CHARACTERIZING THE SPECTRA OF CARDINALITIES OF BRANCHES OF KUREPA TREES

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Abstract. We give a complete characterization of the sets of cardinals that in a suitable forcing extension can be the Kurepa spectrum, that is, the set of cardinalities of branches of Kurepa trees. This answers a question of the first named author.

1. Introduction

A tree is a Kurepa tree if it is of height ω_1, each of its levels is countable, and it has more than ω_1-many cofinal (that is of order type ω_1) branches. In this paper we study the possible values of the branch spectrum of Kurepa trees, i.e. the set

$$\mathrm{Sp}_{\omega_1} = \{\lambda : \text{ there exists a Kurepa tree } T \text{ s.t. } |\mathcal{B}(T)| = \lambda\} \subseteq [\omega_2, 2^{\omega_1}]$$

(where $\mathcal{B}(T)$ stands for the set of cofinal branches of T).

The spectrum is related to the model theoretical spectrum of maximal models of $\mathcal{L}_{\omega_1,\omega}$-sentences [SS17]. Also canonical topological and combinatorial structures are associated with branches of Kurepa trees possessing a remarkably wide range of nonreflecting properties [Kos05]. For higher Kurepa trees (of weakly compact height) the consistency strength of certain types of the branch spectrum was studied in [HM19].

It was first shown by Silver that the Kurepa Hypothesis (i.e. the existence of a Kurepa tree) is independent [Sil67], or see [Kun83, Ch VIII, 3.]. Moreover the non-existence of Kurepa trees is equiconsistent with the existence of an inaccessible cardinal [Kun83, Ch VII, Ex. B8.].

Questions about the possible values of the spectrum were addressed by Jin and Shelah in [JS92]. They proved (assuming an inaccessible cardinal) that consistently there are only Kurepa trees with ω_3-many cofinal branches while $2^{\omega_1} = \omega_4$.

Building on ideas of Jin and Shelah, the first named author provided a sufficient condition for a set to be equal to Sp_{ω_1} in a forcing extension in [Poo]. Formally, it was shown that if **GCH** holds, and $0, 1 \notin S$ is a set of ordinals such that S satisfies either

<u>Case A:</u>

(i) $2 \in S$,

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(ii) $\{\sup C : C \in [S]^{\leq \omega_1}\} \subseteq S$,
(iii) $(\forall \alpha \in S) : \ (\omega \leq \mathrm{cf}(\alpha) < \omega_2) \ \rightarrow \ (\alpha + 1 \in S)$,

or

Case B:
(i) $\exists$ an inaccessible κ,
(ii) $\{\sup C : C \in [S]^{<\kappa}\} \subseteq S$,
(iii) $(\forall \alpha \in S) : \ (\omega \leq \mathrm{cf}(\alpha) < \kappa) \ \rightarrow \ (\alpha + 1 \in S)$,

then in a forcing extension we have $\{\alpha : \ \aleph_\alpha \in \mathrm{Sp}_{\omega_1}\} = S$ (cardinals are only collapsed in Case B, from (ω_1, κ)). It can be easily seen that if $\mathrm{cf}(\mu) = \omega$ and $(\mathrm{Sp}_{\omega_1} \cap \mu)$ is cofinal in μ, then there exists a Kurepa tree with μ-many branches, as the union of countably many Kurepa trees is a Kurepa tree, and it is not difficult to see that the same holds if $\mathrm{cf}(\mu) = \omega_1$, therefore Case A / (ii) and Case B / ((ii)) are in fact necessary. However, it remained a question whether the last clauses can be dropped.

In this paper as the main result we prove that assuming **CH** + $(2^{\omega_1} = \omega_2)$ conditions (i), (ii) (in both cases) are in fact sufficient by forcing a model of $\{\alpha : \aleph_\alpha \in \mathrm{Sp}_{\omega_1}\} = S$. Also, we can arbitrarily prescribe 2^{ω_1} to be any cardinal $\lambda \geq \sup(\mathrm{Sp}_{\omega_1})$ if in Case A the equality $\lambda^{<\omega_2} = \lambda$ holds, or in Case B $\lambda^{<\kappa} = \lambda$ holds too.

Moreover, when we do not want Kurepa trees with ω_2-many cofinal branches, we prove that the inaccessible is necessary by verifying that if ω_2 is a successor in L, then there exists a Kurepa tree with only ω_2-many cofinal branches in V. It was known that these assumptions imply that there exists a Kurepa tree even in $L[A]$ for some $A \subseteq \omega_1$ [Kun83, Ch VII, Ex. B8.] (possibly having more than ω_2-many cofinal branches in V). Our proof not only utilizes countable elementary submodels of initial segments of $L[A]$, but the nodes of the tree are such elementary submodels, and each cofinal branch uniquely corresponds to an initial segment of $L[A]$.

2. Preliminaries, notations

Under ordinals we always mean Neumann ordinals. For a fixed cardinal χ we will use the notation $\mathcal{H}(\chi)$ for the collection of sets of hereditary size less than χ, i.e.

$$\mathcal{H}(\chi) = \{x : \ |\mathrm{trcl}(x)| < \chi\},$$

where $\mathrm{trcl}(x)$ stands for the transitive closure of x. In terms of forcing we will use the notations of [Kun13], e.g. $p \leq q$ means that p is the stronger. If it is clear from the context and won't make any confusion we will identify the set x in the ground model with its canonical name $\check{x}$. For a set A the symbol $\mathcal{P}(A)$ denotes the powerset of A, and $[A]^\lambda$ stands for $\{X \in \mathcal{P}(A) : \ |X| = \lambda\}$. For a function $s = \{\langle \beta, s(\beta)\rangle : \ \beta \in \mathrm{dom}(s)\}$ we will also use the following notation and refer to s as

$$\langle s_\beta : \ \beta \in \mathrm{dom}(s)\rangle.$$

Under a sequence we mean a function defined on a set of ordinals. For sequences s, t the relation $s = t \restriction \mathrm{dom}(s)$ (or equivalently $s \subseteq t$) will be also denoted by $s \lhd t$.

Definition 2.1. A tree $\langle T, \prec_T\rangle$ is a partially ordered set (poset) in which for each $x \in T$ the set

$$T_{\prec x} = \{y \in T : \ y \prec_T x\}$$

is well ordered by $\prec_T$.

Definition 2.2. The height of x in the tree T is the order type of $T_{\prec x}$

$$\mathrm{ht}(x) = \mathrm{otp}(T_{\prec x}).$$

Definition 2.3. For each ordinal α the restriction of T to α is

$$T_{<\alpha} = \{t \in T : \ \mathrm{ht}(t) < \alpha\}.$$

Definition 2.4. The height of the tree T (in symbols $\mathrm{ht}(T)$), is the least β such that

$$\nexists t \in T : \ \mathrm{ht}(t) = \beta.$$

We will need the following lemma [Kun83, Ch II. Thm. 1.6.] which we will refer to as the Δ-system Lemma.

Lemma 2.5. *Let κ be an infinite cardinal, let $\theta > \kappa$ be regular, and satisfy $\forall \alpha < \theta$ $(|\alpha^{<\kappa}| < \theta)$. Assume that $|\mathcal{A}| \geq \theta$, and $\forall x \in \mathcal{A}$ $(|x| < \kappa)$. Then there is a $\mathcal{D} \subseteq \mathcal{A}$, such that $|\mathcal{D}| = \theta$, and $\mathcal{D}$ forms a Δ-system, i.e. there is a kernel set y such that*

$$\forall x \neq x' \in \mathcal{D} : \quad x \cap x' = y.$$

3. The forcing

Now we can state our main theorem.

Theorem 3.1. *Let $S_\bullet$ be a set of infinite cardinals such that $\omega, \omega_1 \notin S_\bullet$. Assume* **CH**, *and that either*

Case 1:

(i) $\omega_2 \in S_\bullet$,
(ii) $2^{\omega_1} = \omega_2$,
(iii) $\{\sup C : \ C \in [S_\bullet]^{<\omega_2}\} \subseteq S_\bullet$,

or

Case 2:

(i) there exists an inaccessible κ such that $S_\bullet \cap (\omega_1, \kappa) = \emptyset$,
(ii) $\{\sup C : \ C \in [S_\bullet]^{<\kappa}\} \subseteq S_\bullet$.

Then there exists a forcing extension $V^{\mathbb{P}}$ such that

$$V^{\mathbb{P}} \models \ S_\bullet = \mathrm{Sp}_{\omega_1}, \quad \text{where } \mathbb{P} \text{ only collapses cardinals in } (\omega_1, \kappa) \text{ in } \underline{\text{Case2}}.$$

The key will be Lemma 3.26. After Lemma 3.29 we will put together the pieces in a short argument. Before these we need some preparation.

Definition 3.2. In Case 1 (i.e. $\omega_2 \in S_\bullet$) define the cardinal κ to be ω_2.

Corollary 3.3. *No cardinal $\mu \notin (\omega_1, \kappa)$ is collapsed.*

Theorem 3.4. *Suppose that all conditions from Theorem 3.1 hold, and κ is defined in Definition 3.2. Assume further that λ is a cardinal which is an upper bound of $S_\bullet$ such that $\lambda^{<\kappa} = \lambda$ (thus $\mathrm{cf}(\lambda) \geq \kappa$). Then there exists a forcing extension $V^{\mathbb{P}}$ with*

$$V^{\mathbb{P}} \models \ (S_\bullet = \{\mu : \ \text{there exists a Kurepa tree } T \text{ s.t. } |\mathcal{B}(T)| = \mu\}) \ \wedge \ (2^{\omega_1} = \lambda).$$

Definition 3.5. Let $S_\bullet^+ = S_\bullet \cup \{\kappa, \lambda\}$.

Definition 3.6. For a cardinal $\theta \in S_\bullet$ let $\mathbb{Q}_\theta$ be the following notion of forcing. The triplet $p = \langle T_p, u_p, \overline{\eta}_p \rangle$ is an element of $\mathbb{Q}_\theta$ iff

(a) T_p is a countable tree of height δ for some $\delta < \omega_1$ on the underlying set $\omega \cdot \delta$, where the β'th level is $[\omega\cdot\beta, \omega\cdot(\beta+1))$, i.e. $T_{p,\leq\beta} \setminus T_{p,<\beta} = [\omega\cdot\beta, \omega\cdot(\beta+1))$ for each $\beta < \delta$,
(b) for each $t \in T_p$ and $\beta < \delta$ there exists $t' \in T_p \setminus T_{p,<\beta}$ s.t. $t \prec_{T_p} t'$,
(c) $u_p \in [\theta]^{\leq\omega}$,
(d) $\overline{\eta}_p = \langle \eta_{p,\alpha} : \ \alpha \in u_p \rangle$, where $\eta_{p,\alpha} \subseteq T_p$ is a branch in $T_{p,<\gamma}$ for some $\gamma \in \{\beta+1 : \ \beta < \delta = \mathrm{ht}(T_p)\}$ (we do it for a technical reason, we also could have stored only the maximal element instead of a chain with a maximal element).

Then $\mathbb{Q}_\theta$ is a poset with the obvious order, i.e. $q \leq p$, if T_q is an end-extension of T_p, formally $T_{q,<\mathrm{ht}(T_p)} = T_p$, and for each $\alpha \in u_p$ the inclusion $\eta_{p,\alpha} \subseteq \eta_{q,\alpha}$ holds.

Let $\underset{\sim}{T}_\theta, \underset{\sim}{\overline{\eta}}_\theta$ be the names for the generic tree and sequence, i.e. denoting the generic filter by $\mathbf{G}_\theta$

$$1_{\mathbb{Q}_\theta} \Vdash \quad \underset{\sim}{T}_\theta = \cup\{T_p : \ p \in \mathbf{G}_\theta\} \qquad \text{and}$$
$$1_{\mathbb{Q}_\theta} \Vdash \quad \underset{\sim}{\overline{\eta}}_\theta = \left\langle \underset{\sim}{\eta}_{\theta,\alpha} = \cup\{\eta_{p,\alpha} : \ p \in \mathbf{G}_\theta\} : \quad \alpha \in \theta \right\rangle.$$

Definition 3.7. For a cardinal $\theta \in S_\bullet$ let $\mathbb{Q}^*_\theta \subseteq \mathbb{Q}_\theta$ be the following subposet.

$$p \in \mathbb{Q}^*_\theta, \text{ iff } \mathrm{ht}(T_p) \text{ is a successor, and } (\forall \alpha \in u_p) : \ \eta_{p,\alpha} \text{ is a branch through } T_p.$$

Definition 3.8. If $\lambda \notin S_\bullet$ then let $\mathbb{Q}_\lambda$ be the countable supported product of $\langle {}^{<\omega_1}2, \lhd \rangle$'s of length λ, i.e.

$$\mathbb{Q}_\lambda = \{p = \langle \eta_\alpha : \alpha \in u_p \rangle : \ (\forall \alpha \in u_p) \ \eta_\alpha \in {}^{<\omega_1}2, \ \text{ for some } u_p \in [\lambda]^{\leq\omega}\}.$$

Definition 3.9. If $\kappa \notin S_\bullet$ (and then $\kappa > \omega_2^V$ is inaccessible), then let $\mathbb{Q}_\kappa$ be the countable supported product of $\langle {}^{<\omega_1}\gamma, \lhd \rangle$'s ($\gamma < \kappa$), a forcing which collapses each cardinal in (ω_1, κ):

$$\mathbb{Q}_\kappa = \{p = \langle \eta_\alpha : \ \alpha \in u_p \rangle : \ (\forall \alpha \in u_p) \ \eta_\alpha \in {}^{<\omega_1}\alpha, \text{ for some } u_p \in [\kappa]^{\leq\omega}\}.$$

Definition 3.10. We define the posets which we will need later.

1) For $S \subseteq S^+_\bullet$ let $\mathbb{P}_S$ be the countable supported product of $\mathbb{Q}_\theta$'s ($\theta \in S$), i.e.

$$\mathbb{P}_S = \{p \text{ is a function} : \ \mathrm{dom}(p) \in [S]^{\leq\omega} \ \wedge \ (\forall \theta \in \mathrm{dom}(p) \ p(\theta) \in \mathbb{Q}_\theta)\}.$$

With a slight abuse of notation for $p \in \mathbb{P}_S$ and $\theta \in S \setminus \mathrm{dom}(p)$ we will mean $1_{\mathbb{Q}_\theta}$ under $p(\theta)$.

2) For $\theta \in S^+_\bullet$, $U \subseteq \theta$ define its restriction from θ to U, i.e.

$$\mathbb{Q}_{\theta,U} = \{p \in \mathbb{Q} : \ u_p \subseteq U\}.$$

3) For $S \subseteq S^+_\bullet$, $\overline{U} = \langle U_\theta : \theta \in S \rangle \in \prod_{\theta\in S} \mathcal{P}(\theta)$ we define $\mathbb{P}_{S,\overline{U}}$ to be $\mathbb{P}$'s restriction to coordinates in U_θ's, i.e.

$$\mathbb{P}_{S,\overline{U}} = \{p \in \mathbb{P}_S : \ (\forall \theta \in S) \ p(\theta) \in \mathbb{Q}_{\theta,U_\theta}\}.$$

4) For $S, S' \subseteq S^+_\bullet$, $\overline{U} = \langle U_\theta : \theta \in S \rangle \in \prod_{\theta\in S} \mathcal{P}(\theta)$, $\overline{U}' = \langle U'_\theta : \theta \in S \rangle \in \prod_{\theta \in S'} \mathcal{P}(\theta)$ we define

- $\overline{U} + \overline{U}' = \langle U_\theta \cup U'_\theta : \ \theta \in S \cup S' \rangle$ (where for $\theta \in S' \setminus S$ under U_θ we mean the empty set, similarly for $\theta \in S \setminus S'$, U'_θ),

- $\overline{U} - \overline{U}' = \langle U_\theta \setminus U'_\theta : \ \theta \in S \rangle$ (here we also mean the empty set under U'_θ if $\theta \in S \setminus S'$),
- $\overline{\mathrm{id}}_S = \langle \theta : \ \theta \in S \rangle$
- for the set X if $\overline{W}_\alpha \in \prod_{\theta \in S} \mathcal{P}(\theta)$ $(\alpha \in X)$ then
$$\sum_{\alpha \in X} \overline{W}_\alpha = \left\langle \bigcup_{\alpha \in X} (W_\alpha)_\theta : \ \theta \in S \right\rangle.$$

5) Let $\mathbb{P} = \mathbb{P}_{S^+_\bullet}$.
6) If $p_0, p_0, \dots, p_n \in \mathbb{P}$ let $\bigwedge_{i \leq n} p_i$ denote the greatest lower bound if exists.
7) For $p \in \mathbb{P}$, and $S \subseteq S^+_\bullet$, $\overline{U} = \langle U_\theta : \theta \in S \rangle \in \prod_{\theta \in S} \mathcal{P}(\theta)$ define $p \restriction \overline{U} \in \mathbb{P}_S$ to be the following restriction of $p \restriction S$ in the obvious fashion
$$\text{for each } \theta \in S : \quad (p \restriction \overline{U})(\theta) = \langle T_{p(\theta)}, u_{p_\theta} \cap U_\theta, \overline{\eta}_p \restriction U_\theta \rangle.$$

Definition 3.11. For $S \subseteq S^+_\bullet$ define the notion of forcing $\mathbb{P}^*$ ($\mathbb{P}^*_S$, $\mathbb{P}^*_{S,\overline{U}}$, resp.) to be the subposet of $\mathbb{P}$ ($\mathbb{P}_S$, $\mathbb{P}_{S,\overline{U}}$, resp.) consisting of elements p for that $p(\theta) \in \mathbb{Q}^*_\theta$ holds for each $\theta \in S_\bullet \cap \mathrm{supp}(p)$.

