COLOURING OF SUCCESSOR OF REGULAR AGAIN SH1163

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Abstract. —[Should rechecked it!]

We get a version of the colouring property Pr_1 proving $\mathrm{Pr}_1(\lambda, \lambda, \lambda, \partial)$ always when $\lambda = \partial^+, \partial$ are regular cardinals and some stationary subset of λ consisting of ordinals of cofinality $< \partial$ do not reflect in any ordinal $< \lambda$.

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§ 0. Introduction

We prove a strong colouring theorem. On Pr_1 see [She94, Ch.III,, §4] and later Rinot citeRi14 and citeSh:1027. see on history there, we intend to say more. The result is incomparable with the one in citeRi14- the assumption on the stationary set is stronger but the the arity - the last parameter ∂ is bigger.

The connection between purely combinatorial theorems and topological constructions is known for many years. Several results in general topology were proved using the property $\mathrm{Pr}_1(\lambda, \mu, \sigma, \theta)$, see recently [JS15], [She19], the latter by improving the existence result on Pr_1. Note that [She97, §4] states more than it proves. s Recall:

Definition 0.1. 1) Assume $\lambda \geq \mu \geq \sigma + \theta_0 + \theta_1, \bar{\theta} = (\theta_0, \theta_1)$, see 0.4(1). Assume further that $\theta_0, \theta_1 \geq \aleph_0$ but σ may be finite

Let $\mathrm{Pr}_1(\lambda, \mu, \sigma, \bar{\theta})$ mean that there is $\mathbf{c} : [\lambda]^2 \to \sigma$ witnessing it, which means:

$(*)_{\mathbf{c}}$ if (a) then (b), where:

- (a) for $\iota = 0, 1, \mathbf{i}_\iota < \theta_\iota$ and $\bar{\zeta}^\iota = \langle \zeta^\iota_{\alpha,i} : \alpha < \mu, i < \mathbf{i}_\iota \rangle$ are sequences of ordinals of λ without repetitions, and $\mathrm{Rang}(\bar{\zeta}^0)$, $\mathrm{Rang}(\bar{\zeta}^1)$ are disjoint and $\gamma < \sigma$
- (b) there are $\alpha_0 < \alpha_1 < \mu$ such that $\forall i_0 < \mathbf{i}_0, \forall i_1 < \mathbf{i}_1, \mathbf{c}\{\zeta^0_{\alpha_0,i_0}, \zeta^1_{\alpha_1,i_1}\} = \gamma$ and $\zeta^0_{\alpha_0,i_0} < \zeta^1_{\alpha_1,i_1}$.

2) Above if $\theta_0 = \theta = \theta_1$ then we may write $\mathrm{Pr}_1(\lambda, \mu, \sigma, \theta)$.

In this paper we prove e.g. that if some stationary $S \subseteq \{\delta < \aleph_2 : \mathrm{cf}(\delta) < \aleph_1\}$ do not reflect then $\mathrm{Pr}_1(\aleph_2, \aleph_2, \aleph_2, \aleph_1)$ holds, which means that countable infinite sequences can be taken in both "sides". Actually, the theorem says that, in particular, $\mathrm{Pr}_1(\lambda, \lambda, \lambda, \partial)$ holds whenever $\partial = \mathrm{cf}(\partial)$ and $\lambda = \partial^+$ and there is S as there.

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Definition 0.2. 1) A filter D on a set I is uniform <u>when</u> for every subset A of I of cardinality $< |I|$, the set $I \setminus A \in D$; all our filters will be uniform
2) A filter D on a set I is weakly θ-saturated <u>when</u> $\theta \geq |I|$ and there is no partition of I to θ sets from D^+,
3) We say the filter D on a set I is θ-saturated <u>when</u> the Boolean algebra $\mathscr{P}(I)/D$ satisfies the θ-c.c.

Fact 0.3. 1) If D is a θ-complete filter on λ and is not θ-saturated then it is not weakly θ-saturated; so those properties are equivalent.
2) If $\theta = \sigma^+$ and D is a θ-complete filter on θ, <u>then</u> D is not weakly θ-saturated.
3) If $n \geq 1$ and $\lambda = \sigma^{+n}$ and D is a (uniform) σ^+-complete filter on λ <u>then</u> D is not weakly σ^{+n}-saturated

Proof. 1) Obvious and well known
2) By [Sol71],
3) Let μ be the minimal cardinal such that D is not μ^+-complete, so clearly $\mu \in [\sigma^+, \lambda]$ hence μ is a successor cardinal. So there is a function f from λ into μ such that for every subset A of μ of cardinality $< \mu$, $f^{-1}(A) = \emptyset \mod D$. Let E be the family of subsets A of μ such that $f^{-1}(A) \in D$. Clearly E is a (uniform) μ-complete filter on μ hence by part (2) is not weakly μ-saturated, let $\langle A_\varepsilon : \varepsilon < \mu \rangle$

be a partition of μ to sets from E^+. Now $\langle f^{-1}(A_\varepsilon) : \varepsilon < \mu \rangle$ witnesses the desired conclusion.

$\square_{0.3}$

Notation 0.4. 1) We denote infinite cardinals by $\lambda, \mu, \kappa, \theta, \partial$ while σ denotes a finite or infinite cardinal. We denote ordinals by $\alpha, \beta, \gamma, \varepsilon, \zeta, \xi$. Natural numbers are denoted by k, ℓ, m, n and $\iota \in \{0, 1, 2\}$
1A) Let D denote a filter on an infinite set $\mathrm{dom}(D)$
2) For a set A of ordinals let $\mathrm{nacc}(A) = \{\alpha \in A : \alpha > \sup(A \cap \alpha)\}$ and $\mathrm{acc}(A) = A \setminus \mathrm{nacc}(A)$. For regular $\lambda > \kappa$ let $S^\lambda_\kappa = \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$.

