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Divide and Conquer: Dividing Lines and Universality

by

SAHARON SHELAH

Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram,
The Hebrew University of Jerusalem, Israel; and
Department of Mathematics, Hill Center - Busch Campus, Rutgers,
The State University of New Jersey, USA

Abstract: We discuss dividing lines (in model theory) and some test questions, mainly the universality spectrum. So there is much on conjectures, problems and old results, mainly of the author and also on some recent results.

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Introduction

THE MAIN THEME of our work in model theory aims at the identification and exploitation of robust, successful and fruitful dividing lines. In this introductory Part I, we will explain in general terms what this means, and how this is done. In particular, we will discuss the use of test problems to find dividing lines. The bulk of the paper, Parts II and III, is devoted mainly to the discussion of a particular test problem, concerning the universality spectrum of a first-order theory. What makes this test problem particularly interesting to us now is the fact that it for long appeared unpromising as a source of good dividing lines, and now appears very promising.

The way we look for dividing lines is via test problems, and these generally involve set-theoretic notions which are in some sense foreign to algebraic first-order model theory but are in the heart of general model theory. For example, we ask how to determine the number of models of a first-order theory in cardinal λ. That is, we consider the function $I(\lambda, T)$ which associates to a first-order theory T and a cardinal λ, the number of models of T of cardinality λ, counted up to isomorphism. It is more useful to think of this as a binary relation $\tilde{I}(f, T)$ where f is a class function on cardinals and T is a first-order theory: the relation holds if

$$f(\cdot) = I(\cdot, T).$$

One side of the question is which functions may occur. The other side of the question is what the function tells us about the theory. We call such a property an external property, in this case, as usual, connected to set theory, but computation (e.g., recursion) theory can serve. We are interested in cases in which the external property has an internal characterization, typically in terms of the behaviour of individual formulas relative to the theory T. By the Compactness Theorem, such characterizations are likely to have a finitary character. This necessarily involves set-theoretic concepts, but the thesis is that it will direct us to more syntactic properties, of interest also to those to whom the original question was not appealing.

One may suggest an elementary analogy. There is a non-standard construction of the real line as the non-standard finite part of the rational line, modulo infinitesimals. This gives, potentially, many real lines and many notions of convergence. So the notion is momentarily "external", until the Cauchy criterion is given as an internal characterization of convergence – with a more combinatorial flavour.

Our test problems, and the associated dividing lines, tend to measure the relative complexity of theories, as expressed by some partial order (actually, a quasi-order at the level of theories). The most direct comparison of theories is by interpretability: $T_1 \leq_i T_2$ if T_1 can be interpreted in T_2. This is much too fine. A dividing line aims to break the theories into two parts, the more simple and the less simple, in a useful way. More generally, we may seek to impose some coarse quasi-order on the theories, hopefully with finitely many classes, and understand something about the structure, notably the height, of the partial order, and the minimal and maximal elements.

In the case of the test problem on the number of models, a natural quasi-order would be derived from the corresponding relation on functions

$$f \leq^* g \Leftrightarrow \text{For sufficiently large } \lambda, f(\lambda) \leq g(\lambda)$$

which allows for some "sporadic" behaviour (notably, for $\lambda = \aleph_0$). In particular, the minimum would then correspond to theories categorical in sufficiently high power, a case treated in Morley's seminal work. The so-called Main Gap refers to the behaviour below the maximum (the thesis being that, if f is below the maximum, then it is quite small), which is crucial for the general case of this test problem. (However, as we will see below, the way in which the quasi-order on theories is derived from the quasi-order on functions is not the most straightforward one.)

The theory developed out of the test problem of the determination of the number of models of given cardinality has been the most successful and productive to date, in most respects. We have considered a number of other test problems which we will refer to in the body of the paper. One, which deserves additional mention here before we focus on the problem of the universality spectrum, is Keisler's order. This is defined directly in terms of a quasi-order on theories, rather than a single test problem and the associated test problems are structural: Is the associated partial order linear, what are the minimum and maximum elements, are there infinitely many classes? We will go into more detail below. Here we speak broadly on a few points of history.

Initially, the study of the low end of Keisler's order produced a new dividing line within stability and a complete understanding of the stable case. Then the subject moved out of focus and languished for several decades with little progress and no concrete ideas as to how to proceed further. New developments have brought the subject back to life, showing in particular that the number of classes is infinite and that infinitely many robust classes appear; serendipitously, new applications to set theory arose, which actually was not a goal of this particular program.

Set-theoretic applications aside, we wish to suggest that the study of the universality spectrum may offer some parallels. While the subject was never near death, the associated test problems appeared both a priori and empirically as a not-so-promising source of fundamental dividing lines. Now as we progress, the associated test problems begin to look quite promising, and this is the main point we wish to discuss. The question at this stage is not, specifically, as to whether we come to new dividing lines, but rather whether we come to any important dividing line at all. (We are not averse to encountering old friends in new places.)

We have one last methodological point to address before entering into details. Namely, the use of set theory, or rather, the malleability of set theory, introduces a further complication. Cardinal arithmetic tends to be a prominent ingredient of our test problems, particularly at the "high end" in terms of complexity of theories, and as we learned starting with Paul Cohen, cardinal arithmetic can be made to behave in almost all ways that can be imagined. In some such universes the class of (A, E), E an equivalence relation, will be maximal. As a result, we do not always want to take the answer to our question in the "real world" but in all possible worlds, or at least all which can be easily reached from the real world (by nice forcing). An alternative is to change the question somewhat to "the maximal number of models of T of cardinality λ pairwise non-embeddable to each other." Then we have a chance to discover useful "intrinsic" properties of our theories. So we build our quasi-orders in two steps, first asking set-theoretic questions about first-order theories, and then repeating the questions in suitable forcing extensions of the set-theoretic universe—or in other ways; if our dividing lines are robust, as we hope, it ought not to matter too much how we deal with the incompleteness of set theory. But we must choose a definite way of proceeding, and whatever way we choose will then require us to make some set-theoretic constructions.

One noteworthy point is that, within the mass of undecidability results associated with cardinal arithmetic, there is a “hidden kernel” of absoluteness, associated with pcf theory (the theory of possible cofinalities). This theory will also make a useful appearance in the body of the present paper, as in much else of the author’s work.

The *universality spectrum* of a class K of structures is the class Univ(K) of (infinite) cardinals in which K has a universal member, i.e., a model of that cardinality such that any other member from this class of this cardinality can be embedded into it; in some cases “cardinality” is replaced by “topological density”. This is a natural notion which arose independently in several contexts (see also Baldwin, 2021).

Below, we shall write on test problems concerning the universality spectrum, but also on test problems of other types, some of a very similar character, others less so.

We focus on the family of elementary classes, i.e., the classes of models of a first-order theory T, and we may write T instead of the class of models of T. This is quite a wide context and is the classical one in model theory and covers many natural examples. We think that maybe we should consider trying to get a complete (or partial) characterization of universality spectra and the associated classes of first-order theories. This is certainly a hard enough challenge.

This topic sits at a crossroads:

(1) We need model theory to analyse the properties of the theories, using known properties of theories, or (which is more exciting) finding new dividing lines among them and investigating them, thus advancing a major theme of model theory which was also very successful in applications. In particular there are classical subjects related to this, like pseudo finite fields.

(2) We need infinite combinatorics to prove relevant properties of the class that suffice for the non-existence of a universal model of cardinality λ.

(3) We are interested also in specific classes, e.g., the classes of linear orders, groups, Abelian groups, graphs and even Banach spaces. Sometimes it suffices to prove that such a class fits a general theorem, but we have to prove that the class fits it. Sometimes the quoted results need to be adapted and sometimes, even more challenging, a really new method and new properties of the class have to be found. So the results should be of interest even to researchers with marginal interest in set theory and model theory.

(4) Last but not least, we need forcing from set theory to prove consistency results, mainly to prove that cardinal arithmetic conditions (such as $\lambda = \mu^{+k}, \mu^{\aleph_0} = \mu$) are not sufficient to give an answer.

We feel the time is ripe for advances in finding a general criterion which is necessary and sufficient for proving that any reasonable class K satisfying it has a universal member in λ, or at least for clarifying the picture.

The article is divided to three parts according to the intended audience such that each can be read independently.

Part I is meant to be accessible both to philosophers and to mathematicians not working in mathematical logic. Part II requires some familiarity with model theory and set theory. Part I is recapitulated very briefly in Part II, and mathematical logicians may prefer to start there. We note that Part II does not require familiarity with classification theory, though such familiarity with that subject, or its applications, casts a great deal of light on the significance of the dividing lines we consider below. The question as to what further theory may develop out of dividing lines associated with the test problems considered here is a separate issue, not addressed in the present paper. Parts I and II describe goals and results, without proof. In Part III, we give some new results with proofs.

A drawback of this organization is that some repetition is unavoidable.

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Part I

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In the following we mention some well-established dividing lines (Section 1), propose a new test question (Section 2) and describe how this new notion fits into and interacts with the established properties and the Keisler order (Sections 3–5), and mention additional goals (Section 6).

In each section, we describe the state of the art and some known theorems; as well as the new terrain, our plans for the proposed projects, in the form of problems and conjectures. We also postulate several theses (informal statements, often expressing the author's personal opinion).

To make this paper converge, we present the knowledge as it is in September 2018. On recent information on the problems, see Shelah (2020b) on T with no universal model in singular cardinalities, Shelah (2020a) on sufficient conditions for $\lambda \notin \mathrm{univ}(T_{\mathrm{feq}})$, $\mathrm{univ}(T_{\mathrm{ceq}})$, Shelah (1999b) on $\lambda \in [\mu, \mu^{\mathrm{cf}(\mu)})$ for μ singular, mainly strong limit. Now Shelah (1999b) will deal also with some questions left open on Abelian groups (see §10B, in particular the table there). See Malliaris and Shelah (2018a, 2019) on Keisler order, and Kaplan et al. (2020) say more on exact saturation (and on continuations). Lastly, a work in preparation with P. Komjath shall deal with the universality spectrum (under embeddability, in uncountable cardinals) for K_H, the class of graphs into which there is no weak embedding of the finite graphs H (see §10D on the countable case).

1. The Aim: The Classification Question

Thesis 1.1. 1) It is worthwhile to classify theories, i.e., find good and meaningful dividing lines, or dichotomies.

2) Good test problems help us to find the right dividing lines.

3) In more detail: We look for properties P of a theory T such that we have relevant information on both sides about those theories which have property P and about those which do not have it. Usually one side (having P, say) tells us that T is analysable, or not too complicated in a certain way, and that we can develop a "positive" theory, i.e., a structure theory for the class of its models. In the other case, i.e., theories not having P, we can prove it has complicated models in a suitable sense.

Let us give names to some aspects of this intuition:

Informal Definition 1.2. We call a candidate for being a dividing line (a pre-dividing line in short):

- *robust*, if it has an internal definition, say a property on relations definable (by first-order formulas with parameters) in $M \in K$, and an equivalent external definition, like having few models up to isomorphism, or that the ultra-powers of any $M \in K$ are "easily λ-saturated", etc.
- (internally) *successful*, if there is a serious structure theory on the positive side. E.g., we have a general definition of non-forking, or of dimension;

- externally *successful*, when it helps proving complicated models exist for T on the up side, but we shall deal here more with the internal side.
- *fruitful*, when the positive theory has applications in parts of mathematics outside model theory.
- *versatile*, if also for contexts not falling in our framework the machinery developed is helpful.

Thesis 1.3. A dividing line (not just a candidate) is one which is robust and successful, and hopefully more. While there are worthwhile properties which do not satisfy these requirements, being such a dividing line is a great bonus:

- it directs us to the right structure theorems developed under the right conditions;
- sometimes it helps us to prove a theorem on all theories by dividing the proof into cases;
- it tells us that some theorems are best possible, i.e., applying only to those with the property;
- we expect that for some classes of interest in other parts of mathematics, the general theorems can help. However, this has not been part of our motivation.

See Figure 1 for some of the known dividing lines and candidates and the implications between them (see http://forkinganddividing.com for an interactive map with examples). Note that some properties look to me excellent candidates to be dividing lines, but the evidence does not exist or is slim, which is why in Figure 1 we have “and candidates”.

Choice 1.4. Here we concentrate on NSOP_4, particularly on simple theories.

The properties stable/superstable have a very strong support (i.e., being robust and successful on the one hand, and attracting researchers and being fruitful, i.e., having applications on the other hand, see e.g. (Hrushovski, 1996; Hrushovski and Loeser, 2016). Also “simple” has strong support, as demonstrated below, as well as being dependent (earlier called NIP) (Simon, 2015) and NTP_2 (Chernikov, 2014).

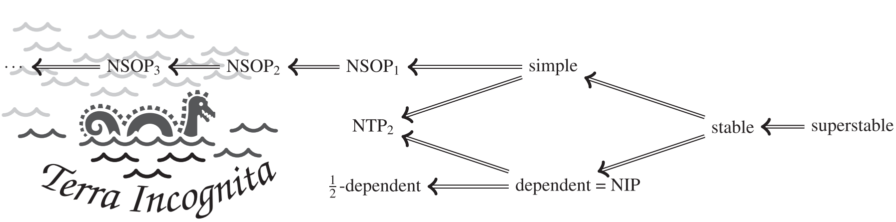

Figure 1. A map of dividing lines and candidates

These are good examples of how successful dividing lines became fruitful. Many technical tools were originally developed for classification theory (etc.) purposes, such as analysing and counting models as test problems. But careful geometric analysis by Hrushovski and many others has led to powerful applications (Hrushovski, 1996, 2012) that created new bridges between model theory, algebraic and Diophantine geometry, and combinatorics (Bouscaren, 1998; Scanlon, 2001; Breuillard et al., 2011). It is natural to believe that developing new dividing lines will also eventually have a similar impact.

A natural way to try to find dividing lines is considering quasi-orders, like $\leq_{\text{univ}}$ discussed below and Keisler's order (see Jerome Keisler, 2017). Now a naturally defined quasi order may be trivial (having only one equivalence class), or too fine, having too many equivalence classes, making too fine a distinction. A possible remedy is to ask about the minimal or maximal such theories, or to change the order to overcome this; we shall discuss some cases.

2. The Test Question: Universality and the Universality Order

We begin by defining the universality spectrum of a class K of models. The reader may suppose that $K = K_T$ is the class of all models of the first-order theory T, as will very often be the case in the present article; alternatively, it may be some particular class of mathematical interest, such as the class of locally finite groups or Banach spaces. (We aim at a general theory applying to a wide range of classes K but may sometimes have to settle for ad hoc adaptations of the main theory to special cases a little outside it.) We write K_λ for the class of models in K of size λ.

The Main Definition 2.1. 1) $M \in K$ is *universal* in λ (for K) if for any $N \in K_\lambda$, N can be embedded (see Section 10 for various alternatives for the meaning of "embedded") into M.

2) $M \in K$ is *universal*, if it is universal in the cardinality $\|M\|$ of M.

3) The *universality spectrum* of K, univ(K) (= univ_K), is the class of cardinals $\lambda \geq \aleph_0$ such that there is a universal $M \in K$ of cardinality λ.

We consider the following test problem, which we call the *universality spectrum problem*:

Problem 2.2. Classify first-order theories T according to the universality spectrum of K_T.

This problem subsumes a number of smaller problems of considerable interest in their own right. If we temporarily ignore the essential point that issues of cardinal arithmetic will lead us (if not force us) to consider forcing extensions of the set-theoretic universe, then these would include the following:

Problem 2.3. (Over-simplified) 1) Which theories T are simplest, in the sense that they have the largest universality spectrum?

2) Which theories T are the most complex, in the sense that they have the smallest universality spectrum?

3) Repeat questions (1) and (2) but working with the universality spectra modulo initial segments.

4) What is the least cardinal λ such that two universality spectra which agree above some cardinal λ' must agree above λ?

In the foregoing, we allude to a common phenomenon in model theory: first-order logic has difficulty making fine distinctions in cardinality greater than the cardinality κ of the language, and even more difficulty in making distinctions above 2^{κ}. For simplicity we take $\kappa = \aleph_0$ so that this threshold should be something like $\aleph_1$ or $2^{\aleph_0}$.

Set theory has no such difficulty, so we will still have to consider points like the cofinality of λ or cardinal arithmetic, but when we do things properly our readiness to change the universe will remove much of this.

However for very small values of λ we anticipate that our test problems may have unusual "sporadic" solutions. The most classical instance of this is the complement to Morley's theorem: on the one hand categoricity in one uncountable cardinal is equivalent to categoricity in all uncountable cardinals; but on the other hand, categoricity in $\aleph_0$ is something else entirely.

Similarly, there is a very interesting theory of countable universality, which we discuss in some detail (§ 10[E]), but we cannot expect it to be closely related to the "generic" theory above—or well above—the cardinality of the theory. And the generic theory is most likely to give dividing lines of the type we seek. Most notably, we will comment on the case of universals in $\aleph_0$ for classes K_H of graphs omitting a given finite graph H, a topic which received considerable attention from combinatorialists as well as logicians (Komjáth and Pach, 1984; Komjáth et al., 1988; Latka, 1994; Cherlin and Shi, 1996; Goldstern and Kojman, 1996; Cherlin et al., 1997; Füredi and Komjáth, 1997; Cherlin et al., 1999; Cherlin and Shi, 2001; Latka, 2003; Cherlin and Shelah, 2007; Cherlin, 2011; Cherlin and Shelah, 2016).

A major problem in this line of research is the following decidability problem: is the collection of finite graphs H such that K_H admits a universal in $\aleph_0$ decidable (Komjáth et al., 1988)? In a series of papers with Cherlin (Cherlin et al., 1999; Cherlin and Shelah, 2007; Cherlin and Shelah, 2016), we devise a general strategy toward a positive answer to the decidability problem just posed. This naturally requires several involved technical investigations in finite graph theory.

But here we concentrate on uncountable (or "sufficiently large") cardinals λ. Some mathematicians are a bit sceptical toward Cantor's paradise, i.e., they are not fond of uncountable cardinals; however, they can still enjoy its fruits, that is, the positive finitary theory for the whole family of K_T's with an "understandable" theory.

We now define a quasi-order on theories in which the smaller theories are the ones for which it is easier to find universal models, and thus the theories are presumably simpler in some respect. Our first version (with superscript 0) sets aside all set-theoretic issues.

Definition 2.4. Let $\leq^0_{\text{univ}}$ be the following partial (quasi) order on the collection of first-order classes: $K_1 \leq^0_{\text{univ}} K_2$ if $\lambda \in \text{univ}(K_2) \Rightarrow \lambda \in \text{univ}(K_1)$, for every sufficiently large cardinal λ.

Problem 2.5. Can we characterize the $\leq^0_{\text{univ}}$ equivalence classes? (T_1 and T_2 are equivalent if $T_1 \leq^0_{\text{univ}} T_2 \leq^0_{\text{univ}} T_1$.) What about the $\leq^0_{\text{univ}}$ maximal theories?

The notion of universality is a natural and interesting notion, which has come up in various contexts. The existence of universal spaces allows for the consideration of mathematical objects as sub-objects of a more concrete one, and thus endows them with a greater wealth of "intrinsic" properties. It also emphasizes the relations of "parts of a whole". Probably the oldest instance of this notion is the observation that the complex field is universal for fields of characteristic 0 and cardinality at most continuum. (This fact was generalized by André Weil (Weil, 1962) to a "universal domain" in algebraic geometry; which is a precursor of the inspirations of the model-theoretic "monster model".) Another classical example is the complete Urysohn space (Urysohn, 1927), which is the unique complete homogeneous separable metric space which embeds every finite metric space. The classical Banach–Mazur Lemma indicates that $C([0, 1])$ is a universal separable Banach space. Another major source of examples is the well-known method of Fraïssé, which subsumes many examples of universal objects, such as the countable random graph (Rado, 1964) and Hall's universal group (Hall, 1959). Another context in which universal objects have arisen is the study of Polish groups and, more generally, Polish spaces, see e.g., (Uspenskii, 1986; Ben-Yaacov, 2014). There has been considerable interest in the question of existence of universal objects among Banach spaces, and more generally, metric and topological spaces of a particular form, e.g., (Bourgain, 1980; Katětov, 1988;

Bell, 2000; Džamonja, 2006; Brech and Koszmider, 2012; Brech and Koszmider, 2013; Džamonja, 2014). Finally, universal objects appear also in the context of sofic groups, see e.g., (Pestov, 2008; Thomas, 2010).

A priori, $\leq^0_{\text{univ}}$ seems like an interesting order, and one would expect that K's (or T's) which are "small" (or even "not maximal") under this partial order should be nice, i.e., "understandable". In short, $\leq^0_{\text{univ}}$ should provide good dividing lines. However, recall the old works of Cantor and Hausdorff on linear orders. The existence of universal models was central in the early stages of model theory, see the classical works (Jónsson, 1956; Jónsson, 1960; Morley and Vaught, 1962). See also Baldwin (2021). In fact, if $\lambda = \mu^+ = 2^\mu$ or just $\lambda = \lambda^{<\lambda} > \aleph_0$ then for any reasonable T, $\lambda \in \text{univ}_T$. Actually, if $\lambda = \lambda^{<\lambda} > \aleph_0$ then there are even so-called homogeneous universal models and saturated models, where each implies universality (which was the original motivation for these investigations). Also if $\lambda = 2^{<\lambda > \aleph_0}$ then there is a universal model. So the problem of the existence of universal models leads to these new notions which became central in model theory. In particular, they motivated the introduction of what is now known as non-forking, an essential tool of classification theory.

Pedantically, as Baldwin has pointed out, Jonsson and Morley-Vaught have spoken about a model of cardinality λ being universal for models of cardinality $<\lambda$. But for the case they were dealing with, $(<\lambda)$-homogeneity, it makes no difference. Moreover, for characterizing the unique universal homogeneous model of cardinality λ, it is clearly preferable to use a weaker version.

Now generally the natural notion of universality, and the one used, is here with the usual embeddings.

In contrast, understanding the existence of universal models directly seems harder and was not studied much by model theorists. Until recently, I too considered it to be a poor candidate for leading us to a dividing line. The only known results were the stable classes (see below) and scattered sufficient conditions for non-existence. There are set-theoretic cases on independence results in specific cases, e.g., see (Kojman and Shelah, 1992a; Shelah, 1996b; Džamonja, 2005; Džamonja, 2011; Shelah, 2016) and references therein, but now I feel that we should reconsider the general theory.

However, we must deal with the fact that $\leq^0_{\text{univ}}$ commits us to whatever version of set theory holds in the "real world" (or in some arbitrary model of set theory, perhaps). Indeed, Gödel told us that possibly $\lambda = 2^{<\lambda}$ for every λ (as it is equivalent to GCH); and in this case, univ_T is maximal, i.e., $\text{univ}_T = \{\lambda : \lambda > \aleph_0\}$.

But Cohen told us that GCH may fail, and, by results of Solovay and Easton (Easton, 1970), cardinal arithmetic can be essentially anything, i.e., there are no non-trivial restriction on $2^{\aleph_0}$ and 2^{λ^+}. This motivates us to refine the order $\leq^0_{\text{univ}}$

by demanding the implication not only to hold "incidentally" (due to, e.g., GCH) in the "present universe", but also in "all" others (meaning "provably") or just in any forcing extension of the present universe. Also, some quite trivial theories have no universal models in $\aleph_1$ when $\aleph_1 < 2^{\aleph_0}$, so in order to get a clean general theory we should restrict to "sufficiently large" λ, and $\lambda \geq 2^{\aleph_0}$, or even $\lambda = \lambda^{\aleph_0}$ is a prudent restriction. The theory for small values of λ may also be of interest (this is certainly the case if $\lambda = \aleph_0$) but with a different character.

Hence, GCH could "sabotage" our test question, but we have an excellent way to avoid GCH: forcing. In fact, forcing allows us to violate GCH at any successor cardinal, while leaving the set of real numbers unchanged.

This leads to the following definition:

Definition 2.6. $T_1 \leq_{\text{univ}} T_2$ if "$\lambda \in \text{univ}(T_2) \Rightarrow \lambda \in \text{univ}(T_1)$" holds also in every larger "universe of sets" (i.e., in every forcing extension where $\lambda \geq 2^{\aleph_0}$, because of 2.5, holds).

The universality spectrum problem is the structural understanding of this quasi-order, or in other words, determination of the classes of the associated equivalence relation and the structure of the corresponding partial order on the classes. Various test problems arise naturally in this case, as well as similar problems involving small cardinals. We concern ourselves initially, as in the case of the Keisler order (see below), with the classes lying at an extreme, that is, very low or very high. *Note that small or low theories have large universality spectra, and large or high theories have small universality spectra*. At the high end, we make the following definition:

Definition 2.7. K is almost $\leq_{\text{univ}}$-maximal if in every forcing extension, K has no universal model in λ for any $\lambda = \lambda^{\aleph_0}$ with $\mu^{++} = \lambda < 2^{\mu}$ for some μ.

While forcing is a very successful and well-established method in set theory, one might wonder why we expect to get a natural and useful notion if we commit ourselves to this specific technique in the definition. The answer is that this detail should not be important: forcing is "good enough", or said otherwise, potential counterexamples are "robust enough":

Thesis 2.8. 1) The order $\leq_{\text{univ}}$ is robust: We should get basically the same results if instead of forcing we talk, e.g., about provability in ZFC or other natural notions of "necessary".

2) But talking about forcing extensions is much more convenient and well defined.

This notion may shed light on the terra incognita of Figure 1. There are other relevant test problems for this area. Most prominent and oldest is the Keisler order $\triangleleft$ and saturation of ultra-powers and its relative $\triangleleft^*$ and $\leq_{\mathrm{SP}}$ (see Section 5).

Problem 2.9. 1) Is there a theory T such that (provably in ZFC), $\mathrm{univ}_T = \{\lambda : \lambda = 2^{<\lambda} > \aleph_0\}$? (i.e., univ_T contains only the cardinals guaranteed by the remark above. So such a theory is $\leq_{\mathrm{univ}}$-maximal.)

2) In particular, is the theory of linear orders $\leq_{\mathrm{univ}}$-maximal, ignoring $\aleph_0$ and $\aleph_1$?

3) Characterize the $<_{\mathrm{univ}}$-maximal theories. Is it a good dividing line or is almost $\leq_{\mathrm{univ}}$-maximality better?

4) Can we at least characterize the almost $\leq_{\mathrm{univ}}$-maximal T's?

The test problems we consider measure notions of complexity which are likely to have something in common, since they follow a common pattern of coarsening interpretability using external set-theoretic notions. At the low end, there are well-established notions of simplicity flowing out more or less directly from the investigation of the number of models up to isomorphism and the existence of saturated models, notably the concept of stability and the more general concept of simplicity which actually goes under the name of simplicity. In the next section, we present these notions in some detail together with our knowledge and expectations for the behaviour of the universality spectrum in these contexts.

In §5 we shall go into some detail regarding recent results on the Keisler order, which currently serves as a rough model for the kind of results we would like to have in the case of the universality spectrum. There is still much that is not understood about the Keisler order, but enough is understood to see that it is a good source of test problems which produce robust dividing lines, many of which are new.

3. Stability

At least in this central case we give the full definition:

Definition 3.1. T is *not stable*, if some formula $\varphi(\bar{x}_n, \bar{y}_n)$ defines on ${}^nM$, for some model M of T, a directed graph which is an infinite linear order on some subset of ${}^nM$.

A priori it is not clear that this is a worthwhile dividing line, but a posteriori stable theories are understandable: They have an internal notion of free amalgamation (called non-forking) and dimension.

We expect stable theories to have a large univ_T, see Shelah (1990a, Ch. III).

There is also a stronger relative called superstable, cf. Shelah (1990a, Ch. III) (and we call strictly stable the theories which are stable but not superstable):

Theorem 3.2. 1) *If T is superstable then: T is $\le_{\text{univ}}$-minimal; and if $\lambda \ge 2^{\aleph_0}$, then T has a universal model of cardinality λ, in fact, a saturated one.*

2) *If T is strictly stable (i.e., stable but not superstable), then T is almost $\le_{\text{univ}}$-minimal, in fact*:

(a) *if $\lambda = \lambda^{\aleph_0}$, then T has a universal model of cardinality λ, in fact, a saturated one*;

(b) *if $\lambda = \sum_n \lambda_n^{\aleph_0}$, equivalently if $(\forall \alpha < \lambda)(|\alpha|^{\aleph_0} \le \lambda)$, then T has a universal model in λ, in fact, a so-called special model*;

(c) *if $2^{\aleph_0} < \mu < \lambda < \mu^{\aleph_0}$ then there are some non-existence results (by Kojman and Shelah* (1995)). *See also §10(B) and §10(C).*

This leads to the following:

Problem 3.3. 1) Are all strictly stable *T*'s $\le_{\text{univ}}$-equivalent?

2) Or does $\le_{\text{univ}}$ reveal dividing lines among them?

Problem 3.4. For superstable *T*'s, so-called regular types lead to a well-behaved dimension theory. Is there a dividing line among strictly stable *T*'s which characterize having a parallel situation (probably for a weak relative of regular types)? See Palacín and Shelah (2018) and references therein.

