}$ and $(*)$ is that here amalgamation fail, so clause (B) of 2.1 is a poor man replacement..

Theorem 2.1. *Assume μ be strong limit of cofinality $\aleph_0$ and $\mathbf{K}$ is μ-nice.*

1) The following conditions are equivalent:

- *(A) there is a universal $G \in \mathbf{K}_\mu$*
- *(B) if $H \in \mathbf{K}_\lambda$ is $\aleph_0$-indecomposable for some $\lambda < \mu$, <u>then</u> there is a sequence $\bar G = \langle G_\alpha : \alpha < \alpha_* \le \mu\rangle$ such that:*
  - *(a) $H \subseteq G_\alpha \in \mathbf{K}_\mu$*
  - *(b) if $G \in \mathbf{K}_\mu$ extend H, <u>then</u> for some α, G is embeddable into G_α over H.*
- *$(B)^+$ We can add in (B)*
  - *(c) if $\alpha_1 < \alpha_2 < \alpha_*$, <u>then</u> $G_{\alpha_1}, G_{\alpha_2}$ cannot be amalgamated over H, that is there are no G, f_1, f_2 such that $H \subseteq G \in \mathbf{K}$ and f_ℓ embeds G_{α_ℓ} into G over H for $\ell = 1, 2$,*
  - *(d) (H, G_α) is an amalgamation pair (see Definition 0.7(1)), moreover a universal amalgamation base (see 0.7(2)).*

2) We can add in part (1):

- *(C) there is G_* such that:*
  - *(a) $G_* \in \mathbf{K}_\mu$ is universal for $\mathbf{K}_{<\mu}$;*
  - *(b) $\mathscr{E}^{\aleph_0}_{G_*,<\mu}$, see see Def. 2.2 below, is an equivalence relation with $\le \mu$ equivalence classes;*
  - *(c) G_* is μ-special, see $(*)^*_G$ below.*

$(C)^+$ *like clause (C) but we add*

(d) *If $G, G_* \in \mathbf{K}_\mu$ are μ-special* *then* *G, G_* are isomorphic, (that is uniqueness).*

Before we prove 2.1, we define (this definition is not just used in the proof but also in phrasing 2.1(2).

Definition 2.2. For $\theta = \mathrm{cf}(\theta) < \mu$ and $M_* \in \mathbf{K}_\mu$: we define:

(A) $\mathrm{IND}^\theta_{M_*,<\mu} = \{N : N \leq_{\mathfrak{k}} M_*$ has cardinality $< \mu$ and is θ-indecomposable$\}$.

(B) $\mathscr{F}^\theta_{M_*,<\mu} = \{f$: for some θ-indecomposable $N = N_f \in K_{<\mu}$ with universe an ordinal, f is an embedding of N into $M_*\}$.

(C) $\mathscr{E}^\theta_{M_*,<\mu} = \{(f_1, f_2) : f_1, f_2 \in \mathscr{F}^\theta_{M_*,<\mu}, N_{f_1} = N_{f_2}$ and there are embeddings g_1, g_2 of M_* into some extension $M \in \mathbf{K}_\mu$ of M_* such that $g_1 \circ f_1 = g_2 \circ f_2\}$.

(D) We say M_* is $\theta - \mathscr{E}^\theta_{M_*,<\mu}$-indecomposably homogeneous (or just M_* is θ-indecomposably homogeneous) when: if $f_1, f_2 \in \mathscr{F}^\theta_{M_*,<\mu}$ and $(f_1, f_2) \in \mathscr{E}^\theta_{M_*,<\mu}$ and $A \subseteq M_*$ has cardinality $< \mu$ then there is $(g_1, g_2) \in \mathscr{E}^\theta_{M_*,<\mu}$ such that $f_1 \subseteq g_1 \wedge f_2 \subseteq g_2$ and $A \subseteq \mathrm{Rang}(g_1) \cap \mathrm{Rang}(g_2)$; it follows that if $\mathrm{cf}(\mu) = \aleph_0$ then for some $g \in \mathrm{aut}(M_*)$ we have $f_2 = g \circ f_1$.

(E) We say that $M_* \in \mathbf{K}_\mu$ is μ-special when it is θ-indecomposably homogeneous and is universal for $\mathbf{K}_{<\mu}$, that is every $M \in \mathbf{K}_{<\mu}$ is embeddable into it.

Remark 2.3. We may consider in 2.1 also $(A)_0 \Rightarrow (A)$ where

$(A)_0$ if $\lambda < \mu, H \subseteq G_1 \in \mathbf{K}_{<\mu}$ and $|H| \leq \lambda$, then for some G_2 we have $G_1 \subseteq G_2 \in \mathbf{K}_{<\mu}$ and (H, G_2) is a (μ, μ, λ)-amalgamation base.