Remark 3.12. The notion of forcing $\mathbb{P}^*$ ($\mathbb{P}^*_S$, $\mathbb{P}^*_{S,\overline{U}}$, resp.) is a dense subposet of $\mathbb{P}$ ($\mathbb{P}_S$, $\mathbb{P}_{S,\overline{U}}$, resp.), therefore forcing with $\mathbb{P}^*$ ($\mathbb{P}^*_S$, $\mathbb{P}^*_{S,\overline{U}}$, resp.) yields the same extensions as forcing with $\mathbb{P}$ ($\mathbb{P}_S$, $\mathbb{P}_{S,\overline{U}}$, resp.).

Claim 3.13. *Let $S \subseteq S^+_\bullet$, $\overline{U} = \langle U_\theta : \theta \in S \rangle$ be fixed. Then the poset $\mathbb{P}_{S,\overline{U}}$ has the κ-cc property.*

Proof. Suppose that $\{p_\alpha : \ \alpha \in \kappa\} \subseteq \mathbb{P}_{S,\overline{U}}$ is an antichain. Working in V', applying the Δ-system lemma (Lemma 2.5) for the system $\{\mathrm{dom}(p_\alpha) : \ \alpha \in \kappa\}$ of countable sets (1) from Definition 3.10), we obtain a set $A \in [\kappa]^\kappa$, such that the $\mathrm{dom}(p_\alpha)$'s $(\alpha \in A)$ form a Δ-system with kernel $K \subseteq S$. Since K is obviously countable, for each α we have that $\langle T_{p_\alpha(\theta)} : \ \theta \in K \rangle$ is a countable sequence of countable trees (by (a) from Definition 3.6). This means that by **CH** we can assume that
$$\langle T_{p_\alpha(\theta)} : \ \theta \in K \rangle = \langle T_{p_\beta(\theta)} : \ \theta \in K \rangle \quad (\forall \alpha, \beta \in A). \tag{3.1}$$
Now applying the Δ-system lemma again for the system
$$U_\alpha = \bigcup_{\theta \in S} \left(\{\theta\} \times u_{p_\alpha(\theta)} \right) \quad (\alpha \in \kappa)$$
yields a set $A' \in [A]^\kappa$ such that the U_α's $(\alpha \in A')$ form a Δ-system with kernel $I \subseteq \bigcup_{\theta \in S} \{\theta\} \times \theta$ (of course, in fact, $I \subseteq \bigcup_{\theta \in K} \{\theta\} \times \theta$). Now by (3.1) it suffices to prove that
$$\exists \alpha \neq \beta \in A' \ \text{ such that (for each } \langle \theta, \delta \rangle \in I) : \ \ \eta_{p_\alpha(\theta),\gamma} = \eta_{p_\beta(\theta),\gamma}, \tag{3.2}$$
for which it is enough to prove
$$\left| \left\{ \left\langle \eta_{p_\alpha(\theta),\gamma} : \ \langle \theta, \gamma \rangle \in I \right\rangle : \ \alpha \in A' \right\} \right| < \kappa. \tag{3.3}$$
Fix $\alpha \in A'$. Now for each $\langle \theta, \gamma \rangle \in I$, if $\theta \in S_\bullet$ then $\eta_{p_\alpha(\theta),\gamma} \in \ [\omega_1]^{<\omega_1}$ (a branch through $T_{p_\alpha(\theta)}$).

This means that (using that I is countable)
$$\left\{ \left\langle \eta_{p_\alpha(\theta),\gamma} : \ \langle \theta, \gamma \rangle \in I, \ \theta \in S_\bullet \right\rangle : \ \alpha \in A' \right\} \subseteq \prod_{\langle \theta, \gamma \rangle \in I, \ \theta \in S_\bullet} [\omega_1]^{<\omega_1}, \tag{3.4}$$

which latter set is of size ω_1 by **CH**. Second, if $\theta = \lambda \in (S^+_\bullet \setminus S_\bullet) \cap S$, then

$$\{\langle \eta_{p_\alpha(\theta),\gamma} : \langle \theta, \gamma\rangle \in I,\ \theta = \lambda\rangle : \alpha \in A'\} \subseteq \prod_{\langle \theta,\gamma\rangle \in I,\ \theta=\lambda} {}^{<\omega_1}2.$$

Finally we have to consider the coordinate $\theta = \kappa$ if $\kappa \in S \setminus S_\bullet$. Then letting $\delta = \sup\{\gamma : \langle \kappa, \gamma\rangle \in I\}$ we have $\delta < \kappa$, because I is countable and κ is inaccessible. Then

$$\{\langle \eta_{p_\alpha(\kappa),\gamma} : \langle \kappa, \gamma\rangle \in I\} \subseteq \prod_{\langle \kappa,\gamma\rangle \in I} {}^{<\omega_1}\delta, \tag{3.5}$$

and since κ is inaccessible, this case $|\prod_{\langle \kappa,\gamma\rangle \in I} {}^{<\omega_1}\delta| < \kappa$. We obtain (using $\omega_1 < \kappa$) that

$$\left|\{\langle \eta_{p_\alpha(\theta),\gamma} : \langle \theta, \gamma\rangle \in I\}\right| \leq \omega_1 \cdot \omega_1 \cdot \left|\prod_{\langle \kappa,\gamma\rangle \in I} {}^{<\omega_1}\delta\right| < \kappa,$$

therefore (3.3) holds. $\square$

Now we make the intuition behind the easy idea of first adding the trees and some branches, and then forcing over the extension precise.

Claim 3.14. *For each $S \subseteq S^+_\bullet$, $\overline{U} = \langle U_\theta : \theta \in S\rangle$ we have*

$$\mathbb{P}_{S,\overline{U}} \lessdot \mathbb{P}_S \lessdot \mathbb{P},$$

i.e. $\mathbb{P}_{S,\overline{U}}$ completely embeds into $\mathbb{P}_S$, which completely embeds into $\mathbb{P}$.

Proof. Since $\mathbb{P} \simeq \mathbb{P}_S \times \mathbb{P}_{S^+_\bullet \setminus S}$, it is enough to prove that $\mathbb{P}_{S,\overline{U}} \lessdot \mathbb{P}_S$.

Assume that $A \subseteq \mathbb{P}_{S,\overline{U}}$ is a maximal antichain in $\mathbb{P}_{S,\overline{U}}$, and let $p \in \mathbb{P}_S \setminus \mathbb{P}_{S,\overline{U}}$. Then there exists $a \in A$, $a' \in \mathbb{P}_{S,\overline{U}}$ such that $a' \leq a$, $a' \leq b \restriction \overline{U}$. But then it is straightforward to check that also a' and b have a common lower bound. $\square$

Definition 3.15. Let $S \subseteq S_\bullet$, $\overline{U} = \langle U_\theta : \theta \in S\rangle$, $\theta_0 \in S$, $U'_{\theta_0} \subseteq \theta_0 \setminus U_{\theta_0}$. Then $\underset{\sim}{\mathbb{Q}}^\circ_{\theta_0,U'_{\theta_0}} = \underset{\sim}{\mathbb{Q}}^\circ_{(S,\overline{U}),\theta_0,U'_{\theta_0}}$ denotes the $\mathbb{P}_{S,\overline{U}}$-name for a notion of forcing which adds the branches $\underset{\sim}{\eta}_{\theta_0,\alpha}$ ($\alpha \in U'_{\theta_0}$) to $\underset{\sim}{T}_{\theta_0}$ in the following way

$$1 \Vdash_{\mathbb{P}_{S,\overline{U}}} \underset{\sim}{\mathbb{Q}}^\circ_{\theta_0,U'_{\theta_0}} = \left\{ \begin{array}{l} p = \langle \overline{\eta}_p, u_p\rangle : \ (u_p \in [U'_{\theta_0}]^{\leq\omega}) \wedge (\overline{\eta}_p = \langle \eta_{p,\alpha} : \ \alpha \in u_p\rangle), \\ \text{such that each } \eta_{p,\alpha} \text{ is a branch of } \underset{\sim}{T}_{\theta_0,<\delta_\alpha} \\ \text{for some } \delta_\alpha \in \{\gamma+1 : \ \gamma < \omega_1\} \end{array} \right\}.$$

If it is clear from the context we will use $\underset{\sim}{\mathbb{Q}}^\circ_{\theta_0,U'_{\theta_0}}$ not mentioning S and $\overline{U}$.

Definition 3.16. Let $S \subseteq S_\bullet$, $\overline{U} = \langle U_\theta : \theta \in S\rangle$, $\theta_0 \in S$.

If $\theta \in S^+_\bullet \setminus S_\bullet$, and $U'_\theta \subseteq \theta \setminus U_\theta$, then define the $\mathbb{P}_{S,\overline{U}}$-name $\underset{\sim}{\mathbb{Q}}_{\theta,U'_\theta} = \underset{\sim}{\mathbb{Q}}^\circ_{\theta,U'_\theta}$ to be the name for $\mathbb{Q}_{\theta,U'_\theta}$.

Definition 3.17. Let $S \subseteq S^+_\bullet$, $\overline{U} = \langle U_\theta : \theta \in S\rangle$, $\overline{U}' = \langle U'_\theta : \theta \in S\rangle \in \prod_{\theta\in S} \mathcal{P}(\theta)$, where $U_\theta \cap U'_\theta = \emptyset$ for each $\theta \in S$. Then $\underset{\sim}{\mathbb{P}}^\circ_{\overline{U}'} = \underset{\sim}{\mathbb{P}}^\circ_{(S,\overline{U}),\overline{U}'}$ denotes the $\mathbb{P}_{S,\overline{U}}$-name for the countably supported product of $\underset{\sim}{\mathbb{Q}}^\circ_{\theta,U'_\theta}$'s ($\theta \in S$), i.e. a notion of forcing which adds the branches $\underset{\sim}{\eta}_{\theta,\alpha}$ $(\alpha \in U'_\theta)$ to $\underset{\sim}{T}_\theta$ for each $\theta \in S \setminus S_\bullet$, and the sequences $\underset{\sim}{\eta}_{\kappa,\alpha}$ $(\alpha \in U'_\kappa)$ if $\kappa \in S \setminus S_\bullet$, $\underset{\sim}{\eta}_{\lambda,\alpha}$ $(\alpha \in U'_\lambda)$ if $\lambda \in S \setminus S_\bullet$:

$$1 \Vdash_{\mathbb{P}_{S,\overline{U}}} \underset{\sim}{\mathbb{P}}^\circ_{\overline{U}'} = \left\{ p \text{ is a function}: \ \mathrm{dom}(p) \in [S]^{\leq\omega} \ \wedge \ (\forall \theta \in \mathrm{dom}(p)\ p(\theta) \in \underset{\sim}{\mathbb{Q}}^\circ_{\theta,U'_\theta})\} \right\}.$$

Again, as in Definition 3.15 if it does not cause any confusion we only use the notation $\underset{\sim}{\mathbb{P}}^\circ_{\overline{U}'}$ not mentioning S and $\overline{U}$.

The following claim is an easy observation.

Claim 3.18. *If $\mathbf{G}$ is a $\mathbb{P}_{S,\overline{U}}$-generic filter over V (where $S \subseteq S^+_\bullet$, $\overline{U} = \langle U_\theta : \theta \in S\rangle$, $\overline{U}' = \langle U'_\theta : \theta \in S\rangle \in \prod_{\theta\in S} \mathcal{P}(\theta)$, and $U_\theta \cap U'_\theta = \emptyset$ for each $\theta \in S$), then with the notation from* [Kun13]

$$\mathbb{P}_{S,\overline{U}+\overline{U}'}/\mathbf{G} = \{p \in \mathbb{P}_{S,\overline{U}+\overline{U}'} : \ \forall q \in \mathbf{G}\ p \not\perp q\},$$

the quotient poset $\mathbb{P}_{S,\overline{U}+\overline{U}'}/\mathbf{G}$ and the evaluation of $\underset{\sim}{\mathbb{P}}^\circ_{\overline{U}'}$ are isomorphic, i.e.

$$V[\mathbf{G}] \models \ \ \underset{\sim}{\mathbb{P}}^\circ_{\overline{U}'}[\mathbf{G}] \simeq \mathbb{P}_{S,\overline{U}+\overline{U}'}/\mathbf{G}.$$

Since $\mathbb{P}_{S,\overline{U}}$ completely embeds into $\mathbb{P}_{S,\overline{U}+\overline{U}'}$ (by Claim 3.14), [Kun13][Lemma V.4.45.] (and [Kun13, Lemma V.4.44.]) implies the following.

Claim 3.19. *Let $S \subseteq S^+_\bullet$, $\overline{U} = \langle U_\theta : \theta \in S\rangle$, $\overline{U}' = \langle U'_\theta : \theta \in S\rangle \in \prod_{\theta\in S} \mathcal{P}(\theta)$, where $U_\theta \cap U'_\theta = \emptyset$ for each $\theta \in S$. Then the canonical embedding from $\mathbb{P}_{S,\overline{U}+\overline{U}'}$ to the iteration $\mathbb{P}_{S,\overline{U}} * (\mathbb{P}_{S,\overline{U}+\overline{U}'}/\mathbf{G})$ is a dense embedding.*

Now putting together Claims 3.18 and 3.19 we have the following, meaning that instead of forcing with $\mathbb{P}_{S,\overline{U}+\overline{U}'}$ we can force with $\mathbb{P}_{S,\overline{U}}$ and then with (the evaluation of) $\underset{\sim}{\mathbb{P}}^\circ_{\overline{U}'}$.

Lemma 3.20. *Let $S \subseteq S^+_\bullet$, $\overline{U} = \langle U_\theta : \theta \in S\rangle$, $\overline{U}' = \langle U'_\theta : \theta \in S\rangle \in \prod_{\theta\in S} \mathcal{P}(\theta)$, where $U_\theta \cap U'_\theta = \emptyset$ for each $\theta \in S$. Then forcing with $\mathbb{P}_{S,\overline{U}+\overline{U}'}$ amounts to the same as forcing with $\mathbb{P}_{S,\overline{U}}$ and then with $\mathbb{P}_{S,\overline{U}+\overline{U}'}/\mathbf{G} \simeq \underset{\sim}{\mathbb{P}}^\circ_{\overline{U}'}$.*

Definition 3.21. If $S \subseteq S^+_\bullet$, $\overline{U} = \langle U_\theta : \theta \in S\rangle$, $\overline{U}' = \langle U'_\theta : \theta \in S\rangle \in \prod_{\theta\in S} \mathcal{P}(\theta)$. Now if $\mathbf{G}$ is generic over $\mathbb{P} = \mathbb{P}_{S^+_\bullet}$ then we define

- $\mathbf{G}_S = \mathbf{G} \cap \mathbb{P}_S$,
- $\mathbf{G}_{S,\overline{U}} = \mathbf{G} \cap \mathbb{P}_{S,\overline{U}}$,
- and $\mathbf{G}^\circ_{\overline{U}'} \subseteq \mathbb{P}^\circ_{\overline{U}'}[\mathbf{G}_{S,\overline{U}}] \in V[\mathbf{G}_{S,\overline{U}}]$ to be the filter given by the canonical mapping from Claims 3.18, 3.19.

The following are basic observations. Roughly speaking, we isolate a dense ω_1-closed subset of a two-step iteration similarly as in [Kun78].

Claim 3.22. $\mathbb{P}^*$ *(and in general each $\mathbb{P}^*_{S,\overline{U}}$) is ω_1-closed, i.e. for each decreasing sequence of type ω has a lower bound. In particular if $\mathbf{G}^* \subseteq \mathbb{P}^*$, (or in general $\mathbf{G}^*_{S,\overline{U}} \subseteq \mathbb{P}^*_{S,\overline{U}}$) is generic over V, then there is no new sequence of ordinals of type ω.*

The last claim and Remark 3.12 obviously implies the following.

