BOREL SETS WITHOUT PERFECTLY MANY OVERLAPPING TRANSLATIONS, II

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Abstract. For a countable ordinal ε and an int6eger $\iota \geq 2$ we construct a Σ^0_2 subset of the Cantor space ${}^\omega 2$ for which one may force $\aleph_\varepsilon$ translations with intersections of size $\geq 2\iota$, but such that it has no perfect set of such translations in any ccc extension. These sets have uncountably many translations with intersections of size $\geq 2\iota$ in ZFC, so this answers [11, Problem 3.4].

1. Introduction

The existence of Borel sets with large squares but no perfect squares was studied and resolved in Shelah [14]. We say that a set $B \subseteq {}^\omega 2 \times {}^\omega 2$ contains a μ–square (perfect square, respectively), if there is a set X of cardinality μ (a perfect set X, respectively) such that $X \times X \subseteq B$. It was shown in [14, Section 1] that

> it is consistent that for every ordinal $\alpha < \omega_1$, there is a Borel subset of ${}^\omega 2 \times {}^\omega 2$ containing an $\aleph_\alpha$–square but no perfect square.

As a matter of fact the problem was given a more complete answer. A rank on models in a countable vocabulary (called here *a splitting rank*, see Definition 2.1) occured to be closely related to the question when we can force Σ^0_2 sets with μ–squares but without perfect squares. The first λ, called λ_{ω_1}, such that there is no model with universe λ, countable vocabulary and countable rank is a cutting point here. Every Σ^1_1 set containing a λ_{ω_1}–square must contain a perfect square. On the other hand for each cardinal $\mu < \lambda_{\omega_1}$ some ccc forcing notion adds a Σ^0_2 set containing a μ–square but no perfect square. The cardinal λ_{ω_1} is quite mysterious: it satisfies $\aleph_{\omega_1} \leq \lambda_{\omega_1} \leq \beth_{\omega_1}$ and (its close relative) cannot be increased by ccc forcing, but not much more is known.

Thinking about subsets of the plane as relations, one may wonder for what kinds of relations we have similar results. Several questions may be reduced to the existence of large squares for special kinds of Borel subsets of ${}^\omega 2 \times {}^\omega 2$. For instance, if $A \subseteq {}^\omega 2$ then a μ–square included in the *spectrum of translation k–disjointness of A,*

$$\mathrm{std}_k(A) = \{(x,y) \in {}^\omega 2 \times {}^\omega 2 : |(A+x) \cap (A+y)| \leq k\},$$

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corresponds to a family of μ many translations of A with intersections of size $\leq k$. So for $k=0$ this would be μ many pairwise disjoint translations. Interest in Borel sets with $\mu \geq \aleph_1$ pairwise disjoint translations but without any perfect set of such translations is motivated by several works in literature. For instance, Balcerzak, Rosłanowski and Shelah [1] studied the σ–ideal of subsets of ${}^\omega 2$ generated by Borel sets with a perfect set of pairwise disjoint translations. A generalization of this direction follows Darji and Keleti [3], Elekes and Steprāns [5], and Zakrzewski [15]. They studied perfectly k–small sets which for finite k can be described as follows. A set $A \subseteq {}^\omega 2$ is perfectly k–small if there is a perfect set $P \subseteq {}^\omega 2$ such that for distinct $x_0, \ldots, x_{k-1} \in P$ the intersection $(A+x_0) \cap \ldots \cap (A+x_{k-1})$ is empty. Elekes and Keleti [4] studied decompositions of the real line into pairwise disjoint Borel pieces so that each piece is closed under addition and in this context they explicitly asks [4, Question 4.5]:

> Suppose that a Borel subset of $\mathbb{R}$ has uncountably many pairwise disjoint translates. Does it also have continuum many pairwise disjoint translates?

If we want to answer the above question by a direct application of [14, Section 1], we could look for a Σ^0_2 set $A \subseteq {}^\omega 2$ such that $\mathrm{std}_0(A)$ contains a large square but no very large square. However, in this situation, $\mathrm{std}_0(A)$ is a Π^0_2 subset of ${}^\omega 2 \times {}^\omega 2$ and, as it was noted in [14, Remark 1.14],

> if $B \subseteq {}^\omega 2 \times {}^\omega 2$ is a Π^0_2 set and it contains an uncountable square, then it contains a perfect square.

Therefore, forcing "a bad Borel set" for std_k must involve adding a Π^0_2 (or more complex) subset of ${}^\omega 2$, a task that at the moment appears substantially more complicated than adding "a bad Σ^0_2 set".

In developing tools to deal with std_k and perfect sets of disjoint translations, we looked into the dual direction. Now, for a set $A \subseteq {}^\omega 2$ and an integer k we consider the *spectrum of translation k–non-disjointness* of A,

$$\mathrm{stnd}_k(A) = \{(x,y) \in {}^\omega 2 \times {}^\omega 2 : |(A+x) \cap (A+y)| \geq k\}.$$

Then a μ–square included in $\mathrm{stnd}_k(A)$ determines a family of μ many pairwise k–overlapping translations. The existence of Borel sets with many, but not too many, pairwise k–overlapping translations was studied in Rosłanowski and Rykov [11] and Rosłanowski and Shelah [13]. In the latter work we carried out arguments fully parallel to that of [14, Section 1] and we showed that, e.g., for $\lambda < \aleph_{\omega_1}$ and an even integer $k \geq 6$ there is a ccc forcing notion $\mathbb{P}$ adding a Σ^0_2 set $B \subseteq {}^\omega 2$ with the property that

- for some $H \subseteq {}^\omega 2$ of size λ, $|(B+h) \cap (B+h')| \geq k$ for all $h, h' \in H$, but
- for every perfect set $P \subseteq {}^\omega 2$ there are $x, x' \in P$ with $|(B+x) \cap (B+x')| < k$.

Our goal in the current article is to analyze the construction of [13] and split it into two steps: first constructing a Σ^0_2 set (in ZFC) and then forcing non-disjoint translations to this set. (A similar analysis for homogeneous sets for analytic colorings was done by Kubiś and Shelah [8].) In addition to better understanding of the connection between the splitting rank and forcing non-disjoint translations, we get an improvement over the older results, extending them to even integers $k \geq 4$. Moreover, our analysis allows us to answer [11, Problem 3.4]: there are Σ^0_2 subsets of ${}^\omega 2$ with uncountably many pairwise 4–non-disjoint translations but with no perfect set of such translations (cf Corollary 5.3). In relation to that problem, let us give an easy construction of a Σ^0_2 set $B^* \subseteq {}^\omega 2 \times {}^\omega 2$ containing an uncountable square but no perfect square. This set, however, does not work for [11, Problem 3.4] as it is not of the form $\mathrm{stnd}_k(A)$.

Fix a bijection $\pi : \omega \times \omega \longrightarrow \omega$ and define a set $B^* \subseteq {}^\omega 2 \times {}^\omega 2$ as follows:

$$\begin{aligned}(x,y) \in B^* \Leftrightarrow \quad & x = y \ \vee \ \big(\exists k \in \omega\big)\big(\forall n \in \omega\big)\big(x(n) = y(\pi(n,k))\big) \ \vee \\ & \big(\exists k \in \omega\big)\big(\forall n \in \omega\big)\big(y(n) = x(\pi(n,k))\big).\end{aligned}$$

Proposition 1.1. (1) *There is an uncountable set $X \subseteq {}^\omega 2$ such that $X \times X \subseteq B^*$.*
(2) *There is no perfect set $P \subseteq {}^\omega 2$ such that $P \times P \subseteq B^*$.*

Proof. (1) We choose inductively a sequence $\langle x_\alpha : \alpha < \omega_1\rangle$ of distinct elements of ${}^\omega 2$ satisfying

($\boxtimes$) $\alpha < \beta < \omega_1 \quad \Rightarrow \quad x_\alpha \neq x_\beta \ \wedge \ \big(\exists k \in \omega\big)\big(\forall n \in \omega\big)\big(x_\alpha(n) = x_\beta(\pi(n,k))\big)$.

So, when arriving to stage $\beta < \omega_1$, we first choose a sequence $\langle y_k : k < \omega\rangle \subseteq {}^\omega 2$ so that

- $\{x_\alpha : \alpha < \beta\} \subseteq \{y_k : k < \omega\}$, and
- $\big(\forall \alpha < \beta\big)\big(\exists n < \omega\big)\big(y_0(n) \neq x_\alpha(\pi(n,0))\big)$.

Next we define

$$x_\beta(i) = y_k(n) \text{ whenever } i = \pi(n,k),$$

Note that x_β satisfies the demand in ($\boxtimes$) (for $\alpha < \beta$).

After the inductive construction is completed, it should be clear that the set $X = \{x_\alpha : \alpha < \omega_1\}$ is uncountable and $X \times X \subseteq B^*$.

(2) Assume towards contradiction that $P \subseteq {}^\omega 2$ is a perfect set such that $P \times P \subseteq B^*$. For $k < \omega$ let

$$\begin{aligned} R_{2k} = \quad & \{(x,y) \in {}^\omega 2 \times {}^\omega 2 : \big(\forall n \in \omega\big)\big(y(n) = x(\pi(n,k))\big)\}, \\ R_{2k+1} = \quad & \{(x,y) \in {}^\omega 2 \times {}^\omega 2 : \big(\forall n \in \omega\big)\big(x(n) = y(\pi(n,k))\big)\}. \end{aligned}$$

These are closed sets and $P \times P \subseteq \bigcup\limits_{\ell<\omega} R_\ell \cup \{(x,x) : x \in P\}$, so by Mycielski theorem [10, Theorem 1, p. 141] (see also [11, Lemma 2.4]), there are a perfect set $P' \subseteq P$ and an increasing sequence of integers $0 = n_0 < n_1 < n_2 < n_3 < \ldots$ such that

($\heartsuit$) for each $k < \omega$, $x, x', y, y' \in P'$ and $\ell \leq 2k+1$, if $x'{\restriction}n_k = x{\restriction}n_k \neq y{\restriction}n_k = y'{\restriction}n_k$, then
$$(x,y) \in R_\ell \ \Leftrightarrow \ (x',y') \in R_\ell.$$

Take distinct $x, y \in P$ and let ℓ be such that $(x,y) \in R_\ell$; by symmetry we may assume that ℓ is even, say $\ell = 2i$. Choose $k > \ell$ such that $x{\restriction}n_k \neq y{\restriction}n_k$ and fix $y' \in P$ such that $y \neq y'$ and $y{\restriction}n_k = y'{\restriction}n_k$. It follows from ($\heartsuit$) that $(x,y') \in R_\ell = R_{2i}$ and hence $y'(n) = x(\pi(n,i)) = y(n)$ for all $n \in \omega$, a contradiction. $\square$

Every uncountable Borel subset B of ${}^\omega 2$ has a perfect set of pairwise non-disjoint translations (just consider a perfect set $P \subseteq B$ and note that for $x, y \in P$ we have $\mathbf{0}, x+y \in (B+x) \cap (B+y)$). The problem of many non-disjoint translations is more interesting if we demand that the intersections have more elements. Note that in ${}^\omega 2$, if $x+b_0 = y+b_1$ then also $x+b_1 = y+b_0$. Hence $x \neq y$ and $|(B+x) \cap (B+y)| < \omega$

imply that $|(B+x)\cap(B+y)|$ is even. Therefore we will look at intersections of size $\geq 2\iota$ and (unlike in [13]) we will manage to deal here with any finite $\iota \geq 2$.

We fully utilize the algebraic properties of $({}^{\omega}2,+)$, in particular the fact that all elements of ${}^{\omega}2$ are self-inverse. Independence results for the general case of Abelian Polish groups is investigated in third paper of the series [12], however we do not carry out any rank analysis there (leaving that aspect open).

Notation: Our notation is rather standard and compatible with that of classical textbooks (like Jech [6] or Bartoszyński and Judah [2]). However, in forcing we keep the older convention that *a stronger condition is the larger one.*

(1) For a set u we let
$$u^{\langle 2\rangle} = \{(x,y)\in u\times u : x\neq y\}.$$
(2) For two sequences η,ν we write $\nu \vartriangleleft \eta$ whenever ν is a proper initial segment of η, and $\nu \trianglelefteq \eta$ when either $\nu \vartriangleleft \eta$ or $\nu=\eta$.
(3) The set of all sequences of length n and with values in $\{0,1\}$ is denoted by ${}^{n}2$ and we let ${}^{\omega>}2 = \bigcup\limits_{n<\omega} {}^{n}2$.
(4) The Cantor space ${}^{\omega}2$ of all infinite sequences with values 0 and 1 is equipped with the natural product topology and the group operation of coordinate-wise addition $+$ modulo 2.
(5) A *tree* is a $\vartriangleleft$–downward closed set of sequences. For a tree $T\subseteq {}^{\omega>}2$ the set of all ω–branches through T is denoted $\lim(T)$.
(6) Ordinal numbers will be denoted by lower case initial letters of the Greek alphabet $\alpha,\beta,\gamma,\delta,\varepsilon$. Finite ordinals (non-negative integers) will be denoted by letters $a,b,c,d,i,j,k,\ell,m,n,J,K,L,M,N$ and ι. For integers $N^s<N^t$, notations of the form $[N^s,N^t)$ are used to denote *intervals of integers.*
(7) The Greek letter λ will stand for an uncountable cardinal.
(8) For a forcing notion $\mathbb{P}$, all $\mathbb{P}$–names for objects in the extension via $\mathbb{P}$ will be denoted with a tilde below (e.g., $\underset{\sim}{\tau}$, $\underset{\sim}{X}$), and $\underset{\sim}{G}_{\mathbb{P}}$ will stand for the canonical $\mathbb{P}$–name for the generic filter in $\mathbb{P}$.

2. Two Ranks from the Past

Let us recall two closely related ranks used in previous papers. They are central for the studies here too.

2.1. **Splitting rank** $\mathrm{rk}^{\mathrm{sp}}$**.** The results recalled in this subsection are quoted from [13, Section 2], however they were first given in [14, Section 1].

Let λ be a cardinal and $\mathbb{M}$ be a model with the universe λ and a countable vocabulary τ.

Definition 2.1. (1) By induction on ordinals δ, for finite non-empty sets $w\subseteq\lambda$ we define when $\mathrm{rk}(w,\mathbb{M})\geq\delta$. Let $w=\{\alpha_0,\ldots,\alpha_n\}\subseteq\lambda$, $|w|=n+1$.

(a) $\mathrm{rk}(w,\mathbb{M})\geq 0$ if and only if for every quantifier free formula $\varphi\in\mathcal{L}(\tau)$ and each $k\leq n$, if $\mathbb{M}\models\varphi[\alpha_0,\ldots,\alpha_k,\ldots,\alpha_n]$ then the set
$$\big\{\alpha\in\lambda : \mathbb{M}\models\varphi[\alpha_0,\ldots,\alpha_{k-1},\alpha,\alpha_{k+1},\ldots,\alpha_n]\big\}$$
is uncountable;

(b) if δ is limit, then $\mathrm{rk}(w,\mathbb{M})\geq\delta$ if and only if $\mathrm{rk}(w,\mathbb{M})\geq\gamma$ for all $\gamma<\delta$;

(c) $\mathrm{rk}(w,\mathbb{M}) \geq \delta+1$ if and only if for every quantifier free formula $\varphi \in \mathcal{L}(\tau)$ and each $k \leq n$, if $\mathbb{M} \models \varphi[\alpha_0,\ldots,\alpha_k,\ldots,\alpha_n]$ then there is $\alpha^* \in \lambda \setminus w$ such that

$$\mathrm{rk}(w\cup\{\alpha^*\},\mathbb{M}) \geq \delta \quad \text{and} \quad \mathbb{M} \models \varphi[\alpha_0,\ldots,\alpha_{k-1},\alpha^*,\alpha_{k+1},\ldots,\alpha_n].$$

By a straightforward induction on δ one easily shows that if $\emptyset \neq v \subseteq w$ then

$$\mathrm{rk}(w,\mathbb{M}) \geq \delta \geq \gamma \implies \mathrm{rk}(v,\mathbb{M}) \geq \gamma.$$

Hence we may define the rank functions on finite non-empty subsets of λ.

Definition 2.2. The rank $\mathrm{rk}(w,\mathbb{M})$ of a finite non-empty set $w \subseteq \lambda$ is defined as:

- $\mathrm{rk}(w,\mathbb{M}) = -1$ if $\neg(\mathrm{rk}(w,\mathbb{M}) \geq 0)$,
- $\mathrm{rk}(w,\mathbb{M}) = \infty$ if $\mathrm{rk}(w,\mathbb{M}) \geq \delta$ for all ordinals δ,
- for an ordinal δ: $\mathrm{rk}(w,\mathbb{M}) = \delta$ if $\mathrm{rk}(w,\mathbb{M}) \geq \delta$ but $\neg(\mathrm{rk}(w,\mathbb{M}) \geq \delta+1)$.

Definition 2.3. For an ordinal ε and a cardinal λ let $\mathrm{NPr}^{\varepsilon}(\lambda)$ be the following statement:

> "there is a model $\mathbb{M}^*$ with the universe λ and a countable vocabulary τ^* such that $1+\mathrm{rk}(w,\mathbb{M}^*) \leq \varepsilon$ for all $w \in [\lambda]^{<\omega} \setminus \{\emptyset\}$."

Let $\mathrm{Pr}^{\varepsilon}(\lambda)$ be the negation of $\mathrm{NPr}^{\varepsilon}(\lambda)$.