Proof. of 2.1

It suffices to prove the following implications:

$(A) \Rightarrow (B)$:

Let $G_* \in \mathbf{K}_\mu$ be universal and choose a sequence $\langle G^*_n : n < \omega\rangle$ such that $G_* = \bigcup_n G^*_n, G^*_n \subseteq G^*_{n+1}, |G^*_n| < \mu$.

Let H be as in 2.1(B) and let $\mathscr{G} = \{g : g$ embed H into G^*_n for some $n\}$. So clearly $|\mathscr{G}| \leq \sum_n |G^*_n|^{|H|} \leq \sum_{\lambda<\mu} 2^\lambda = \mu$, (an over-kill).

Let $\langle g^*_\alpha : \alpha < \alpha_* \leq \mu\rangle$ list $\mathscr{G}$ and let (G_α, g_α) be such that (exist by renaming):

$(*)_1$ (a) $H \subseteq G_\alpha \in \mathbf{K}_\mu$;

(b) g_α is an isomorphism from G_α onto G_* extending g^*_α.

It suffices to prove that $\bar{G} = \langle G_\alpha : \alpha < \alpha_*\rangle$ is as required in clause (B). Now clause (B)(a) holds by $(*)_1(a)$ above. As for clause (B)(b), let G satisfy $H \subseteq G \in \mathbf{K}_{\leq\mu}$, hence there is an embedding g of G into G_*. We know that $g(H) \subseteq G = \bigcup_n G_n$ hence $\langle g(H) \cap G_n : n < \omega\rangle$ is $\subseteq$-increasing with union $g(H)$; but $g(H)$ by the assumption on H is $\aleph_0$-indecomposable, hence $g(H) = g(H) \cap G^*_n \subseteq G^*_n$ for some n. This implies $g{\restriction}H \in \mathscr{G}$ and so for some $\alpha < \alpha_*$ we have $g = g^*_\alpha$. Hence $g^{-1}_\alpha g$

is an embedding of G into G_* extending $(g_\alpha{\restriction}H)^{-1}(g{\restriction}H) = (g^*_\alpha)^{-1}(g^*_\alpha) = \mathrm{id}_H$ as promised.

$(B) \Rightarrow (B)^+$:

What about $(B)^+(c)$? while $\bar G$ does not necessarily satisfy it, we can "correct it", e.g. we choose u_α, v_α and if $\alpha \notin \cup\{v_\beta : \beta < \alpha\}$ also G'_α by induction on $\alpha < \alpha_*$ such that (the idea is that if $\beta \in v_\alpha$ then $\beta > \alpha$ and G_β is discarded being embeddable into some G'_α and G'_α is the "corrected" member):

$(*)^2_\alpha$ (a) $G_\alpha \subseteq G'_\alpha \in \mathbf{K}_\mu$ if $\alpha \notin \cup\{v_\beta : \beta < \alpha\}$;
(b) $u_\alpha \subseteq \alpha$ and $v_\alpha \subseteq \alpha_* \backslash (\alpha + 1)$;
(c) if $\beta < \alpha$ then $u_\beta = u_\alpha \cap \beta$ and $u_\alpha \cap v_\beta = \emptyset$;
(d) if $\alpha = \beta + 1$ then $\beta \in u_\alpha$ iff $\beta \notin \cup\{v_\gamma : \gamma < \beta\}$;
(e) if $\alpha \notin \cup\{v_\gamma : \gamma < \alpha\}$, then:
- $\bullet_1$ $\gamma \in v_\alpha$ iff ($\gamma > \alpha$ and) G_γ is embeddable into G'_α over H
- $\bullet_2$ if $\gamma \in \alpha_* \backslash (\alpha + 1) \backslash (\cup\{v_\beta : \beta \le \alpha$) then G_γ is not embeddable over H into any G' satisfying $G'_\alpha \subseteq G' \in \mathbf{K}$;

(f) if $\alpha = \beta + 1$ and $\beta \notin u_\alpha$ then $v_\beta = \emptyset$.

Why this suffices?

Because if we let $u_{\alpha_*} = \alpha_* \backslash (\cup\{v_\gamma : \gamma < \alpha_*\})$, then $\langle G'_\alpha : \alpha \in u_{\alpha_*}\rangle$ is as required; but we elaborate.

First, for clause $(B)^+$(c) assume that $\alpha < \beta$ are from u_{α_*}. As $\beta \notin v_\alpha$, by $(*)^2_\alpha(e)\bullet_2$ we know that G_β is not embeddable into any extension of G'_α over H; but as $G_\beta \subseteq G'_\beta$ clearly also G'_β is not embeddable into any extension of G'_α over H. Renaming this means that G'_α, G'_β cannot be amalgamated over H, as promised.