§ 1. A colouring theorem

Our aim is to prove

Theorem 1.1. *1)* $\mathrm{Pr}_1(\lambda, \lambda, \theta, \partial)$ *and moreover* $\mathrm{Pr}_1(\lambda, \lambda, \lambda, \partial)$ *holds provided that:*

(a) $\lambda = \partial^+, \partial = \mathrm{cf}(\partial) \geq \theta = \mathrm{cf}(\theta) > \aleph_0$
(b) *there is a stationary subset* $\mathscr{W}$ *of* λ *consisting of ordinals of cofinality* $< \partial$ *reflecting in no ordinal* $< \lambda$
2) Clause (b) of the assumption above follows

Remark 1.2. 1) The case of θ colours, i.e. proving only $\mathrm{Pr}_1(\lambda, \lambda, \theta, \theta)$ is easier so we prove it first.
5) By monotonicity of Pr_1 in θ, if clause (b) of 1.1 holds for some regular $\theta' \in (\theta, \partial)$ this suffice
2) Can we weaken clause (b) of 1.1 replacing "reflecting in no ordinal $< \lambda$ by "reflecting in no ordinal of cofinlaity ∂?

The answer seem yes provided that we add

(a) there is a sequence $\langle e_\alpha : \alpha \notin \mathscr{W} \rangle$ such that ($\mathscr{W}$ is as above and) e_α is a club of α of order type $< \partial$ and for $\alpha \in E_\beta$ from $\mathscr{W}$ we have $e_\alpha = \alpha \cap e_\beta$
(b) there is no $ex\theta$-complete not ∂^+-complete uniform weakly ∂-saturated filter on λ.

Proof. Stage A: We begin exactly as in earlier proofs. We let $(\kappa_1, \kappa_2) = (\theta, \lambda)$. Let $S \subseteq S^\lambda_\partial$ be stationary and $h : \lambda \to \lambda$ be such that $\alpha < \lambda \Rightarrow h(\alpha) < 1 + \alpha, h{\restriction}(\lambda \backslash S)$ is constantly zero and $S^*_\gamma := \{\delta \in S : h(\delta) = \gamma\}$ is a stationary subset of λ for every $\gamma < \lambda$. Let $F_\iota : \lambda \to \kappa_\iota$ for $\iota = 1, 2$ be such that for every $(\varepsilon_1, \varepsilon_2) \in (\kappa_1 \times \kappa_2)$ the set $W_{\varepsilon_1,\varepsilon_2}(\beta) = \{\gamma \in S^*_\beta : F_\iota(\gamma) = \varepsilon_\iota \text{ for } \iota = 1, 2\}$ is a stationary subset of λ for every $\beta < \lambda$.

For $\iota = 1, 2$ and $\rho \in {}^{\omega >}\lambda$ let $F_\iota(\rho) = \langle F_\iota(\rho(\ell)) : \ell < \ell g(\rho) \rangle$.

$\odot_0$ without loss of generalityif $\delta \in \mathscr{W}$ the δ is divisible by ∂.

Let $\bar{e} = \langle e_\alpha : \alpha < \lambda \rangle$ be such that

$\odot_1$ (a) if $\alpha = 0$ then $e_\alpha = \emptyset$
(b) if $\alpha = \beta + 1$ then $e_\alpha = \{\beta\}$
(c) if α is a limit ordinal then e_α is a club of α of order type $\mathrm{cf}(\alpha)$ disjoint to S^λ_∂ hence to S.
(d) if δ is a limit ordinal and $\delta \notin \mathscr{W}$ then e_δ is disjoint to $\mathscr{W}$.

In other cases (not here) instead h we use a sequence $\langle h_\alpha : \alpha < \lambda \rangle$ of functions, $h_\alpha : e_\alpha \to \theta$ and use e.g $\langle h_{\gamma_\ell(\beta,\alpha)}(\gamma_{\ell+1}(\beta, \alpha)) : \ell < k(\beta, \alpha) \rangle$ and ρ_h, but this is not necessary here.

Now (using $\bar{e}$) for $\alpha < \beta < \lambda$, let

$$\gamma(\beta, \alpha) := \min\{\gamma \in e_\beta : \gamma \geq \alpha\}.$$

Let us define $\gamma_\ell(\beta, \alpha)$:

$$\gamma_0(\beta, \alpha) = \beta,$$

$$\gamma_{\ell+1}(\beta, \alpha) = \gamma(\gamma_\ell(\beta, \alpha), \alpha) \text{ (if well defined).}$$

If $\alpha < \beta < \lambda$, let $k(\beta,\alpha)$ be the maximal $k < \omega$ such that $\gamma_k(\beta,\alpha)$ is defined (equivalently is equal to α) and let $\rho_{\beta,\alpha} = \rho(\beta,\alpha)$ be the sequence

$$\langle \gamma_0(\beta,\alpha), \gamma_1(\beta,\alpha), \ldots, \gamma_{k(\beta,\alpha)-1}(\beta,\alpha)\rangle.$$

Let $\gamma_{\ell t}(\beta,\alpha) = \gamma_{k(\beta,\alpha)-1}(\beta,\alpha)$ where ℓt stands for last.

Let

$$\rho_h = \langle h(\gamma_\ell(\beta,\alpha)) : \ell < k(\beta,\alpha)\rangle$$

and we let $\rho(\alpha,\alpha)$ and $\rho_h(\alpha,\alpha)$ be the empty sequences. Now clearly:

$\odot_2$ if $\alpha < \beta < \lambda$ then $\alpha \leq \gamma(\beta,\alpha) < \beta$

hence

$\odot_3$ if $\alpha < \beta < \lambda, 0 < \ell < \omega$, and $\gamma_\ell(\beta,\alpha)$ is well defined, then

$$\alpha \leq \gamma_\ell(\beta,\alpha) < \beta$$

and

$\odot_4$ if $\alpha < \beta < \lambda$, then $k(\beta,\alpha)$ is well defined and letting $\gamma_\ell := \gamma_\ell(\beta,\alpha)$ for $\ell \leq k(\beta,\alpha)$ we have

$$\alpha = \gamma_{k(\beta,\alpha)} < \gamma_{\ell t}(\beta,\alpha) = \gamma_{k(\beta,\alpha)-1} < \cdots < \gamma_1 < \gamma_0 = \beta$$

$$\text{and } \alpha \in e_{\gamma_{\ell t}(\beta,\alpha)}$$

i.e. $\rho(\beta,\alpha)$ is a (strictly) decreasing finite sequence of ordinals, starting with β, ending with $\gamma_{\ell t}(\beta,\alpha)$ of length $k(\beta,\alpha)$.