Of course, while the unstable theories are less "analysable" than stable ones, not all of them are equally complicated. There are two separate "reasons" for being unstable, leading usually to two separate lines of research, that is:

Theorem 3.5. (Shelah, 1990a, Ch. II) *If T is unstable, then at least one of the following occurs*:

(A) *T has the independence property, i.e., some formula $\varphi(\bar{x}_n, \bar{y}_n)$ defines in some $M \in K_T$ a graph which has a random graph as an induced subgraph*;

(B) *T has the strict order property, i.e., some formula $\varphi(\bar{x}_n, \bar{y}_n)$ defines in some $M \in K_T$ a partial order with infinite chains.*

Now if T falls under (B), then it is almost $\le_{\text{univ}}$-maximal, by Kojman and Shelah (1992a), and so we turn to "lacking the strict order property" which seemed to me an excellent candidate for a dividing line, and I have considered several suitable test problems (to show the property to be robust); but alas, no evidence so far. However, there are weaker properties for which there is evidence: simple and to some extent $NSOP_n$ for $n \leq 4$. In fact, we know that SOP_4 is enough for almost $\leq_{univ}$-maximalility (Shelah, 1996b), and so, with respect to our test question, it seems reasonable not to look beyond $NSOP_4$ theories. Given the many scattered results on the "positive" side for particular $NSOP_4$ theories, we think that now the time is ripe for relooking at the full classification problem. We first turn to simple theories.

Related directions on which we shall not say more (in part II) are:

(1) classification on a predicate (Pillay and Shelah, 1985; Shelah, 1986; Shelah and Usvyatsov, 2019)

(2) Game equivalence (Hyttinen and Tuuri, 1991; Hyttinen et al., 1993; Hyttinen and Shelah, 1994, 1995, 1999; Havlin and Shelah, 2007; Shelah, 2008; Palacín and Shelah, 2018).

4. On Simple Theories

Simple theories were originally introduced in Shelah (1980b) as a natural candidate for a dividing line for which some results on stable theories can be generalized. In the 1990s, due to the natural examples found (see below) and Kim (1996) and Kim and Pillay (1998), the study of simple theories become a very active area. Many properties of non-forking independence generalize from stable theories to this context (see also Grossberg et al., 2002; Kim, 2014). Furthermore, simplicity is a robust dividing line (see 1.2), though not in an ideal way. First a definition:

Definition 4.1. We say a class $K \subseteq K_T$ is *dense in* λ (for T) if whenever $M \in K_T$ has cardinality λ, then for some $N \in K$ of cardinality λ we have $M \subseteq N$.

Theorem 4.2. *T is simple iff necessarily, if $\aleph_0 < \mu = \mu^{<\mu} < \lambda = \lambda^{\aleph_0} < 2^\mu$ and some generalization of Martin's axiom for μ holds, then the set of μ-saturated $M \in K_T$ is dense in λ.*

There are important natural examples for simple theories: the random graph, pseudo-finite fields (i.e., theories of ultraproducts of finite fields), ACFA (algebraically closed fields with a generic automorphism), as well as smoothly approximable structures (a class of well-behaved ω-categorical countable structures, see Cherlin and Hrushovski, 2003).

In fact, ACFA (the algebraically closed fields with an automorphism) is a great example of a simple unstable theory whose analysis has part in various achievements outside of logic, and, in particular, to a new proof of the Manin–Mumford conjecture (Hrushovski, 2001).

Another highly non-trivial result states that ACFA is the limit theory of the Frobenius automorphism. In other words, for every sentence φ in the language of fields with a distinguished symbol for an automorphism, ACFA implies φ iff φ is true for almost all structures of the form (K_q, σ_q) where q is a prime power, K_q is the algebraic closure of the finite field with q elements and σ_q is the map $x \mapsto x^q$ (see Hrushovski, 2004).

How does simplicity relate to $\leq_{\text{univ}}$?

Conjecture 4.3. 1) The order $\leq_{\text{univ}}$ separates simple from non-simple theories.

2) There is a relative of Theorem 4.2 for $\leq_{\text{univ}}$.

3) There is a reasonable characterization of $\leq_{\text{univ}}$ among simple theories.

A weak support for this is: there are forcing results on the existence of models in $\lambda^+ < 2^\lambda$ universal for λ, for every simple T. Presently some non-simple T have similar results (see Shelah, 1993b, 1996b; Džamonja and Shelah, 2004a). However, for the theory T_{rg} of random graphs and relatives, there are strong forcing results for the non-existence of universal models, cf. (Mekler, 1990; Shelah, 1990b).

On recent results see Shelah (2016).

We believe that this line of research will shed more light on simple theories. There are several long-standing questions (such as the "stable forking" conjecture), and it is clear that among simple theories there are several subclasses that exhibit different behaviour. Uncovering new dividing lines may help us to understand this structure and discover new interesting phenomena.

For example, T_{rg} is $\leq_{\text{univ}}$-minimal among the unstable T's; so we are led to the following problem:

Problem 4.4. 1) Characterize the theories which are $\leq_{\text{univ}}$-equivalent to T_{rg}.

2) Is this a reasonable dividing line?

3) Prove that the simple theories of fields mentioned above (pseudo-finite fields, ACFA), and more generally all the simple theories of fields, are $\leq_{\text{univ}}$-equivalent to T_{rg}.

As is well-known, superstable fields are algebraically closed (Macintyre, 1971; Cherlin and Shelah, 1980). However, although stable fields have long been conjectured to be separably closed, this is still open. For unstable classes the situation is even less clear (while there are precise conjectures, see the beginning of Kaplan et al., 2011). One hopes that the tools developed in our context will help to advance the situation at least in the class of fields with simple theories.

5. The Keisler Order

Many times, various kinds of limit models shed light on the original one. For example, the real and the p-adic field shed light on the rational (field); and the universal sofic groups on finite permutation groups. Being universal is a weak version; being saturated (or κ-saturated) says "whatever may happen already happened", and it is central in model theory. A general way to construct such countable structures is using Fraïssé limits, a construction well known to logicians and combinatorialists alike. Another one is considering ultra-powers of models; this motivates Keisler's order (Keisler, 1967).

Definition 5.1. $T_1 \trianglelefteq T_2$ iff for any set I and so-called regular ultrafilter D on I if $M_\ell \in K_{T_\ell}$ for $\ell = 1, 2$ and M_2^I/D is $|I|^+$-saturated then so is M_1^I/D.

In an initial bloom in the 1960s and 1970s, the stable case was sorted out (exactly two equivalence classes!); but afterwards little progress was made until the topic was revived in the last decade. Recently:

Theorem 5.2. (Malliaris and Shelah, 2016b) *The property* SOP_2 *implies* $\triangleleft$*-maximality.*

Note that proofs on $\triangleleft$ involve constructing ultrafilters with precise properties (separating one theory from another). While those problems are of great interest per se, there is a relative $\triangleleft^*$ of $\triangleleft$ that does not lead to such problems, but it is suspected to be too fine.

Using the order $\triangleleft^*$, we can show the following:

Theorem 5.3. (Assuming an instance of GCH) *(one direction (Džamonja and Shelah*, 2004a; *Shelah and Usvyatsov*, 2008b*) and the other (Malliaris and Shelah, 2017)) T is $\triangleleft^*$-maximal iff T is* SOP_2 *(so* SOP_2 *is a robust pre-dividing line!).*

Thesis 5.4. 1) It would be ideal if the order $\leq_{\mathrm{univ}}$ (as well as $\triangleleft$, $\triangleleft^*$) had finitely many equivalence classes (or at least countable) or just in some way nicely describable.

2) If this fails, we may try to find a relative for which this holds.

Unfortunately, this hope is shattered, at least for $\triangleleft$, in (1) above:

Theorem 5.5. (Malliaris and Shelah, 2018b) *There is an infinite $\triangleleft$-decreasing sequence of simple theories. Similarly for $\leq_{\mathrm{SP}}$ by Shelah and Ulrich (2018).*

This seems to indicate that this line will help to understand simple theories. This motivates:

Problem 5.6. 1) Prove that any $\lhd$-maximal theory has the SOP_2.

2) Characterize the $NSOP_2$ theories T such that: For every large enough μ, for every $\lambda > \mu$ there is a regular ultrafilter D on λ such that: if M a model of T, then M^λ/D is not μ^+-saturated.

So $NSOP_2$ is robust, but is it successful? (Recall 1.2) If Problem 5.6 is answered positively, then we have:

Problem 5.7. Develop a positive/structure theory for $NSOP_2$ (but see Malliaris and Shelah, 2016a).

Lately there were some advances in this direction for $NSOP_1$, related to non-forking and Kim-non-forking (see Chernikov and Ramsey, 2016; Kaplan and Ramsey, 2020; Kruckman and Ramsey, 2018; Kaplan et al., 2018). Note that it is open whether $NSOP_1$, $NSOP_2$ and $NSOP_3$ are in fact distinct for complete first-order theories.

Conjecture 5.8. The order $\trianglelefteq$ has continuum many equivalence classes.

If so, this may indicate that even $\trianglelefteq$ is too fine for our purposes.

6. Other Directions

Considered a breakthrough, a triumph for model theory is Malliaris and Shelah (2016b), which received a lot of attention (Hartnett, 2017; Juhász, 2017; Whipple, 2017). It includes a very surprising result: $\mathfrak{p} = \mathfrak{t}$, recalling $\mathfrak{p}$ and $\mathfrak{t}$ are "cardinal characteristics of the continuum". These invariants measure the size of the continuum from various perspectives. These cardinals are defined combinatorially or by algebraic properties, an area which, until then, had absolutely no relation to model theory; moreover, the question whether $\mathfrak{p} = \mathfrak{t}$ had been open since the 1940s and was generally expected to be (consistently) false.

In light of this development, we propose:

Problem 6.1. Understand the connection to model theory for other cardinal characteristics of the continuum.

We expect this direction will have further connections with finite (and infinite) combinatorics, including graph theory (which already has proven to have close connections to model theory in the past). In particular generalizations of Ramsey theory, Vapnik–Chervonenkis dimension, graph regularity cf. (Sauer, 1972; Szemerédi, 1975; Kechris et al., 2005; Gowers, 2007; Malliaris and Shelah, 2014). It is natural to look at other test problems which we hope will lead to worthwhile dividing lines.

Let us just list a few:

- λ-exact saturation and λ-PC-exact saturation (see Kaplan et al., 2017, 2019). The conjecture is that any simple theory has λ-PC-exact saturation for all singular λ, or at least for all strong limit cardinals λ. There is evidence (by work in progress, see Malliaris and Shelah, 2017) that "singular PC-exact saturation" may turn out to be a good test problem for a dividing line between simple and $NSOP_2$.
- The quite old question about the order $\leq_{SP}$, related to Theorem 4.2, see Shelah and Ulrich (2018).

We shall not elaborate on:

- Classification over a predicate
- Game equivalence, see end of §3
- $\leq_{SP}$ (Shelah, 1980b; 1981; Shelah and Ulrich, 2018)
- Borel reducability, Borel completeness (Laskowski and Shelah, 2015)
- Recent advances on universality.

Thesis 6.2. When dealing with dividing lines, there are good reasons to expect that answers to the proposed questions will have applications in several fields of mathematics.

Part II

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7. Introduction to Part II

The universality spectrum of a class K of structures is the class of infinite cardinals in which K has a universal member, i.e., cardinals λ such that there is a model M of size λ such that any other member of K of cardinality can be embedded into M. In some cases, "cardinality" is replaced by "density". This is a natural notion which arose independently in several contexts. We focus here on the family of elementary classes, i.e., the class of models of a first-order theory T. This rather wide context is the classical one in model theory and covers many natural examples.

Our main test questions is: which theories have universal models in certain cardinalities? Another concerns the classical Keisler order (which just recently moved to the front line, receiving considerable attention from the mathematical community). Our expectation is that these questions will lead us to find *new dividing lines* between "understandable" and "chaotic" theories. The classes we are mainly interested in are unstable but without the strict order property (i.e., no formula $\varphi(\bar{x}_n, \bar{y}_n)$ defines partial orders with an infinite chain), so they are "far" from, e.g., the class of real-closed fields, and even generally so-called dependent T's; and they include the class of simple theories. Such dividing lines have been a major theme in model theory, with many successful applications to other areas of mathematics (such as pseudo-finite fields). However, the question itself also connects with other areas of mathematics.

The subject is at a crossroads. We need finite combinatorics to analyse the local behaviour of finite patterns of definable relations. We need infinite combinatorics to prove that certain properties of classes imply that there cannot be universal models. We are interested in specific classes, e.g., the classes of linear orders, groups, Abelian groups, graphs and Banach spaces. While the basic question for such a specific class may be solved by a general criterion, it may also be necessary to find specific arguments or even *develop new methods*. So the results should be of interest also to researchers outside of model theory and set theory. We will also employ the set-theoretic method of forcing, which will prove consistency results that can show that the obtained results are best possible (and thus really constitute a dividing line among theories).

We feel the time is ripe for trying to find a general criterion which is necessary and sufficient for proving that any reasonable class satisfying it has a universal member in a given cardinal. But even "just" developing the dividing line for *T*'s with minimal universality spectrum, or even "just" for the so-called simple theories, would certainly be major breakthroughs and hard challenges.

7.1 Preliminaries

We assume the reader has some familiarity with model theory (but not necessarily classification theory) and set theory (at least some forcing). Note that there are two expected kinds of interaction with mathematics outside mathematical logic:

(1) dealing with the existence of universal models for specific classes (e.g., groups);

(2) finding new dividing lines and developing their positive/structure theory, and consider the place of such classes in the map of model-theoretic properties (cf. Figure 1). Such progress has found outside applications. Furthermore, we expect interaction with finite as well as infinite combinatorics.

How are classification theory and the universality spectrum connected? We consider the universality spectrum as an excellent test problem, important and natural per se, and we hope it will lead us to find new dividing lines and develop new perspectives in classification theory and applications.

The main test problem for classical classification theory (Shelah, 1990a) was the *main gap conjecture* on the number of non-isomorphic models, but for unstable theories, we expect other kinds of structure theory relevant to other problems. We expect our test problems to drive us to discover the relevant positive/structure theory.

We shall also consider other test questions — the well-established Keisler order (see Malliaris and Shelah, 2016b, 2016c), on saturation of ultrapowers, but also $\leq_{\mathrm{SP}}$ and the exact saturation problem. As a prototypical algebraic example, we shall consider *ACFA*, the algebraically closed fields with a distinguished automorphism, PAC fields (= every variety which is indecomposable [even in bigger fields] has a solution).

Note that being saturated is central in model theory and problems about the existence of saturated models drove us to introduce forking, which has been central also for model theorists without interest in set theory. The existence of universal models is a fundamental problem, considered in various contexts, which is model theoretic in nature and stood high in the beginning of model theory, see the classical (Jónsson, 1956, 1960; Morley and Vaught, 1962), but received less attention in later developments. We believe that now is the time for making real advances.

§8 presents the main test problem. In §9 we present classifications related to the universality spectrum, most notably Keisler's order with the relevant recent breakthroughs, and to other dividing lines, also on other test problems. In §10 we turn to countable universality ($\lambda = \aleph_0$), notably in the case of graphs. §11 talks about connections with combinatorics (both finite and infinite). In §12 we turn to results on the positive side: the consistency of existence (under a strong negation of GCH) of universal models for certain theories and families of theories.

In each section, we describe the state of the art and some known theorems, as well as the new terrain, our plans for the future, in the form of problems and conjectures. We also postulate several theses (including informal statements, often expressing my personal opinion).

8. A New Test Problem: Universality

Our main test problem is the universality spectrum of a class of models, but first we fix some notation:

Convention 8.1. 1) Fix a vocabulary τ (countable for simplicity); K will denote a class of τ-models.

2) If not said otherwise, an embedding of $M_1 \in K$ into $M_2 \in K$ is an isomorphism from M_1 onto a submodel of M_2.

3) K_λ is the class of $M \in K$ of cardinality λ.

4) If not said otherwise, our class is elementary, i.e., it is the class K_T of models of a first-order theory T in the vocabulary τ, and for transparency we assume[1] that it has the joint embedding property (JEP) and the amalgamation property (AP) and (see below) if T is complete then it has elimination of quantifiers.

5) We will generally not distinguish between the class of T-models and T itself.

---

1 Why this assumption? Because otherwise in general the existence results for $\lambda = 2^{<\lambda} > \aleph_0$ fail and we run into problems of a different character.

6) Let Th(M) be the theory of M, i.e., the set of first-order sentences that M satisfies.

A few comments to this convention: Elementary embeddings are central in model theory. But with respect to our problem, we can expand the vocabulary by adding predicates for every first-order formula; then the notions of embedding and elementary embedding coincide and T has amalgamation and the JEP. So elementary universality (universality with respect to elementary embeddings) can be viewed as a special case of universality.

In some cases, we should consider other notions of embeddability and/or replace cardinality e.g., by density (most notably in Banach spaces).

Many interesting classes are not elementary (e.g., locally finite groups, Banach spaces), and we shall consider some of them later. But we believe that it will help to first focus on the first-order context and already understanding the first-order T's forms a hard enough challenge.

Convention 8.2. For transparency:

- In the general definitions and results (but not the specific examples) we assume that the theory T is complete.
- Formulas if not said otherwise are quantifier free and denoted by φ, ψ, ϑ.

Now there are many yardsticks to measure the complexity of a theory T, and many possible test questions. We concentrate on the following:

The Main Definition 8.3. (For the test problem).

1) We say that $M \in K$ is universal in λ (for the class K) if for any $N \in K_\lambda$, N can be embedded into M.

2) We say that $M \in K$ is universal when it is universal in the cardinality $\|M\|$ of M.

3) Let $\mathrm{univ}_K = \mathrm{univ}(K)$, the universality spectrum of K, be the class of cardinals $\lambda > \aleph_0$ such that there is a universal $M \in K$ of cardinality λ (we shall/may speak about $\lambda = \aleph_0$ separately).

4) Set $K_1 \leq^0_{\mathrm{univ}} K_2$ if $\mathrm{univ}(K_1) \supseteq \mathrm{univ}(K_2)$.

5) Let $\mathrm{univ}_K(\cdot)$ be the class function $\mathrm{univ}_K(\lambda) = \mathrm{Min}\{\mu$: there is a sequence $\bar{M} = \langle M_\alpha : \alpha < \mu \rangle, M_\alpha \in K_\lambda$ and $\bar{M}$ is universal$\}$, i.e., every $N \in K_\lambda$ can be embedded into some M_α.

We may have chosen to use $\mathrm{univ}'_K(\lambda)$ instead of $\mathrm{univ}_K(\lambda)$ where $\mathrm{univ}'_K(\lambda) = \mathrm{Min}\{\mu$: there is $M \in K_\mu$ universal for $K_\lambda\}$ and even $\mathrm{univ}_K(<\lambda) = \mathrm{Min}$

$\{\mu$: there is $M \in K_\mu$ universal for models from $K_{<\lambda}\}$. Note that for $\mu < \lambda^{+\omega}$, we have $\text{univ}_K(\lambda) = \mu \Leftrightarrow \text{univ}'_K(\lambda) = \mu$ so for the most interesting case here, $\mu = \lambda^+$, there is no difference.

We will also consider (see Problem 10.21) the case of universal models in $\aleph_0$ for the class of graphs omitting a given one, but in general we concentrate on uncountable cardinals λ. Some mathematicians are a bit sceptical toward the *Cantor paradise*, i.e., they are not fond of uncountable cardinals; however, they can still enjoy its fruits, that is, the positive finitary theory for the whole family of K_T's with a "low, understandable" place in the map of all theories.

Until recently, I have not considered universality spectrum to be a good candidate for a dividing line. There were scattered results: sufficient conditions for non-existence and various independence results, but now we feel the time is ripe for dealing with a general theory.

We shall discuss below also other related dividing lines.

The notion of universality is a natural and interesting notion, which has come up in various contexts, see comments above after Problem 2.5.

Now Gödel told us that maybe $\lambda = 2^{<\lambda}$ for every λ (as it follows from GCH); so in this case, univ_T is always maximal, i.e., $\{\lambda : \lambda > \aleph_0\}$. But Cohen told us GCH may fail. This leads to refine the order $\leq_{\text{univ}}$ by demanding it not only in the present universe but in others. The obvious choice is in forcing extensions. Also, some quite trivial T have no universal in $\aleph_1$ when $\aleph_1 < 2^{\aleph_0}$, so we will usually require $\lambda \geq 2^{\aleph_0}$, though we may consider specific small values of λ separately.

Definition 8.4. 1) For Θ a class of cardinals we define: Let $\leq_{\text{univ},\Theta}$ be the following quasi order: $T_1 \leq_{\text{univ},\Theta} T_2$ <u>iff</u> in every forcing extension $\mathbf{V}_0$ of our universe and for every cardinal $\lambda \in \Theta$ (that is $\mathbf{V}_0 \models \lambda \in \Theta$) we have that $\lambda \in \text{univ}(T_2) \Rightarrow \lambda \in \text{univ}(T_1)$. We may omit Θ if it is $\{\lambda : \lambda \geq 2^{\aleph_0}\}$.

2) K is almost $\leq_{\text{univ}}$-maximal if for any $\mathbf{V}_0$ as above it has no universal model in λ for every λ, μ with $\mu^{++} = \lambda = \lambda^{\aleph_0} < 2^\mu$.

3) K is pseudo $\leq_{\text{univ}}$-maximal when in any $\mathbf{V}_0$ as above, if $\mu^{++} = \lambda = \lambda^{\aleph_0}, \lambda^+ < 2^\mu$ then no $M \in K_T$ of cardinality λ^+ is universal for λ, T.

What is the rationale for choosing this definition of almost $\leq_{\text{univ}}$-maximal/minimal? Why $\lambda = \mu^{+2}$ and not $\lambda = \mu^+$ or $\lambda = \mu^{+3}$? Restricting ourselves to such λ's, we can prove many cases of $\leq_{\text{univ}}$-maximality (or -minimality), although we are far from a complete answer, see later. E.g., demanding $\lambda = \lambda^{\aleph_0}$ makes all stable T form one equivalence class. But for $\lambda < \lambda^{\aleph_0}$ there are some independence results, and it seems that we get very fine distinctions. It is of course still desirable to find what occurs at the other cardinals. Maybe advances will make us change the restrictions on λ, or even eliminate them.

Note that $\leq_{\mathrm{univ},\Theta}$ is a notation which helps us express what we know, e.g.,

- T $\aleph_0$-stable $\Rightarrow$ T is $\leq_{\mathrm{univ},\{\lambda:\lambda\geq\aleph_0\}}$-minimal.
- T superstable $\Rightarrow$ T is $\leq_{\mathrm{univ},\{\lambda:\lambda\geq 2^{|T|}\}}$-minimal.
- T stable $\Rightarrow$ T is $\leq_{\mathrm{univ},\{\lambda:\lambda=\lambda^{|T|}\}}$-minimal.

While forcing is a very successful and well-established method in set theory, one might wonder why we expect to get a natural and useful notion if we commit ourselves to this specific technique in the definition. The answer is that this choice should not be important: Forcing is "good enough", or said otherwise, potential counterexamples are "robust enough":

Thesis 8.5. The order $\leq_{\mathrm{univ}}$ is robust: We should get basically the same results if instead of forcing we talk, e.g., about provability in ZFC or other natural notions of "necessary".

(But talking about forcing extensions is much more convenient and well defined.)

9. Our Aim: The Classification Question and Universality Spectrum

9.1 Dividing lines

Underlying this aim is the following thesis:

Thesis 9.1. 1) It is worthwhile to classify theories, i.e., finding dividing lines or dichotomies.

2) Good test problems help us to find the right dividing lines.

In more detail: We look for properties P of a theory T such that we have relevant information on both sides: about those theories which have property P and about those which do not have it. Usually one side (having P, say) tells us that T is analysable, or not too complicated in a certain way, and that we can develop a "positive" theory, i.e., a structure theory for these theories. Theories that fall on the other side will usually fail the above structure theory in a strong way, and in general exhibit various non-structure properties.

We have given some names to this intuition in Definition 1.2.

Examples illustrating those notions (others are covered later):

$\bullet_1$ Robustness:

(a) T is classifiable, (that is, T is superstable, NDOP, NOTOP and shallow) <u>iff</u> $\dot{I}(\aleph_\alpha,T)<\beth_{\omega_1}(\alpha)$ in every forcing extension (see Shelah, 1990a, Ch.

XII, Th. 6.1), as all the properties mentioned there are absolute which say that e.g., extending the universe by forcing does not change their truth value. If you do not like the forcing extension

(b) small IE: recalling $\mathrm{IE}(\lambda,T)=\sup\{\mathscr{M}|:\mathscr{M}\subseteq K_\lambda^T$ is a set of pairwise non-embeddable models$\}$, we get a trichotomy (see Shelah, 1990a, Ch. XIII, Th. 2.1, p. 627)

- (α) T is superstable, NDOP, NOTOP and shallow iff $\mathrm{IE}(\aleph_\alpha,T)<\beth_{\omega_1}(\alpha)$ for every α.
- (β) T is unsuperstable or with DOP or with OTOP then $\mathrm{IE}(\aleph_\alpha,T)=2^{\aleph_\alpha}$ for every $\alpha = 0$.
- (γ) if T is superstable NDOP, NOTOP and deep (=not shallow) then IE ($\aleph_\alpha$, T) is the first beautiful cardinal if $\aleph_\alpha$ is at least this cardinal and is $2^{\aleph_\alpha}$ before it, see more Baldwin (2021).

(c) T is stable iff every so-called $\aleph_1$-resplendent model is saturated (see Shelah, 2011).

(d) T is SOP_2 iff T is $\triangleleft^*$-maximal (see Malliaris and Shelah, 2016a).

$\bullet_2$ (Internal) successfulness; we know a lot on superstable and on stable, simple, dependent and NTP_2.

$\bullet_3$ External successfulness, non-structure results: mainly many non-isomorphic models as above; and constructing complicated models for infinitary logics; see Hyttinen and Tuuri (1991), Hyttinen et al. (1993), Hyttinen and Shelah (1994, 1995, 1999), Laskowski and Shelah (2001, 2003) and Shelah (2008).

$\bullet_4$ Fruitfulness: Hrushovski's work using stability of separably closed fields.

$\bullet_5$ Versatility: There is much on it in the so-called neostability works. A nice example is of versatility is the class of K^{lf} of locally finite groups (and K_{exlf} the existentially closed sub-class). On the one hand Macintyre and Shelah (1976) use Shelah (1990a, Ch. VIII) to prove $I(\lambda, K) = 2^\lambda$ for uncountable λ, though K_{exlf} is not first order; so a point for externally successful. On the other hand, by Shelah (2017a) there is a dense set of definable types, so a rudimentary non-forking frame holds, hence we can construct canonical existentially closed extension of $G \in K_{\mathrm{lf}}$ without changing the cardinality.

See more Eklof and Mekler (2002) on black boxes which start in Shelah (1990a, Ch. VIII), also see Göbel and May (1990) about constructing "elevators".

The general program of the search for dividing lines and investigation of the structural consequences and non-structural consequences is for me "classification theory", a two-faced Janus; but many model theorists refer only to the structure side.

How do we envision the positive/structure theory? A thesis underlying our approach is:

Thesis 9.2. Trying to crack good test problems will drive us to the right inside theory.

We consider the stable/unstable dividing line a success (discussed in more detail in §9(C) below). Many technical tools (such as non-forking) were originally developed for addressing classification theory questions, such as existence of saturated models and counting models up to isomorphism and defining regular types, dimension and p-weights, see Shelah (1990a, V, §4). However, continuing this and works of Zilber, careful geometric analysis has led to powerful applications (e.g., Hrushovski, 1996; Hrushovski and Zilber, 1996) and building bridges with other areas of mathematics, such as algebraic and Diophantine geometry, and combinatorics. Attempts to generalize these tools to wider (unstable) contexts has led to major progress and breakthroughs in e.g., simple theories (also discussed in section II) and more applications in other areas of mathematics.

It is natural to try to generalize stability, but we feel that there is something missing from the emerging picture, even in the case of simple theories where this was highly successful. Perhaps investigating new test questions and examining the tools that come out of this investigation will shed more light on it. For example, the free amalgamation (given by non-forking) loses its strong uniqueness property outside of the stable context. It is natural to conjecture that uniqueness and existence of amalgamation (of finite diagrams, cf. 14.1) are the keys to a better understanding of different classes of theories. These properties are at heart of the hopeful investigation of simple theories with respect to our test question (see §12).

It is not a priori clear that there should be robust, even successful dividing lines at all. Some appealing (at least to me) candidate test problems have not led to very good dividing lines, at least so far. But for instance, the properties stable/superstable (see Definition 9.10) have a very strong support (i.e., being robust and successful on the one hand, and versatile and fruitful; i.e., attracting researchers and applications on the other hand). Also the property of being simple has strong support, see below, as well as dependent (=NIP) (Simon, 2015) and NTP_2 (Chernikov, 2014).

