SOME VARIATIONS ON THE SPLITTING NUMBER

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1. Introduction

While not included in the Cichon diagram, the cardinal invariant $\mathfrak{s}$, the splitting number, has been the source of considerable interest. Any of the surveys of cardinal invariants — such as [3], [8] or [2] — will provide ample justification for this assertion. Of course, $\mathfrak{s}$ is defined to be the least cardinal of a family $\mathcal{S}$ of infinite subsets of ω such that for any infinite $X \subseteq \omega$ there is $S \in \mathcal{S}$ such that $|S \cap X| = |X \setminus S|$. The article [6] introduces a modification of the splitting number obtained by what can be considered a localization of the concept. The authors of [6] define the pair splitting number $\mathfrak{s}_{\text{pair}}$ to be the least cardinal of a family $\mathcal{S}$ of subsets of ω such that for any infinite, pairwise disjoint family of pairs $X \subseteq [\omega]^2$ there is $S \in \mathcal{S}$ such that $|S \cap x| = |x \setminus S|$ for infinitely many $x \in X$. The authors establish connections between $\mathfrak{s}_{\text{pair}}$ and well known cardinal invariants of the continuum, as well as with the covering number of the finite chromatic ideal consisting of graphs, considered as sets of pairs of integers, with finite chromatic number.

The present work will continue these explorations by expanding the definitions of [6] beyond pairs. It was already shown in [6] that generalizing $\mathfrak{s}_{\text{pair}}$ to, for example, $\mathfrak{s}_{\text{triple}}$ in the obvious way does not create a new concept. The goal of the research to be presented here is that generalizing from pairs to finite sets does introduce a new concept. Generalizing splitting to what may be called balanced splitting has been examined in [4]. Some connections between the research under consideration here and that of [4] will also be established.

2. Definitions and basic results

This section will introduce some cardinal invariants very similar to those introduced in [6]. Indeed, they are so similar that it will be shown they are, in fact, the same. It was mentioned in the introduction that it was shown in [6] that if $\mathfrak{s}_n$ is defined to be the least cardinal of a family $\mathcal{S}$ of subsets of ω such that for any infinite, pairwise disjoint family of pairs $X \subseteq [\omega]^n$ there is $S \in \mathcal{S}$ such that $S \cap x \neq \varnothing \neq x \setminus S$ for infinitely many $x \in X$ then $\mathfrak{s}_{\text{pair}} = \mathfrak{s}_{\text{n}}$. It will be shown that the same holds if one considers splitting n-sized sets into k pieces.

Definition 2.1. *The reader unwilling to part with von Neumann's definition of ordinals is warned that if F is any function and x a subset of its domain then $F(x)$ will be used to denote the image of x under F; this will be used even when there is a slight danger that some confusion between an ordinal thought of as a point and a set may arise. For a function $f : \omega \to \omega$ define $\mathfrak{s}_{k,f}$ to be the least cardinal λ such that there is a family $\mathcal{F} \subseteq k^\omega$ of cardinality λ such that for each sequence of pairwise disjoint sets of integers $\{a_n\}_{n\in\omega}$ such that $|a_n| = f(n)$ there is $F \in \mathcal{F}$ such that $F(a_n) = k$ for infinitely many n. The notation $\mathfrak{s}_{k,m}$ will be used to denote $\mathfrak{s}_{k,f}$ when f is constant with value m.*

Lemma 2.1. *If $f \leq^* g$ then $\mathfrak{s}_{k,g} \leq \mathfrak{s}_{k,f}$.*

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Proof. Let $\mathcal{F} \subseteq k^\omega$ be such that $|\mathcal{F}| = \mathfrak{s}_{k,f}$ and for each sequence of pairwise disjoint sets $\{a_n\}_{n\in\omega}$ such that $|a_n| = f(n)$ there is $F \in \mathcal{F}$ such that $F(a_n) = k$ for infinitely many n. Given $\{b_n\}_{n\in\omega}$ such that $|b_n| = g(n)$ let $b_n^* \subseteq b_n$ be such that $|b_n^*| = f(n)$ and $F \in \mathcal{F}$ such that $F(b_n^*) = k$ for infinitely many n. Clearly the same holds for $F(b_n) = k$. □

Lemma 2.2. *If*

(a) f, g and h are functions from ω from ω
(b) h is increasing
(c) $g(n) = f(h(n))$ for all n

then $\mathfrak{s}_{k,f} \leq \mathfrak{s}_{k,g}$.

Proof. Suppose that $|\mathcal{F}| = \mathfrak{s}_{k,g}$ and for each sequence of pairwise disjoint sets $\{a_n\}_{n\in\omega}$ such that $|a_n| = g(n)$ there is $F \in \mathcal{F}$ such that $F(a_n) = k$ for infinitely many n. Then if $\{b_n\}_{n\in\omega}$ is sequence of pairwise disjoint sets such that $|b_n| = f(n)$ then $\{b_{h(n)}\}_{n\in\omega}$ is sequence of pairwise disjoint sets such that $|b_{h(n)}| = g(n)$ and so there is $F \in \mathcal{F}$ such that $F(b_{h(n)}) = k$ for infinitely many n and hence $F(b_n) = k$ for infinitely many n. □

Theorem 2.1. *If f and g are unbounded functions from ω to ω then $\mathfrak{s}_{k,g} = \mathfrak{s}_{k,f}$.*

Proof. Since f and g are both unbounded, it is possible to find an increasing h and a function e such that $f(n) < e(n)$ and such that $e(n) = g(h(n))$ for all n. By Lemma 2.1 it follows that $\mathfrak{s}_{k,e} \leq \mathfrak{s}_{k,f}$ and by Lemma 2.2 it follows that $\mathfrak{s}_{k,g} \leq \mathfrak{s}_{k,e}$. Hence $\mathfrak{s}_{k,g} \leq \mathfrak{s}_{k,f}$. The symmetry of the hypothesis implies that $\mathfrak{s}_{k,g} = \mathfrak{s}_{k,f}$. □

Theorem 2.1 justifies the following definition.

Definition 2.2. *$\mathfrak{s}_{k,\infty}$ will be used to denote $\mathfrak{s}_{k,f}$ when f is any unbounded function.*

Note that if f is any bounded function then a simple re-indexing argument shows that $\mathfrak{s}_{k,f} = \mathfrak{s}_{k,m}$ where $m = \limsup_n f(n)$. Hence the cardinals $\mathfrak{s}_{k,f}$ can be replaced by the cardinals $\mathfrak{s}_{k,\infty}$ and $\mathfrak{s}_{k,m}$ for $m \in \omega$. Some simple relationships bewteen these cardinals are easily established.

Lemma 2.3. *$\mathfrak{s}_{2,m} = \mathfrak{s}_{2,m^2}$ for all $m \geq 2$.*

Proof. By Lemma 2.1 it follows that $\mathfrak{s}_{2,m} \geq \mathfrak{s}_{2,m^2}$ for all $m \geq 2$. Now suppose that $|\mathcal{F}| = \mathfrak{s}_{2,m^2}$ and for each sequence of pairwise disjoint sets $\{a_n\}_{n\in\omega}$ such that $|a_n| = m^2$ there is $F \in \mathcal{F}$ such that $F(a_n) = 2$ for infinitely many n. If $\mathfrak{s}_{2,m} > \mathfrak{s}_{2,m^2}$ then there is a sequence of pairwise disjoint sets $\{b_n\}_{n\in\omega}$ such that $|b_n| = m$ and for each $F \in \mathcal{F}$ there is $F^* : \omega \to 2$ such that for all but finitely many $n \in \omega$ the restriction of F to b_n has constant value $F^*(n)$.

Since $|\{F^* \mid F \in \mathcal{F}\}| < \mathfrak{s}_{2,m}$ there is a sequence of pairwise disjoint sets $\{c_n\}_{n\in\omega}$ such that $|c_n| = m$ for each n and such that for each $F \in \mathcal{F}$ for all but finitely many $n \in \omega$ the restriction of F^* to c_n has constant value. Then let $d_n = \bigcup_{m\in c_n} b_m$ and note that the d_n are pairwise disjoint elements of $[\omega]^{m^2}$ and F is eventually constant on each d_n. This contradicts the choice of $\mathcal{F}$. □

Corollary 2.1. *$\mathfrak{s}_{2,m} = \mathfrak{s}_{2,k}$ for all $m, k \geq 2$.*

Lemma 2.4. *$\mathfrak{s}_{2,2} \geq \mathfrak{s}_{m,m}$ for all $m \geq 2$.*

Proof. Let $\mathcal{F} \subseteq 2^\omega$ be a family such that $|\mathcal{F}| = \mathfrak{s}_{2,2}$ and for each sequence of pairwise disjoint sets $\{a_n\}_{n\in\omega}$ such that $|a_n| = 2$ there is $F \in \mathcal{F}$ such that $F(a_n) = 2$ for infinitely many n. For any indexed family $\vec{F} = \{F_i\}_{i\in k} \subseteq \mathcal{F}$ and $\vec{\sigma} = \{\sigma_j\}_{j\in m}$ a family of distinct elements of 2^k define $H_{\vec{F},\vec{\sigma}} \in m^\omega$ by

$$H_{\vec{F},\vec{\sigma}}(n) = j \text{ if } (\forall i \in k)\ \sigma_j(i) = F_i(n)$$

and let

$$\mathcal{F}^* = \left\{ H_{\vec{F},\vec{\sigma}} \;\middle|\; \vec{F} = \{F_i\}_{i\in k} \subseteq \mathcal{F} \text{ and } \vec{\sigma} = \{\sigma_j\}_{j\in m} \text{ are distinct elements of } 2^k \right\}.$$

Clearly $|\mathcal{F}^*| = \mathfrak{s}_{2,2}$.

Now suppose that $\{a_n\}_{n\in\omega}$ are pairwise disjoint elements of $[\omega]^m$. Let $\vec{F} = \{F_i\}_{i\in k} \subseteq \mathcal{F}$ be such that if

$$\mathcal{S}_n = \left\{ a_n \cap \bigcap_{i\in k} F_i^{-1}\{\sigma(i)\} \right\}_{\sigma\in 2^k}$$

then $\limsup_n |\mathcal{S}_n|$ is maximal. Note that this means that $\lim_n |\mathcal{S}_n| = m$ because if there are infinitely many $b_n \in \mathcal{S}_n$ such that $|b_n| \geq 2$ then there is then $F \in \mathcal{F}$ such that $F(b_n) = 2$ for infinitely many n contradicting the maximality of $\limsup_n |\mathcal{S}_n|$. For all but finitely many n there are $\vec{\sigma}_n = \{\sigma_{n,i}\}_{i\in m} \subseteq 2^k$ such that for each $j \in a_n$ there is $i \in m$ such that

$$\{j\} = \bigcap_{\ell\in k} F_\ell^{-1}\{\sigma_{n,i}(\ell)\}.$$

Let $\vec{\sigma}$ be such that $\vec{\sigma}_n = \vec{\sigma}$ for infinitely many n. Then for each such n it follows that $H_{\vec{F},\vec{\sigma}}(a_n) = m$. $\square$

The following is a refinement of Theorem 1.3 of [6] due to Kamo.

Corollary 2.2. *If $2 \leq m \leq k < \omega$ then $\mathfrak{s}_{m,k} = \mathfrak{s}_{2,2}$.*

Proof. From Lemma 2.4 and Corollary 2.1 it follows that $\mathfrak{s}_{2,2} \geq \mathfrak{s}_{m,m} \geq \mathfrak{s}_{m,k} \geq \mathfrak{s}_{2,k} = \mathfrak{s}_{2,2}$. $\square$

Hence the only question that remains to be addressed is whether $\mathfrak{s}_{2,\infty} = \mathfrak{s}_{2,2}$. It will be shown in the next section that it is consistent for these cardinals to be different. But it is worth pointing out a connection to cardinals that have been studied elsewhere. The following is Definition 2.2 from [4].

Definition 2.3 ([4])**.** *If S and X are infinite subsets of ω say that S* bisects *X* in the limit *if*

$$\lim_{n\to\infty} \frac{|S \cap X \cap n|}{|X \cap n|} = 1/2$$

and for ϵ such that $0 < \epsilon < 1/2$ say that S ϵ-almost bisects *X if for all but finitely many $n \in \omega$*

$$\frac{|S \cap X \cap n|}{|X \cap n|} \in (1/2 - \epsilon, 1/2 + \epsilon).$$

Then define $\mathfrak{s}_{1/2}$ to be the least cardinal of a family $\mathcal{S}$ such that for all $X \in [\omega]^{\aleph_0}$ there is an element of $\mathcal{S}$ that bisects X. Define $\mathfrak{s}_{1/2\pm\epsilon}$ to be the least cardinal of a family $\mathcal{S}$ such that for all $X \in [\omega]^{\aleph_0}$ there is an element of $\mathcal{S}$ that ϵ-almost bisects X.