(Note that $\mathrm{NPr}_{\varepsilon}$ of [13, Definition 2.4] differs from our $\mathrm{NPr}^{\varepsilon}$: "$\sup\{\mathrm{rk}(w,\mathbb{M}^*) : \emptyset \neq w \in [\lambda]^{<\omega}\} < \varepsilon$ " there is replaced by "$1+\mathrm{rk}(w,\mathbb{M}^*) \leq \varepsilon$" here.)

Proposition 2.4. (1) $\mathrm{NPr}^1(\aleph_1)$.
(2) *If* $\mathrm{NPr}^{\varepsilon}(\lambda)$*, then* $\mathrm{NPr}^{\varepsilon+1}(\lambda^+)$.
(3) *If* $\mathrm{NPr}^{\varepsilon}(\mu)$ *for* $\mu<\lambda$ *and* $\mathrm{cf}(\lambda)=\omega$*, then* $\mathrm{NPr}^{\varepsilon}(\lambda)$.
(4) *If* $\alpha<\omega_1$*, then* $\mathrm{NPr}^{\alpha}(\aleph_\alpha)$ *but* $\mathrm{Pr}^{\alpha}(\beth_{\omega_1})$ *holds.*

Definition 2.5. Let $\tau^{\otimes} = \{R_{n,j} : n,j<\omega\}$ be a fixed relational vocabulary where $R_{n,j}$ is an n–ary relational symbol (for $n,j<\omega$).

Definition 2.6. Assume that $\varepsilon<\omega_1$ and λ is an uncountable cardinal such that $\mathrm{NPr}^{\varepsilon}(\lambda)$. By this assumption, we may fix a model $\mathbb{M}(\varepsilon,\lambda) = \mathbb{M} = (\lambda, \{R^{\mathbb{M}}_{n,j}\}_{n,j<\omega})$ in the vocabulary $\tau^{\otimes}$ with the universe λ such that:

$(\circledast)_{\mathrm{a}}$ for every n and a quantifier free formula $\varphi(x_0,\ldots,x_{n-1}) \in \mathcal{L}(\tau^{\otimes})$ there is $j<\omega$ such that for all $\alpha_0,\ldots,\alpha_{n-1} \in \lambda$,

$$\mathbb{M} \models \varphi[\alpha_0,\ldots,\alpha_{n-1}] \Leftrightarrow R_{n,j}[\alpha_0,\ldots,\alpha_{n-1}],$$

$(\circledast)_{\mathrm{b}}$ the rank of every singleton is at least 0,
$(\circledast)_{\mathrm{c}}$ $1+\mathrm{rk}(v,\mathbb{M}) \leq \varepsilon$ for every $v \in [\lambda]^{<\omega} \setminus \{\emptyset\}$.

For a nonempty finite set $v \subseteq \lambda$ let $\mathrm{rk}^{\mathrm{sp}}(v) = \mathrm{rk}(v,\mathbb{M})$, and let $\mathbf{j}(v)<\omega$ and $\mathbf{k}(v)<|v|$ be such that $R_{|v|,\mathbf{j}(v)}, \mathbf{k}(v)$ witness the rank of v. Thus letting $\{\alpha_0,\ldots,\alpha_k,\ldots\alpha_{n-1}\}$ be the increasing enumeration of v and $k=\mathbf{k}(v)$ and $j=\mathbf{j}(v)$, we have

$(\circledast)_{\mathrm{e}}$ if $\mathrm{rk}^{\mathrm{sp}}(v) \geq 0$, then $\mathbb{M} \models R_{n,j}[\alpha_0,\ldots,\alpha_k,\ldots,\alpha_{n-1}]$ but there is no $\alpha \in \lambda\setminus v$ such that

$$\mathrm{rk}^{\mathrm{sp}}(v\cup\{\alpha\}) \geq \mathrm{rk}^{\mathrm{sp}}(v) \quad \text{and} \quad \mathbb{M} \models R_{n,j}[\alpha_0,\ldots,\alpha_{k-1},\alpha,\alpha_{k+1},\ldots,\alpha_{n-1}],$$

$(\circledast)_{\mathrm{f}}$ if $\mathrm{rk}^{\mathrm{sp}}(v) = -1$, then $\mathbb{M} \models R_{n,j}[\alpha_0,\ldots,\alpha_k,\ldots,\alpha_{n-1}]$ but the set

$$\big\{\alpha \in \lambda : \mathbb{M} \models R_{n,j}[\alpha_0,\ldots,\alpha_{k-1},\alpha,\alpha_{k+1},\ldots,\alpha_{n-1}]\big\}$$

is countable.

We may and will also require that for $j = \mathbf{j}(v)$, $n = |v|$ we have:

$(\circledast)_{\rm g}$ for every $\beta_0, \ldots, \beta_{n-1} < \lambda$

$$\text{if } \mathbb{M} \models R_{n,j}[\beta_0, \ldots, \beta_{n-1}] \text{ then } \beta_0 < \ldots < \beta_{n-1}.$$

The choices above define functions $\mathbf{j} : [\lambda]^{<\omega} \setminus \{\emptyset\} \longrightarrow \omega$, $\mathbf{k} : [\lambda]^{<\omega} \setminus \{\emptyset\} \longrightarrow \omega$, and $\mathrm{rk}^{\rm sp} : [\lambda]^{<\omega} \setminus \{\emptyset\} \longrightarrow \{-1\} \cup (\varepsilon + 1)$.

2.2. **Non-disjontness rank** ndrk_ι**.** Here we recall the rank measuring the easiness of building large sets of pairwise overlapping translations of a given Σ^0_2 set. The definitions and results given here are quoted after [13, Section 3]. Let us point out that Definition 2.8 is a slightly modified version of [13, Definition 3.5] – we added demand (f) here. The addition is needed for the precise rank considerations when our ranks are finite (to eliminate "disturbances in rank" by not important factors). It does not change the proofs of the facts quoted here, however.

We assume the following.

Assumption 2.7. (1) $T_n \subseteq {}^{\omega>}2$ is a tree with no maximal nodes (for $n < \omega$);
(2) $B = \bigcup\limits_{n<\omega} \lim(T_n)$, $\bar{T} = \langle T_n : n < \omega \rangle$ and $2 \leq \iota < \omega$;
(3) there are distinct $\rho_0, \rho_1 \in {}^\omega 2$ such that $\big|(\rho_0 + B) \cap (\rho_1 + B)\big| \geq 2\iota$.

Definition 2.8. Let $\mathbf{M}_{\bar{T},\iota}$ consist of all tuples

$$\mathbf{m} = (\ell_{\mathbf{m}}, u_{\mathbf{m}}, \bar{h}_{\mathbf{m}}, \bar{g}_{\mathbf{m}}) = (\ell, u, \bar{h}, \bar{g})$$

such that:

(a) $0 < \ell < \omega$, $u \subseteq {}^\ell 2$ and $2 \leq |u|$;
(b) $\bar{h} = \langle h_i : i < \iota \rangle$, $\bar{g} = \langle g_i : i < \iota \rangle$ and for each $i < \iota$ we have

$$h_i : u^{\langle 2 \rangle} \longrightarrow \omega \quad \text{ and } \quad g_i : u^{\langle 2 \rangle} \longrightarrow \bigcup_{n<\omega} (T_n \cap {}^\ell 2);$$

(c) $g_i(\eta, \nu) \in T_{h_i(\eta,\nu)} \cap {}^\ell 2$ for all $(\eta, \nu) \in u^{\langle 2 \rangle}$, $i < \iota$;
(d) if $(\eta, \nu) \in u^{\langle 2 \rangle}$ and $i < \iota$, then $\eta + g_i(\eta, \nu) = \nu + g_i(\nu, \eta)$;
(e) for any $(\eta, \nu) \in u^{\langle 2 \rangle}$, there are no repetitions in the sequence $\langle g_i(\eta, \nu), g_i(\nu, \eta) : i < \iota \rangle$;
(f) there are $\langle F(\eta) : \eta \in u \rangle$ and $\langle G_i(\eta, \nu) : i < \iota \ \wedge \ (\eta, \nu) \in u^{\langle 2 \rangle} \rangle$ such that

$$\begin{aligned} \eta \vartriangleleft F(\eta) \in {}^\omega 2 \ \text{ and } \ & g_i(\eta, \nu) \vartriangleleft G_i(\eta, \nu) \in \lim\big(T_{h_i(\eta,\nu)}\big) \\ \text{and } \ & F(\eta) + G_i(\eta, \nu) = F(\nu) + G_i(\nu, \eta) \end{aligned}$$

(for $i < \iota$, $(\eta, \nu) \in u^{\langle 2 \rangle}$).

Note that by Assumption 2.7(3) the family $\mathbf{M}_{\bar{T},\iota}$ is not empty.

Definition 2.9. Assume $\mathbf{m} = (\ell, u, \bar{h}, \bar{g}) \in \mathbf{M}_{\bar{T},\iota}$ and $\rho \in {}^\ell 2$. We define $\mathbf{m} + \rho = (\ell', u', \bar{h}', \bar{g}')$ by

- $\ell' = \ell$, $u' = \{\eta + \rho : \eta \in u\}$,
- $\bar{h}' = \langle h'_i : i < \iota \rangle$ where $h'_i : (u')^{\langle 2 \rangle} \longrightarrow \omega$ are such that $h'_i(\eta + \rho, \nu + \rho) = h_i(\eta, \nu)$ for $(\eta, \nu) \in u^{\langle 2 \rangle}$,
- $\bar{g}' = \langle g'_i : i < \iota \rangle$ where $g'_i : (u')^{\langle 2 \rangle} \longrightarrow \bigcup\limits_{n<\omega} (T_n \cap {}^\ell 2)$ are such that

$$g'_i(\eta + \rho, \nu + \rho) = g_i(\eta, \nu) \ \text{ for } (\eta, \nu) \in u^{\langle 2 \rangle}.$$

Also if $\rho \in {}^\omega 2$, then we set $\mathbf{m} + \rho = \mathbf{m} + (\rho{\restriction}\ell)$.

Observation 2.10. (1) *If $\mathbf{m} \in \mathbf{M}_{\bar{T},\iota}$ and $\rho \in {}^{\ell_{\mathbf{m}}}2$, then $\mathbf{m}+\rho \in \mathbf{M}_{\bar{T},\iota}$.*

(2) *For each $\rho \in {}^{\omega}2$ the mapping $\mathbf{M}_{\bar{T},\iota} \longrightarrow \mathbf{M}_{\bar{T},\iota} : \mathbf{m} \mapsto \mathbf{m}+\rho$ is a bijection.*

Definition 2.11. Assume $\mathbf{m},\mathbf{n} \in \mathbf{M}_{\bar{T},\iota}$. We say that $\mathbf{n}$ *extends* $\mathbf{m}$ ($\mathbf{m} \sqsubseteq \mathbf{n}$ in short) if and only if:

- $\ell_{\mathbf{m}} \leq \ell_{\mathbf{n}}$, $u_{\mathbf{m}} = \{\eta\restriction\ell_{\mathbf{m}} : \eta \in u_{\mathbf{n}}\}$, and
- for every $(\eta,\nu) \in (u_{\mathbf{n}})^{\langle 2\rangle}$ such that $\eta\restriction\ell_{\mathbf{m}} \neq \nu\restriction\ell_{\mathbf{m}}$ and each $i<\iota$ we have
$$h_i^{\mathbf{m}}(\eta\restriction\ell_{\mathbf{m}},\nu\restriction\ell_{\mathbf{m}}) = h_i^{\mathbf{n}}(\eta,\nu) \quad \text{and} \quad g_i^{\mathbf{m}}(\eta\restriction\ell_{\mathbf{m}},\nu\restriction\ell_{\mathbf{m}}) = g_i^{\mathbf{n}}(\eta,\nu)\restriction\ell_{\mathbf{m}}.$$

Definition 2.12. We define a function $\mathrm{ndrk}_\iota : \mathbf{M}_{\bar{T},\iota} \longrightarrow \mathrm{ON}\cup\{\infty\}$ declaring inductively when $\mathrm{ndrk}_\iota(\mathbf{m}) \geq \alpha$ (for an ordinal α).

- $\mathrm{ndrk}_\iota(\mathbf{m}) \geq 0$ always;
- if α is a limit ordinal, then $\mathrm{ndrk}_\iota(\mathbf{m}) \geq \alpha \Leftrightarrow (\forall\beta<\alpha)(\mathrm{ndrk}_\iota(\mathbf{m}) \geq \beta)$;
- if $\alpha=\beta+1$, then $\mathrm{ndrk}_\iota(\mathbf{m}) \geq \alpha$ if and only if for every $\nu \in u_{\mathbf{m}}$ there is $\mathbf{n} \in \mathbf{M}_{\bar{T},\iota}$ such that $\ell_{\mathbf{n}} > \ell_{\mathbf{m}}$, $\mathbf{m} \sqsubseteq \mathbf{n}$ and $\mathrm{ndrk}_\iota(\mathbf{n}) \geq \beta$ and
$$|\{\eta \in u_{\mathbf{n}} : \nu \vartriangleleft \eta\}| \geq 2;$$
- $\mathrm{ndrk}_\iota(\mathbf{m}) = \infty$ if and only if $\mathrm{ndrk}_\iota(\mathbf{m}) \geq \alpha$ for all ordinals α.

We also define
$$\mathrm{ndrk}_\iota(\bar{T}) = \sup\{\mathrm{ndrk}_\iota(\mathbf{m}) : \mathbf{m} \in \mathbf{M}_{\bar{T},\iota}\}.$$

Lemma 2.13. *[See [13, Lemma 3.10]]*

(1) *The relation $\sqsubseteq$ is a partial order on $\mathbf{M}_{\bar{T},\iota}$.*
(2) *If $\mathbf{m},\mathbf{n} \in \mathbf{M}_{\bar{T},\iota}$ and $\mathbf{m} \sqsubseteq \mathbf{n}$ and $\alpha \leq \mathrm{ndrk}_\iota(\mathbf{n})$, then $\alpha \leq \mathrm{ndrk}_\iota(\mathbf{m})$.*
(3) *The function ndrk_ι is well defined.*
(4) *If $\mathbf{m} \in \mathbf{M}_{\bar{T},\iota}$ and $\rho \in {}^{\omega}2$ then $\mathrm{ndrk}_\iota(\mathbf{m}) = \mathrm{ndrk}_\iota(\mathbf{m}+\rho)$.*
(5) *If $\mathbf{m} \in \mathbf{M}_{\bar{T},\iota}$, $\nu \in u_{\mathbf{m}}$ and $\mathrm{ndrk}_\iota(\mathbf{m}) \geq \omega_1$, then there is an $\mathbf{n} \in \mathbf{M}_{\bar{T},\iota}$ such that $\mathbf{m} \sqsubseteq \mathbf{n}$, $\mathrm{ndrk}_\iota(\mathbf{n}) \geq \omega_1$, and*
$$|\{\eta \in u_{\mathbf{n}} : \nu \vartriangleleft \eta\}| \geq 2.$$
(6) *If $\mathbf{m} \in \mathbf{M}_{\bar{T},\iota}$ and $\infty > \mathrm{ndrk}_\iota(\mathbf{m}) = \beta > \alpha$, then there is $\mathbf{n} \in \mathbf{M}_{\bar{T},\iota}$ such that $\mathbf{m} \sqsubseteq \mathbf{n}$ and $\mathrm{ndrk}_\iota(\mathbf{n}) = \alpha$.*
(7) *If $\mathrm{ndrk}_\iota(\bar{T}) \geq \omega_1$, then $\mathrm{ndrk}_\iota(\bar{T}) = \infty$.*
(8) *Assume $\mathbf{m} \in \mathbf{M}_{\bar{T},\iota}$ and $u' \subseteq u_{\mathbf{m}}$, $|u'| \geq 2$. Put $\ell' = \ell_{\mathbf{m}}$, $h'_i = h_i^{\mathbf{m}}\restriction(u')^{\langle 2\rangle}$ and $g'_i = g_i^{\mathbf{m}}\restriction(u')^{\langle 2\rangle}$ (for $i<\iota$), and let $\mathbf{m}\restriction u' = (\ell',u',\bar{h}',\bar{g}')$. Then $\mathbf{m}\restriction u' \in \mathbf{M}_{\bar{T},\iota}$ and $\mathrm{ndrk}_\iota(\mathbf{m}) \leq \mathrm{ndrk}_\iota(\mathbf{m}\restriction u')$.*

Directly from Definition 2.12 we get the following observation.

Observation 2.14. *If $\mathbf{m} \in \mathbf{M}_{\bar{T},\iota}$ and $\mathrm{ndrk}_\iota(\mathbf{m}) \geq \alpha+1$, then there is $\mathbf{n} \sqsupseteq \mathbf{m}$ such that $\ell_{\mathbf{n}} = \ell_{\mathbf{m}}+1$ and $\mathrm{ndrk}_\iota(\mathbf{n}) \geq \alpha$.*

Proposition 2.15 (See [13, Proposition 3.11])**.** *The following conditions are equivalent.*

(a) $\mathrm{ndrk}_\iota(\bar{T}) \geq \omega_1$.
(b) $\mathrm{ndrk}_\iota(\bar{T}) = \infty$.
(c) *There is a perfect set $P \subseteq {}^{\omega}2$ such that*
$$(\forall\eta,\nu \in P)\big(|(B+\eta)\cap(B+\nu)| \geq 2\iota\big).$$

Proposition 2.16. *Assume* $\mathrm{ndrk}_\iota(\bar{T}) \le \varepsilon$. *If there is* $A \subseteq {}^\omega 2$ *of cardinality* λ *such that*

$$\big(\forall \eta, \nu \in A\big)\big(|(B+\eta)\cap(B+\nu)| \ge 2\iota\big),$$

then $\mathrm{NPr}^{1+\varepsilon}(\lambda)$.