Claim 3.23. *Forcing with $\mathbb{P}$ (or $\mathbb{P}_{S,\overline{U}}$) doesn't add new sequence of ordinals of type ω, and for a given generic filter $\mathbf{G} \subseteq \mathbb{P}$*

$$\mathcal{H}(\omega_1)^V = \mathcal{H}(\omega_1)^{V[\mathbf{G}]} = \mathcal{H}^{V[\mathbf{G}_{S,\overline{U}}]}.$$

Lemma 3.24. *Let $\mathbf{G} \subseteq \mathbb{P}_{S,\overline{U}}$ generic over V, $B \in V[\mathbf{G}]$ where $B \subseteq \mathcal{H}(\omega_1)$. Then (in V) there exist $S_* \subseteq S$, $|S_*| < \kappa$ and $\overline{W}_* = \langle W^*_\gamma : \gamma \in S_* \rangle \in \prod_{\gamma \in S_*} [U_\gamma]^{<\kappa}$, such that $B \in V[\mathbf{G}_{S_*,\overline{W}_*}]$.*

Proof. Choose $p \in \mathbf{G}$ forcing that $B \subseteq \mathcal{H}(\omega_1)$, and a nice $\mathbb{P}_{S,\overline{U}}$-name for B, obtaining for each $x \in \mathcal{H}(\omega_1)$ an antichain $A_x \subseteq \mathbb{P}_{S,\overline{U}}$ deciding about $x \in B$. Then by κ-cc we have that each $|A_x| < \kappa$, the set $S_* = \bigcup_{x \in \mathcal{H}(\omega_1)} \bigcup_{a \in A_x} \mathrm{dom}(a)$ is of size less than κ (as κ is either inaccessible, or ω_2). Also for $\theta \in S_*$ the set $W^*_\theta = \bigcup_{x \in \mathcal{H}(\omega_1)} \bigcup_{a \in A_x} u_{a(\theta)}$ is smaller that κ. Now it is easy to see that $\overline{W}_* = \langle W^*_\gamma : \gamma \in S_* \rangle$ is as claimed. □

Then the following immediately follows from the ω_1-closedness, and κ-cc.

Claim 3.25. *Forcing with $\mathbb{P}$ doesn't collapse ω_1, and cardinals at least κ. Moreover, if $\mathbf{G} \subseteq \mathbb{P}$ is generic, then*

$$V[\mathbf{G}] \models \text{"}\kappa = \omega_2\text{"}.$$

Lemma 3.26. *Let $T \in V[\mathbf{G}_{S,\overline{U}_*}]$ be a Kurepa tree, $S' \subseteq S$ ($S' \in V$). Then, if $b \in V[\mathbf{G}_{S,\overline{U}_*+\mathrm{id}_{S'}}]$ is a branch of T, then there exists a finite set $S'' \subseteq S'$, and $\overline{U}_\bullet = \langle U^\bullet_\theta : \theta \in S'' \rangle$ s.t. each $U^\bullet_\theta$ is finite, and b is in the model obtained by adding these finitely many $\eta_{\theta,\alpha}$'s ($\theta \in S''$, $\alpha \in U^\bullet_\theta$) to $V[\mathbf{G}_{S,\overline{U}_*}]$, i.e.*

$$b \in V[\mathbf{G}_{S,\overline{U}_*+\overline{U}_\bullet}].$$

Proof.

Let $\dot{T} \in V$ be a $\mathbb{P}_{S,\overline{U}_*}$-name for T. Define

$$\mathbb{P}' = \mathbb{P}_{S,\overline{U}_*+\overline{\mathrm{id}}_{S'}}. \tag{3.6}$$

Suppose that $p_* \in \mathbb{P}'$ forces that $\dot{b} \in V$ is a $\mathbb{P}'$-name for a counterexample (i.e. forcing that for no such $\overline{U}_\bullet$ there exists a $\mathbb{P}_{\overline{U}_*+\overline{U}_\bullet}$-name $\dot{b}'$ - which is of course also a $\mathbb{P}'$-name - with $\dot{b}' = \dot{b}$). Let χ be large enough, and let $\langle N_0, \in \rangle \prec \langle \mathcal{H}(\chi), \in \rangle$ be countable s.t. $p_*, \dot{b}, \dot{T}, S, S', \overline{V}, \mathbb{P}_{S,\overline{U}_*} \in N_0$.

Let $\delta_\bullet = N_0 \cap \omega_1$. Define the countable set N_1 to be such that $N_0 \in N_1$, and $\langle N_1, \in \rangle \prec \langle \mathcal{H}(\chi), \in \rangle$. Let X be set of the indices of the new branches added to $\langle \underset{\sim}{T}_\theta : \theta \in S' \rangle$ by $\mathbf{G}_{S,\overline{U}_*+(\mathrm{id}_{S'})}$ that are in $V[\mathbf{G}_{S,\overline{U}_*+\overline{\mathrm{id}}_{S'}}] \setminus V[\mathbf{G}_{S,\overline{U}_*}]$, and belong to N_0, i.e.

$$X = N_0 \cap \{\langle \theta, \alpha \rangle : (\theta \in S') \wedge (\alpha \in \theta \setminus U^*_\theta)\}. \tag{3.7}$$

We fix an enumeration of X and define also the sequence of the first n indices from this countable set, and as well for each n the one-length sequence consisting only the n'th, that is

$$\text{let } \langle \langle \varrho_n, \xi_n \rangle : n \in \omega, n > 0 \rangle \text{ enumerate } X \text{ (starting from 1)},$$

$$\begin{aligned} \overline{W}_n = \; & \langle W_{n,\theta} : \theta \in S' \cap N_0 \rangle, \\ & \text{where } W_{n,\theta} = \{\alpha : \langle \theta, \alpha \rangle = \langle \varrho_j, \xi_j \rangle \text{ for some } j \leq n\} \\ \overline{w}_n = \; & \langle w_{n,\theta} : \theta \in S' \cap N_0 \rangle \\ & \text{where } w_{n,\theta} = \{\xi_n\} \text{ if } \theta = \varrho_n, \; w_{n,\theta} = \emptyset \text{ otherwise.} \end{aligned} \tag{3.8}$$

Observe that if $p \in \mathbb{P} \cap N_0$, then each $\theta \in \mathrm{dom}(p)$ is an element of N_0 since $\mathrm{dom}(p)$ is countable (by Definition 3.10), and similarly $T_{p(\theta)}, u_{p(\theta)} \subseteq N_0$ (by Definitions 3.6 – 3.9).

Working in V we will construct an N_0-generic condition in $\mathbb{P}'$, which will derive us to a contradiction. It is enough to prove the following claim.

Claim 3.27. *There exists a sequence $\langle \overline{p}_n : n \in \omega \rangle \in V$, $p'_0 \in \mathbb{P}_{S,\overline{U}*}$ and a sequence $\overline{q} = \langle q_n : n \in \omega \rangle$ such that the following holds.*

$\boxplus_1$ *$\overline{p}_0 = \langle p_{0,l} : l \in \omega \rangle$ is such that*
- *(a) $p_{0,0} = p_* \restriction \overline{U}_*$,*
- *(b) $p_{0,l} \in N_0 \cap \mathbb{P}_{S,\overline{U}_*}$ for each $l \in \omega$,*
- *(c) $\langle p_{0,l} : l \in \omega \rangle$ is $\leq_{\mathbb{P}}$-decreasing,*
- *(d) $\overline{p}_0 \in N_1$,*
- *(e) letting $\mathbf{G}_0 = \{p \in \mathbb{P}_{S,\overline{U}*} \cap N_0 : (\exists l)\ p \geq p_{0,l}\}$, the filter $\mathbf{G}_0$ is $\mathbb{P}_{S,\overline{U}_*}$-generic over N_0.*

$\boxplus_2$ *$p'_0 \in \mathbb{P}_{S,\overline{U}*}$ satisfies the following*
- *(a) p'_0 is a lower bound of $p_{0,l}$ for each $l \in \omega$ (hence forces a value to $\underset{\sim}{T}_{\theta,<\delta_\bullet}$ for each $\theta \in S \cap N_0$),*
- *(b) p'_0 forces a value to $\underset{\sim}{T}_{\theta,\leq\delta_\bullet}$ for each $\theta \in S \cap N_0$ such that for every $\delta_\bullet$-branch B in $\underset{\sim}{T}_{\theta,<\delta_\bullet}$ the inclusion $B \in N_1$ implies that B has an upper bound in $\underset{\sim}{T}_{\theta,\leq\delta_\bullet}$,*
- *(c) p'_0 forces a value to $\dot{T}_{\leq\delta_\bullet}$.*

$\boxplus_3$ *for every $n > 0$ the sequence $\overline{p}_n = \langle p_{n,l} : l \in \omega \rangle$ has the following properties.*
- *(a) $\forall l \in \omega\ p_{n,l} \in N_0 \cap \mathbb{P}_{S,\overline{U}_*+\overline{w}_n}$,*
- *(b) $p_{n,l} \restriction \overline{U}_* \in \mathbf{G}_0$*
- *(c) $\langle p_{n,l} : l \in \omega \rangle$ is $\leq_{\mathbb{P}}$-decreasing,*
- *(d) $\overline{p}_n \in N_1$,*
- *(e) letting*

$$\mathbf{G}_n = \{p \in \mathbb{P}_{S,\overline{U}*+\overline{W}_n} \cap N_0 : (\exists l_0, l_1, \ldots, l_n)\ p \geq \bigwedge_{j=0}^{n} p_{j,l_j}\},$$

the filter $\mathbf{G}_n$ is $\mathbb{P}_{S,\overline{U}_*+\overline{W}_n}$-generic over N_0.

$\boxplus_4$ *For the sequence $\overline{q} = \langle q_n : n \in \omega \rangle$*
- *(a) $q_n \in N_0 \cap \mathbb{P}_{S,\overline{U}_*+\overline{\mathrm{id}}_{S'}}$ for each $n \in \omega$,*
- *(b) $q_0 = p_*$,*
- *(c) $\langle q_n : n \in \omega \rangle$ is $\leq_{\mathbb{P}}$-decreasing,*
- *(d) $\forall n$: $q_n \restriction (\overline{U}_* + \overline{W}_n) \in \mathbf{G}_n$,*
- *(e) Let $\langle \dot{B}_n : n \in \omega \rangle$ enumerate the branches of $\dot{T}_{<\delta_\bullet}$ which has an upper bound in $\dot{T}_{\leq\delta_\bullet}$ (forced by p'_0). Then $q_{n+1} \wedge p'_0$ forces that $\dot{b} \neq B_n$, which will be guaranteed by the following requirement:*
  There exist $\delta < \delta_\bullet$, $t \neq t' \in \dot{T}_{\leq\delta} \setminus \dot{T}_{<\delta}$, such that p'_0 forces B_n's δ'th level to be t', and q_{n+1} forces $t \in \dot{b}$, i.e.

$$(3.9)\qquad \begin{array}{l} p'_0 \Vdash\ \dot{B}_n \cap (\dot{T}_{\leq\delta} \setminus \dot{T}_{<\delta}) = \{t'\} \\ \text{and} \\ q_{n+1} \Vdash\ \dot{b} \cap (\dot{T}_{\leq\delta} \setminus \dot{T}_{<\delta}) = \{t\}. \end{array}$$

(Observe that the latter is a statement in N_0.)

Before proving Claim 3.27 we argue why this claim implies Lemma 3.26. First, the claim gives the following condition in $\mathbb{P}_{S,\overline{U}_*+\overline{\mathrm{id}}_{S'}}$. For each $n \in \omega$ let η_{ϱ_n,ξ_n} be the branch in $\underset{\sim}{T}_{\varrho_n,<\delta_\bullet}$ represented by the sequence $\overline{p}_n$, i.e.

$$\eta_{\varrho_n,\xi_n} = \cup\{\eta_{p_{n,l}(\varrho_n),\xi_n} : \ l \in \omega\}, \tag{3.10}$$

and note that $\eta_{\varrho_n,\xi_n} \in N_1$ $(n \in \omega)$ by $\boxplus_3/(d)$. Therefore by $\boxplus_2/(b)$ we can extend each η_{ϱ_n,ξ_n} to a branch η'_{ϱ_n,ξ_n} in $(T_{p'_0(\varrho_n)})_{<\delta_\bullet+1}$. Define the function $p_\bullet$ to be the extension of p'_0 by the η_{ϱ_n,ξ_n}'s in the obvious way: (Note that by $\boxplus_2$ we have $S\cap N_0 \subseteq \mathrm{dom}(p'_0) \subseteq S$, and for each $\theta \in S\cap N_0$ the inclusion $U^*_\theta \cap N_0 \subseteq u_{p'_0(\theta)} \subseteq U^*_\theta$.) Define $p_\bullet$ to be function on $\mathrm{dom}(p'_0)$ such that if $\theta \notin N_0 \cap S'$, then $p_\bullet(\theta) = p'_0(\theta)$, and for $\theta \in N_0 \cap S'$ define $p_\bullet(\theta)$ to be the following proper extension of $p'_0(\theta)$. Let $u_{p_\bullet(\theta)} = u_{p_0(\theta)} \cup (\theta \cap N_0)$, and if $\alpha \notin u_{p'_0(\theta)}$ (when necessarily $\alpha \notin U^*_\theta$) and (by (3.8)) choose $n > 0$ so that

$$\langle \theta, \alpha \rangle = \langle \varrho_n, \xi_n \rangle, \text{ and let } \eta_{p_\bullet(\theta),\alpha} = \eta'_{\varrho_n,\xi_n}, \tag{3.11}$$

otherwise

$$\eta_{p_\bullet(\theta),\alpha} = \eta_{p_0(\theta),\alpha} \qquad (\text{if } \alpha \in U^*_\theta). \tag{3.12}$$

Observe that as η'_{ϱ_n,ξ_n} was a cofinal branch in $(T_{p_\bullet(\varrho_n)})_{<\delta_\bullet+1} = (T_{p'_0(\varrho_n)})_{<\delta_\bullet+1}$ our function $p_\bullet$ is indeed a condition in $\mathbb{P}_{S,\overline{U}_*+\overline{\mathrm{id}}_{S'}}$. Moreover, the following shows that $\forall n \in \omega$ $p_\bullet \leq q_n$. Fix $n \in \omega$, then using $\boxplus_4/(d)$ we have $q_n \restriction (\overline{U}_* + \overline{W}_n) \in \mathbf{G}_n$, i.e. there exist $l_0, l_1, \dots l_n \in \omega$, such that $\bigwedge_{j=0}^{n} p_{j,l_j} \leq_{\mathbb{P}} q_n \restriction (\overline{U}_* + \overline{W}_n)$. This means that

$$\bigwedge_{j=0}^{n} p_{j,l_j} \leq q_n \restriction (\overline{U}_* + \overline{W}_0) = q_n \restriction (\overline{U}_*),$$

and

$$\text{for each } 0 < j \leq n \ \ \eta_{q_n(\varrho_j),\xi_j} \subseteq \eta_{p_{j,l_j}(\varrho_j),\xi_j} \subseteq \eta'_{\varrho_j,\xi_j} = \eta_{p_\bullet(\varrho_j),\xi_j}.$$

On the other hand, for $j > n$ we have (recalling $\overline{q} = \langle q_n : \ n \in \omega\rangle$ is $\leq_{\mathbb{P}}$-decreasing by $\boxplus_4$) that

$$\eta_{q_n(\varrho_j),\xi_j} \subseteq \eta_{q_j(\varrho_j),\xi_j} \subseteq \eta'_{\varrho_j,\xi_j} = \eta_{p_\bullet(\varrho_j),\xi_j},$$

therefore $p_\bullet \leq q_n$, indeed.

Now assuming $p_\bullet \in \mathbf{G}_{S,\overline{U}_*+\overline{id}_{S'}}$ will easily yield a contradiction: First recall that p_* (and therefore as well q_0 and $p_\bullet$) forced that $\dot{b}$ is a branch through $\dot{T}$. Then $\boxplus_2/(c)$ implies that p'_0, thus $p_\bullet$ as well determines $\dot{T}_{\leq\delta_\bullet}$, and $p_\bullet$ forces (by $\boxplus_4/(e)$) that each element of the $\delta_\bullet$'th level of $\dot{T}$ is the upper bound of B_i for some $i \in \omega$. This means that

$$p_\bullet \Vdash \ (\exists i \in \omega) \ \dot{b} \cap \dot{T}_{<\delta_\bullet} = B_i,$$

while at the same time

$$(q_i \wedge p'_0) \Vdash \dot{b} \neq B_i,$$

since (3.9) holds.

This together with $p_\bullet \leq q_i, p'_0$ gives the contradiction. Now we can turn to the proof of the claim.

Proof. (Claim 3.27)

For the construction of each sequence $\overline{p}_n$ and each q_n we will work in N_1. This will need a lot of preparation.