Proposition 2.1. *$\mathfrak{s}_{2,\infty} \leq \mathfrak{s}_{1/2\pm\epsilon}$ if $0 < \epsilon < 1/2$.*

Proof. Let $\mathcal{S}$ be a family of cardinality $\mathfrak{s}_{1/2\pm\epsilon}$ such that for all $X \in [\omega]^{\aleph_0}$ there is an element of $\mathcal{S}$ that ϵ-almost bisects X. It suffices to show that if $\{a_n\}_{n\in\omega}$ is any family of pairwise disjoint finite sets such that $|a_n| > \sum_{i\in n} |a_i|(1/2 + \epsilon)$ then there is $S \in \mathcal{S}$ such that

$$(\exists^\infty n)\ a_n \cap S \neq \varnothing \neq a_n \setminus S. \tag{1}$$

Given such a family $\{a_n\}_{n\in\omega}$ let $A = \bigcup_n a_n$ and let $S \in \mathcal{S}$ be such that for all but finitely many $n \in \omega$

$$\frac{|S \cap A \cap n|}{|A \cap n|} \in (1/2 - \epsilon, 1/2 + \epsilon).$$

If (1) fails it can be assumed that there are infinitely many n such that $a_n \subseteq S$. But for any such n if $m = \max(a_n)$ then

$$\frac{|S \cap A \cap m|}{|A \cap n|} \geq \frac{|a_n|}{\sum_{i\in n} |a_i|} > 1/2 + \epsilon.$$

$\square$

Of course, $\mathfrak{s} \leq \mathfrak{s}_{2,\infty} \leq \mathfrak{s}_{1/2\pm\epsilon} \leq \mathfrak{s}_{1/2}$ and in Theorem 2.4 of [4] it is shown that $\mathfrak{s}_{1/2}$ is no greater than $\mathbf{non}(\mathcal{N})$, the least cardinal of a non-Lebesgue null set. A companion to this is the following, which is one of various inequalities established in Proposition 0.1 of [6].

Proposition 2.2. $\mathfrak{s}_{2,2} \leq \mathbf{non}(\mathcal{N})$.

The natural question about possible equality is easily answered by the following. The inequality $\mathfrak{s}_{2,\infty} < \mathfrak{s}_{2,2}$ is much harder and is the main result to established in the current work.

Proposition 2.3. *It is consistent that* $\mathfrak{s}_{2,2} \neq \mathbf{non}(\mathcal{N})$.

Proof. Since $\mathbf{non}(\mathcal{N}) = \aleph_2$ after adding $\aleph_2$ Cohen reals, it suffices to show that if $\mathbb{C}$ is Cohen forcing and

$$1 \Vdash_{\mathbb{C}} \text{“}\{\dot{a}_n\}_{n\in\omega} \text{ are pairwise disjoint pairs ”}$$

then there if $F : \omega \to 2$ such that $1 \Vdash_{\mathbb{C}}$ “$(\exists^\infty n)\ F(\dot{a}_n) = 2$”. Now apply the argument that Cohen forcing does not add a dominating real. For the reader who would appreciate the details, let $\{p_n\}_{n\in\omega}$ enumerate $\mathbb{C}$. Construct inductively $\{b_{n,j}\}_{j\in n}$ such that

(a) $b_{n,j} \cap b_{m,i} = \varnothing$ unless $(n, j) = (m, i)$
(b) there is some $q_{n,j} \leq p_j$ such that $q_{n,j} \Vdash_{\mathbb{C}}$ “$\dot{a}_\ell = b_{n,j}$” for some ℓ.

To carry out the construction suppose that $\{b_{n,j}\}_{j\in n}$ and $\{b_{n+1,i}\}_{i\in k}$ have been constructed for some $k \in n + 1$. Let

$$B = \bigcup_{m\leq n} \bigcup_{j\in m} b_{m,j} \cup \bigcup_{m\leq k} b_{n+1,m}.$$

There is then some $q \leq p_k$ and $\ell > n + 1$ such that $q \Vdash_{\mathbb{C}}$ “$\dot{a}_\ell \cap B = \varnothing$”. Let $q_{n+1,k} \leq q$ and $b_{n+1,k}$ be such that $q_{n+1,k} \Vdash_{\mathbb{C}}$ “$\dot{a}_\ell = b_{n+1,k}$”.

Now let $F : \omega \to 2$ be such that $F(b_{n,j}) = 2$ for all n and j. To see that $1 \Vdash_{\mathbb{C}}$ “$(\exists^\infty n)\ F(\dot{a}_n) = 2$” suppose that there are p and k such that $p \Vdash_{\mathbb{C}}$ “$(\forall n \geq k)\ F$ is constant on $\dot{a}_n$”. If $p = p_j$ let ℓ be greater than both j and k. Then $F(b_{\ell,j}) = 2$ and $q_{\ell,j} \leq p$ and $q_{\ell,j} \Vdash_{\mathbb{C}}$ “$\dot{a}_\ell = b_{\ell,j}$” yielding a contradiction. □

3. Combinatorial content of consistency of $\mathfrak{s}_{2,\infty} < \mathfrak{s}_{2,2}$

The goal of this section to introduce the combinatorial results that will be used in the proof that it is consistent that $\mathfrak{s}_{2,2} = \aleph_2$ and $\mathfrak{s}_{2,\infty} = \aleph_1$. The forcing to be used is a countable support iteration of creature forcing partial orders about which the interested reader can find more in [7], although the reader familiar with [1] should have no trouble following the argument. The argument to be used will rely on a fusion over finite subsets of the support; so this section will look at the structures that result when obtaining finite approximations to the fusion argument. Although not logically necessary, it may be useful to some readers to jump ahead and look at Definition 4.2 before continuing to the results leading to Theorem 3.1 which will play a key role in establishing Theorem 5.1. In reading Definition 3.4 it may help to jump ahead to Fact 4.1.

Definition 3.1. *Define* $\mathbf{Ramsey}_v(k) = r$ *if* r *is the least integer such that* $r \to (k)^4_v$. *Note that the Canonical Ramsey Theorem* [5] *shows that if* v *is sufficiently large then for any* $Z : [r]^2 \to \omega$ *there is* $B \in [r]^k$ *such that* Z *is canonical on* $[B]^2$ *in that one of the following four options holds:*

(1) Z *is constant on* $[B]^2$
(2) Z *is one-to-one on* $[B]^2$
(3) there is a one-to-one $Z^* : B \to \omega$ *such that* $Z(a) = Z^*(\min(a))$ *for* $a \in [B]^2$
(4) there is a one-to-one $Z^* : B \to \omega$ *such that* $Z(a) = Z^*(\max(a))$ *for* $a \in [B]^2$.

Let $\mathbf{Ramsey}^n_v$ *be the* n*-fold iteration of* $\mathbf{Ramsey}_v$ *defined inductively by*

$$\mathbf{Ramsey}^{n+1}_v(k) = \mathbf{Ramsey}_v(\mathbf{Ramsey}^n_v(k)).$$

The following obvious fact will often be used without further explanation.

Fact 3.1. $\mathbf{Ramsey}_v(m) \to (m)^2_v$.

Definition 3.2. *Construct an increasing sequence of integers* $\{e_j\}_{j\in\omega}$ *inductively. Let* $e_0 = 0$ *and* $e_1 \geq \mathbf{Ramsey}_{6^6}(5)$ *and now suppose that* e_k *has been defined. Define* $I_j = [e_j, e_{j+1} - 1]$ *and let*

$$u_j = \left(\prod_{i\in j}\binom{e_{i+1}-e_i}{2}\right)^k = \left|\left(\prod_{i\in j}[I_i]^2\right)\right|^j.$$

Note that if e_k *has been defined then only* I_{k-1} *and* u_{k-1} *have been defined up to this point. Let* M_k *be so large that if* W *is a function from* M_k *to the family of partial functions from* u_{k-1} *to* $4 \times k$ *then there is* $\mathcal{M} \in [M_k]^{2u_{k-1}}$ *such that* W *is constant on* $\mathcal{M}$*. Let* b_k *be so large that*

$$b_k > (4k)^{M_k} > 2^{u_{k-1}} \tag{2}$$

and let $E_{k,0} = b_k$ *and then define*

$$E_{k,j+1} = 2\,\mathbf{Ramsey}_{b_k}^{F^2_{k,j}}(E_{k,j}) \tag{3}$$

where $F_{k,j}$ *satisfies*

$$F_{k,j} > u_{k-1}\prod_{\ell\leq j} E^k_{k,\ell} > b_k. \tag{4}$$

Then let $e_{k+1} = e_k + E_{k,kM_k}$.

Fact 3.2. $E_{k,j+1} - F_{k,j} \geq E_{k,j}$.

Definition 3.3. *If* $P \subseteq [I_k]^2$ *then define* $\|P\|$ *to be the greatest integer* j *such that there is* $X \subseteq I_k$ *such that* $|X| = E_{k,j}$ *and* $[X]^2 \subseteq P$*. (The* k *is implicit in the definition of* $\|P\|$*, but the dependence on this will be suppressed to simplify the already cumbersome notation.) Let*

$$\mathbb{U} = \bigcup_{k\in\omega}\prod_{j\in k}[I_j]^2.$$

If $T \subseteq \mathbb{U}$ *is a subtree and* $t \in T$ *let* $\mathbf{succ}_T(t) = \left\{x \in [I_{|t|}]^2 \mid t^\frown x \in T\right\}$ *and let* $\|t\|_T = \|\mathbf{succ}_T(t)\|$*. Then let* $\mathbb{P}$ *consist of all trees* $T \subseteq \mathbb{U}$ *such that*

$$(\forall k \in \mathbb{N})\ \left|\left\{t \in T \;\middle|\; \|t\|_T < M^3_{|t|}k\right\}\right| < \aleph_0. \tag{5}$$

Let $\mathbb{P}_\gamma$ *be the countable support iteration of length* γ *of the partial order* $\mathbb{P}$*. (This will only play a role later in §4.)*

Definition 3.4. *Let* J *and* K *be positive integers. For* $T \subseteq \mathbb{U}$ *define* $T[K] = \{t \in T \mid |t| = K\}$*. If* $j \in J$ *and* $k \in K$ *and* $\Theta \in \mathbb{U}[K]^J$ *define*

$$\Theta \restriction (k,j) = (\Theta(i) \restriction k)_{i\in j}.$$

For Θ *and* Θ^* *in* $\mathbb{U}[K]^J$ *and* $k \in K$ *and* $j \in J$ *define*

$$\Theta \sim_{k,j} \Theta^*$$

if $\Theta \restriction (k+1, j) = \Theta^* \restriction (k+1, j)$ *and* $\Theta \restriction (k, j+1) = \Theta^* \restriction (k, j+1)$*. For* $\mathcal{U} \subseteq \mathbb{U}[K]^J$ *and* $k \leq K$ *and* $j \leq J$ *define*

$$\mathcal{U}[k,j] = \{\Theta \restriction (k,j) \mid \Theta \in \mathcal{U}\}$$

and

$$B(k,j,\Theta,\mathcal{U}) = \{\Theta^*(k,j) \mid \Theta \sim_{k,j} \Theta^*\} \subseteq [I_k]^2$$

where $\Theta^*(k,j)$ *will serve to abbreviate the more cumbersome* $\Theta^*(j)(k)$*. For* $\Theta \in \mathcal{U}[k,j]$ *define*

$$\mathcal{U}\langle\Theta\rangle = \{\Theta^* \in \mathcal{U} \mid \Theta \restriction (k,j) = \Theta^* \restriction (k,j)\}.$$

It is worth noting that the asymmetry of the definition of $\sim_{k,j}$ point to the fact that this play a role in an iteration rather than a product of the $\mathbb{P}$.

Definition 3.5. *For $\mathcal{U} \subseteq \mathbb{U}[K]^J$ and $k \in K$ define*

$$\|\mathcal{U}\|_k = \min_{\Theta \in \mathcal{U}, j \in J} \|B(k, j, \Theta, \mathcal{U})\|.$$

For $k \in K$ define $\mathcal{U}$ to be k-organized if $\|B(k, j, \Theta, \mathcal{U})\| = \|\mathcal{U}\|_k + j$ whenever $\Theta \in \mathcal{U}$ and $j \in J \leq K$.

Fact 3.3. *If $\mathcal{U} \subseteq \mathbb{U}[K]^J$ and $k \leq K$ and $j \leq J$ then $|\mathcal{U}[k, j]| \leq u_{k-1}$.*

Fact 3.4. *If $\mathcal{U} \subseteq \mathbb{U}[K]^J$ and $\mathcal{U}$ is k-organized $j \leq J$ then the number of $\sim_{k,j}$ equivalence classes in $\mathcal{U}$ is bounded by $F_{k,\|\mathcal{U}\|_k+j}$.*

Definition 3.6. *If $\mathcal{U} \subseteq \mathbb{U}[K]^J$ for $k \leq K$ define $\mathcal{U}^* \sqsubseteq_k \mathcal{U}$ if $\mathcal{U}^* \subseteq \mathcal{U}$ and*

$$\mathbf{Ramsey}^{F_{k,j}}_{b_k}(\|B(k, j, \Theta, \mathcal{U}^*)\|) \geq \|B(k, j, \Theta, \mathcal{U})\|$$

for all $j \in J$ and $\Theta \in \mathcal{U}^*$. For $k_0 \leq k_1$ define $\mathcal{U}^* \sqsubseteq_{k_0,k_1} \mathcal{U}$ if $\mathcal{U}^* \sqsubseteq_k \mathcal{U}$ holds provided that $k_0 \leq k < k_1$.