Proof. This was implicitly shown by the proof of [13, Proposition 3.11($(d) \Rightarrow (a)$)], but let us repeat this argument.

Assume $\langle \eta_\alpha : \alpha < \lambda\rangle$ is a sequence of distinct elements of ${}^\omega 2$ such that

$$\big(\forall \alpha < \beta < \lambda\big)\big(|(B+\eta_\alpha)\cap(B+\eta_\beta)| \ge 2\iota\big).$$

Let $\tau = \{R_{\mathbf{m}} : \mathbf{m} \in \mathbf{M}_{\bar{T},\iota}\}$ be a (countable) vocabulary where each $R_{\mathbf{m}}$ is a $|u_{\mathbf{m}}|$–ary relational symbol. Let $\mathbb{M} = \big(\lambda, \{R^{\mathbb{M}}_{\mathbf{m}}\}_{\mathbf{m}\in\mathbf{M}_{\bar{T},\iota}}\big)$ be the model in the vocabulary τ, where for $\mathbf{m} = (\ell, u, \bar{h}, \bar{g}) \in \mathbf{M}_{\bar{T},\iota}$ the relation $R^{\mathbb{M}}_{\mathbf{m}}$ is defined by

$$\begin{aligned} R^{\mathbb{M}}_{\mathbf{m}} = \quad & \Big\{(\alpha_0,\ldots,\alpha_{|u|-1}) \in {}^{|u|}\lambda : \{\eta_{\alpha_0}{\restriction}\ell,\ldots,\eta_{|u|-1}{\restriction}\ell\} = u \text{ and} \\ & \quad \text{for distinct } j_1, j_2 < |u| \text{ there are } G_i(\alpha_{j_1},\alpha_{j_2}) \text{ (for } i<\iota) \text{ such that} \\ & \quad g_i(\eta_{\alpha_{j_1}}{\restriction}\ell, \eta_{\alpha_{j_2}}{\restriction}\ell) \vartriangleleft G_i(\alpha_{j_1},\alpha_{j_2}) \in \lim\big(T_{h_i(\eta_{\alpha_{j_1}}\restriction\ell,\eta_{\alpha_{j_2}}\restriction\ell)}\big) \text{ and} \\ & \quad \eta_{\alpha_{j_1}} + G_i(\alpha_{j_1},\alpha_{j_2}) = \eta_{\alpha_{j_2}} + G_i(\alpha_{j_2},\alpha_{j_1}) \ \Big\}. \end{aligned}$$

We will show that the model $\mathbb{M}$ witnesses $\mathrm{NPr}^{1+\varepsilon}(\lambda)$.

Claim 2.16.1. (1) *If* $\alpha_0,\alpha_1,\ldots,\alpha_{j-1} < \lambda$ *are distinct,* $j \ge 2$*, then for sufficiently large* $\ell < \omega$ *there is* $\mathbf{m} \in \mathbf{M}_{\bar{T},\iota}$ *such that*

$$\ell_{\mathbf{m}} = \ell, \quad u_{\mathbf{m}} = \{\eta_{\alpha_0}{\restriction}\ell,\ldots,\eta_{\alpha_{j-1}}{\restriction}\ell\} \quad \textit{and} \quad \mathbb{M} \models R_{\mathbf{m}}[\alpha_0,\ldots,\alpha_{j-1}].$$

(2) *Assume that* $\mathbf{m} \in \mathbf{M}_{\bar{T},\iota}$, $j < |u_{\mathbf{m}}|$, $\alpha_0,\alpha_1,\ldots,\alpha_{|u_{\mathbf{m}}|-1} < \lambda$ *and* $\alpha^* < \lambda$ *are all pairwise distinct and such that* $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0,\ldots,\alpha_j,\ldots,\alpha_{|u_{\mathbf{m}}|-1}]$ *and* $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0,\ldots,\alpha_{j-1},\alpha^*,\alpha_{j+1},\ldots\alpha_{|u_{\mathbf{m}}|-1}]$. *Then for every sufficiently large* $\ell > \ell_{\mathbf{m}}$ *there is* $\mathbf{n} \in \mathbf{M}_{\bar{T},\iota}$ *such that* $\mathbf{m} \sqsubseteq \mathbf{n}$ *and*

$$\ell_{\mathbf{n}} = \ell, \quad u_{\mathbf{n}} = \{\eta_{\alpha_0}{\restriction}\ell,\ldots,\eta_{\alpha_{|u_{\mathbf{m}}|-1}}{\restriction}\ell, \eta_{\alpha^*}{\restriction}\ell\} \quad \textit{and} \quad \mathbb{M} \models R_{\mathbf{n}}[\alpha_0,\ldots,\alpha_{|u_{\mathbf{m}}|-1},\alpha^*].$$

Proof of the Claim. (1) For distinct $j_1, j_2 < j$ let $G_i(\alpha_{j_1},\alpha_{j_2}) \in B$ (for $i < \iota$) be such that

$$\eta_{\alpha_{j_1}} + G_i(\alpha_{j_1},\alpha_{j_2}) = \eta_{\alpha_{j_2}} + G_i(\alpha_{j_2},\alpha_{j_1})$$

and there are no repetitions in the sequence $\langle G_i(\alpha_{j_1},\alpha_{j_2}), G_i(\alpha_{j_2},\alpha_{j_1}) : i < \iota\rangle$. (Remember, $x \in (B+\eta_{\alpha_{j_1}})\cap(B+\eta_{\alpha_{j_2}})$ if and only if $x+(\eta_{\alpha_{j_1}}+\eta_{\alpha_{j_2}}) \in (B+\eta_{\alpha_{j_1}})\cap(B+\eta_{\alpha_{j_2}})$, so the choice of $G_i(\alpha_{j_1},\alpha_{j_2})$ is possible by the assumptions on η_α's.) Suppose that $\ell < \omega$ is such that for any distinct $j_1, j_2 < j$ we have $\eta_{\alpha_{j_1}}{\restriction}\ell \neq \eta_{\alpha_{j_2}}{\restriction}\ell$ and there are no repetitions in the sequence $\langle G_i(\alpha_{j_1},\alpha_{j_2}){\restriction}\ell, G_i(\alpha_{j_2},\alpha_{j_1}){\restriction}\ell : i < \iota\rangle$. Now let $u = \{\eta_{\alpha_{j'}}{\restriction}\ell : j' < j\}$, and for $i < \iota$ let $g_i(\eta_{\alpha_{j_1}}{\restriction}\ell, \eta_{\alpha_{j_2}}{\restriction}\ell) = G_i(\alpha_{j_1},\alpha_{j_2}){\restriction}\ell$, and let $h_i(\eta_{\alpha_{j_1}}{\restriction}\ell, \eta_{\alpha_{j_2}}{\restriction}\ell) < \omega$ be such that $G_i(\alpha_{j_1},\alpha_{j_2}) \in \lim\big(T_{h_i(\eta_{\alpha_{j_1}}\restriction\ell,\eta_{\alpha_{j_2}}\restriction\ell)}\big)$. This defines $\mathbf{m} = (\ell, u, \bar{h}, \bar{g}) \in \mathbf{M}_{\bar{T},\iota}$ and easily $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0,\ldots,\alpha_{j-1}]$.

(2) Similar to (1). $\square$

Now the proof of the Proposition will be an immediate consequence of the following Claim.

Claim 2.16.2. *If* $\mathbf{m} \in \mathbf{M}_{\bar{T},\iota}$ *and* $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0,\ldots,\alpha_{|u_{\mathbf{m}}|-1}]$, *then*

$$\mathrm{rk}(\{\alpha_0,\ldots,\alpha_{|u_{\mathbf{m}}|-1}\}, \mathbb{M}) \le \mathrm{ndrk}_\iota(\mathbf{m}) \le \varepsilon.$$

Proof of the Caim. By induction on β we show that *for every* $\mathbf{m} \in \mathbf{M}_{\bar{T},\iota}$ and *all* distinct $\alpha_0, \ldots, \alpha_{|u_{\mathbf{m}}|-1} < \lambda_{\omega_1}$ such that $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \ldots, \alpha_{|u_{\mathbf{m}}|-1}]$:

$$\beta \leq \mathrm{rk}(\{\alpha_0, \ldots, \alpha_{|u_{\mathbf{m}}|-1}\}, \mathbb{M}) \text{ implies } \beta \leq \mathrm{ndrk}_\iota(\mathbf{m}).$$

STEPS $\beta = 0$ AND β IS LIMIT: Straightforward.

STEP $\beta = \gamma + 1$: Suppose $\mathbf{m} \in \mathbf{M}_{\bar{T},\iota}$ and $\alpha_0, \ldots, \alpha_{|u_{\mathbf{m}}|-1} < \lambda$ are such that $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \ldots, \alpha_{|u_{\mathbf{m}}|-1}]$ and $\gamma + 1 \leq \mathrm{rk}(\{\alpha_0, \ldots, \alpha_{|u_{\mathbf{m}}|-1}\}, \mathbb{M})$. Let $\nu \in u_{\mathbf{m}}$, so $\nu = \eta_{\alpha_j} \restriction \ell_{\mathbf{m}}$ for some $j < |u_{\mathbf{m}}|$. Since $\gamma + 1 \leq \mathrm{rk}(\{\alpha_0, \ldots, \alpha_{|u_{\mathbf{m}}|-1}\}, \mathbb{M})$ we may find $\alpha^* \in \lambda \setminus \{\alpha_0, \ldots, \alpha_{|u_{\mathbf{m}}|-1}\}$ such that $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \ldots, \alpha_{j-1}, \alpha^*, \alpha_{j+1}, \ldots, \alpha_{|u_{\mathbf{m}}|-1}]$ and $\mathrm{rk}(\{\alpha_0, \ldots, \alpha_{|u_{\mathbf{m}}|-1}, \alpha^*\}, \mathbb{M}) \geq \gamma$. Taking sufficiently large ℓ we may use Claim 2.16.1(2) to find $\mathbf{n} \in \mathbf{M}_{\bar{T},\iota}$ such that $\mathbf{m} \sqsubseteq \mathbf{n}$, $\ell_{\mathbf{n}} = \ell$ and $\mathbb{M} \models R_{\mathbf{n}}[\alpha_0, \ldots, \alpha_{|u_{\mathbf{m}}|-1}, \alpha^*]$ and $|\{\eta \in u_{\mathbf{n}} : \nu \lhd \eta\}| \geq 2$. By the inductive hypothesis we have also $\gamma \leq \mathrm{ndrk}_\iota(\mathbf{n})$. Now we may easily conclude that $\gamma + 1 \leq \mathrm{ndrk}_\iota(\mathbf{m})$. □

□

Definition 2.17. Assume $\mathbf{m}, \mathbf{n} \in \mathbf{M}_{\bar{T},\iota}$.

(1) We say that $\mathbf{m}$, $\mathbf{n}$ *are essentially the same* ($\mathbf{m} \doteqdot \mathbf{n}$ in short) if and only if:
- $\ell_{\mathbf{m}} = \ell_{\mathbf{n}}$, $u_{\mathbf{m}} = u_{\mathbf{n}}$ and
- for each $(\eta, \nu) \in (u_{\mathbf{m}})^{\langle 2 \rangle}$ we have

$$\big\{\{g_i^{\mathbf{m}}(\eta,\nu), g_i^{\mathbf{m}}(\nu,\eta)\} : i < \iota\big\} = \big\{\{g_i^{\mathbf{n}}(\eta,\nu), g_i^{\mathbf{n}}(\nu,\eta)\} : i < \iota\big\},$$

and for $i, j < \iota$:
if $g_i^{\mathbf{m}}(\eta,\nu) = g_j^{\mathbf{n}}(\eta,\nu)$, then $h_i^{\mathbf{m}}(\eta,\nu) = h_j^{\mathbf{n}}(\eta,\nu)$,
if $g_i^{\mathbf{m}}(\eta,\nu) = g_j^{\mathbf{n}}(\nu,\eta)$, then $h_i^{\mathbf{m}}(\eta,\nu) = h_j^{\mathbf{n}}(\nu,\eta)$.

(2) We say that $\mathbf{n}$ *essentially extends* $\mathbf{m}$ ($\mathbf{m} \sqsubseteq^* \mathbf{n}$ in short) if and only if:
- $\ell_{\mathbf{m}} \leq \ell_{\mathbf{n}}$, $u_{\mathbf{m}} = \{\eta \restriction \ell_{\mathbf{m}} : \eta \in u_{\mathbf{n}}\}$, and
- for every $(\eta, \nu) \in (u_{\mathbf{n}})^{\langle 2 \rangle}$ such that $\eta \restriction \ell_{\mathbf{m}} \neq \nu \restriction \ell_{\mathbf{m}}$ we have

$$\big\{\{g_i^{\mathbf{m}}(\eta\restriction\ell_{\mathbf{m}}, \nu\restriction\ell_{\mathbf{m}}), g_i^{\mathbf{m}}(\nu\restriction\ell_{\mathbf{m}}, \eta\restriction\ell_{\mathbf{m}})\} : i < \iota\big\} = \big\{\{g_i^{\mathbf{n}}(\eta,\nu)\restriction\ell_{\mathbf{m}}, g_i^{\mathbf{n}}(\nu,\eta)\restriction\ell_{\mathbf{m}}\} : i < \iota\big\},$$

and for $i, j < \iota$:
if $g_i^{\mathbf{m}}(\eta\restriction\ell_{\mathbf{m}}, \nu\restriction\ell_{\mathbf{m}}) = g_j^{\mathbf{n}}(\eta,\nu)\restriction\ell_{\mathbf{m}}$, then $h_i^{\mathbf{m}}(\eta\restriction\ell_{\mathbf{m}}, \nu\restriction\ell_{\mathbf{m}}) = h_j^{\mathbf{n}}(\eta,\nu)$,
if $g_i^{\mathbf{m}}(\eta\restriction\ell_{\mathbf{m}}, \nu\restriction\ell_{\mathbf{m}}) = g_j^{\mathbf{n}}(\nu,\eta)\restriction\ell_{\mathbf{m}}$, then $h_i^{\mathbf{m}}(\eta\restriction\ell_{\mathbf{m}}, \nu\restriction\ell_{\mathbf{m}}) = h_j^{\mathbf{n}}(\nu,\eta)$.

3. CUTE $\mathcal{YZR}$ AND FORCING NONDISJOINT TRANSLATIONS

In this section we give a property of $\bar{T}$ allowing us to force many (but not too many) overlapping translations of the corresponding Σ^0_2 set. Conditions in the forcing notions come from finite approximations (*bricks*) suitably placed on finite subsets of λ. An amalgamation property, cute $\mathcal{YZR}$ systems and the splitting rank on λ will all help with the ccc of the forcing notion.

Definition 3.1. Let $0 < \varepsilon < \omega_1$. *A* $\mathcal{YZR}(\varepsilon)$*–system*[1] is a tuple $s = (X^s, \bar{r}^s, \bar{\jmath}^s, \bar{k}^s) = (X, \bar{r}, \bar{\jmath}, \bar{k})$ such that

$(*)_1$ X is a nonempty set of ordinals,
$(*)_2$ $\bar{r} : [X]^{<\omega} \setminus \{\emptyset\} \longrightarrow \varepsilon + 1$, $\bar{\jmath} : [X]^{<\omega} \setminus \{\emptyset\} \longrightarrow \omega$, and $\bar{k} : [X]^{<\omega} \setminus \{\emptyset\} \longrightarrow \omega$,
$(*)_3$ if $\emptyset \neq u \subseteq w \in [X]^{<\omega}$, then $\bar{r}(u) \geq \bar{r}(w)$,
$(*)_4$ $\bar{r}(\{a\}) > 0$ for all $a \in X$,

[1] $\mathcal{YZR}$ are the initials of the first author daughter who really wanted to be in this paper

$(*)_5$ if $\emptyset \neq w \in [X]^{<\omega}$, $w = \{a_0, \ldots, a_{n-1}\}$ (the increasing enumeration) then $\bar{k}(w) < n$ and there is no $b \in X \setminus w$ such that

$$|w \cap b| = \bar{k}(w) \quad \text{and} \quad \bar{\jmath}\big((w \setminus \{a_{\bar{k}(w)}\}) \cup \{b\}\big) = \bar{\jmath}(w) \quad \text{and} \quad \bar{r}\big(w \cup \{b\}\big) = \bar{r}(w).$$

We say that the system s is *finite* if the set X^s is finite.

Example 3.2. With the choices of $\mathbf{j}, \mathbf{k}$ and $\mathrm{rk}^{\mathrm{sp}}$ as described in Definition 2.6 (for ε and λ as there), the *finite* $\mathcal{YZR}(\varepsilon)$*–system associated with a set* $w \in [\lambda]^{<\omega}$ is $s(w) = (w, \bar{r}, \bar{\jmath}, \bar{k})$ defined as follows. First, fix an enumeration $\{v^*_i : i < i^*\} = \{v \subseteq w : v \neq \emptyset \ \wedge \ \mathrm{rk}^{\mathrm{sp}}(v) = -1\}$. Let $J = \max\big(\mathbf{j}(v) : \emptyset \neq v \subseteq w\big) + 1$. For $\emptyset \neq v \subseteq w$ we define

- $\bar{r}(v) = 1 + \mathrm{rk}^{\mathrm{sp}}(v)$, and $\bar{k}(v) = \mathbf{k}(v)$, and
- if $\mathrm{rk}^{\mathrm{sp}}(v) \geq 0$, then $\bar{\jmath}(v) = \mathbf{j}(v)$, and
- $\bar{\jmath}(v^*_i) = J + i$ for $i < i^*$.