Note that if $\alpha \in S, \alpha < \beta$ then $\gamma_{\ell t}(\beta,\alpha) = \alpha + 1$.

Also

$\odot_5$ if δ is a limit ordinal and $\delta < \beta < \lambda$, <u>then</u> for some $\alpha_0 < \delta$ we have: $\alpha_0 \leq \alpha < \delta$ <u>implies</u>:

- (i) for $\ell < k(\beta,\delta)$ we have $\gamma_\ell(\beta,\delta) = \gamma_\ell(\beta,\alpha)$
- (ii) $\delta \in \mathrm{nacc}(e_{\gamma_{\ell t}(\beta,\delta)}) \Leftrightarrow \delta = \gamma_{k(\beta,\delta)}(\beta,\delta) = \gamma_{k(\beta,\delta)}(\beta,\alpha) \Leftrightarrow \neg[\gamma_{k(\beta,\delta)}(\beta,\delta) = \delta > \gamma_{k(\beta,\delta)}(\beta,\alpha)]$
- (iii) $\rho(\beta,\delta) \trianglelefteq \rho(\beta,\alpha)$; i.e. is an initial segment
- (iv) $\delta \in \mathrm{nacc}(e_{\gamma_{\ell t}(\beta,\delta)})$ (here always holds if $\delta \in S$) implies:
  - $\rho(\beta,\delta)\hat{\ }\langle\delta\rangle \trianglelefteq \rho(\beta,\alpha)$ hence
  - $\rho_h(\beta,\delta)\hat{\ }\langle h(\beta,\delta)(\delta)\rangle \trianglelefteq \rho_h(\beta,\alpha)$.
- (v) if $\mathrm{cf}(\delta) = \partial$ <u>then</u> we have $\gamma_{\ell t}(\beta,\delta) = \delta + 1$ so $\delta \in \mathrm{nacc}(e_{\gamma_{\mathrm{lt}}(\beta,\delta)})$
- (vi) if $\mathrm{cf}(\delta) = \partial$ and $\delta \in e_\gamma$, <u>then</u> necessarily $\gamma = \delta + 1$.

Why? Just let

$$\alpha_0 = \mathrm{Max}\{\sup(e_{\gamma_\ell(\beta,\delta)} \cap \delta) + 1 : \ell < k(\beta,\delta) \text{ and } \delta \notin \mathrm{acc}(e_{\gamma_\ell(\beta,\delta)})\}.$$

Notice that if $\ell < k(\beta,\delta) - 1$ then $\delta \notin \mathrm{acc}(e_{\gamma_\ell(\beta,\delta)})$ follows.

Note that the outer maximum (in the choice of α_0) is well defined as it is over a finite non-empty set of ordinals. The inner sup is on the empty set (in which case we get zero) or is the maximum (which is well defined) as $e_{\gamma_\ell(\beta,\delta)}$ is a closed subset of $\gamma_\ell(\beta,\delta), \delta < \gamma_\ell(\beta,\delta)$ and $\delta \notin \mathrm{acc}(e_{\gamma_\ell(\beta,\delta)})$ - as this is required. For clauses (v), (vi) recall $\delta \in S^\lambda_\partial$ and $e_\gamma \cap S^\lambda_\partial = \emptyset$ when γ is a limit ordinal and $e_\gamma = \{\gamma - 1\}$ when γ is a successor ordinal.

$\odot_6$ (a) if $\alpha < \beta < \lambda, \ell < k(\beta,\alpha), \gamma = \gamma_\ell(\beta,\alpha)$ then $\rho(\beta,\alpha) = \rho(\beta,\gamma)\hat{\ }\rho(\gamma,\alpha)$ and $\rho_h(\beta,\alpha) = \rho_h(\beta,\gamma)\hat{\ }\rho_h(\gamma,\alpha)$

(b) if $\alpha_0 < \ldots < \alpha_k$ and $\rho(\alpha_k,\alpha_0) = \rho(\alpha_k,\alpha_{k-1})\hat{\ }\ldots\hat{\ }\rho(\alpha_1,\alpha_0)$ then this holds for any sub-sequence of $\langle \alpha_0,\ldots,\alpha_k\rangle$.

$\odot_7$ let F'_ι be $F_\iota \circ h$ for $\iota = 1,2$; so F'_1 is a function from λ into ∂ and F'_2 is a function from λ into λ.

$\odot_8$ if $\alpha < \beta$ are from $\mathscr{W}$ then $\gamma_{\mathrm{lt}}(\beta,\alpha) = \alpha + 1$

[Why? by the assumptions on $\mathscr{W}$].]

Stage B:

Let

$\boxplus_2$ $\mathbf{T} = \{\bar t : \bar t = \langle t_\alpha : \alpha < \lambda\rangle$ satisfies $t_\alpha \in [\lambda]^{<\partial}$ and $t_\alpha \subseteq \lambda \backslash \alpha\}$.

$\boxplus_3$ for $\varepsilon < \partial$ and $\bar t \in \mathbf{T}$ let $A_{\bar t,\varepsilon}$ be the set of $\gamma < \lambda$ such that for some (α_0,α_1) we have:

(a) $\alpha_0 < \alpha_1 < \lambda$ and[1] $(\zeta,\xi) \in t_{\alpha_0} \times t_{\alpha_1} \Rightarrow \zeta < \xi$

(b) for every $(\zeta,\xi) \in t_{\alpha_0} \times t_{\alpha_1}$ for some ℓ we have:

(α) $\ell < k(\xi,\zeta)$

(β) $\gamma_\ell(\xi,\zeta) = \gamma$

(γ) if $k < k(\xi,\zeta)$ then $F'_1(\gamma) \geq F'_1(\gamma_k(\xi,\zeta))$ and $F'_1(\gamma) \geq \varepsilon$

(δ) if $k < \ell$ then $F'_1(\gamma_k(\xi,\zeta)) < F'_1(\gamma)$.