It should be noted that the superscript in $\mathbf{Ramsey}^{F_{k,j}}_{b_k}$ is correct and is not intended to be $\mathbf{Ramsey}^{F^2_{k,j}}_{b_k}$ as in Definition 3.2. The reason for this will become clear in Lemma 3.13.

Fact 3.5. *If $\mathcal{U}_{F_{K_0,0}} \sqsubseteq_k \mathcal{U}_{F_{K_0,0}-1} \sqsubseteq_k \mathcal{U}_{F_{K_0,0}-2} \sqsubseteq_k \dots \sqsubseteq_k \mathcal{U}_1 \sqsubseteq_k \mathcal{U}$ then $\|\mathcal{U}_{F_{K_0,0}}\|_k \geq \|\mathcal{U}\|_k - 1$ for all $k \geq K_0$.*

Lemma 3.1. *If $T \subseteq ([I_k]^2)^J$ is thought of as a tree and $P : T \to b_k$ then there is $T^* \subseteq T$ such that P is constant on T^* and $\|t\|_T \geq \mathbf{Ramsey}_{b_k}(\|t\|_{T^*})$ for each $t \in T^*$ (with a slight abuse of the $\|t\|_T$ notation, since the tree T is not a subset of $\mathbb{U}$).*

Proof. This is a standard induction argument on J using Definition 3.1 and Fact 3.1. □

Lemma 3.2. *If $\mathcal{U} \subseteq \mathbb{U}[k+1]^J$ and $|\mathcal{U}[k, J]| = 1$ and $P : \mathcal{U} \to b_k$ then there is $\mathcal{U}^* \sqsubseteq_k \mathcal{U}$ such that P is constant on $\mathcal{U}^*$.*

Proof. Note that the hypothesis yields that $\mathcal{U}$ can be identified with the subset of $([I_{k+1}]^2)^J$ consisting of all σ such that if $\mathcal{U}[k, J] = \{\Theta\}$ then $\Theta \sqcup \sigma \in \mathcal{U}$ where $\Theta \sqcup \sigma(i) = \Theta(i)^\frown\sigma(i)$ for $i \in J$. Now apply Lemma 3.1. □

Lemma 3.3. *If*

- $J \leq K_0 \leq K_1$
- $\mathcal{U} \subseteq \mathbb{U}[K_1]^J$
- $|\mathcal{U}[K_0, J]| = 1$
- $P : \mathcal{U} \to b_{K_0}$

then there is $\mathcal{U}^* \sqsubseteq_{K_0,K_1} \mathcal{U}$ such that P is constant on $\mathcal{U}^*$.

Proof. Proceed by induction on $K_1 - K_0$ using Lemma 3.2. If $K_1 = K_0+1$ this is precisely Lemma 3.2. If $K_1 > K_0+1$ and the result is true for $K_1 - K_0$ then $\mathcal{U}\langle\Theta\rangle$ and $P \restriction \mathcal{U}\langle\Theta\rangle$ satisfy the inductive hypothesis for each $\Theta \in \mathcal{U}[K_0 + 1, J]$. Hence there is $P^*(\Theta) \in b_{K_0} < b_{K_0+1}$ and $\mathcal{U}^*_\Theta \sqsubseteq_{K_0+1,K_1} \mathcal{U}$ such that P has constant value $P^*(\Theta)$ on $\mathcal{U}^*_\Theta$. Apply Lemma 3.2 for $\mathcal{U}[K_0 + 1, J]$ and P^* to get $\mathcal{U}^{**} \sqsubseteq_{K_0} \mathcal{U}[K_0 + 1]$ such that P^* is constant on $\mathcal{U}^{**}$. Then let

$$\mathcal{U}^* = \bigcup_{\Theta \in \mathcal{U}^{**}} \mathcal{U}^*_\Theta.$$

□

Definition 3.7. *$\mathcal{U} \subseteq \mathbb{U}[K]^J$ and $Z : \mathcal{U} \to \omega$ define $C(\mathcal{U}, Z)$ to be the set of all 4-tuples (k, j, Θ, ℓ) such that $k \in K$, $j \in J$, $\Theta \in \mathcal{U}$ and $\ell \in 3$ and there is $Z_{k,j,\Theta}$ such that:*

(1) if $\ell = 0$ then $Z_{k,j,\Theta} : B(k, j, \Theta, \mathcal{U}) \to \omega$ is one-to-one and $Z_{k,j,\Theta}(a) = Z(\Theta^*)$ for any Θ^* such that $\Theta^* \sim_{k,j} \Theta$ and $\Theta^*(k, j) = a$
(2) if $\ell = 1$ then $Z_{k,j,\Theta} : \bigcup B(k, j, \Theta, \mathcal{U}) \to \omega$ is one-to-one and $Z_{k,j,\Theta}(\min(a)) = Z(\Theta^*)$ for any Θ^* such that $\Theta^* \sim_{k,j} \Theta$ and $\Theta^*(k, j) = a$
(3) if $\ell = 2$ then $Z_{k,j,\Theta} : \bigcup B(k, j, \Theta, \mathcal{U}) \to \omega$ is one-to-one and $Z_{k,j,\Theta}(\max(a)) = Z(\Theta^*)$ for any Θ^* such that $\Theta^* \sim_{k,j} \Theta$ and $\Theta^*(k, j) = a$.

Observe that $Z_{k,j,\Theta}$ and ℓ are invariant under the $\sim_{k,j}$ equivalence relation. In each case, let $R(Z, k, j, \Theta, \mathcal{U})$ be the image of either $Z_{k,j,\Theta}$ on $B(k, j, \Theta, \mathcal{U})$ if $\ell = 0$ or $Z_{k,j,\Theta}$ on $\bigcup B(k, j, \Theta, \mathcal{U})$ in the other two cases. Say that $C(\mathcal{U}, Z)$ is a front in $\mathcal{U}$ if for every $\Theta \in \mathcal{U}$ there are $k_\Theta(Z) \leq K$, $j_\Theta(Z) \in J$ and $\ell_\Theta(Z) \in 3$ such that $(k_\Theta(Z), j_\Theta(Z), \Theta, \ell_\Theta(Z)) \in C(\mathcal{U}, Z)$

Fact 3.6. *$R(Z, k, j, \Theta, \mathcal{U})$ depends only on the $\sim_{k,j}$ equivalence class of Θ.*

Lemma 3.4. *If $\mathcal{U} \subseteq \mathbb{U}[k + 1]^J$ and $|\mathcal{U}[k, J]| = 1$ and $Z : \mathcal{U} \to \omega$ then there is $\mathcal{U}^* \sqsubseteq_k \mathcal{U}$ such that either Z is constant on $\mathcal{U}^*$ or $C(\mathcal{U}^*, Z)$ is a front in $\mathcal{U}^*$.*

Proof. Identifying $\mathcal{U}$ with a subset of $([I_k]^2)^J$ as in Lemma 3.2 proceed by induction on J, the case $J = 1$ following from Definition 3.1. Now assume the result true for J and suppose that $T \subseteq ([I_k]^2)^{J+1}$. Let S be of maximal cardinality such that $[S]^2 = \{t(0) \mid t \in T\}$ and for each $a \in [S]^2$ let $T_a \subseteq ([I_k]^2)^J$ be the set of all t such that $a^\frown t \in T$. Apply the induction hypothesis to each T_a and $P_a : T_a \to \omega$ defined by $P_a(t) = P(a^\frown t)$. This yields $T^*_a \subseteq T_a$ such that $\|t\|_{T_a} \leq \mathbf{Ramsey}_{b_k}(\|t\|_{T^*_a})$ for each $t \in T^*$ and such that either P_a is constant on T^*_a or one of the three alternatives of Definition 3.7 holds.

Define $Q : [S]^2 \to 2$ by $Q(a) = 0$ if and only if P_a is constant on T^*_a. By Fact 3.1 there is $S^* \subseteq S$ such that $|S^*| \geq \mathbf{Ramsey}_{b_k}(|S|)$ and S^* is homogeneous for Q. If S^* is 1-homogeneous then let $T^* = \{a^\frown t \mid a \in [S^*]^2 \ \& \ t \in T^*_a\}$ and there is nothing further to do. On the other hand, if S^* is 0-homogeneous then let $P^*(a)$ be the constant value of P_a on T^*_a for each $a \in [S^*]^2$. By Definition 3.1 it is then possible to find $S^{**} \subseteq S^*$ such that $\mathbf{Ramsey}_{b_k}(|S^{**}|) \geq |S^*|$ — and hence $\mathbf{Ramsey}^2_{b_k}(|S^{**}|) \geq |S|$ — such that P^* is either constant on S^{**} or one of the three alternatives holds. Now let $T^* = \{a^\frown t \mid a \in [S^{**}]^2 \ \& \ t \in T^*_a\}$ □

Lemma 3.5. *If*

- *$J \leq K_0 \leq K_1$*
- *$\mathcal{U} \subseteq \mathbb{U}[K_1]^J$*
- *$|\mathcal{U}[K_0, J]| = 1$*
- *$Z : \mathcal{U} \to \omega$*

then there is $\mathcal{U}^* \sqsubseteq_{K_0,K_1} \mathcal{U}$ such that either Z is constant on $\mathcal{U}^*$ or $C(\mathcal{U}^*, Z)$ is a front in $\mathcal{U}^*$.

Proof. Proceed by induction on $K_1 - K_0$ using Lemma 3.4. □

Lemma 3.6. *If*

- *$K_0 \leq K_1$*
- *$\mathcal{U} \subseteq \mathbb{U}[K_1]^J$*
- *$\varnothing \neq B^*(k, j, \Theta) \subseteq B(k, j, \Theta, \mathcal{U})$ provided that $K_0 \leq k \leq K_1$, $j < J$ and $\Theta \in \mathcal{U}$*

then there is $\mathcal{U}^* \subseteq \mathcal{U}$ such that

- *$B(k, j, \Theta, \mathcal{U}^*) = B^*(k, j, \Theta)$ provided that $K_0 \leq k \leq K_1$, $j < J$ and $\Theta \in \mathcal{U}^*$*
- *$\mathcal{U}^*[K_0, J] = \mathcal{U}[K_0, J]$.*

Proof. Define $\mathcal{U}_{k,j} \subseteq \mathcal{U}$ by induction for k and j such that $K_0 \leq k \leq K_1$ and $j < J$.

(1) Let $\mathcal{U}_{K_0,0}$ be the set of all $\Theta \in \mathcal{U}$ such that $\Theta(K_0, 0) \in B^*(K_0, 0, \Theta)$.
(2) If $\mathcal{U}_{k,0}$ has been defined, let $\mathcal{U}_{k+1,0}$ be the set of all $\Theta \in \mathcal{U}_{k,0}$ such that

$$\Theta(k + 1, 0) \in B^*(k + 1, 0, \Theta).$$

(3) If $\mathcal{U}_{K_0,j}$ has been defined and $j < J-1$, let $\mathcal{U}_{K_0,j+1}$ to consist of the set of all $\Theta \in \mathcal{U}_{K_0,j}$ such that

$$\Theta(K_0, j+1) \in B^*(K_0, j+1, \Theta).$$

(4) It now suffices to define $\mathcal{U}_{k+1,j+1}$ assuming that both $\mathcal{U}_{k,j+1}$ and $\mathcal{U}_{k+1,j}$ have been defined. In this case define $\mathcal{U}_{k+1,j+1}$ to consist of the set of all $\Theta \in \mathcal{U}_{k,j+1} \cap \mathcal{U}_{k+1,j}$ such that

$$\Theta(k+1, j+1) \in B^*(k+1, j+1, \Theta).$$

Before proceeding, observe that if $k \geq K_0$ and $j < J_1$ then $\mathcal{U}_{k,j}$ is $\sim_{k,j}$ invariant. Let $\mathcal{U}^* = \mathcal{U}_{K_1,J-1}$. To see that this satisfies the lemma it will be shown by induction on (k,j) that if $K_0 \leq k$ and $j < J$ then $B(k,j,\Theta,\mathcal{U}_{k,j}) = B^*(k,j,\Theta)$ if $\Theta \in \mathcal{U}_{k,j}$ and $\mathcal{U}_{k,j} \neq \varnothing$.