(It should be clear that the above conditions define a $\mathcal{YZR}(\varepsilon)$–system indeed.)

Definition 3.3. (1) Assume q, s are $\mathcal{YZR}(\varepsilon)$–systems. A *quasi–embedding of q in s* is an increasing injection $\varphi : X^q \longrightarrow X^s$ such that for all nonempty finite $v \subseteq X^q$ we have
   - $\bar{r}^s\big(\varphi[v]\big) = \bar{r}^q(v)$ and $\bar{k}^s\big(\varphi[v]\big) = \bar{k}^q(v)$, and
   - if $\bar{r}^q(v) > 0$, then $\bar{\jmath}^s\big(\varphi[v]\big) = \bar{\jmath}^q(v)$.

(2) If $w \subseteq X^s$, then an increasing injection $\varphi : w \longrightarrow X^s$ is *a quasi–embedding* if it is a quasi embedding of the (naturally defined) restricted $\mathcal{YZR}(\varepsilon)$–system $s{\restriction}w$ into s.

(3) A $\mathcal{YZR}(\varepsilon)$–system S is *cute* if $X^S = \omega$ and for every finite $\mathcal{YZR}(\varepsilon)$–system q and an $M < \omega$, there is a quasi–embedding φ of q in S with $\mathrm{rng}(\varphi) \subseteq [M, \omega)$.

Theorem 3.4. *For every $0 < \varepsilon < \omega_1$ there exists a cute $\mathcal{YZR}(\varepsilon)$–system.*

Proof. Assume $0 < \varepsilon < \omega_1$. Let $\mathcal{S}$ consist of all finite $\mathcal{YZR}(\varepsilon)$–systems $s = (N^s, \bar{r}^s, \bar{\jmath}^s, \bar{k}^s)$ such that $0 < N^s < \omega$. For $q, s \in \mathcal{S}$ we will say that s *extends* q, in short $q \preceq s$, if and only if $N^q \leq N^s$, $\bar{r}^q \subseteq \bar{r}^s$, $\bar{\jmath}^q \subseteq \bar{\jmath}^s$, and $\bar{k}^q \subseteq \bar{k}^s$.

Claim 3.4.1. *The relation $\preceq$ is a partial order on $\mathcal{S}$. As a matter of fact, $(\mathcal{S}, \preceq)$ is the Cohen forcing notion.*

Claim 3.4.2. *Suppose that $s \in \mathcal{S}$ and $q = (X^q, \bar{r}^q, \bar{\jmath}^q, \bar{k}^q)$ is a finite $\mathcal{YZR}(\varepsilon)$–system. Then there are $t \succeq s$ and an increasing injection $\varphi : X^q \longrightarrow [N^s, N^t)$ such that for each nonempty $v \subseteq X^q$ we have*

$$\bar{r}^t\big(\varphi[v]\big) = \bar{r}^q(v) \text{ and } \bar{\jmath}^t\big(\varphi[v]\big) = \bar{\jmath}^q(v) \text{ and } \bar{k}^t\big(\varphi[v]\big) = \bar{k}^q(v).$$

Proof of the Claim. Without loss of generality, $X^q = N < \omega$. Let $N^t = N^s + N$ and let $\varphi : X^q \longrightarrow [N^s, N^t) : m \mapsto N^s + m$. We also let

$$J_0 = \max\big(\mathrm{rng}(\bar{\jmath}^s) \cup \mathrm{rng}(\bar{\jmath}^q)\big) + 1 \quad \text{and} \quad J_1 = J_0 + (2^{N^s} - 1) \cdot (2^N - 1)$$

and we fix a bijection

$$\psi : \big\{u \subseteq N^t : u \cap N^s \neq \emptyset \neq u \cap [N^s, N^t)\big\} \longrightarrow [J_0, J_1).$$

Now, to define $\bar{r}^t, \bar{\jmath}^t$ and $\bar{k}^t$ we put for $u \subseteq N^t$:

- if $u \subseteq N^s$, then $\bar{r}^t(u) = \bar{r}^s(u)$, $\bar{\jmath}^t(u) = \bar{\jmath}^s(u)$ and $\bar{k}^t(u) = \bar{k}^s(u)$,

- if $u \subseteq [N^s, N^t)$, then $\bar{r}^t(u) = \bar{r}^q\big(\varphi^{-1}[u]\big)$, $\bar{\jmath}^t(u) = \bar{\jmath}^q\big(\varphi^{-1}[u]\big)$ and $\bar{k}^t(u) = \bar{k}^q\big(\varphi^{-1}[u]\big)$,
- if $u \cap N^s \neq \emptyset \neq u \cap [N^s, N^t)$, then $\bar{r}^t(u) = 0$, $\bar{k}^t(u) = 0$ and $\bar{\jmath}^t(u) = \psi(u)$.

This completes the definition of $t = (N^t, \bar{r}^t, \bar{\jmath}^t, \bar{k}^t)$. To verify that $t \in \mathcal{S}$ note that clauses $(*)_1$–$(*)_4$ of Definition 3.1 follow immediately from our choices.

Let us argue that 3.1$(*)_5$ is satisfied too. Suppose that $\emptyset \neq u \subseteq N^t$ and $u = \{a_0, \ldots, a_{n-1}\}$ is the increasing enumeration. Straightforward from the definitions above, $\bar{k}(u) < n$. Now,

- if $u \cap N^s \neq \emptyset \neq u \cap [N^s, N^t)$, then no other $u' \subseteq N^t$ satisfies $\bar{\jmath}(u') = \bar{\jmath}(u)$. At the same time $(u \setminus \{a\}) \cup \{b\} \neq u$ for $a \in u$ and $b \notin u$.
- If $u \subseteq N^s$, then
  - for every $b \in N^s \setminus u$, by $(*)_5$ for s, either $|u \cap b| \neq \bar{k}^s(u) = \bar{k}^t(u)$ or $\bar{\jmath}^t(u \setminus \{a_{\bar{k}^t(u)}\} \cup \{b\}) = \bar{\jmath}^s(u \setminus \{a_{\bar{k}^s(u)}\} \cup \{b\}) \neq \bar{\jmath}^s(u) = \bar{\jmath}^t(u)$ or $\bar{r}^t(u \cup \{b\}) = \bar{r}^s(u \cup \{b\}) < \bar{r}^s(u) = \bar{r}^t(u)$,
  - for every $b \in [N^s, N^t)$ we have
    - $\bar{r}^t(u \cup \{b\}) = 0 < \bar{r}^t(u)$ when $n = 1$ and
    - $\bar{\jmath}^t(u \setminus \{a_{\bar{k}^t(u)}\} \cup \{b\}) \neq \bar{\jmath}^s(u) = \bar{\jmath}^t(u)$ when $n > 1$.
- If $u \subseteq [N^s, N^t)$, then
  - for every $b \in [N^s, N^t) \setminus u$, by $(*)_5$ for q, either $|u \cap b| = |\varphi^{-1}[u] \cap \varphi^{-1}(b)| \neq \bar{k}^q(\varphi^{-1}[u]) = \bar{k}^t(u)$ or $\bar{\jmath}^t(u \setminus \{a_{\bar{k}^t(u)}\} \cup \{b\}) = \bar{\jmath}^q(\varphi^{-1}[u \setminus \{a_{\bar{k}^t(u)}\} \cup \{b\}]) \neq \bar{\jmath}^q(\varphi^{-1}[u]) = \bar{\jmath}^t(u)$ or $\bar{r}^t(u \cup \{b\}) = \bar{r}^q(\varphi^{-1}[u \cup \{b\}]) < \bar{r}^q(\varphi^{-1}[u]) = \bar{r}^t(u)$,
  - for every $b \in N^s$ we have
    - $\bar{r}^t(u \cup \{b\}) = 0 < \bar{r}^t(u)$ when $n = 1$ and
    - $\bar{\jmath}^t(u \setminus \{a_{\bar{k}^t(u)}\} \cup \{b\}) \neq \bar{\jmath}^q(\varphi^{-1}[u]) = \bar{\jmath}^t(u)$ when $n > 1$.

Consequently, in any possible case there is no $b \in N^t \setminus u$ such that

$$|u \cap b| = \bar{k}^t(u) \quad \text{and} \quad \bar{\jmath}^t(u \setminus \{a_{\bar{k}^t(u)}\} \cup \{b\}) = \bar{\jmath}^t(u) \quad \text{and} \quad \bar{r}^t(u \cup \{b\}) = \bar{r}^t(u).$$

Therefore, $q \in \mathcal{S}$ and easily it is as required. $\square$

Let $\langle q_i : i < \omega \rangle$ list with infinite repetitions all elements of $\mathcal{S}$. Use Claim 3.4.2 to construct a sequence $\langle s_i : i < \omega \rangle$ such that for all $i < \omega$:

- $s_i \in \mathcal{S}$, $s_i \preceq s_{i+1}$,
- for some increasing injection $\varphi_i : N^{q_i} \longrightarrow [N^{s_i}, N^{s_{i+1}})$ we have
  $$\bar{r}^{s_{i+1}}\big(\varphi_i[v]\big) = \bar{r}^{q_i}(v) \text{ and } \bar{\jmath}^{s_{i+1}}\big(\varphi[v]\big) = \bar{\jmath}^{q_i}(v) \text{ and } \bar{k}^{s_{i+1}}\big(\varphi[v]\big) = \bar{k}^{q_i}(v)$$
  for all $\emptyset \neq v \subseteq N^{q_i}$.

Then let $S = (\omega, \bar{r}^S, \bar{\jmath}^S, \bar{k}^S)$ be defined by

$$\bar{r}^S = \bigcup_{i<\omega} \bar{r}^{s_i}, \quad \bar{\jmath}^S = \bigcup_{i<\omega} \bar{\jmath}^{s_i}, \quad \bar{k}^S = \bigcup_{i<\omega} \bar{k}^{s_i}.$$

Plainly, S is a cute $\mathcal{YZR}(\varepsilon)$–system. $\square$

Assumption 3.5. In the rest of this section we assume that

- $2 \leq \iota < \omega$, and $\bar{c} = \langle c_m : m < \omega \rangle \subseteq \omega$,
- $T_m \subseteq {}^{\omega>}2$ (for $m < \omega$) are trees with no maximal nodes, $\bar{T} = \langle T_m : m < \omega \rangle$, and $B = \bigcup\limits_{m<\omega} \lim(T_m)$,

- there are pairwise different $\rho_0,\rho_1,\rho_2\in {}^\omega 2$ such that
$$\big|\big(\rho_j+B\big)\cap\big(\rho_{j'}+B\big)\big|\geq 2\iota$$
for $j,j'<3$,
- $\mathbf{M}_{\bar{T},\iota}$ is defined as in Definition 2.8 and
- $S=(\omega,\bar{r},\bar{\jmath},\bar{k})$ is a cute $\mathcal{YZR}(\varepsilon)$–system, $0<\varepsilon<\omega_1$.

Definition 3.6. (1) An $(S,\iota,\bar{T},\bar{c})$*–brick* is a tuple
$$\mathfrak{b}=(w^{\mathfrak{b}},n^{\mathfrak{b}},\bar{\eta}^{\mathfrak{b}},\bar{h}^{\mathfrak{b}},\bar{g}^{\mathfrak{b}},\mathcal{M}^{\mathfrak{b}})=(w,n,\bar{\eta},\bar{h},\bar{g},\mathcal{M})$$
such that

$(\boxplus)_1$ $w\in[\omega]^{<\omega}$, $|w|\geq 3$, $0<n<\omega$.

$(\boxplus)_2$ $\bar{\eta}=\langle\eta_a:a\in w\rangle$ is a sequence of linearly independent vectors in ${}^n2$ (over the field $\mathbb{Z}_2$); so in particular $\eta_a\in {}^n2$ are pairwise distinct non-zero sequences (for $a\in w$).

$(\boxplus)_3$ $\bar{h}=\langle h_i:i<\iota\rangle$, where $h_i:w^{\langle 2\rangle}\longrightarrow\omega$, and $c_{h_i(a,b)}\leq n$ for $(a,b)\in w^{\langle 2\rangle}$ and $i<\iota$, and $\bar{g}=\langle g_i:i<\iota\rangle$, where $g_i:w^{\langle 2\rangle}\longrightarrow\bigcup\limits_{m<\omega}(T_m\cap {}^n2)$ for $i<\iota$.

$(\boxplus)_4$ Letting $n^*=n$, $u^*=\{\eta_a:a\in w\}$, $h^*_i(\eta_a,\eta_b)=h_i(a,b)$ and $g^*_i(\eta_a,\eta_b)=g_i(a,b)$ we have $(n^*,u^*,\bar{h}^*,\bar{g}^*)\in\mathbf{M}_{\bar{T},\iota}$.

$(\boxplus)_5$ $\mathcal{M}$ consists of all $\mathbf{m}\in\mathbf{M}_{\bar{T},\iota}$ such that for some ℓ_*,w_* we have

$(\boxplus)_5^{\mathfrak{a}}$ $w_*\subseteq w$, $3\leq|w_*|$, $0<\ell_{\mathbf{m}}=\ell_*\leq n$, and for each $(a,b)\in(w_*)^{\langle 2\rangle}$ and $i<\iota$ we have $c_{h_i(a,b)}\leq\ell_*$,

$(\boxplus)_5^{\mathfrak{b}}$ $u_{\mathbf{m}}=\{\eta_a\restriction\ell_*:a\in w_*\}$ and $\eta_a\restriction\ell_*\neq\eta_b\restriction\ell_*$ for distinct $a,b\in w_*$,

$(\boxplus)_5^{\mathfrak{c}}$ $\bar{h}_{\mathbf{m}}=\langle h^{\mathbf{m}}_i:i<\iota\rangle$, where
$$h^{\mathbf{m}}_i:(u_{\mathbf{m}})^{\langle 2\rangle}\longrightarrow M\omega:(\eta_a\restriction\ell_*,\eta_b\restriction\ell_*)\mapsto h_i(a,b),$$

$(\boxplus)_5^{\mathfrak{d}}$ $\bar{g}_{\mathbf{m}}=\langle g^{\mathbf{m}}_i:i<\iota\rangle$, where
$$g^{\mathbf{m}}_i:(u_{\mathbf{m}})^{\langle 2\rangle}\longrightarrow\bigcup_{m<\omega}(T_m\cap {}^{\ell_*}2):(\eta_a\restriction\ell_*,\eta_b\restriction\ell_*)\mapsto g_i(a,b)\restriction\ell_*$$

In the above situation we will write $\mathbf{m}=\mathbf{m}(\ell_*,w_*)=\mathbf{m}^{\mathfrak{b}}(\ell_*,w_*)$.

$(\boxplus)_6$ If $\mathbf{m}(\ell,w_0),\mathbf{m}(\ell,w_1)\in\mathcal{M}$, $\rho\in {}^\ell 2$ and $\mathbf{m}(\ell,w_0)\doteqdot\mathbf{m}(\ell,w_1)+\rho$, then the order isomorphism $\pi:w_0\longrightarrow w_1$ is a quasi–embedding and $(\eta_a\restriction\ell)+\rho=\eta_{\pi(a)}\restriction\ell$ for all $a\in w_0$.

$(\boxplus)_7$ If $\mathbf{m}(\ell_*,w_*)\in\mathcal{M}$, $a\in w_*$, $|a\cap w_*|=\bar{k}(w_*)$, $\bar{r}(w_*)=0$, and $\mathbf{m}(\ell_*,w_*)\sqsubseteq^*\mathbf{n}\in\mathcal{M}$, then $|\{\nu\in u_{\mathbf{n}}:(\eta_a\restriction\ell_*)\trianglelefteq\nu\}|=1$.

(2) Suppose that $t_m=T_m\cap {}^{n\geq}2$ and $c_m\leq n$ for $m<M<\omega$. Let $\bar{t}=\langle t_m:m<M\rangle$ and $\bar{d}=\bar{c}\restriction M$. An $(S,\iota,\bar{T},\bar{c})$–brick $\mathfrak{b}$ such that $n^{\mathfrak{b}}=n$, $h^{\mathfrak{b}}_i(a,b)<M$ for all $(a,b)\in\big(w^{\mathfrak{b}}\big)^{\langle 2\rangle}$ and $i<\iota$ will be also called an $(S,\iota,\bar{t},\bar{d})$*–brick.*

(3) For bricks $\mathfrak{b}_0,\mathfrak{b}_1$ we write $\mathfrak{b}_0\Subset\mathfrak{b}_1$ if and only if
- $w^{\mathfrak{b}_0}\subseteq w^{\mathfrak{b}_1}$, $n^{\mathfrak{b}_0}\leq n^{\mathfrak{b}_1}$, and
- $\eta^{\mathfrak{b}_0}_a\trianglelefteq\eta^{\mathfrak{b}_1}_a$ for all $a\in w^{\mathfrak{b}_0}$, and
- $h^{\mathfrak{b}_1}_i\restriction(w^{\mathfrak{b}_0})^{\langle 2\rangle}=h^{\mathfrak{b}_0}_i$ and $g^{\mathfrak{b}_0}_i(a,b)\trianglelefteq g^{\mathfrak{b}_1}_i(a,b)$ for $i<\iota$ and $(a,b)\in(w^{\mathfrak{b}_0})^{\langle 2\rangle}$.