Recall that $X \subseteq N_0$ denoted the indices of branches added by forcing with $\mathbb{P}_{S,\overline{U}_*+\overline{\mathrm{id}}_{S'}} \cap N_0$ but missing from $V[\mathbf{G}_{S,\overline{U}_*}]$ (3.7), and that for each condition p,

$\theta \in S_\bullet$, and $\delta < \omega_1$ the δ'th level of $T_{p(\theta)}$ is (a subset of) $[\omega \cdot \delta, \omega \cdot (\delta+1))$. Define $E \subseteq N_0$ as follows.

$$\begin{array}{rl} e \in E \ \text{ iff } & e \in N_0, \text{ and } e = (u_e, \overline{\eta}_e), \text{ where } u_e \in [X]^{\leq \omega}, \\ & \overline{\eta}_e = \langle \eta_{e,\theta,\alpha} : \ \langle \theta, \alpha \rangle \in u_e \rangle, \ \text{ such that} \\ & \eta_{e,\theta,\alpha} \subseteq \omega \cdot (\delta_{\theta,\alpha}+1) \text{ for some } \delta_{\theta,\alpha} < \omega_1 \end{array} \tag{3.13}$$

Definition 3.28. For each n, $p \in \mathbb{P}_{S,\overline{U}_*+\overline{W}_n}$, and $e \in E$, <u>if</u> for each $\langle \theta, \alpha \rangle \in u_e$ we have $\theta \in \operatorname{dom}(p)$, and for each $i < n$ $\langle \varrho_i, \xi_i \rangle \notin u_e$ holds then define $p \frown e$ as

$$\begin{array}{l} \operatorname{dom}(p \frown e) = \operatorname{dom}(p), \\ u_{(p\frown e)(\theta)} = u_{p(\theta)} \cup \{\alpha : \ \langle \theta, \alpha \rangle \in u_e\}) \ \ (\forall \theta \in \operatorname{dom}(p \frown e)), \\ \eta_{(p\frown e)(\theta),\alpha} = \begin{cases} \eta_{p(\theta),\alpha}, & \text{if } \alpha \in u_{p(\theta)}, \\ \eta_{e,\theta,\alpha}, & \text{if } \langle \theta, \alpha \rangle \in u_e, \end{cases} \\ \text{if this is a condition in } \mathbb{P} \text{ (i.e. for each } \langle \theta, \alpha \rangle \in u_e \\ \eta_{e,\theta,\alpha} \text{ is a cofinal branch of } (T_{p(\theta)})_{<\delta+1} \text{ for some } \delta \leq \operatorname{ht}(T_{p(\theta)}), \\ \\ \quad \text{otherwise } p \frown e = \emptyset. \end{array} \tag{3.14}$$

Let $\mathcal{D}$ denote the set of dense subsets of $\mathbb{P}_{S,\overline{U}_*+\overline{\mathrm{id}}_{S'}}$. Fix an enumeration

$$\langle \langle J_i, \varepsilon_i \rangle : \ i \in \omega \rangle \in N_1 \quad \text{of} \ \ (\mathcal{D} \cap N_0) \times E,$$

and let $k(D,e)$ denote the index of the pair $\langle D, e \rangle$ (i.e.

$$J_{k(D,e)} = D, \ \varepsilon_{k(D,e)} = e), \text{ then we also have } k \in N_1, \text{ of course.} \tag{3.15}$$

Fix a function $g \in N_0$

$$\begin{array}{rl} g : & \mathbb{P}_{S,\overline{U}_*+\overline{\mathrm{id}}_{S'}} \times \mathcal{D} \ \to \ \mathbb{P}_{S,\overline{U}_*+\overline{\mathrm{id}}_{S'}} \\ \text{with} & \forall p, D : \\ & \bullet_1 g(p,D) \in D, \\ & \bullet_2 \ g(p,D) \leq p, \end{array} \tag{3.16}$$

$$\text{(Then } g \in N_0 \text{ obviously implies } (p, D \in N_0 \ \Rightarrow \ g(p,D) \in N_0).)$$

We will have to define also the auxiliary sequence $\overline{r} = \langle r_l : \ l \in \omega \rangle$ with the following property:

- $\circledast_1$ $\overline{r} \in N_1$,
- $\circledast_2$ for each l $r_l \in \mathbb{P}_{S,\overline{U}_*} \cap N_0$,
- $\circledast_3$ for each l $p_{0,l+1} \leq r_l \leq p_{0,l}$,
- $\circledast_4$ if there exists $p \in \mathbb{P}_{S,\overline{U}^*}$ such that $p \leq p_{0,l}$, and $p \frown \varepsilon_l$ is a condition extending $p_{0,l}$ in $\mathbb{P}_{S,\overline{U}^*+\overline{\mathrm{id}}_{S'}}$, then r_l is such that.

Now we can construct the $p_{0,i}$'s (and r_i's). Let $p_{0,0} = p_* \restriction \overline{U}_*$. For obtaining the $p_{0,l}$'s proceed as follows. Assume we have defined $p_{0,0}, p_{0,1}, \ldots, p_{0,l-1}$ (and as well the r_i's for $i < l-1$). Now if there exists $p \in \mathbb{P}_{S,\overline{U}^*}$ $p \leq p_{0,l-1}$, s.t. $p \frown \varepsilon_{l-1} \neq \emptyset$ but a condition extending $p_{0,l-1}$, then let $r_{l-1} \in N_0$ be such a p (recall that $\varepsilon_{l-1} \in E \subseteq N_0$ by (3.13)), otherwise define $r_{l-1} = p_{0,l} = p_{0,l-1}$. Lastly, in the former case define $p_{0,l} = g(r_{l-1}, D_{l-1}) \restriction \overline{U}_*$. It is clear from the construction and the definition of g that $p_{0,l-1} \leq r_{l-1} \leq p_{0,l}$, and $r_{l-1}, p_{0,l} \in N_0$, and since every object as well as the series $\langle \varepsilon_i : \ i \in \omega \rangle$ are elements of N_1, we obtain $\overline{p}_0, \overline{r}_0 \in N_1$, too.

Finally, it is straightforward to check that the filter $\mathbf{G}_0$ generated by the $p_{0,l}$'s meets every dense subset $D \in N_0$ of $\mathbb{P}_{S,\overline{U}_*}$. Fixing such a D

$$D' = \{p \in \mathbb{P}_{S,\overline{U}_*+\overline{\mathrm{id}}_{S'}} : \ p \restriction \overline{U}_* \in D\}$$

is clearly a dense subset of $\mathbb{P}_{S,\overline{U}_*+\overline{\mathrm{id}}_{S'}}$ belonging to N_0. This means that if $e \in E$ is the empty sequence, then there exists $i \in \omega$, such that $J_i = D'$, and $\varepsilon_i = e$, therefore $p_{0,i+1} \in D$.

For p_0', first consider the condition $p_0'' \in N_1$ consisting of only the generic trees given by $\mathbf{G}_0$ (for each $\theta \in \mathrm{dom}(p_0'') = N_0 \cap S$ the tree $T_{p_1'(\theta)} = \cup\{T_{p(\theta)} : p \in \mathbf{G}_0\}$ is of height $\delta_\bullet$, but $u_{p_0''(\theta)=\emptyset}$). Then let $p_0''' \in \mathbb{P}_{S,\overline{U}_*}$, $p_0''' \leq p_0''$ be an extension so that for each $\theta \in S' \cap N_0$ the tree $T_{p_2'}(\theta)$ satisfies that for each branch B through $(T_{p_0'''}(\theta))_{<\delta_\bullet} = T_{p_0''(\theta)}$, if $B \in N_1$, then there is an upper bound of B in $T_{p_0'''(\theta)}$. This can be done since N_1 is countable. Moreover, we choose the other part of p_0''' so that for each $\theta, \alpha \in N_0$, if $\alpha \in U^*_\theta$ the chain $\eta_{p_0'''(\theta),\alpha}$ (with a top element) contains the chain $\cup\{\eta_{p(\theta),\alpha} : \ p \in \mathbf{G}_0\}$ which is given by $\mathbf{G}_0$ at this coordinate. This can be done as $\cup\{\eta_{p(\theta),\alpha} : \ p \in \mathbf{G}_0\} \in N_1$, since $\mathbf{G}_0, \overline{p}_0 \in N_1$. Then clearly $p_0''' \leq p_{0,l}$ for each $l \in \omega$.

Finally, for the last item of $\boxplus_2$ first recall that $\mathbb{P}^*_{S,\overline{U}_*}$ is an ω_1-closed dense subposet of $\mathbb{P}_{S,\overline{U}_*}$ by Remark 3.11. Then if a countable increasing sequence in $\mathbb{P}^*_{S,\overline{U}_*}$ (where a first element stronger than p_0''') decides more and more about the $\delta_\bullet$'th level of $\dot{T}$, then choosing p_0' to be an upper bound will work (e.g. choose an enumeration $\langle \dot{t}_i : \ i \in \omega\}$ of the $\delta_\bullet$'th level of $\dot{T}$, let $\langle s_i : \ i \in \omega\rangle$ enumerate $\dot{T}_{<\delta_\bullet}$ in type ω, and let r_j decide whether the j'th ordered pair in the countable set $\{s_i : \ i \in \omega\} \times \{\dot{t}_i : i \in \omega\}$ is in $\leq_{\dot{T}}$).

The next step is to construct the $\overline{p}_i$'s ($i > 0$) and the q_n's. This will be done simultaneously by induction. The induction is carried out in V, but each step can be done in N_1, which will guarantee that each $\overline{p}_n \in N_1$.

It is straightforward to check that choosing $q_0 = p_*$ would satisfy our requirements, as e.g. $p_{0,0} = p_* \restriction \overline{U}_*$. Then fixing $n > 0$, and assuming that $\overline{p}_i, q_i$ are constructed for each $i < n$, first we construct q_n. Recall that $q_{n-1} \restriction (\overline{U}_* + \overline{W}_{n-1}) \in \mathbf{G}_{n-1}$ (by $\boxplus_4/(d)$).

Recall the definition of the set E (3.13), and let

$$E_{n-1} = \{e \in E : \ \forall i < n \ \langle \varrho_i, \xi_i\rangle \notin e\}.$$

Using that $p_* \in \mathbb{P}_{S,\overline{U}_*+\overline{\mathrm{id}}_{S'}}$ forced that $\dot{b}$ is not an element of $V[G_{S,\overline{U}_*+\overline{W}_{n-1}}]$, i.e. there is no $\mathbb{P}_{S,\overline{U}_*+\overline{W}_{n-1}}$-name of it, we argue that

$$\begin{aligned} D = \{ \quad & p \in \mathbb{P}_{S,\overline{U}_*+\overline{W}_{n-1}} : \ \exists e, e' \in E_{n-1} \ (p ^\frown e \leq q_{n-1}, \ p ^\frown e' \leq q_{n-1}) \wedge \\ & (\exists \delta < \omega_1, \ t \neq t' \in \dot{T}_{\leq\delta} \setminus \dot{T}_{<\delta} : \quad (p ^\frown e \Vdash t \in \dot{b}) \wedge (p ^\frown e' \Vdash t' \in \dot{b}))\} \end{aligned}$$

is dense in $\mathbb{P}_{S,\overline{U}_*+\overline{W}_{n-1}}$ under $q_{n-1} \restriction (\overline{U}_* + \overline{W}_{n-1})$. Indeed, assume on the contrary that $q' \in \mathbb{P}_{S,\overline{U}_*+\overline{W}_{n-1}}$, $q' \leq q_{n-1} \restriction (\overline{U}_* + \overline{W}_{n-1})$ is such that that D has no element under q'. Now for every $\delta < \omega_1$, consider the set

$$\begin{aligned} D_\delta = \{ \quad & p \in \mathbb{P}_{S,\overline{U}_*+\overline{W}_{n-1}} : \quad (p \leq q') \ \wedge \ (\exists e \in E_{n-1} : \ [p ^\frown e \leq q_{n-1}] \wedge \\ & \wedge \ [\exists t_{p,e,\delta} \in \dot{T}_{\leq\delta} \setminus \dot{T}_{<\delta} : \ p ^\frown e \ \Vdash t_{p,e,\delta} \in \dot{b}])\} \end{aligned},$$

which is dense under q' in $\mathbb{P}_{S,\overline{U}_*+\mathrm{id}S'}$. Now since for each $\delta < \omega_1$ the sets D and D_δ are disjoint, for $p \in D_\delta$ the witnessing $t_{p,e,\delta}$ doesn't depend on e, therefore $q' \wedge q_{n-1}$ forces that $\dot{b}$ is in $V[\mathbf{G}_{S,\overline{U}_*+\overline{W}_{n-1}}]$ (i.e. forces that the $\mathbb{P}_{S,\overline{U}_*+\overline{W}_{n-1}}$-name $\{\langle p, t_{p,\delta}\rangle : \ p \in D_\delta, \ \delta < \omega_1\}$ and $\dot{b}$ are equal).
Then as our set $D \in N_0$ is indeed dense we have that there exists a condition $q'' \in \mathbf{G}_{n-1} \cap D$, witnessed by $t \neq t'$ and e, e'. Finally, if $t \in B_n$ then define $q_n = q'' \frown e'$, otherwise we can let $q_n = q'' \frown e$, which are both stronger conditions than q_{n-1} by the definition of D. It is straightforward to check $\boxplus_4$.

As q_n is already defined (and so are $\overline{p}_i, q_i$ for each $i < n$), we turn to the definition of $\overline{p}_n$, which we will do similarly to that of $\overline{p}_0$. Let $p_{n,0} = q_n \restriction (\overline{U}_* + \overline{w}_n)$, assume that $p_{n,0}, p_{n,1}, \dots, p_{n,l-1}$ are already chosen.
If $\varepsilon_{l-1} \notin E_{n-1}$, then $p_{n,l} = p_{n,l-1}$, otherwise proceed as follows. Choose the sequence $\overline{e} = \overline{e}(n, l-1) = \langle e_i : \ 1 \leq i \leq n+1\rangle \in E^{n+1\setminus\{0\}}$ and the sequence $\overline{m} = \overline{m}(n, l-1) = \langle m_i : \ i \leq n\rangle \in \omega^{n+1}$ with the property

1) $e_{n+1} = \varepsilon_{l-1}$ and $m_n = l-1$,
2) for each $i < n+1$

$$J_{m_i} = D \quad \wedge \quad \text{"}e_i = \ (e_{i+1} \ \text{ plus } \ (\eta_{p_{i,m_i}(\varrho_i),\xi_i} \text{ attained on } \langle \varrho_i, \xi_i\rangle))\text{"}. \tag{3.17}$$

Provided that the e_j's are defined for $j > i$, and as well each m_j for $j \geq i$, let $e_i \in E$ be the element with $u_{e_i} = u_{e_{i+1}} \cup \{\langle \varrho_i, \xi_i\rangle\}$, $\overline{\eta}_{e_i} \supseteq \overline{\eta}_{e_{i+1}}$, $\eta_{e,\varrho_i,\xi_i} = \eta_{p_{i,m_i}(\varrho_i),\xi_i}$, and let $m_{i-1} = k(D, e_i)$. Observe that by our procedure, and by the definition of the function k (3.15) we have $e_1 = \varepsilon_{m_0}$, and also

$$\eta_{e_1,\varrho_n,\xi_n} = \eta_{p_{n,l-1}(\varrho_n),\xi_n}. \tag{3.18}$$

At some point later we will use the following fact, hence it is worth to note that for each i, $1 \leq i \leq n$

$$\overline{e}(i, m_i) \subseteq \overline{e}(n, l-1), \text{ and } \overline{m}(i, m_i) \subseteq \overline{m}(n, l-1). \tag{3.19}$$

Finally consider the condition r_{m_0} (from $\circledast_1 - \circledast_4$): if $r_{m_0} \frown e_1$ is a not a condition in $\mathbb{P}_{S,\overline{U}_*+\mathrm{id}\restriction S'}$, then let $p_{n,l} = p_{n,l-1}$, otherwise first define the auxiliary condition

$$r_\bullet = g(r_{m_0} \frown e_1, D), \tag{3.20}$$

and note that in this case $\eta_{(r_{m_0} \frown e_1)(\varrho_n),\xi_n} = \eta_{p_{n,l-1}(\varrho_n),\xi_n}$ by (3.18), therefore by the properties of g we obtain

$$\eta_{r_\bullet(\varrho_n),\xi_n} \supseteq \eta_{p_{n,l-1}(\varrho_n),\xi_n}. \tag{3.21}$$

Recall that $p_{n,l-1} \restriction \overline{U}_* \in \mathbf{G}_0$ by our induction hypotheses $\boxplus_3$, and it can be seen from the construction of $p_{0,j}$'s that in this case $p_{0,m_0+1} = r_\bullet \restriction \overline{U}_* \in \mathbf{G}_0$. Therefore by (3.21) we have that $(r_\bullet \restriction \overline{U}_* + \overline{w}_n) \wedge p_{n,l-1}$ is a condition in $\mathbb{P}_{\overline{U}_*+\overline{w}_n}$, and let

$$p_{n,l} = (r_\bullet \restriction \overline{U}_* + \overline{w}_n) \wedge p_{n,l-1}.$$