The result is immediate for $\mathcal{U}_{K_0,0}$. Now assume that $B(k,j,\Theta,\mathcal{U}_{k,j}) = B^*(k,j,\Theta)$ for some $\Theta \in \mathcal{U}_{k,j}$. Let $a \in B^*(k+1,j,\Theta)$. Since $B^*(k+1,j,\Theta) \subseteq B(k+1,j,\Theta,\mathcal{U})$ there is some $\Theta^* \sim_{k+1,j} \Theta$ such that $\Theta^*(k+1,j) = a$. But then $\Theta^* \sim_{k,j} \Theta$ and hence, since $\mathcal{U}_{k,j}$ is $\sim_{k,j}$ invariant, it follows that $\Theta^* \in \mathcal{U}_{k,j}$. But by virtue of the fact that $\Theta^* \in \mathcal{U}_{k,j}$ and $\Theta^*(k+1,j) \in B^*(k+1,j,\Theta)$ it follows that $\Theta^* \in \mathcal{U}_{k+1,j}$. This shows that $\mathcal{U}_{k+1,j} \neq \varnothing$ and that $B^*(k+1,j,\Theta) = B(k+1,j,\Theta,\mathcal{U}_{k+1,j})$. A similar argument works for $a \in B^*(k,j+1,\Theta)$. □

Lemma 3.7. *Suppose that*

- $J \leq K_0 \leq K_1$
- $\mathcal{U} \subseteq \mathbb{U}[K_1]^J$

There is then $\mathcal{U}^* \subseteq \mathcal{U}$ such that $\mathcal{U}^*$ is k-organized and $\|\mathcal{U}^*\|_k \geq \|\mathcal{U}\|_k - J$ for $k \geq K_0$.

Proof. Let $n(k)$ be maximal such that Since $\|\mathcal{U}\|_k \geq n(k) + J$ for all $k \geq K_0$. Then simply choose $B^*(k,j,\Theta) \subseteq B(k,j,\Theta,\mathcal{U})$ such that $\|B^*(k,j,\Theta)\|_k = n(k) + j$ and apply Lemma 3.6. □

Lemma 3.8. *Suppose that*

- $J \leq K_0 \leq K_1$
- $\mathcal{D}$ *is a family of subsets of* $\mathbb{U}[K_1]^J$
- *for all* $\mathcal{U} \subseteq \mathbb{U}[K_1]^J$ *and* $\Theta \in \mathcal{U}[K_0, J]$ *there is* $\mathcal{U}^* \sqsubseteq_{K_0,K_1} \mathcal{U}\langle\Theta\rangle$ *such that* $\mathcal{U}^* \in \mathcal{D}$.

Then for any $\mathcal{U} \subseteq \mathbb{U}[K_1]^J$ there is $\bar{\mathcal{U}} \subseteq \mathcal{U}$ such that

- $\bar{\mathcal{U}}[K_0, J] = \mathcal{U}[K_0, J]$
- $\bar{\mathcal{U}}\langle\Theta\rangle \in \mathcal{D}$ *for each* $\Theta \in \mathcal{U}[K_0, J]$
- $\|\bar{\mathcal{U}}\|_k \geq \|\mathcal{U}\|_k - 1$ *for all* $k \geq K_0$.

Proof. Let $\{\Theta_i\}_{i=0}^u$ enumerate $\mathcal{U}[K_0, J]$ where $u \leq u_{K_0-1} \leq F_{K_0,0}$. Then construct inductively $\mathcal{U}_i$ such that

- $\mathcal{U} = \mathcal{U}_0$
- $\bar{\mathcal{U}}_i[K_0, J] = \mathcal{U}[K_0, J]$
- $\bar{\mathcal{U}}_{i+1}\langle\Theta_i\rangle \in \mathcal{D}$
- $\mathcal{U}_{i+1} \sqsubseteq_{K_0,K_1} \mathcal{U}_i$.

Letting $\bar{\mathcal{U}} = \mathcal{U}_u$ it follows from Fact 3.5 that$\|\bar{\mathcal{U}}\|_k \geq \|\mathcal{U}\|_k - 1$ for all $k \geq K_0$ and hence $\bar{\mathcal{U}}$ satisfies the lemma. □

Lemma 3.9. *Suppose that*

(a) $J \leq K_0 \leq K_1$
(b) $\mathcal{U} \subseteq \mathbb{U}[K_1]^J$
(c) $Z : \mathcal{U} \to \omega$.

There is then $\mathcal{U}^* \subseteq \mathcal{U}$ such that:

(d) $\|\mathcal{U}^*\|_k \geq \|\mathcal{U}\|_k - 1$ *for* $k \geq K_0$
(e) $\mathcal{U}^*[K_0, J] = \mathcal{U}[K_0, J]$

(f) for each $\Theta \in \mathcal{U}[K_0, J]$ either Z is constant on $\mathcal{U}^*\langle\Theta\rangle$ or $C(\mathcal{U}^*\langle\Theta\rangle, Z)$ is a front in $\mathcal{U}^*\langle\Theta\rangle$.

Proof. Let $\mathcal{D}$ consist of all $\mathcal{V} \subseteq \mathcal{U}$ such that either Z is constant on $\mathcal{V}$ or $C(\mathcal{V}, Z)$ is a front in $\mathcal{V}$. By Lemma 3.5 it follows that $\mathcal{D}$ satisfies the hypotheses of Lemma 3.8 and, hence, there is $\bar{\mathcal{U}} \subseteq \mathcal{U}$ such that

- $\bar{\mathcal{U}}[K_0, J] = \mathcal{U}[K_0, J]$
- $\bar{\mathcal{U}}\langle\Theta\rangle \in \mathcal{D}$ for each $\Theta \in \mathcal{U}[K_0, J]$
- $\|\bar{\mathcal{U}}\|_k \geq \mathcal{U}\|_k - 1$for $k \geq K_0$.

as required. □

Lemma 3.10. *Suppose that $|S_0| = |S_1| = \mathbf{Ramsey}_2(2m)$ and $Z_i : [S_i]^2 \to \omega$. Suppose also that $Z_i(a) \neq Z_i(b)$ if $a \cap b = \varnothing$. There are then $S_i^* \subseteq S_i$ such that $|S_0^*| = |S_1^*| = m$ and $Z_0([S_0^*]^2) \cap Z_1([S_1^*]^2) = \varnothing$.*

Proof. For $x \subseteq \omega$ let $\{x[i]\}_{i\in|x|}$ enumerate x in increasing order. Let $\Psi : S_0 \to S_1$ be a bijection and define $P : [S_0]^4 \to 2$ by $P(x) = 0$ if and only if $Z_0(\{x[0], x[1]\}) = Z_1(\{\Psi(x[2]), \Psi(x[3])\})$. It is easy to see that the hypothesis on the Z_i rules out the possibility that there is a 0-homogeneous set for P of cardinality greater than 4. Let S^* be homogenous for P of cardinality $2m$ and let $S_0^* = \{S^*[i]\}_{i\in m}$ and $S_1^* = \Psi(S^* \setminus S_0^*)$. □

Corollary 3.1. *Suppose that*

- *$Z_i : [S_i]^2 \to \omega$ for $i \in b_k$ and $Z_i(a) \neq Z_i(b)$ if $a \cap b = \varnothing$.*
- *$|S_i| \geq \mathbf{Ramsey}_2^{F_{k,j}}(2m)$ for each $i \in F_{k,j}$.*

There are then $S_i^* \subseteq S_i$ such that

- *$|S_i^*| = m$ for each i*
- *$Z_i([S_i^*]^2) \cap Z_j([S_j^*]^2) = \varnothing$ if $i < j < F_{k,j}$.*

Proof. Use that $b_k < F_{k,j}$. □

Lemma 3.11. *Suppose that $|S| = \mathbf{Ramsey}_2(m)$ and $Z_i : [S]^2 \to \omega$. Suppose also that $Z_0(a) \neq Z_1(a)$ for all $a \in [S]^2$ and that one of the following three options holds:*

- *each Z_i is one-to-one*
- *$Z_i(x) = Z_i(y)$ if and only if $\min(x) = \min(y)$ for each i*
- *$Z_i(x) = Z_i(y)$ if and only if $\max(x) = \max(y)$ for each i*

There is then $S^* \subseteq S$ such that $|S^*| = m$ and $Z_0([S^*]^2) \cap Z_1([S^*]^2) = \varnothing$.

Proof. Define $P : [S]^4 \to 2$ by $P(x) = 0$ if and only if there are a and b in $[x]^2$ such that $Z_0(a) = Z_1(b)$. It suffices to show that no 0-homogeneous subset of P can have cardinality 5. Three cases need to be considered.

If each Z_i is one-to-one and $w \in [S]^5$ is 0-homogeneous. Let a and b be in $[w]^2$ such that $Z_0(a) = Z_1(b)$ and let $x \in [w]^4$ be such that $a \cup b \subseteq x \subseteq w$. There is then $x' \in [w]^4$ such the isomorphism taking x to x' moves precisely one of a or b. This yields a contradiction to the assumption that each Z_i is one-to-one.

If $Z_i(x) = Z_i(y)$ if and only if $\min(x) = \min(y)$ for each i let $Z_i^* : S \to \omega$ be such that $Z_i(a) = Z_i^*(\min(a))$ and note that $Z_0^*(s) \neq Z_1^*(s)$ for all $s \in S$ and each Z_i^* is one-to-one. However, if w is 0-homogenous for P and $|w| \geq 7$ then it is possible to find $x \in [w]^4$ such that $Z_0^*(x) \cap Z_1^*(x) = \varnothing$ contradicting that w is 0-homogenous.

The case that $Z_i(x) = Z_i(y)$ if and only if $\max(x) = \max(y)$ for each i is handled similarly. □

Note that in the last part of the proof of Lemma 3.11 if it were the case that $Z_0(x) = Z_0^*(\min(x))$ and $Z_1(x) = Z_1^*(\max(x))$ then the argument to get x from w would fail since it might be possible that $Z_0^*(s) = Z_1^*(s)$ without violating the hypothesis that $Z_0(a) \neq Z_1(a)$ for all $a \in [S]^2$.

Lemma 3.12. *Suppose that*

- $J \leq K_0 \leq K_1$
- $\mathcal{U} \subseteq \mathbb{U}[K_1]^J$
- $Z_i : \mathcal{U} \to \omega$ *for* $i \in 2$.

There is then $\mathcal{U}^* \subseteq \mathcal{U}$ *such that* $\|\mathcal{U}^*\|_k \geq \|\mathcal{U}\|_k - (J+1)$ *for* $k \geq K_0$ *and such that if:*

- $K_0 \leq k_0 < k_1 < K_1$ *or* $K_0 \leq k_0 = k_1 < K_1$ *and* $j_0 < j_1$
- $(k_i, j_i, \Theta_i, \ell_i) \in C(\mathcal{U}^*, Z_i)$ *for* $i \in 2$

then $R(Z_0, k_0, j_0, \Theta_0, \mathcal{U}^*) \cap R(Z_1, k_1, j_1, \Theta_1, \mathcal{U}^*) = \varnothing$. *Moreover, if* $\Theta \in \mathcal{U}[K_0, J]$ *and* Z_0 *has constant value* v *on* $\mathcal{U}^*\langle\Theta\rangle$ *then* $v \notin R(Z_1, k_1, j_1, \Theta, \mathcal{U}^*)$.

Proof. Begin by using Lemma 3.7 to find $\mathcal{U}^* \subseteq \mathcal{U}$ such that $\|\mathcal{U}^*\|_k \geq \|\mathcal{U}\|_k - J$ and $\mathcal{U}^*$ is k-organized. for each $k \geq K_0$. Let V be the set of all v such that there is some $\Theta \in \mathcal{U}[K_0, J]$ and Z_0 has constant value v on $\mathcal{U}^*\langle\Theta\rangle$. Then let

$$R_{k,j} = \bigcup_{\Theta \in \mathcal{U}^*} \left(\bigcup_{K_0 \leq k^* < k} \bigcup_{j^* \in j} R(Z_0, k^*, j^*, \Theta, \mathcal{U}^*) \cup \bigcup_{j^* < j} R(Z_0, k, j^*, \Theta, \mathcal{U}^*) \right) \cup V$$

and note that $|R_{k,j}| < F_{k, \|\mathcal{U}^*\|_k + j}$ by Fact 3.4 and Fact 3.6. Let $Z = (Z_1)_{k,j,\Theta}$ as defined in Definition 3.7 and let ℓ be such that $(k, j, \Theta, \ell) \in C(\mathcal{U}^*, Z_1)$. Consider two cases.

Case 1. *If* $\ell = 0$ *then let*

$$B^*(k, j, \Theta) = B(k, j, \Theta, \mathcal{U}^*) \setminus Z^{-1}(R_{k,j})$$

Case 2. *If* $\ell = 1$ *or* $\ell = 2$ *then let*

$$B^*(k, j, \Theta) = \left[\left(\bigcup B(k, j, \Theta, \mathcal{U}^*) \right) \setminus Z^{-1}(R_{k,j}) \right]^2$$

Using Fact 3.2, it follows that $\|B^*(k, j, \Theta)\|_k \geq \|\mathcal{U}^*\|_k + j - 1$. Now apply Lemma 3.6. □

Corollary 3.2. *Suppose that:*

- $J \leq K_0 \leq K_1$
- $\mathcal{U} \subseteq \mathbb{U}[K_1]^J$
- $Z_i : \mathcal{U} \to \omega$ *for* $i \in 2$ *are such that* $Z_0(\Theta) \neq Z_1(\Theta)$ *for all* $\Theta \in \mathcal{U}$.