There are some properties which we believe are dividing lines but remain candidates, and unfortunately, we have only weak evidence so far, and some have none, such as $NSOP_n$ for $n > 4$. Maybe they are just approximations to the right properties, which are yet to be found. To clarify and show some of the most important dividing lines currently known consider:

See Figure 1 for some of the known dividing lines and candidates, and the implications between them (see http://forkinganddividing.com for an interactive map with examples).

Thesis 9.3. Generalizing structure theory which works for specific cases is a fruitful approach. But it may miss some phenomena which are not apparent in the usual specific cases (not all group theory is generalizing some theorems on Abelian groups). In particular, in model theory, looking at the known examples may be misleading. A serious test problem is a good way to discover them.

9.2 Universality order and dividing lines

Considering the universality spectrum leads us to a specific area in the diagram of dividing lines, which is currently a part of the "terra incognita". There are other relevant (and related) test problems, the most prominent and old of them is Keisler's order $\triangleleft$ on saturation of ultrapowers and its relative $\triangleleft^*$.

Let us elaborate on what is currently known on these connections.

By Shelah (1980a), consistently $\aleph_1 < 2^{\aleph_0}$ and there is a universal linear order in $\aleph_1$, but by Kojman and Shelah (1992a) we have:

Theorem 9.4. 1) *The class of linear orders (and thus many other T's) is almost* $\leq_{\rm univ}$*-maximal, (even* $\lambda = \mu^{++} < 2^\mu$ *suffices and less), that is, it is* $\leq_{{\rm univ},\{\lambda:(\exists\mu)(\lambda=\mu^{++}=\lambda^{\aleph_0})\}}$*-maximal.*

2) *Moreover, the* StOP *(= the strict order property) suffices (*StOP *means that some formula* $\varphi(\bar{x}_n, \bar{y}_n)$ *defines a partial order in every model of T, and it has an infinite chain in some model of T).*

This is annoying! We know the answer for many cardinals but not for all. This leads to:

Problem 9.5. 1) Is there a T such that ${\rm univ}_T = \{\lambda : \lambda = 2^{<\lambda} > \aleph_0\}$, provably in ZFC, that is, are there $\leq_{\rm univ}$-maximal T's, and for them ${\rm univ}_T$ is the minimal possible class (by the classical results).

2) Can we characterize the $<_{\rm univ}$-maximal T's (cf. Definition 8.4)? Is this a good dividing line?

3) Can we at least characterize the almost $\leq_{\rm univ}$-maximal T's? Does this give a good dividing line?

4) Is there a better notion of "almost $\leq_{\rm univ}$-maximal" which leads to good dividing lines?

The StOP seems a good candidate for being a dividing line, and we had believed it may match with being almost $\leq_{\rm univ}$-maximal. However (see Shelah, 1996b, §2):

Theorem 9.6. *The* SOP_4 *(see definition below) suffices in Theorem 9.4.*

Definition 9.7. 1) For $n \geq 3$, T has SOP_n if there is a formula $\varphi(\bar{x},\bar{y})$ that defines on some model of T a directed graph with an infinite chain and no cycles of length $\leq n$.

2) We write NSOP_n for the negation of SOP_n.

A prototypical class for SOP_n (for $n \geq 3$) is the class of directed graphs with no ($\leq n$)-cycle.

The SOP_n strengthen instability (the order property, Definition 9.10) and are natural approximations to the strict order property. They look like candidates for being good dividing lines, but so far we have little evidence, unlike stability (see Baldwin, 1988; Shelah, 1990a; Shelah, 1996b, §2). So is the SOP_4 the right answer for universality? Here we have some encouraging signs and weak consistency results for some SOP_3 theories, but quite recently, it was proved:

Theorem 9.8. *The results of Theorem 9.4 hold for any T with the so-called olive property (see Shelah, 2016).*

Out of compassion for the reader, we heed advice to delay to the end of §11 the rather involved definition of the olive property (Shelah, 2016).

Is the olive property really a good candidate for a dividing line? Formally, it is of the same form as unstable or SOP_4, but it looks cumbersome and there are many variants with the same properties (with respect to universality). Hence, it is not a reasonable candidate; so why consider it in the first place? First, it is a property with serious consequences on the universality spectrum. Second, examples show that there are (not complete) T's which satisfy the olive property, are NSOP_4 but SOP_3. So NSOP_4 is not the relevant dividing line for us.

In fact, one such example is the class of all groups known to have NSOP_4 and SOP_3 (Shelah and Usvyatsov, 2006) and previously believed to have more universal objects. But alas, this class has the olive property (Shelah, 2016) (exemplified by quantifier free formulas), hence surprisingly, there are "very few" universal groups. More precisely, Theorem 9.4 applies to the class of all groups.

Problem 9.9. Are there consistency results showing all NSOP_3 theories T are strictly $<_{\mathrm{univ}}$-below T_{dlo}?

We still believe there is a nice dividing line lurking here. Certainly, we believe that for "nicer" T's we can get more universals.

9.3 Stable theories

So we are led by Theorems 9.4, 9.6, 9.8 to classes which in a sense have no partial order, so we first consider:

Definition 9.10. 1) T is not stable iff T has the order property which means: some formula $\varphi(\bar{x},\bar{y})$ defines on some model of T a directed graph with an infinite chain (e.g., for the class of random graph the formula $\varphi((x_0,y_0),(x_1,y_1)) \equiv x_0 R y_0$).

2) T is not superstable iff there are $\varphi_n(x,\bar{y}_{k(n)})$ for $n \in \mathbb{N}$ and $M \in K_T$ and $b_\eta \in M, \bar{a}_\eta \in {}^{k(n)}M$ for $\eta \in \cup_n {}^n\mathbb{N}$ such that $M \vDash \varphi_n[b_\eta, \bar{a}_\rho]$ iff $\rho \triangleleft \eta$.

3) T is called strictly stable when it is stable and not superstable.

Some examples:

1) Prototypical superstable T's are: algebraically closed fields of fixed characteristic, T_{eq} (the theory of one equivalence relation), the theory of planar graphs, and $T_{\mathrm{eq}(I)}$, for I a countable well ordering, where:

(*) for a countable linear order I, $T_{\mathrm{eq}(I)}$ says E_s is an equivalence relation for $s \in I$ and E_t refines E_s for $s <_I t$.

2) One prototypical example of a strictly stable theory is the theory of separably closed fields of positive characteristic.
3) Another prototypical strictly stable T is the theory of free groups. This is a deep result of Sela (Sela, 2013). This theory is still under investigation, and several advanced model-theoretic questions about it remain unsolved, see Pillay (2008, 2009).

Now stable K_T are understandable; they have an internal notion of independence or non-forking (see below), and dimension. When applied to other parts of mathematics, e.g., replacing, say, the family of varieties (i.e., sets definable by equations) by a bigger collection of definable sets (definable by first-order formulas), with better closure properties, enabling e.g., results on the Mordell–Lang conjecture (Hrushovski, 1996; Bouscaren, 1998; Scanlon, 2001) (building on Hrushovski and Zilber, 1996) and on approximate groups (Hrushovski, 2012) continued by Breuillard et al. (2012) and non-Archimedean geometry (Hrushovski and Loeser, 2016). For simple K_T's we have similar but weaker results (Grossberg et al., 2002; Casanovas, 2011) and applications (e.g., Cherlin and Hrushovski, 2003).

Why do we say stable T's are analysable? Stable theories admit a notion of free amalgamation (also called "independence") arising from non-forking, i.e., we say

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that $\{A_s : s \in S\}$ is independent over A inside M when they are subsets of M and the type of A_s over $\cup\{A_t : t \neq s\} \cup A$ does not fork over A. This notion is particularly well behaved:

Theorem 9.11. • *Preservation: Independence is preserved by elementary extensions and elementary submodels (of the ambient model).*

- *Basis free*: $\{A_s : s \in I\}$ *is independent over* A *iff* $\{A_s \cup A : s \in I\}$ *is independent over* A.
- *Localness*: $\{A_s : s \in I\}$ *is independent over* A *iff* $A_s, \cup_{t \in u} A_t$ *is independent over* A *for every* $s \in I$ *and finite* $u \subseteq I \setminus \{s\}$.
- *Symmetry*: B, C *are independent over* A *in* M *iff* C, B *are independent over* A *in* M.
- *Transitivity: If* $A_0 \subseteq A_1 \subseteq A_2$ *and* B, $A_{\ell+1}$ *are independent over* A_ℓ *in* M *for* $\ell = 0, 1$ *then* B, A_2 *are independent over* A_0 *in* M.
- *Finitary character*: B, C *are independent over* A *in* M *iff* B', C' *are independent over* A *for every finite* $B' \subseteq B$, $C' \subseteq C$.
- *Countable base: If* B *is finite and* $C \subseteq M$ *then for some countable* $A \subseteq C$ *the sets* B, C *are independent over* A.
- *Monotonicity: If* $A'_s \subseteq A_s$ *and* $\vee_{s \in I} A \subseteq A' \subseteq A'_s \subseteq A_s$ *and* $\{A_s : s \in I\}$ *is independent over* A *then* $\left\{A'_s : s \in I\right\}$ *is independent over* A'. *(Follows by finite character and localness when* $A = A'$.*)*

Given such a well-behaved notion of amalgamation, one expects that, for stable classes, the universality spectrum is nice. Indeed Shelah (1990a, Ch. II) and Kojman and Shelah (1993):

Theorem 9.12. 1) *If* T *is superstable, then*:

(*a*) T *is* $\leq_{\mathrm{univ},\left\{\lambda:\lambda \geq 2^{\aleph_0}\right\}}$*-minimal*;

(b) if $\lambda \geq 2^{\aleph_0}$ *then* T *has a universal model of cardinality* λ*, in fact, a saturated one.*

2) If T *is strictly stable (i.e., stable not superstable), then* T *is almost* $\leq_{\mathrm{univ}}$*-minimal, in fact*:

(a) if $\lambda = \lambda^{\aleph_0}$*, then* T *has a universal model of cardinality* λ*, in fact, a saturated one*;

(b) if $\lambda = \sum_n \lambda_n^{\aleph_0}$*, equivalently if* $(\forall \alpha < \lambda)(|\alpha|^{\aleph_0} \leq \lambda)$*, then* T *has a universal model of cardinality* λ*, in fact, when* $\lambda^{\aleph_0} > \lambda$*, a so-called special one.*

3) *If* T *is strictly stable (as in Definition 9.7(2)) and* $2^{\aleph_0} < \mu < \lambda < \mu^{\aleph_0}$ *and e.g.,* $\lambda = \mu^{++}$ *then* $\lambda \notin \mathrm{univ}_T$.

Also, the so-called oak property has similar consequences, see Džamonja and Shelah (2006), Shelah (2017b).

Problem 9.13. 1) Are all strictly stable T's pairwise $\leq_{\text{univ}}$-equivalent?

2) Or does $\leq_{\text{univ}}$ reveal dividing lines among them?

3) Can we in 9.12(3) allow $\lambda = \mu^+$? Can we just require $\lambda \in (\mu, \mu^{\aleph_0})$? Maybe it is more a forcing problem.

We need on the one hand to prove more on the relevant forcing; and on the other hand, further investigate strictly stable T's (which is of independent interest, cf. e.g., the theory of free groups).

Problem 9.14. 1) Find sufficient conditions on T_1, T_2 for $\lambda \in \text{univ}(T_1) \Leftrightarrow \lambda \in \text{univ}(T_2)$ at least when "λ is reasonable", meaning:

(*) $\aleph_0 < \mu = \mu^{<\mu} < \lambda = \lambda^{<\mu} < 2^\mu$ for some μ.

2) In particular, for T_1= the theory of random graphs.

Why do we call those cardinals "reasonable"? Because for those cardinals we have a way to force complementary consistency results, not involving large cardinals.

9.4 Simple theories

Simple theories were originally introduced in Shelah (1980b) as a natural candidate for a dividing line having a reasonable theory of non-forking. In the 1990s, due to the natural examples found (e.g., smoothly approximable structures (Cherlin and Hrushovski, 2003), ACFA: algebraically closed fields with a generic automorphism (Hrushovski, 2004)) and the discovery that non-forking behaves well in simple theories (Kim, 1996; Kim and Pillay, 1998), the study of simple theories was a very active area. Many properties of non-forking independence generalize from stable theories to this context. For more, see (Grossberg et al., 2002; Casanovas, 2011; Kim, 2014). In recent years, the study of dependent =NIP theories has been more popular than the study of simple ones (although there is still research on simple theories e.g., Shami, 2015), but we believe that there is much more to investigate in simple theories.

Definition 9.15. 1) T is non-simple when there is a (quantifier free for our T) formula $\varphi(\bar{x},\bar{y})$ and model M of T and $\bar{b}_\eta \in {}^{\ell g(\bar{x})}M, \bar{a}_\eta \in {}^{\ell g(\bar{y})}M$ for $\eta \in \cup_n {}^n\mathbb{N}$ such that:

(a) if $\eta \in {}^{\omega>}\omega$ then $\varphi\left(b_\eta, \bar{a}_{\eta|'n}\right)$: for $n < \ell g(\eta)$;

(b) if $\eta \in {}^{\omega>}\omega$ then $\left\{\varphi\left(\bar{x}, \bar{a}_{\eta^\wedge\langle n\rangle}\right) : n < \omega\right\}$ are pairwise contradictory.

2) T is dependent iff it is not independent which means that some formula $\varphi(\bar{x}_n, \bar{y}_n)$, in some $M \in K_T$ defines a graph on ${}^nM$ such that any finite graph is embedded into it.

3) A prototypical simple T is the theory T_{rg} of random graphs.

4) T_{ceq} seem to me a minimal non-simple theory; fully: a model M is a model of T^0_{ceq} iff P^M, Q^M is a partition of the model, E^M is an equivalence relation on Q^M and $F^M : Q^M \times P^M \to P^M$ satisfies for each $b \in P^M$ the function $x \mapsto F^M(x, b)$ is a choice function for E^M. (Earlier a relative T_{feq} was used, for our purpose there is no difference.)

5) T_{dlo}, the theory of dense linear order, is prototypical for dependent. Other examples include the theory of the real field (and each p-adic field).

There has been considerable amount of work using non-forking (as said, a kind of free amalgamation). In fact, amalgamation has been (naturally) of crucial importance in proofs of (consistent) existence of universal objects.

Let us give some examples of positive results of this kind, which signify the role of simple theories (and also some of the other dividing lined mentioned above) and various kinds of amalgamation in the picture.

Assume (see below) $\mu = \mu^{<\mu}$, first for $\lambda = \mu^+$. By Shelah (1990b) we can force that there is a universal graph in λ while $\lambda < 2^\mu$ not collapsing cardinals, nor adding sequences of length $<\mu$. Moreover, if $\mu = \mu^{<\mu} < \lambda < \chi = \chi^\lambda$ (and GCH holds in $[\mu, \chi)$) then we can force a universal graph in λ, not collapsing cardinals and preserving ${}^{\mu>}\mathrm{Ord}$. Subsequently, Mekler (Mekler, 1990) proves that, for universal T closed under submodels without algebraicity and with $\mathscr{P}^-(3)$-amalgamation (see Definition 14.1), there is a universal model of cardinality λ for $\lambda = \mu^+$, and that there is one for every λ assuming both $\mathscr{P}^-(3)$ and $\mathscr{P}^-(4)$ amalgamation. This is what the proof gives; it is not clear for which T's we cannot force this.

Can simple theories be $\leq_{\mathrm{univ}}$-maximal? This is unsettled, but Shelah (1996b, §1) gives the consistency of "weak universal". The consistency of the existence of "weak universal" where.

Definition 9.16. We say K has a weak universal in $\lambda < 2^{<\lambda}$ when there is $M \in K_{\lambda^+}$ into which every $N \in K_\lambda$ can be embedded (equivalently, in our context of $\mathrm{univ}_K(\lambda) \leq \lambda^+$).

This does not register in the universality spectrum but comes fairly close in terms of the function $\mathrm{univ}_T(\cdot)$ taking λ to the minimal cardinality of a λ-universal model. So $\mathrm{univ}_T(\lambda) = \lambda$ corresponds to the universality spectrum, and $\mathrm{univ}_T(\lambda) = \lambda^+$ is a close approximation of interest when $\lambda^+ < 2^\lambda$. Moreover, the same result applies also to $T_{\rm tfr}$, (the class of triangle free graphs) and $T_{\rm ceq}$ see Definition 9.15(4). Both are not simple and are NSOP_4. But $T_{\rm ceq}$ is NSOP_2 while $T_{\rm tfr}$ has even SOP_3. This is a weaker form of "there is a universal", and although it is not what we really want, it still indicates that there is a difference. So certainly this tells us that being simple is relevant, but not fully satisfying (from this point of view).

Problem 9.17. 1) Can we force $\lambda < 2^{<\lambda}$ + "$\lambda \in \mathrm{univ}_T$ for all simple T"?

2) Is there a simple T such that: if $\aleph_0 < \mu = \mu^{<\mu} < \lambda = \mathrm{cf}(\lambda) = \lambda^{\aleph_0} = \chi^{++} < 2^\mu$ then $\lambda \notin \mathrm{univ}_T$? This may require adding $\mu^+ < \lambda$.

3) What are the $\le_{\rm univ}$-minimal unstable T's? Should they be similar enough to $T_{\rm rg}$?

4) What are the $\le_{\rm univ}$-maximal among simple T's? This should throw new light on the family of simple T's.

5) What about other dividing lines for non-simple T's? Recently, there has been work on NSOP_2 theories being a robust dividing line (discussed in detail in the next subsection), and on NSOP_1, see §9(F).

6) Are all non-simple T almost $\le_{\rm univ}$-maximal? Or at least pseudo $\le_{\rm univ}$-maximal? Seems too optimistic, still.

9.5 Keisler's order and saturation of ultrapowers

Definition 9.18. We define SOP_n for $n = 1, 2$.

(1) T has SOP_2 iff there is $\varphi(x,\bar y)$, $M \in K_T$ and $\bar a_\eta \in {}^{\ell g(\bar y)}M, b_\eta \in M$ for η a finite sequence of zero's and one's such that:
   - (a) $\varphi(b_\eta, \bar a_{\eta \restriction \ell})$ for $\ell < \ell g(\eta)$;
   - (b) $\varphi(x, \bar a_{\eta_0}), \varphi(x, \bar a_{\eta_1})$ are contradictory when $\eta\hat{\ }\langle \ell \rangle \trianglelefteq \eta_\ell$ for $\ell = 0, 1$.

(2) T has SOP_1 <u>iff</u> as above but in (b) we add $\eta\hat{\ }\langle 1 \rangle = \eta_1$.

We define the partial order $\lhd$ and its relative $\lhd^*$ as follows:

Definition 9.19. Let D be an ultrafilter on a set I.

(1) We define $N = M^I/D$ as the model with the same vocabulary (= kind) with set of elements $\{f/D : f \in {}^I M\}$ where $f_1/D = f_2/D$ means that $\{s \in I : f_1(s) = f_2(s)\} \in D$, and e.g., $N \models f/D = F(f_1/D, \ldots, f_n/D)$ <u>iff</u> $\{s \in I : f(s) = F^M(f_1(s), \ldots, f_n(s))\} \in D$.

(2) We say D is regular when, if $f_\alpha/D \in M^I/D$ for $\alpha < |I|$ then for some finite $w_s \subseteq M$ for $s \in I$, we have $\alpha < |I| \Rightarrow \{s \in I : f_\alpha(s) \in w_s\} \in D$.

(3) $T_1 \trianglelefteq T_2$ iff for any set I and regular ultrafilter D on I if $M_\ell \in K_{T_\ell}$ and M_2^I/D is $|I|^+$-saturated then so is M_1^I/D (this order is known as the *Keisler order*).

(4) $T_1 \trianglelefteq^* T_2$ iff there is a model M in which T_1, T_2 are interpreted such that if $N \models \mathrm{Th}(M)$ and N_1, N_2 are the models of T_1, T_2 interpreted in N correspondingly and N_2 is λ^+-saturated then so is N_1.

In an initial bloom in the 1960s and 1970s, the stable case was sorted out (exactly two equivalence classes!); but afterwards little progress had been made on the Keisler order, until recently.

Note that $\triangleleft^*$ is more model theoretic, as for a proof of $\ntrianglelefteq$ we need to build suitable ultrafilters on the relevant infinite sets; however, we suspect that $\triangleleft^*$ is making too fine a distinction. Ideally, we would have finitely many equivalence classes; however, even for $\triangleleft$ any such hopes were shattered by (Malliaris and Shelah, 2018b), where it was proved that there is an infinite decreasing $\triangleleft$-chain of T's. Still:

Problem 9.20. Are there continuum many pairwise $\triangleleft$-incomparable T's?

Like in the case of $\leq_{\mathrm{univ}}$, the StOP seemed a candidate for $\triangleleft$-maximality, but alas, SOP_4 suffices at least for almost $\leq_{univ}$-maximal (Shelah, 1996b; Shelah and Usvyatsov, 2006). However by Džamonja and Shelah (2004b) and Shelah and Usvyatsov (2008b) if T is NSOP_2 then T is not $\triangleleft^*$-maximal. Well, assuming an instance of GCH.) The proof seems like a beginning of a positive theory.

Indeed, more recently, Malliaris and Shelah (2017, 7.14) proves the other direction: if T is SOP_2, then it is $\triangleleft^*$-maximal, a further suggestion of a positive theory. Together, we conclude that SOP_2 is a *robust* dividing line.

Interestingly enough, the results above were preceded by a breakthrough (Malliaris and Shelah, 2016b) telling us a somewhat weaker result: $\mathrm{SOP}_2 \Rightarrow \triangleleft$-maximal, and also, out of the blue, that $\mathfrak{p} = \mathfrak{t}$, i.e., that two classical cardinal characteristics of the continuum are equal. This is another example of how investigating "good" dividing lines may lead to interesting and unexpected discoveries in other areas of mathematics.

Problem 9.21.

(1) Prove that any NSOP_2-theory is not $\triangleleft$-maximal.

(2) Characterize the $NSOP_2$ theories T such that for every large enough μ, for every $\lambda > \mu$ there is a regular ultrafilter D on λ such that: if M a model of T, then M^λ/D is not μ^+-saturated.

Problem 9.22. Develop a structure theory for $NSOP_2$. That is, show that $NSOP_2$ is a *successful* dividing line.

9.6 $NSOP_1$ and exact saturation

Lately, there have been serious advances on $NSOP_1$, making $NSOP_1$ a successful dividing line. By Chernikov and Ramsey (2016), any theory satisfying a certain amalgamation criterion is $NSOP_1$, but it was not clear that the converse holds (i.e., that this amalgamation criterion holds for $NSOP_1$ theories). In Kaplan and Ramsey (2020), an independence relation called Kim-non-forking was introduced. This relation is defined in a similar way to non-forking, with the difference that dividing is now required to be witnessed by generic sequences. In particular, Kim-non-forking satisfies the so-called *independence theorem*. By Kaplan et al. (2018) Kim-non-forking satisfies local character in $NSOP_1$ (and also a dual phenomenon). Kruckman and Ramsey (2018) studies expansions of $NSOP_1$ theories; e.g., under mild assumptions, any $NSOP_1$ theory can be expanded to one with built-in Skolem functions. Examples of $NSOP_1$ theories include $T_{\rm feq}$ (Shelah, 1993b), $T_{\rm ceq}$ (see Definition 9.15(4)) the generic binary function (Kruckman and Ramsey, 2018), certain fields (e.g., ω-free PAC fields (Chatzidakis, 2002; Chernikov and Ramsey, 2016)), vector spaces with a generic bi-linear form (Chernikov and Ramsey, 2016) and the generic projective plane (Conant and Kruckman, 2017).

Another serious candidate for being a good test problem is:

Definition 9.23.

(1) T has the λ-exact saturation property when T has a λ-saturated model which is not λ^+-saturated.

(2) T has the λ-PC-exact[2] saturation property when for every $T_1 \supseteq T$ there is a model M of T which can be expanded to a model of T_1, it is λ-saturated, and it is not λ^+-saturated.

Model theoretically these give rise to reasonable test problems. Now, exact saturation was first dealt with in Shelah (2015a), where it was shown that for

2 PC refers to projective class.

unstable T's it holds for every regular $\lambda(>\aleph_0)$, and that for stable T's it holds for every λ; so we may as well assume that T is unstable and λ singular. For dependent theories T, the relevant dividing line turns out to be "T is distal" (Kaplan et al., 2017). Again, this result was reached by stages: Shelah (2015a) gives a sufficient condition for existence of universal models for dependent theories, and a sufficient condition for nonexistence for really complicated theories, e.g., Peano arithmetic.

However, for the PC-exact version the situation is different. On the one hand, SOP_2 implies that T does not have PC-exact saturation (for λ singular) by Malliaris and Shelah (2017); on the other hand for T simple, exact saturation holds for cardinals for which Jensen's square holds (Kaplan et al., 2017).

Conjecture 9.24. Any simple T has singular PC-exact saturation, at least for λ strong limit.

But more exciting is:

Problem 9.25. Is "singular PC-exact saturation" a good test problem, pointing to a dividing line between simple and $NSOP_2$? Is the right characterization $NSOP_2$ or $NSOP_1$?

This may involve continuing the work on $NSOP_1$, as well as developing it.

Back to $\leq_{\mathrm{univ}}$ we think that T_{ceq} is the prototypical and simplest among non-simple theories; it is $NSOP_1$, but is it actually minimal? Now (see Shelah, 1993b and Džamonja and Shelah, 2004a) we can force for it "there is a universal in μ^+ for μ of cardinality $\mu^{++} < 2^{\mu}$", similar to the case of simple theories. Again, still this gives only an indication (and not a proof) that it is not maximal. On the other hand, it is prototypical for the oak property, which gives sufficient conditions for non-existence near a singular cardinal. Like strictly stable T's; "the oak property" was introduced in Džamonja and Shelah (2006); see more in Shelah (2017b). The oak property, though of the right form, like SOP_n, $n \geq 3$ and StOP is as of now not supported by robust evidence for being a robust dividing line.

Problem 9.26.
(1) Can we force fully $\mu^+ \in \mathrm{univ}(T_{\mathrm{ceq}})$, $\mu = \mu^{<\mu}$, $\mu^+ < 2^{\mu}$.
(2) Is T_{ceq} really prototypical for TP_2 theories? Is it $\leq_{\mathrm{univ}}$-minimal among them?
(3) Can we prove for T_{ceq} non-existence results like in the case of SOP_4 theories?

Note that the family NTP_2 is too big for our program, as it includes theories that are clearly too complicated (by the present yardstick), like T_{dlo}, the theory of dense linear orders. So we will need to restrict ourselves to a subfamily.

10. Specific Classes (Not Necessarily Elementary)

10.1 Examples

We consider some classical classes. For which of those specific classes can we prove non-existence of universals?

(A) linear orders: recall that $K_{\text{linear order}}$ is almost $\leq_{\text{univ}}$-maximal but consistently $\aleph_1 < 2^{\aleph_0}$ and there is a universal linear order in $\aleph_1$; we will later discuss the remaining cardinals;

(B) Boolean algebras and ordered fields (which are like linear orders for non-existence);

(C) the theory of the random graph is $\leq_{\text{univ}}$-minimal among the unstable theories; so for reasonable cardinals (cf. Problem 9.14) they may have universals. But, e.g., for $\lambda = \beth_\omega^+ < 2^{\beth_\omega}$ we have only weak consistency results: some graph of cardinality $\lambda^+ < 2^{\beth_\omega}$ is universal for λ (see Džamonja and Shelah, 2003; Cummings et al., 2017);

(D) T_{trf}, the theory of triangle free graphs, we know only of cases (of consistency) of weak universality (as in Shelah, 1993b; Džamonja and Shelah, 2004a).

(E) K_{groups}, the class of groups has the non-existence results of $K_{\text{linear order}}$ (for universality), though it has more amalgamation properties (in our framework, is $NSOP_4$). We now denote (see Shelah and Usvyatsov, 2008a)

(F) Abelian groups, there are several natural versions, a major one is the class of torsion-free Abelian groups under pure embeddings (i.e., non-divisibility by a natural number is preserved), it seems a prototypical case for strictly stable T's in general and specifically to trees with ω- levels; on this we have much to say (see Fuchs, 1970a, 1973; Kojman and Shelah, 1995; Shelah, 1997 and §10(B));

We shall consider more examples below.

Note that many relevant results in ZFC have roots in "difficulties" in extending independence results (see e.g., §10(B) below) thus providing a case for:

Thesis 10.1. Even if you do not like independence results, you better look at them, as you will not even consider your desirable ZFC results when they are camouflaged by the litany of many independence results. Once forcing gets the rubbish out of the way, you can try to find diamonds.

Of course, independence has interest per se; still in a given problem in general a solution in ZFC is for me preferable over an independence result. But if it gives a method of forcing (so relevant to a series of problems), the independence result is preferable (of course, we assume there are no other major differences; the depth of the proof would be of first importance).