Then clearly $p_{n,l} \leq p_{n,l-1}$, and $p_{n,l} \restriction \overline{U}_* \in \mathbf{G}_0$. From $\boxplus_3$ it only remained to check that (d) and (e) also hold. Since the whole construction of $\overline{p}_n$ took place in N_1 ($k \in N_1$ and so is the enumeration $\langle\langle J_i, \varepsilon_i\rangle : \ i \in \omega\rangle$, $g \in N_0$), $\overline{p}_n \in N_1$ obviously follows. Verifying the genericity of $\mathbf{G}_n$ goes similarly as of $\mathbf{G}_0$. Let $D \subseteq \mathbb{P}_{S,\overline{U}_*+\overline{W}_n}$, $D \in N_0$ be a fixed dense set, and $e' \in E$ be the empty sequence. Now, if we choose l so that $J_{l-1} = D' = \{p \in \mathbb{P}_{S,\overline{U}_*+\mathrm{id}_{S'}} : \ p \restriction \overline{U}_* + \overline{W}_n \in D\}$, $\varepsilon_{l-1} = e'$, then it follows from the construction of $p_{k,j}$'s, that of $\overline{m} = \overline{m}(n, l-1)$ and $\overline{e} = \overline{e}(n, l-1)$, and from (3.19) that

$$p_{i,m_i+1} = (r_\bullet \restriction \overline{U}_* + \overline{w}_i) \wedge p_{i,m_i} \quad \text{if } 1 \leq i \leq n,$$

and

$$p_{0,m_0+1} = g(r_{m_0} {}^\frown e_1) \restriction \overline{U}_*,$$

therefore

$$\bigwedge_{i \leq n} p_{i,m_i} \leq g(r_{m_0} {}^\frown e_1) \restriction (\overline{U}_* + \overline{W_n}) \in D'.$$

$\square$(Claim 3.27)

$\square$(Lemma 3.26)

Lemma 3.29. *Let $T \in V[\mathbf{G}_{S,\overline{U}_*}]$ be a Kurepa tree, $S' \subseteq S \cap S_\bullet$ ($S' \in V$), $\mathbf{G}^\circ_{\overline{\mathrm{id}}_{S'} - \overline{U}_*} \subseteq \mathbb{P}^\circ_{\overline{\mathrm{id}}_{S'} - \overline{U}_*}$ be generic over $V[\mathbf{G}_{S,\overline{U}_*}]$. Suppose that $b \in V[\mathbf{G}_{S,\overline{U}_*}][\mathbf{G}^\circ_{S',(\overline{\mathrm{id}}_{S'} - \overline{U}_*)}] \setminus V[\mathbf{G}_{S,\overline{U}_*}]$ is a new branch of T, and suppose that $\gamma \geq \kappa$ is a cardinal, and for each $\theta \in S'$ the inequality $|\theta \setminus U^*_\theta| \geq \gamma$ holds. Then the filter $\mathbf{G}^\circ_{\overline{\mathrm{id}}_{S'} - \overline{U}_*}$ adds at least $|\gamma|$-many new branches to T.*

Proof. W.l.o.g. we can assume that $T \subseteq \omega_1$, and λ is a cardinal (in $V[\mathbf{G}_{S,\overline{U}_*}]$). First we will choose a system $\overline{W}_0 = \langle W_{0,\theta} : \theta \in S' \rangle \in \prod_{\theta \in S'} \mathcal{P}(\theta)$ with $(\forall \theta \in S')$ $|W_{0,\theta}| < \kappa$, and $b \in V[\mathbf{G}_{S,\overline{U}_*}][\mathbf{G}^\circ_{\overline{W}_0}]$: since $b \in V[\mathbf{G}_{S,\overline{U}_*}][\mathbf{G}^\circ_{\overline{\mathrm{id}}_{S'} - \overline{U}_*}]$, $S' \in V$ we can use Lemma 3.20 and obtain that $b \in V[\mathbf{G}_{S,\overline{U}_*}][\mathbf{G}^\circ_{\overline{\mathrm{id}}_{S'} - \overline{U}_*}] = V[\mathbf{G}_{S,\overline{U}_* + \overline{\mathrm{id}}_{S'}}]$. And because $b \subseteq \mathcal{H}(\omega_1)^V$, applying Lemma 3.24 with S, and $\overline{U} = \overline{U}_* + \overline{\mathrm{id}}_{S'}$, there exists $S_* \subseteq S$, $\overline{W}_* \in \prod_{S_* \setminus S'} \mathcal{P}(U_\theta) \times \prod_{\theta \in S_* \cap S'} \mathcal{P}(\theta)$ with

$$b \in V[\mathbf{G}_{S_*,\overline{W}_*}] \subseteq V[\mathbf{G}_{S,\overline{U}_* + \overline{W}_*}] = V[\mathbf{G}_{S,\overline{U}_*}][\mathbf{G}^\circ_{\overline{W}_* - \overline{U}_*}],$$

where $|S_*| < \kappa$, and $|W^*_\theta| < \kappa$ for each $\theta \in S_*$. Then fixing $\overline{W}_0 \in \prod_{\theta \in S'} \mathcal{P}(\theta)$ so that $W_{0,\theta} = W^*_\theta \setminus U^*_\theta$ if $\theta \in S_*$, and $W_{0,\theta} = \emptyset$ for $\theta \in S \setminus S_*$ has the required properties.

Now, as $|W_{0,\theta}| < \kappa \leq \gamma$, and $\gamma \leq |\theta \setminus U^*_\theta|$ for each $\theta \in S'$ we can fix for each $\alpha < \gamma$ a system $\overline{W}_\alpha = \langle W_{\alpha,\theta} : \theta \in S' \rangle \in \prod_{\theta \in S'} \mathcal{P}(\theta \setminus U^*_\theta)$ such that for every $\theta \in S'$

(i) $W_{\alpha,\theta} \cap W_{\beta,\theta} = \emptyset$ for every $\alpha < \beta < \gamma$,
(ii) $|W_{0,\theta}| = |W_{\alpha,\theta}|$ for each $\alpha < \gamma$.

For each $0 < \alpha < \gamma$ define the bijections

$$\pi_\alpha : \bigcup_{\theta \in S'} \{\theta\} \times W_{0,\theta} \to \bigcup_{\theta \in S'} \{\theta\} \times W_{\alpha,\theta}$$

where $\pi_\alpha \restriction \{\theta\} \times W_{0,\theta}$ is a bijection to $\{\theta\} \times W_{\alpha,\theta}$. Then clearly each π_α induces an automorphism $\hat{\pi}_\alpha \in V[\mathbf{G}_{S,\overline{U}_*}]$ of $\mathbb{P}^\circ_{\overline{W}_0}$ and $\mathbb{P}^\circ_{\overline{W}_\alpha}$. Moreover, $\hat{\pi}_\alpha$ induces a natural operation $\hat{\pi}^*_\alpha$ from the class of $\mathbb{P}^\circ_{\overline{W}_0}$-names to the class of $\mathbb{P}^\circ_{\overline{W}_\alpha}$-names. Now fix a $\mathbb{P}^\circ_{\overline{W}_0}$-name $\dot{b}_0 \in V[\mathbf{G}_{S,\overline{U}_*}]$ for our new branch $b \in V[\mathbf{G}_{S,\overline{U}_*}][\mathbf{G}^\circ_{\overline{W}_0}]$, and choose an element $p_\bullet \in \mathbb{P}^\circ_{\overline{W}_0}$ forcing that $\dot{b}_0$ is a new branch, i.e.

$$V[\mathbf{G}_{S,\overline{U}_*}] \models \quad p_\bullet \Vdash \ \dot{b}_0 \in \mathcal{B}(T) \setminus \mathcal{B}^{V[\mathbf{G}_{S,\overline{U}_*}]}(T). \tag{3.22}$$

Let $\mathbb{P}^\circ_\bullet = \mathbb{P}^\circ_{\sum_{\alpha<\gamma} \overline{W}_\alpha}$, i.e. adding the branches $\bigcup_{\alpha\in\gamma} W_{\alpha,\theta}$ to $\underset{\sim}{T}_\theta$ for each $\theta \in S'$, which is of course equal to the countably supported product of $\mathbb{P}^\circ_{\overline{W}_\alpha}$'s ($\alpha < \gamma$), and let $\mathbf{G}^\circ_\bullet$ denote the generic filter $\mathbf{G}^\circ_{\mathrm{id}_{S'}-\overline{U}_*} \cap \mathbb{P}^\circ_\bullet$.

We will show that in $V[\mathbf{G}_{S,\overline{U}_*}][\mathbf{G}^\circ_\bullet] \subseteq V[\mathbf{G}_{S,\overline{U}_*}][\mathbf{G}^\circ_{\mathrm{id}_{S'}-\overline{U}_*}]$ there are at least γ-many new branches of T, i.e.

$$\left|\mathcal{B}(T) \cap \left(V[\mathbf{G}_{S,\overline{U}_*}][\mathbf{G}^\circ_\bullet] \setminus V[\mathbf{G}_{S,\overline{U}_*}]\right)\right| \geq \lambda,$$

by arguing that

$\otimes_1$ for any $\alpha < \gamma$ (in $V[\mathbf{G}_{S,\overline{U}_*}]$)

$$\hat{\pi}_\alpha(p_\bullet) \Vdash_{\mathbb{P}^\circ_\bullet} \hat{\pi}^*_\alpha(\dot{b}_0) \notin V[\mathbf{G}_{S,\overline{U}_*}][\mathbf{G}^\circ_{\bullet,<\alpha}]$$

(where $\mathbf{G}^\circ_{\bullet,<\alpha}$ stands for $\mathbf{G}^\circ_\bullet \cap \mathbb{P}^\circ_{\sum_{\beta<\alpha} \overline{W}_\beta}$), and

$\otimes_2$ $|\{\alpha < \gamma : \ \hat{\pi}_\alpha(p_\bullet) \in \mathbf{G}^\circ_\bullet\}| = \gamma$.

This will complete the proof of Lemma 3.29.

First we will prove $\otimes_2$, for which recall that we assumed that γ is a cardinal, and choose a system of uncountable regular cardinals $\{\rho_\beta : \ \beta < \chi < \gamma\}$, and a partition $\langle I_\beta : \ \beta < \chi \rangle$ of γ with $\mathrm{otp}(I_\beta) = \rho_\beta$ for each $\beta < \chi$ (i.e. $I_\beta \cap I_\delta = \emptyset$ for $\beta < \delta < \rho$, and $\bigcup_{\beta<\rho} I_\beta = \gamma$). Then it is enough to verify

$$(\forall \beta < \chi) \quad |\{\alpha \in I_\beta : \ \hat{\pi}_\alpha(p_\bullet) \in \mathbf{G}^\circ_\bullet\}| = \rho_\beta, \tag{3.23}$$

which can be seen by a standard density argument: Fix $\beta < \varrho$, $\alpha \in I_\beta$, then it suffices to show that

$$D_{\beta,\alpha} = \{p \in \mathbb{P}^\circ_\bullet : \ p \leq \hat{\pi}_\delta(p_\bullet) \ \text{ for some } \ \delta > \alpha, \ \delta \in I_\beta\} \text{ is dense,}$$

which obviously holds by the regularity of the uncountable $\rho_\beta = |I_\beta|$ (since for $\delta \in I_\beta$ we have $\hat{\pi}_\delta(p_\bullet) \in \mathbb{P}^\circ_{\overline{W}_\delta}$, $\mathbb{P}^\circ_\bullet$ is the countably supported product of $\mathbb{P}^\circ_{\overline{W}_\alpha}$'s ($\alpha < \gamma$), and $I_\beta \subseteq \gamma$).

For $\otimes_1$ first consider $\mathbb{P}^\circ_\bullet$ as the product of $\mathbb{P}^\circ_{\sum_{\beta<\gamma,\beta\neq\alpha} \overline{W}_\beta}$ and $\mathbb{P}^\circ_{\overline{W}_\alpha}$. We will need the following claim.

Claim 3.30. *For each $p \in \mathbb{P}^\circ_{\overline{W}_\alpha}$, $p \leq \hat{\pi}_\alpha(p_\bullet)$ there exist $q_0, q_1 \in \mathbb{P}^\circ_{\overline{W}_\alpha}$ $q_0, q_1 \leq p$, and the incomparable elements t_0, t_1 of the tree T such that*

$$V[\mathbf{G}_{S,\overline{U}_*}][\mathbf{G}^\circ_{\bullet,\gamma\setminus\{\alpha\}}] \models \quad (q_i \Vdash_{\mathbb{P}^\circ_{\overline{W}_\alpha}} \ t_i \in \hat{\pi}^*_\alpha(\dot{b}_0)) \quad \textit{for each } i \in \{0,1\},$$

where $\mathbf{G}^\circ_{\bullet,\gamma\setminus\{\alpha\}} = \mathbf{G}^\circ_\bullet \cap \mathbb{P}^\circ_{\sum_{\beta<\gamma,\beta\neq\alpha} \overline{W}_\beta}$.

Before proving the claim we verify that $\otimes_1$ follows from it. In fact

$$\hat{\pi}_\alpha(p_\bullet) \Vdash_{\mathbb{P}^\circ_\bullet} \ \hat{\pi}^*_\alpha(\dot{b}_0) \notin V[\mathbf{G}_{S,\overline{U}_*}][\mathbf{G}^\circ_{\bullet,\gamma\setminus\{\alpha\}}].$$

Since $\mathbf{G}^\circ_\bullet \subseteq \mathbb{P}^\circ_\bullet$ is generic over $V[\mathbf{G}_{S,\overline{U}_*}]$, and $\mathbb{P}^\circ_\bullet$ can be identified with

$$\left(\mathbb{P}^\circ_{\sum_{\beta<\gamma,\beta\neq\alpha} \overline{W}_\beta}\right) \times \mathbb{P}^\circ_{\overline{W}_\alpha},$$

by [Kun13, Lemma V.1.1] $\mathbf{G}^\circ_{\bullet,\gamma\setminus\{\alpha\}} = \mathbf{G}^\circ_\bullet \cap \mathbb{P}^\circ_{\sum_{\beta<\gamma,\beta\neq\alpha} \overline{W}_\beta}$ is generic over $V[\mathbf{G}_{S,\overline{U}_*}]$, and $\mathbf{G}^\circ_{\bullet,\alpha} = \mathbf{G}^\circ_\bullet \cap \mathbb{P}^\circ_{\overline{W}_\alpha}$ is generic over $V[\mathbf{G}_{S,\overline{U}_*}][\mathbf{G}^\circ_{\bullet,\gamma\setminus\{\alpha\}}]$. For each branch $c \in$

$V[\mathbf{G}_{S,\overline{U}_*}][\mathbf{G}^\circ_{\bullet,\gamma\setminus\{\alpha\}}]$ of T define (in $V[\mathbf{G}_{S,\overline{U}_*}][\mathbf{G}^\circ_{\bullet,\gamma\setminus\{\alpha\}}]$)

$$D_c = \{q \in \mathbb{P}^\circ_{\overline{W}_\alpha} :\ \exists t \in T \setminus c \text{ such that } q \Vdash_{\mathbb{P}^\circ_{\overline{W}_\alpha}} t \in \hat{\pi}^*_\alpha(\dot{b}_0)\},$$

which is dense under $\hat{\pi}_\alpha(p_\bullet)$ by Claim 3.30, since for a fixed $p \in \mathbb{P}^\circ_{\overline{W}_\alpha}$ at most one t_i can be in the branch c.