Second for clause $(B)^+$(d), let $\alpha \in u_{\alpha_*}$ and we have to prove that the pair (G'_α, H) is a universal (μ,κ)-amalgamation base where κ is the cardinality of H. So assume $G' \in \mathbf{K}_\mu$ extend G'_α; recally that we are assuming that $\langle G_\alpha : \alpha < \alpha_*\rangle$ is as in clause (B), hence there are $\beta < \alpha_*$ and an embedding f of G' into G_β ove H; we shall prove that $\beta = \alpha$ hence (recalling $G_\alpha \subseteq G'_\alpha$) f embed G' into G'_α over H thus finishing.

If $\beta \in u_{\alpha_*} \setminus \{\alpha\}$ then $f \restriction G'_\alpha$ embed G'_α into G'_β over H contradiction to $(B)^+$(c) which we have already proved.

If $\beta \in \alpha_* \backslash u_{\alpha_*}$ then for some γ we have $\beta \in v_\gamma$ hence $\gamma < \beta$ and G_β is embeddable into G'_γ over H; hence G' is embeddable into G'_γ over H. As in the previous sentence necessarily $\gamma = \alpha$ and we are done.

Why can we carry the induction?

For $\alpha = 0, \alpha$ limit we have nothing to do because u_α is determined by $(*)^2_\alpha(b)$ and $(*)^2_\alpha(c)$. For $\alpha = \beta + 1$, if $\beta \in \bigcup\limits_{\gamma<\beta} v_\gamma$ we have nothing to do, in the remaining case we choose $G'_{\beta,i} \in \mathbf{K}_\mu$ by induction on $i \in [\alpha, \alpha_*]$, increasing continuous with i. For $i = 0$ let $G'_{\beta,\alpha} = G_\alpha$ and for limit i let $G'_{\beta,i} = \cup\{G'_{\beta,j} : j < i\}$. Then choose $G'_{\beta,i+1}$ to make clause (e) true. That is, first if $G'_{\beta,i}$ has an extension into which G_i is embeddable over H, then there is such an extension of cardinality μ; and choose $G'_{\beta,i+1}$ as such an extension.

Second, if $G'_{\beta,i}$ has no extension into which G_i is embeddable over H, then we let $G'_{\beta,i+1} = G'_{\beta,i}$.

Lastly, let $G'_\alpha = G'_{\alpha,\alpha_*}$ and $u_\alpha = u_\beta \cup \{\alpha\}$ and $v_\alpha = \{i : i \in \alpha_*, i > \alpha, i \notin \cup\{v_\gamma : \gamma < \beta\}$ and G_i is embeddable into G'_β over $H\}$.

$\underline{(B)^+ \Rightarrow (A)}$:

We prove below more: there is something like "special model", i.e. part (2), that is $(B)^+ \Rightarrow (C)^+$.

$\underline{(C)^+ \Rightarrow (C) \Rightarrow (A)}$

This is trivial so we are left with proving the following.

$\underline{(B)^+ \Rightarrow (C)^+}$:

Let $\mathbf{K}^{\rm spc}_\mu$ be the class of G such that:

$(*)^3_G$
- (a) $G \in \mathbf{K}_\mu$
- (b) if $H \subseteq G, H \in \mathbf{K}_{<\mu}$ then there are $\aleph_0$-indecomposable $H_n \subseteq G$ for $n < \omega$ with union of cardinality $< \mu$ such that $H \subseteq \cup\{H_n : n < \omega\}$; hence there are $\aleph_0$-indcomposable $G_n \subseteq G$ for $n < \omega$ such that $G_n \in \mathbf{K}_{<\mu}$, $G_n \subseteq G_{n+1}$ and $G = \cup\{G_n : n < \omega\}$
- (c) if $H \subseteq G$ is $\aleph_0$-indecomposable of cardinality $< \mu$ <u>then</u> the pair (G, H) is an universal $(\mu, < \mu)$-amalgamation base (see Definition 0.7(2));
- (d) if $H \subseteq G$ is $\aleph_0$-indecomposable of cardinality $< \mu, H \subseteq H' \in \mathbf{K}_{<\mu}, H'$ is $\aleph_0$-indecomposable[2], and G, H' are compatible over H (in $\mathbf{K}_{\le\mu}$), <u>then</u> H' is embeddable into G over H.

Now we can finish by proving $(*)_4 + (*)_5$ below.