We define:

$\boxplus_4$ $D = \{A \subseteq \lambda : A$ includes $A_{\bar t,\varepsilon}$ for some $\bar t \in \mathbf{T}, \varepsilon < \partial\}$.

Now note:

$\boxplus_5$ (a) if $\bar s,\bar t \in \mathbf{T}, \varepsilon \leq \zeta < \partial$ and $(\forall \alpha < \lambda)(s_\alpha \subseteq t_\alpha)$, then $A_{\bar t,\zeta} \subseteq A_{\bar s,\varepsilon}$

(b) if $\bar s \in \mathbf{T}, \varepsilon < \partial, g$ is an increasing function from λ to λ and $\bar t = \langle t_\alpha : \alpha < \lambda\rangle$ is defined by $t_\alpha = s_{g(\alpha)}$ then $A_{\bar t,\varepsilon} \subseteq A_{\bar s,\varepsilon}$.

[Why? Read the definitions.]

$\boxplus_6$ (a) the intersection of any $< \partial$ members of D is a member of D, equivalently includes the set $A_{\bar t,\zeta}$ for some $\bar t \in \mathbf{T}, \zeta < \partial$

(b) for every $\beta < \lambda$ for some $\bar t \in \mathbf{T}, A_{\bar t,0} \subseteq [\beta,\lambda)$

[1] If we choose to add here "$t_{\alpha_0} \subseteq \alpha_1$", then we would a problem in proving clause $\boxplus_5(b)$.

(c) if $\bar{t} \in \mathbf{T}$ and $\alpha < \lambda \Rightarrow t_\alpha \neq \emptyset$ then $\cap\{A_{\bar{t},\varepsilon} : \varepsilon < \partial\} = \emptyset$
(d) D is upward closed.
(e) λ belongs to D

[Why? For clause (a) assume $A_\varepsilon \in D$ for $\varepsilon < \varepsilon(*) < \partial$ then for some $\zeta_\varepsilon < \partial$ and $\bar{t}_\varepsilon \in \mathbf{T}$ we have $A_\varepsilon \supseteq A_{\bar{t}_\varepsilon,\zeta_\varepsilon}$. Define $t_\alpha = \bigcup\{t^\varepsilon_\alpha : \varepsilon < \varepsilon(*)\}$ for $\alpha < \lambda$ and $\zeta = \sup\{\zeta_\varepsilon : \varepsilon < \varepsilon(*)\}$; as the cardinal ∂ is regular, clearly $|t_\alpha| \leq \sum\limits_{\varepsilon<\varepsilon(*)} |t^\varepsilon_\alpha| < \partial$ and obviously $t_\alpha \subseteq [\alpha, \lambda)$ hence $\bar{t} = \langle t_\alpha : \alpha < \lambda\rangle \in \mathbf{T}$ and similarly $\zeta < \partial$. Easily $A_{\bar{t},\zeta} \subseteq A_{\bar{t}_\varepsilon,\zeta_\varepsilon}$ for every $\varepsilon < \varepsilon(*)$, see $\boxplus_5(a)$ so we are done proving clause (a). For clause (b) define $t_\alpha = \{\beta + \alpha + 1\}$ and recalling $\boxplus_3(b)(\beta)$ and $\odot_4$ check that $A_{\bar{t},0} \subseteq [\beta, \lambda)$. Also clause (c) obviously holds because $\gamma \in A_{\bar{t},\varepsilon} \Rightarrow F'_1(\gamma) \geq \varepsilon$ by $\boxplus_3(b)(\gamma)$ and F'_1 is a function from λ to ∂ and clauses (d),(e) hold trivially by the definition.]

$\boxplus_7$ (a) $\emptyset \notin D$
(b) D is a filter on λ, equivalently $A_{\bar{t},\varepsilon} \neq \emptyset$ for every $\bar{t}, \varepsilon$; also D is uniform ∂-complete, not ∂^+-complete.

[Why? Clause (a) is a major point, proved in Stage C below. That is, by $\boxplus_6(a),(d)$ the only missing point is to show $A_{\bar{t},\zeta} \neq \emptyset$, (in fact, $|A_{\bar{t},\zeta}| = \lambda$). For clause (b) by (a) and $\boxplus_6(a),(d),(e), D$ is a ∂-complete filter and the statement that D is uniform holds by $\boxplus_6(b)$ and not ∂^+-complete holds by $\boxplus_6(c)$.]

Note also

$\boxplus_8$ D is not weakly ∂-saturated.

[Why? By $\boxplus_7$ and clause (b) in the assumptions of the theorem. That is it is known that if D fail this statement (and has the properties listed before) that there is no S in in clause (b) of the theorem. transversality

Stage C:

In this stage we accomplish the proof of the missing point in $\boxplus_7(a)$ from above, so we shall prove "$A_{\bar{t},\varepsilon}$ is non-empty (in fact, has cardinality λ)" when:

$\boxplus$ (a) $t_\alpha \subseteq \lambda\backslash\alpha$ for $\alpha < \lambda$
(b) $|t_\alpha| < \partial$
(c) $\varepsilon < \partial$.

To start we note that:

$(*)_1$ without loss of generality $t_\alpha \neq \emptyset$ and $\alpha < \min(t_\alpha)$.

[Why? First, recalling $\boxplus_5(a)$ we can replace $\bar{t}$ by $\bar{t} = \langle t_\alpha \cup \{\alpha\} : \alpha < \lambda\}$, so we may assume that each t_α is not empty. Second, let $\bar{t}' = \langle t'_\alpha : \alpha < \lambda\rangle, t'_\alpha = t_{\alpha+1}$, so easily $\bar{t}'$ satisfies $(*)_1$ and $A_{\bar{t}',\varepsilon} \subseteq A_{\bar{t},\varepsilon}$ by clause $\boxplus_5(b)$.]