There is then $\mathcal{U}^* \subseteq \mathcal{U}$ *such that*

- $\|\mathcal{U}^*\|_k \geq \|\mathcal{U}\|_k - 1$ *for all* $k \geq K_0$
- *if* $\ell \in 3$ *and* $(k, j, \Theta, \ell) \in C(\mathcal{U}^*, Z_i)$ *for* $i \in 2$ *then* $R(Z_0, k, j, \Theta, \mathcal{U}^*) \cap R(Z_1, k, j, \Theta, \mathcal{U}^*) = \varnothing$ *provided that* $k \geq K_0$.

Proof. By Lemma 3.8 and Lemma 3.6 it suffices to show that if:

- $\Theta \in \mathcal{U}[K_0, J]$
- $\ell \in 3$
- $K_0 \leq k < K_1$
- $j < J$
- $\Theta^* \in \mathcal{U}\langle\Theta\rangle$
- $(k, j, \Theta^*, \ell) \in C(\mathcal{U}, Z_i)$ for each $i \in 2$

that there is $B^* \subseteq B(k, j, \Theta^*, \mathcal{U})$ such that $\mathbf{Ramsey}_2(\|B^*\|_k) \geq \|B(k, j, \Theta, \mathcal{U})\|_k$ and

$$R(Z_0, k, j, \Theta, \mathcal{U}^*) \cap R(Z_1, k, j, \Theta, \mathcal{U}^*) = \varnothing.$$

This follows directly from Lemma 3.11. □

Lemma 3.13. *Suppose that*

(a) $J \leq K_0 \leq K_1$

(b) $\mathcal{U} \subseteq \mathbb{U}[K_1]^J$
(c) $Z_i : \mathcal{U} \to \omega$ *for* $i \in 2$

There is then $\mathcal{U}^* \subseteq \mathcal{U}$ *such that* $\|\mathcal{U}^*\|_k \geq \|\mathcal{U}^*\|_k - 1$ *for* $k \geq K_0$ *and if*

$$(\forall i \in 2)\ (k, j, \Theta_i, \ell_i) \in C(\mathcal{U}^*, Z_i) \tag{6}$$

$$\Theta_0 \not\sim_{k,j} \Theta_1 \tag{7}$$

then $R(Z_0, k, j, \Theta_0, \mathcal{U}^*) \cap R(Z_1, k, j, \Theta_1, \mathcal{U}^*) = \varnothing$.

Proof. By Lemma 3.8 and Lemma 3.6 it suffices to show that if:

- $\Theta \in \mathcal{U}[K_0, J]$
- $\ell \in 2$
- $K_0 \leq k < K_1$
- $j < J$

then there are $B(\Theta^*) \subseteq B(k, j, \Theta^*, \mathcal{U})$ for each $\Theta^* \in \mathcal{U}\langle\Theta\rangle$ such that

$$\mathbf{Ramsey}_{b_k}^{F_{k,j}}(\|B^*(\Theta^*)\|/2) \geq 2\|B(k, j, \Theta, \mathcal{U})\| \tag{8}$$

and such that if

- $(k, j, \Theta_i, \ell) \in C(\mathcal{U}, Z_i)$ for each $i \in 2$
- $\Theta_0 \not\sim_{k,j} \Theta_1$

then $R(Z_0, k, j, \Theta_0, \mathcal{U}^*) \cap R(Z_1, k, j, \Theta_1, \mathcal{U}^*) = \varnothing$. This follows by applying Corollary 3.1 and Fact 3.4 to each pair of Θ_0 and Θ_1 in $\mathcal{U}[K_0, J]$ such that $\Theta_0 \not\sim_{k,j} \Theta_1$ since $B(k, j, \Theta, \mathcal{U})$ depends only on the $\sim_{k,j}$ equivalence class of Θ and there are no more than $|\mathcal{U}[k, J]|^2 \leq F_{k,j}$ such pairs. Moreover, the Hypothesis of Corollary 3.1 is satisfied since $(k, j, \Theta_i, \ell_i) \in C(\mathcal{U}^*, Z_i)$ and so the Z_i are not constant on $B(k, j, \Theta, \mathcal{U}^*)$. The factor of 1/2 in (8) is not a problem because of the factor of 2 in (3) in Definition 3.2. □

Lemma 3.14. *Suppose that*
(a) $J \leq K_0 \leq K_1$
(b) $\mathcal{U} \subseteq \mathbb{U}[K_1]^J$
(c) $Z_\ell : \mathcal{U} \to \omega$ *for* $\ell \in 2$ *are such that* $Z_0(\Theta) \neq Z_1(\Theta)$ *for all* $\Theta \in \mathcal{U}$.

There is then $\mathcal{U}^* \subseteq \mathcal{U}$ *such that* $\|\mathcal{U}^*\|_k \geq \|\mathcal{U}\|_k - (2J + 4)$ *for* $k \geq K_0$ *and if* $(k_i, j_i, \Theta_i, \ell_i) \in C(\mathcal{U}, Z_i)$ *for each* $i \in 2$ *and one of following three options holds:*
(1) $k_0 = k_1 = k$ *and* $j_0 = j_1 = j$ *and* $\Theta_0 \not\sim_{k,j} \Theta_1$
(2) $k_0 \neq k_1$ *or* $j_0 \neq j_1$
(3) $\Theta_0 \sim_{k,j} \Theta_1$ *and* $\ell_0 = \ell_1$

then

$$R(Z_0, k_0, j_0, \Theta_0, \mathcal{U}^*) \cap R(Z_1, k_1, j_1, \Theta_1, \mathcal{U}^*) = \varnothing$$

and, moreover, if $i_0 \neq i_1$ *and* $\Theta \in \mathcal{U}[K_0, J]$ *and* Z_{i_0} *has constant value* v *on* $\mathcal{U}^*\langle\Theta\rangle$ *then* $v \notin R(Z_{i_1}, k_{i_1}, j_{i_1}, \Theta, \mathcal{U}^*)$.

Proof. Apply Lemma 3.13 to get (1) and Corollary 3.2 to get (3). Then apply Lemma 3.12 twice, once for the pair (Z_0, Z_1) and again for (Z_1, Z_0) to get (2). □

Corollary 3.3. *Suppose that*
(a) $J \leq K_0 \leq K_1$
(b) $\mathcal{U} \subseteq \mathbb{U}[K_1]^J$
(c) $Z : \mathcal{U} \to [\omega]^M$ *and* $Z_m(\Theta) = Z(\Theta)[m]$ *for* $m \in M$ *(see the notation in Lemma 3.10).*

There is then $\mathcal{U}^* \subseteq \mathcal{U}$ *such that* $\|\mathcal{U}^*\|_k \geq \|\mathcal{U}\|_k - M^2(2J + 4)$ *for* $k \geq K_0$ *and if* $m_i \in M$ *and* $(k_i, j_i, \Theta_i, \ell_i) \in C(\mathcal{U}, Z_{m_i})$ *and one of following three options holds:*

(1) $k_0 = k_1 = k$ *and* $j_0 = j_1 = j$ *and* $\Theta_0 \not\sim_{k,j} \Theta_1$
(2) $k_0 \neq k_1$ *or* $j_0 \neq j_1$
(3) $\Theta_0 \sim_{k,j} \Theta_1$ *and* $\ell_0 = \ell_1$

then

$$R(Z_{m_0}, k_0, j_0, \Theta_0, \mathcal{U}^*) \cap R(Z_{m_1}, k_1, j_1, \Theta_1, \mathcal{U}^*) = \varnothing$$

and, moreover, if $m_0 \neq m_1$ *are in* M *and* $\Theta \in \mathcal{U}[K_0, J]$ *and* Z_{m_0} *has constant value* v *on* $\mathcal{U}^*\langle\Theta\rangle$ *then* $v \notin R(Z_{m_1}, k_{m_1}, j_{m_1}, \Theta, \mathcal{U}^*)$.

Proof. For each pair $(m, m') \in M^2$ apply Lemma 3.14. It needs to be noted that $Z_m(\Theta) \neq Z_{m'}(\Theta)$ for all $\Theta \in \mathcal{U}$ and $m \neq m'$ by the definition of the Z_m and hence the hypotheses of Lemma 3.14 are satisfied. □

Theorem 3.1. *Suppose that*

- $J < K_0 \leq K_1$
- $\mathcal{U} \subseteq \mathbb{U}[K_1]^J$
- $Z : \mathcal{U} \to [\omega]^{M_{K_0}}$.

There are then $\mathcal{U}^* \subseteq \mathcal{U}$ *and disjoint* A *and* B *such that*

- $\|\mathcal{U}^*\|_k \geq \|\mathcal{U}\|_k - (M_{K_0}^2(2J+4) + M + 1)$
- $\mathcal{U}[K_0] = \mathcal{U}^*[K_0]$
- $A \cap Z(\Theta) \neq \varnothing \neq B \cap Z(\Theta)$ *for all* $\Theta \in \mathcal{U}^*$.

Proof. Let $M = M_{K_0}$. Let $Z_m(\Theta)$ the m^{th} element of $Z(\Theta)$. Using Lemma 3.9 inductively, find $\mathcal{U}_m$ for each $m \leq M$ such that

- $\mathcal{U}_0 = \mathcal{U}$
- $\|\mathcal{U}_m\|_k \geq \|\mathcal{U}\|_k - m$
- $\mathcal{U}_m[K_0, J] = \mathcal{U}[K_0, J]$
- for each $\Theta^* \in \mathcal{U}[K_0]$ either Z_m is constant on $\mathcal{U}_{m+1}\langle\Theta^*\rangle$ or $C(\mathcal{U}_{m+1}, Z_m)$ is a front in $\mathcal{U}_{m+1}$.

Hence $\|\mathcal{U}_M\|_k \geq \|\mathcal{U}\|_k - M$.

For each $\Theta \in \mathcal{U}_M$ let $\Psi_\Theta : M \to (J+1) \times 4$ be the mapping defined by

$$\Psi_\Theta(m) = \begin{cases} (j_\Theta(Z_m), \ell_\Theta(Z_m)) & \text{if this is defined as in Definition 3.7} \\ (J, 3) & \text{otherwise.} \end{cases}$$

Note that $4(J+1)^M \leq 4K_0^M < b_{K_0}$. Hence, it is possible to use Lemma 3.3 in conjunction with Lemma 3.8 to find $\bar{\mathcal{U}} \subseteq \mathcal{U}_M$ such that for each $\Theta \in \mathcal{U}[K_0, J]$ and $m \in M$ there are $j^*_\Theta(m)$ and $\ell^*_\Theta(m)$ such that such that $\Psi_{\Theta^*}(m) = (j^*_\Theta(m), \ell^*_\Theta(m))$ for each m and $\Theta^* \in \bar{\mathcal{U}}\langle\Theta\rangle$ and such that $\|\bar{\mathcal{U}}\|_k \geq \|\mathcal{U}\|_k - (M+1)$ for $k \geq K_0$. Then use Corollary 3.3 to get the conclusion of that corollary to hold on $\mathcal{U}^* \subseteq \bar{\mathcal{U}}$ and $\|\bar{\mathcal{U}}\|_k \geq \|\mathcal{U}\|_k - (M^2(2J+4) + M + 1)$.

Let W be the function defined on M such that $W(m)$ is the function from $\mathcal{U}^*[K_0, J]$ to $4 \times K_0$ (noting that $J + 1 \leq K_0$) defined by $W(m)(\Theta) = (\ell^*_\Theta(m), j^*_\Theta(m))$. Referring to Definition 3.2, let $\mathcal{M} \subseteq M$ and $W^* : \mathcal{U}^*[K_0, J] \to 4 \times K_0$ be such that $W(m) = W^*$ for all $m \in \mathcal{M}$ and note that

$$|\mathcal{M}| \geq 2u_{K_0-1} \geq |\mathbb{U}[K_0]|^J \geq 2|\mathcal{U}[K_0, J]|.$$

Let the coordinate functions of W^* be given by $W^* = (W^*_\ell, W^*_j)$.