Remark 3.7. (1) Note that in $(\boxplus)_5$ of 3.6, the set w_* is not determined uniquely by $\mathbf{m}$ and we may have $\mathbf{m}^{\mathfrak{b}}(\ell,w_0)=\mathbf{m}^{\mathfrak{b}}(\ell,w_1)$ for distinct $w_0,w_1\subseteq w$.

(2) If $w_*\subseteq w^{\mathfrak{b}}$ has at least 3 elements, then $\mathbf{m}^{\mathfrak{b}}(n^{\mathfrak{b}},w_*)\in\mathcal{M}^{\mathfrak{b}}$.

(3) We will use $(S, \iota, \bar{t}, \bar{d})$–bricks for $\bar{t} = \langle t_m : m < M\rangle$ and $\bar{d} = \langle d_m : m < M\rangle$ (see Definition 3.6(2)) even if full $\bar{T}, \bar{c}$ are not defined. In these cases we mean *for* $\bar{T} = \langle T_m : m < \omega\rangle$ *where for* $m < M$ *we have*

$$T_m = \Big\{\nu \in {}^{\omega>}2 : \nu{\restriction}n \in t_m \ \wedge\ \big(\forall k < \ell g(\nu)\big)\big(n \le k \ \Rightarrow\ \nu(k) = 0\big)\Big\}$$

and $T_m = {}^{\omega>}2$ *when* $M \le m < \omega$, *and some* $\bar{c}$ *such that* $c_m = d_m \le n$ *whenever* $m < M$.

Observation 3.8. *Assume* $\mathfrak{b}$ *is an* $(S, \iota, \bar{T}, \bar{c})$*–brick. Then:*

(1) $n^{\mathfrak{b}} \ge |w^{\mathfrak{b}}|$.
(2) *If* $w^* \subseteq w^{\mathfrak{b}}$ *and* $|w^*| \ge 3$ *then there is a unique* $(S, \iota, \bar{T}, \bar{c})$*–brick* $\mathfrak{b}^*$ *such that* $w^{\mathfrak{b}^*} = w^*$, $n^{\mathfrak{b}^*} = n^{\mathfrak{b}}$ *and* $\mathfrak{b}^* \Subset \mathfrak{b}$.
We may write $\mathfrak{b}^* = \mathfrak{b}{\restriction}w^*$ *then.*
(3) *If* $\mathbf{m} = (n^*, u^*, \bar{h}^*, \bar{g}^*)$ *is as given by 3.6*$(\boxplus)_4$, *then* $\mathbf{m} = \mathbf{m}^{\mathfrak{b}}(n^{\mathfrak{b}}, w^{\mathfrak{b}}) \in \mathcal{M}^{\mathfrak{b}}$.
(4) *If* $w_0 \subseteq w$, $\mathbf{m}^{\mathfrak{b}}(\ell, w) \in \mathcal{M}^{\mathfrak{b}}$ *and* $3 \le |w_0|$, *then* $\mathbf{m}^{\mathfrak{b}}(\ell, w_0) \in \mathcal{M}^{\mathfrak{b}}$.
(5) *If* $\varphi : w^{\mathfrak{b}} \longrightarrow \omega$ *is a quasi–embedding (into* S*) then there is a unique* $(S, \iota, \bar{T}, \bar{c})$*–brick* $\mathfrak{b}^*$ *such that*
   - $w^{\mathfrak{b}^*} = \varphi[w^{\mathfrak{b}}]$, $n^{\mathfrak{b}^*} = n^{\mathfrak{b}}$, $\mathcal{M}^{\mathfrak{b}^*} = \mathcal{M}^{\mathfrak{b}}$, *and*
   - $\eta^{\mathfrak{b}}_a = \eta^{\mathfrak{b}^*}_{\varphi(a)}$, $h^{\mathfrak{b}}_i(a, b) = h^{\mathfrak{b}^*}_i(\varphi(a), \varphi(b))$ *and* $g^{\mathfrak{b}}_i(a, b) = g^{\mathfrak{b}^*}_i(\varphi(a), \varphi(b))$ *for all relevant* a, b, i.

   This $\mathfrak{b}^*$ *will be denoted* $\varphi(\mathfrak{b})$.

Definition 3.9. We say that $\bar{T}$ has $(\bar{c}, S)$*–controlled amalgamation property* if there is a sequence $\bar{\mathfrak{b}} = \langle \mathfrak{b}_n : n < \omega\rangle$ of $(S, \iota, \bar{T}, \bar{c})$–bricks such that

(1) $\mathfrak{b}_n \Subset \mathfrak{b}_{n+1}$ for each $n < \omega$,
(2) $\bigcup\limits_{n<\omega} w^{\mathfrak{b}_n} = \omega$ and $\lim\limits_{n\to\infty} n^{\mathfrak{b}_n} = \infty$,
(3) IF
   (a) $n < \omega$, $u \subseteq w \subseteq w^{\mathfrak{b}_n}$, $3 \le |w|$,
   (b) $\bar{k}\big(v \cup \{\delta\}\big) \ne |\delta \cap v|$ whenever $v \subseteq u$ and $\delta \in w \setminus u$ and $\bar{r}\big(v \cup \{\delta\}\big) = 0$,
   (c) $\pi_0, \pi_1 : w \longrightarrow \omega$ are quasi–embeddings (into S) such that $\pi_0(a) = \pi_1(a)$ for $a \in u$ and $\pi_0[w \setminus u] \cap \pi_1[w \setminus u] = \emptyset$,

   THEN there is a $K < \omega$ and a quasi–embedding $\pi : \mathrm{rng}(\pi_0) \cup \mathrm{rng}(\pi_1) \longrightarrow w^{\mathfrak{b}_K}$ (into S) such that

$$(\pi \circ \pi_0)(\mathfrak{b}_n{\restriction}w) \Subset \mathfrak{b}_K{\restriction}(\pi \circ \pi_0[w]) \quad \text{and} \quad (\pi \circ \pi_1)(\mathfrak{b}_n{\restriction}w) \Subset \mathfrak{b}_K{\restriction}(\pi \circ \pi_1[w]),$$

(4) IF
   (a) $n < \omega$, $w \subseteq w^{\mathfrak{b}_n}$, $3 \le |w|$,
   (b) $\pi_0 : w \longrightarrow \omega$ is a quasi–embedding (into S) and $\mathrm{rng}(\pi_0) \subseteq u \in [\omega]^{<\omega}$,

   THEN there is a $K < \omega$ and a quasi–embedding $\pi : u \longrightarrow w^{\mathfrak{b}_K}$ (into S) such that $(\pi \circ \pi_0)(\mathfrak{b}_n{\restriction}w) \Subset \mathfrak{b}_K{\restriction}(\pi \circ \pi_0[w])$.

The name of the $(\bar{c}, S)$–controlled *amalgamation property* comes from the third part of the demand. This demand is taylored to guarantee that if

- $\mathfrak{b}_0, \mathfrak{b}_1$ are $(S, \iota, \bar{T}, \bar{c})$–bricks, $n^{\mathfrak{b}_0} = n^{\mathfrak{b}_1}$, $|w^{\mathfrak{b}_0}| = |w^{\mathfrak{b}_1}|$ and
- the order isomorphism $\pi : w^{\mathfrak{b}_0} \longrightarrow w^{\mathfrak{b}_1}$ is a quasi–embedding (into S), $\pi(a) = a$ for $a \in w^{\mathfrak{b}_0} \cap w^{\mathfrak{b}_1}$, and $\pi(\mathfrak{b}_0) = \mathfrak{b}_1$, and
- $\mathfrak{b}_0, \mathfrak{b}_1$ satisfy a condition ensuring we would not violate demand 3.6(1)$(\boxplus)_7$,

then there is an $(S,\iota,\bar{T},\bar{c})$–brick $\mathfrak{b}$ and a quasi embedding $\pi^*: w^{\mathfrak{b}_0}\cup w^{\mathfrak{b}_1}\longrightarrow\omega$ such that $\pi^*(\mathfrak{b}_0)\Subset\mathfrak{b}$ and $\pi^*(\mathfrak{b}_1)\Subset\mathfrak{b}$. Such $\mathfrak{b}$ may be thought of as an amalgamation of $\mathfrak{b}_0,\mathfrak{b}_1$ over $w^{\mathfrak{b}_0}\cap w^{\mathfrak{b}_1}$.

In the next section we will construct $\bar{T}$ with the $(\bar{c},S)$–controlled amalgamation property. Here we show the main reason to consider such $\bar{T}$ and the associated Σ^0_2 sets.

Theorem 3.10. *Assume that*

(1) $2\leq\iota<\omega$, *and* $\bar{c}=\langle c_m:m<\omega\rangle\subseteq\omega$,
(2) $T_m\subseteq{}^{\omega>}2$ *(for $m<\omega$) are trees with no maximal nodes,* $\bar{T}=\langle T_m:m<\omega\rangle$, *and* $B=\bigcup\limits_{m<\omega}\lim(T_m)$,
(3) $S=(\omega,\bar{r},\bar{\jmath},\bar{k})$ *is a cute* $\mathcal{YZR}(\varepsilon)$*–system,* $0<\varepsilon<\omega_1$,
(4) $\bar{T}$ *has* $(\bar{c},S)$*–controlled amalgamation property , and*
(5) $\mathrm{NPr}^\varepsilon(\lambda)$ *holds true.*

Then there is a ccc forcing notion $\mathbb{P}$ of size λ such that

$$\Vdash_{\mathbb{P}} \text{“ there is a sequence } \langle\eta_\alpha:\alpha<\lambda\rangle \text{ of distinct elements of } {}^\omega 2 \text{ such that } \big|(\eta_\alpha+B)\cap(\eta_\beta+B)\big|\geq 2\iota \text{ for all } \alpha,\beta<\lambda \text{ ”.}$$

Proof. Let a sequence $\bar{\mathfrak{b}}=\langle\mathfrak{b}_n:n<\omega\rangle$ of $(S,\iota,\bar{T},\bar{c})$–bricks witness the $(\bar{c},S)$–controlled amalgamation property for $\bar{T}$.

First, let us assume that λ is uncountable (we will comment on the countable case at the end of the proof). Let $\mathbb{M}=\big(\lambda,\{R^{\mathbb{M}}_{n,j}\}_{n,j<\omega}\big)$ be the model fixed in Definition 2.6, let $\mathrm{rk}^{\rm sp}$ be the associated rank and let $\mathbf{j},\mathbf{k}:[\lambda]^{<\omega}\setminus\{\emptyset\}\longrightarrow\omega$ be the "witness functions" fixed there.

A condition in $\mathbb{P}$ is a tuple

$$p=(u^p,n^p,\bar{\eta}^p,\bar{h}^p,\bar{g}^p)$$

such that $u^p\in[\lambda]^{<\omega}$, $3\leq|u^p|$ and for some quasi–embedding $\varphi:u^p\longrightarrow\omega$ of the system $s(u^p)$ associated with u^p (see Definition 3.2) into S and for some $N<\omega$ we have

- $\varphi[u^p]\subseteq w^{\mathfrak{b}_N}$, $n^p\leq n^{\mathfrak{b}_N}$, $\mathbf{m}^{\mathfrak{b}_N}(n^p,\varphi^p[u])\in\mathcal{M}^{\mathfrak{b}_N}$,
- $\bar{\eta}^p=\langle\eta^p_\alpha:\alpha\in u^p\rangle$ and $\eta^p_\alpha=\eta^{\mathfrak{b}_N}_{\varphi(\alpha)}{\restriction}n^p$,
- $\bar{h}^p=\langle h^p_i:i<\iota\rangle$, where
- $h^p_i:\big(u^p\big)^{\langle 2\rangle}\longrightarrow\omega$ are such that $h^p_i(\alpha,\beta)=h^{\mathfrak{b}_N}_i(\varphi(\alpha),\varphi(\beta))$,
- $\bar{g}^p=\langle g^p_i:i<\iota\rangle$, where $g^p_i:\big(u^p\big)^{\langle 2\rangle}\longrightarrow\bigcup\limits_{m<\omega}(T_m\cap{}^{n^p}2)$ are such that $g^p_i(\alpha,\beta)=g^{\mathfrak{b}_N}_i(\varphi(\alpha),\varphi(\beta)){\restriction}n_p$.

(For φ and N as above we say that they *witness* $p\in\mathbb{P}$.)

A condition $q\in\mathbb{P}$ *is stronger than* $p\in\mathbb{P}$ ($p\leq q$ in short) if and only if

- $u^p\subseteq u^q$, $n^p\leq n^q$, and
- $\eta^p_\alpha\trianglelefteq\eta^q_\alpha$ for all $\alpha\in u^p$, and
- $h^q_i{\restriction}(u^p)^{\langle 2\rangle}=h^p_i$ and $g^p_i(\alpha,\beta)\trianglelefteq g^q_i(\alpha,\beta)$ for $i<\iota$ and $(\alpha,\beta)\in(u^p)^{\langle 2\rangle}$.

Clearly, $(\mathbb{P},\leq)$ is a partial order of size λ.

Claim 3.10.1. (1) *Suppose that $u\subseteq\lambda$ is a finite set with at least 3 elements and $\varphi:u\longrightarrow\omega$ is a quasi–embedding of $s(u)$ into S. Assume $\varphi[u]\subseteq w^{\mathfrak{b}_N}$ and $n\leq n^{\mathfrak{b}_N}$ is such that $\mathbf{m}^{\mathfrak{b}_N}(n,\varphi[u])\in\mathcal{M}^{\mathfrak{b}_N}$. Then there is a unique condition $p=p(n,\varphi,N)\in\mathbb{P}$ such that $n^p=n$ and φ and N witness $p\in\mathbb{P}$.*

(2) *Assume that* $\emptyset \neq u_0 \subseteq u_1 \subseteq \lambda$, u_1 *finite, and* $\varphi : u_1 \longrightarrow \omega$ *is a quasi–embedding into* S. *Suppose* n_0, n_1, K_0, K_1 *are such that* $p(n_0, \varphi{\restriction}u_0, K_0)$ *and* $p(n_1, \varphi, K_1)$ *are well defined and* $n_0 \leq n_1$, $K_0 \leq K_1$. *Then* $p(n_0, \varphi{\restriction}u_0, K_0) \leq p(n_1, \varphi, K_1)$.

Claim 3.10.2. $\mathbb{P}$ *has the Knaster property.*

Proof of the Claim. Suppose that $\langle p_\xi : \xi < \omega_1 \rangle$ is a sequence of pairwise distinct conditions from $\mathbb{P}$ and let

$$p_\xi = \big(u^\xi, n^\xi, \bar{\eta}^\xi, \bar{h}^\xi, \bar{g}^\xi\big)$$

where $\bar{\eta}^\xi = \langle \eta^\xi_\alpha : \alpha \in u^\xi \rangle$, $\bar{h}^\xi = \langle h^\xi_i : i < \iota \rangle$, and $\bar{g}^\xi = \langle g^\xi_i : i < \iota \rangle$. Let φ_ξ and N_ξ witness $p_\xi \in \mathbb{P}$.

Use the standard Δ–system cleaning procedure to find an uncountable set $A \subseteq \omega_1$ such that the following demands $(\oplus)_1$–$(\oplus)_4$ are satisfied.

$(\oplus)_1$ $\{u^\xi : \xi \in A\}$ forms a Δ–system with kernel u.

$(\oplus)_2$ If $\xi, \varsigma \in A$, then $|u^\xi| = |u^\varsigma|$ and $n^\xi = n^\varsigma$.

$(\oplus)_3$ If $\xi < \varsigma$ are from A and $\pi : u^\xi \longrightarrow u^\varsigma$ is the order isomorphism, then
  (a) $\pi(\alpha) = \alpha$ for $\alpha \in u^\xi \cap u^\varsigma$,
  (b) if $\emptyset \neq v \subseteq u^\xi$, then $\mathrm{rk}^{\rm sp}(v) = \mathrm{rk}^{\rm sp}(\pi[v])$, $\mathbf{j}(v) = \mathbf{j}(\pi[v])$ and $\mathbf{k}(v) = \mathbf{k}(\pi[v])$,
  (c) $\eta^\xi_\alpha = \eta^\varsigma_{\pi(\alpha)}$ (for $\alpha \in w_\xi$),
  (d) $g_i(\alpha, \beta) = g_i(\pi(\alpha), \pi(\beta))$ and $h_i(\alpha, \beta) = h_i(\pi(\alpha), \pi(\beta))$ for $(\alpha, \beta) \in (w_\xi)^{\langle 2 \rangle}$ and $i < \iota$,

and

$(\oplus)_4$ $\mathrm{rng}(\varphi_\xi) = \mathrm{rng}(\varphi_\varsigma) = w$ and $N_\xi = N_\varsigma = N$ for $\xi, \varsigma \in A$.

Note that then also

$(\oplus)_5$ if $\xi \in A$, $v \subseteq u$ and $\delta \in u^\xi \setminus u$ are such that $\mathrm{rk}^{\rm sp}\big(v \cup \{\delta\}\big) = -1$, then $\mathbf{k}\big(v \cup \{\delta\}\big) \neq |\delta \cap v|$.