Proof. (Claim 3.30) First we argue that the statement holds in $V[\mathbf{G}_{S,\overline{U}_*}]$, i.e. for each $p \in \mathbb{P}^\circ_{\overline{W}_\alpha}$, $p \leq \hat{\pi}_\alpha(p_\bullet)$ there exist $q_0, q_1 \in \mathbb{P}^\circ_{\overline{W}_\alpha}$ $q_0, q_1 \leq p$, and the incomparable elements t_0, t_1 of the tree T such that

$$V[\mathbf{G}_{S,\overline{U}_*}] \models \quad (q_i \Vdash_{\mathbb{P}^\circ_{\overline{W}_\alpha}} t_i \in \hat{\pi}^*_\alpha(\dot{b}_0)) \quad \text{for each } i \in \{0,1\}. \tag{3.24}$$

Now (3.22) implies that

$$V[\mathbf{G}_{S,\overline{U}_*}] \models \quad \hat{\pi}_\alpha(p_\bullet) \Vdash_{\mathbb{P}^\circ_{\overline{W}_\alpha}} \hat{\pi}^*_\alpha(\dot{b}_0) \in \left(\mathcal{B}(T) \setminus \mathcal{B}^{V[\mathbf{G}_{S,\overline{U}_*}]}(T)\right)$$

since $\dot{b}_0 \in V[\mathbf{G}_{S,\overline{U}_*}]$ is a $\mathbb{P}^\circ_{\overline{W}_0}$-name and $T \in V[\mathbf{G}_{S,\overline{U}_*}]$. Suppose that $p \leq \hat{\pi}_\alpha(p_\bullet)$ is a counterexample, but then for the set

$$b' = \{t \in T :\ \exists q \in \mathbb{P}^\circ_{\overline{W}_\alpha},\ q \leq p \text{ s.t. } q \Vdash t \in \hat{\pi}^*_\alpha(\dot{b}_0)\} \in V[\mathbf{G}_{S,\overline{U}_*}]$$

we have $p \Vdash \hat{\pi}^*_\alpha(\dot{b}_0) = b'$ (since $\hat{\pi}_\alpha(p_\bullet)$ forced that $\hat{\pi}^*_\alpha(\dot{b}_0)$ is a cofinal branch in T), a contradiction. Finally, fixing $p \leq \hat{\pi}_\alpha(p_\bullet)$, if $q_0, q_1 \in \mathbb{P}^\circ_{\overline{W}_\alpha}$ $q_0, q_1 \leq p$, and the incomparable elements $t_0, t_1 \in T$ are such that (3.24) holds, then

$$V[\mathbf{G}_{S,\overline{U}_*}][\mathbf{G}^\circ_{\bullet,\gamma\setminus\{\alpha\}}] \models \quad (q_i \Vdash_{\mathbb{P}^\circ_{\overline{W}_\alpha}} t_i \in \hat{\pi}^*_\alpha(\dot{b}_0)) \quad \text{for each } i \in \{0,1\},$$

since if $q_i \in \mathbf{H} \subseteq \mathbb{P}^\circ_{\overline{W}_\alpha}$ is generic over $V[\mathbf{G}_{S,\overline{U}_*}][\mathbf{G}^\circ_{\bullet,\gamma\setminus\{\alpha\}}]$, and $t_i \notin \hat{\pi}^*_\alpha(\dot{b}_0)[\mathbf{H}]$ (for some $i \in \{0,1\}$), then $\mathbf{H}$ is generic over $V[\mathbf{G}_{S,\overline{U}_*}]$ too, and the same holds in $V[\mathbf{G}_{S,\overline{U}_*}][\mathbf{H}]$. $\square$

It is left to argue why Lemma 3.26 and Lemma 3.29 complete the proof of Theorem 3.1 (and Theorem 3.4). Suppose that $T \in V[\mathbf{G}]$ is a Kurepa tree (where $\mathbf{G} \subseteq \mathbb{P} = \mathbb{P}_{S^+_\bullet, \mathrm{id}_{S^+_\bullet}}$ is generic), and assume on the contrary that $|\mathcal{B}^{V[\mathbf{G}]}(T)| \notin S_\bullet$. We can also assume that $T \subseteq \mathcal{H}(\omega_1)^V$, and by Lemma 3.24 there exists $S_* \subseteq S^+_\bullet$, $|S_*| < \kappa$, $\overline{W}_* = \langle W^*_\theta :\ \theta \in S_*\rangle \in \prod_{\theta \in S_*} [\theta]^{<\kappa}$ such that $T \in V[\mathbf{G}_{S_*,\overline{W}_*}]$. For estimating $(2^{\omega_1})^{V[\mathbf{G}_{S_*,\overline{W}_*}]}$ first a straightforward calculation yields that $|\mathbb{P}_{S_*,\overline{W}_*}| < \kappa$: Since $|\mathbb{P}_{S_*,\langle\emptyset:\ \theta \in S_*\rangle}| = (|S_*||\omega_1|)^\omega$ which is either $(\omega_1 \cdot \omega_1)^\omega = \omega_1 < \omega_2$ (if $\kappa = \omega_2$, by **CH**), or $\gamma^\omega < \kappa$ (for some $\gamma < \kappa$, if κ is inaccessible). Thus recalling the definition of $\mathbb{Q}_{\theta,W^*_\theta}$'s, the fact $\sum_{\theta \in S_*} |W^*_\theta| < \kappa$ as κ is regular, and $\sup W^*_\kappa < \kappa$ (if $\kappa \in S_*$) we have the following (in both cases regardless of whether $\kappa = (\omega_2)^V$, or an inaccessible)

$$|\mathbb{P}_{S_*,\overline{W}_*}| = |\mathbb{P}_{S_*,\langle\emptyset:\ \theta \in S_*\rangle}| \cdot \left((\omega_1) \cdot \left(\sum_{\theta \in S_* \setminus \{\kappa\}} |W^*_\theta|\right)\right)^\omega \cdot (|W^*_\kappa| \cdot \sup W^*_\kappa)^\omega < \kappa.$$

At this point we have to discuss the two cases (i.e. whether $\kappa \in S_\bullet$) differently, arguing that in both cases there are branches outside $V[\mathbf{G}_{S_*,\overline{W}_*}]$.

If $\kappa = \omega_2 \in S_\bullet$, then as

$$V \models\ |\mathbb{P}_{S_*,\overline{W}_*}|^{\omega_1 \cdot |\mathbb{P}_{S_*,\overline{W}_*}|} = \omega_2,$$

we have

$$V[\mathbf{G}_{S_*,\overline{W}_*}] \models\ 2^{\omega_1} = \omega_2,$$

therefore as $|\mathcal{B}^{V[\mathbf{G}]}(T)| \notin S_\bullet$, there are branches of T in $V[\mathbf{G}]$ not in $V[\mathbf{G}_{S_*,\overline{W}_*}]$. On the other hand, if $\kappa \notin S_\bullet$ is inaccessible, then we obtain that

$$V[\mathbf{G}_{S_*,\overline{W}_*}] \models\ |\mathcal{B}(T)| \leq 2^{\omega_1} < \kappa,$$

and as κ remains a cardinal in $V[\mathbf{G}]$ (by Claim 3.25), and

$$V[\mathbf{G}] \models\ |\mathcal{B}(T) \cap V[\mathbf{G}_{S_*,\overline{W}_*}]| = \omega_1,$$

we conclude that this case there also must be branches of T not in $V[\mathbf{G}_{S_*,\overline{W}_*}]$ as T is a Kurepa tree in $V[\mathbf{G}]$. Now let $\overline{R} \in \prod_{\theta \in S^+_\bullet \setminus S_\bullet} \mathcal{P}(\theta)$, $R_\theta = \theta \setminus W^*_\theta$, then

$$\mathbb{P} = \mathbb{P}_{S^+_\bullet,\overline{\mathrm{id}}_{S^+_\bullet}} \simeq (\mathbb{P}_{S_*,\overline{\mathrm{id}}_{S_*}-\overline{R}}) \times (\mathbb{P}_{S_*\cap(S^+_\bullet\setminus S_\bullet),\overline{R}}) \times (\mathbb{P}_{S^+_\bullet\setminus S_*,\overline{\mathrm{id}}_{S^+_\bullet\setminus S_*}}),$$

and there are no new sequences of type ω in $V[\mathbf{G}]$ (by Claim 3.23), and the second component is ω_1-closed, the third component has an ω_1-closed dense subset (which thus remain ω_1-closed in $V[\mathbf{G}_{S_*,\overline{\mathrm{id}}_{S_*}-\overline{R}}]$) we obtain that each branch of T is added by $\mathbf{G}_{S_*,\overline{\mathrm{id}}_{S_*}-\overline{R}} = \mathbf{G} \cap \mathbb{P}_{S_*,\overline{\mathrm{id}}_{S_*}-\overline{R}}$ (since an ω_1-closed forcing do not add new branches to Kurepa trees [Kun13, Lemma V.2.26]). We only have to derive a contradiction from

$$V[\mathbf{G}_{S_*,\overline{\mathrm{id}}_{S_*}-\overline{R}}] \models\ |\mathcal{B}(T)| \notin S_\bullet.$$

Now letting $\partial = |\mathcal{B}^{V[\mathbf{G}_{S_*,\overline{\mathrm{id}}_{S_*}-\overline{R}}]}(T)| \notin S_\bullet$, $S^-_* = S_* \cap S_\bullet \cap \partial$, $S^+_* = (S_* \cap S_\bullet) \setminus S^-_*$ by Lemma 3.20 we have

$$V[\mathbf{G}_{S_*,\overline{\mathrm{id}}_{S_*}-\overline{R}}] = V[\mathbf{G}_{S_*,\overline{W}_*+\overline{\mathrm{id}}_{S^-_*}}][\mathbf{G}^\circ_{\overline{\mathrm{id}}_{S^+_*}-\overline{W}_*}].$$

As $\partial \notin S^-_*, S^+_*$, it is enough to prove that in $V[\mathbf{G}_{S_*,\overline{W}_*+\overline{\mathrm{id}}_{S^-_*}}]$ there are less than ∂-many branches of T, because if $\mathbf{G}^\circ_{\overline{\mathrm{id}}_{S^+_*}-\overline{W}_*}$ adds new branches, then adds $\min(S^+_*)$-many new branches by Lemma 3.29 (since each $|W^*_\theta| < \kappa \leq \min(S_\bullet) \leq \min(S^+_*)$).

Now if $\partial = \kappa$, then $S^-_* = \emptyset$, we are done, so we can assume that $\partial > \kappa$, and $\sup S^-_* \geq \kappa$. As $|S_*| < \kappa$ (in V), and our conditions (<u>Case1</u>/(iii), or <u>Case2</u>/(ii)) states that then $\sup(S_* \cap S_\bullet \cap \partial) \in S_\bullet$ implying $\sup S^-_* < \partial$. Therefore using that $W^*_\theta \subseteq \theta$ we get $\sum_{\theta \in S^-_*} |W^*_\theta| \leq |\sup S^-_*|^2 < \partial$. Now by Lemma 3.26 for each branch b of T in $V[\mathbf{G}_{S_*,\overline{W}_*+\overline{\mathrm{id}}_{S^-_*}}] = V[\mathbf{G}_{S_*,\overline{W}_*}][\mathbf{G}^\circ_{(\overline{\mathrm{id}}_{S^-_*})-\overline{W}_*}]$ there exist $\theta_0, \theta_1, \ldots, \theta_{n-1}$, $U^\bullet_{\theta_0}, U^\bullet_{\theta_1}, \ldots, U^\bullet_{\theta_{n-1}}$ finite such that $b \in V[\mathbf{G}_{S_*,\overline{W}_*}][\mathbf{G}^\circ_{\overline{U}_\bullet}]$. Therefore, as $|\mathbb{P}^\circ_{\overline{U}_\bullet}| = \omega_1^n = \omega_1$, counting the nice $\mathbb{P}^\circ_{\overline{U}_\bullet}$-names of subsets T for each possible n, sequence of θ's, and $\overline{U}_\bullet$

$$\mathcal{B}(T) \cap (V[\mathbf{G}_{S_*,\overline{W}_*}][\mathbf{G}^\circ_{(\overline{\mathrm{id}}_{S^-_*})-\overline{W}_*}] \setminus V[\mathbf{G}_{S_*,\overline{W}_*}]) \leq (|\sup S^-_*|^{<\omega} \cdot \omega_1^{\omega_1})^{V[\mathbf{G}_{S_*,\overline{W}_*}]} \leq \sup S^-_*,$$

which is smaller than ∂, a contradiction.

For $V[\mathbf{G}] \models\ 2^{\omega_1} = \lambda$ we only need to show that $2^{\omega_1} \leq \lambda$. But a similar straightforward calculation yields that $\mathbb{P} = \mathbb{P}_{S^+_\bullet,\overline{\mathrm{id}}_{S^+_\bullet}}$ is of cardinality λ, and then (using κ-cc and the equality $\lambda^{<\kappa} = \lambda$) by counting the possible nice names for subsets of ω_1 we obtain the desired inequality.

Remark 3.31. If $S_\bullet$ also satisfies

$$\forall \mu \in S_\bullet : \ \mathrm{cf}(\mu) < \kappa \ \rightarrow \ \mu^+ \in S_\bullet, \tag{3.25}$$

and **GCH** holds in V then $S_\bullet \setminus \{\lambda\}$ is the spectrum for the Jech-Kunen trees in $V[\mathbf{G}]$. (A tree T of height ω_1 and power ω_1 is a Jech-Kunen tree if $\omega_1 < |\mathcal{B}(T)| < 2^{\omega_1}$.) For more on Jech-Kunen trees see also [JS93], [JS92], [JS94]. Note that **CH** in the final model implies that the product of countably many Jech-Kunen trees is a Jech-Kunen tree, so is the diagonal product of ω_1-many Jech Kunen trees, hence (3.25) cannot be dropped.

One can obtain similar cardinal arithmetic conditions for Sp_μ with μ large.

4. The necessity of the inaccessible cardinal

In this section we prove that if ω_2 is not an element of the spectrum, then ω_2 is inaccessible in L. The idea of using transitive collapses of elementary submodels of constructible sets as nodes of a tree goes back to Solovay's original unpublished argument for the consistency strength of the negation of the Kurepa Hypothesis. Although the next proof is deemed to be well-known, for the sake of completeness we include the proof as there is probably no known source to cite.

Theorem 4.1. *Suppose that ω_2^V is a successor in L. Then there exists a Kurepa tree T with $\mathcal{B}^V(T) = \omega_2$.*

Proof. We will use an extension of L, an inner model between L and V, what serves as the motivation for the following definition of relative constructibility, which can be found in e.g. [Kan03].

Definition 4.2. For a set A define $L[A] = \bigcup_{\alpha \in ON} L_\alpha[A]$ by transfinite recursion as follows. $L_0[A] = \emptyset$, $L_{\alpha+1}[A] = \mathrm{def}_A(L_\alpha[A])$, and α limit $L_\alpha[A] = \bigcup_{\beta<\alpha} L_\beta[A]$ (where $\mathrm{def}_Y(X)$ are the subsets of X that can be defined in the structure $(X, \in\restriction (X \times X), Y \cap X)$ by parameters from X, see [Kan03, Chapter 1, §3]).

The following is standard easy exercise, but for the sake of completeness we include the proof.

Claim 4.3. *There exists a set $A \subseteq \omega_1$ such that $\omega_1^{L[A]} = \omega_1$, $\omega_2^{L[A]} = \omega_2$.*

Proof. If $\omega_2^V = (\lambda^+)^L$, where $|\lambda| = \omega_1$, then in a single subset A of ω_1 we can code a well-ordering of ω_1 in type λ, and also for each $\alpha < \omega_1$ a well-ordering of ω in type α in the obvious fashion, and such that L can read this coding (implying $\omega_1^{L[A]} = \omega_1$, $\omega_2^{L[A]} = \omega_2$): First let $\langle X_\alpha : \ \alpha \leq \omega_1 \rangle \in L$ be a set of pairwise disjoint sets of ω_1 with $|X_\alpha|^L = \omega$ for each $\alpha < \omega_1$, and $|X_{\omega_1}|^L = \omega_1$, then for each $\alpha < \omega_1$ we can code the well ordering X_α in order type α, and the well ordering of X_{ω_1} in type λ in a subset A' of $\bigcup_{\alpha \leq \omega_1} X_\alpha^2 \subseteq \omega_1^2$. Finally, taking the preimage of this set under a bijection $f \in L$ between ω_1 and ω_1^2, i.e. $A = f^{-1}(A')$ works. $\square$

We have to recall a classical Lemma [Kan03, Theorem 3.3]. Recall that $\mathcal{L}_\in(R_A)$ stands for the (first-order) language of set theory extended by the unary predicate R_A.

Lemma 4.4. *There is a sentence $\sigma \in \mathcal{L}_\in(R_A)$ such that for every transitive set N*

$$(N, \in, X \cap N) \models \sigma \ \textit{implies} \ N = L_\gamma[X] \ \textit{for some limit} \ \gamma.$$

In particular, if $M \prec (L_\beta[X], \in, X \cap L_\beta[X])$, where β is a limit ordinal and π is the collapsing isomorphism from M onto the transitive set $\mathrm{ran}(\pi)$*, then the Mostowski collapse*

$$\mathrm{ran}(\pi) = L_\gamma[\{\pi(x) : \ x \in M \cap X\}]$$

for some $\gamma \leq \beta$.

The following is immediate.

Claim 4.5. *For each infinite ordinal β and $Y \subseteq L_\beta[X]$, if $Y \in L[X]$ and $X \subseteq L_\beta[X]$, then $\mu = (|\beta|^+)^{L[X]}$ implies $Y \in L_\mu[X]$.*

(Working in $L[X]$, if $Y \in L_\gamma[X]$, then let $M \prec L_\gamma[X]$ with $\{Y\} \cup L_\beta[X] \subseteq M$, $|M| = |L_\beta[X]|$, and apply the lemma recalling that $\pi \restriction L_\beta[X]$ is the identity.)

Now we can turn to the definition of the tree T, which will be defined by its branches.

Recall that there exists a definable well-order on $L[A]$, which is downward absolute to almost every initial segment of $L[A]$ (to the ones indexed by limit ordinals) [Kan03, Theorem 3.3]:

Lemma 4.6. *There exists a formula $\varphi \in \mathcal{L}_\in(R_A)$ (i.e. in the language of set theory extended with the unary relation symbol A) which define a well-ordering on $(L[A], \in, A)$, moreover if δ is a limit ordinal, $x, y \in L_\delta[A]$, then*

$$(L[A], \in, A) \models \varphi(x, y) \quad \iff \quad (L_\delta[A], \in, A \cap L_\delta[A]) \models \varphi(x, y).$$

From now on '$x <_{L[A]} y$' abbreviates $\varphi(x, y)$.