Then for $\Theta \in \mathcal{U}^*[K_0, J]$ and $m \in \mathcal{M}$ define

$$R^*_m(\Theta) = \bigcup_{\Theta^* \in \mathcal{U}^*\langle\Theta\rangle} R(Z_m, k_{\Theta^*}(Z_m), W^*_j(\Theta), \Theta^*, \mathcal{U}^*).$$

Let $\mathcal{Y} = \left\{\Theta \in \mathcal{U}^*[K_0, J] \;\middle|\; W^*_j(\Theta) = J\right\}$ or, in other words, $\mathcal{Y}$ consists of those $\Theta \in \mathcal{U}^*[K_0, J]$ such that Z_m is constant on $\mathcal{U}_M\langle\Theta\rangle$ for each $m \in \mathcal{M}$. Let $Y(\Theta, m)$ be this constant value and define

$Y_\Theta = \{Y(\Theta, m) \mid m \in \mathcal{M}\}$. Observing that $|Y_\Theta| = |\mathcal{M}| \geq 2|\mathcal{U}[K_0, J]|$ it is easy to find disjoint A_0 and B_0 such that $A_0 \cap Y_\Theta \neq \varnothing \neq B_0 \cap Y_\Theta$ for $\Theta \in \mathcal{Y}$.

Finally, let $\mathcal{Y}^* = \mathcal{U}^*[K_0, J] \setminus \mathcal{Y}$ let $m_a \in \mathcal{M}$ and $m_b \in \mathcal{M}$ be distinct and define

$$A = A_0 \cup \left(\bigcup_{\Theta \in \mathcal{Y}^*} R^*_{m_a}(\Theta) \right) \ \&\ B = B_0 \cup \left(\bigcup_{\Theta \in \mathcal{Y}^*} R^*_{m_b}(\Theta) \right)$$

To see that $\mathcal{U}^*$ and A and B satisfy the conclusion, two points need to be verified. The first is that if $\Theta^* \in \mathcal{U}^*$ then $Z(\Theta^*) \cap A \neq \varnothing \neq Z(\Theta^*) \cap B$. Let $\Theta = \Theta^* \restriction (K_0, J)$. If $\Theta \in \mathcal{Y}$ then it follows that $A_0 \cap Y_\Theta \neq \varnothing \neq B_0 \cap Y_\Theta$ and hence $A_0 \cap Z(\Theta^*) \neq \varnothing \neq B_0 \cap Z(\Theta^*)$. On the other hand, if $\Theta \in \mathcal{Y}^*$ then

$$Z_{m_a}(\Theta^*) \in R(Z_{m_a}, k_{\Theta^*}(Z_{m_a}), W^*_j(\Theta), \Theta^*, \mathcal{U}^*) \subseteq R^*_{m_a}(\Theta) \subseteq A$$

and the result follows. The same argument works for B.

The final point that needs to be checked is that $A \cap B = \varnothing$. The fact that $A_0 \cap B_0 = \varnothing$ follows from the construction. The fact that

$$\left(\bigcup_{\Theta \in \mathcal{Y}^*} R^*_{m_a}(\Theta) \right) \cap (A_0 \cup B_0) = \varnothing = \left(\bigcup_{\Theta \in \mathcal{Y}^*} R^*_{m_b}(\Theta) \right) \cap (A_0 \cup B_0)$$

follows immediately from the last clause of Corollary 3.3. The fact that $R^*_{m_a}(\Theta) \cap R^*_{m_b}(\Theta') = \varnothing$ for all Θ and Θ' in $\mathcal{Y}^*$ will be shown to follow from the first part of Corollary 3.3.

To see this it has to be shown that if $\Theta_i \in \mathcal{Y}^*$ for $i \in 2$ and $\Theta^*_i \in \mathcal{U}\langle \Theta_i \rangle$ then

$$R(Z_{m_a}, k_{\Theta^*_0}(Z_{m_a}), W^*_j(\Theta_0), \Theta^*_0, \mathcal{U}^*) \cap R(Z_{m_b}, k_{\Theta^*_1}(Z_{m_b}), W^*_j(\Theta_1), \Theta^*_1, \mathcal{U}^*) = \varnothing.$$

If $k_{\Theta^*_0}(Z_{m_a}) \neq k_{\Theta^*_0}(Z_{m_b})$ or if $W^*_j(\Theta_0) \neq W^*_j(\Theta_1)$ then Corollary 3.3 can be directly applied, so assume that $k_{\Theta^*_0}(Z_{m_a}) = k_{\Theta^*_0}(Z_{m_b}) = k$ and $W^*_j(\Theta_0) = W^*_j(\Theta_1) = w$. If $\Theta^*_0 \not\sim_{k,w} \Theta^*_1$ then again Corollary 3.3 can be directly applied, so it may be assumed that $\Theta^*_0 \sim_{k,w} \Theta^*_1$. In this case it must be verified that $\ell^*_{\Theta^*_0}(Z_{m_a}) = \ell^*_{\Theta^*_1}(Z_{m_b})$ and for this it suffices to show that $W^*_\ell(\Theta_0) = W^*_\ell(\Theta_1)$. But this follows from the fact that $\Theta^*_0 \sim_{k,w} \Theta^*_1$ implies that $\Theta_0 = \Theta_1$ and hence $W^*_\ell(\Theta_0) = W^*_\ell(\Theta_1)$ as required. □

4. The iteration

Definition 4.1. *If T and S are in $\mathbb{P}$ define $T \leq_n S$ is $T \subseteq S$ and $\|t\|_T < nM^3_{|t|}$ implies that $\mathbf{succ}_T(t) \subseteq S$. As usual, if p and q are in $\mathbb{P}_{\omega_2}$ and $\Gamma \subseteq \omega_2$ is finite, define $p \leq_{\Gamma,n} q$ if $p \leq q$ and*

$$(\forall n \in \Gamma)\ p \restriction \gamma \Vdash_{\mathbb{P}_\gamma} \text{“}p(\gamma) \leq_n q(\gamma)\text{”}.$$

For trees T and S, perhaps finite, define $T \prec_n S$ if $T \subseteq S$ and $T[n] = S[n]$ (see Definition 3.4).

Define $p \prec_{\Gamma,n} q$ if $p \leq q$ and $p \restriction \gamma \Vdash_{\mathbb{P}_\gamma}$ “$p(\gamma) \prec_n q(\gamma)$” for each $n \in \Gamma$.

Lemma 4.1. *If*

- $p \in \mathbb{P}_{\omega_2}$
- Γ *is a finite subset of* ω_2
- $k \in \omega$ *and* $n \in \omega$
- $p \Vdash_{\mathbb{P}_{\omega_2}}$ “$\dot{x} \in V$”

then there is some K, a finite X and $q \prec_{\Gamma,k} p$ such that:

(1) $q \Vdash_{\mathbb{P}_{\omega_2}}$ “$\dot{x} \in X$”
(2) $q \restriction \gamma \Vdash_{\mathbb{P}_\gamma}$ “ *if* $k \leq |t| \leq K$ *then* $\|t\|_{q(\gamma)} \geq \|t\|_{p(\gamma)} - 1$” *for all* $\gamma \in \Gamma$ *(see Definition 3.3)*
(3) $q \restriction \gamma \Vdash_{\mathbb{P}_\gamma}$ “ *if* $|t| \geq K$ *then* $\|t\|_{q(\gamma)} \geq nM^3_{|t|}$” *for all* $\gamma \in \Gamma$.

Proof. A standard rank argument can be used. For $T \in \mathbb{U}$ and $t \in T$ define $\mathbf{rank}(t) = 0$ if $\|t\|_T \geq nM^3_{|t|}$ and define $\mathbf{rank}(t) \leq r + 1$ if

$$\|\{a \in \mathbf{succ}_T(t) \mid \mathbf{rank}(t^\frown a) \leq n\}\| \geq \|t^\frown a\|_T - 1.$$

This shows that if $\|t\|_T \geq 1$ for $t \supseteq t^*$ then $\mathbf{rank}(t^*)$ is defined. A standard induction yields the result for the iteration. □

Lemma 4.2. *Suppose that*

(a) $K_0 \leq K_1 \in \omega$
(b) $\{Y_j\}_{K_0 \leq j < K_1}$ *are such that* $\prod_{\ell=K_0}^{j-1} |Y_\ell| < b_j$ *for each* j
(c) $T \in \mathbb{P}$
(d) $y_t \in \prod_{j=K_0}^{K_1-1} Y_j$ *for each* $t \in T[K_1]$.

Then there are $T^* \prec_{K_0} T[K_1]$ *and* $y_t \in \prod_{j=K_0}^{\ell-1} Y_j$ *for* $t \in T^*[\ell]$ *where* $K_0 \leq \ell < K_1$ *such that*

(e) *if* $t \in T^*$ *and* $K_0 \leq |t| < K_1$ *then* $\|t\|_{T^*} \geq \|t\|_T - 1$
(f) *if* $t \subseteq s \in T^*[K_1]$ *then* $y_t \subseteq y_s$.

Proof. Proceed by induction on $d = K_1 - K_0$. If $d = 0$ — in other words, $K_1 = K_0$ — then let $T^* = T$ and note that the $y_t \in Y_{K_0-1}$ are already defined. Given the result for d suppose that $K_1 - K_0 = d+1$ and that $T \in \mathbb{P}$, $\{Y_j\}_{K_0 \leq j < K_1}$ are such that $\prod_{\ell=K_0}^{j-1} |Y_\ell| < b_j$ for each j along with $y_t \in \prod_{j=K_0}^{K_1-1} Y_j$ for each $t \in T[K_1]$ are given. Then $|\prod_{\ell=K_0}^{K_1-2} Y_\ell| < b_{K_1-1}$ and it is possible, using Fact 3.1, to find $X_t \subseteq \mathbf{succ}_T(t)$ for each $t \in T[K_1 - 1]$ such that $\|X_t\| \geq \|t\|_T - 1$ and such that there is $y_t \in \prod_{\ell=K_0}^{K_1-2} |Y_\ell|$ such that $y_t = y_{t^\frown w} \restriction [K_0, K_1 - 1]$ for each $w \in X_t$. Now apply the induction hypothesis to $\{y_t\}_{t \in T[K_1-1]}$. □

Notation 4.1. *For a tree* T *and* $t \in T$ *let* $T\langle t\rangle = \{s \in T \mid s \subseteq t \text{ or } t \subseteq s\}$.

Definition 4.2. *Let* Γ *be a finite subset of* ω_2 *enumerated in the ordinal ordering as* $\Gamma = \{\gamma_j\}_{j \in J}$ *and* $p \in \mathbb{P}_{\omega_2}$. *For* $j \leq J$ *and* $\sigma \in \mathbb{U}^j$ *define*[1] p_σ *by induction on* j *as follows:*

(a) $p_\varnothing = p \restriction \gamma_0$
(b) *if* $i = |\sigma|$ *and* p_σ *is defined and* $t \in \mathbb{U}$ *then*
  (i) $p_{\sigma^\frown t} \restriction \gamma_i = p_\sigma$
  (ii) $p_{\sigma^\frown t}(\gamma_i) = p(\gamma_i)\langle t\rangle$
  (iii) $p_{\sigma^\frown t}(\gamma) = p(\gamma)$ *if* $\gamma_i < \gamma < \gamma_{i+1}$ *where* γ_J *is defined to be* ω_2.

The $p(\gamma)$ *are, of course,* $\mathbb{P}_\gamma$ *names, but the reader will not be reminded of this by dots in forcing statements. Note that it may well be the case that* $p_\sigma \notin \mathbb{P}_{\omega_2}$. *Indeed,* $p_\sigma \in \mathbb{P}_{\omega_2}$ *precisely if*

$$(\forall j \in \mathbf{domain}(\sigma))\ p_{\sigma\restriction j} \in \mathbb{P}_{\omega_2} \text{ and } p_{\sigma\restriction j} \Vdash_{\mathbb{P}_{\gamma_j}} \text{“}\sigma(j) \in p(\gamma_j)\text{”} \tag{9}$$

for every j *in the domain of* σ.

A condition p *will be called* (Γ, K, N)*-determined if there is* $\mathcal{U} \subseteq \mathbb{U}[K]^J$ *such that for each* $k \leq K$, $j \leq J$ *and each* $\sigma \in \mathcal{U}[k,j]$

(c) $p_\sigma \in \mathbb{P}_{\omega_2}$
(d) $p_\sigma \restriction \gamma_j \Vdash_{\mathbb{P}_{\gamma_j}}$ “$\dot{p}(\gamma_j)[k] = T(k, j, \sigma, \mathcal{U})$” *for all* $\sigma \in \mathcal{U}[k,j]$
(e) *if* $\sigma \in \mathcal{U}[K,j]$ *then* $p_\sigma \Vdash_{\mathbb{P}_{\gamma_j}}$ “$(\forall t \in \dot{p}(\gamma_j))$ *if* $|t| \geq K$ *then* $\|t\|_{\dot{p}(\gamma_j)} \geq NM^3_{|t|}$”

Say that $\mathcal{U}$ *determines* p *for* (Γ, K, N).

Note that the requirements (d) and (c) are quite strong since they require that if $\sigma \in \mathcal{U}[k,j]$ the condition p_σ decides information about $p(\gamma_j)[k]$ whereas this usually requires extending p_σ to some $p_{\bar\sigma}$ where $\bar\sigma \in \mathcal{U}[\bar k, j]$ for some $\bar k > k$. The following fact will not play an explicit role in the coming arguments but may help understanding the construction.