Now

$(*)_2$ we can find $\mathscr{U}^{\rm dn}_1, \varepsilon^{\rm dn}$ such that:
(a) $\mathscr{U}^{\rm dn}_1 \subseteq \mathscr{W}$ is stationary in λ, see stage A on S^*_0
(b) $\alpha < \delta \in \mathscr{U}^{\rm dn}_1 \Rightarrow t_\alpha \subseteq \delta$
(c) $\varepsilon^{\rm dn} < \partial$

(d) if $\delta \in \mathscr{U}_1^{\rm dn}$ <u>then</u> for arbitrarily large $\alpha < \delta$ we have $\zeta \in t_\alpha \Rightarrow \mathrm{Rang}(F_1(\rho_h(\delta,\zeta))) \subseteq \varepsilon^{\rm dn} < \kappa_1 = \partial$.

[Why? Clearly $E_0 = \{\delta < \lambda : \delta$ is a limit ordinal such that $\alpha < \delta \Rightarrow t_\alpha \subseteq \delta\}$ is a club of λ. For every $\delta \in \mathscr{W} \cap E_0$ and $\alpha < \delta$ we can find $\varepsilon^{\rm dn}_{\delta,\alpha}$ as in clauses (c),(d) of $(*)_2$ (because $|t_\delta| < \partial$) and so recalling that $\mathrm{cf}(\delta) < \partial$ it follows that there is $\varepsilon^{\rm dn}_\delta$ such that $\delta = \sup\{\alpha < \delta : \varepsilon^{\rm dn}_{\delta,\alpha} = \varepsilon^{\rm dn}_\delta\}$. Then recalling $\lambda = \mathrm{cf}(\lambda) > \theta$ we can choose $\varepsilon^{\rm dn}$ such that the set $\mathscr{U}_1^{\rm dn} = \{\delta \in \mathscr{W} \cap E_0 : \varepsilon^{\rm dn}_\delta = \varepsilon^{\rm dn}\}$ is stationary. So $(*)_2$ holds indeed.]

$(*)_3$ We can find $\mathscr{U}_1^{\rm up}, \alpha_1^*, \varepsilon^{\rm up}$ such that:
- (a) $\mathscr{U}_1^{\rm up} \subseteq S_0^*$ is stationary
- (b) $h{\restriction}\mathscr{U}_1^{\rm up}$ is constantly 0, actually follows by (a), see Stage A
- (c) $\alpha_1^* < \lambda$ satisfies $\alpha_1^* < \min(\mathscr{U}_1^{\rm up})$ and $\varepsilon^{\rm up} < \partial$
- (d) if $\delta \in \mathscr{U}_1^{\rm up}$ and $\alpha \in [\alpha_1^*, \delta)$ and $\beta \in t_\delta$ <u>then</u>:
  - $\rho_{\beta,\delta}{}^\frown\langle\delta\rangle \trianglelefteq \rho_{\beta,\alpha}$
  - $\mathrm{Rang}(F_1(\rho_h(\beta,\delta))) \subseteq \varepsilon^{\rm up}$.

[Why? For every $\delta \in S_0^* \subseteq S$ and $\zeta \in t_\delta$ let $\alpha_{1,\delta,\zeta} < \delta$ be such that $(\forall\alpha)(\alpha \in [\alpha_{1,\delta,\zeta}, \delta) \Rightarrow \rho_{\zeta,\delta}{}^\frown\langle\delta\rangle \trianglelefteq \rho_{\zeta,\alpha})$, it exists by $\odot_5$ of Stage A.

Let

- $\alpha_{1,\delta} = \sup\{\alpha_{1,\delta,\zeta} : \zeta \in t_\delta\}$
- $\varepsilon^{\rm up}_\delta = \sup\{F_1'(\gamma_\ell(\zeta,\delta))(\ell)+1 : \zeta \in t_\delta$ and $\ell < k(\zeta,\delta)\} = \cup\{\sup\mathrm{Rang}(F_1(\rho_h(\zeta,\delta)))+1 : \zeta \in t_\delta\}$; as $\mathrm{cf}(\delta) = \partial = \mathrm{cf}(\partial) > \theta$ and $\theta = \mathrm{cf}(\theta) > |t_\delta|$, necessarily $\alpha_{1,\delta} < \delta$ and $\varepsilon^{\rm up}_\delta < \theta$.

Lastly, there are $\alpha_1^* < \lambda$ and $\varepsilon^{\rm up} < \kappa_1 = \theta$ and $\mathscr{U}_1^{\rm up} \subseteq S_0^*$ as required by using Fodor lemma. So $(*)_3$ holds indeed.]

Now let $E = \{\delta < \lambda : \delta$ is a limit ordinal $> \alpha_1^*$ such that $\delta = \sup(\mathscr{U}_1^{\rm dn} \cap \delta)$ and $\alpha < \delta \Rightarrow t_\alpha \subseteq \delta\}$, it is a club of λ because $\alpha_1^* < \lambda$ by $(*)_3(c)$ and $\mathscr{U}_1^{\rm dn}$ is an unbounded subset of λ by $(*)_2(a)$, and t_α is a subset of λ of cardinality $< \theta$ hence is bounded.

Choose $\varepsilon(*) = \max\{\varepsilon^{\rm up}+1, \varepsilon^{\rm dn}+1, \varepsilon+1\}$ where ε is from $\boxplus(c)$, so $\varepsilon(*) < \theta$ and choose $\delta_2 \in E \cap S$ such that $F_1'(\delta_2) = \varepsilon(*)$. Next choose $\alpha_2 \in \mathscr{U}_1^{\rm up}\backslash(\delta_2+1)$ and let $\alpha^* \in (\alpha_1^*, \delta_2)$ be large enough such that $\zeta \in (\alpha^*, \delta_2) \wedge \xi \in t_{\alpha_2} \Rightarrow \rho(\xi,\delta_2){}^\frown\langle\delta_2\rangle \triangleleft \rho(\xi,\zeta)$. Now choose $\delta_1 \in \mathscr{U}_1^{\rm dn} \cap (\alpha^*, \delta_2)$ and $\alpha^{**} \in (\alpha^*, \delta_1)$ be such that $\alpha \in (\alpha^{**}, \delta_1) \wedge \xi \in t_{\alpha_2} \Rightarrow \rho(\xi,\delta_1){}^\frown\langle\delta_1\rangle \triangleleft \rho(\xi,\alpha)$.