$(*)_4$ if $G \in \mathbf{K}_{\le\mu}$ <u>then</u> some $H \in \mathbf{K}^{\rm spc}_\lambda$ extend G

We break the proof to some stages, $(*)_{4.3}$ gives the desired conclusion of $((*)_4$.

$(*)_{4.0}$ if $G \in \mathbf{K}_{\le\mu}$ then for some $H, \bar H$ we have
- (a) $G \subseteq H \in \mathbf{K}_\mu$
- (b) $\bar H = \langle H_n : n < \omega\rangle$
- (c) $H_n \subseteq H_{n+1} \subseteq H$
- (d) $H = \cup\{H_n : n < \omega\}$
- (e) each H_n is $\aleph_0$-indecomposable
- (f) (not really needed) if $K \subseteq H_n, |K| \le \partial$ and $2^\partial \le |H_n|$ then there is a subgroup L of H_n extending K which is $\Theta^{\rm orth}_\partial$-indecomposable.

[Why? by 1.5(1),(2)].

$(*)_{4.1}$ if $N_1 \in \mathbf{K}_{\le\mu}$ <u>then</u> there is N_2 such that
- (a) $N_2 \in \mathbf{K}_\mu$
- (b) $N_1 \subseteq N_2$
- (c) if $H \in {\rm IDC}_{{\rm cf}(\mu)}(N_1)$ <u>then</u> (N_2, H) is a universal $(\mu, < \mu)$-amalgamation base.

Why? by 1.6 it is enough to deal with one such H, which is O.K. by clause (d) of Def 1.3, recalling "universal $(\mu, < \mu)$-amalgamation base" by (B)$^+$ which we are assuming.

$(*)_{4.2}$ like $(*)_{4.1}$ but in clause (c) is replaced by

[2]The $\aleph_0$-indecomposability is not always necessary, but we need it sometimes.

(c)' if $H_1 \in \mathrm{IDC}_{\mathrm{cf}(\mu)}(N_1)$ and $H_1 \subseteq H_2 \in \mathbf{K}_{<\mu}$ (and, we may add, H_2 is $\aleph_0$-indecomposable) then <u>either</u> N_2, H_1 are incompatible over H_1 in $\mathbf{K}_{\leq\mu}$ <u>or</u> H_2 is embeddable into N_2 over H_1

[Why? Again it is enough to deal with one pair (H_1, H_2)] which is done by hand.]

$(*)_{4.3}$ If $N_1 \in \mathbf{K}_{\leq\mu}$ then there is N_2 such that

- (a) $N_2 \in \mathbf{K}_\mu$
- (b) $N_1 \subseteq N_2$
- (c) if $H \in \mathrm{IDC}_{\mathrm{cf}(\mu)}(N_2)$ <u>then</u> (N_2, H) is a universal $(\mu, <\mu)$-amalgamation base
- (d) if $H_1 \in \mathrm{IDC}_{\mathrm{cf}(\mu)}(N_2)$ and $H_1 \subseteq H_2 \in \mathbf{K}_{<\mu}$ (and, we may add, H_2 is $\aleph_0$-indecomposable) then <u>either</u> N_2, H_1 are incompatible over H_1 in $\mathbf{K}_{\leq\mu}$ <u>or</u> H_2 is embeddable into N_2 over H_1

[Why? We choose $L_\varepsilon \in \mathbf{K}_\mu$ by induction on $\varepsilon \leq \mathrm{cf}((\mu)$, such that

- (a) $L_\alpha \in \mathbf{K}_\mu$
- (b) $\langle L_\beta : \beta \leq \alpha\rangle$ is increasing continuous
- (c) $G_1 \subseteq L_0$
- (d) if $\alpha = 3\beta + 1$ then L_α relate to $L_{3\beta}$ as N_2 relate to N_1 is $(*)_{4.0}$
- (e) if $\alpha = 3\beta + 2$ then L_α relate to $L_{3\beta+1}$ as N_2 relate to N_1 is $(*)_{4.1}$
- (f) if $\alpha = 3\beta + 3$ then L_α relate to $L_{3\beta+2}$ as N_2 relate to N_1 is $(*)_{4.2}$

There is no problem to carry the induction and note that: if $N \subseteq L_{\mathrm{cf}(\mu)}$ is $\mathrm{cf}(\mu)$-indecomposable the for some $\varepsilon < \mathrm{cf}(\mu)$ we have $N \subseteq L_\varepsilon$. Now then $N_2 = L_{\mathrm{cf}(\mu)}$ is as required in $(*)_{4.3}$ hence in $(*)_4$.