As occurs often in the author's papers, references to pcf theory appear.

This paper is also a case of the following thesis:

Thesis 10.2. Assumption of cases of the negation of GCH at singular (more generally $pp\lambda > \lambda^+$) are "good", "helpful" assumptions; i.e., traditionally uses of GCH proliferate mainly not from conviction but as you can prove many theorems assuming $2^{\aleph_0} = \aleph_1$ but very few from $2^{\aleph_0} > \aleph_1$, but assuming $2^{\beth_\omega} > \beth_\omega^+$ is helpful in some proofs, and see §10(B) on $2^{\beth_\omega} > \beth_\omega^{++}$.

10.2 Abelian groups

The class of Abelian groups (for usual embeddings) has universal models in every cardinality λ; they are suitable divisible ones, so we better restrict ourselves to so-called reduced ones, i.e., Abelian group G with no divisible non-trivial subgroup. Another natural notion of embedding is pure embedding, that is, embedding preserving non-divisibility by natural numbers; again we better consider only reduced Abelian groups. In both cases because any such G is the direct sum of divisible(G) and reduced(G), where divisible(G) $= \cup\{H : H \subseteq G$ is divisible$\}$ and reduced(G) is not unique but is isomorphic to G/divisible(G).

Now if π is a pure embedding of G_1 into G_2, then it induces a pure embedding of divisible(G_1) into divisible (G_2) and of reduced(G_1) into reduced (G_2).

There are many interesting classes of Abelian groups, but we shall consider here only (all with $\subseteq$ the used embedding or with $\subseteq_{\rm pr}$, the pure one):

- $K_{\rm rtf}$, reduced torsion free
- $K_{{\rm rs}(p)}$, reduced separable p-group, the separable means that every element has finite height, equivalently belongs to a finite direct summand.

In particular, we shall not deal here with

- $K^{{\rm r}(p)}$, the class of reduced Abelian p-groups, because in every cardinal λ, there is no universal member, see Kojman and Shelah (1995, 3.1) and Fact 10.6, (though we could fix the height to some $\alpha < \lambda^+$, but have not dealt with this)

- classes depending on a sequence $\bar{t}$ of natural number (see Shelah, 1997, 2001)
- $K_{\aleph_1-\text{free}}$, the class of $\aleph_1$-free Abelian groups (see Shelah, 1997).

Note that this class fails amalgamation "badly", hence $\lambda=\lambda^{\aleph_0}\Rightarrow \lambda\notin \text{univ}(K_{\aleph_1-\text{free}})$; see Shelah (1997). Note also $(K_{\text{rtf}},\subseteq)$ fail amalgamation but not so badly; that is, for enough existentially closed ones, it has amalgamation, see [169, §1] or Claim 10.9, but we shall not elaborate on $K_{\aleph_1-\text{free}}$.

We also shall barely mention the class (see Shelah, 1996c)

- $(<\lambda)$-stable one.

There are closely related "combinatorial classes".

Thesis 10.3. General Abelian groups and trees with $\omega+1$ levels behave in universality theorems like stable non-superstable theories.

The simplest example of such a class is the class K^{tr}:= of normal trees $\mathscr{T}$ with $(\omega+1)$-levels, i.e., (up to isomorphism) sub-trees $\mathscr{T}\subseteq {}^{\omega\geq}\alpha$ for some α, with the relations $\eta E^0_n\nu := \eta\restriction n=\nu\restriction n$ and $<_{\mathscr{T}}$ being an initial segment. For K^{tr} we know that $\mu^+<\lambda=\text{cf}(\lambda)<\mu^{\aleph_0}$ implies there is no universal for K^{tr}_λ (by Kojman and Shelah, 1992b). Classes as K^{rtf} or $K^{\text{rs}(p)}$ (reduced separable Abelian p-groups) are similar (though they are not elementary classes) were first considered with pure embeddings (by Kojman and Shelah, 1995). But it is not less natural to consider usual embeddings (remembering they, the [Abelian] groups under consideration, are reduced). Though the problem is the more natural it is harder to treat. The problem is that the invariant has been defined using divisibility, and so under non-pure embedding those seemed to be erased.

Unfortunately, most results in Shelah (1997) are only almost in ZFC as they use extremely weak assumptions from pcf, assumptions whose independence is not known. So practically it is not so tempting to try to remove them as they may be true, and it is unreasonable to try to prove independence results before independence results on pcf have advanced.

Then, in Shelah (1996c) the non-existence of universals is proved, restricting ourselves to $\lambda>2^{\aleph_0}$ and $(<\lambda)$-stable groups (see there). These restrictions hurt the generality of the theorem; because of the first requirement we lose some cardinals. The second requirement changes the class to one which is not established among Abelian group theorists (though to me it looks natural). The aim in Shelah (1997) was to eliminate those requirements or show that they are necessary.

Now Shelah (1997) gives an explanation of some of the earlier difficulties: considering K^{x}_{λ} (for $\mathrm{x} \in \{\mathrm{rtf}, \mathrm{rs}(p)\}$), we let $\mu = \min\{\mu : \mu^{\aleph_0} \geq \lambda\}$ and let $K^{\mathrm{x}}_{\lambda,\mu}$ be the class of $G \in K^{\mathrm{x}}_{\lambda}$ with density μ by the metric $d(x,y) = \inf\{\frac{1}{2^n} : n!$ divide $x - y\}$. The natural combinatorial problem is about $K^{\mathrm{tr}}_{\lambda,\mu} = \{\mathscr{T} \in K^{\mathrm{tr}}_{\lambda}$: the level n of $\mathscr{T}, P^{\mathscr{T}}_n$ has cardinality $\leq \mu$ for $n < \omega\}$.

Parallelly we define:

- $K^{\mathrm{rs}(p)}_{\lambda,\mu} = \{G \in K^{\mathrm{rs}(p)}_{\lambda} : | G/p^n G | \leq \mu$ for every $n\}$
- $K^{\mathrm{rs}(p)}_{\langle \lambda_\alpha : \alpha \leq \omega \rangle} = \{G \in K^{\mathrm{rs}(p)}_{\lambda_\omega} : p^n G/p^{n+1} G$ has dimension $\leq \lambda_n$ for every $n\}$ (the proof in [(Shelah, 1997), 1.1] gives a reduction to this case).

In [167, 1.1=pg.271], or see Proposition 10.7, it is proved[3] that $(K^{\mathrm{rs}(p)}_{\lambda}, \subseteq)$ can be replaced by $K^{\mathrm{rp}(p)}_{\lambda,\mu}$, and moreover by $K^{\mathrm{rs}(p)}_{\langle \lambda_n : n < \omega \rangle ^\frown \langle \lambda \rangle}$, where $\wedge_n \lambda_n = \lambda_n^{\aleph_0} < \mu$.

The problem (of the existence of universals for $K^{\mathrm{rs}(p)}$) is not like looking for $K^{\mathrm{tr}}_{\lambda}$ but for $K^{\mathrm{tr}}_{\lambda,\mu}$ or even $K^{\mathrm{tr}}_{\langle \lambda_\alpha : \alpha \leq \omega \rangle}$ (where $K^{\mathrm{tr}}_{\lambda,\mu} = K^{\mathrm{tr}}_{\langle \mu : n < \omega \rangle ^\frown \langle \lambda \rangle}$) when

- $\lambda_n \leq \lambda_{n+1} < \mu = \Sigma_n \lambda_n$ and $\mu < \lambda = \lambda_\omega = \mathrm{cf}(\lambda) \leq \mu^{\aleph_0}$;

where

$\oplus$ for $\bar{\lambda} = \langle \lambda_n : n \leq \omega \rangle$ non-decreasing, $\lambda \leq \Pi_n \lambda_n$, $K^{\mathrm{tr}}_{\bar{\lambda}}$ is $\{T : T$ a tree with $\omega + 1$ levels, in level $n < \omega$ there are λ_n elements$\}$.

For $K^{\mathrm{rs}(p)}$ this, or 10.7, reduction is proved fully (see Shelah, 2017b); for K^{rtf} this is proved just for the natural examples.

In [167, §2], we define, in addition to $K^{\mathrm{tr}}_{\bar{\lambda}}$, also $K^{\mathrm{fc}}_{\langle \lambda_\alpha : \alpha \leq \omega \rangle}$. In the second, we have $\omega + 1$ kinds of elements and for each n a function from the ω-th kind to the n-th kind. We can interpret a tree $\mathscr{T} \in K^{\mathrm{tr}}$ as a member of the second example: $P^{\mathscr{T}}_{\alpha} = \{x : x \text{ is of level } \alpha\}$ and

$$F_n(x) = y \Leftrightarrow x \in P^{\mathscr{T}}_{\omega} \ \& \ y \in P^{\mathscr{T}}_{n} \ \& \ y <_{\mathscr{T}} x.$$

Now $K^{\mathrm{fc}}_{\lambda,\mu} = K^{\mathrm{fc}}_{\langle \mu : n < \omega \rangle ^\frown \langle \lambda \rangle}$ is defined as it is easier to get the non-existence theorems. But this is not one of the classes we considered originally.

In [167, §3] we return to K^{rtf} (reduced torsion-free Abelian groups with the usual embedding) and prove the non-existence of universal ones in λ if $2^{\aleph_0} < \mu^+ < \lambda = \mathrm{cf}(\lambda) < \mu^{\aleph_0}$ and an additional very weak set-theoretic assumption

3 Note that (Shelah, 1997, 1.1) the statement is referred to and proved rightly but stated wrongly.

(the consistency of its failure is not known). In Shelah (2001, 1.6) the non-existence result for $K^{\rm rtf}_\lambda$ for λ as above is fully proved, i.e., with no extra assumption. For $K^{{\rm rs}(p)}_\lambda$, the parallel result is proved only almost in ZFC: we need $\lambda \geq \beth_\omega$, as we quote the RGCH, see Shelah (2000). But recall that for the class of $\aleph_1$-free Abelian groups we have non-existence for $\lambda = \lambda^{\aleph_0}$.

We have noted above that for $K^{\rm rtf}_\lambda$ requiring $\lambda \geq 2^{\aleph_0}$ is reasonable because we can prove (i.e., in ZFC) that there is no universal member. What about $K^{{\rm rs}(p)}_\lambda$? By [167, §1] or 10.7 we should look at $K^{\rm tr}_{\langle \lambda_i : i \leq \omega\rangle}, \lambda_\omega = \lambda < 2^{\aleph_0}, \lambda_n < \aleph_0$.

In [167, §4] we prove the consistency of the existence of universals for $K^{\rm tr}_{\langle \lambda_i : i \leq \omega\rangle}$ when $\lambda_n \leq \aleph_0, \lambda_\omega = \lambda < 2^{\aleph_0}$ but weakly, i.e., of cardinality λ^+; this is not the original problem but it seems to be a reasonable variant, and more seriously, it shoots down the hope to use the present methods of proving non-existence of universals. Anyhow, this is $K^{\rm tr}_{\langle \lambda_i : i \leq \omega\rangle}$ not $K^{{\rm rs}(p)}_{\lambda_\omega}$, so we proceed to reduce this problem to the previous one under a mild variant of MA.

The reader should remember that the consistency of e.g., the other side, i.e., of (*) below is much easier to obtain, even in a wider context (just add many Cohen reals) where

(*) $2^{\aleph_0} > \lambda > \aleph_0$ and there is no M such that $M \in K^{{\rm rs}(p)}$ is of cardinality $< 2^{\aleph_0}$ and universal for $K^{{\rm rs}(p)}_\lambda$.

As in [167, §4] the problem for $K^{{\rm rs}(p)}_\lambda$ was reasonably resolved for $\lambda < 2^{\aleph_0}$ (and for $\lambda = \lambda^{\aleph_0}$, see (Kojman and Shelah, 1995)), we then, in [167, §5] turn to $\lambda > 2^{\aleph_0}$ (and μ, λ_n) as in ($\oplus$) above.

How above do we overcome the problem of "divisibiity is not preserved? and how does pcf help"? As in an earlier proof we use $\langle C_\delta : \delta \in S\rangle$ guessing clubs for λ (see references or §11(B)), so C_δ is a subset of δ (hence the invariant depends on the representation of G, but this disappears when we divide by suitable ideal on λ). What we do is, rather than trying to code a subset of C_δ (for $\bar{G} = \langle G_i : i < \lambda\rangle$ a representation or filtration of the structure G as the union of an increasing continuous sequence of structures of smaller cardinality) by an element of G, we do it, say, by some sequence $\bar{x} = \langle x_t : t \in {\rm Dom}(I)\rangle$, I an ideal on Dom(I) (so really we use $\bar{x}/I$). At first glance if Dom(I) is infinite we cannot list *a priori* all possible such sequences for a candidate H for being a universal member, as their number is $\geq \lambda^{\aleph_0} = \mu^{\aleph_0}$. But we can find a family

$$\mathscr{F} \subseteq \{\langle x_t : t \in A\rangle : A \subseteq {\rm Dom}(I),\ A \notin I,\ x_t \in \lambda\}$$

of cardinality $<\mu^{\aleph_0}$ such that, for any $\bar{x}=\langle x_t : t\in \mathrm{Dom}(I)\rangle$, for some $\bar{y}\in\mathscr{F}$ we have $\bar{y}=\bar{x}\restriction\mathrm{Dom}(\bar{y})$.

As in [167, §3] there is such $\mathscr{F}$ except when some set-theoretic statement related to pcf holds. This statement is extremely strong, also in the sense that we do not know how to prove its consistency at present. But again, it seemed unreasonable to try to prove its consistency before the pcf problem was dealt with. Of course, we may try to improve the combinatorics to avoid the use of this statement but were naturally discouraged by the possibility that the pcf statement can be proved in ZFC; thus we would retroactively get the non-existence of universals in ZFC. However, in Shelah (2001), this pcf assumption for K_λ^{rtf} is eliminated but for $K_\lambda^{\mathrm{rs}(p)}$ is only replaced by $\lambda>\beth_\omega$, using the RGCH.

Further results are in [167, §6–9§].

An interesting aspect of the above and of Shelah (2001), is the use of pcf for ideals defined using algebra. From another aspect it is probably better to replace K^{fc} (fc stands for function) by K^{fd} defined by:

⊞

(A) $M\in^{K^{\mathrm{fd}}}$ iff:
  (a) $\langle P_\alpha^M : \alpha\le\omega\rangle$ is a partition
  (b) $F_{n,\ell}^M$ is a function from P_ω^M into P_n^M for $n, \ell<\omega$
  (c) $\langle F_{n,\ell}^M(a) : \ell<\omega\rangle$ is eventually constant for every $n<\omega$

(B) $K_{\lambda,\mu}^{\mathrm{fd}}, K_{\langle\lambda_\alpha:\alpha\le\omega\rangle}^{\mathrm{fd}}$ are defined naturally

The point is:

Claim 10.4. $\mathrm{univ}(K_\lambda^{\mathrm{tr}})\le\mathrm{univ}(K_\lambda^{\mathrm{rtf}})$.
$\mathrm{univ}(K_\lambda^{\mathrm{tr}})\le\mathrm{univ}(K_\lambda^{\mathrm{rs}(p)})\le\mathrm{univ}(K_\lambda^{\mathrm{fd}})$.
$\mathrm{univ}(K_\lambda^{\mathrm{tr}})\le\mathrm{univ}(K_\lambda^{\mathrm{fc}})\le\mathrm{univ}(K_\lambda^{\mathrm{fd}})$.

See Shelah (1997).

Question 10.5.
(1) Can we prove the consistency of: μ strong limit of cofinality $\aleph_0$, $\lambda=\mu^+<2^\mu$ and $\mathrm{univ}(K_\lambda^{\mathrm{tr}})=1$ and even $\mathrm{univ}(K_\lambda^{\mathrm{fd}})=1$?
(2) Prove that $\lambda\le\mathrm{univ}_{\mathrm{group}}$ iff $\lambda=2^{<\lambda}>\aleph_0$.

Maybe it will help to formally divide to cases and give a table clarifying the situation (see Table 1).

Table 1. See Table 2 for Legend

Cardinality	Class						
	K_*^{fc}	K^{tr}	K_*^{tr}	$(K^{rtf}, \subseteq_{pr})$	$(K^{rtf}, \subseteq)$	$(K^{rs(p)}, \subseteq_{pr})$	$(K^{rs(p)}, \subseteq)$
$\lambda = \aleph_0$	×	✓	✓	×	×	✓	✓
$\lambda = \lambda^{\aleph_0}$	✓ 10.11(1)	✓ 10.11(1)	✓	✓ (Kojman and Shelah, 1995, 3.5)	✓ [177, 1.2(1)]	✓ (Kojman and Shelah, 1995, 3.2)	✓
$\aleph_0 < \lambda < 2^{\aleph_0}$	× 10.11 (2)	⊞ for λ singular $>\aleph_1$, by 10.10(2) (A)(a). × for $\lambda = \mathrm{cf}(\lambda) > \aleph_1$, by 10.10 (2)(A)(a). Con(✓) for $\aleph_1$ 10.11 (4) Con(×) see 10.11(3)	Con(✓*), by (Shelah, 1997, 4.7). Con(×), by 10.11(3). Con(✓) for $\aleph_1$, by 10.11 (4), (Shelah, 1980a)	× (Kojman and Shelah, 1995, 3.7)	× (Shelah, 2001, 1.2)	⊞ 10.10(2)(A)(d)	Con(×),10.11 (3) Con(✓*), (Shelah, 1997, §4).
$2^{\aleph_0} + \mu^+ < \lambda < \mu^{\aleph_0}$ and λ regular	Con(×) 10.10 (2)(A) (b)	× 10.10 (2)(A)(a)	Con(×)10.10(2)(A)(b)	× (Kojman and Shelah, 1995, 3.8)	× (Shelah, 2001, 1.6)	× (Kojman and Shelah, 1995, 3.3)	×10.11(5)
$2^{\aleph_0} + \mu^+ < \lambda < \mu^{\aleph_0}$ and λ singular	Con(×) 10.10 (2) (A)(b)	Con(×)10.10 (2)(A)(a)	Con(×) 10.10(2)(A)(b)	⊠	⊠	⊠	⊠
$\mu^+ = \lambda < \mu^{\aleph_0}$ and $(\forall \chi < \mu)(\chi^{\aleph_0} < \mu)$	Con(×) e.g. $\lambda = \beth_\omega^+$	Con(×) 10.10(2)	Con(×)	Con(×)	Con(×)	Con(×)	Con(×)
$\mathrm{cf}(\lambda) = \aleph_0$ and $(\forall \chi < \lambda)(\chi^{\aleph_0} < \lambda)$	✓ 10.11(1)	✓ 10.11(1)	✓ 10.11(2)	✓ (Shelah, 2017b, 1.2(2))	✓ (Shelah, 2017b, 1.2(2))	✓	✓ (Shelah, 2017b)

Table 2. Legend for Table 1

$\times$	There is no universal model
$\boxtimes$	There is no universal model except possibly when some extreme pcf condition holds
✓	There is a universal model
✓*	There is a weak universal model for λ of cardinality $\lambda^+ < 2^\lambda$
$\rightarrow$	A solution on the left serves also on the right
$\leftarrow$	A solution on the right serves also on the left
Pcf	Under mild pcf conditions
$K^* K*$	Stands for $K_{\bar\lambda}$, $\bar\lambda = \langle \lambda_n : n < \omega \rangle$ such that (a) or (b), where (a) $\Sigma_n \lambda_n = \mu < \lambda = \lambda_\omega < \mu^{\aleph_0}$ for some μ, (b) $\lambda_n = \lambda_\omega = \lambda$.

10.2.1 Fulfilling some points

Fact 10.6. (Kojman and Shelah, 1995, Th.3.1) For every λ, $\lambda \notin \text{univ}(K^{r(p)})$ recalling $K^{r(p)} = \{G : G$ a reduced Abelian p-group$\}$.

Proof. Recall

(A) For $G \in K^{r(p)}$ and $x \in G'$ defined rk, ht$(x, G) = \sup\{\text{rk}(y, G) + 1 : y \in G$, $G = py = x\} \in \{\text{Ord}\}$. So ht$(x, G) = \infty \Leftrightarrow x = 0_G \Leftrightarrow \text{ht}(x,G) < |G|^+$. Let ht$(G) = \sup\{\text{ht}(x,G) + 1 : x \in G\backslash\{0_G\}\}$

(B) For every $\alpha < \lambda^+$, $\alpha \geq 2$ there is $G \in K_\lambda^{r(p)}$ with rk$(G) = \alpha$

(C) If $g \in \text{Hom}(G_1, G_2)$ then ht$(x, G_2) \leq \text{ht}(g(x), G_2)$

(D) Together $\text{univ}(K_\lambda^{r(p)}) \geq \lambda^+$. □

Proposition 10.7. (Shelah, 1997, 1.1) *Assume* $\mu = \Sigma_{n<\omega} \lambda_n = \text{limsup}_n \lambda_n$, $\mu \leq \lambda \leq \mu^{\aleph_0}$, *G is a reduced separable (Abelian) p-group,* $|G| = \lambda$ *and* $\lambda_n(G) = $ *dim* $((p^n G)[p]/(p^{n+1} G)[p]) \leq \mu$ *(this is a vector space over* $\mathbb{Z}/p\mathbb{Z}$ *hence the dimension is well defined).*

<u>Then</u> *there is H, a reduced separable p-group such that* $|H| = \lambda$, H *extends G (not necessarily purely) and* $(p^n H)[p]/(p^{n+1} H)[p]$ *is a group of cardinality* λ_n.

Remark 10.8. So, for *H* the invariants from Kojman and Shelah (1995) are trivial.

Proof. (See Fuchs, 1970b). We can find z_i^n $(n < \omega, i < \lambda_n(G) \leq \mu)$ such that:

(a) $z_i^n \in G$ has order p^n,

(b) $B = \sum_{n,i} \langle z_i^n \rangle_G$ is a direct sum,

(c) B is dense in G (in the topology induced by the norm

$$\| x \| = \min\{2^{-n} p^n \text{divides} x \text{in} G\}).$$

For each $n < \omega$ and $i < \lambda_n(G)$ $(\leq \mu)$ choose $\eta_i^n \in \prod_{m<\omega} \lambda_m$, pairwise distinct such that: for $(n^1, i^1) \neq (n^2, i^2)$ for some $n(*)$ we have:

$$\lambda_n \geq \lambda_{n(*)} \Rightarrow \eta_{i^1}^{n^1}(n) \neq \eta_{i^2}^{n^2}(n).$$

Let H be generated by G, x_i^m (for $i < \lambda_m, m < \omega$), $y_i^{n,k}$ (for $i < \lambda_n, n < \omega, n \leq k < \omega$) freely except for:

(α) the equations of G.

(β) $y_i^{n,n} = z_i^n$.

(γ) $p y_i^{n,k+1} - y_i^{n,k} = x_{\eta_i^n(k)}^k$.

(δ) $p^{n+1} x_i^n = 0$,

(ε) $p^{k+1} y_i^{n,k} = 0$.

Check. □

Implicit in Shelah (2001).

Claim 10.9. *Letting* $K_{\text{ftf}} = \{G \in K_{\text{rtf}}$: *G is full (see* [169, Definition 1.3(3)])}.

(1) K_λ^{ftf} *is dense in* λ *and it satisfies*:

 (a) *every member of* K_λ^{rtf} *can be extended to a full one of cardinality* $\lambda + 2^{\aleph_0}$

 (b) *if* M_1, $M_2 \in K_{\text{rtf}}$ *and* M_1 *is full and h embeds* M_1 *into* M_2, *h is a pure embedding.*

(2) $(K_\lambda^{\text{ftf}}, \subseteq) = (K_\lambda^{\text{ftf}}, \subseteq_{\text{pr}})$ *and it has amalgamation.*

$$\text{univ}(K_\lambda^{\text{ftf}}, \subseteq) = \text{univ}(K_\lambda^{\text{rtf}}, \subseteq) = \text{univ}(K_\lambda^{\text{ftf}}, \subseteq_{pure}) \leq \text{univ}(K_\lambda^{\text{rtf}}, \subseteq).$$

4) *If* $\lambda = \lambda^{\aleph_0}$ *then* $(K_\lambda^{\text{ftf}}, \subseteq)$ *has a universal member*, *(even a homogeneous universal one)*

5) *Like part (4) when* $\lambda = \Sigma_{n<\omega} \lambda_n$ *when* $\lambda_n = \lambda_n^{\aleph_0} < \lambda_{n+1}$

Claim 10.10. *1) If (A) then (B) where*:

(A) *(a)* $\mu = \mu^{<\mu} < \theta = \text{cf}(\theta) \leq \lambda < \chi = \chi^\mu$ *and* θ *has weak club guessing for* μ *or at least for some* $\sigma \in [\mu, \lambda)$, *see* §*11(B)*

(b) $\mathbb{Q}$ *is the forcing of adding* χ *many* μ*-Cohens.*

(c) $\mathbb{Q} \lessdot \mathbb{P}$ *and* $\mathbb{P}$ *satisfies the* μ^{++}*-c.c., and* $\mathbf{V}^{\mathbb{P}}$ *and* $\mathbf{V}$ *have the same cardinals.*

(d) $\mathbf{V}^{\mathbb{P}} \models$ "$\mathrm{cf}(\mu) = \aleph_0$" *so if* $|\mathbb{P}| = \chi$ *then* $\Vdash_{\mathbb{P}}$ "$\mu^{\aleph_0} = \chi$".

(B) *(a)* $\mathrm{univ}(K^{\mathrm{tr}}_{\lambda}) \geq \chi$

(b) $\mathrm{univ}(K^{\mathrm{tr}}_{\lambda,\mu}) \geq \chi$.

2) Assume $\mu < \theta = \mathrm{cf}(\theta) \leq \lambda \leq \mu^{\aleph_0}$ *and* θ *has club guessing for* μ *(even weakly, see* §*11(B), easy to get consistently)*:

(A) *if* $\mathrm{cf}([\lambda]^{\theta}, \subseteq) < \mu^{\aleph_0}$ *(e.g* $\lambda = \theta < \mu^{\aleph_0}$*)* <u>*then*</u>

⊞ *(a)* $\mathrm{univ}(K^{\mathrm{tr}}_{\lambda}), \mathrm{univ}(K^{\mathrm{fc}}_{\lambda}) \geq \mu^{\aleph_0}$

(b) $\mathrm{univ}(K^{\mathrm{tr}}_{\lambda,\mu}), \mathrm{univ}(K^{\mathrm{fc}}_{\lambda,\mu}) \geq \mu^{\aleph_0}$

(c) $\mathrm{univ}(K^{\mathrm{rtf}}_{\lambda}) \geq \mu^{\aleph_0}$

(d) $\mathrm{univ}(K^{\mathrm{rs}(p)}_{\lambda}) \geq \mu^{\aleph_0}$

(B) *if cov*$(\lambda, \theta^+, \theta^+, \theta) < \mu^{\aleph_0}$, <u>*then*</u> *clause (a), (b), (c) of* ⊞ *holds.*

3) All the standard models for violating the SCH *fit under part (1).*

Proof. 1) By clause (A) of (2).

2) See Shelah (2001) for K^{rtf}_{λ} use 10.10.

3) Clear. □

Recall $K^{\mathrm{fc}}_{\lambda}, K^{\mathrm{fd}}_{\lambda}$ are defined after ⊕, in ⊞ (before Claim 10.4) respectively.

Claim 10.11.

(1) *Assume* $\lambda = \sum_n \lambda_n$ *where* $\lambda_n = (\lambda_n)^{\aleph_0}$. *Then* $\mathrm{univ}(K^{\mathrm{tr}}_{\lambda}) = \mathrm{univ}(K^{\mathrm{fc}}_{\lambda}) = \mathrm{univ}(K^{\mathrm{fd}}_{\lambda}) = 1$.

(2) $K^{\mathrm{fc}}_{\lambda}$ *has no universal element when* $\lambda < 2^{\aleph_0}$, *moreover, this holds for* $K^{\mathrm{fc}}_{\bar{\lambda}}$, *where* $\bar{\lambda} = \langle \lambda_n : n \leq \omega \rangle$, $\aleph_0 \leq \lambda_\omega < 2^{\aleph_0}$, $\lambda_n \leq \lambda_\omega$ *for all* $n < \omega$, *and* $\lambda_n \geq 2$ *for infinitely many* $n < \omega$.

(3) *If* $\aleph_0 < \lambda < \mu$ *and* $\mathbb{P}$ *is the forcing of adding* μ *Cohen reals, then in* $\mathbf{V}^{\mathbb{P}}$ *we have* $\mathrm{univ}(K^{\mathrm{tr}}_{\lambda}) \geq \mu$ *and even* $\mathrm{univ}(K^{\mathrm{tr}}_{\bar{\lambda}}) \geq \mu$ *for every* $\bar{\lambda}$ *as above when* $\lambda_n \geq n$*; moreover* $\mathrm{univ}(K^{\mathrm{sp}(p)}_{\lambda}) \geq \mu$*; mentioned in* [167, §0, after ⊕, page 5].