[Why this is possible? First as $\alpha^{**} > \alpha^*$ it is enough to have $\alpha \in (\alpha^{**}, \delta_1) \Rightarrow \rho(\delta_2,\delta_1){}^\frown\langle\delta_1\rangle \triangleleft \rho(\delta_2,\alpha)$. Second here $\mathrm{cf}(\delta_1) < \partial$ however this condition holds because $\delta_1 \in \mathscr{U}_1^{dn} \subseteq \mathscr{W}$ so necessarily $\gamma_{\rm lt}(\delta_2,\delta_1) = \delta_1+1$ by $\odot_8$).

Next let $\ell_* < \ell g(\rho(\alpha_2,\delta_1)$ be such that:

$(*)_4$
- $\gamma_* = F_1(\rho_h(\alpha_2,\delta_1))(\ell_*) = \max\mathrm{Rang}F_1(\rho_h(\alpha_2,\delta_1))$
- under this restriction ℓ_* is minimal.

Now let $\gamma_* = \rho(\alpha_2,\delta_1)(\ell_*)$.

Lastly, choose $\alpha_1 \in (\alpha^{**}, \delta_1)$ which is as in $(*)_2(d)$ with respect to δ_1, i.e. such that:

$(*)_5$ if $\zeta \in t_{\alpha_1}$ then $\mathrm{Rang} F_1(\rho_h(\delta_1,\zeta)) \subseteq \varepsilon^{\mathrm{dn}}$.

Now we shall prove that the pair (α_1,α_2) is as required. So let $(\zeta,\xi) \in t_{\alpha_1} \times t_{\alpha_2}$; now by our choices

$(*)_6$ $\rho(\xi,\zeta) = \rho(\xi,\alpha_2)\hat{\ }\rho(\alpha_2,\delta_2)\hat{\ }\rho(\delta_2,\delta_1)\hat{\ }\rho(\delta_1,\zeta)$ and $\rho(\alpha_2,\delta_1) = \rho(\alpha_2,\delta_2)\hat{\ }\rho(\delta_2,\delta_1)$

So

$(*)_7$ $\mathrm{Rang}(F_1(\rho_h(\xi,\alpha_2))) \subseteq \varepsilon^{\mathrm{up}} \le \varepsilon(*)$

[Why? by $(*)_3$(a), the choice of $\alpha_2 \in \mathscr{U}_1^{\mathrm{up}}$ and ξ being from t_{α_2}]

$(*)_8$ $\mathrm{Rang}(F_1(\rho_h(\delta_1,\zeta)) \subseteq \varepsilon^{\mathrm{dn}} \le \varepsilon(*)$

[Why by $(*)_2$(d) and the choice of α_1 (and ζ being a member of t_{α_1}]

$(*)_9$ $\varepsilon(*) = F_1 \circ h(\delta_2) \in \mathrm{Rang}(F_1(\rho_h(\alpha_2,\delta_1)))$, see $(*)_6$ and (before and after) $\odot_1$.

[Why? Recall that δ_2 was chosen in $E \cap S$ such that $F_1'(\delta_2) = \varepsilon(*)$.]

Hence

$(*)_{10}$ in $\boxplus_3(b)$ for our $\bar{t}$ and the pair (α_1,α_2), our γ_* (chosen before $(*)_5$) is gotten, witnessing $\gamma_* \in A_{\bar{t},\varepsilon(*)} \subseteq A_{\bar{t},\varepsilon}$ as first $\varepsilon < \varepsilon(*)$, by the choice of $\varepsilon(*)$, and second if $(\zeta,\xi) \in t_{\alpha_1} \times t_{\alpha_2}$ then $\ell = \ell g(\rho(\xi,\alpha_2)) + \ell_*$ is as required in $\boxplus_3(b)$ for $\bar{t}$ by $(*)_6 - (*)_9$

So we are done proving $\boxplus_7(a)$.

Stage D: By $\boxplus_8$

$\circledast_1$ there is $F_* : \lambda \to \theta$ such that $\varepsilon < \theta \Rightarrow F_*^{-1}(\{\varepsilon\}) \ne \emptyset \mod D$.

We first deal with the easier version with θ colours, i.e. proving $\mathrm{Pr}_1(\lambda,\lambda,\theta,\theta)$.

We now define the colouring $\mathbf{c}_1 : [\lambda]^2 \to \theta$ by:

$\circledast_2$ if $\alpha < \beta < \lambda$ then $\mathbf{c}_1\{\alpha,\beta\}$ is $F_*(\gamma_{\ell(\beta,\alpha)}(\beta,\alpha))$ where $\ell(\beta,\alpha) = \min\{\ell < k(\beta,\alpha) : F_1'(\gamma_\ell(\beta,\alpha)) = \max \mathrm{Rang}(F_1'(\rho(\beta,\alpha)))\}$.

To prove that the colouring $\mathbf{c}_1$ really witnesses $\mathrm{Pr}_1(\lambda,\lambda,\theta,\theta)$, our task is to prove:

$\circledast_3$ given $\bar{t} \in \mathbf{T}$ and $\iota < \theta$ there are $\alpha < \beta$ such that:
- $\zeta \in t_\alpha \wedge \xi \in t_\beta \Rightarrow \mathbf{c}_1\{\zeta,\xi\} = \iota$.

[Why does $\circledast_3$ holds? Let $B_\iota = \{\gamma < \lambda : F_*(\gamma) = \iota\}$. By the choice of F_* we know that $B_\varepsilon \ne \emptyset \mod D$. Focus on $A_{\bar{t},\varepsilon}$ for the specific $\bar{t} \in \mathbf{T}$ and any $\varepsilon < \theta$. Since $A_{\bar{t},\varepsilon} \in D$ we conclude that $B_\varepsilon \cap A_{\bar{t},\varepsilon} \ne \emptyset$.