$(*)_5$ (a) if $G_1, G_2 \in \mathbf{K}^{\mathrm{spc}}_\mu$ then G_1, G_2 are isomorphic;

(b) if $G_1, G_2 \in \mathbf{K}^{\mathrm{spc}}_\mu, H \in \mathbf{K}$ is $\aleph_0$-indecomposable and f_ℓ embeds H into G_ℓ, for $\ell = 1, 2$, and this diagram can be completed, (i.e. there are $G \in \mathbf{K}_\mu$ and embedding $g_\ell : G_\ell \to G_*$ such that $g_1 \circ f_1 = g_2 \circ f_2$) <u>then</u> there is h such that:

- (α) h is an isomorphism from G_1 onto G_2;
- (β) $h \circ f_1 = f_2$;

Why? Clause (a) follows from clause (b) using as H the trivial group. For clause (b), let $\mathscr{F} = \mathscr{F}[G_1, G_2]$ be the set of f such that:

- (a) f is an isomorphism from $G_{1,f} \in \mathrm{IDC}_{\mathrm{cf}(\mu)}(G_1)$ onto $G_{2.f} \in \mathrm{IDC}_{\mathrm{cf}(\mu)}(G_2)$
- (b) G_1, G_2 are f-compatible in $\mathbf{K}_\mu$ which means that there is $G \in \mathbf{K}_\mu$ and embeddings g_ℓ of G_ℓ into G for $\ell = 1, 2$ such that $g_2 \circ f = g_1 {\restriction} G_{1,f}$.

First $\mathscr{F}$ is non-empty (the function f with domain $f_1(H)$ and range $f_2(H)$ will do.) Second use the hence and forth argument; here we use $\mathrm{cf}(\mu) = \aleph_0$.

$\square_{2.1}$

Remark 2.4. 1) Can we prove for strong limit singular μ of uncountable cofinality κ a parallel result? Well we have to consider the following game:

(*) the game is defined by:

- (a) a play last θ moved
- (b) in the ε move, first Player I choose $M_\varepsilon \in \mathbf{K}_{<\mu}$ and then player II choose $N_\varepsilon \in \mathbf{K}_{<\mu}$
- (c) $M_\varepsilon \in \mathbf{K}_{<\mu}$ if ε is non-limit then M_ε is $\mathrm{cf}(\mu)$-indecomposable
- (d) $\langle M_\zeta : \zeta \leq \varepsilon\rangle$ is increasing continuous

(e) $M_\varepsilon \subseteq N_\varepsilon \subseteq M_{\varepsilon+1}$
(f) in the end of the play, the player II wins iff for every limit ordinal $\varepsilon < \mathrm{cf}(\mu), M_\varepsilon$ is an amalgamation base inside $\mathbf{K}_{<\mu}$

Now if player II does not lose then we can imitate the proof above; this should be clear. Does the existence of a universal mmeber of $\mathbf{K}_\mu$ implies this? we hope to return to this elsewhere.
2) The proof works for any a.e.c. $\mathfrak{k}$ with $\mathrm{LST}_{\mathfrak{k}} < \mu$. But we may wonder: can we weaken the demand on $\mathfrak{k}$. Actually we can: there is no need of smoothness (that is: if $\langle M_\alpha : \alpha \le \delta\rangle$ is $\le_{\mathfrak{k}}$-increasing then $\cup\{M_\alpha : \alpha < \delta\} \le_{\mathfrak{k}} M_\delta$). Moreover while we need the existence of an upper bound for any $\le_{\mathfrak{k}}$-increasing sequence, its being the union can be demanded only for the cofinality $\mathrm{cf}(\mu)$.
3) We may add a version fixing $\bar\lambda$

§ 3. Universal in $\beth_\omega$

In §2 we have characterized when there are special models in $\mathbf{K}$ of cardinality, e.g. $\beth_\omega$. We try to analyze a related combinatorial problem. Our intention is to first investigate $\mathfrak{k}_{\rm fnq}$ (the class structures consisting of a set and a directed family of equivalence relations on it, each with a finite bound on the size of equivalence classes). So $\mathfrak{k}_{\rm fnq}$ is similar to $\mathbf{K}_{\rm lf}$ but seems easier to analyze. We consider some partial orders on $\mathfrak{k} = \mathfrak{k}_{\rm fnq}$.

First, under the substructure order, $\leq_1 = \subseteq$, this class fails amalgamation. Second, another order, $\leq_2$ demanding a Tarski-Vaught condition (see below)TV. However using $\leq_3$ where we have a similar demand for countably many points, finitely many equivalence relations, we have amalgamation. This is naturally connected to locally finite groups, see 3.6, 3.7.