(4) *If* $2^{\aleph_\ell} = \aleph_\ell$ *for* ℓ = 1, 2 *then for some c.c.c. forcing notion* $\mathbb{P}$ *of cardinality* $\aleph_2$ *in* $\mathbf{V}^{\mathbb{P}}$ *we have* $\mathrm{univ}(K^{\mathrm{tr}}_{\aleph_1}) = 1$*; see Shelah (1980a, 2.7).*

(5) *If and* $\lambda < 2^{\aleph_0}$ <u>*then*</u> $\mathrm{univ}(K^{\mathrm{rs}(p)}_{\lambda}) \leq \mu^{\aleph_0}$*; (Why? By Shelah, 1997, 5.9 + Shelah, 2001, 2.6 as in Shelah, 2001, 2.7).*

Proof. Easy. E.g.

2) Let $u_* = \{n < \omega : \lambda_n \geq 2\}$, an infinite set. For every $u \subseteq u_*$ we define the model M_u as follows:

(a) The universe of M_u is the set $\{(n, i) : n < \omega, i \in \{0, 1\}$ and $n \in u \Rightarrow i = 0\} \cup \{(\omega, 0), (\omega, 1)\}$ (b) $P_n^{M_u} = \{(n,i) : i < 2\}$
(c) $F_n^{M_u}((\omega,i))$ is $(n, 1)$ if $n \in u$, and $(n, 0)$ if $n \notin u$.

Clearly,

$$M_u \in K^{\rm fc}_{\aleph_0}$$

- If $N \in K^{\rm fc}$, then the set $\{u \subseteq u_* : M_u \text{ embeddable in } N\}$ is of cardinality $\leq \| N \|$. □

We are done.

10.3 Continuity

In [**167**, §9] we turn to a closely related class: the class of metric spaces with (one to one) continuous embeddings; similar results hold for it. We also phrase a natural criterion for deducing the non-existence of universals from one class to another.

(G) A very active and successful area concerns complete metric spaces and Banach spaces; this area is known as continuous model theory. The analogous problems there are usually considerably more difficult.

The basic idea is to consider a Banach space or just model M with metric $\mathbf{d}_M$ which is a complete metric space under the metric $\mathbf{d}_M$; so relations $R_{\mathbf{d},\leq r} = \{(x, y) : \mathbf{d}_M(x,y) \leq r\}$, $R_{\mathbf{d},\geq r} = \{(x,y) : \mathbf{d}_M(x,y) \geq r\}$ for $r \in \mathbb{R}_{\geq 0}$; and individual constants 0_M, and countably many other closed relations. Taking ultra-products, "throwing" away elements of infinite distance from 0 and dividing by the equivalence relation of being of distance $\leq 1/(n + 1)$ for every n things are fine, but we preserve only some formulas and sentences.

For model theorists, cardinality is replaced by topological density, and "M embeddable into N" means M is isometric to a submodel of N; Banach space theorists prefer isomorphic which means if $f(a) = b$ and $\varepsilon \in (0, 1)$, then for some $\xi \in (0, 1)_{\mathbb{R}}$ we have

$$(b \in M \wedge \mathbf{d}_M(b,a) < \xi) \Rightarrow \mathbf{d}_N(f(b),f(a)) < \varepsilon,$$

or ξ-isomorphism for $\xi \in (0, 1)_{\mathbb{R}}$, where ξ-isomorphism means that if $a \neq b \in M$, then

$$\mathbf{d}_N(g(a),g(b))/\mathbf{d}_M(a,b)\in[\xi,1/\xi].$$

On categoricity, the first step, see Ben-Yaacov (2005) and Shelah and Usvyatsov (2011); the first uses isometry, the second an almost isometric (that is $(1+\varepsilon)$-isomorphic for every $\varepsilon\in\mathbb{R}_{>0}$). But for our purpose central is (Shelah and Usvyatsov, 2006) showing that the class of Banach spaces has a version of SOP_4 (even SOP_n, for all n), and deduce it is almost $\leq_{\mathrm{univ}}$-maximal under isometric embeddings.

Central problems are:

Problem 10.12. Prove "categoricity spectrum" is nice under isomorphisms.

Problem 10.13. Develop the parallel of superstable in this context.

See more in Shelah and Usvyatsov (2006, 2008a) and Ben-Yaacov and Usvyatsov (2010).

In particular, Shelah and Usvyatsov (2006) shows that the class of Banach spaces is almost $\leq_{\mathrm{univ}}$-maximal for isometries; more recently, Brech and Koszmider (2012) proved no universal Banach space of density the continuum (assuming the continuum is far from GCH). More general negative results on isomorphic embeddings are obtained in Džamonja (2014). Recently there has been interest from the operator algebras community, and some similar preliminary results have been obtained, see Farah et al. (2020).

Some general results were applied to Eberlein spaces, see e.g., Džamonja (2005, 2006).

Now, Shelah and Steprāns (2016) addresses the case $\lambda=\aleph_1=\mathbf{d}<2^{\aleph_0}$, some other classes and notions of universality, universality of functions, and also distinctions of existence of universals for closely related classes.

Definition 10.14.

(1) For $\mathbf{k}\in[2,\aleph_0]$ let $\mathrm{Un}^1_{\mathbf{k}}$ be the statement: there is $\mathbf{c}:[\omega_1]^2\to\mathbf{k}$ such that for any $\mathbf{d}:[\omega_1]^2\to\mathbf{k}$ there is a one-to-one function $\pi:\omega_1\to\omega_1$ such that $\alpha<\beta<\omega_1\Rightarrow\mathbf{d}\{\alpha,\beta\}=\mathbf{c}\{\pi(\alpha),\pi(\beta)\}$.

(2) Let $\mathrm{Un}^2_{\mathbf{k}}$ be defined similarly but there is also a permutation $\varkappa$ of $\mathbf{k}$ such that $\alpha<\beta<\omega_1\Rightarrow\mathbf{d}\{\alpha,\beta\}=\varkappa(\mathbf{c}\{\pi(\alpha),\pi(\beta)\})$.

Can we prove:

Problem 10.15. Can we distinguish $\mathrm{Un}^1_{\aleph_0}$ and $\mathrm{Un}^2_{\mathbf{k}}$?

10.4 Locally finite groups

The class K_{lfg} of locally finite groups is the class of groups G such that every finitely generated subgroup of G is finite, see on it the books (Giorgetta and Shelah, 1984) and (Kegel and Wehrfritz, 2000). Characteristic of the area is mixing finite group theory and infinite combinatorics. By Grossberg and Shelah (1983) we have:

Theorem 10.16. *If $\lambda = \lambda^{\aleph_0}$, then K_{lfg} has no universal in λ.*

This result is very different from the results cited above, the reason being that K_{lfg} is not a first-order class; moreover, amalgamation fails very badly in this class. The first cardinal which is not covered by Theorem 10.16 under GCH is $\beth_\omega$.

Problem 10.17.

(1) Does K_{lfg} have a universal member in $\beth_\omega$?

(2) Assume μ is strong limit (i.e., $\theta < \mu \Rightarrow 2^\theta < \mu$) uncountable and of cofinality $\aleph_0$, i.e., $\mu = \sum_n \mu_n, \aleph_0 < \mu_n < \mu_{n+1}$. Does $\mu \in \mathrm{Univ}(K_{\mathrm{lfg}})$?

Note that, if $\lambda = \mu^{++}$, then no group of cardinality λ is universal even for locally finite groups or cardinality λ, (Shelah, 2016). Concerning Problem 10.17, by Grossberg and Shelah (1983) we have:

Theorem 10.18. *If κ is a very large cardinal, specifically a so-called compact cardinal and $\mu > \kappa > \theta$ is as in Problem 10.17(2), then $\mu \in Univ(K_{lfg})$.*

A very special cardinal is $\aleph_1$.

Problem 10.19. Can we prove $\aleph_1 \notin \mathrm{Univ}(K_{\mathrm{lfg}})$?

This, in particular, would require use of finite group theory. As in earlier cases, if $\beth_\omega < \lambda < 2^{\beth_\omega}$, then in many cases we can prove $\lambda \notin \mathrm{Univ}(K_{\mathrm{lfg}})$, but note:

Problem 10.20.

(1) Assume $\mu = \Sigma\lambda_n$, $\lambda_n = \lambda_n^{\aleph_0} < \mu_{n+1}$ and $\mu < \lambda < \mu^{\aleph_0}$. Prove $\lambda \notin \mathrm{Univ}(K_{\mathrm{lfg}})$.

(2) What about singular not strong limit cardinals?

By Shelah (1983) we know that having $G \in K_{\mathrm{lfg}}$ of cardinality $\mu = \Sigma_{n<\omega} 2^{\lambda_n} = \Sigma_{n<\omega} \lambda_n$ is equivalent to having something like a special model.

10.5 Countable graphs and countable density

Universality in $\aleph_0$ has its mysteries, which attracted the attention of logicians (see Part I and e.g., Cherlin et al., 1999; Cherlin and Tallgren, 2007), combinatorialists (see e.g., Komjáth et al., 1988; Füredi and Komjáth, 1997) and algebraists alike (see e.g., Hall, 1959 and Donkin, 2006). One of the most studied cases of universality in $\aleph_0$ is the case of graphs, and specifically of classes of countable graphs defined by forbidding a finite subgraph (see Cherlin, 2011 for a general discussion). A characteristic well-known case is the case of triangle-free graphs (Henson, 1971), where a universal object is known to exist by the usual method of Fraïssé. Corresponding to the usual graph-theoretic distinction between subgraphs (weak embeddings) and induced subgraphs (strong embeddings), we have here two notions of universality for a class K of countable graphs, one with respect to weak embeddings (i.e., every $A \in K$ embeds weakly into the proposed universal object) and one with respect to strong embeddings (i.e., every $A \in K$ embeds strongly into the proposed universal object).

In this context, one usually prefers to prove the existence of universal objects in the strong form, and the nonexistence in the weak form, taking special note of the rare instances where a weakly universal object exists but a strongly universal one does not. The main problem in this line of inquiry is the following:

Problem 10.21. (Cherlin, 2011).

(1) Given a finite graph H, letting K_H be the class of countable graphs omitting H as a subgraph in the weak sense, when does K_H admits a universal object in the strong sense?

(2) Is the class of finite graphs H such that K_H admits a universal object (in the strong sense) *decidable*?

This problem has been studied by many scholars, from different backgrounds, see e.g., (Komjáth and Pach, 1984; Komjáth et al., 1988; Latka, 1994; Cherlin and Shi, 1996; Goldstern and Kojman, 1996; Cherlin et al., 1997; Cherlin and Shi, 2001; Latka, 2003). One of the most general results in the area was due to Komjáth (Komjáth, 1999), who proved that for any incomplete 2-connected H the class K_H does not admit a universal graph.

Now Cherlin et al. (1999) and Cherlin and Shelah (2007, 2016) present a general strategy which lead to fundamental progress toward a positive answer to the decidability problem above. Crucial to this strategy is the discovery of a fundamental connection with the model-theoretic algebraic closure operator, where we say that $a \in G$ is algebraic over a finite subset $B \subseteq G$ if the orbit of a under $Aut_A(G)$ (automorphisms of G fixing A pointwise) is finite. Now, fixing a connected graph H, obviously K_H has the joint embedding property; let G be an existentially closed $\aleph_0$-homogeneous graph in K_H. It turns out that:

$\boxplus$ The following three conditions are intimately related at both a theoretical and an empirical level:

(A) the algebraic closure operation associated to G is (uniformly) locally finite, in the sense that the algebraic closure of a set of size n is bounded in size by a function of n;
(B) there is a strongly universal H-free graph;
(C) there is a weakly universal H-free graph.

(These three conditions are successively weaker and not much different in practice.)

This breakthrough allowed for serious progress in Problem 10.21, for example Cherlin and Shelah (2007) gave a *complete characterization* of the finite trees (in the graph-theoretic sense) T such that K_T admits a universal object. In the case of trees, all blocks have at most two vertices. Also, the opposite case was considered, in which all blocks have at least three vertices. We now have an explicit list of the finite graphs of this type: exactly the ones for which K_H admits a universal object. The general case should be a mix (rather than a union) of the case of trees and this opposite case. Of course, this would be the best possible result.

In order to settle the conjecture, we need several crucial technical results, which naturally involves investigating deeply the structure of finite graphs. Our advancement has led to the following specific conjecture, which in many respects is truly the missing piece of our venue:

Conjecture 10.22. The Solidity Graph conjecture: if K_H has a universal member in $\aleph_0$ then every block of H (i.e., a maximal 2-connected subgraph) is complete.

Apart from its inherent theoretical interest, the question of universality in $\aleph_0$ and the study of relative classes K_H's as above connects crucially to our main line of investigation. In fact, we conjecture:

Conjecture 10.23. Investigating K_H's as above (and relative generalizations to hypergraphs, and combinatorial structures in general) will be a source of interesting examples of unstable theories, as in the case of the involved discovery of infinite chains in Keisler's order (see Malliaris and Shelah, 2018b).

Finally, the study of graphs forbidding a given substructure hopefully will help us in building new examples of T's. Another source of such examples is $0-1$ laws (see e.g., Kolaitis et al., 1987; Shelah and Spencer, 1988; Shelah, 1996a; Baldwin and Shelah, 1997; Baldwin and Shelah, 1998; Shelah, 2002, 2015b).

11. Combinatorics

11.1 Finite combinatorics

In this section, we discuss the combinatorial ingredients of the theory of universality and connections with established themes in combinatorics, both finite and infinite. It is no surprise that infinite combinatorics will play a crucial role, including club guessing and pcf theory, which are particularly relevant (see e.g., Shelah, 1980a; Kojman and Shelah, 1992a; Shelah, 1993a, 2013b), but there are also strong connections with finite combinatorics, which we wish to stress, such as structural Ramsey theory, the Szemeredi Regularity Lemma, and probabilistic methods (see e.g., Shelah, 1980a; Kojman and Shelah, 1992a; Shelah, 1993a, 2013b).

Why is this? In principle because the dividing lines we mainly are looking for use finitary properties of the family of subsets defined by mainly one formula $\varphi(\bar{x}_n, \bar{a}_n)$, in a model of T, varying the parameter $\bar{a}_n$. Let us look at some examples of connections.

(1) Ramsey's theorem
Ramsey's theorem stands behind Ehrenfeucht–Mostowski models (Ehrenfeucht and Mostowski, 1956); their aim was to find models with many automorphisms and this is where indiscernible sequences originate; this was used by the seminal work of Morley. In fact, indiscernible sequences are one of the most important tools in model theory. Generalizations of Ramsey are also very important. The proof that theories with the independence property have complicated models in terms of Karp complexity in Laskowski and Shelah (2003) uses the Ramsey theorem for ordered graphs; see more generally (Shelah, 2010).

(2) Stability theory and the number of types
At the beginning of stability theory, a crucial point was that the order property (one of the definitions of being unstable) is equivalent to some rank being large (and having "many" φ-types over a set). This was done in Shelah (1971) quoting a combinatorial result from Erdős and Makkai (1966); now Shelah (1972) gives stronger infinitary results solving problems from the list of Erdös-Hajnal, and Hodges (1981) gives the finitary theorem. This is also connected to counting the number of local types equivalently φ-types.

Another connection to counting types is the proof that: if for some model M of T, $\lambda = |\mathbf{S}(M)|$ (i.e., the number of complete 1-types over M) is too large (essentially no linear order of cardinality $\|M\|$ has $\geq\lambda$ cuts) then T has the independence property for some φ. This leads me to interest in the finitary question: if $\mathscr{P}\subseteq\mathscr{P}(n)$ and $|\mathscr{P}|$ big enough, then there are k-independent sets from $\mathscr{P}$, see Vapnik and Červonenkis (1971) and Sauer (1972), and VC-dimension. Via the infinite theorem we knew Shelah (1971) that: if some $\varphi(\bar{x},\bar{y})$ has the independence property in (a complete T) then also some $\varphi(x,\bar{y})$ does. Now this needs forcing (as if GCH holds, the infinitary theorem is vacuous), and considerably later a finitary proof was found by Laskowski (1992) with explicit bounds.

(3) Graph regularity
Szemerédi's lemma is suspected to be relevant as well as results like Szemerédi (1975), Gowers (2007) and Gowers (2013). So far, we have an application in the other direction (i.e., improving the general combinatorial results under model-theoretic assumptions). For example, in graph with no complete k-half graphs the bounds in the Szemerédi Lemma can be much improved (see Lovász and Szegedy, 2010; Malliaris and Shelah, 2014). Other examples include better regularity lemmas assuming distality or dependence = NIP, see e.g., (Chernikov and Starchenko, n. d., 2016).

(4) Uniform definability over finite sets
In Chernikov and Simon (2015) a long-standing open problem was solved: if T is dependent, then for every $\varphi(x,\bar{y})$ there is $\psi(\bar{y},\bar{z})$ such that: if A is a finite subset of $M\in K_T$ and $a\in M$ then $\mathrm{tp}_{\varphi}(a, A, M)$ is definable by $\psi(\bar{y},\bar{c})$ for some $\bar{c}\in {}^{\ell g(\bar{z})}A$. They have essentially used tools from both model theory and finite combinatorics (in the form of the (p, k)-theorem, see Matoušek, 2004).

11.2 Infinite combinatorics

We would like to give a taste of the infinite combinatorics involved. After Shelah (1980a), surprisingly Kojman and Shelah (1992a) prove that if e.g., $\lambda = \mu^{++} < 2^{\mu}$ then $\lambda \notin \mathrm{univ}(T_{\mathrm{dlo}})$ and more. How is this done?

A major point is:

Definition 11.1.

(1) We say λ has club guessing for S when some $\bar{C}$ witnesses it, which means:

  (a) S is a stationary subset of λ;

(b) $\bar{C} = \langle C_\delta : \delta \in S \rangle$, where C_δ is a club of δ of order type cf(δ), the cofinality of δ;

(c) if E is a club of λ, then for stationarily many $\delta \in S$ we have $C_\delta \subseteq E$.

(2) Replacing S by κ means κ is a cardinal $<\lambda$ and S is a stationary subset of $\{\delta < \lambda : \mathrm{cf}(\delta) = \mathrm{cf}(\kappa)\}$ and demanding $\mathrm{otp}(C_\delta) = \kappa$.

The point is that, unlike e.g., Jensen's diamond, cases of this principle are provable in ZFC (see Shelah (1994, Ch. III) and for part (3), see Shelah (1993a) and more in Shelah (2013b, 0.11), or Shelah (1999a, 1.3(b)(β))).

Theorem 11.2.

(1) *If $\lambda > \kappa$ are regular cardinals and $\kappa^+ < \lambda$ then λ has club guessing for κ.*

(2) *Moreover this holds for any such stationary $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \mathrm{cf}(\kappa)\}$.*

(3) *Moreover we can (for many such sets S) find a sequence $\bar{C} = \langle C_\delta : \delta \in S \rangle$, otp($C_\delta$) = κ, witnessing λ has a club guessing sequence for κ such that*:

$$\oplus \text{ if } \alpha < \lambda \text{ then } \{\delta \in S : \alpha \in \mathrm{nacc}(C_\delta)\} \text{ has cardinality } < \lambda.$$

(4) *If $\lambda = \kappa^{++}$, κ not necessarily regular, then λ has club guessing for κ. Moreover, we can take S as in 11.2(2) and demand*

$$\alpha \in C_{\delta_1} \cap C_{\delta_2} \Rightarrow C_{\delta_1} \cap \alpha = C_{\delta_2} \cap \alpha.$$

Definition 11.3. We say that $\bar{C}$ is an $\check{I}_\kappa[\lambda]$-club guessing when it is as in 11.1(3), allowing κ to be singular; we add "weakly" when we omit $\delta \in S \Rightarrow \delta = \sup(C_\delta)$ so C_δ is just a club of $\sup(C_\delta)$.

Now Kojman and Shelah (1992a) proves:

Theorem 11.4.

(1) *If λ has club guessing for κ and $2^\kappa > \lambda$ then $\lambda \notin univ(T_{dlo})$.*

(2) *This holds for any T with the strict order property.*

Later, in Shelah (1996b) we continue this proving it for a wider family of T-s:

Theorem 11.5. *If T has the SOP$_4$ a, there is a weak $\check{I}_\kappa[\lambda]$-club guessing $\bar{C}$, and $2^\kappa > 2^{<\lambda}$ the univ(T) $\geq 2^\kappa$.*

Problem 11.6. Assume $\lambda = \mu^+ < 2^\mu$ and $\kappa < \mu \Rightarrow 2^\kappa \le \lambda$. Can we prove $\lambda \notin$ univ (T) for T as above, i.e., with SOP_4 or with the olive property?

For singular λ, this was proved for most λ's, but the general case is still open. Recall.

Theorem 11.7. (Kojman and Shelah, 1992a, 4.3) *Suppose* $\theta = \text{cf}(\theta) < \theta < \ < \kappa$ *are regular cardinals,* $\kappa < \mu$ *and there is a binary tree* $T \subseteq {}^{<\theta}2$ *of size* $<\kappa$ *with* $>\mu^* := cov(\mu, \kappa^+, \kappa^+, \kappa)$ *branches of length* θ*. Then.*

$(*)_{\mu,\kappa}$ *There is no linear order of size* μ *which is universal for linear orders of size* κ *(namely, such that every linear order of size* κ *is embedded in it).*

Maybe it is time to say what the olive property is. It means:

$\boxplus$ There are $(\varphi_0, \varphi_1, \psi)$ and a model $\mathfrak{C}$ of T such that:

- (a) for some $m, \varphi_0 = \varphi_0(\bar{x}_{[m]}), \varphi_1 = \varphi_1(\bar{x}_{[m]}, \bar{y}_{[m]}), \psi = \psi(\bar{x}_{[m]}), \bar{y}_{[m]}, \bar{z}_{[m]})$ are quantifier free formulas (and $\bar{x}_{[m]}, \bar{y}_{[m]}, \bar{z}_{[m]}$ are m-tuples of variables, see Notation 0.10 below)
- (b) for every k and $\bar{f} = \langle f_\alpha : \alpha < k\rangle$ where f_α is a function from α to $\{0, 1\}$ we can find a model M of T and $\bar{a}_\alpha \in {}^m M$ for $\alpha < k$ such that:
  - (α) $\varphi_1[\bar{a}_\alpha, \bar{a}_\beta]$ for $\alpha < \beta < k$ when $1 = f_\beta(\alpha)$.
  - (β) $\psi[\bar{a}_\alpha, \bar{a}_\beta, \bar{a}_\gamma]$ when $\alpha < \beta < \lambda$ and $f_\gamma \upharpoonright [\alpha, \beta]$ is constantly 0.
- (c) there are no $\bar{a}_i \in {}^m M$ for $\ell = 0,1,2,3$ such that the following conditions are satisfied in M:
  - (α) $\varphi_0[\bar{a}_0, \bar{a}_i]$ for $\ell = 1,2,3, \varphi[\bar{a}_i, \bar{a}_\ell]$ for $\ell = 1, 2$ and $\varphi_0[\bar{a}_2, \bar{a}_3]$.
  - (β) $\psi[\bar{a}_0, \bar{a}_2, \bar{a}_3]$.

12. Consistency Results

12.1 Forcing

Naturally, as possibly $\le^0_{\text{univ}}$ is trivial (when GCH holds), we need consistency results. In particular, we have to apply (and develop!) forcing methods.

The first relevant forcing is Shelah (1980a), aiming to prove that some results of Shelah (1990a, Ch. VIII) were best possible. First, it introduces oracle c.c.c. forcing that $T = \text{Th}({}^\omega 2, E_n)_{n<\omega}$ has a universal model in $\aleph_1$ and $\aleph_1 < 2^{\aleph_0}$; this method was inspired by omitting types arguments in model theory. Second, introducing proper forcing (in fact, with oracle) we force that there is a universal linear order in $\aleph_1$ while $\aleph_1 < 2^{\aleph_0}$. This is done proving that there is a non-meagre $A \subseteq \mathbb{R}$, $|A| = \aleph_1$, that realizes as many cuts as possible.

Problem 12.1.

(1) Can we generalize those results to $\lambda = \mu^+ < 2^\mu$, $\mu = \mu^{<\mu}$, at least for μ inaccessible or even a suitable large cardinal?

(2) Similarly for suitably defined λ^+-oracle forcing preserving $\mathrm{non}_\lambda(\lambda - \text{meagre}) = \mu^+$.

(3) Or in the inverse direction, if $\aleph_1 < \lambda < 2^{<\lambda}$, then there is no universal linear order.

The forcing in Shelah (1990b) started with $\mu = \mu^{<\mu} < \lambda < \chi = \chi^\lambda$, a sequence $\langle A_\alpha : \alpha < \chi\rangle$ of members of $[\lambda]^\lambda$ which is say μ-almost disjoint (i.e., $\alpha < \beta < \chi \Rightarrow |A_\alpha \cap A_\beta| < \mu$) gotten by Baumgartner (1976) and then used a ($<\mu$)-support iteration $\mathbf{q} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \chi, \beta < \chi\rangle$. Now $\mathbb{Q}_0$ forces a graph $\underset{\sim}{\mathbf{G}}_0$ on λ by conditions of cardinality $<\mu$, so this is just adding λ many μ-Cohens. Next for $\alpha > 0$, $\underset{\sim}{H}_\alpha$ is a $\mathbb{P}_\alpha$-name of a graph with universe (= set of nodes) λ, and $\mathbb{Q}_\alpha$ adds an embedding of $\underset{\sim}{H}_\alpha$ into $\underset{\sim}{\mathbf{G}}_0 \upharpoonright A_\alpha$. The problem is to choose $\mathbb{Q}_\alpha$ such that we can prove that the μ^+-c.c. is preserved; Mekler (1990) continues this, proving it for many theories (universal theories with no algebraicity and $\mathscr{P}^-(3)$, $\mathscr{P}^-(4)$ -amalgamation). We naturally ask (see Shelah, 2013a):

Problem 12.2.

(1) For $\aleph_0 < \mu = \mu^{<\mu} \ll \lambda \ll \chi = \chi$ find a ($<\mu$)-complete ($<\mu$)-support iteration for forcing notions which are suitably μ^+-c.c. forcing: making $\lambda \in \mathrm{univ}(T)$ for a maximal set of T's; (Or at least for all simple T's.)

(2) Develop a framework for distinguishing T's, e.g., the $T^{\mathrm{hgr}}_{n,k}$ (the theory of $(k+1)$-hyper-graphs not complete on any $n+1$ nodes).

This requires "hair splitting" distinction of forcing. Close to this are Shelah (1993b) and Džamonja and Shelah (2004a), where we restrict ourselves to $\lambda = \mu^+$ and force a sufficient condition which covers T_{ceq} and triangle free graphs. However, those works prove only weak universality i.e., the M universal for $\lambda = \mu^+$ was of cardinality $\lambda^+ < 2^\mu$. It makes sense that for the case $\lambda = \mu^+$ we shall get stronger consistency results.

Problem 12.3.

(1) Are there T and λ such that

(a) There is no universal model in K for λ.

(b) There is a family of λ^+ models in K of cardinality λ which together are universal.
(c) $\lambda^+ \ll 2^\lambda$.

(2) Can we force the full result for $\lambda = \mu^+$ for $T = T_{\rm ceq}$? For T = triangle-free graphs, for simple theories?

To approach this, let us describe the way this is accomplished in the forcing. We start with $\mu = \mu^{<\mu}$ and for $\mu^{++} < 2^\mu = 2^{\mu^+}$. Now we use a $(<\mu)$-support iteration of length μ^{++}, $\mathbf{q} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \mu^{++}, \beta < \mu^{++} \rangle$ where $\mathbb{Q}_\beta$ adds a model $\underset{\sim}{M}_i$ of T with set of elements λ which is universal for all "old" models of T of cardinality λ (so without loss of generality with set of elements λ), old meaning from the universe $\mathbf{V}^{\mathbb{P}_\alpha}$. Clearly this is easier than first forcing the hopeful universal and then try to embed new models into it. Also $\lambda = \mu^+$ should help us.

Problem 12.4. Find the dividing lines suitable for such forcing.

There is a different approach in Shelah and Steprāns (2016) for $\mu = \aleph_0$. The forcing with finite support necessarily adds many μ-Cohens. So if we like to not add Cohen reals or even retain $\mathfrak{d} = \aleph_1$ we have to use CS iteration. Now it is natural to start with a universe $\mathbf{V}$ satisfying CH, so we have a saturated random graph H_0, but after forcing it cannot be saturated but we retain a weak property being "random saturated" or "Cohen saturated" which follows from the forcing preserving "a set of reals is non-null/non-meagre". The forcing method used relies on iterating well-known reals, such as Miller and Laver reals, and alternating this with the PID forcing which adds no new reals.

Problem 12.5. 1) Generalize those methods to other cardinals $\mu > \aleph_0$.

2) Find the maximal family of theories for which this applies.