Fix an ordinal $\gamma \in B_\iota \cap A_{\bar{t},\varepsilon}$. By the very definition of $A_{\bar{t},\varepsilon}$ in $\boxplus_3$ we choose $\alpha < \beta < \lambda$ and $\gamma \in B_\iota$ such that for every $(\zeta,\xi) \in t_\alpha \times t_\beta$ there exists $\ell < k(\xi,\zeta)$ for which $\gamma_\ell(\xi,\zeta) = \gamma$ and $F_1'(\gamma) \ge F_1'(\gamma_k(\xi,\zeta))$ whenever $k < k(\xi,\zeta)$ and $F_1(\gamma) \ge \varepsilon$ and $F_1'(\gamma) > F_1'(\gamma_k(\xi,\zeta))$ whenever $k < \ell$. Let $\ell(\xi,\zeta)$ be this ℓ, in fact, this ℓ is unique (for the pair (ζ,ξ)).

Now $\mathbf{c}_1\{\zeta,\xi\} = F_*(\gamma_{\ell(\xi,\zeta)}(\xi,\zeta))$ (by $\circledast_2$) which equals $F_*(\gamma)$ (by the choice of $\ell(\xi,\zeta)$) which equals ι (since $\gamma \in B_\iota$). Hence $\circledast_3$ holds and we finish Stage D.]

Stage E: The full theorem: the case of λ colors; so from now on we can assume $\theta = \partial$.

Let h', h'' be functions from θ into θ, ω respectively such that the mapping $\zeta \mapsto (h'(\zeta), h''(\zeta))$ is onto $\theta \times \omega$ and moreover each such pair is gotten θ times.

We have to define a colouring $\mathbf{c}_2 : [\lambda]^2 \to \lambda$ exemplifying $\mathrm{Pr}_1(\lambda, \lambda, \lambda, \theta)$.

This is done as follows using h', h'' and F_* from $\circledast_1$:

$\oplus_1$ for $\alpha < \beta < \lambda$ we let

- $\bullet_1$ $\zeta = \zeta(\beta, \alpha) := h'(\mathbf{c}_1\{\beta, \alpha\})$, necessarily $< \theta$
- $\bullet_2$ $n = n(\beta, \alpha) := h''(\mathbf{c}_1\{\beta, \alpha\})$, necessarily $< \omega$
- $\bullet_3$ $m = m(\beta, \alpha)$ is the n-th member of $\{k < k(\beta, \alpha) : F_1'(\gamma_k(\beta, \alpha)) = \zeta\}$ when there is such m and is zero otherwise
- $\bullet_4$ we define $\mathbf{c}_2$ as follows: for $\alpha < \beta, \mathbf{c}_2\{\alpha, \beta\}$ is $F_2'(\gamma_{m(\beta,\alpha)}(\beta, \alpha))$ recalling that F_2', a function from λ to λ is from $\odot_2$ from the end of stage A.

To prove that $\mathbf{c}_2$ indeed exemplifies $\mathrm{Pr}_1(\lambda, \lambda, \lambda, \theta)$ it suffice to prove (this is the task of the rest of the proof)

$\oplus_2$ assume $\bar{t} \in \mathbf{T}$ and $j_* < \lambda$ and we shall find $\alpha < \beta$ such that $t_\alpha \subseteq \beta$ and $(\zeta, \xi) \in t_\alpha \times t_\beta \Rightarrow \mathbf{c}_2\{\zeta, \xi\} = j_*$.

Toward this:

$\oplus_3$ (a) we apply $(*)_3$ to our $\bar{t}$, getting $\varepsilon^{\mathrm{up}}, \mathscr{U}_1^{\mathrm{up}}, \alpha_1^*$ as there

(b) we apply $(*)_2$ to our $\bar{t}$ getting $\mathscr{U}_1^{\mathrm{dn}}, \varepsilon^{\mathrm{dn}}$

(c) let $\varepsilon^{\mathrm{md}} = \max\{\varepsilon^{\mathrm{up}} + 1, \varepsilon^{\mathrm{dn}} + 1\}$.

We can find $g_2, \mathscr{U}_2^{\mathrm{up}}, \gamma_*, \alpha_2^*, m_2^*$ such that:

$\oplus_4$ (a) $\gamma_* < \lambda$ satisfies $F_2(\gamma_*) = j_*$ and $F_1(\gamma_*) = \varepsilon^{\mathrm{md}}$

(b) $\mathscr{U}_2^{\mathrm{up}} \subseteq S^*_{\gamma_*}$ is stationary hence $\delta \in \mathscr{U}_2^{\mathrm{up}} \Rightarrow F_2'(\delta) = F_2(h(\delta)) = F_2(\gamma_*) = j_* \wedge F_1'(\delta) = F_1(h(\delta)) = F_1(\gamma_*) = \varepsilon^{\mathrm{md}}$

(c) g_2 is a function with domain $\mathscr{U}_2^{\mathrm{up}}$ such that $\delta \in \mathscr{U}_2^{\mathrm{up}} \Rightarrow \delta < g_2(\delta) \in \mathscr{U}_1^{\mathrm{up}}$

(d) α_2^* satisfies $\alpha_1^* < \alpha_2^* < \min(\mathscr{U}_2^{\mathrm{up}})$

(e) if $\delta \in \mathscr{U}_2^{\mathrm{up}}$ and $\alpha \in [\alpha_2^*, \delta)$ and $\beta \in t_{g_2(\delta)}$ then

- $\rho(g_2(\delta), \delta)\hat{\ }\langle\delta\rangle \trianglelefteq \rho(g_2(\delta), \alpha)$ hence
- $\rho_{\beta,\delta}\hat{\ }\langle\delta\rangle \trianglelefteq \rho_{\beta,\alpha}$

(f) m_2^* satisfies: for every $\delta \in \mathscr{U}_2^{\mathrm{up}}$, it is the cardinality of the set $\{\ell < k(g_2(\delta), \delta) : F_1'(\gamma_\ell(g_2(\delta), \delta)) = \varepsilon^{\mathrm{md}}\}$ which may be zero.