[Why? Suppose $\mathrm{rk}^{\rm sp}\big(v \cup \{\delta\}\big) = -1$ and $k = \mathbf{k}\big(v \cup \{\delta\}\big) = |\delta \cap v|$, $j = \mathbf{j}\big(v \cup \{\delta\}\big)$. For $\varsigma \in A$ let $\pi_\varsigma : u^\xi \longrightarrow u^\varsigma$ be the order isomorphism and let $\delta_\varsigma = \pi_\varsigma(\delta)$. By $(\oplus)_3$ we know that $k = \mathbf{k}\big(v \cup \{\delta_\varsigma\}\big) = |\delta_\varsigma \cap v|$ and $j = \mathbf{j}\big(v \cup \{\delta_\varsigma\}\big)$. Therefore, letting $v \cup \{\delta\} = \{a_0, \ldots, a_{n-1}\}$ be the increasing enumeration, for every $\varsigma \in A$ we have $\mathbb{M} \models R_{n,j}[a_0, \ldots, a_{k-1}, \delta_\varsigma, a_{k+1}, \ldots, a_{n-1}]$. Hence the set

$$\{b < \lambda : \mathbb{M} \models R_{n,j}[a_0, \ldots, a_{k_1}, b, a_{k+1}, \ldots, a_{n-1}]\}$$

is uncountable, contradicting $(\circledast)_{\mathrm{f}}$ of 2.6.]

Let us argue that for distinct ξ, ς from A the conditions p_ξ, p_ς are compatible. So let $\xi, \varsigma \in A$, $\xi < \varsigma$. Let $v^* = u^\xi \cup u^\varsigma$ and let $s(v^*)$ be the finite $\mathcal{YZR}(\varepsilon)$–system associated with v^* (see Definition 3.2). Since S is cute, it includes a copy of $s(v^*)$, so there is a quasi–embedding $\psi : v^* \longrightarrow \omega$ of $s(v^*)$ into S. Then, remembering $(\oplus)_4$, we may choose two quasi–embeddings $\pi_0, \pi_1 : w \longrightarrow \mathrm{rng}(\psi)$ such that $\pi_0(\alpha) = \pi_1(\alpha)$ for $\alpha \in \varphi_\xi[u]$ and $\pi_0 \circ \varphi_\xi = \psi{\restriction}u^\xi$ and $\pi_1 \circ \varphi_\varsigma = \psi{\restriction}u^\varsigma$. Apply Definiton 3.9(3) to $N, \varphi_\xi[u], w, \pi_0, \pi_1$ to choose K and a quasi–embedding $\pi : \mathrm{rng}(\psi) \longrightarrow w^{\mathfrak{b}_K}$ such that

$$(\pi \circ \pi_0)(\mathfrak{b}_N{\restriction}w) \Subset \mathfrak{b}_K{\restriction}(\pi \circ \pi_0[w]) \quad \text{and} \quad (\pi \circ \pi_1)(\mathfrak{b}_N{\restriction}w) \Subset \mathfrak{b}_K{\restriction}(\pi \circ \pi_1[w]).$$

Then the condition $p(n^{\mathfrak{b}_K}, \pi \circ \psi, K)$ is a common upper bound of p_ξ, p_ς (remember Claim 3.10.1(2)). $\square$

Claim 3.10.3. *The following sets are open dense in $\mathbb{P}$:*
$D_\alpha = \{p \in \mathbb{P} : \alpha \in u^p\}$ *for* $\alpha < \lambda$, *and*
$D^n = \{p \in \mathbb{P} : n^p > n\}$ *for* $n < \omega$.

Proof of the Claim. To show the density of D_α suppose that $p \in \mathbb{P}$ and $\alpha \in \lambda \setminus u^p$. Let a quasi-embedding $\varphi : u^p \longrightarrow \omega$ and $N < \omega$ witness $p \in \mathbb{P}$. Let $w = \mathrm{rng}(\varphi)$. Put $v^* = u^p \cup \{\alpha\}$ and let $s(v^*)$ be the finite $\mathcal{YZR}(\varepsilon)$–system associated with v^*. Since S is cute, it includes a copy of $s(v^*)$, so there is a quasi–embedding $\psi : v^* \longrightarrow \omega$ of $s(v^*)$ into S. Applying 3.9(4) to w, $\pi_0 = \psi \circ \varphi^{-1}$ and $u = \mathrm{rng}(\psi)$ we may find $K < \omega$ and a quasi–embedding $\pi : u \longrightarrow w^{\mathfrak{b}_K}$ such that

$$(\pi \circ \pi_0)\big(\mathfrak{b}_N{\restriction}w\big) \Subset \mathfrak{b}_k{\restriction}(\pi \circ \pi_0[w]).$$

Then the condition $p(n^{\mathfrak{b}_K}, \pi \circ \psi, K)$ belongs to D_α and it is stronger than p.

To argue that D^n is dense (for $n < \omega$) suppose that $p \in \mathbb{P}$ and $n^p \leq n$. Let $\varphi : u^p \longrightarrow \omega$ and $N < \omega$ witness that $p \in \mathbb{P}$. Let $M > n$ be such that $n^{\mathfrak{b}_M} > n$. Then the condition $p(n^{g\mathfrak{b}_M}, \varphi, M)$ belongs to D^n and it is stronger than p. □

Now, for $(\alpha, \beta) \in \lambda^{\langle 2 \rangle}$ we define $\mathbb{P}$–names $\underset{\sim}{\eta}_\alpha$ and $\underset{\sim}{g}_i(\alpha, \beta)$ by

$$\Vdash_{\mathbb{P}} \text{“}\underset{\sim}{\eta}_\alpha = \bigcup\{\eta^p_\alpha : \alpha \in u^p \wedge p \in \underset{\sim}{G}\} \text{ and } \underset{\sim}{g}_i(\alpha,\beta) = \bigcup\{g^p_i(\alpha,\beta) : \alpha,\beta \in u^p \wedge p \in \underset{\sim}{G}\}\text{”}.$$

By the definition of the order of $\mathbb{P}$ and by Claim 3.10.3 we easily see that

$$\begin{aligned} \Vdash_{\mathbb{P}} \quad \text{“ } & \langle \underset{\sim}{\eta}_\alpha : \alpha < \lambda \rangle \subseteq {}^{\omega}2 \text{ are pairwise distinct,} \\ & \underset{\sim}{g}_i(\alpha,\beta) \in \bigcup_{m<\omega} \lim(T_m) \text{ for } (\alpha,\beta) \in \lambda^{\langle 2\rangle},\ i < \iota, \\ & \underset{\sim}{\eta}_\alpha + \underset{\sim}{\eta}_\beta = \underset{\sim}{g}_i(\alpha,\beta) + \underset{\sim}{g}_i(\beta,\alpha) \text{ for } (\alpha,\beta) \in \lambda^{\langle 2\rangle},\ i < \iota\text{”}. \end{aligned}$$

Hence $\mathbb{P}$ is as required. □

4. Existence of Σ^0_2 sets with the amalgamation property

Here we will prove our main result: there exists $\bar{T}$ with the amalgamation property (over a cute $\mathcal{YZR}(\varepsilon)$–system) and with the nondisjointness rank ndrk_ι bounded by
omega $\cdot$ (*vare* + 2) + 2. For this $\bar{T}$ (or rather $\bigcup\limits_{m<\omega} \lim(T_m)$) we may force many 2ι–non-disjoint translations without adding a perfect set of such translations.

Definition 4.1. Assume that $\bar{T}^*, \bar{t}, M, \iota, n$ are such that

- $2 \leq \iota < \omega$, and $M, n < \omega$, and
- $\bar{t} = \langle t_m : m < M \rangle$ where each $t_m \subseteq {}^{n\geq}2$ is a tree with maximal nodes of length n (for $m < M$).

Let $\bar{T}^* = \langle T^*_m : m < \omega \rangle$ where for $m < M$ we have

$$T^*_m = \Big\{\nu \in {}^{\omega>}2 : \nu{\restriction}n \in t_m \ \wedge\ \big(\forall k < \ell g(\nu)\big)\big(n \leq k \ \Rightarrow\ \nu(k) = 0\big)\Big\}$$

and $T^*_m = {}^{\omega>}2$ when $M \leq m < \omega$.

(1) We say that $\bar{t}$ is (M, ι, n)–*usable* if, letting $B = \bigcup\limits_{m<M} \lim(T^*_m)$ [*sic*], there are pairwise different $\rho_0, \rho_1, \rho_2 \in {}^{\omega}2$ such that

$$\big|\big(\rho_j + B\big) \cap \big(\rho_{j'} + B\big)\big| \geq 2\iota$$

for $j, j' < 3$.

(2) We define $\mathbf{M}^n_{\bar t,\iota}$ as the set of all tuples $\mathbf{m} = (\ell_{\mathbf{m}}, u_{\mathbf{m}}, \bar h_{\mathbf{m}}, \bar g_{\mathbf{m}}) \in \mathbf{M}_{\bar T^*,\iota}$ such that $\ell_{\mathbf{m}} \le n$ and $\mathrm{rng}(h^{\mathbf{m}}_i) \subseteq M$ for each $i < \iota$. (Remember, $\mathbf{M}_{\bar T^*,\iota}$ was defined in Definition 2.8 (for $\bar T^*,\iota$)).

Observation 4.2. *If $\mathbf{m} \in \mathbf{M}^n_{\bar t,\iota}$ and $\rho \in {}^{\ell_{\mathbf{m}}}2$, then $\mathbf{m} + \rho \in \mathbf{M}^n_{\bar t,\iota}$ (see Definition 2.9).*

Lemma 4.3 (See [11, Lemma 2.3]). *Let $0 < \ell < \omega$ and let $\mathcal{B} \subseteq {}^{\ell}2$ be a linearly independent set of vectors (in $({}^{\ell}2, +)$ over $(2, +_2, \cdot_2)$). If $\mathcal{A} \subseteq {}^{\ell}2$, $|\mathcal{A}| \ge 5$ and $\mathcal{A} + \mathcal{A} \subseteq \mathcal{B} + \mathcal{B}$, then for a unique $x \in {}^{\ell}2$ we have $\mathcal{A} + x \subseteq \mathcal{B}$.*

Theorem 4.4. *Assume $0 < \varepsilon < \omega_1$ and let $2 \le \iota < \omega$. Let $S = (\omega, \bar r, \bar{\bar j}, \bar k)$ be a cute $\mathcal{YZR}(\varepsilon)$–system. Then there is a sequence $\bar T = \langle T_m : m < \omega\rangle$ of trees $T_m \subseteq {}^{\omega>}2$ without maximal nodes and a sequence $\bar c = \langle c_m : m < \omega\rangle$ of integers such that*

1. *$\bar T$ has $(\bar c, S)$–controlled amalgamation property, and*
2. *$\varepsilon \le \mathrm{ndrk}_\iota(\bar T) \le \omega \cdot (\varepsilon + 2) + 2$ (the ordinal multiplication).*

Proof. We will mix the forcing construction of [13] with the arguments of [11], getting our result for all $\iota \ge 2$. Let $\mathcal{P}_\iota$ be the collection of all tuples

$$p = \big(w^p, n^p, M^p, \bar\eta^p, \bar t^p, \bar d^p, \bar h^p, \bar g^p, \mathcal{M}^p, \bar\rho^p\big) = \big(w, n, M, \bar\eta, \bar t, \bar d, \bar h, \bar g, \mathcal{M}, \bar\rho\big)$$

such that the following demands $(\boxtimes)_1$–$(\boxtimes)_7$ are satisfied.

$(\boxtimes)_1$ $w \in [\omega]^{<\omega}$, $|w| \ge 3$, $0 < n, M < \omega$.

$(\boxtimes)_2$ $\bar t = \langle t_m : m < M\rangle$ is (M, ι, n)–usable, so in particular $\emptyset \ne t_m \subseteq {}^{n\ge}2$ (for $m < M$) is a tree in which all terminal branches are of length n.

$(\boxtimes)_3$ $\bar d = \langle d_m : m < M\rangle$, where $0 < d_m \le n$ for $m < M$.

$(\boxtimes)_4$ $\mathfrak{b}(p) = (w^p, n^p, \bar\eta^p, \bar h^p, \bar g^p, \mathcal{M}^p)$ is an $(S, \iota, \bar t, \bar d)$–brick (cf Definition 3.6(2) and Remark 3.7(3)).

$(\boxtimes)_5$ $\bar\rho = \langle \rho_{i,a,b} : i < \iota,\ a, b \in w,\ a < b\rangle \subseteq {}^n2$ and

$$g_i(a, b) = \eta_a + \rho_{i,a,b} \text{ and } g_i(b, a) = \eta_b + \rho_{i,a,b}$$

whenever $a < b$ are from w and $i < \iota$.

$(\boxtimes)_6$ the list

$$\bar\eta{}^\frown\bar\rho = \langle \eta_a : a \in w\rangle{}^\frown\langle \rho_{i,a,b} : i < \iota,\ a, b \in w,\ a < b\rangle$$

is a list of linearly independent vectors (in $({}^n2, +, \cdot)$ over $(2, +_2, \cdot_2)$); in particular they are pairwise distinct,

$(\boxtimes)_7$ if $m < M$ then

$$t_m \cap {}^n2 \subseteq \{g_i(a, b) : (a, b) \in w^{\langle 2\rangle}\ \wedge\ i < \iota\},$$

and $t_m \cap t_{m'} \cap {}^n2 = \emptyset$ whenever $m < m' < M$.

For $p, q \in \mathcal{P}_\iota$ we declare that $p \preccurlyeq q$ if and only if

- $n^p \le n^q$, $M^p \le M^q$, and $t^p_m = t^q_m \cap {}^{n^p\ge}2$ and $d^p_m = d^q_m$ for all $m < M^p$, and
- $\mathfrak{b}(p) \Subset \mathfrak{b}(q)$.

It is straightforward to verify that $(\mathcal{P}_\iota, \preccurlyeq)$ is a nonempty partial order.

Claim 4.4.1. *Assume $p = \big(w, n, M, \bar\eta, \bar t, \bar d, \bar h, \bar g, \mathcal{M}, \bar\rho\big) \in \mathcal{P}_\iota$. Suppose that $\nu^0_i, \nu^1_i \in \bigcup\limits_{m<M} (t_m \cap {}^n2)$ (for $i < \iota$) are such that*

(a) *there are no repetitions in $\langle \nu^0_i, \nu^1_i : i < \iota\rangle$, and*

(b) $\nu^0_i+\nu^1_i=\nu^0_j+\nu^1_j$ *for* $i<j<\iota$.

Then

(A) *if* $\iota\geq 3$ *then for some* $a,b\in w$ *we have*
$$\big\{\{\nu^0_i,\nu^1_i\}:i<\iota\big\}=\big\{\{g_i(a,b),g_i(b,a)\}:i<\iota\big\}.$$
(B) *If* $\iota=2$ *then for some* $a,b\in w$ *we have*
$$\big\{\nu^0_0,\nu^1_0,\nu^0_1,\nu^1_1\big\}=\big\{g_0(a,b),g_0(b,a),g_1(a,b),g_1(b,a)\big\}.$$

Proof of the Claim. For $a>b$ from w and $i<\iota$ we will write $\rho_{i,a,b}$ for $\rho_{i,b,a}$. With this notation, all elements of $\bigcup\limits_{m<M}(t_m\cap{}^n2)$ are of the form $\eta_a+\rho_{i,a,b}$ for some $i<\iota$ and $(a,b)\in w^{\langle 2\rangle}$.

Let $i<j<\iota$ and let $a,b,c,d,a',b',c',d',i_0,i_1,i'_0,i'_1$ be such that $\nu^0_i=\eta_a+\rho_{i_0,a,b}$, $\nu^1_i=\eta_c+\rho_{i_1,c,d}$, $\nu^0_j=\eta_{a'}+\rho_{i'_0,a',b'}$, $\nu^1_j=\eta_{c'}+\rho_{i'_1,c',d'}$. Since $\nu^0_i+\nu^1_i=\nu^0_j+\nu^1_j$ we have then
$$\eta_a+\rho_{i_0,a,b}+\eta_c+\rho_{i_1,c,d}=\eta_{a'}+\rho_{i'_0,a',b'}+\eta_{c'}+\rho_{i'_1,c',d'}.$$
If $a\neq c$ then it follows from $(\boxtimes)_6$ that $\{a,c\}=\{a',c'\}$ (so either $a=a'$, $c=c'$ or $a=c'$, $c=a'$), and
$$\rho_{i_0,a,b}+\rho_{i_1,c,d}=\rho_{i'_0,a',b'}+\rho_{i'_1,c',d'}.$$
Then, still assuming $a\neq c$, we consider relationships among ρ's above getting four possible subcases.

If $\rho_{i_0,a,b}=\rho_{i_1,c,d}$ then $a\in\{a,b\}=\{c,d\}\ni c$ and also $\rho_{i'_0,a',b'}=\rho_{i'_1,c',d'}$ so $a'\in\{a',b'\}=\{c',d'\}\ni c'$. Moreover, $i_0=i_1$ and $i'_0=i'_1$. Thus, remembering that $\{a,c\}=\{a',c'\}$, we get in this case:

$(\rightrightarrows)^{a,c}_{i,j}$ if $a=a'$ and $c=c'$, then also $c=b=b'$, $a=d=d'$ and
$$\begin{array}{ll}\nu^0_i=\eta_a+\rho_{i_0,a,c}, & \nu^1_i=\eta_c+\rho_{i_0,a,c},\\ \nu^0_j=\eta_a+\rho_{i'_0,a,c}, & \nu^1_j=\eta_c+\rho_{i'_0,a,c},\end{array}$$
if $a=c'$ and $c=a'$, then also $a=d=b'$ and $c=d'=b$ and
$$\begin{array}{ll}\nu^0_i=\eta_a+\rho_{i_0,a,c}, & \nu^1_i=\eta_c+\rho_{i_0,a,c},\\ \nu^0_j=\eta_c+\rho_{i'_0,a,c}, & \nu^1_j=\eta_a+\rho_{i'_0,a,c}.\end{array}$$
If $\rho_{i_0,a,b}\neq\rho_{i_1,c,d}$ then $\{\rho_{i_0,a,b},\rho_{i_1,c,d}\}=\{\rho_{i'_0,a',b'},\rho_{i'_1,c',d'}\}$ and analysis as above provides that there are only two possible cases.