12.2 Strong limit singular μ

Cardinals near a singular μ, in particular for μ a strong limit, raise special problems. For example, even forcing $2^\mu > \mu^+$ while still controlling the situation below μ has been a serious problem, and necessarily involves "large cardinals", cf. Silver (1975), Macintyre (1977), Magidor (1977), Foreman and Woodin (1991), Gitik (2010) and Abraham and Magidor (2010).

Can we get further consistency results for $\lambda \in (\mu, 2^\mu)$, e.g., like Martin's Axiom or like consistency results on universals? There are some results; Mekler and Shelah (1989) proves that for such λ's we can get uniformization results; by Gitik and Shelah (1998) we can get consistency of the density of base product spaces.

And Džamonja and Shelah (2003) proves that we can force $\lambda = \mu^+ < 2^\mu$, cfμ) = $\aleph_0$ and there is a graph on $\mu^{++} < 2^\mu$ universal for graph in λ^+. Subsequently Cummings et al. (2017) also for the case cf(μ) > $\aleph_0$ and gives a general forcing axiom.

Problem 12.6.
(1) Can we get $\lambda \in \text{univ}_{\text{graph}}$ above?
(2) For which T's does this work?
(3) Generally: state and prove consistency of relevant forcing axioms.

Part III

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13. Introduction to Part III

This part assumes more familiarity with model theory; we shall speculate, point out some answers and give some proofs. Here we shall consider simple theories, suggest relevant properties which hopefully lead to good dividing lines and prove that all simple theories have weak consistency results on "$\text{univ}_{T/K}(\lambda)$ being small" better than in Shelah (1996b, §2).

We had thought for long that maybe NSOP_4 is the right dividing line for universality. An important case is the class of groups (which has various amalgamation properties hence NSOP_4, though it has SOP_3, like triangle-free graphs (see Shelah and Usvyatsov, 2006)) so expect it will have at least weak universality consistency results (like Shelah, 1996b), and even had some typed proofs. Alas, mathematics is a harsh mistress, and proved me wrong on both accounts; by Shelah (2016) group theory is almost $\leq_{\text{univ}}$-maximal; in fact there is a property called "olive property", which implies this (and it is satisfied by the class of groups). However, this property seems ad hoc, and it is doubtful that it is a good candidate for a successful dividing line.

Still, all this tells us that hidden there is a family of theories which has a good structure theory, its models are somehow no more complicated than random graphs, and has the external definition of being almost $\leq_{\text{univ}}$-maximal (or something similar). Just as the main gap theorem (Shelah, 1990a, Ch., Th. 6.1) found the family of theories with models coding set-theoretic information like stationary sets rather than just (generalized) (Shelah, 1987), (earlier see Shelah, 1985; Shelah, 1990a, Ch. XII, XIII).

14. Speculations

14.1 Hyper amalgamation and simple theories

We feel we have considerable knowledge on simple classes, but much remains to be done; consider:

Definition 14.1. Let us fix a theory T.

(1) Here always $\mathscr{P} \subseteq \mathscr{P}(\theta)$ is closed under intersections of two; and includes $\{\{i\} : i < \theta\} \cup \{\emptyset\}$.

(2) $\bar{M}$ is a $(\theta, \mathscr{P})$-*problem* (for T) when: $\bar{M} = \langle M_u : u \in \mathscr{P} \rangle$ satisfies: M_u is a model of T, $\|M\| \leq \theta$; and $M_u \prec M_v$ for $u \subseteq v \in \mathscr{P}$; and $M_u \cap M_v = M_{u \cap v}$ for $u, v \in \mathscr{P}$; and if $u, v_0, \ldots, v_{n-1} \subset v \in \mathscr{P}$ then $\text{tp}(M_u, \cup\{M_{v_l} : l < n\}, M_v)$ does not fork over $\cup\{M_{v_l \cap u} : l < n\}$ inside M_v, where tp means type.

(3) For $\bar{M}$ as above we say $(N, \bar{f})$ is a *solution* of $\bar{M}$ when:

 (a) N is a model of T

$$\bar{f} = \langle f_u : u \in \mathscr{P} \rangle$$

 (b) f_u is an embedding of M_u into N

$$u \subseteq v \in \mathscr{P} \Rightarrow f_u \subseteq f_v$$

 (c) If $u, v_0, \ldots, v_{n-1} \subset v \in \mathscr{P}$, then $\text{tp}\left(f_u(M_u), \cup \left\{f_{v_l}(M_{v_l}) : l < n\right\}, N\right)$ does not fork over $\cup\left\{f_{v_l}(M_{v_l \cap u}) : l < n\right\}$.

(4) T has $(\theta, \mathscr{P})$-existence when every $(\theta, \mathscr{P})$-problem has a solution.

(5) $\bar{M}$ has $(\theta, \mathscr{P})$-uniqueness when $\bar{M}$ is a $(\theta, \mathscr{P})$-problem and: <u>if</u> $(N_\iota, \bar{f}_\iota)$ is a solution of $\bar{M}$ for $\iota = 1, 2$, <u>then</u> there is a $N \in K_T$ and embedding g_ι of N_ι into N for $\iota = 1, 2$ such that $u \in \mathscr{P} \Rightarrow g_1 \circ (f_1 \restriction M_u) = g_2 \circ (f_2 \restriction M_u)$.

(6) We say T has $(\mathscr{P},\mu)$-uniqueness when every $(\theta,\mathscr{P})$-problem $\bar{M}$ has uniqueness provided that: (1) $\Sigma\{\| M \|: u\in\mathscr{P}\}$ is $\leq \mu$. Omitting μ means "for some μ".

Problem 14.2.

(1) Sort out the implications between those properties for any simple theory T, in particular for $\mathscr{P}=\mathscr{P}^-(n)$.

(2) Prove that they determine the place of T under $\leq_{\text{univ}}$ assuming "reasonable" cardinal arithmetic.

Question 14.3. Can we have existence of decompositions such that the existence problems is local, so in Definition 14.1, θ finite is enough?

Recall that in the main gap theorem (see $\bullet_1(b)(\gamma)$ after Thesis 9.1), it is proved for superstable T with none of the three relevant order properties that every model M of T is prime over $\cup\{M_\eta:\eta\in\mathscr{T}\}$ where $\mathscr{T}\subseteq^{\omega>}\| M \|$ is a subtree and $\langle M_\eta:\eta\in\mathscr{T}\rangle$ is a non-forking tree of models. Hence for any T, either T has many models coding complicated objects like stationary sets, or all of its models can be described by suitable dimensions, the machinery developed along the way is not less important. Its proof Shelah (1990a, Ch. 12) uses such $(\theta,\mathscr{P})$-problems.

In our case, we cannot expect anything so strong; but we try to go in this direction, so let us describe a scenario for analysing models of T, assume for transparency that T is super-simple; and M is a model of T of cardinality $\lambda\geq\theta\geq\ |T|$. We can find $\bar{N}=\langle N_u:u\in[\lambda]^{<\aleph_0}\rangle$ which is a $(\theta,[\lambda]^{<\aleph_0})$-problem and $\{M\prec\cup N_u:u\in[\lambda]^{<\aleph_0}\}$ (see more with a proof later).

Now if T has $(\theta,\mathscr{P}^-(n))$-existence holds for every n, then irrespective of what is T, we can reduce the problem of "$\lambda\in\text{univ}_T$"to "there is a universal $f:[\lambda]^{<\aleph_0}\to 2^{\aleph_0}$". Restricting ourselves to "reasonable λ's" we can use only $f:[\lambda]^{<\aleph_0}\to\{0,1\}$, which can be translated to $\lambda\in\text{univ}(T_{\text{un}(<\omega)})$, where $K_{T_{\text{un}(<\omega)}}=$ the class of $\tau_{\text{un}(<\omega)}$-models where $\tau_{\text{un}(<\omega)}=\{R_n:n<\omega\}$, R_n an n-place predicate; hopefully all such T's are equivalent to $T_{\text{un}(<\omega)}$.

Now T being just simple (rather than super-simple) is a burden, and generally moving from a scenario to reality will have its complications. Anyhow let $\mathbf{n}=\mathbf{n}_{ex}(T)\leq\omega$ be such that $(\theta,\mathscr{P}^-(n))$-existence iff $n<1+\mathbf{n}$, and $\mathbf{n}=\mathbf{n}(T)_{uq}\leq\omega$ be such that $(\theta,\mathscr{P}^-(n))$-uniqueness iff $n<1+\mathbf{n}$. We describe how the structure theory is used but expect this will lead us to finding finitary, syntactical characterization too.

Relatives of such properties should be relevant also to the $\leq_{\text{SP}}$-problem (see Shelah and Ulrich, 2018), whereas Keisler's order seems to be of a different character (see Malliaris and Shelah, 2019). By such arguments we can prove that any simple T is not pseudo $\leq_{\text{univ}}$-maximal.

Discussion 14.4. Though the existence properties above are nice and natural, they do not provide the right dividing lines for the problems we are interested in, they are just first approximations. Why?

Naturally, we assume:

$(*)_1$ $\lambda < 2^{<\lambda}$.

Let $\mu = \min\{\mu : \lambda \leq 2^{\mu}\}$. Now if $\mu^{|T|} \geq \lambda$ this is connected to the SCH, which introduces extra set-theoretic complications, so we shall concentrate on the case:

$(*)_2$ $\mu = \mu^{|T|}$ and even $\mu = \mu^{<\mu}$, $\mu > 2^{|T|}$ for transparency (the point is that it is natural when we try to force).

To analyse $M \in K(\lambda)$ by Claim 16.2 below we can consider an NF-diagram $\bar{M} = \langle M_u : u \in [\lambda]^{\leq 2^{|T|}} \rangle$, we may replace $2^{|T|}$ e.g., by $|T|$. Now choose $\bar{\eta} = \langle \eta_\alpha : \alpha < \lambda \rangle \in {}^{\lambda}(\mu^2)$, the η_α's pairwise distinct. After mild forcing letting $\bar{\mathbf{a}}_u$ list the elements of M_u, we may assume:

$(*)_3$ $(\bar{M}, \bar{\eta})$ is tree indiscernible, i.e., $(a) \Rightarrow (b)$, where
- (a) (α) $\alpha(\eta, \iota) < \lambda$ for $n < n(*)$, $\iota < 2$
  - (β) if $n < m < n(*)$ and $\iota < 2$ then $\alpha(n, \iota) \neq \alpha(m, \iota)$.
  - (γ) $\varepsilon < \mu$ and.
  - (δ) if $\iota < 2$ then $\langle \eta_{\alpha(n,\iota)} \restriction \varepsilon : \eta < n(*) \rangle$ is without repetitions.
  - (ε) $\eta_{\alpha(n,0)} \restriction \varepsilon = \eta_{\alpha(n,1)} \restriction \varepsilon$ for $n < n(*)$
- (b) there is f such that:
  - (α) f is an isomorphism from $M_{\{\alpha(n,0) : n < n(*)\}}$ onto $M_{\{\alpha(n,1) : n < n(*)\}}$.
  - (β) for $u \subseteq n(*)$, f maps $\mathbf{a}_{\{\alpha(n,0) : n \in u\}}$ to $\mathbf{a}_{\{\alpha(n,1) : n \in u\}}$.

Hopeful Theorem 14.5. *If (A) then (B) where*:

(A) (a) $\kappa < \mu = \mu^{<\mu} < \lambda = \lambda^{2^\kappa} < \lambda < \chi = 2^\mu = \chi^\lambda$.

(b) $\mathscr{A} \subseteq [\lambda]^\lambda$ is $\mu -$ AD *of cardinality* χ.

(B) *there is a forcing notion* $\mathbb{P}$ *such that*:

(a) $\mathbb{P}$ *is* ($<\mu$)*-complete*, μ^+*-c.c. (even* $*^{\omega}_{\mu}$ *holds, see Shelah*, *2013a) and has cardinality* 2^μ.

(b) *in* $\mathbf{V}^{\mathbb{P}}$, *for any complete simple first-order theory T cardinality* $\leq\kappa$, *we have* $\text{univ}_T(\lambda) \leq \lambda^+$.

Discussion 14.6. It is enough to prove:

(*) there is $\mathbb{P}$ such that:
  (a) as in (B)(a)
  (b) for every complete simple first-order T of cardinality $\leq\kappa$, in $\mathbf{V}^{\mathbb{P}}$ there is $M \in K_{T,\lambda}$ into which every $M' \in K_{T,\lambda}$ can be embedded.

We hope to continue this.

14.2 Non-existence of universals

For $T = T_{\text{dlo}}$ (or T which is SOP_4 or have the olive property) we know it is almost $\leq_{\text{univ}}$-maximal, but maybe it is even $\leq_{\text{univ}}$-maximal. There are two separate cases: singular cardinals and regular cardinals.

For the singular case, trying to look for an exact reference, I see (as I recall) that the results in Kojman and Shelah (1992a) for regular $\lambda > \aleph_1$, use linear order with many cuts in the proof but not as an assumption in the theorem; however I discover, for singular λ, there is such a demand in the theorem. So it says less than what I attributed to it (so my memory improves the result, but alas, not the proof). For singular λ there are not only pcf demands but:

$\boxplus_\lambda$ there is a linear order I of cardinality $<\lambda$ with $>\lambda$ Dedekind cuts.

and moreover.

$\boxplus^+_\lambda$ for arbitrarily large regular $\kappa<\lambda$, there is a tree with κ nodes and $>\lambda$ κ-branches.

Why is not required for regular λ? If $\kappa = \min\{\kappa : 2^\kappa > \lambda\}$ and $\lambda < 2^{<\lambda}$ then $\kappa < \lambda$ hence $2^{<\kappa}$ is the sum of $<\lambda$ cardinals $\leq\lambda$ hence $2^{<\kappa} \leq \lambda$. Let $\langle \nu_\alpha : \alpha < \lambda^+\rangle$ be a sequence of pairwise distinct members of ${}^\kappa 2$ and let $\langle \eta_\zeta : \zeta < 2^{<\kappa}\rangle$ list with no repetitions ${}^{\kappa>}2$. If $2^{<\kappa} < \lambda$ we are done, so assume $2^{<\kappa} = \lambda$. Recalling λ is regular, for every $\alpha < \lambda^+$ there is $\varepsilon(\alpha) < \lambda$ such that $\{\nu_\alpha \upharpoonright \zeta : \zeta < \kappa\} \subseteq \{\eta_\zeta) : \zeta < \varepsilon(\alpha)$. So for some $\varepsilon < \lambda$ the set $\alpha < \lambda^+$ " $\varepsilon(\alpha) = \varepsilon$ has cardinality λ^+, hence the closure under initial segment of $\{\eta_\zeta : \zeta < \varepsilon\}$ exemplify $\boxplus_\lambda$. But for λ singular, this is not the case, so Kojman and Shelah (1992a) was correct to distinguish but does not explicate the natural problem below; and Shelah (2016) (as not said there) makes a real advance on the singular case: For T with (any variant of) the olive property we have only pcf obstacles. So it is natural to ask:

Question 14.7. Is the use of $\boxplus^+_\lambda$ for singular λ, in proving $N\text{univ}_{T_{\text{dlo}}}(\lambda)$ for $\lambda \notin u$, necessary?

This is clarified to some extent by:

Claim 14.8. If (A) + (B) then (C) <u>when</u>:
(A) (a) *κ is a Mahlo cardinal.*
  (b) *$\lambda > \kappa$ is strong limit singular of cofinality κ.*

(c) $\kappa_i = \kappa_i^{<\kappa_i}$ *is increasing for* $i < \kappa$ *with limit* κ.
(d) $\lambda_i = \lambda_i^{\kappa_i} \in (\kappa, \lambda)$ *is increasing with limit* λ
(e) $\lambda_i > \prod_{j<i} 2^{\lambda_j}$
(f) for limit $\delta < \kappa$ we have $2^{\sup\{\kappa_i : i < \delta\}} = (\sum\{\kappa_i : i < \delta\})^+$ and $\prod_{i<\delta}\lambda_i = (\Sigma\{\lambda_i : i < \delta\})^+$.

(B) (a) let $\mathbb{Q}_i = \mathrm{Cohen}_{\kappa_i}(\lambda_i)$, the forcing of adding λ_i many κ_i-Cohens so $\mathbb{Q}_i$ satisfies the κ_i^+-c.c. and is κ_i-complete of cardinality λ_i.
(b) $\mathbb{P} = \prod\limits_{i<\kappa} \mathbb{Q}_i$ is the product with the Easton support.

(C) in $\mathbf{V}^{\mathbb{P}}$ we have:
(a) cardinals and cofinalities are the same as in $\mathbf{V}$.
(b) $\kappa_i = \mathrm{cf}(\kappa_i)$ increasing with $i < \kappa = \mathrm{cf}(\kappa)$ and $\lambda_i > \kappa > \kappa_i$.
(c) $2^{\kappa_i} = \lambda_i, 2^{<\kappa} = \sum\limits_{i<\kappa} \lambda_i = \lambda$ and $\theta < \kappa_0 \Rightarrow 2^\theta = (2^\theta)^{\mathbf{V}} < \kappa$.
(d) λ is singular, $\lambda = \sum\limits_{i<\kappa} 2^{\kappa_i}$ hence $\kappa = \mathrm{cf}(\lambda)$ and $\lambda < 2^\kappa$.
(e) if I is a linear order of cardinality $<\lambda$, <u>then</u> I has $\leq\lambda$ cuts.
(f) if $\theta = \mathrm{cf}(\theta) < \lambda$ and $\mu \in [\theta, \lambda)$, <u>then</u> $\mathrm{trp}_\theta(\mu) \leq (\mu^\theta)^{\mathbf{V}} < \lambda$.

Proof. 1) We are assuming κ is a Mahlo cardinal so, recall.

$(*)_1$ (A) $p \in \mathbb{Q}_i$ <u>iff.</u>
(a) p is a function
(b) $\mathrm{dom}(p) \in [\lambda_i]^{<\kappa_i}$
(c) if $\alpha \in \mathrm{dom}(p)$ then $p(\alpha) = {}^{\kappa_i >}2$.
(B) order natural.
(C) the generic of $\mathbb{Q}_i$ is $\bar{\underset{\sim}{\eta}}_i = \langle \underset{\sim}{\eta}_{i,\alpha} : \alpha < \lambda_i \rangle, \underset{\sim}{\eta}_{i,\alpha} \in {}^{(\kappa_i)}2$.

Now

$(*)_2$ (a) for $u \subseteq \lambda_i$ let $\mathbb{Q}_{i,u} = \{p \in \mathbb{Q}_i : \mathrm{dom}(p) \subseteq u\}$.
(b) for $v \subseteq \kappa$ and $\bar{u} = \langle u_i : i \in v \rangle$ with $u_i \subseteq \lambda_i$ let ${}_{\bar{u}} = \prod_{i\in v} \mathbb{Q}_{i,u_i}$ product with Easton support.
(c) for $u \subseteq \lambda_i$ we have $\mathbb{Q}_{i,u} \lessdot \mathbb{Q}_i$.
(d) for $v \subseteq \kappa$ and $\bar{u}$ as above, $\mathbb{P}_{\bar{u}} \lessdot \mathbb{P}$.

The cardinal arithmetic is well known so should be clear. Clause (e) of (C) of the claim follows from clause (f). To prove it, i.e., concerning $\mathrm{trp}_\theta(\mu)$ where $\theta = \mathrm{cf}(\theta) \leq \mu < \lambda$, toward contradiction assume $\mu < \lambda$, $\theta = \mathrm{cf}(\theta) \leq \mu$ and $p \Vdash_{\mathbb{P}}$ "$\mathscr{T}$ is a subtree of ${}^{\theta>}\mu$ with $\leq\mu$ nodes and $>\lambda$ many θ-branches". As $\Vdash_{\mathbb{P}} \lambda = \Sigma\{2^{\kappa_i} : i < \kappa\}, \lambda_i = 2^{\kappa_i}$ increasing with i, necessarily $\theta \geq \kappa$, so $\lambda > \mu \geq \theta \geq \kappa$.

As $\mathbb{P}$ satisfies the κ^+-c.c., we can find $\bar{u} = \langle u_i : i < \kappa \rangle, u_i \in [\lambda_i]^{\leq\mu}$ such that $p \in \mathbb{P}_{\bar{u}}$ and $\underset{\sim}{\mathscr{T}}$ is a $\mathbb{P}_{\bar{u}}$-name. Now $\mathbb{P}_{\bar{u}}$ has cardinality $\leq \mu^\kappa$ which is $<\lambda$ because λ is a strong limit cardinal. As $\theta > \kappa$ or $\theta = \kappa$ recalling κ is Mahlo clearly $\mathbb{P}$ satisfies the θ-c.c. so the forcing notion $\mathbb{P}/\mathbb{P}_{\bar{u}}$ satisfies the θ-c.c. hence it does not add θ-branches to $\underset{\sim}{\mathscr{T}}$, contradiction. □

For regular λ the problem is different. Necessarily, λ is a successor cardinal, say $\lambda = \mu^+$ so (see earlier) $2^{<\mu} \leq \lambda$. We may need better guessing of clubs (see Shelah, 1994, Ch. III), and probably division to cases will help. A very different problem but with related character appears when for $\lambda \in (\mu, \mu^{\aleph_0})$ where $\mu = \Sigma \mu_n, \mu_n^{\aleph_0} < \mu_{n+1}$ for $n < \omega$ and T is strictly stable. The most puzzling case is $\lambda = \mu^+$ unlike e.g., $\lambda = \mu^{++} < \mu^{\aleph_0}$, in the case $\lambda = \mu^+$ we do now know strong enough guessing of clubs.

The arguments above may lead to a nice complete non-existence answer, but it may well reveal a hidden dividing line among those theories.

14.3 Back to dividing lines

First, while the olive property (see Theorem 9.8) is not satisfactory, probably there is a weaker, neater, "right" property which will give necessary and sufficient condition, at least for almost $\leq_{\text{univ}}$-maximality. This calls us to analyse the proof in Shelah (2016) and/or try to find a better proof; all this is trying to find the dividing line from up.

We can try to look from below. A weak version of having universal concern.

Question 14.9. Given T, when does $(A) \Rightarrow (B)$?

(A) $\mu = \mu^{<\mu} > |T|, \lambda = \mu^+, \bar{M} = \langle M_\alpha : \alpha < 2^\lambda \rangle$ list the models of T with universe λ and E_α is a thin enough club of λ related to M_α (e.g., definable in $\mathfrak{B}_\alpha = (\mathscr{H}(\beth_2^+(\mu)), \in, <_*, \theta, \mu, M_\alpha)$, where $<_*$ is a well-ordering of $\mathscr{H}(\beth_2^+(\mu))$; let $E_\alpha = \{\delta < \lambda : \text{the Skolem Hull } \mathfrak{B}_{\alpha,\delta} \text{ of } \{\iota : \iota < \delta\} \text{ in } \mathscr{H}(\beth_2^+(\mu)) \text{ satisfies } \mathfrak{B}_{\alpha,\delta} \cap \lambda = \delta\}$)

(B) there are $(\mathbb{Q}, M)$ such that:

- (a) $\mathbb{Q}$ is $(<\mu)$-complete, μ^+-c.c. moreover a $*^\varepsilon_\mu$-forcing notion, (Shelah, 2013a)
- (b) $\underset{\sim}{M}$ is a $\mathbb{Q}$-name of a model of T with universe λ
- (c) $\Vdash_{\mathbb{Q}}$ "M_α is embeddable into $\underset{\sim}{M}$", moreover, if $\delta \in E \wedge \text{otp}$ $(\delta_1 \cap E_\alpha) = \beta$ then the embedding maps $M_\alpha \delta$ into $M(\lambda + \lambda\beta)$.

This is the point of Shelah (1993b, 1996b), Džamonja and Shelah (2004a), so we get a positive answer for every simple T, and for some non-simple T, but those theories are the parallel of "simple T with trivial forcing notions".

Problem 14.10. Does the theory of non-forking for NSOP_1 theories suffice to get such a result? This is fine but $K_{\text{triangle-free}}$ is a SOP_3 theory for which that answer to Question 14.9 is positive.

Problem 14.11. Is there a relevant definition of non-forking describing a best positive answer for Problem 14.10?

14.4 Forcing near regular

The case we concentrate on here is $\lambda \in [\mu, \chi)$, $\mu = \mu^{<\mu} < \lambda < \chi = \chi^{\lambda}$ and even GCH holds between μ and λ; an important special case is $\lambda = \mu^{+}$. The forcing of Shelah (1990b) being ($<\mu$)-support, μ^{+}-c.c. has some limitation. It is persuasive to remember Shelah and Steprāns (2016), which allows us to preserve for $\mu = \aleph_0$, to preserve $=\aleph_1$ and even "${}^{\omega}2$ is non-null"; alternatively "${}^{\omega}2$ is non-meagre". So the universal model can have, even in $\mathbf{V}^{\mathbb{P}\alpha}$, some distinct remnants of having been saturated (in the original $\mathbf{V}$).

Problem 14.12.

(1) Develop iterated forcing for such universality problems.

(2) In particular, force "$\lambda = \mu^{+} < 2^{\mu}$ and $\lambda \in \mathrm{univ}(K_{\text{triangle}-\text{free}})$".

(3) For which T's this work?

(4) Can we, in Question 14.9, use Q a creature forcing (at least for μ inaccessible)?

14.5 Back to $\aleph_1$

Clearly, the situation in $\aleph_1 < 2^{\aleph_0}$ is different.

We would like to characterize the (countable complete) T's with $\mathbf{D}(T) = \{\mathrm{tp}(\bar{a}, \emptyset, M) : M \in K_T, \bar{a} \in {}^{\omega >}M\}$ countable such that consistently T has a universal member. By Kojman and Shelah (1992a) for some such T's this fails so the problem is delicate; hence we may first concentrate on the superstable case; for one such theory this is done in Shelah (1980a).

An interesting case is K_{lfg}, locally finite groups; the results cited earlier do not cover this case, as it is not first order and fails amalgamation for countable models, still have the relevant case of the olive property. A proof of non-existence will require finding finite groups behaving as in the example above, so require using at least some finite groups theory. But we may try to work as in §(5D), to force existence recalling that we may use the so called creature forcing (see Rosłanowski and Shelah, 1999); that requires a choice of so called norms; again those norms has to be related to finite group theory.

14.6 Non-simple theories

As hinted earlier, we feel that $T_{\rm ceq}$ should be a test case. It is not simple but seems to us to be the "simplest non-simple". This may indicate trying to prove things for all $\rm NSOP_1$ theories, which seems reasonable, but we like to include also $T_{\rm trf}$, the theory of triangle-free graphs which is not even $\rm NSOP_3$. It seems reasonable that, for many cardinals, $\lambda = {\rm cf}(\lambda) \in (\mu, 2^\mu)$ we shall succeed to prove $\lambda \notin {\rm univ}(T_{\rm ceq})$, still for this we probably need to have better club guessing and/or better ways to use them.

But maybe we better start also from the other direction. Recall we have gotten a weak consistency result for the existence of a weak universal for $T_{\rm ceq}$ (it is of cardinality λ^+, is universal for $\lambda = \mu^+$ and $\lambda^+ < 2^\mu$). We should try to get a real universal in λ, maybe using creature forcing for inaccessibles. Of course, all this will be preliminary work for achieving this for the class of T's similar enough to $T_{\rm ceq}$. This will require developing a new non-forking theory under quite hard conditions (for $\rm NSOP_1$ there is one but not for $\rm NSOP_3$).

15. Consistency for Simple T

Claim 15.1. *Assume T is a complete simple first-order theory. T has $\mathscr{P}$-existence*[4] *when*:

(*) $\mathscr{P}$ *for some* $n, \ell(*), \bar{v}$ *we have*:

(a) $\mathscr{P} \subseteq \mathscr{P}(I)$ *is downward closed where* $I_\iota = [\iota n, \iota n + n)$ *for* $\iota = 0,1,2$ *and* $I = \bigcup_\iota I_\iota$

$$\mathscr{P}_1 = \mathscr{P}(I_0 \cup I_1) \cap \mathscr{P}$$

$$\mathscr{P}_2 = \mathscr{P}(I_0 \cup I_2) \cap \mathscr{P}$$

(b) *for some* $\langle v_\ell : \ell < \ell(*) \rangle$ *we have* $\mathscr{P} = \cup \{ \mathscr{P}(v_\ell) : \ell < \ell(*) \} \cup \mathscr{P}_1 \cup \mathscr{P}_2$ *and* $\ell < k < \ell(*) \Rightarrow v_\ell \cap v_k \subseteq I_0$.

Proof. By induction on $\ell(*)$.

If $\ell(*) = 0$ or $\ell(*) = 1$ by basic properties of non-forking.

4 See Definitions 14.1 and 15.

If $\ell(*) > 1$ let $J = I_0 \cup (I_1 \backslash v_0) \cup (I_2 \backslash v_0)$ and $\mathscr{P}_* = \mathscr{P}(J)$. By the induction hypothesis we can find a non-forking $\bar{N} = \langle N_u : u \in \mathscr{P}_* \rangle$ such that $u \in \mathscr{P}_* \cap \mathscr{P} \Rightarrow N_u = M_u$. By the case $\ell(*) = 1$, there is a non-forking diagram $\bar{M}' = \langle M'_u : u \subseteq I_0 \cup v_0 \rangle$ such that $u \in \mathscr{P}(I_0 \cup v_0) \cap \mathscr{P} \neq M'_u = M_u$. Applying twice the basic properties we are done. $\square$

Discussion 15.2. Can we apply 15.1 to prove the "hopeful theorem" 14.5?