Definition 3.1. Let $\mathbf{K} = \mathbf{K}_{\rm fnq}$ be the class of structures M such that (the vocabulary is defined implicitly and is $\tau_{\mathbf{K}}$, i.e. depends just on $\mathbf{K}$):

(a) P^M, Q^M is a partition of M, P^M non-empty;

(b) $E^M \subseteq P^M \times P^M \times Q^M$ (is a three-place relation) and we write $aE^M_c b$ for $(a,b,c) \in E^M$;

(c) for $c \in Q^M, E^M_c$ is an equivalence relation on P^M with $\sup\{|a/E^M_c| : a \in P^M\}$ finite (see more later);

(d) $Q^M_{n,k} \subseteq (Q^M)^n$ for $n, k \geq 1$

(e) if $\bar{c} = \langle c_\ell : \ell < n \rangle \in {}^n(Q^M)$ we let $E^M_{\bar{c}}$ be the closure of $\bigcup\limits_\ell E_\ell$ to an equivalence relation;

(f) ${}^n(Q^M) = \bigcup\limits_{k \geq 1} Q^M_{n,k}$;

(g) if $\bar{c} \in Q^M_{n,k}$ then $k \geq |a/E^M_{\bar{c}}|$ for every $a \in P^M$.

Definition 3.2. We define some partial order on $\mathbf{K}$.
1) $\leq_1 = \leq^1_{\mathbf{K}} = \leq^1_{\rm fnq}$ is being a sub-model.
2) $\leq_3 = \leq^3_{\mathbf{K}} = \leq^3_{\rm fnq}$ is the following: $M \leq_3 N$ iff:

(a) $M, N \in \mathbf{K}$

(b) $M \subseteq N$

(c) if $A \subseteq N$ is countable and $A \cap Q^N$ is finite, <u>then</u> there is an embedding of $N{\restriction}A$ into M over $A \cap M$ or just a one-to-one homomorphism.

3) $\leq_2 = \leq^2_{\mathbf{K}} = \leq^2_{\rm fnq}$ is defined like $\leq_3$ but in clause (c), A is finite.

Claim 3.3. *1) $\mathbf{K}$ is a universal class, so $(\mathbf{K}, \subseteq)$ is an a.e.c.*
2) $\leq^3_{\mathbf{K}}, \leq^2_{\mathbf{K}}, \leq^1_{\mathbf{K}}$ are partial orders on $\mathbf{K}$.
3) $(\mathbf{K}, \leq^2_{\mathbf{K}})$ is an a.e.c.
4) $(\mathbf{K}, \leq^3_{\mathbf{K}})$ has disjoint amalgamation.

Proof. 1),2),3) Easy.
4) By 3.4 below. $\square_{3.3}$

Claim 3.4. *If $M_0 \le^1_{\mathbf{K}} M_1, M_0 \le^3_{\mathbf{K}} M_2$ and $M_1 \cap M_2 = M_0$, then $M = M_1 + M_2$, the disjoint sum of M_1, M_2 belongs to $\mathbf{K}$ and extends M_ℓ for $\ell = 0,1,2$ and even $M_1 \le^3_{\rm fnq} M$ and $M_0 \le^2_{\mathbf{K}} M_1 \Rightarrow M_2 \le^2_{\mathbf{K}} M$ when:*

- $(*)$ *$M = M_1 +_{M_0} M_2$ means M is defined by:*
  - (a) *$|M| = |M_1| \cup |M_2|$;*
  - (b) *$P^M = P^{M_1} \cup P^{M_2}$;*
  - (c) *$Q = Q^{M_1} \cup Q^{M_2}$;*
  - (d) *we define E^M by defining E^M_c for $c \in Q^M$ by cases:*
    - (α) *if $c \in Q^{M_0}$ then E^M_c is the closure of $E^{M_1}_\ell \cup E^{M_2}_\ell$ to an equivalence relation;*
    - (β) *if $c \in Q^{M_\ell} \backslash Q^{M_0}$ and $\ell \in \{1,2\}$ then E^M_c is defined by*
      - *$aE^M_c b$ iff $a = b \in P^{M_{3-\ell}} \backslash M_0$ or $aE^{M_\ell}_c b$ so $a,b \in P^{M_\ell}$;*
  - (e) *$Q^M_{n,k}$ is the union of $Q^{M_1}_{n,k}, Q^{M_2}_{n,k}$ and the set of $\bar c$ satisfying*
    - (α) *$\bar c \in {}^n(Q^M)$*
    - (β) *$\bar c \notin ({}^n(Q^{M_1})) \cup {}^n(Q^{M_2})\}$,*
    - (γ) *for some $\bar c_0, \bar c_1, \bar c_2, n_1, n_2 k_1, k_2$ we have:*
      - $(\bullet_1)$ *$\bar c_\iota$ is a sequence of elements of M_ι, not from M_0 when $\iota > 0$*
      - $(\bullet_2)$ *$\bar c = \bar c_0 {}^\frown \bar c_1 {}^\frown \bar c_2$*
      - $(\bullet_3)$ *$\bar c_0 {}^\frown \bar c_\iota \in Q^{M_\iota}_{n_1,k_1}$ for $\iota = 1,2$*
      - $(\bullet_4)$ *$k_1 + k_2 \le k$*