$(\downdownarrows)^{a,c}_{i,j}$ If $\rho_{i_0,a,b}=\rho_{i'_0,a',b'}$ then we must also have $a=c'$ and
$$\begin{array}{ll}\nu^0_i=\eta_a+\rho_{i_0,a,c}, & \nu^1_i=\eta_c+\rho_{i_1,a,c},\\ \nu^0_j=\eta_c+\rho_{i_0,a,c}, & \nu^1_j=\eta_a+\rho_{i_1,a,c}.\end{array}$$
$(\swarrow\!\!\!\!\searrow)^{a,c}_{i,j}$ If $\rho_{i_0,a,b}=\rho_{i'_1,c',d'}$ then we must also have $a=a'$ and
$$\begin{array}{ll}\nu^0_i=\eta_a+\rho_{i_0,a,c}, & \nu^1_i=\eta_c+\rho_{i_1,a,c},\\ \nu^0_j=\eta_a+\rho_{i_1,a,c}, & \nu^1_j=\eta_c+\rho_{i_0,a,c}.\end{array}$$
Now about what happens if $a=c$ (and $a'=c'$). We easily eliminate the possibility of $\rho_{i_0,a,b}=\rho_{i_1,c,d}$. Considering all other options we get the following.

$(\downdownarrows)^{a,a'}_{i,j}$ If $\rho_{i_0,a,b}=\rho_{i'_0,a',b'}$ then
$$\begin{array}{ll}\nu^0_i=\eta_a+\rho_{i_0,a,a'}, & \nu^1_i=\eta_a+\rho_{i_1,a,a'},\\ \nu^0_j=\eta_{a'}+\rho_{i_0,a,a'}, & \nu^1_j=\eta_{a'}+\rho_{i_1,a,a'}.\end{array}$$

$(\bowtie)^{a,a'}_{i,j}$ If $\rho_{i_0,a,b} = \rho_{i'_1,c',d'}$ then

$$\nu^0_i = \eta_a + \rho_{i_0,a,a'}, \qquad \nu^1_i = \eta_a + \rho_{i_1,a,a'},$$
$$\nu^0_j = \eta_{a'} + \rho_{i_1,a,a'}, \qquad \nu^1_j = \eta_{a'} + \rho_{i_0,a,a'}.$$

Thus we see that, for each $i < j < \iota$ we have

$(\heartsuit)_{i,j}$ there are $a < b$ from w and $i_0, i_1 < \iota$ such that

$$\{\nu^0_i, \nu^1_i, \nu^0_j, \nu^1_j\} = \{g_{i_0}(a,b), g_{i_0}(b,a), g_{i_1}(a,b), g_{i_1}(b,a)\}.$$

This immediately gives us the assertion of (B). If $\iota \geq 3$ then considering triples $i < j < k < \iota$ and $(\heartsuit)_{i,j} + (\heartsuit)_{i,k} + (\heartsuit)_{k,j}$ we get from the linear independence declared in $(\boxtimes)_6$ that

$(\heartsuit)^+$ for some $a < b$ from w, for every $i < \iota$ we have

$$\nu^0_i, \nu^1_i \in \{g_j(a,b), g_j(b,a) : j < \iota\} = \{\eta_a + \rho_{j,a,b}, \eta_b + \rho_{j,a,b} : j < \iota\}.$$

By the same linear independence,

- the sum $g_{i_0}(a,b) + g_{j_0}(b,a)$ (where $i_0 \neq j_0$) can be equal to only one other sum of two elements of $\{g_j(a,b), g_j(b,a) : j < \iota\}$, namely $g_{i_0}(b,a) + g_{j_0}(a,b)$,
- the sum $g_{i_0}(a,b) + g_{j_0}(a,b)$ (for $i_0 \neq j_0$) can be equal to only one other sum of two elements of $\{g_j(a,b), g_j(b,a) : j < \iota\}$, namely $g_{i_0}(b,a) + g_{j_0}(b,a)$.

Therefore, if $\iota \geq 3$ then for a, b given by $(\heartsuit)^+$,

$$\{\{\nu^0_i, \nu^1_i\} : i < \iota\} = \{\{g_j(a,b), g_j(b,a)\} : j < \iota\}$$

and the assertion of (A) follows. $\square$

Claim 4.4.2. *Let* $p = (w^p, n^p, M^p, \bar{\eta}^p, \bar{t}^p, \bar{d}^p, \bar{h}^p, \bar{g}^p, \mathcal{M}^p, \bar{\rho}^p) \in \mathcal{P}_\iota$. *Assume that* $\mathbf{m} \in \mathbf{M}^{n^p}_{\bar{t}^p,\iota}$ *is such that*

(i) $|u_{\mathbf{m}}| \geq 5$, *and*

(ii) $d_{h^{\mathbf{m}}_i(\eta,\nu)} \leq \ell_{\mathbf{m}}$ *for all* $(\eta,\nu) \in (u_{\mathbf{m}})^{\langle 2 \rangle}$ *and* $i < \iota$.

Then for some $\rho \in {}^{\ell_{\mathbf{m}}}2$ *and* $\mathbf{n} \in \mathcal{M}^p$ *we have* $(\mathbf{m} + \rho) \doteqdot \mathbf{n}$.

Proof of the Claim. Suppose $\mathbf{m} \in \mathbf{M}^{n^p}_{\bar{t}^p,\iota}$ satisfies (i) and (ii). By Definitions 4.1(2) + 2.8(f) there is $\mathbf{m}^+ \in \mathbf{M}^{n^p}_{\bar{t}^p,\iota}$ such that $\mathbf{m} \sqsubseteq \mathbf{m}^+$, $\ell_{\mathbf{m}^+} = n^p$ and $|u_{\mathbf{m}^+}| = |u_{\mathbf{m}}|$. If $(\eta,\nu) \in (u_{\mathbf{m}^+})^{\langle 2 \rangle}$ then for all $i < \iota$:

$$g^{\mathbf{m}^+}_i(\nu,\eta), g^{\mathbf{m}^+}_i(\eta,\nu) \in \bigcup_{m < M^p} t^p_m \cap {}^{n^p}2 \quad \text{and} \quad g^{\mathbf{m}^+}_i(\nu,\eta) + g^{\mathbf{m}^+}_i(\eta,\nu) = \nu + \eta.$$

Let us consider **the case when** $\iota \geq 3$. Then by Claim 4.4.1(A), for every $(\eta,\nu) \in (u_{\mathbf{m}^+})^{\langle 2 \rangle}$ there are $a < b$ from w^p such that

$(*)^{\nu,\eta}_{a,b}$ $\quad \{\{g^{\mathbf{m}^+}_i(\nu,\eta), g^{\mathbf{m}^+}_i(\eta,\nu)\} : i < \iota\} = \{\{g^p_i(a,b), g^p_i(b,a)\} : i < \iota\}.$

In particular, $\eta + \nu = \eta^p_a + \eta^p_b$. Using Lemma 4.3 we may conclude that for some $x \in {}^{n^p}2$ we have $u_{\mathbf{m}^+} + x \subseteq \{\eta^p_c : c \in w^p\}$. The linear independence of η^p_c's implies that if $\eta, \nu \in (u_{\mathbf{m}^+})^{\langle 2 \rangle}$ and $(*)^{\nu,\eta}_{a,b}$ holds, then $\{\eta + x, \nu + x\} = \{\eta^p_a, \eta^p_b\}$. By $(\boxplus)_7$, $g^p_i(a,b)$ ($g^{\mathbf{m}^+}_i(\nu,\eta)$, respectively) determines $h^p_i(a,b)$ ($h^{\mathbf{m}^+}_i(\nu,\eta)$, respectively). So easily $(\mathbf{m}^+ + x) \doteqdot \mathbf{n}^+$ for some $\mathbf{n}^+ \in \mathcal{M}^p$ and

$$\mathbf{m} = \mathbf{m}^+ \restriction \ell_{\mathbf{m}} \doteqdot (\mathbf{n}^+ \restriction \ell_{\mathbf{m}}) + x \restriction \ell_{\mathbf{m}}$$

and $\mathbf{n}^+ \restriction \ell_{\mathbf{m}} \in \mathcal{M}^p$ (remember assumption (ii) for $\mathbf{m}$).

Now, consider **the case when** $\iota = 2$. By Claim 4.4.1(B), we know that for every $(\eta, \nu) \in (u_{\mathbf{m}^+})^{\langle 2 \rangle}$

$(**)^{\nu,\eta}$ there are $a < b$ from w^p such that

$$\big\{g_0^{\mathbf{m}^+}(\nu,\eta), g_0^{\mathbf{m}^+}(\eta,\nu), g_1^{\mathbf{m}^+}(\nu,\eta), g_1^{\mathbf{m}^+}(\eta,\nu)\big\} = \big\{g_0^p(a,b), g_0^p(b,a), g_1^p(a,b), g_1^p(b,a)\big\}.$$

Define functions $\chi : [u_{\mathbf{m}^+}]^2 \longrightarrow 2$ and $\Theta : [u_{\mathbf{m}^+}]^2 \longrightarrow [w^p]^2$ as follows. Suppose $\{\eta, \nu\} \in u_{\mathbf{m}^+}$.

- If $\eta + \nu = \eta_a^p + \eta_b^p$, $a, b \in w^p$, then $\chi(\{\eta,\nu\}) = 1$ and $\Theta(\{\eta,\nu\}) = \{a,b\}$.
- If $\eta + \nu = \eta_a^p + \eta_b^p + \rho_{0,a,b}^p + \rho_{1,a,b}^p$, $a, b \in w^p$, then $\chi(\{\eta,\nu\}) = 0$ and $\Theta(\{\eta,\nu\}) = \{a,b\}$.
- If $\eta + \nu = \rho_{0,a,b}^p + \rho_{1,a,b}^p$, $a, b \in w^p$, then $\chi(\{\eta,\nu\}) = 0$ and $\Theta(\{\eta,\nu\}) = \{a,b\}$.

It follows from $(**)^{\nu,\eta}$ and the linear independence of $\bar\eta^p{}^\frown\bar\rho^p$ (see $(\boxtimes)_6$) that exactly one of the cases described above holds for $\eta + \nu$.

$(***)_1$ If $\eta_0, \eta_1, \eta_2 \in u_{\mathbf{m}^+}$ are pairwise distinct and $\chi(\{\eta_0,\eta_1\}) = \chi(\{\eta_1,\eta_2\}) = 1$, then $\Theta(\{\eta_0,\eta_1\}) \neq \Theta(\{\eta_1,\eta_2\})$ and $\chi(\{\eta_0,\eta_2\}) = 1$.

Why? Assume $\chi(\{\eta_0,\eta_1\}) = \chi(\{\eta_1,\eta_2\}) = 1$. Then both $\eta_0 + \eta_1$ and $\eta_1 + \eta_2$ are sums of two elements of $\{\eta_c^p : c \in w^p\}$. Hence $\eta_0 + \eta_2$ is a sum of some elements of $\{\eta_c^p : c \in w^p\}$ and therefore $\chi(\{\eta_0,\eta_2\}) \neq 0$ (as the terms of $\bar\eta^p{}^\frown\bar\rho^p$ are linearly independent). Now, if we had $\Theta(\{\eta_0,\eta_1\}) = \Theta(\{\eta_1,\eta_2\}) = \{a,b\}$, then

$$\eta_0 + \eta_1 = \eta_a^p + \eta_b^p = \eta_1 + \eta_2$$

and hence $\eta_0 = \eta_2$, a contradiction.

$(***)_2$ If $\eta_0, \eta_1, \eta_2 \in u_{\mathbf{m}^+}$ are pairwise distinct and $\chi(\{\eta_0,\eta_1\}) = \chi(\{\eta_0,\eta_2\}) = 0$, then $\Theta(\{\eta_0,\eta_1\}) = \Theta(\{\eta_0,\eta_2\}) = \Theta(\{\eta_1,\eta_2\})$ and $\chi(\{\eta_1,\eta_2\}) = 1$.

Why? First note that if we had $\Theta(\{\eta_0,\eta_1\}) \neq \Theta(\{\eta_0,\eta_2\})$ then $\eta_1 + \eta_2 = (\eta_0 + \eta_1) + (\eta_0 + \eta_2)$ would be a sum of four elements of $\{\rho_{i,a,b}^p : i < 2,\ a < b \text{ from } w\}$ and possiby some elements of $\{\eta_c^p : c \in w^p\}$. This is clearly impossible and thus $\Theta(\{\eta_0,\eta_1\}) = \Theta(\{\eta_0,\eta_2\})$, say it is $\{a,b\}$. Since $\eta_0 + \eta_1 \neq \eta_0 + \eta_2$ we immediately conlude that one of them is $\eta_a^p + \eta_b^p + \rho_{0,a,b}^p + \rho_{1,a,b}^p$ and the other is $\rho_{0,a,b}^p + \rho_{1,a,b}^p$. Consequently,

$$\eta_1 + \eta_2 = (\eta_0 + \eta_1) + (\eta_0 + \eta_2) = \eta_a^p + \eta_b^p.$$

$(***)_3$ If $\eta_0, \eta_1, \eta_2, \eta_3 \in u_{\mathbf{m}^+}$ are pairwise distinct and $\chi(\{\eta_0,\eta_1\}) = \chi(\{\eta_0,\eta_2\}) = 0$, then $\chi(\{\eta_0,\eta_3\}) = 1$.

Why? Assume towards contradiction that $\chi(\{\eta_0,\eta_3\}) = 0$. It follows from $(***)_2$ that then

$$\Theta(\{\eta_1,\eta_2\}) = \Theta(\{\eta_0,\eta_1\}) = \Theta(\{\eta_0,\eta_2\}) = \Theta(\{\eta_0,\eta_3\}) = \Theta(\{\eta_2,\eta_3\})$$

and $\chi(\{\eta_1,\eta_2\}) = \chi(\{\eta_2,\eta_3\}) = 1$. Thus, letting $\{a,b\} = \Theta(\{\eta_2,\eta_3\})$, we have $\eta_2 + \eta_3 = \eta_a^p + \eta_b^p = \eta_1 + \eta_2$, a contradiction.

$(***)_4$ $\chi(\{\eta_0,\eta_1\}) = 1$ for all distinct $\eta_0, \eta_1 \in u_{\mathbf{m}^+}$.

Why? Suppose towards contradiction that $\chi(\{\eta_0,\eta_1\}) = 0$. It follows from $(***)_3$ that there is at most one $\eta \in u_{\mathbf{m}^+} \setminus \{\eta_0,\eta_1\}$ such that $\chi(\{\eta_0,\eta\}) = 0$, and there is at most one $\eta \in u_{\mathbf{m}^+} \setminus \{\eta_0,\eta_1\}$ such that $\chi(\{\eta_1,\eta\}) = 0$. Since $|u_{\mathbf{m}^+}| \geq 5$ we may choose $\eta_2 \in u_{\mathbf{m}^+} \setminus \{\eta_0,\eta_1\}$ such that $\chi(\{\eta_0,\eta_2\}) = \chi(\{\eta_1,\eta_2\}) = 1$. Then, however, we get an immediate contradiction with $(***)_1$.

Consequently,

$$\big(\forall \eta,\nu\in u_{\mathbf{m}^+}\big)\big(\exists a,b\in w^p\big)\big(\eta+\nu=\eta^p_a+\eta^p_b\big),$$

and we may get our desired conclusion similarly to the case of $\iota\geq 3$. $\square$

Claim 4.4.3. *Assume that*

(a) $p\in\mathcal{P}_\iota$ *and* $u\subseteq w\subseteq w^p$, $|w|\geq 3$, *and* $w^*\in\big[\omega\setminus w^p\big]^{<\omega}$,
(b) $\bar{k}\big(v\cup\{d\}\big)\neq|d\cap v|$ *whenever* $v\subseteq u$ *and* $d\in w\setminus u$ *and* $\bar{r}\big(v\cup\{d\}\big)=0$,
(c) $\pi_0,\pi_1:w\longrightarrow w^*$ *are quasi–embeddings (into* S*) such that* $\pi_0(a)=\pi_1(a)$ *for* $a\in u$ *and* $\pi_0[w\setminus u]\cap\pi_1[w\setminus u]=\emptyset$.