Discussion 15.3. 1) Assume $\theta < \mu = \mu^{<\mu} < \lambda$, T is simple, $2^{|T|} \leq \theta$ for transparency. Let $\mathbb{Q}_0 = \mathbb{Q}^0_{\lambda,\mu} = \mathbb{Q}_{\lambda,\mu,T}$ be defined as follows:

(A) $p \in \mathbb{Q}$ <u>iff:</u>

$$p = \langle M_{p,u} : u \in [p]^{\leq |T|} \rangle$$

$$v_p \in [\lambda]^{<\mu}$$

- (a) $M_{p,u}$ has universe $\mathscr{P}(u) \times \theta$
- (b) p is a non-forking system of models of T

(B) $\leq_{\mathbb{Q}}$ is defined naturally

(C) the generic is $\langle N_u : u \in [\lambda]^{\leq \theta} \rangle$.

2) Assume that $\underline{T} = \langle (\lambda_i, T_i) : i < i(*) \rangle$ each (μ_i, λ_i, T_i) is as above for each i. Let $\mathbb{Q}_{\mu,\underline{T}}$ be the product $\Pi\left\{\mathbb{Q}^0_{\lambda_0,\mu,T_i} : i < i(*)\right\}$ with support $<\mu$ and $\langle N_{0,i,u} : u \in [\lambda_i]^{\leq |T_i|, i < i(*) SSY} \rangle$ the generic, as above.

Claim 15.4. *Assume* $\mu = \mu^{<\mu} < \lambda < \chi = \chi^\lambda$, let $\bar{T} = \langle (\lambda_i, T_i) : i < i(*) \rangle$ *be as in Definition 15.3(2).*

(1) $\mathbb{Q}_{\lambda_i,\mu,T_i}$ satisfies $*^\omega_\mu$.

(2) So does $\mathbb{Q}_{\mu,\bar{T}}$ the product $\Pi\{\mathbb{Q}_{\lambda_i,\mu,T_i} : i < i(*)\}$ with support $<\mu$.

Proof. Should be clear. $\square$

Hypothesis 15.5. 1) $\aleph_0 < \mu = \mu^{<\mu} < \lambda_* \leq \chi = \chi^{<\lambda_*}$.

2) $\mathscr{A}_\lambda \subseteq [\lambda]^\lambda$ for $\lambda \in (\mu, \lambda_*)$ is $\mu-$AD of cardinality χ (justified by Baumgartner, 1976).

3) $\langle A^\bullet_\alpha : \alpha \in [1,\chi) \rangle$ list $\cup_\lambda \mathscr{A}_\lambda$ with no repetitions with $A^\bullet_\alpha \in \mathscr{A}_{\lambda_{i(\alpha)}}, i(\alpha) \leq i(*)$ and is the increasing function from $\lambda_{\iota(\alpha)}$ onto $A^\bullet_\alpha$.

4) $\bar{T} = \langle (\lambda_i, T_i) : i < i(*) \leq \chi \rangle$ list the pairs (λ, T) such that:

• $\lambda_i \in [\mu^+, \lambda_*)$, $T_i \in \mathscr{H}(\lambda_*)$ is simple complete first order $\theta_i = \theta(i) = 2^{|T_i|}$.

Definition 15.6. Let **Q** be the class of **q** consisting of:

(a) $\alpha_{\mathbf{q}} = \alpha(\mathbf{q}) = \ell g(\mathbf{q})$, the length

(b) $\langle \mathbb{P}_\alpha \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \alpha_{\mathbf{q}} \beta < \alpha_{\mathbf{q}} \rangle$ is $(<\mu)$-support iteration

(c) $\mathbb{Q}_0 = \mathbb{Q}_{\mu, \bar{T}}$ and $\bar{N} = \{\underset{\sim}{N}_{0,i,u} : i < i(*), u \in [\lambda_i]^{\leq |T_i|}\}$ and $\mathbb{P}_{\mathbf{q}} = \mathbb{P}_{\alpha(\mathbf{q})}$ and $\bar{T}, \bar{N}$ are as in 15.3(2)

(d) for $\alpha \in (0, \chi)$

(α) $i(\alpha) = i(\alpha) < i(*)$.

(β) $\underset{\sim}{M} = \langle \underset{\sim}{M}_{\alpha,u} : u \in [\lambda_{i(\alpha)}]^{\leq \theta(i(\alpha))} \rangle$ is a $\mathbb{P}_\alpha$-name of a non-forking diagram of models of T_i such that $\underset{\sim}{M}_{\alpha,u}$ has universe $\mathscr{P}(u) \times \theta(i_\alpha)$

(e) for $\alpha \in [1, \lg(\mathbf{q}))$, $\underset{\sim}{\mathbb{Q}}_\alpha$ is defined by (the order is inclusion)

(*) $p \in \mathbb{Q}_\alpha$ <u>iff</u>
- p is a function with domain $\in [\lambda_{i(\alpha)}]^{<\mu}$
- if $\beta \in \mathrm{dom}(p)$ then $p(\beta) < \mu$
- every $u \in [\mathrm{dom}(p)]^{<\theta(i(\alpha))}$ the set $f_p = \{\langle (v,\zeta), (h''_\alpha(v), \zeta) \rangle : v \subseteq u, \zeta < \mu\}$ is a function; moreover, an isomorphism from $\underset{\sim}{M}_{\alpha,u}$ onto $\underset{\sim}{N}_{0,i(\alpha),h''_\alpha(u)}$ mapping $\underset{\sim}{M}_{\alpha,v}$ onto $\underset{\sim}{N}_{0,i(\alpha),h''_\alpha(v)}$ for every $v \subseteq u$

(f) $\mathbb{P}'_\alpha$ is a dense subset of $\mathbb{P}_\alpha$ defined by induction on α by:

(*) $\mathbb{P}'_\alpha$ is the set of $p \in \mathbb{P}_\alpha$ such that for some $I = I_p$, $u = u_p$ we have:
(α) $u \in [\lambda_*]^{<\mu}$ and $I \subseteq \mathbf{i}(*)$.
(β) assuming $\mathrm{dom}(p) \neq \emptyset$ we have.
($\bullet_1$) $0 \in \mathrm{dom}(p)$ and $\mathrm{dom}(p(0)) = I$.
($\bullet_2$) $i \in \mathrm{dom}(p(0)) \Rightarrow \mathrm{dom}(p(i)) = u \cap \lambda_i$.
($\bullet_3$) if $0 < \alpha \in \mathrm{dom}(p)$ then $(\alpha) \in I$ and $\mathrm{dom}(p(\alpha)) = u \cap \lambda_{(\alpha)}$.
($\bullet_4$) if $0 < \alpha \in \mathrm{dom}(p)$ and $v \in [u_p]^{\leq |T_i|}$ then the $\mathbb{P}_\alpha$-name $\underset{\sim}{M}_{\alpha,v}$ is defined by maximal antichains included in $\mathbb{P}'_\alpha$ and for transparency we fix it. Note that $p(\alpha)$ is an object, not just a $\mathbb{P}_\alpha$-name.

Hopeful Theorem 15.7. If $\mathbf{q} \in \mathbf{Q}$ then the forcing notion $\mathbb{P}_{\mathbf{q}} = \mathbb{P}_{\mathbf{q},\ell g(\mathbf{q})}$ is $(<\mu)$-complete satisfying the μ^+-c.c. (and even $*^\omega_\mu$?).

Discussion 15.8. How do we intend to prove 15.7? Clearly:

$(*)_1$ for $\alpha \le \alpha_{\mathbf{q}}$, $\mathbb{P}_\alpha$ is $(<\mu)$-complete; moreover, for any increasing sequence $\langle p_\alpha : \alpha < \delta\rangle$ of members of $\mathbb{P}'_\alpha$ of length $\delta < \mu$ the union naturally defined is a lub.

$(*)_2$ (a) let $\mathbf{T}$ be the set of triples $(I, \mathscr{U}, \mathscr{W})$ such that $I \subseteq i(*), \mathscr{U} \subseteq \lambda_*,, \mathscr{W} \subseteq \alpha_{\mathbf{q}}$, each of cardinality $\le\mu$ ordered naturally.

(b) for $(I, \mathscr{U}, \mathscr{W}) \in \mathbf{T}$ we define $\mathbb{P} = \mathbb{P}'_{I,\mathscr{U},\mathscr{W}}$ by induction on $\alpha = \cup \{\beta+1 : \beta \in \mathscr{W}\}$ as follows: $p \in \mathbb{P}$ iff:

(α) $p \in \mathbb{P}'_\alpha$ as witnessed by I_p, u_p.

(β) $I_p \subseteq I, u_p \subseteq \mathscr{U}, \mathrm{dom}(p) \subseteq \mathscr{W}$.

(γ) if $\beta \in \mathrm{dom}(p) \setminus \{0\}$ then the maximal antichains in the definition of $p(\beta)$, see 15.6(f)(*)(β)($\bullet_4$), are included in $\mathbb{P}_{I,\mathscr{U},\mathscr{W}\cap\beta}$.

$(*)_3$ we say a triple $(I, \mathscr{U}, \mathscr{W})$ as above is $\mathbf{q}$-closed when: if $\alpha \in \mathscr{W}$ and $v \subseteq \mathscr{U}$ has cardinality $\le |T_{\mathbf{i}(\alpha)}|$ then the maximal antichains defining $\underset{\sim}{M}_{\alpha,v}$ are $\subseteq \mathbb{P}_{I,\mathscr{U},\mathscr{W}\cap\alpha}$.

$(*)_4$ if $(I_1 \mathscr{U},_1 \mathscr{W},_1) \in \mathbf{T}$, then there is a triple $(I_2, \mathscr{U}_2, \mathscr{W}_2) \in \mathbf{T}$ above $(I_1, \mathscr{U}_1, \mathscr{W}_1)$.

[Why? Should be clear.]

$(*)_5$ if $p \in \mathbb{P}'_{\mathbf{q}}$, then $p \in \mathbb{P}_{I,\mathscr{U},\mathscr{W}}$ for some $(I, \mathscr{U}, \mathscr{W}) \in \mathbf{T}$

$(*)_6$ if $\langle (I_\varepsilon, \mathscr{U}_\varepsilon, \mathscr{W}_\varepsilon) : \varepsilon < \delta\rangle$ is an increasing sequence of closed members of $\mathbf{T}$ and $\mathrm{cf}(\delta) = \mu$, then the union is a closed member of $\mathbf{T}$.

[Why? Think!]

Now to finish:

$(*)_7$ assume $p_\zeta \in \mathbb{P}_{\mathbf{q}}$ for $\zeta < \mu^+$ and we should prove then for some $\zeta_1 < \zeta_2 < \mu^+$, the conditions p_{ζ_1}, p_{ζ_2} are compatible.

$(*)_{7.1}$ (a) without loss of generality $p_\zeta \in \mathbb{P}'_{\mathbf{q}}$.

(b) there is a sequence $\langle (I_\zeta, \mathscr{U}_\zeta, \mathscr{W}_\zeta) : \zeta < \mu^+ \rangle$ of closed members of $\mathbf{T}$ such that $p_\zeta \in \mathbb{P}_{I_{\zeta+1}, \mathscr{U}_{\zeta+1}, \mathscr{W}_{\zeta+1}}$.

We intend to continue in (Shelah, 1999b).

16. Non-forking Systems

We give a representation theorem for models of a simple T. As we are interested in universality, we stress the case where we do not mind increasing the model, though we shall comment on the other case (in the end, 16.10). Naturally, the representation helps for proving results on consistency of existence of universal models. The next steps should be, on the one hand to deal with consistency results (of existence) and on the other hand connect those with §15, that is with the $\mathscr{P}^-(n)$-existence properties. But those issues are outside the scope of this work.

Hypothesis 16.1.
(1) T is simple, hence $\kappa(T) \leq |T|^+$.
(2) $\mathfrak{C} = \mathfrak{C}_T$ is a monster model for T.
(3) We have $\theta \geq |T|$, $\theta^+ \geq \kappa = \text{cf}(\kappa) \geq \kappa(T)$ and (but we shall mention when we use it) $\theta \leq 2^{|T|}$ (e.g., $\theta = |T|$, $\kappa = \theta^+$).

Claim 16.2. *Assume $M_* \prec \mathfrak{C}_T$ has cardinality $\leq\theta$ and $\mathbf{a} = \langle \bar{a}_\alpha : \alpha < \lambda \rangle, \bar{a}_\alpha \in {}^{\omega >}(\mathfrak{C}_T)$ for $\alpha < \lambda$ with no repetitions for transparency.*

We can find $= (\bar{I}, \bar{u}, \bar{M}) = (\bar{I}_{\mathrm{s}}, \bar{u}_{\mathrm{s}}, \bar{M}_{\mathrm{s}})$ such that:

$\boxplus$

(a) *$i_{\mathrm{s}} = i(\mathrm{s}) \leq \kappa$ and $\bar{I} = \langle I_i : i < i_{\mathrm{s}} \rangle$ and I_i is a set of ordinals of cardinality $\leq\lambda$ disjoint to $I_{<i} := \bigcup_{j<i} I_j$, (even of order type $\leq\lambda$), and $I_0 = \{0\}$*

(b) *$\bar{u} = \langle \bar{u}_i : i < i_{\mathrm{s}} \rangle$ and $\bar{u}_i = \langle u_s = u_{i,s} : s \in I_i \rangle$*

(c) *$u_{i,s} \subseteq I_{<i}$ has cardinality $<\kappa$* and *$t \in u_s \Rightarrow u_t \subseteq u_s$*

(d) *$\bar{M} = \langle \bar{M}_i : i < i_* \rangle$ and $\bar{M}_i = \langle M_s = M_{i,s} : s \in I_i \rangle$ and $M \prec \mathfrak{C}_T$ where $M = \cup \{M_s : s \in I_{<i(\mathbf{s})}\}$*

(e) *(α) $M_t \prec \mathfrak{C}_T$ has cardinality $\leq\theta$,*
(β) $M_0 \prec M_t$.
(γ) $s \in u_t \Rightarrow M_s \prec M_t$ for $t \in I_i$

(f) *(non-forking) if $t \in I_i$ then* tp*(M_t, $\cup \{M_s : s \in I_{<i}$* or *$s \in I_i \backslash \{t\}\}$) does not fork over $\cup \{M_s : s \in u_t\}$*

$$\bar{a}_\alpha \in \cup \left\{ {}^{\omega >}(M_s) : s \in I_{<i(\mathrm{s})} \right\}$$

(g) *<u>notation</u> let $I = I_{<i_{\mathrm{s}}}$ and let $u_s^+ = u_s \cup \{s\}$ and for $s \in I_i$ let $A_s = \cup \{M_t : t \subset s$ so $t \in I_{<i}\}$*

(h) *(α) if the theory T is stable in θ, then the model $(M_s, a)_{a \in A_s}$ is saturated, for every $s \in I_i$*
(β) if $\sigma > |T|$ and $\theta^{<\sigma} = \theta = 2^{|T|}$ (or just $|\mathbf{D}(T)|$ has cardinality $\leq\theta$) then the model M_s is σ-saturated, for every $s \in I_i$.

1A) We can in part (1) replace clauses (e)(α), (i) by:
(e) (α)′ if $t \in I_i$ then M_t has cardinality $\beth_{i+1}(T)$.

(i)′ if $s \in I_i$ then the model $(M_s, a)_{a \in A_s}$ is $(\beth_i(T))^+$-saturated and even $(\beth_i(T))^+$-resplendent, recalling ⊞*(h),*

(j) *[follows] if p is a complete type over M_s where $s \in I_i$ and $t \in I_j$, $j > i$ and $s \subseteq t$ then there is $b \in M_t$ which realizes p and moreover it realizes in M_t over A_t a complete type not forking over M_s; (we can add this to part (1) when T is stable in θ).*

1B) We can in part (1) replace clauses (e)(α), (i) by:

(e) $(\alpha)''$ if $t \in I_i$ then M_t has cardinality $|T|$.

(i)″ we can choose $\mathbf{b}_s$ for $s \in I$ and E, $\langle (f_{s(1),s(2)}, g_{s(1),s(2)}) : (s(1), s(2)) \in E \rangle$ such that:

(α) *$\mathbf{b}_s$ list the elements of M_s for $s \in I$ and is of length $|T|$.*

(β) *E is an equivalence relation on I defined by: $s_1 E s_2$ iff there is a tuple (i, f, g) such that:*

$$s_1, s_2 \in I_i$$

(i) *f is a one-to-one order preserving from u_{s_1} onto u_{s_2} recalling that I is a set of ordinals*

(ii) *if $t \in u_{S_1}$ and $j < I_{<i}$ then $t \in I_j$ iff $f(t) \in I_j$ and f maps u_t onto $u_{f(t)}$*

(iii) *if $r, t \in u_{s_1}$ then $r \in u_t$ iff $f(r) \in u_{f(t)}$*

(iv) *g is an elementary mapping (of $\mathfrak{C}_T$) such that for every $t \in u_{s_1}$, f maps $\mathbf{b}_t$ to $\mathbf{b}_{f(t)}$*

(γ) *if sEt then we let $(f_{s,t}, g_{s,t})$ be the unique pair (f, g) which are as above.*

δ) *we have that if sEt then there is an elementary mapping g extending $g_{s,t}$ and mapping $\mathbf{b}_s$* to *$\mathbf{b}_t$, necessarily unique.*

(ε) *E is indeed an equivalence relation and it has, at most, 2^θ equivalence classes.*

2) If T is super-simple that is $\kappa(T) = \aleph_0$, and the pair $[\theta, \sigma)$ satisfies $\theta = 2^{|T|}$ and $\sigma = |T|^+$ or just $\sigma > |T|$, $\theta^{<\sigma} = \theta$

(A) *renaming, in part (1) above without loss of generality $\bar{M} = \langle M_u : u \in [\lambda]^{<\aleph_0} \rangle$ i.e., $I_i = [\lambda]^i$ for $i < \kappa = \aleph_0$. hence $M_u \cap M_v = M_{u \cap v}$*

(B) *there is a (nice) EM - blueprint Φ for the class of linear orders (see e.g., [Shed, §1]) such that:*

  (a) *τ_T has cardinality $2^{|T|}$*

  (b) *for every finite linear order J the model $\mathrm{EM}_{\tau(T)}(J, \Phi)$ is a σ-saturated model of T of cardinality θ,*

  (c) *if in addition T is stable (hence superstable) and I is a linear order then $\mathrm{EM}_{\tau(T)}(I, \Phi)$ is saturated, Moreover if $J \subseteq I$ is finite*

(c) *then the model* $(\mathrm{EM}_{\tau(T)}(J, \Phi), c)_{c \in A(J)}$ *is saturated, where* $A(J) = \cup\{\mathrm{EM}(J', \Phi) : J' \subseteq J, J' \neq J\}$

(d) *above the type of* $\mathrm{EM}_{\tau(T)}(J, \Phi)$ over $\cup\{\mathrm{EM}(J', \Phi) : J' \subset I$, *but* $I \nsubseteq J'\}$ *does not fork over* $\cup\{\mathrm{EM}(J', \Phi) : J' \subset J\}$; *see more* (Shelah, 2009)

3) *If* $(A)_{\lambda,\kappa}$ *then for some* $\mathscr{P}$ *we have* $(B)_{\lambda,\theta,\kappa,\mathscr{P}}$ *which implies* $(C)_{\lambda,\theta,\kappa,\mathscr{P}}$ *where*:

$(A)_{\lambda,\kappa}$ *if* $\alpha < \kappa$ *then* $|\alpha|^{\aleph_0} < \kappa$, (e.g., $\kappa = (|T|^{\aleph_0})^+$).

$(B)_{\lambda,\theta,\kappa,\mathscr{P}}$ *we have*[5]

(a) *$\mathscr{P}$ is a cofinal subset of* $[\lambda]^{<\kappa}$ *which is well founded, see 16.3(3) below*
(b) *$\mathscr{P}$ is closed under finite unions and finite intersection and* $\emptyset \in \mathscr{P}$
(c) *any* $\mathscr{P}' \subseteq \mathscr{P}$ *of cardinality* $<\kappa$ *has an upper bound, follows*
(d) *for any* $\aleph_1$*-directed subset of* $\mathscr{P}$ *of cardinality* $<\kappa$, *its union belongs to* $\mathscr{P}$
(e) *similarly for intersection (and no need to bound the cardinality)*

$(C)_{\lambda,\theta,\kappa,\mathscr{P}}$ *(recall* $M_*, \bar{a}_\alpha$ *are from the beginning of 16.2 and* θ, σ *are from 16.1) we have*

(a) $\bar{M} = \langle M_u : u \in \mathscr{P}\rangle$ *and* $M = \cup\{M_u : u \in \mathscr{P}\}$ *include* $\cup\{\bar{a}_\alpha : \alpha < \lambda\}$
(b) $M_u \prec \mathfrak{C}_T$ *has cardinality* θ, *and it is* σ*-saturated whenever* $\theta^{<\sigma} = \theta$, $\sigma > |T|$
(c) *if* $u \subseteq v$ *are from* $\mathscr{P}$ then $M_u \prec M_v$
(d) (α) $\mathrm{tp}(M_s, \cup\{M_t : t \in \mathscr{P}, s \nsubseteq t\}$ *does not fork over* $\cup\{M_t : t \in \mathscr{P}, t \subseteq s, t \neq s\}$

(β) *if* $|T|$ *is stable, then the type which* M_u *realizes over* $A_u = M_v : v \in \mathscr{P}, v \subset u\{\}$ *has a unique complete extension over* $\cup\{M_v : v \in \mathscr{P}, v \subseteq u, v \neq u\}$ *which does not fork over* A_u; *in fact the type of* $\cup\{M_v : v \in \mathscr{P}, u \nsubseteq v\}$ *over* $\cup\{M_v : v \in \mathscr{P}, v \subseteq u, v \neq u\}$ *is finitely satisfiable in* A_u, *actually this follows, see the proof.*

(e) if $a \in M$ then for some $u \in \mathscr{P}$ we have $a \in M_u$[6] and moreover[7] $a \in M_v \Leftrightarrow u \subseteq v$ for every $v \in \mathscr{P}$
(f) if $u, v_i \in \mathscr{P}$ for $i < i_*$ and $i_* < \omega$ and $u = \cap\{v_i : i < i_*\}$ then $M_u = \cap\{M_{v_i} : i < i_*\}$
(g) (α) if the theory T is stable in θ and $u \in \mathscr{P}$, then the model $(M_u, a)_{a \in A_u}$ is saturated,

5 We can gain "closure under intersection of any sub-family", but the price is that we lose closure under finite unions.
6 if $\langle a_\alpha : \alpha < \lambda\rangle$ list the elements of M with no repetitions, and we are allowed to change P we can get $a_\alpha \in M_u$ iff $a \in u$. In this case, in clause (e) any ordinal i_* is OK.
7 The problem in the proof is that if e.g., $a \in (M_u \cup M_v) \backslash (M_u \cup M_v)$ we need that there is a minimal $w \in P$ and $u \subseteq w \wedge v \subseteq w$ which holds here as P is closed under finite unions.

(β) if $\sigma > |T|$ and $\theta^{<\sigma} = \theta$ then the model M_s is σ-saturated, for every $s \in I_i$.

Remark 16.3.

(1) Concerning clause (C) of 16.2(3) we have thought it nice to have $\mathscr{P} = [\lambda]^{<\kappa}$, but even for strictly stable T this is impossible (let $u_n \subseteq u_{n+1}$ for $n < \omega$ and $a \in M$ satisfies $\mathrm{tp}(a, M_{n+1})$ forks over M_n for every n).

(2) In 16.2(3) the well foundedness of $\mathscr{P}$ means that we can define the function $\mathrm{Dp} : \mathscr{P} \to \mathrm{Ord}$ by deciding $\mathrm{Dp}(u) = \varepsilon$ iff ε is the minimal ordinal β such that for every $v \in \mathscr{P}$ satisfying $v \subset u$ we have $\mathrm{Dp}(v)$ is well defined and $<\alpha$.

(3) Concerning 16(2)(B)(c), so the case T is superstable.

(4) If $(A)_{\lambda,\kappa}$ holds and we construct $\mathscr{P}$ as in the proof of 16(3), then $u \in P \Rightarrow \mathrm{Dp}_P(u) < \kappa$ because $(*)_{\mathscr{P}}$ if $u \in \mathscr{P}$ then the set $\{v \in \mathscr{P} : v \subseteq u\}$ has cardinality $<\kappa$.

Proof. 1) By induction on $i < \kappa$ we try to choose $S_i = (\bar{I}^i, \bar{u}^i, \bar{M}^i)$ increasing with i (in the natural sense, so $\bar{I}^i = \langle I_j : j < i \rangle$, $\bar{u}^i = \langle u_{j,s} : j < i, s \in I_j \rangle$, $\bar{M}^i = \langle M_{j,s} : j < i, s \in I_j \rangle$) satisfying the clauses (a)-(f), (h) of $\boxplus$and in addition:

$\oplus$ If $\alpha < \lambda$ and $\bar{a}_\alpha \notin \cup\{M_s : s \in I_{<i}\}$, then $\mathrm{tp}(\bar{a}_\alpha, \cup\{M_s : s \in I_{\leq i}\})$ forks over $\cup\{M_s : s \in I_{<i}\}$.

For $i = 0$, $\mathbf{s}_i$ is well defined, so there is nothing to do.

For $i = 1$ let S_1 be defined by $I_0 = \{0\}$, $M_* = M_0 \prec \varsigma_T$ of cardinality θ (and $u_0 = \emptyset$).

For i a limit ordinal $\bar{\mathbf{s}}_i$ is well defined.

For $i = j + 1$, we try to choose $(\alpha_{j,\varepsilon}, u_{j,\varepsilon}, M_{j,\varepsilon})$ by induction on $\varepsilon < \lambda$ such that:

$(*)_1$ $M_{j,\varepsilon} = \prec \varsigma_T$ has cardinality $\leq\theta$.

In step ε, first choose $\alpha_{j,\varepsilon}$ as the minimal ordinal $\alpha < \lambda$ such that:

$(*)_2$ (a) $s \in I_{<j}$ implies $\bar{a}_\alpha \notin^{\omega>} (M_s)$.

(b) $\xi < \varepsilon$ implies $\bar{a}_\alpha \notin^{\omega>} (M_{j,\xi})$;

(c) $\mathrm{tp}(\bar{a}_\alpha, \cup\{M_s : s \in I_{<j}\} \cup \cup\{M_{\varepsilon,\xi} : \xi < \varepsilon\})$ does not fork over $\cup\{M_s : s \in I_{<j}\}$.

(d) if T is stable and $\kappa > |T|$ then the type above is stationary over $\cup\{M_s : s \in I_{<j}\}$.

If there is no such α, let $\varepsilon_j = \varepsilon$ and we do not continue.

Second,

$(*)_3$ choose $u_{j,\varepsilon}$ such that:

(a) as a subset of $I_{<j}$ (which is already well defined)
(b) it is of cardinality $<\kappa(T)$
(c) $\mathrm{tp}(\bar{a}_{\alpha_{j,\varepsilon}}, \cup\{M_s : s \in I_{<j}\})$ does not fork over $\cup\{M_s : s \in u_{\varepsilon,j}\}$
(d) is as in $(*)_2(d)$ above

Why is this possible? by the definition of $\kappa(T)$. By the transitivity property of non-forking:

$\bullet_0$ $\mathrm{tp}(\bar{a}_\alpha, \cup\{M_s : s \in I_{<j}\} \cup \{M_{\varepsilon,\zeta} : \zeta < \varepsilon\})$ does not fork over $\cup\{M_s : s \in u_{\varepsilon,j}\}$.

Third, choose $M_{j,s}$ such that.

$\bullet_1$ $M_{j,\varepsilon} \prec \mathfrak{C}_T$ has cardinality $\leq\theta$.

$\bullet_2$ $\cup\{M_s : s \in u_{\varepsilon,j} \cup \bar{a}_\alpha\} \subseteq M_{j\restriction\varepsilon}$.

Fourth, w.l.o.g. (possible by the extension property of non-forking):

$\bullet_3$ $\mathrm{tp}(M_{j,\varepsilon} \cup \cup\{M_s : s \in I_{<j}\} \cup \{M_{\varepsilon,\xi} : \xi < \varepsilon\})$ does not fork over $\cup\{M_s : s \in u_{\varepsilon,j}\} \cup \bar{a}_\alpha$.

By the transitivity of non-forking (and $\bullet_0 - \bullet_3$) we have $\mathrm{tp}(M_{j,\varepsilon} \cup \cup\{M_s : s \in I_{<j}\} \cup \{M_{\varepsilon,\xi} : \xi < \varepsilon\})$ does not fork over $\cup\{M_s : s \in u_{\varepsilon,j}\} \cup \bar{a}_\alpha$.