We will take Skolem hulls many times, thus we need to introduce the following variant of this standard notion.

Definition 4.7. Let $(M, \in, X, \partial)$, $M \subseteq L[A]$ be a set model of the language $\mathcal{L}_\in(R_A, c_\partial)$ with $\emptyset \in M$, $M' \subseteq M$ such that the well-ordering formula $\varphi \in \mathcal{L}_\in(R_A)$ from Lemma 4.6 is absolute to M, i.e.

$$(\forall x, y \in M) : \ (L[A], \in, A) \models \varphi(x, y) \ \text{ iff } \ (M, \in, X) \models \varphi(x, y), \tag{4.1}$$

e.g. when $(M, \in, X) = (L_\zeta[A], \in, A \cap L_\zeta[A])$ for some limit ordinal ζ. Then the Skolem-hull of M' in $(M, \in, X, \partial)$ (in symbols, $\mathfrak{H}^{(M,\in,X,\partial)}(M')$) is the closure of M' under the functions $f_\psi^{(M,\in,X,\partial)}$ for each formula $\psi(v_0, v_1, \ldots, v_{n_\psi}) \in \mathcal{L}_\in(R_A, c_\partial)$ with $n_\psi + 1$ free variables, where the function $f_\psi^{(M,\in,X,\partial)}$ satisfies the following.

$$f_\psi^{(M,\in,X,\partial)} : M^{n_\psi} \to M$$

is defined so that for every $\langle x_1, x_2, \ldots, x_{n_\psi} \rangle \in M^{n_\psi}$:

$$\text{if } \exists y! \in M \ \text{ s.t. } (M, \in, X, \partial) \models \psi(y, x_1, x_2, \ldots, x_{n_\psi}),$$

$$\text{then let } f_\psi^{(M,\in,X,\partial)}(x_1, x_2, \ldots, x_{n_\psi}) \text{ be the unique such } y,$$

$$\text{otherwise let } f_\psi^{(M,\in,X,\partial)}(x_1, x_2, \ldots, x_{n_\psi}) = \emptyset.$$

Then the fact that for each formula ψ' we can define the formula saying that y is the least y (w.r.t. the well-order given by φ) satisfying $\psi'(y, x_1, x_2, \ldots x_{n_{\psi'}})$ together with the Tarski-Vaught criterion implies that the closure is an elementary submodel of M, in symbols, $M' \prec (M, \in, X, \partial)$.

Observe that this closure only depends on the isomorphism class of $(M, \in, X, \partial)$ by the absoluteness of the well-ordering formula φ (4.1).

Choose $\xi < \omega_2$ such that

$$\text{(4.2)}\quad \xi \text{ is the minimal ordinal } (\forall \alpha < \omega_1)\ \exists f_\alpha \in L_\xi[A] \text{ bijection between } \omega \text{ and } \alpha$$

(which can be done due to Corollary 4.5, in fact $\xi = \omega_1$, but we won't use this equality, hence we don't argue that).

Now we will define an operation which assigns for each $\delta \in [\xi, \omega_2)$ the ordinal $\delta' < \omega_2$ in the following way. We would like to choose δ' so that in $L_{\delta'}[A]$ it is true that for each set x there exists a surjection from ω_1 to x, and for $\delta'' \neq \delta'$ the structures $(L_{\delta'}[A], \in, A, \delta)$ and $(L_{\delta''}[A], \in, A, \delta)$ cannot be elementarily equivalent.

Definition 4.8. Fix $\delta \in [\xi, \omega_2)$, and define δ' to be the least ordinal such that

a) $\delta \in L_{\delta'}[A]$,
b) for each $x \in L_{\delta'}[A]$ there is a bijection $f \in L_{\delta'}[A]$ between ω_1 and x,
c) taking the sentence σ from Lemma 4.4 $(L_{\delta'}[A], \in, A) \models \sigma$.

(Using Claim 4.5 and $(|L_\alpha[A]| = |\alpha|)^{L[A]}$ for $\alpha \geq \omega$ it is easy to see that we can do this closure operation, and there is such a $\delta' < \omega_2$.) Then we have

$$\text{(4.3)}\qquad (\delta' \text{ is a limit }) \bigwedge\ (L_{\delta'}[A] \models\ \text{'}\omega_1 \text{ is the largest cardinal'}),$$

and also the desired uniqueness by our next claim.

Claim 4.9. *There is a statement $\sigma' \in \mathcal{L}_\in(R_A, c_\partial)$ such that for each $\delta \in [\xi, \omega_2)$ $(L_{\delta'}[A], \in, A, \delta) \models \sigma'$, moreover, for each $\delta > \omega_1$ and $\delta'' > \delta$*

$$((L_{\delta''}[A], \in, A, \delta) \models \sigma') \ \Rightarrow\ (\delta'' = \delta').$$

Proof. First define $\sigma'' = \sigma \wedge (\forall y \exists f : \omega_1 \to y \text{ bijection})$ and let σ' be the following sentence

$$\sigma' = \sigma'' \wedge \big(\neg(\exists X)\ (X \text{ is transitive}) \wedge (\sigma'')^X \wedge (\delta \in X)\big)$$

(where under ψ^X we always mean the formula $\psi \in \mathcal{L}_\in(R_A, c_\partial)$ relativized to X, and σ is from Lemma 4.4). $\square$

Now fix $\delta \in [\xi, \omega_2)$, and for each ordinal $0 < \alpha < \omega_1$ define $M_{\delta,\alpha}$ to be the Skolem-hull

$$\text{(4.4)}\qquad M_{\delta,\alpha} = \mathfrak{H}^{(L_{\delta'}[A], \in, A, \delta)}(\alpha) \ \ (\text{for each } \alpha < \omega_1),$$

Also define

$$\text{(4.5)}\qquad M_{\delta,0} = \emptyset.$$

Then

$$\text{(4.6)}\qquad M_{\delta,\alpha} \prec (L_{\delta'}[A], \in, A, \delta) \ \ (\text{for each } \alpha > 0).$$

Observe that whenever $M^* \prec (L_{\delta'}[A], \in, A, \delta)$ we have for the Skolem functions from Definition 4.7 that $f_\psi^{(L_{\delta'}[A], \in, A, \delta)} \restriction (M^*)^{n_\psi} = f_\psi^{(M^*, \in, A \cap M^*, \delta)}$, hence

$$\text{(4.7)}\quad \forall M' \subseteq M^* \prec (L_{\delta'}[A], \in, A, \delta) :\ \ \mathfrak{H}^{(L_{\delta'}[A], \in, A, \delta)}(M') = \mathfrak{H}^{(M^*, \in, A \cap M^*, \delta)}(M').$$

Now as we defined $\langle M_{\delta,\alpha} :\ \alpha < \omega_1 \rangle$ note that

$$\text{(4.8)}\qquad (M \prec (L_{\delta'}[A], \in, A, \delta)) \wedge (|M| = \omega) \ \rightarrow\ (M \cap \omega_1 \in \omega_1),$$

in particular

$$\text{(4.9)}\qquad M_{\delta,\alpha} \cap \omega_1 \in \omega_1,$$

since (4.2) together with $\xi \le \delta < \delta'$ implies that in $L_{\delta'}[A]$ there is an enumeration of each ordinal less than ω_1 (and $M_{\delta,\alpha}$ is countable). This implies that

$$(C_\delta = \{\alpha < \omega_1 : \ M_{\delta,\alpha} \cap \omega_1 = \alpha\} \text{ is a club in } \omega_1) \ \wedge \ (0 \in C_\delta).$$

It is easy to see that

$$\forall \alpha < \omega_1 : \ M_{\delta,\alpha} = M_{\delta,\min(C_\delta \setminus \alpha)}. \tag{4.10}$$

For later use we verify the following statement.

Claim 4.10.

$$\bigcup_{\alpha<\omega_1} M_{\delta,\alpha} = L_{\delta'}[A].$$

Proof. Since the union of an increasing chain of elementary submodels is an elementary submodel, we have $M_{\omega_1} = \bigcup_{\alpha<\omega_1} M_{\delta,\alpha} \prec (L_{\delta'}[A], \in, A, \delta)$. Now recall, that in $L_{\delta'}[A]$ every set x admits a surjection from ω_1 onto x, therefore $\omega_1 \subseteq M_{\omega_1}$ implies that M_{ω_1} is transitive. Then by Lemma 4.4 and $M_{\omega_1} \models \sigma$ we have $M_{\omega_1} = L_{\delta''}[A]$ for some $\delta'' > \delta$. But then either $M_{\omega_1} \in L_{\delta'}[A]$, or $M_{\omega_1} = L_{\delta'}[A]$, and because the former would contradict Claim 4.9, we arrive at our conclusion. $\square$

For each $\alpha \in C_\delta$ and $\beta < \omega_1$, if $\alpha = \max(C_\delta \cap (\beta+1))$, then let $N_{\delta,\beta,\alpha}$ be the range of the Mostowski-collapse $\pi_{\delta,\alpha}$ of $(M_{\delta,\alpha}, \in)$, and let $A_{\delta,\beta,\alpha} = \pi_{\delta,\alpha}(A)$, $\partial_{\delta,\beta,\alpha} = \pi_{\delta,\alpha}(\delta)$:

$$\pi_{\delta,\alpha} : M_{\delta,\alpha} \to N_{\delta,\beta,\alpha}, \tag{4.11}$$

which is of course not only an isomorphism between $(M_{\delta,\alpha}, \in)$ and $(N_{\delta,\beta,\alpha}, \in)$, but witnesses

$$(M_{\delta,\alpha}, \in, A \cap M_{\delta,\alpha}, \delta) \simeq (N_{\delta,\beta,\alpha}, \in, A_{\delta,\beta,\alpha}, \partial_{\delta,\beta,\alpha}). \tag{4.12}$$

Now we are ready to construct the tree T. For a fixed $\delta \in [\xi, \omega_2)$, $\alpha \in C_\delta$, $\beta < \omega_1$, if $0 < \alpha = \max(C_\delta \cap (\beta+1))$ holds then we define

$$t_{\delta,\beta,\alpha} = (N_{\delta,\beta,\alpha}, \in, A_{\delta,\beta,\alpha}, \partial_{\delta,\beta,\alpha}), \tag{4.13}$$

i.e. the structure $(N_{\delta,\beta,\alpha}, \in)$ extended by the one-place relation for the image of $A \in M_{\delta,\alpha}$ under the collapsing isomorphism, and the constant symbol for $\partial_{\delta,\beta,\alpha}$. For $\max(C_\delta \cap (\beta+1)) = 0$ let $t_{\delta,\beta,0} = \emptyset$.
Observe that given $t = t_{\delta,\beta,\alpha}$ we can decode α from t, as α is the first uncountable ordinal of t.

Definition 4.11. Define

$$T = \{(\beta, t_{\delta,\beta,\alpha}) : \ \delta \in [\xi, \omega_2), \beta < \omega_1, \ \alpha = \max(C_\delta \cap (\beta+1))\},$$

with the partial order $(\beta_0, t_{\delta_0,\beta_0,\alpha_0}) \le_T (\beta_1, t_{\delta_1,\beta_1,\alpha_1})$ iff either $\alpha_0 = 0$ (thus $t_{\delta_0,\beta_0,\alpha_0}$ is the empty structure), or

- (i) $\beta_0 \le \beta_1$, and
- (ii) taking the Skolem-hull M of α_0 in

  $$t_{\delta_1,\beta_1,\alpha_1} = (N_{\delta_1,\beta_1,\alpha_1}, \in, A_{\delta_1,\beta_1,\alpha_1}, \partial_{\delta_1,\beta_1,\alpha_1})$$

  (i.e. $M = \mathfrak{H}^{t_{\delta_1,\beta_1,\alpha_1}}(\alpha_0)$ is isomorphic to $t_{\delta_0,\beta_0,\alpha_0}$:

  $$(M, \in, A_{\delta_1,\beta_1,\alpha_1} \cap M, \partial_{\delta_1,\beta_1,\alpha_1}) \simeq (N_{\delta_0,\beta_0,\alpha_0}, \in, A_{\delta_0,\beta_0,\alpha_0}, \partial_{\delta_0,\beta_0,\alpha_0})),$$

  and

(iii) if $\alpha_0 < \alpha_1$, then there is no proper elementary submodel $M \prec (N_{\delta_1,\beta_1,\alpha_1}, \in, A_{\delta_1,\beta_1,\alpha_1}, \partial_{\delta_1,\beta_1,\alpha_1})$ with

$$\alpha_0 \cup \{\alpha_0\} \subseteq M, \text{ and}$$

$$M \cap \alpha_1 \subseteq \beta_0.$$

Roughly speaking, in level β we have (isomorphism types of) initial segments M of models of the form $(L_{\Delta'}[A], \in, A, \Delta)$ (for some $\Delta \in [\xi, \omega_2)$), such that $M \cap \omega_1 \leq \beta$, and M is maximal w.r.t. this condition. We need to check that T is a tree, its levels are countable, and that it has only ω_2-many branches even in V.

The following claim is a standard calculation, but for the sake of completeness we include the proof.

Claim 4.12. *Let $\delta \in [\xi, \omega_2)$ be fixed, $\beta_0 \leq \beta_1 < \omega_1$, let $\alpha_1 = \max(C_\delta \cap (\beta_1 + 1))$, $\alpha_0 = \max(C_\delta \cap (\beta_0 + 1))$. Then $(\beta_0, t_{\delta,\beta_0,\alpha_0}) \leq_T (\beta_1, t_{\delta,\beta_1,\alpha_1})$.*

Moreover, the embedding $\pi_{\beta_0,\beta_1} : N_{\delta,\beta_0,\alpha_0} \to N_{\delta,\beta_1,\alpha_1}$ is unique.

Proof. First observe that by (4.4) and (4.7) for $\delta \in [\xi, \omega_2)$, $\alpha_0 < \alpha_1$

$$\mathfrak{H}^{(M_{\delta,\alpha_1}, \in, A, \delta)}(\alpha_0) = \mathfrak{H}^{(L_{\delta'}[A], \in, A, \delta)}(\alpha_0) = M_{\delta,\alpha_0},$$

therefore since $\beta_1 < \omega_1$ is such that $\alpha_1 = \max(C_\delta \cap (\beta_1 + 1))$, then applying (the restriction of) the collapsing isomorphism π_{δ,α_1} to the left side, we obtain

$$(\mathfrak{H}^{(N_{\delta,\beta_1,\alpha_1}, \in, A_{\delta,\beta_1,\alpha_1}, \partial_{\delta,\beta_1,\alpha_1})}(\alpha_0), \in) \simeq (M_{\delta,\alpha_0}, \in)$$

and because $\beta_0 < \beta_1$ is such that $\alpha_0 = \max(C_\delta \cap (\beta_0 + 1))$, then applying the isomorphism π_{δ,α_0} to the right side (which fixes α_0) we obtain

$$(\mathfrak{H}^{(N_{\delta,\beta_1,\alpha_1}, \in, A_{\delta,\beta_1,\alpha_1}, \partial_{\delta,\beta_1,\alpha_1})}(\alpha_0), \in) \simeq (N_{\delta,\alpha_0,\beta_0}, \in).$$

Finally, since $\pi_{\delta,\alpha_1}(A) = A_{\delta,\beta_1,\alpha_1}$, $\pi_{\delta,\alpha_0}(A) = A_{\delta,\beta_0,\alpha_0}$, and $\pi_{\delta,\alpha_1}(\delta) = \partial_{\delta,\beta_1,\alpha_1}$, $\pi_{\delta,\alpha_0}(\delta) = \partial_{\delta,\beta_0,\alpha_0}$, we have

$$(\mathfrak{H}^{N_{\delta,\beta_1,\alpha_1}}(\alpha_0), \in A_{\delta,\beta_1,\alpha_1}, \partial_{\delta,\beta_1,\alpha_1})$$

$$\text{is isomorphic to } (N_{\delta,\beta_0,\alpha_0}, \in, A_{\delta,\beta_0,\alpha_0}, \partial_{\delta,\beta_0,\alpha_0}),$$

therefore (ii) holds. The uniqueness easily follows from the facts that the embedding of $(N_{\delta,\beta_0,\alpha_0}, \in, A_{\delta,\beta_0,\alpha_0}, \partial_{\delta,\beta_0,\alpha_0})$ has to fix the ordinals less than α_0, and elementary embeddings uniquely extend to Skolem-hulls.