Proof. Clearly M is a well defined structure, extends M_0, M_1, M_2 and satisfies clauses (a),(b),(c) of Definition 3.1. There are two points to be checked: $a \in P^M, \bar c \in Q^M_{n,k} \Rightarrow |a/E^M_{\bar c}| \le k$ and ${}^n(Q^M) = \bigcup\limits_{k \ge 1} Q^M_{n,k}$

$(*)_1$ if $a \in P^M$ and $\bar c \in Q^M_{n,k}$ then $|a/E^M_{\bar c}| \le k$.

Why? If $\bar c \in Q^M_{n,k} \backslash (Q^{M_1}_{n,k} \cup Q^{M_2}_{n,k})$ this holds by the definition, so assume $\bar c \in Q^{M_\iota}_{n,k}, \iota \le 2$. If this fails, then there is a finite set $A \subseteq M$ such that $\bar c \subseteq A, a \in A$ and letting $N = M{\restriction}A$ we have $|a/E^N_{\bar c}| > k$. By $M_0 \le^1_{\mathbf{K}} M_1, M_0 \le^3_{\mathbf{K}} M_2$ (really $M_0 \le^2_{\mathbf{K}} M_2$ suffice) there is a one-to-one homomorphism f from $A \cap M_2$ into M_0. Let $B' = (A \cup M_1) \cup f(A \cap M_2)$ and $N' = M{\restriction}B$ and let $g = f \cup \mathrm{id}_{A \cap M_1}$. So g is a homomorphism from N onto N' and $g(a)/E^{N'}_{g(\bar c)}$ has $> k$ members, which implies $g'(a)/E^{M_1}_{g'(\bar c)}$ has $> k$ members. Also $g(\bar c) \in Q^{M_1}_{n,k}$. (Why? If $\iota = 1$ trivially, if $\iota = 2$ by the choice of f, contradiction to $M \in \mathbf{K}$.)]

$(*)_2$ if $\bar c \in {}^n(Q^M)$ then $\bar c \in \bigcup\limits_k Q^M_{n,k}$.

Why? If $\bar c \in M_1$ or $\bar c \subseteq M_2$, this is obvious by the definition of M, so assume that they fail. By the definition of the $Q^M_{n,k}$'s we have to prove that $\sup\{a/E^M_{\bar c} : a \in P^M\}$ is finite. Toward contradiction assume this fails for each $k \ge 1$ there is $a_k \in P^M$ such that $a_k/E^M_{\bar c}$ has $\ge k$ elements hence there is a finite $A_k \subseteq M$ such that $a_k/E^{M{\restriction}A_k}_{\bar c}$ has $\ge k$ elements. Let $A = \bigcup\limits_{k \ge 1} A_k$, so A_k is a countable subset of M and we continue as in the proof of $(*)_1$.

Additional points (not really used) are proved like $(*)_2$:

$(*)_3$ $M_1 \leq^3_{\mathbf{K}} M$;
$(*)_4$ $M_0 \leq^2_{\mathbf{K}} M_1 \Rightarrow M_2 \leq^2_{\mathbf{K}} M$;
$(*)_5$ $M_1 +_{M_0} M_2$ is equal to $M_2 +_{M_0} M_1$.

$\square_{3.4}$

Claim 3.5. *1) If $\lambda = \lambda^{<\mu}$ and $M \in \mathbf{K}$ has cardinality $\leq \lambda$ then there is N such that:*

(a) *$N \in \mathbf{K}_\lambda$ extend M;*
(b) *if $N_0 \leq^3_{\mathbf{K}} N_1$ and N_0 has cardinality $< \mu$ and f_0 embeds N_0 into N, then there is an embedding f_1 of N_1 into N extending f_0.*

2) For every $M \in \mathbf{K}$ we can define an equivalence relation $E + E_{\mathbf{K}}$ on the class $\{N \in \mathbf{K} : M \leq_2 N\}$ with $\leq 2^{\|M\|^{\aleph_0}}$-equivalence classes such that; if N_1, N_2 are E -equivalence then they can be amalgamated over M (in $(\mathbf{K}, \leq_2)$.
3) If μ is strong limit then $(\mathbf{K}, \leq_2)$ is μ-nice.