Fact 4.1. *If* $\mathcal{U}$ *determines* p *for* (Γ, K, N) *as in Definition 4.2 and* $k < K$ *and* $j < J$ *and* $\Theta \in \mathcal{U}$ *then*

$$B(k, j, \Theta, \mathcal{U}) = \mathbf{succ}_{T(k,j,\Theta,\mathcal{U})}(\Theta(0,j)^\frown\Theta(1,j)^\frown\Theta(2,j)^\frown \dots {}^\frown\Theta(k-1,j))$$

where $B(k, j, \Theta, \mathcal{U})$ *is defined in Definition 3.4.*

[1] To be completely precise, p_σ should be denoted by $p_{\sigma,\Gamma}$ but the dependence on Γ will always be clear.

Lemma 4.3. *Suppose that $p \in \mathbb{P}_{\omega_2}$, $\Gamma \in [\omega_2]^J$, $\dot{x}$ is a $\mathbb{P}_{\omega_2}$ name for a finite set of integers and $K_0 > J = |\Gamma|$ and N are in ω. There are then q, $K_1 \in \omega$ and $\mathcal{U} \subseteq \mathbb{U}[K_1]^J$ such that:*

(a) $\mathcal{U}$ determines q for (Γ, K_1, N)
(b) $q \prec_{\Gamma,K_0} p$,
(c) $q \restriction \gamma \Vdash_{\mathbb{P}_\gamma}$ "$(\forall t \in q(\gamma))$ if $|t| \geq K_0$ then $\|t\|_{q(\gamma)} \geq \|t\|_{p(\gamma)} - 1$" for all $\gamma \in \Gamma$
(d) there is $Z : \mathcal{U} \to [\omega]^{<\aleph_0}$ such that $q_\sigma \Vdash_{\mathbb{P}_{\omega_2}}$ "$\dot{x} = Z(\sigma)$" for all $\sigma \in \mathcal{U}[K_1, J]$.

Proof. Proceed by induction on $J = |\Gamma|$. In order to prove the general case a stronger induction hypothesis is required: There is $\bar{K} \in \omega$ such that for all $K_1 \geq \bar{K}$ there is $q \leq p$ and $\mathcal{U} \subseteq \mathbb{U}[K_1]^J$ such that conditions (a), (b), (c) and (d) are all satisfied for $\dot{x}$, q, K_0, $K_1, \mathcal{U}$ and Γ. If $\Gamma = \varnothing$ there is nothing to do and if $|\Gamma| = 1$, Lemma 4.1 can be applied.

Now suppose that the general result has been established if $|\Gamma| = J$ and that Γ is given such that $|\Gamma| = J + 1$. Let γ be the minimum element of Γ and let $\tilde{\Gamma} = \Gamma \setminus \{\gamma\}$. Then find $\tilde{q} \leq p \restriction \gamma$ such that:

(i) there is $\tilde{K}$ such that $\tilde{q} \Vdash_{\mathbb{P}_\gamma}$ " the induction hypothesis holds for $\dot{x}/\mathbb{P}_\gamma$ and $\tilde{\Gamma}$ and $\tilde{K}$"
(ii) there is K' such that $\tilde{q} \Vdash_{\mathbb{P}_\gamma}$ "$\|t\|_{p(\gamma)} > (N + 1)M^3_{|t|}$ if $|t| \geq K'$".

Let $\bar{K} = \max(\tilde{K}, K')$ and suppose that $K_1 \geq \bar{K}$.

Using the stronger induction hypothesis and Lemma 4.1 it is possible to find and $q^* \leq \tilde{q}$ and $T \subseteq \mathbb{U} \restriction K_1$ such that

(10) $$q^* \Vdash_{\mathbb{P}_\gamma} \text{"}T \subseteq p(\gamma) \text{ and } (\forall s \in T) \text{ if } K_0 \leq |s| < K_1 \text{ then } \|s\|_T \geq \|s\|_{p(\gamma)} - 1\text{"}$$

and, moreover, for each $t \in T[K_1]$ there are $\dot{T}_t$, $\dot{q}_t$, Z_t and $\mathcal{U}_t$ such that for each $t \in T[K_1]$

(11) $$q^* \Vdash_{\mathbb{P}_\gamma} \text{"}p(\gamma)\langle t\rangle \supseteq \dot{T}_t\text{"}$$

(12) $$q^* \Vdash_{\mathbb{P}_\gamma} \text{"}(\forall s \in \dot{T}_t) \text{ if } |s| \geq k \text{ then } \|s\|_{\dot{T}_t} \geq NM^3_{|s|}\text{"}$$

(13) $$q^* * \dot{T}_t * \dot{q}_t \Vdash_{\mathbb{P}_{\omega_2}} \text{"}K_1, Z_t, \mathcal{U}_t \text{ witness that conditions (a), (b), (c) and (d) of Lemma 4.3 hold."}$$

Let q be defined by letting $q(\gamma) = \bigcup_{t \in T[K_1]} \dot{T}_t$ and having $q^* * \dot{T}_t \Vdash_{\mathbb{P}_{\gamma+1}}$ "$q \restriction [\gamma + 1, \omega_2) = \dot{q}_t$". Observe that (12), (10) and (ii) together imply that $q \Vdash_{\mathbb{P}_\gamma}$ "$(\forall s \in q(\gamma))$ if $K_1 \leq |s|$ then $\|s\|_T \geq NM^3_{|s|}$". Hence (e) of Definition 4.2 holds.

Let $Y_\ell = \mathcal{P}(\mathbb{U}[\ell]^J)$ and $y_t(\ell) = \mathcal{U}_t[\ell, J]$ if $K_0 \leq \ell < K_1$ and $t \in T[K_1]$. It follows that $y_t(\ell) \in Y_\ell$ and, from the fact that $J < K_0$ and Inequality (2) of Definition 3.2, it follows that $\prod_{\ell=K_0}^{j-1} |Y_\ell| < b_j$. Then using Lemma 4.2 it is possible to find to find $T^* \prec_{K_0} T$ such that the following hold:

(iii) if $t \in T^*$ and $K_0 \leq |t| \leq K_1$ then $\|t\|_{T^*} \geq \|t\|_T - 1$.
(iv) $q^* * T\langle t\rangle \Vdash_{\mathbb{P}_{\gamma+1}}$ "$\mathcal{U}_t[\ell, J] = \mathcal{U}_s[\ell, J]$" if $t \subseteq s \in T^*[K_1]$ and $\ell \leq |t|$.

Then let $\mathcal{U} = \{t^\frown\sigma \mid t \in T^*[K_1] \text{ and } \sigma \in \mathcal{U}_t[K_1, J]\}$. Then let $Z(t^\frown\sigma) = Z_t(\sigma)$ for $t \in T^*[K_1]$ and $\sigma \in \mathcal{U}_t$.

It needs to be verified that (c) and (d) of Definition 4.2 hold. To see that (c) holds note that

(14) $$(\forall t \in T^*[K])(\forall \sigma \in \mathcal{U}_t[\ell, j])\ q^* * T^*\langle t\rangle \Vdash_{\mathbb{P}_{\gamma+1}} \text{"}q_\sigma = (\dot{q}_t)_\sigma \in \mathbb{P}_{\omega_2}/\mathbb{P}_{\gamma+1}\text{"}$$

and hence it suffices to show that if $K_0 \leq \ell < K_1$ and $t \in T^*[\ell]$ and $\sigma \in \mathcal{U}_t[\ell, j]$ then

$$q^* * T^*\langle t\rangle \Vdash_{\mathbb{P}_{\gamma+1}} \text{"}q_\sigma = (\dot{q}_t)_\sigma \in \mathbb{P}_{\omega_2}/\mathbb{P}_{\gamma+1}\text{"}$$

To this end, let $\tilde{p} \leq q^* * T^*\langle t\rangle$ and note that it can be assumed that there is $s \in T^*[K_1]$ such that $\tilde{p} \leq q^* * T^*\langle s\rangle$ and such that $t \subseteq s$. From (iv) it follows that

$$q^* * T^*\langle t\rangle \Vdash_{\mathbb{P}_{\gamma+1}} \text{"}\sigma \in \mathcal{U}_t[\ell, J] = \mathcal{U}_s[\ell, J]\text{"}$$

and by combining this with (14) it follows that

$$q^* * T^*\langle s\rangle \Vdash_{\mathbb{P}_{\gamma+1}} \text{“}q_\sigma \in \mathbb{P}_{\omega_2}/\mathbb{P}_{\gamma+1}\text{”}$$

and, since $t \subseteq s$ and $\tilde{q} \leq q^* * T^*\langle s\rangle$ it follows that $\tilde{q} \Vdash_{\mathbb{P}_{\gamma+1}}$ “$q_\sigma \in \mathbb{P}_{\omega_2}/\mathbb{P}_{\gamma+1}$” as required.

It now follows from the definition of q and $\mathcal{U}$ that (d) of Definition 4.2 holds and hence $\mathcal{U}$ determines q for (Γ, K, N). Conditions (b) and (d) of the lemma are immediate. To see that (c) of the lemma holds use (10) and (iii). □

Lemma 4.4. *If*

(1) $K_0 \leq K_1$
(2) $\mathcal{U}^* \subseteq \mathcal{U} \subseteq \mathbb{U}[K_1]^J$
(3) $\mathcal{U}$ *determines* p *for* (Γ, K_1, N)
(4) $\|\mathcal{U}^*\|_k \geq nM_k^3$ *for* $k \geq K_0$
(5) $\mathcal{U}^*[K_0, J] = \mathcal{U}[K_0, J]$

then there is q *such that*

(6) $\mathcal{U}^*$ *determines* q *for* (Γ, K_1, N)
(7) $q \leq_{\Gamma,n} p$
(8) $q \prec_{\Gamma,K_0} p$

Proof. Proceed by induction on J. If $J = 1$ and $\Gamma = \{\gamma\}$ then there is $T \subseteq \mathbb{U}[K_1]$ such that $p \restriction \gamma \Vdash_{\mathbb{P}_\gamma}$ “$T = p(\gamma)[K_1]$” and $\mathcal{U} = T$; or, to be more precise, this is true if $\mathbb{U}[K]$ is identified with $\mathbb{U}[K]^1$. The hypothesis on $T^* = \mathcal{U}^*$ implies that $T^*[K_0] = T[K_0]$ and $\|t\|_{T^*} \geq nM_{|t|}^3$ for each $t \in T^*$ such that $|t| \geq K_0$. The conclusion follows by letting q be $\bigcup_{t\in T^*} T\langle t\rangle$.

Assuming the result holds for J suppose that $\mathcal{U}^* \subseteq \mathcal{U} \subseteq \mathbb{U}[K_1]^{J+1}$ and $\Gamma \in [\omega_2]^{J+1}$ and and that $\mathcal{U}$ determines p for (Γ, K_1, N) and the hypotheses are satisfied. Let $T = \mathcal{U}[K_1, 1]$ and let γ_0 be the first element of Γ. For each $t \in T$ let

$$\mathcal{U}_t = \left\{\Theta \in \mathbb{U}[K_1]^J \mid (t, \Theta) \in \mathcal{U}\right\}$$

and let

$$\mathcal{U}_t^* = \left\{\Theta \in \mathbb{U}[K_1]^J \mid (t, \Theta) \in \mathcal{U}^*\right\}$$

and note that

$$p \restriction \gamma_0 * t \Vdash_{\mathbb{P}_{\gamma_0+1}} \text{“}\mathcal{U}_t \text{ determines } p/(p \restriction \gamma_0 * t) \text{ for } \Gamma \setminus \gamma_0, K_1 \text{ and } N\text{”}$$

because, given $\sigma \in \mathcal{U}_t[k, j]$ it follows that $(t \restriction k)^\frown\sigma \in \mathcal{U}[k, j+1]$ and hence

$$\left(p_{(t\restriction k)^\frown\sigma}\right) \restriction \gamma_j \Vdash_{\mathbb{P}_{\gamma_{j+1}}} \text{“}p(\gamma_{j+1}) = T(k, j+1, (t \restriction k)^\frown\sigma, \mathcal{U}) = T(k, j, \sigma, \mathcal{U}_t)\text{”}.$$