For (iii) suppose that $\alpha_0 < \alpha_1$, and note that

$$(N_{\delta,\beta_1,\alpha_1}, \in) \models \text{ '}\alpha_1 \text{ is the least uncountable ordinal, } \alpha_0 \text{ is countable'},$$

and for $M \prec (N_{\delta,\beta_1,\alpha_1}, \in, A_{\delta,\beta_1,\alpha_1}, \partial_{\delta,\beta_1,\alpha_1})$ if $\alpha_0 \cup \{\alpha_0\} \subseteq M$ then consider the corresponding submodel $M' \prec (M_{\delta,\alpha_1}, \in, A, \delta)$, for which $M' \supseteq M_{\delta,\alpha_0+1}$. But (recalling (4.8)) since $\max(C_\delta \cap (\beta_0 + 1)) = \alpha_0$ we obtain $\beta_0 \cup \{\beta_0\} \subseteq M' \subseteq M_{\delta,\alpha_1}$, that can happen only if β_0 is smaller than the least uncountable ordinal in $N_{\delta,\beta_1,\alpha_1}$, α_1. But then $\beta_0 \in M \cap \alpha_1$. $\square$

The next claim will verify that T is a tree of height ω_1 (for the transitivity of $\leq_T$ use the claim two times).

Claim 4.13. *For a fixed $\delta_1 \in [\xi, \omega_2)$, $\beta_0 \leq \beta_1 < \omega_1$, let $\alpha_1 = \max(C_{\delta_1} \cap (\beta_1 + 1)$, and fix arbitrary $\alpha_0 \in \omega_1$, $\delta_0 \in [\xi, \omega_2)$. Then $(\beta_0, t_{\delta_0,\beta_0,\alpha_0}) \leq_T (\beta_1, t_{\delta_1,\beta_1,\alpha_1})$ iff $t_{\delta_0,\beta_0,\alpha_0} = t_{\delta_1,\beta_0,\max(C_{\delta_1 \cap (\beta_0+1)})}$.*

Proof. We only have to check the 'only if' part, but first observe that Definition 4.11 clearly implies that up to isomorphism there exists only one t for which $(\beta_0, t) \leq (\beta_1, t_{\delta_1,\beta_1,\alpha_1})$. Now the claim is the consequence of the fact that $t_{\delta_*,\beta_0,\alpha_*} \neq t_{\delta_{**},\beta_0,\alpha_{**}}$ implies that they are not isomorphic as structures of the language $\mathcal{L}_\in(R_A, c_\partial)$: For transitive sets N and N' with $X, \partial \in N$, $X', \partial' \in N'$ the structures $(N, \in, X, \partial)$, $(N', \in, X', \partial')$ are isomorphic if and only if $N = N'$, $X = X'$ and $\partial = \partial'$ (since by the uniqueness of the Mostowski collapse we know that $(N, \in) \simeq (N', \in)$ iff $N = N'$). $\square$

Lemma 4.14. *For each $\beta < \omega_1$ the β'th level of T is countable.*

Proof. By Claim 4.13 we have that the β'th level of T is

$$T_{\leq\beta} \setminus T_{<\beta} = \{(\beta, t_{\delta,\beta,\alpha}) : \ \delta \in [\xi, \omega_2), \ \alpha = \max(C_\delta \cap (\beta+1))\}\}.$$

For a fixed $\delta \in [\xi, \omega_2)$ fix $\alpha = \max(C_\delta \cap (\beta+1))$ too, and consider the structure

$$t_{\delta,\beta,\alpha} = (N_{\delta,\beta,\alpha}, \in, A_{\delta,\beta,\alpha}, \partial_{\delta,\beta,\alpha}),$$

where $N_{\delta,\beta,\alpha}$ is the Mostowski collapse of $(M_{\delta,\alpha}, \in)$ (by the isomorphism $\pi_{\delta,\alpha}$), and $A_{\delta,\beta,\alpha} = A \cap \alpha$. Now (4.6) states $M_{\delta,\alpha} \prec (L_{\delta'}, \in, A)$ then (recalling $M_{\delta,\alpha} \cap \omega_1 = \alpha$, and $\pi_{\delta,\alpha} \restriction \alpha = \mathrm{id}_\alpha$) by Lemma 4.4

$$N_{\delta,\beta,\alpha} = L_\gamma[A \cap \alpha]$$

for some $\gamma = \gamma(\delta, \alpha) \in (\alpha, \omega_1)$. Now we determine an upper bound γ_α for the set $\{\gamma(\delta, \alpha) : \delta \in [\xi, \omega_2) \ \wedge \ \alpha \in C_\delta\}$. If we have such a bound for each possible $\alpha \leq \beta$, then letting γ_∞ denote $\sup\{\gamma_\alpha : \ \alpha \leq \beta\}$, we get

$$\{t_{\delta,\beta,\alpha}) : \ \delta \in [\xi, \omega_2), \ \alpha = \max(C_\delta \cap (\beta+1))\}\} \subseteq$$
$$\{(L_\gamma[A \cap \alpha], \in, A \cap \alpha, \partial) : \ \gamma \leq \gamma_\infty, \ \alpha \leq \beta, \ \partial < \gamma\},$$

which latter set is obviously countable, this will finish the proof of the lemma.

So fix $\alpha \leq \beta$ and $\delta \in [\xi, \omega_2)$ such that $\alpha \in C_\delta$. Now we have two cases depending on whether there is any $(\text{cardinal})^{L[A\cap\alpha]}$ in (α, ω_1). If $\lambda \in (\alpha, \omega_1)$ is a cardinal in the inner model $L[A \cap \alpha]$, then for each δ if $\alpha = \max(C_\delta \cap (\beta+1))$, then the transitive set $N_{\delta,\beta,\alpha}$ cannot contain λ, as $M_{\delta,\alpha}$ sees ω_1 as the largest cardinal, and $\pi_{\delta,\alpha}(\omega_1) = \alpha$. This case choosing $\gamma_\alpha = \lambda$ works.

On the other hand, if $(|\alpha|^+)^{L[A\cap\alpha]} = \omega_1$, then we first prove that $\alpha \in C_\delta$ implies $(|\alpha| = \omega)^{L[A\cap\alpha]}$: otherwise in $M_{\delta,\alpha}$, and in $N_{\delta,\beta,\alpha}$ each ordinal less than α are countable, thus as well in $L[A \cap \alpha]$. Then it is easy to see that the condition

$$(\lambda \text{ is the unique cardinal in } (\omega, \omega_1^V))^{L[A\cap\lambda]}$$

cannot hold for two different λ's, therefore α can be defined in $L[A]$. But then using Claim 4.5 with $X = A \cap \alpha$ we have that for each $\zeta \in (\alpha, \omega_1)$ there is a bijection $f_\zeta \in L_{\omega_1}[A \cap \alpha]$ between α and ζ, therefore α can be defined also in $L_{\delta'}[A]$, and $M \prec (L_{\delta'}[A], \in)$ implies $\alpha \in M$, contradicting that $M_{\delta,\alpha} \cap \omega_1 = \alpha$ (which holds by $\alpha \in C_\delta$). Then $(|\alpha| = \omega)^{L[A\cap\alpha]}$ and Claim 4.5 implies that there is an ordinal $\lambda < \omega_1$ such that there exists a bijection between α and ω in $L_\lambda[A \cap \alpha]$, implying

$$N_{\delta,\beta,\alpha} = L_{\gamma(\delta,\alpha)}[A \cap \alpha] \subsetneq L_\lambda[A \cap \alpha],$$

since α is uncountable in $N_{\delta,\beta,\alpha}$. This case

$$\{\gamma(\delta, \alpha) : \delta \in [\xi, \omega_2) \ \wedge \ \alpha \in C_\delta\} \subseteq \gamma_\alpha = \lambda,$$

which completes the proof of Lemma 4.14.

$\square$

Now T is obviously a Kurepa tree by the following fact and lemma.

Fact 4.15. *The sequence $\langle B_\delta : \ \delta \in [\xi, \omega_2) \rangle$ lists pairwise distinct cofinal branches in T, where*

$$B_\delta = \{ (\beta, t_{\delta, \beta, \max(C_\delta \cap (\beta+1))}) : \ \beta < \omega_1 \}.$$

Proof. We only need to prove that $B_\delta \neq B_\gamma$ if $\delta \neq \gamma$. But according to the second statement of Claim 4.12 for each $\beta < \beta' < \omega_1$ there is a unique elementary embedding of $t_{\delta, \beta', \max(C_\delta \cap (\beta'+1))}$ to $t_{\delta, \beta, \max(C_\delta \cap (\beta+1))}$, therefore there is a unique direct-limit of this elementary chain, isomorphic to $\bigcup_{\alpha \in C_\delta} M_{\delta, \alpha}$, which is $(L_{\delta'}[A], \in, A, \delta)$ by Claim 4.10. $\square$

It is only left to prove that each branch of T is of the form B_δ for some $\delta \in [\xi, \omega_2)$ (even in V). The following lemma will complete the proof of Theorem 4.1.

Lemma 4.16. *Let $B \subseteq T$ a cofinal branch in T, $B \in V$. Then $B = B_{\delta_\bullet}$ for a unique $\delta_\bullet \in [\xi, \omega_2)$.*

Proof. Let $t_{\delta_\beta, \beta, \alpha_\beta} = (N_{\delta_\beta, \beta, \alpha_\beta}, \in, A_{\delta_\beta, \beta, \alpha_\beta}, \partial_{\delta_\beta, \beta, \alpha_\beta})$ denote the element in $B \cap (T_{\leq \beta} \setminus T_{<\beta})$. Working in V first we define the following bonding maps: for $\gamma \leq \beta < \omega_1$ let

$$\pi_{\gamma, \beta} : N_{\delta_\gamma, \gamma, \alpha_\gamma} \to N_{\delta_\beta, \beta, \alpha_\beta}$$

be the unique elementary embedding (combining Claim 4.13, and the second statement of Claim 4.12). Since elementary submodels of an elementary submodel are elementary submodels, $\pi_{\beta', \beta} \circ \pi_{\beta'', \beta'}$ is an elementary embedding for each $\beta'' \leq \beta' \leq \beta < \omega_1$, therefore by the uniqueness

$$(\forall \beta'' \leq \beta' \leq \beta < \omega_1) : \ \pi_{\beta', \beta} \circ \pi_{\beta'', \beta'} = \pi_{\beta'', \beta}. \tag{4.14}$$

This elementary chain allows us to define the limit $D = (N_{\omega_1}, \mathbf{E}, A_{\omega_1}, \partial_{\omega_1})$ of the directed system $\{ t_{\delta_\beta, \beta, \alpha_\beta}, \ \pi_{\beta', \beta} : \ \ \beta' \leq \beta < \omega_1 \}$.

Let $\pi_\beta : N_{\delta_\beta, \beta, \alpha_\beta} \to N_{\omega_1}$ be the embedding, $N_\beta = \mathrm{ran}(\pi_\beta)$ (hence $N_{\omega_1} = \bigcup_{\beta < \omega_1} N_\beta$).

First note that $(N_{\omega_1}, \mathbf{E})$ is well-founded, otherwise there would be an infinite $\mathbf{E}$-decreasing chain in the embedded image of $N_{\delta_\beta, \beta, \alpha_\beta}$ for some (in fact, every large enough) β, contradicting that $(N_{\delta_\beta, \beta, \alpha_\beta}, \in)$ is well-founded. Now (by the $\mathbf{E}$-extensionality in N_{ω_1}) we can assume that N_{ω_1} is a Mostowski collapse, i.e. $(N_{\omega_1}, \mathbf{E}) = (N_{\omega_1}, \in)$. Then it is easy to see that if $\beta < \omega_1$ for the elementary embedding $\pi_\beta : N_{\delta_\beta, \beta, \alpha_\beta} \to N_{\omega_1}$ we have $\pi_\beta \restriction \alpha_\beta = \mathrm{id}_{\alpha_\beta}$, and $\pi_\beta(\alpha_\beta) = \omega_1$, thus (recalling that $A_{\delta_\beta, \beta, \alpha_\beta} = A \cap \alpha_\beta$) we obtain $(N_{\omega_1}, \mathbf{E}, A_{\omega_1}, \partial_{\omega_1}) = (N_{\omega_1}, \in, A, \delta_\bullet)$ for some $\delta_\bullet \in (\omega_1, \omega_2)$. Now we can use Lemma 4.4 (since $(N_{\delta_\beta, \beta, \alpha_\beta}, \in, A_{\delta_\beta, \beta, \alpha_\beta}) \models \sigma$), there exists $\zeta > \delta_\bullet$ such that

$$N_{\omega_1} = L_\zeta[A],$$

and then

$$(N_{\omega_1}, \in, A, \delta_\bullet) = (L_\zeta[A], \in, A, \delta_\bullet).$$

Now because the formula $\sigma' \in \mathcal{L}_\in(R_A, c_\partial)$ from Claim 4.9 holds in $(L_{\delta'}[A], \in, A, \delta)$ (for each $\delta \in [\xi, \omega_2)$) (for our mapping $\delta \mapsto \delta'$ from Definition 4.8) and therefore also in $M_{\delta, \alpha}$'s, $N_{\delta, \beta, \alpha}$'s ($\delta \in [\xi, \omega_2)$), so it must hold in $(N_{\omega_1}, \in, A, \delta_\bullet)$, which means that $\delta_\bullet \geq \xi$, and $\zeta = \delta'_\bullet$, i.e.

$$(N_{\omega_1}, \in, A, \delta_\bullet) = (L_{\delta'_\bullet}[A], \in, A, \delta_\bullet),$$

Finally, we have to prove that for each $\beta < \omega_1$

$$t_{\delta_\beta,\beta,\alpha_\beta} = (N_{\delta_\beta,\beta,\alpha_\beta}, \in, A_{\delta_\beta,\beta,\alpha_\beta}, \partial_{\delta_\beta,\beta,\alpha_\beta}) = t_{\delta_\bullet,\beta,\max(C_{\delta_\bullet}\cap(\beta+1))}$$

by arguing (having β fixed) that for a large enough γ

$$(\beta, t_{\delta_\bullet,\beta,\max(C_{\delta_\bullet}\cap(\beta+1))}) \leq_T (\gamma, t_{\delta_\gamma,\gamma,\alpha_\gamma}).$$

Let $\alpha = \max(C_{\delta_\bullet} \cap (\beta+1))$, $\alpha' = \min(C_{\delta_\bullet} \setminus (\beta+1))$, $\beta' = \alpha'$, and consider the models $M_{\delta_\bullet,\alpha}, M_{\delta_\bullet,\alpha'} \prec (L_{\delta'_\bullet}[A], \in, A, \delta_\bullet)$. Choose $\gamma \geq \beta'$, $\gamma < \omega_1$ so that $N_\gamma = \pi_\gamma[N_{\delta_\gamma,\gamma,\alpha_\gamma}] \supseteq M_{\delta_\bullet,\alpha'}$. Then

$$\alpha_\gamma \geq \alpha' > \beta + 1, \text{ and} \tag{4.15}$$

$\alpha' \cup \{\omega_1\} \subseteq N_\gamma \prec (L_{\delta'_\bullet}[A], \in, A, \delta_\bullet)$ with (4.7) imply

$$\mathfrak{H}^{(N_\gamma,\in,A\cap N_\gamma,\delta_\bullet)}(\alpha) = \mathfrak{H}^{(L_{\delta'_\bullet}[A],\in,A,\delta_\bullet)}(\alpha) = M_{\delta_\bullet,\alpha}.$$

Therefore in $(N_\gamma, \in, A\cap N_\gamma, \delta_\bullet) \simeq (N_{\delta_\gamma,\gamma,\alpha_\gamma}, \in, A_{\delta_\gamma,\gamma,\alpha_\gamma}, \partial_{\delta_\gamma,\gamma,\alpha_\gamma})$ there is an elementary submodel isomorphic to $(M_{\delta_\bullet,\alpha}, \in, A \cap M_{\delta_\bullet,\alpha}, \delta_\bullet)$, which latter is isomorphic to $(N_{\delta_\bullet,\beta,\alpha}, \in, A \cap \alpha, \partial_{\delta_\bullet,\beta,\alpha})$, thus (ii) from Definition 4.11 holds.
Similarly, using also (4.10) and the definitions of α, α'

$$\mathfrak{H}^{(N_\gamma,\in,A\cap N_\gamma,\delta_\bullet)}(\alpha+1) = M_{\delta_\bullet,\alpha+1} = M_{\delta_\bullet,\alpha'} \supseteq \alpha' \supseteq \beta \cup \{\beta\},$$

and since the isomorphism between $(N_\gamma, \in, A \cap N_\gamma, \delta_\bullet)$ and $(N_{\delta_\gamma,\gamma,\alpha_\gamma}, \in, A_{\delta_\gamma,\gamma,\alpha_\gamma}, \partial_{\delta_\gamma,\gamma,\alpha_\gamma})$ fixes the ordinals less than or equal to α' we obtain

$$\mathfrak{H}^{(N_{\delta_\gamma,\gamma,\alpha_\gamma},\in,A_{\delta_\gamma,\gamma,\alpha_\gamma},\partial_{\delta_\gamma,\gamma,\alpha_\gamma})}(\alpha+1) \supseteq \beta \cup \{\beta\}.$$

Therefore recalling (4.15) we obtain that (iii) (of Definition 4.11) holds as well. □

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