What is the connection to $\mathbf{K}_{\rm lf}$? the following explain (see [KW73])

Definition 3.6. 1) For a group $G \in \mathbf{K}_{\rm lf}$ we define $M = {\rm fnq}_G \in \mathbf{K}_{\rm fnq}$ as follows:

(a) P^M is the set of elements of G
(b) $Q^M = \{(c, 1) : c \in G\}$, a copy of G
(c) E^M is the set of triples $(a, b, (c, 1)$ such that $a, b, c \in G$ and for some $n, m \in \mathbb{Z}$ we have $G \vdash c^n a c^m = b$.

2) For $M \in \mathbf{K}$ we define $G = {\rm grp}_M$ as the subgroup of ${\rm sym}(P^M)$ consisting of the permutations π of P^M such that for some finite sequence $\bar{c}$ of elements of Q^M we have: for every $x \in P^M$ we have $\pi(x) E^M_{\bar{c}} x$.

Discussion 3.7. The problem is that cases of amalgamation in $(\mathbf{K}, \leq_2)$ cannot be lifted to one in $\mathbf{K}_{\rm lf}$

References

[BG03] Gábor Braun and Rüdiger Göbel, *Outer automorphisms of locally finite p-groups*, J. Algebra **264** (2003), no. 1, 55–65.

[CK62] Chen-Chung Chang and H. Jerome Keisler, *Model theories with truth values in a uniform space*, Bulletin of the American Mathematical Society **68** (1962), 107–109.

[CS20] Samuel M. Corson and Saharon Shelah, *Strongly bounded groups of various cardinalities*, Proc. Amer. Math. Soc. **148** (2020), no. 12, 5045–5057, arXiv: 1906.10481. MR 4163821

[DG93] Manfred Dugas and Rüdiger Göbel, *On locally finite p-groups and a problem of Philip Hall's*, J. Algebra **159** (1993), no. 1, 115–138.

[GS83] Rami P. Grossberg and Saharon Shelah, *On universal locally finite groups*, Israel J. Math. **44** (1983), no. 4, 289–302. MR 710234

[GS84] Donato Giorgetta and Saharon Shelah, *Existentially closed structures in the power of the continuum*, Ann. Pure Appl. Logic **26** (1984), no. 2, 123–148. MR 739576

[Hic78] Ken Hickin, *Complete universal locally finite groups*, Transactions of the American Mathematical Society **239** (1978), 213–227.

[Jón56] Bjarni Jónsson, *Universal relational systems*, Mathematica Scandinavica **4** (1956), 193–208.

[Jón60] ———, *Homogeneous universal relational systems*, Mathematica Scandinavica **8** (1960), 137–142.

[KW73] Otto H. Kegel and Bertram A.F. Wehrfritz, *Locally finite groups*, xi+210.

[MS76] Angus J. Macintyre and Saharon Shelah, *Uncountable universal locally finite groups*, J. Algebra **43** (1976), no. 1, 168–175. MR 0439625

[MV62] M. D. Morley and R. L. Vaught, *Homogeneous and universal models*, Mathematica Scandinavica **11** (1962), 37–57.

[She] Saharon Shelah, *LF groups, aec amalgamation, few automorphisms*, arXiv: 1901.09747.

[She09] ———, *Classification theory for elementary abstract classes*, Studies in Logic (London), vol. 18, College Publications, London, 2009, [Title on cover: Classification theory for abstract elementary classes], Mathematical Logic and Foundations arXiv: 0705.4137 Ch. I of [Sh:h]. MR 2643267

[She17a] ———, *Existentially closed locally finite groups (Sh312)*, Beyond first order model theory, CRC Press, Boca Raton, FL, 2017, arXiv: 1102.5578, pp. 221–298. MR 3729328

[She17b] ———, *Universal structures*, Notre Dame J. Form. Log. **58** (2017), no. 2, 159–177, arXiv: math/0405159. MR 3634974

[She20] ———, *Density of indecomposable locally finite groups*, Rend. Semin. Mat. Univ. Padova **144** (2020), 253–270. MR 4186458

[Shear] ———, *Divide and Conquer: Dividing lines on universality*, Theoria (to appear).

[ST97] Saharon Shelah and Simon Thomas, *The cofinality spectrum of the infinite symmetric group*, J. Symbolic Logic **62** (1997), no. 3, 902–916, arXiv: math/9412230. MR 1472129

[SZ79] Saharon Shelah and Martin Ziegler, *Algebraically closed groups of large cardinality*, J. Symbolic Logic **44** (1979), no. 4, 522–532. MR 550381

[Tho86] Simon Thomas, *Complete universal loclly finite groups of large cardinality*, 277–301.

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