In other words, the condition $p \restriction \gamma_0^\frown(p(\gamma_0)\langle t \restriction k\rangle) \in \mathbb{P}_{\gamma_0+1}$ forces the following

$$((p/(p \restriction \gamma_0 * t))_\sigma) \restriction [\gamma_0 + 1, \gamma_j) \Vdash_{\mathbb{P}_{\gamma_{j+1}}/\mathbb{P}_{\gamma_0+1}} \text{“}p/(p \restriction \gamma_0 * t))_\sigma(\gamma_{j+1}) = T(k, j, \sigma, \mathcal{U}_t)\text{”}$$

as required to show that $p/(p \restriction \gamma_0 * t)$ is forced to be determined by $\mathcal{U}_t$. It also easy to see that that

$$(\forall k \geq K_0)\ (p \restriction \gamma_0) * (p(\gamma_0)\langle t \restriction k\rangle) \Vdash_{\mathbb{P}_{\gamma_0+1}} \text{“}\|\mathcal{U}_t^*\|_k \geq n\text{”}$$

and hence the induction hypothesis yields a $\mathbb{P}_{\gamma_0+1}$ name q_t such that

$$(p \restriction \gamma_0) * (p(\gamma_0)\langle t \restriction k\rangle) \Vdash_{\mathbb{P}_{\gamma_0+1}} \text{“}\mathcal{U}_t^* \text{ determines } q_t\text{”}$$

$$(p \restriction \gamma_0) * (p(\gamma_0)\langle t \restriction k\rangle) \Vdash_{\mathbb{P}_{\gamma_0+1}} \text{“}(\forall\gamma \in \Gamma \setminus \{\gamma_0\})\ q_t(\gamma) \prec_{K_0} p(\gamma)\ \&\ q_t(\gamma) \leq_n p(\gamma)\text{”}.$$

Let $T^* = \{\Theta(1) \mid \Theta \in \mathcal{U}^*\}$ and note that $T^*[K_0] = T[K_0]$. Let q be defined by setting $q \restriction \gamma_0 = p \restriction \gamma_0$, letting $q(\gamma_0) = T^*$ and letting $q \restriction [\gamma_0 + 1, \omega_2)$ be the name determined by letting

$$(q \restriction \gamma_0) * t \Vdash_{\mathbb{P}_{\gamma_0+1}} \text{“}q \restriction [\gamma_0 + 1, \omega_2) = q_t\text{”}.$$

□

5. Proof of the Main Theorem

Theorem 5.1. *It is consistent that $\mathfrak{s}_{2,2} = \aleph_2$ and $\mathfrak{s}_{2,\infty} = \aleph_1$.*

Proof. It should be clear that $\mathbb{P}$ is proper and ω^ω-bounding. Moreover, the choice of the $E_{k,j}$ guarantees that if $P \subseteq [I_k]^2$ and $\|P\| \geq j+1$ and $X \subseteq I_k$ then either $\|P \cap [X]^2\| \geq j$ or $\|P \setminus [X]^2\| \geq j$ and this implies that

(15) $$1 \Vdash_{\mathbb{P}_{\omega_2}} \text{“}\mathfrak{s}_{2,2} = \aleph_2\text{”}.$$

Using Observation (15) it suffices to show that $1 \Vdash_{\mathbb{P}_{\omega_2}}$ "$\mathfrak{s}_{2,\infty} = \aleph_1$" so suppose that

$$p \Vdash_{\mathbb{P}_{\omega_2}} \text{“}\dot{\mathcal{Z}} \in [\omega]^{<\aleph_0} \ \& \ \limsup_{z\in\dot{\mathcal{Z}}} |z| = \infty\text{”}.$$

It suffices to show that there are $p_n \in \mathbb{P}_{\omega_2}$, $\Gamma_n \in [\omega_2]^n$, positive integers $K_{0,n} \leq K_{1,n}$, finite sets A_n and B_n and name $\dot{z}_n$ such that:

a) $p_0 = p$
b) $A_n \cap B_n = \varnothing$
c) $\min(A_{n+1} \cup B_{n+1}) > \max(A_n \cup B_n)$
d) $p_n \Vdash_{\mathbb{P}_{\omega_2}}$ "$\dot{z}_n \in \dot{\mathcal{Z}}$ & $\min(\dot{z}_n) > n$"
e) $p_n \Vdash_{\mathbb{P}_{\omega_2}}$ "$\dot{z}_n \cap A_n \neq \varnothing \neq \dot{z}_n \cap B_n$"
f) $K_{0,n+1} \geq K_{1,n} \geq K_{0,n}$
g) $p_{n+1} \leq_{\Gamma_n,n} p_n$
h) $p_n \restriction \gamma \Vdash_{\mathbb{P}_\gamma}$ " if $|t| \geq K_{1,n}$ then $\|t\|_{p_n(\gamma)} \geq (n+2)M^3_{|t|}$" for $\gamma \in \Gamma_n$
i) $\Gamma_{n+1} \supseteq \Gamma_n$
j) $\bigcup_n \Gamma_n = \bigcup_n \mathbf{domain}(p_n)$.

Given this, a standard fusion argument establishes that there is $q \leq p_n$ for all n. Using c) and b) it is possible to define $A = \bigcup_{n\in\omega} A_n$ and $B = \bigcup_{n\in\omega} B_n$ such that $A \cap B = \varnothing$ and, using e), such that

$$q \Vdash_{\mathbb{P}_{\omega_2}} \text{“}(\forall n)(\exists z \in \mathcal{Z}) \ \min(z) \geq n \ \& \ z \cap A \neq \varnothing \neq z \cap B\text{”}$$

thus establishing that $\mathfrak{s}_{2,\infty} = \aleph_1$ after forcing with $\mathbb{P}_{\omega_2}$ over a model of $2^{\aleph_0} = \aleph_1$.

To carry out the inductive construction, suppose that $p_n \in \mathbb{P}_{\omega_2}$, $\Gamma_n \in [\omega_2]^n$, $K_{1,n}$, A_n, B_n and z_n have been constructed. Let $\Gamma = \Gamma_{n+1} \supseteq \Gamma_n$ be chosen according to some scheme that will guarantee that j) will be satisfied. Use Lemma 4.1 to find $K_{0,n+1}$ and $q \prec_{\Gamma_n,K_{1,n}} p_n$ such that

(k) $q \restriction \gamma \Vdash_{\mathbb{P}_\gamma}$ " if $|t| \geq K_{0,n+1}$ then $\|t\|_{q(\gamma)} \geq (n+5)M^3_{|t|}$" for all $\gamma \in \Gamma_n$
(l) $q \restriction \gamma \Vdash_{\mathbb{P}_\gamma}$ " if $K_{1,n} \leq |t| \leq K_{0,n+1}$ then $\|t\|_{q(\gamma)} \geq \|t\|_{p_n(\gamma)} - 1$" for all $\gamma \in \Gamma_n$

and observe that this implies that $q \leq_{\Gamma_n,n+1} p_n$. Let $\dot{z}$ be a name such that

$$1 \Vdash_{\mathbb{P}_{\omega_2}} \text{“}z \in \dot{\mathcal{Z}} \ \& \ \min(\dot{z}) > n \ \& \ |\dot{z}| \geq M_{K_{0,n+1}-1}\text{”}.$$

Using Lemma 4.3 find $\bar{q} \prec_{\Gamma_n,K_{0,n+1}} q$, $K_{1,n+1} \geq K_{0,n+1}$ and $\mathcal{U} \subseteq \mathbb{U}[K_{1,n+1}]^n$ such that:

(m) $\mathcal{U}$ determines $\bar{q}$ for $(\Gamma_n, K_{1,n+1}, n+3)$
(n) $\bar{q} \restriction \gamma \Vdash_{\mathbb{P}_\gamma}$ "$(\forall t \in \bar{q}(\gamma))$ if $|t| \geq K_{0,n+1}$ then $\|t\|_{q(\gamma)} \geq \|t\|_{p(\gamma)} - 1$" for all $\gamma \in \Gamma_n$
(o) there is $Z : \mathcal{U} \to [\omega]^{<\aleph_0}$ such that $\bar{q}_\sigma \Vdash_{\mathbb{P}_{\omega_2}}$ "$\dot{z} = Z(\sigma)$" for all $\sigma \in \mathcal{U}[K_{1,n+1}, n]$.

Observe that it follows from (k) and (n) that $\|\mathcal{U}\|_k \geq M^3_{K_{0,n+1}}(n+3)$ if $k \geq K_{0,n+1}$. Now use Theorem 3.1 to find $\mathcal{U}^* \subseteq \mathcal{U}$ and disjoint A_{n+1} and B_{n+1} such that

(16) $$(\forall k \geq K_{0,n+1}) \ \|\mathcal{U}^*\|_k \geq \|\mathcal{U}\|_k - (M^2_{K_{0,n+1}}(2n+4) + M_{K_{0,n+1}} + 2) \geq M^3_{K_{0,n+1}}(n+3) - (M^2_{K_{0,n+1}}(2n+4) + M_{K_{0,n+1}} + 1) \geq M^3_{K_{0,n+1}}(n+2)$$

and such that $\mathcal{U}[K_{0,n+1}] = \mathcal{U}^*[K_{0,n+1}]$ and $A_{n+1} \cap Z(\Theta) \neq \varnothing \neq B_{n+1} \cap Z(\Theta)$ for all $\Theta \in \mathcal{U}^*$.

Then apply Lemma 4.4 to find $p_{n+1} \leq_{\Gamma_n, n+2} \bar{q}$ and $p_{n+1} \prec_{\Gamma_n, K_{0,n+1}} \bar{q}$ such that $\mathcal{U}^*$ determines p_{n+1} for $(\Gamma_n, K_{1,n+1}, n+3)$ and note that this implies that $p_{n+1} \Vdash_{\mathbb{P}_{\omega_2}}$ "$A_{n+1} \cap Z(\Theta) \neq \varnothing \neq_{n+1} B \cap Z(\Theta)$" for all $\Theta \in \mathcal{U}^*$ and, hence, by (o) that $p_{n+1} \Vdash_{\mathbb{P}_{\omega_2}}$ "$A_{n+1} \cap \dot{z} \neq \varnothing \neq_{n+1} B \cap \dot{z}$".

□

6. Some more cardinal invariants

Those readers who have followed the proof of Theorem 5.1 may well be asking themselves whether better results are possible. In order to formulate a precise questions along these lines it is worth introducing some new cardinal invariants that incorporate ideas already found in the definitions of $\mathfrak{s}_{1/2\pm\epsilon}$ and $\mathfrak{s}_{1/2\pm\epsilon}$.

Definition 6.1. *For $\epsilon > 0$ define $\mathfrak{s}_{k,\epsilon}$ to be the least cardinal of a family $\mathcal{F} \subseteq k^\omega$ such that for each infinite, pairwise disjoint family $\mathcal{A} \subseteq [\omega]^{<\aleph_0}$ whose elements have unbounded cardinality there is $F \in \mathcal{F}$ such that for infinitely many $a \in \mathcal{A}$*

$$\frac{1-\epsilon}{k} < \frac{|a \cap F^{-1}(j)|}{|a|} < \frac{1+\epsilon}{k}$$

for all $j \in k$. Define $\mathfrak{s}_{k,0}$ to be the least cardinal of a family $\mathcal{F} \subseteq k^\omega$ such that for each infinite, pairwise disjoint family $\mathcal{A} \subseteq [\omega]^{<\aleph_0}$ whose elements have unbounded cardinality there is $F \in \mathcal{F}$ such that

$$\liminf_{a \in \mathcal{A}} \left(\max_{j \in k} \left(\frac{|a \cap F^{-1}(j)|}{|a|} - 1/k \right) \right) = 0.$$

Other variations of the splitting cardinals also come to mind.

Definition 6.2. *Let $2 \leq m \leq k$ and let $\mathfrak{s}^*_{m,k}$ be the least cardinal of a family $\mathcal{F} \subseteq m^\omega$ such that for any infinite, pairwise disjoint family $\mathcal{A} \subseteq [\omega]^k$ there is $F \in \mathcal{F}$ such that for any non-empty $x \subseteq m$ there are infinitely many $a \in \mathcal{A}$ such that $F[a] = x$.*

Finally, recall that $\mathfrak{s}_{\omega_1}$ is the following stronger version of the statement $\mathfrak{s} = \aleph_1$: There is a family $\{S_\xi\}_{\xi \in \omega_1}$ such that for each infinite $X \subseteq \omega$ there is $\beta \in \omega_1$ such that $|S_\alpha \cap X| = \aleph_0 = |X \setminus S_\alpha|$ for all $\alpha > \beta$. The following definition extends this to the current context.

Definition 6.3. *Define $\mathfrak{s}^{\omega_1}_{k,m}$ to be the assertion that there is family $\{f_\eta\}_{\eta \in \omega_1}$ such that $f_\eta : \omega \to k$ and for each infinite, pairwise disjoint family $\mathcal{A} \subseteq [\omega]^m$ there is $\beta \in \omega_1$ such that $f_\eta[a] = k$ for infinitely many $a \in \mathcal{A}$ and each $\eta > \beta$.*

It can easily be checked that the proof of Corollary 2.1 shows that $\mathfrak{s}^{\omega_1}_{2,m}$ holds for some m if and only if $\mathfrak{s}^{\omega_1}_{2,2}$ holds. However, the proof of Lemma 2.4 does not seem to extend to show that $\mathfrak{s}^{\omega_1}_{k,m}$ holds for some k and m if and only if $\mathfrak{s}^{\omega_1}_{2,2}$ holds. These questions will be considered in a forthcoming paper.

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