Clearly, for some $\varepsilon_j \leq \lambda$, $\alpha_{j,\varepsilon}$ is well defined iff $\varepsilon < \varepsilon_j$. Let $I_j = \{\alpha_{j,\varepsilon} : \varepsilon < \varepsilon_j\}$ and $M_{\alpha_{j,\varepsilon}} = M_{j,\varepsilon}$, $u_{\alpha_{j,s}} = u_{j,\varepsilon}$ for $\varepsilon < \varepsilon_j$. So $\mathbf{s}_i$ is well defined.

So we have carried out the induction on $i \leq \kappa$ and so $\mathbf{s} = \mathbf{s}_\kappa$ is well defined and satisfies clauses $(a) - (f)$ and (h) of $\boxplus$; what about clause (g)? Let $\alpha < \lambda$, so as $\kappa = \mathrm{cf}(\kappa) \geq \kappa(T)$, there is $j < \kappa$ such that $\mathrm{tp}(\bar{a}_\alpha, \cup\{M_s : s \in I_{<j}\})$ does not fork over $\cup\{M_s : s \in u_{\varepsilon,j}\}$. By monotonicity of non-forking, $\alpha \in \{\alpha_{j,\varepsilon} : \varepsilon < \varepsilon_j\}$ hence for some $\varepsilon < \varepsilon_1$, $\alpha = \alpha_{j,\varepsilon}$, hence $\bar{a}_\alpha \in {}^{\omega>}(M_{\alpha_{j,\varepsilon}})$, as promised.

1A), 1B) The proof is similar.

2) The first clause holds by part (1) for $\theta = 2^{|T|}$. For the second clause, let τ^+ be $\tau(T) \cup \{F_{n,\alpha} : n < \omega, \alpha < 2^{|T|}\}$ where the $F_{n,\alpha}$ is an n-place function symbol, not from $\tau(T)$ and they are pairwise distinct. Choose $a_\varepsilon \in M_{\{\varepsilon\}} \backslash M_\emptyset$. We can find $M^+_{\{\varepsilon\}}$, a τ^+-expansion of $M_{\{\varepsilon\}}$ such that $|M_{\{\varepsilon\}} = \{F_{1,\alpha}(a_\varepsilon) : \alpha < 2^{|T|}\}$.

Generally by induction of n for every $u \in [I]^n$ letting $\varepsilon_0 < \cdots < \varepsilon_{n-1}$ list u, we choose a τ^+-model M^+_u expanding M_u, extending M^+_w for $w \subset u$ such that $|M_u| = \left\{F^{M^+_u}_{n,\alpha}(a_{\varepsilon_0}, \ldots, a_{\varepsilon_{n-1}}) : \alpha < 2^{|T|}\right\}$. Now (see [Shed, 1.18 = Lc13(3), page.12], [Shed, 1.17 = Lc2, page.11]) there is a Φ such that clauses (c) and (d) follow.

3) $(A)_{\lambda,\kappa}$ <u>implies that for some</u> $\mathscr{P}$ <u>we have</u> $(B)_{\lambda,\kappa}$<u>.</u>

First choose $H{:}^{\omega}\lambda \to \lambda$ such that there is no ω-decreasing sequence $\langle u_n : n < \omega\rangle$ of H-closed subsets of λ, see Erdös-Hajnal (for being self-contained, just let $S_\alpha \in [\lambda]^{\aleph_0}$ for $\alpha < \alpha_*$ be pairwise almost disjoint, and $\{S_\alpha : \alpha < \alpha_*\}$ is maximal under those conditions [that is it is a so called MAD family]; choose H such that for any countable $u < [\lambda]^{\aleph_0}$, if $u \cap S_\alpha$ is infinite then $S_\alpha \subseteq \{H(\eta) : \eta \in {}^\omega u\}$). As the values of H only on sequences with no repetitions suffice, e.g., without loss of generality for some one-to-one function h from ${}^{\omega >}\lambda$ onto λ, we have:

$(*)_1$ if $\eta \in {}^n\lambda$ and $h(\eta) = \alpha$ then:

(a) $\alpha \notin \{\eta(\ell) : \ell < n\}$

- (a) if $\nu = \nu_0 \hat{}\ \nu_1 \hat{}\ \ldots \in {}^\omega\lambda$ and $\nu_{2i} = \nu_{2i+1}\eta\ \hat{}\ \langle i \rangle$ then $H(\nu) = \alpha$
- (b) we have $\beta_\ell = \eta(\ell + 1)$ for $\ell < n$ when we define β_ℓ by induction on ℓ by:
  - (α) $\beta_0 = \alpha$.
  - (β) $\beta_{\ell+1} = H(\langle \beta_\ell, \beta_\ell, \ldots \rangle)$.

Now we choose our families.

$(*)_2$ Let

- (a) $\mathscr{P}_0$ be the family of sets $u \in [\lambda]^{<\kappa}$ which are H-closed, that is $\{H(\eta) : \eta \in {}^\omega u\} \subseteq u$.
- (b) $\mathscr{P}$ is the family of finite unions of members of $\mathscr{P}_0$

$(*)_3$ the family $\mathscr{P}_0$ satisfies all the demands except being closed under finite unions.

[Why? being cofinal by $(*)_4$, $\emptyset \in \mathscr{P}_0$ by $(*)_4$, also the rest are easy.]

$(*)_4$ $\mathscr{P}$ is a cofinal subset of $[\lambda]^{<\kappa}$ (so satisfies (B)(c) and the first clause of (B)(a))

[Why: Given $u_1 \in [\lambda]^{<\kappa}$, let u_2 be its H-s closure, (this mean that we choose v_i by induction on $i \leq \omega_1$ by $v_i = \{\alpha : \alpha \in u_1$ or for some $j < i, \alpha \in v_j$ or $(\exists \eta \in {}^\omega(v_j))[\alpha = H(\eta)]\}$. Now $u_2 = v_{\omega_1}$ is a subset of λ and it has cardinality $<\kappa$ because we are assuming $\alpha < \kappa \Rightarrow |\alpha|^{\aleph_0} < \lambda$. Now clearly $u_1 \subseteq u_2 \in \mathscr{P}_0 \subseteq \mathscr{P}$ so we are done.]

$(*)_5$ $\emptyset \in \mathscr{P}_0 \subseteq \mathscr{P}$, (so the last clause of (C)(b) holds)

[Why? Because $\emptyset$ is an H-closed subset of λ of cardinality $<\kappa$.]

$(*)_6$ $\mathscr{P}$ is closed under finite intersection, (the second clause of (C)(b))

[Why? Clearly $\mathscr{P}_0$ is closed under finite intersection, equivalently intersection of two. Now if $u_1, u_2 \in \mathscr{P}$ then for $\iota = 1, 2$ let $u_\iota = \cup\{u_{\iota,\ell} : \ell < n_\iota\}$ where $u_{\iota,\ell} \in \mathscr{P}_0$ for $\ell < n_\iota$. Now letting $v_{\ell,m} = u_{1,\ell} \cap u_{2,m}$ we have:

$v_{\ell,m} \in \mathscr{P}_0$ and $u_1 \cup u_2 = \cup\{u_{\ell,m} : \ell < n_1, m < n_2\}$. Now think.]

$(*)_7$ $\mathscr{P}$ is closed under finite unions, (so the first clause of (C)(b) holds).

[Why? This is obvious].

$(*)_8$ $\mathscr{P}$ is closed under $\aleph_1$-directed unions (so (C)((d) holds)

Why? Let I be an $\aleph_1$-directed partial order and let $\langle u_s : s \in I \rangle$ be a $\subseteq$-increasing sequence of members of $\mathscr{P}$; let $\langle u_{s,\ell} : \ell < k_s \rangle$ be a finite sequence of members of

$\mathscr{P}_0$ whose union is u_s. For each $s \in I$ let m_s be minimal such that there are t,w such that $s \le t \in I, w \subseteq k_t$ and $u_s \subseteq \cup\{u_{t,\ell} : \ell \in w\}$ and $|w| = m_s$. Clearly $m_s \le k_s$.

$(*)_{8.1}$ without loss of generality $m_s = k_s$.

[Why? We can choose for each $s \in I$ a pair (t_s, w_s) witnessing the value of m_s. Now let J be the partial order whose elements are those of I and its order is $\le_J$ is $\{((s_1,s_2) : t_{s_1} \le_I s_2\}$. Now check that $(J, \langle u_{t_s} : s \in J\rangle)$ has all the desired properties and it suffice to prove that $\{u_s : s \in J\} \in \mathscr{P}$].

Next

$(*)_{8.2}$ without loss of generality $\langle k_s : s \in I\rangle$ is constantly k_*.

[Why? because I is $\aleph_1$-directed].

$(*)_{8.3}$ if $s <_I t$ and $k < k_*$ then for some $\ell < k_*$ we have $u_{s,k} \subseteq u_{t,\ell}$

[Why? if not then we get contradiction to $(*)_1$].

So if $s <_I t$ then for some function $h_{t,s} : k_* \to k_*$ we have $\ell < k_* \Rightarrow u_{s,\ell} \subseteq u_{t,h_{t,s(\ell)}}$. By the choice of k_*, clearly this function is one-to one hence it is onto. Fixing $s_* \in I$ without loss of generality $\langle h_{t,s} : t \in I, s_* <_I t\rangle$ is constant so renaming all the function $h_{t,s}$ are the identity. So for $k < k_*$ the set $u_k = \cup\{u_{t,k} : t \in \mathscr{P}, s_* <_I t\}$ belongs to $\mathscr{P}_0$ hence $\cup\{u_k : k < k_*\} \in \mathscr{P}$, and we are done proving $(*)_8$ holds.]

$(*)_9$ $\mathscr{P}$ is closed under $\aleph_1$-directed intersections (no bound on the size; this is (C)(e)).

[Why? similarly to the proof of $(*)_8$].

$(*)_{10}$ $\mathscr{P}$ is well founded, (the second clause of (C)(a))

[Why? Toward contradiction assume that $\mathscr{P}$ is not well founded, so there is a strictly $\subseteq$-decreasing sequence $\langle u_n : n < \omega\rangle$ of members of $\mathscr{P}$. So we can choose $\alpha_n \in u_n \backslash u_{n+1}$ for $n < \omega$ and choose for each n a finite sequence $\langle u_{n,k} : k < k_n\rangle$ of members of $\mathscr{P}_0$ with union u_n. Now let $\eta_n \in \Pi_{m \le n} k_m$ be defined by: $\eta(m)$ is the minimal $k < k_m$ such that $\alpha_n \in u_{m,k}$, clearly well defined. Next by König lemma there is $\eta \in \Pi_{m < \omega} k_m$ such that for every $\ell < \omega$ there is $n(\ell) \in (\ell, \omega)$ satisfying $\eta \restriction \ell = \eta_{n(\ell)} \restriction \ell$. Now for $m < \omega$ let $v_m = \cap\{u_{n,\eta(n)} : n < m\}$, obviously it belongs to $\mathscr{P}_0$. Clearly $\langle v_m : m < \omega\rangle$ is a $\subseteq$-decreasing sequence. Also $\alpha_{n(\ell)}$ belongs to v_ℓ but does not belongs to $u_{n(\ell)+1}$ hence not to $u_{n(\ell)+1,\eta(n(\ell)+1)}$ hence not to $v_{n(\ell)+2}$. As the v_m-s are all from $\mathscr{P}_0$ we get a contradiction to the choice of H and $\mathscr{P}_0$.]

Now check.

$(B)_{\lambda,\theta,\kappa,\mathscr{P}} \Rightarrow (C))_{\lambda,\theta,\kappa,\mathscr{P}}$

Let $\bar{u}_* = \langle u^*_\alpha : \alpha < \lambda\rangle$ be such that:

$(*)_1$ (a) $\bar{u}_* = \langle u^*_\alpha : \alpha < \alpha_*\rangle$.

(b) $u^*_\alpha \subseteq \alpha$ be of cardinality $<\kappa$.

(c) $\mathrm{tp}(\bar{a}_\alpha, \cup\{\bar{a}_\beta : \beta < \alpha\})$ does not fork over $\cup\{\bar{a}_\beta : \beta \in u^*_\alpha\}$.

[Why such u_α exists? because $\kappa(T) \le \kappa$].

Now

$(*)_2$ for $u{\in}\mathscr{P}$ let $w_u = w[u] = \{\alpha{\in}u : u^*_\alpha{\subseteq}u\}$.

Easily.

$(*)_3$ We have.

(*) if $v \in [\lambda]^{<\kappa}$ then for some $u{\in}\mathscr{P}$ we have $v \subseteq w_u$.

(*) if $u \subseteq v$ are from $\mathscr{P}$ then $w_u \subseteq w_v$.

(*) $\lambda = \cup\{w_u : u{\in}\mathscr{P}\}$.

$(*)_4$ let $\langle u_\alpha : \alpha < \alpha_* \rangle$ list $\mathscr{P}$ such that $\alpha < \beta < \alpha_* \Rightarrow \mathrm{Dp}_{\mathscr{P}}(u_\alpha) \leq \mathrm{Dp}_{\mathscr{P}}(u_\beta)$; see 16.3(3), (recalling $\mathscr{P}$ is well founded); so $u_\alpha \subseteq u_\beta \Rightarrow \alpha \leq \beta$.

We shall now choose M_{u_α} for $\alpha < \alpha_*$ by induction on α such that:

$(*)_5$ (a) $M_{u_\alpha} \prec \mathfrak{C}_T$ has cardinality θ recalling $\theta \geq |T|$.

(b) M_{u_α} include the sets $A_{u_\alpha} = A[u_\alpha] = \cup\{M_{u_\beta} : \beta < \alpha, u_\beta \subset u_\alpha\}$ and $A^*_{u_\alpha} = A^*[u_\alpha] = \cup\{\bar{a}_\beta : \beta{\in}w[u_\alpha]\}$.

(c) the complete type, which M_{u_α} realizes over $\cup\{M_{u_\beta} : \beta < \alpha\} \cup \{\bar{a}_\beta : \beta < \lambda\}$, does not fork over $A[u_\alpha] \cup A^*[u_\alpha]$.

(d) if T is stable in θ then the model $(M, a)_{a \in A[u_\alpha]}$ is saturated.

(e) if $|\mathbf{D}(T)| \leq \theta = \theta^{<\sigma}$ then M_{u_α} is σ-saturated.

(f) moreover, if $u \subseteq v$ are from $\mathscr{P}$ and $p(x)$ is a type over M_u of cardinality $<\sigma$ then some $a \in M_v$ realizes $p(x)$ and $\mathrm{tp}(a, A_v)$ does not fork over M_u; (this help for the case T is stable, to prove that M is saturated).

There is no problem to carry the induction and let $M = \cup\{M_u : u{\in}\mathscr{P}\}$. Now by the non-forking calculus we can prove all the required clauses, In particular, for clause $(d)(\beta)$ we use Shelah (1990a, Ch. XII, 3.5, p. 608) recalling the definitions Shelah (1990a, 2.1, p. 598, 3.2, p. 604). Concerning clauses (C)(e), if $a \in M$ let α be the first ordinal α such that $a{\in}M_{u_\alpha}$, and u_α is as required because if $a \in u_\beta$ then necessarily $\beta \geq \alpha$. The point is that $v = u_\alpha \cap u_\beta$ belongs to $\mathscr{P}$ for some γ we have $u_\gamma = v$, but $Mu_\gamma = M_{u_\alpha} \cap M_{u_\beta}$ and we ca finish. Reduce to finite subsets of $\mathscr{P}(n)$. $\square$

Formalizing clause (C) of 16.2(3)):

Definition 16.4.

(1) Let $\mathscr{M}_1 = \mathscr{M}_{T,\theta,\kappa,1}$ be the class of three sorted structures $\mathbf{m}$ consisting of (so $I = I_{\mathbf{m}}$ etc)

(a) first sort I, (will serve as an index set)

(b) second sort P with $E_1 \subseteq I \times P$ (will serve as coding a family of subsets of I)

(c) for every $u \in P$ the set $(u) = \mathrm{set}_{\mathbf{m}}(u) = \{s \in I : sE_{\mathbf{m}}u\}$ has cardinality $<\kappa$,

(d) for $i < \kappa$, G_i is a unary function with domain $P\backslash\{c_1\}$ (see clause (e) below) such that $\langle G_i(u) : i < \kappa\rangle$ list (u) possibly with repetitions (if κ is a successor cardinal ∂^+ then it is enough to have G_i for $i < \partial$)

(e) the partial order $\subseteq_{\mathbf{m}}$ on P such that $u \subseteq_{\mathbf{m}} v$ iff (u, $v \in P$ and) $(u) \subseteq (v)$ and $(u) = set(v) \Rightarrow u = v$ and $c_1 \in P$ satisfies $(c_1) = \emptyset$

(f) the partial order $\subseteq_{\mathbf{m}}$ is well founded, and the partial order $\subseteq_{\mathbf{m}}$ is directed

(g) the family $\{(u) : u \in P\}$ is closed under finite intersections, finite unions and $\emptyset \in \mathscr{P}$

(h) a third sort is M, a τ_T-model and $E_2 \subseteq M \times P$

(i) for every $u \in P$ let M_u be $M \upharpoonright \{a \in M : aE_2u\}$, and $M = \cup \{M_u : u \in P_{\mathbf{m}}\}$

(j) M_u is a model of T of cardinality θ

(k) if $u \subseteq_{\mathbf{m}} v$ then $M_u \prec M_v$

(l) if set$(u) = \cap \{u_\varepsilon : \varepsilon < \zeta\}$ then $M_u = \cap\{M_{u_\varepsilon} : \varepsilon < \zeta\}$

(m) $F_\varepsilon(\varepsilon < \theta)$ are unary functions such that $\langle F_\varepsilon(u) : \varepsilon < \theta\rangle$ list the elements of M_u

(n) non-forking: the parallel of 16.2(3)(C)(d) holds, that is: (α) $\text{tp}(M_s, \cup\{M_t : t \in \mathscr{P}, s \not\subseteq t\}, M_{\mathbf{m}})$ does not fork over $A_s = \cup \{M_t : t \in \mathscr{P}, t \subseteq s, t \neq s\}$. ($\beta$) if $|T|$ is stable, then the type which M_u realizes over $A_u = M_v : v \in \mathscr{P}, v \subset u\{\}$ has a unique complete extension over $\cup\{M_v : v \in \mathscr{P}, v \subseteq u, v \neq u\}$ which does not fork over A_u; in fact the type of $\cup\{M_v : v \in \mathscr{P}, u \not\subseteq v\}$ over $\cup\{M_v : v \in \mathscr{P}, v \subseteq u, v \neq u\}$ is finitely satisfiable in A_u, actually this follows,

(o) saturation:

(α) if the theory T is stable in θ, then the model $(M_u, a)_{a \in A_u}$ is saturated, for every $u \in \mathscr{P}$. If T is stable we can make the choice of $\langle F_\varepsilon(u) : \varepsilon < \theta\rangle$ canonical, as in 16.2(3)(C)(g), this just help in 16.5(1A) below.

(β) if $\sigma > |T|$ and $|\mathbf{D}(T)| \leq \theta^{<\sigma} = \theta$ then the model M_u is σ-saturated, for every $u \in \mathscr{P}$

(2) Let $\mathscr{M}_0$ be the set of $\mathbf{x} \in \mathscr{M}_1$ such that

(a) its set of elements is included in θ

(b) (for notational convenience) we fix $c_1^{\mathbf{x}}, M_{c_1}^{\mathbf{x}}$ and so its set of elements, (otherwise the class below will fail the JEP).

(c) in $P_{\mathbf{x}}$ there is a $\subseteq_{\mathbf{x}}$-maximal member $\max_{\mathbf{x}} = \max(\mathbf{x})$

(3) Let $\mathscr{M}_2$ be the class of models $\mathbf{m}$ of the following form (it vocabulary $\tau(\mathscr{M}_2)$ is implicitly defined):

(a) I, $\subseteq_{\mathbf{m}}$, P, G_i are as in part (1)

(b) $\langle P_{\mathbf{x}} : \mathbf{x} \in \mathscr{M}_0 \rangle$ is a partition of $P_{\mathbf{m}}$

(c) if $u \in P_{\mathbf{x}}$ $\mathbf{x}$ is isomorphic to $\mathbf{m} \upharpoonright (\mathrm{set}_{\mathbf{m}}(u) \cup \{v \in P_{\mathbf{m}} : v \subseteq_{\mathbf{m}} u\}$

(d) if $u \subseteq_{\mathbf{m}} v$ and $u \in P^{\mathbf{m}}_{\mathbf{x}}, v \in P^{\mathbf{m}}_{\mathbf{y}}$ then the isomorphisms commute

(4) Let $\leq_{\mathscr{M}_1}$ be the following partial order on $\mathscr{M}_1$: $\mathbf{m} \leq_{\mathcal{M}_1} \mathbf{n}$ iff

(a) $\mathbf{m}, \mathbf{n} \in \mathscr{M}_1$

(b) $\mathbf{m}$ is a submodel of $\mathbf{n}$

(c) if $v \in P^{\mathbf{m}}$ and $u \subseteq_{\mathbf{n}} v$ then $u \in P^{\mathbf{m}}$

4A) We define $\leq_{\mathcal{M}_2}$ similarly.

5) For $\mathbf{m} \in \mathscr{M}_1$ let its reduct $(\mathbf{m}) \in \mathscr{M}_2$ be naturally defined.

6) We define $\mathscr{M}_3, \mathscr{M}_4, \leq_{\mathscr{M}_3}, \leq_{\mathscr{M}_4}$ like $\mathscr{M}_1, \mathscr{M}_2, \leq_{\mathscr{M}_1}, \leq_{\mathscr{M}_2}$ just omitting the demand that $(P_{\mathbf{m}}, \subseteq_{\mathbf{m}})$ is directed

Claim 16.5. 1) *$\mathscr{M}_0$ has cardinality 2^θ* hence $\leq 2^{2^{|T|}}]$ *when we assume $\theta \leq 2^{|T|}$.*

1A) If T is stable, then we can have $\mathscr{M}_0$ have cardinality $\leq 2^{|T|}$ see 16.2(3)(C)(g). 16.4(1)(e).

2) *$(\mathscr{M}_1, \leq_{\mathscr{M}_1})$ is an aec with the LST-number is $\leq \theta$.*

3) *$(\mathscr{M}_2, \leq_{\mathscr{M}_2})$ is an aec with the LST-number $\leq \theta$.*

4) *For every* $\mathbf{m} \in \mathscr{M}_1$, its *$\mathscr{M}_2$-reduct rdc$(\mathbf{m})$ really belongs to $\mathscr{M}_2$ and also if* $\mathbf{m} \leq_{\mathscr{M}_1} \mathbf{n}$ *then rdc* $(\mathbf{m}) \leq_{\mathscr{M}_2} \mathrm{rdc}(\mathbf{n})$, *see Def 16.4(5).*

5) *The results above holds for $\mathscr{M}_3, \mathscr{M}_4, \leq_{\mathscr{M}_3}, \leq_{\mathscr{M}_4}$.*

Proof. Easy. □

Claim 16.6.

(1) *$(\mathscr{M}_1, \leq_{\mathscr{M}_1})$ has the amalgamation property and the* JEP *(see in Def 16.4 (1)(e)).*

(2) *Similarly $(\mathscr{M}_2, \leq_{\mathscr{M}_2})$*

(3) *if* $\mathbf{n} \in \mathscr{M}_2$ *then for some* $\mathbf{m} \in \mathscr{M}_1$ *we have rdc*$(\mathbf{m}) = \mathbf{n}$

(4) *In parts (1),(2) we moreover have the $\mathscr{P}^-(3)$- amalgamation*

(5) *If T is stable, then the results above holds for $\mathscr{M}_3, \mathscr{M}_4, \leq_{\mathscr{M}_3}, \leq_{\mathscr{M}_4}$*

Proof. By the non-forking calculus and Shelah (1990a, Ch. XII, §2, 3). □

Claim 16.7. *Assume $\partial < \theta$ and $\chi^{<\partial} = \chi$.*

(i) *If* $\mathbf{m}_1 \in \mathscr{M}_1$ *is of cardinality $\leq \chi$ then there is* $\mathbf{m}_2 \in \mathscr{M}_1$ *of cardinality χ which is $\leq_{\mathscr{M}_1}$-above* $\mathbf{m}_1$ *and is* $(<\partial)$*-homogeneous and (recalling (16.4(1)(e))) is $<\partial$-universal*

(ii) *Similarly for $\mathscr{M}_2$.*

Proof. By 16.6(1) and 16.5(2,(3). □

A central case is when we try to analyse a model M of cardinality $\lambda = \lambda^{<\kappa}$ which is in the interval $(\mu = \mu, 2^\mu)$ when $|T| < \mu = \mu^{<\mu} < \lambda < 2^\mu$.

Claim 16.8. *Assume $\kappa \geq \kappa(T)$, $\theta^+ \geq \kappa$, $\theta \geq |T|$, $\mu = \mu^{<\kappa} < \lambda \leq 2^\mu$ and the triple $(\bar{I}, \bar{u}, \bar{M})$ is as in 16.2 and for $s \in I_{<i_*}$ we let $\mathbf{b}_s = \langle b_{s,j} : j < \theta \rangle$ list the elements of M_s, so $s \in u_t \Rightarrow \mathrm{Rang}(\bar{b}_s) \subseteq \mathrm{Rang}(\bar{b}_t)$).*

Then we can find a sequence $\langle (\zeta_\alpha, s_\alpha) : \alpha < \lambda \rangle$ such that:

$$\zeta_\alpha < \theta^+$$

(a) $s_\alpha \in I$ and $\bar{a}_\alpha \subseteq M_{s_\alpha}$, (recall I is a set of ordinals)

(b) if $\zeta_\alpha = \zeta_\beta$, then $\mathrm{otp}\left(u^+_{s_\alpha}\right) = \mathrm{otp}\left(u^+_{s_\beta}\right)$ and letting $h_{\beta,\alpha}$ be the unique order preserving function from $u^+_{s_\alpha}$ onto $u^+_{s_\beta}$ we have:

(α) $h_{\beta,\alpha}(s_\alpha) = s_\beta$ and $h_{\beta,\alpha}(s) = t \wedge i < i_* \Rightarrow (s \in I_i \equiv t \in I_i)$ and $\wedge^2_{i=1} h_{\beta,\alpha}(s_\ell) = t_\ell \Rightarrow (s_1 \in u_{t_1} \equiv s_2 \in u_{t_2})$.

(β) $\bar{\mathbf{b}}_{s_\alpha}, \bar{\mathbf{b}}_{s_\beta}$ realize the same type over M_*.

(γ) moreover, if $h_{\beta,\alpha}(r_1) = r_2$ then $\bar{\mathbf{b}}_{s_\alpha} {}^\wedge \bar{\mathbf{b}}_{r_1}, \bar{\mathbf{b}}_{s_\beta} {}^\wedge \bar{\mathbf{b}}_{r_2}$ realize the same type over M_*.

(δ) $h_{\beta,\alpha}$ is the identity on $u_{s_\beta} \cap u_{s_\alpha}$.

Remark 16.9. Note that presently there may be $s_1 \neq s_2 \in I, u_{s_1} = u_{s_2}$. Overcoming this we get $2^{<\kappa}$ at least. For well foundations use $F : {}^\omega\lambda \to \lambda$.

Proof. Let $\langle \eta_s : s \in I \rangle$ be a sequence with no repetition of members of ${}^\mu 2$, etc., or quote (Engelking and Karłowicz, 1965).

Discussion 16.10. As promised, we now comment on the case we like to fix in 16.2: the model M a priori; toward this we assume that $\kappa = \theta^+$, so $\kappa > |T|$. So the changes are:

(a) in the beginning of 16.2 we fix M of cardinality λ such that $M_* \prec M \prec \mathfrak{C}_T$ and let $\langle a_\alpha : \alpha < \lambda \rangle$ list the elements of M (so $\bar{a}_\alpha = \langle a_\alpha \rangle$), and in the statement of 16.2(1):
  - (a) in 16.2(1)(d) M is the given one so clause (g) is redundant
  - (b) we omit 16.2(1)(i)

(c) in the proof of 16.2(1)
  - (a) We replace $\mathfrak{C}_t$ by M
  - (b) for $i = j + 1$ in the ε-th step we choose $(\alpha_{j,\varepsilon}, u_{j,\varepsilon}, M_{j,je},)$ such that: $\alpha_{j,\varepsilon} < \lambda$ and $a_\alpha \notin \cup \{M_v : v \in I_{<i}\} \cup \{M_{\alpha_{j,\zeta}} : \zeta < \varepsilon\}$ and the other

conditions (but $M_{j,\varepsilon} \prec M$) and modulo this $\alpha_{j,\varepsilon}$ is minimal. If T is stable we can add the stationarity of the type.

(c) in the end, instead of considering a_α we use a suitable triple.

(d) the changes in the other parts of 16.2 are similar

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