[Why? First choose γ_* as in clause (a) of $\oplus_4$ (possible by the choice of F_0, F_1, F_2 in the beginning of Stage A; hence $\delta \in S_{\gamma_*} \Rightarrow F_2'(\delta) = F_2(h(\delta)) = F_2(\gamma_*) = j_*$ and $F_1'(\delta) = F_1(h(\delta)) = F_1(\gamma_*) = \varepsilon^{\mathrm{md}}$ (by the choice of F_1' in $\odot_7$ recalling the definitions of h, F_1'). Second, define $g' : S^*_{\gamma_*} \to \mathscr{U}_1^{\mathrm{up}}$ such that $\delta \in S^*_{\gamma_*} \Rightarrow \delta < g'(\delta) \in \mathscr{U}_1^{\mathrm{up}}$. Third, for each $\delta \in S^*_{\gamma_*} \backslash (\alpha_1^* + 1)$, find $\alpha'_{2,\delta} < \delta$ above α_1^* and $m_{2,\delta}$ such that the parallel of clauses (e),(f) (with g' here instead of g_2 there) of $\oplus_4$ holds. Fourth, use Fodor lemma to get a stationary $\mathscr{U}_2^{\mathrm{up}} \subseteq S^*_{\gamma_*}$ such that $\langle(\alpha'_{2,\delta}, m_{2,\delta}) : \delta \in \mathscr{U}_2^{\mathrm{up}}\rangle$ is constantly (α_2^*, m_2^*) and lastly let $g_2 = g' {\restriction} \mathscr{U}_2^{\mathrm{up}} \backslash (\alpha_2^* + 1)$. Now it is easy to check that $\oplus_4$ holds indeed.]

Next

$\oplus_5$ if $\delta \in \mathscr{U}_2^{\rm up}$ <u>then</u>:

(a) $F'_1(\delta) = \varepsilon^{\rm md}$

(b) if $\alpha \in [\alpha^*_2, \delta), \xi \in t_{g_2(\delta)}$ <u>then</u> $u = \{\ell < k(\xi,\alpha) : F'_1(\gamma_\ell(\xi,\alpha)) = \varepsilon^{\rm md}\}$ has $> m^*_2$ members and if ℓ is the m^*_2-th member of u then $\gamma_\ell(\xi,\alpha) = \delta$.

Why? Clause (a) holds by $\oplus_4(a),(b)$. For clause (b) use clause (a) and the demands on m^*_2. That is

(a) $\rho(\xi,\alpha) = \rho(\xi, g_2(\delta))\hat{\ }\rho(g_2(\delta),\delta)\hat{\ }\rho(\delta,\alpha)$
[Why? by $(*)_3, \oplus_4(e)$]

(b) $\mathrm{Rang}(\rho_h(\alpha, g_2(\delta))) \subseteq \varepsilon^{\rm up} \subseteq \varepsilon^{\rm md}$
[Why? by $(*)_2$]

(c) the set $\{\ell < k(g_2(\delta),\delta) : F'_1(\gamma_\ell(g_2(\delta),\delta)) = \varepsilon^{\rm md}\}$ has m^*_2 members
[why? by $\oplus_4(f)$]

(d) $F'_1(\gamma_0(\delta,\alpha)) = F'_1(\delta) = \varepsilon^{\rm md}$
[Why? by $\oplus_4(a),(b)$]]

(e) if ℓ_* is the m^*_2-th member of $\{\ell : F_1(\gamma_\ell(\xi,\alpha)) = \varepsilon^{\rm md}\}$ then $\gamma_{\ell_*}(\xi,\alpha) = \delta$
[Why? putting the above together]

So $\oplus_5$ holds indeed.

Now choose $\varepsilon(*) < \theta$ such that $h'(\varepsilon(*)) = \varepsilon^{\rm md}$ and $h''(\varepsilon(*)) = m^*_2$.

Next, let $E = \{\delta < \lambda : \delta$ limit ordinal $> \alpha^*_2$ such that $\delta = \sup(\mathscr{U}_1^{\rm dn} \cap \delta)$ and $\alpha < \delta \Rightarrow g_2(\alpha) < \delta\}$.

Lastly,

$\oplus_6$ choose $\delta_1 < \delta_2$ such that

(a) $\delta_1 \in \mathscr{U}_1^{\rm dn} \cap E$

(b) $\delta_2 \in \mathscr{U}_2^{\rm up} \cap E\backslash(\delta_1 + 1)$

(c) $\mathbf{c}_1\{\delta_2,\delta_1\} = \varepsilon(*)$,

[Why does such a pair (δ_1,δ_2) exist? By Stage D applied to $\bar{s} = \langle s_\alpha : \alpha < \lambda\rangle$ where $s_\alpha = \{\min(\mathscr{U}_1^{\rm dn} \cap E\backslash\alpha), \min(\mathscr{U}_2^{\rm up} \cap E\backslash\alpha)\}$.

That is, we can find ordinals $\alpha < \beta < \lambda$ such that: for every $(\zeta,\xi) \in (s_\alpha \times s_\beta)$ we have $\mathbf{c}_1\{\xi,\zeta\} = \varepsilon^{\rm md}$.

Let $\delta_1 = \min(\mathscr{U}_1^{\rm dn} \cap E \setminus \alpha$ and let $\delta_2 = \min(\mathscr{U}_1^{\rm up} \cap E \setminus \beta$.

So $(\delta_1,\delta_2) \in (s_\alpha \times s_\beta)$ hence clearly $\delta_1 < \delta_2$, $\mathbf{c}_1\{\delta_1,\delta_2\} = \varepsilon(*)$, $\delta_1 \in \mathscr{U}_1^{\rm dn} \cap E$ and $\delta_1 \in \mathscr{U}_1^{\rm up} \cap E$. So the pair (δ_1,δ_2) is as promised in in $\oplus_6$]

Now let $\beta = g_2(\delta_2)$ and choose $\alpha \in \mathscr{U}_1^{\rm dn} \cap \delta_1\backslash(\alpha^*_2 + 1)$. Easy to check that α,β are as required.

So we have finished proving Theorem 1.1. $\square_{1.1}$

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