Then there is $q\in\mathcal{P}_\iota$ *such that* $p\preccurlyeq q$, $w^q=w^*\cup w^p$ *and*

$$\pi_0\big(\mathfrak{b}(p)\restriction w\big)\Subset\mathfrak{b}(q)\restriction(\pi_0[w])\quad\textit{and}\quad\pi_1\big(\mathfrak{b}(p)\restriction w\big)\Subset\mathfrak{b}(q)\restriction(\pi_1[w]).$$

Proof of the Claim. Let $N=|w^p|+|w^*|$, $K=\big(|w^p|+|w^*|\big)^2$, and

$$K^*=\Big|\big(w^p\cup w^*\big)^{\langle 2\rangle}\setminus\Big(\big(w^p\big)^{\langle 2\rangle}\cup\big(\pi_0[w]\big)^{\langle 2\rangle}\cup\big(\pi_1[w]\big)^{\langle 2\rangle}\Big)\Big|.$$

Fix **injections**

$$\psi_0:w^p\cup w^*\longrightarrow[n^p,n^p+N),\quad \psi_1:\iota\times\big(w^p\cup w^*\big)^{\langle 2\rangle}\longrightarrow[n^p+N,n^p+N+\iota\cdot K)$$

and

$$\varphi:\big(w^p\cup w^*\big)^{\langle 2\rangle}\setminus\Big(\big(w^p\big)^{\langle 2\rangle}\cup\big(\pi_0[w]\big)^{\langle 2\rangle}\cup\big(\pi_1[w]\big)^{\langle 2\rangle}\Big)\longrightarrow[M^p,M^p+K^*).$$

Define:
$w^q=w^p\cup w^*$, $n^q=n^p+N+\iota\cdot K$, $M^q=M^p+K^*$,
$\bar{\eta}^q=\langle\eta^q_a:a\in w^q\rangle$ and

- if $a\in w^p$ then $\eta^q_a\restriction n^p=\eta^p_a$, $\eta^q_a(\psi_0(a))=1$ and $\eta^q_a(\ell)=0$ for all other $\ell\in[n^p,n^q)$,
- if $j<2$, $a=\pi_j(c)\in\pi_j[w]$, $c\in w$, then $\eta^q_a\restriction n^p=\eta^p_c$, $\eta^q_a(\psi_0(a))=1$ and $\eta^q_a(\ell)=0$ for all other $\ell\in[n^p,n^q)$ (note that by assumption (c), if $a=\pi_j(c)$, $c\in u$, then also $a=\pi_{1-j}(c)$, so there is no ambiguity here),
- if $a\in w^*\setminus(\pi_0[w]\cup\pi_1[w])$, then $\eta^q_a(\psi_0(a))=1$ and $\eta^q_a(\ell)=0$ for all other $\ell<n^q$,

$\bar{h}^q=\langle h^q_i:i<\iota\rangle$ and for $i<\iota$ and $(a,b)\in(w^q)^{\langle 2\rangle}$:

- if $(a,b)\in(w^p)^{\langle 2\rangle}$, then $h^q_i(a,b)=h^p_i(a,b)$,
- if $j<2$, $(a,b)\in\big(\pi_j[w]\big)^{\langle 2\rangle}$, and $a=\pi_j(c)$, $b=\pi_j(d)$ where $c,d\in w$, then $h^q_i(a,b)=h^p_i(c,d)$,
- if $(a,b)\in\big(w^q\big)^{\langle 2\rangle}\setminus\Big(\big(w^p\big)^{\langle 2\rangle}\cup\big(\pi_0[w]\big)^{\langle 2\rangle}\cup\big(\pi_1[w]\big)^{\langle 2\rangle}\Big)$, then $h^q_i(a,b)=\varphi(a,b)$,

$\bar{\rho}^q=\langle\rho^q_{i,a,b}:i<\iota,\ a,b\in w^q,\ a<b\rangle$ and for $i<\iota$ and $a<b$ from w^q:

- if $a,b\in w^p$ then $\rho^q_{i,a,b}\in{}^{n^q}2$ is such that $\rho^p_{i,a,b}\trianglelefteq\rho^q_{i,a,b}$, $\rho^q_{i,a,b}(\psi_1(i,a,b))=1$ and $\rho^q_{i,a,b}(\ell)=0$ for all other $\ell<n^q$,
- if $j<2$, $(a,b)\in\big(\pi_j[w]\big)^{\langle 2\rangle}$, and $a=\pi_j(c)$, $b=\pi_j(d)$ where $c,d\in w$, then $\rho^q_{i,a,b}\in{}^{n^q}2$ is such that $\rho^p_{i,c,d}\trianglelefteq\rho^q_{i,a,b}$, $\rho^q_{i,a,b}(\psi_1(i,a,b))=1$ and $\rho^q_{i,a,b}(\ell)=0$ for all other $\ell<n^q$,
- if $(a,b)\in\big(w^q\big)^{\langle 2\rangle}$ is not covered by the cases above, then $\rho^q_{i,a,b}(\psi_1(i,a,b))=1$ and $\rho^q_{i,a,b}(\ell)=0$ for all other $\ell<n^q$,

$\bar{t}^q = \langle t^q_m : m < M^q\rangle$ is such that

$$t^q_m = \{g^q_i(a,b){\restriction} n : n \le n^q,\ i < \iota,\ (a,b) \in \big(w^q\big)^{\langle 2\rangle},\ h^q_i(a,b) = m\},$$

$\bar{d}^q = \langle d^q_m : m < M^q\rangle$, where $d^q_m = d^p_m$ if $m < M^p$ and $d^q_m = n^q$ if $M^p \le m < M^q$, $\bar{g}^q$ is defined by condition $(\boxtimes)_5$ and $\mathcal{M}^q$ is defined by Definition 3.6$(\boxplus)_5$.

The verification that $q = \big(w^q, n^q, M^q, \bar{\eta}^q, \bar{t}^q, \bar{d}^q, \bar{h}^q, \bar{g}^q, \mathcal{M}^q\big) \in \mathcal{P}_\iota$ is quite straightforward. The only non trivial part is checking conditions $(\boxplus)_6$ and $(\boxplus)_7$ of Definition 3.6. For $(\boxplus)_6$, assume that $\mathbf{m}^{\mathfrak{b}(q)}(\ell, w_0), \mathbf{m}^{\mathfrak{b}(q)}(\ell, w_1) \in \mathcal{M}^q$ are such that $\mathbf{m}^{\mathfrak{b}(q)}(\ell, w_0) \doteqdot \mathbf{m}^{\mathfrak{b}(q)}(\ell, w_1) + \rho$. If for some $(a,b) \in (w_0)^{\langle 2\rangle}$ and $i < \iota$ we have $h^q_i(a,b) \ge M^p$, then for every $c \in w_0 \setminus \{a,b\}$ and an $i < \iota$, either $M^p \le h^q_i(a,c) = \varphi(a,c)$ or $M^p \le h^q_i(b,c) = \varphi(b,c)$. Since φ is one-to-one, the values of $h^q_i(a,c), h^q_i(b,c)$ determine c then and we immediately conlude that $w_0 = w_1$. Suppose now that $h^q_i(a,b) < M^p$ for all $(a,b) \in (w_0)^{\langle 2\rangle}$. Then, both for $j = 0$ and $j = 1$, neccessarily either we have $w_j \subseteq w^p$ or $w_j \subseteq \pi_0[w]$ or $w_j \subseteq \pi_1[w]$. If $w_0 \cup w_1 \subseteq w^p$ then we may set $\ell^* = \min(n^p, \ell)$ and apply condition $(\boxplus)_6$ for p and $\mathbf{m}^{\mathfrak{b}(p)}(\ell^*, w_0)$, $\mathbf{m}^{\mathfrak{b}(p)}(\ell^*, w_1)$. If $w_0 \subseteq w^p$ and $w_1 \subseteq \pi_j[w]$, then we first note that for each $n \in [n^p, n^q)$ there is at most one $a \in w^q$ such that $\eta^q_a(n) = 1$. Therefore, in the current situation, $\eta^q_a(n) = 0$ whenever $n^p \le n \le \ell$, $a \in w_0 \cup w_1$. So we set $\ell^* = \min(n^p, \ell)$ and again apply condition $(\boxplus)_6$ for p and $\mathbf{m}^{\mathfrak{b}(p)}(\ell^*, w_0)$, $\mathbf{m}^{\mathfrak{b}(p)}(\ell^*, \pi_j^{-1}\big[w_1\big])$. Similarly in other cases.

To show $(\boxplus)_7$ of Definition 3.6, suppose towards contradiction that $\mathbf{m}^{\mathfrak{b}(q)}(\ell_0, w_0)$ and $\mathbf{m}^{\mathfrak{b}(q)}(\ell_1, w_1)$ from $\mathcal{M}^q$ and $a \in w_0$ are such that $\mathbf{m}^{\mathfrak{b}(q)}(\ell_0, w_0) \sqsubseteq^* \mathbf{m}^{\mathfrak{b}(q)}(\ell_1, w_1)$, $\bar{r}(w_0) = 0$, $|a \cap w_0| = \bar{k}(w_0)$ and $1 < |\{b \in w_1 : (\eta^q_a{\restriction}\ell_0) \lhd \eta^q_b\}|$. We may assume that $4 \le |w_0| + 1 = |w_1|$ and $\{b_0, b_1\} = \{b \in w_1 : (\eta^q_a{\restriction}\ell_0) \lhd \eta^q_b\}$. Necessarily, $\ell_0 < \ell_1 \le n^q$ and therefore $h^q_i(a,b) < M^p$ for all $(a,b) \in (w_0)^{\langle 2\rangle}$ and $i < \iota$. Consequently, either $w_0 \subseteq w^p$ or $w_0 \subseteq \pi_0[w]$ or $w_0 \subseteq \pi_1[w]$. If also $w_1 \subseteq w^p$ or $w_1 \subseteq \pi_0[w]$ or $w_1 \subseteq \pi_1[w]$, then for any distinct $a, b \in w_1$ we have $\eta^q_a{\restriction} n^p \ne \eta^q_b{\restriction} n^p$ and we may assume $\ell_0 < \ell_1 \le n^p$. Then we may use π_0^{-1}, π_1^{-1}, as appropriate, to copy (if needed) both w_0 and w_1 to w^p and get easy contradiction with $(\boxplus)_7$ for p.

So suppose otherwise, that is neither of the inclusions

$$w_1 \subseteq w^p, \quad w_1 \subseteq \pi_0[w], \quad w_1 \subseteq \pi_1[w]$$

holds true. We may replace w_0 with $\pi_j^{-1}[w_0]$ and ℓ_0 with $\min(n^p, \ell_0)$, so without loss of generality $w_0 \subseteq w^p$ and $\ell_0 \le n^p$. Now, for all $b \in w_1 \setminus \{b_0, b_1\} \ne \emptyset$ we have

$$h^q_0(b, b_0), h^q_0(b, b_1) < M^p \quad \text{ while } \quad h^q_0(b_0, b_1) \ge M^p.$$

This is only possible if $b_0 \in \pi_j[w] \setminus \pi_{1-j}[w]$ and $b_1 \in \pi_{1-j}[w] \setminus \pi_j[w]$ (say $b_j \in \pi_j[w] \setminus \pi_{1-j}[w]$) and $w_1 \setminus \{b_0, b_1\} \subseteq \pi_0[w] \cap \pi_1[w] = \pi_0[u] = \pi_1[u]$. But then letting $v = \pi_0^{-1}\big[w_1 \setminus \{b_0, b_1\}\big]$ and considering $\mathbf{m}^{\mathfrak{b}(p)}(\ell_0, v \cup \{\pi_0^{-1}(b_0)\})$ we get (by 3.6$(\boxplus)_6$ for p) that the order isomorphism from w_0 onto $v \cup \{\pi_0^{-1}(b_0)\}$ is a quasi–embedding mapping a to $\pi_0^{-1}(b_0) \in w \setminus u$. This immediately contradicts assumption (b) of the Claim (applied to v and $d = \pi_0^{-1}(b_0)$).

Thus $q \in \mathcal{P}_\iota$ indeed. It should be clear that $p \preccurlyeq q$ and q is as required. $\square$

Claim 4.4.4. *Let $p \in \mathcal{P}_\iota$, $k < \omega$. Then there is $q \in \mathcal{P}_\iota$ such that $p \preccurlyeq q$, $k \in w^q$, $n^q > k$ and $M^q > k$.*

Proof of the Claim. By the cuteness of S, we may find a quasi-embedding $\varphi : w^p \longrightarrow \omega$ such that $\max(w^p) < \min(\mathrm{rng}(\varphi))$. Let $w^* \in [\omega \setminus w^p]^{<\omega}$ be such that $k \in w^p \cup w^*$, $\mathrm{rng}(\varphi) \subseteq w^*$ and $|w^*| > k$. Applying (the proof of) Claim 4.4.3 to p, $u = w = w^p$, w^* and $\pi_0 = \pi_1 = \varphi$ we will get $q \in \mathcal{P}_\iota$ as there. (Note that the assumption 4.4.3(b) is satisfied vacuously.) This q has the properties that $p \preccurlyeq q$, and $w^q = w^p \cup w^*$, and $M^q = M^p + |w^p| \cdot |w^*| > k$. Since $|w^q| > k$ we also have $n^q > k$ (remember Observation 3.8(1)). $\square$

Claim 4.4.5. *Assume that*

(a) $p \in \mathcal{P}_\iota$ *and* $w \subseteq w^p$, $|w| \geq 3$,
(b) $\pi_0 : w \longrightarrow \omega$ *is a quasi–embedding (into S) and* $\mathrm{rng}(\pi_0) \subseteq w^+ \in [\omega]^{<\omega}$.

Then there are $q \in \mathcal{P}_\iota$ *and* $\pi : w^+ \longrightarrow w^q$ *such that*

- $p \preccurlyeq q$ *and* π *is a quasi–embedding, and*
- $(\pi \circ \pi_0)(\mathfrak{b}(p){\restriction}w) \Subset \mathfrak{b}(q){\restriction}\big(\pi\big[\pi_0[w]\big]\big)$.

Proof of the Claim. Since S is cute we may find a quasi embedding $\pi : w^+ \longrightarrow \omega$ such that $\mathrm{rng}(\pi) \cap w^p = \emptyset$. Apply Claim 4.4.3 to $w, w, \mathrm{rng}(\pi), \pi \circ \pi_0, \pi \circ \pi_0$ here standing for w, u, w^*, π_0, π_1 there. (Note that the assumption 4.4.3(b) is satisfied vacuously.) $\square$

Claim 4.4.6. *Assume that*

(a) $p \in \mathcal{P}_\iota$, $u \subseteq w \subseteq w^p$, $3 \leq |w|$,
(b) $\bar{k}\big(v \cup \{d\}\big) \neq |d \cap v|$ *whenever* $v \subseteq u$ *and* $d \in w \setminus u$ *and* $\bar{r}\big(v \cup \{d\}\big) = 0$,
(c) $\pi_0, \pi_1 : w \longrightarrow \omega$ *are quasi–embeddings (into S) such that* $\pi_0(a) = \pi_1(a)$ *for* $a \in u$ *and* $\pi_0[w \setminus u] \cap \pi_1[w \setminus u] = \emptyset$.

Then there are $q \in \mathcal{P}_\iota$ *and a quasi–embedding* $\pi : \mathrm{rng}(\pi_0) \cup \mathrm{rng}(\pi_1) \longrightarrow w^q$ *such that* $p \preccurlyeq q$ *and*

$$(\pi \circ \pi_0)(\mathfrak{b}(p){\restriction}w) \Subset \mathfrak{b}(q){\restriction}(\pi \circ \pi_0[w]) \quad \textit{and} \quad (\pi \circ \pi_1)(\mathfrak{b}(p){\restriction}w) \Subset \mathfrak{b}(q){\restriction}(\pi \circ \pi_1[w]).$$

Proof of the Claim. Using the cuteness of S we first pick a quasi embedding $\pi^+ : \mathrm{rng}(\pi_0) \cup \mathrm{rng}(\pi_1) \longrightarrow \omega$ such that $\mathrm{rng}(\pi^+) \cap w^p = \emptyset$. Then apply Claim 4.4.3 to $u, w, \mathrm{rng}(\pi^+), \pi^+ \circ \pi_0, \pi^+ \circ \pi_1$ here standing for u, w, w^*, π_0, π_1 there. $\square$

Using Claims 4.4.4 (for (ii)), 4.4.6 (for (iii)) and 4.4.5 (for (iv)), and employing a suitable bookkeeping device we may inductively choose a sequence $\langle p_\ell : \ell < \omega\rangle \subseteq \mathcal{P}_\iota$ such that

(i) $p_\ell \preccurlyeq p_{\ell+1}$ for all $\ell < \omega$,
(ii) for every $k < \omega$ there is an $\ell < \omega$ such that $k \in w^{p_\ell}$, $n^{p_\ell} > k$, and $M^{p_\ell} > k$,
(iii) if (a) $k < \omega$, $u \subseteq w \subseteq w^{p_k}$, $3 \leq |w|$, and
  (b) $\bar{k}\big(v \cup \{d\}\big) \neq |d \cap v|$ whenever $v \subseteq u$ and $d \in w \setminus u$ and $\bar{r}\big(v \cup \{d\}\big) = 0$,
  (c) $\pi_0, \pi_1 : w \longrightarrow \omega$ are quasi–embeddings (into S) such that $\pi_0(a) = \pi_1(a)$ for $a \in u$ and $\pi_0[w \setminus u] \cap \pi_1[w \setminus u] = \emptyset$,
  then there is an $\ell < \omega$ and a quasi–embedding $\pi : \mathrm{rng}(\pi_0) \cup \mathrm{rng}(\pi_1) \longrightarrow w^{p_\ell}$ such that

$$(\pi \circ \pi_0)(\mathfrak{b}(p_k){\restriction}w) \Subset \mathfrak{b}(p_\ell){\restriction}(\pi \circ \pi_0[w]) \quad \text{and} \quad (\pi \circ \pi_1)(\mathfrak{b}(p_k){\restriction}w) \Subset \mathfrak{b}(p_\ell){\restriction}(\pi \circ \pi_1[w]),$$

(iv) if (a) $k < \omega$, $w \subseteq w^{p_k}$, $3 \leq |w|$, and
  (b) $\pi_0 : w \longrightarrow \omega$ is a quasi–embedding and $\mathrm{rng}(\pi_0) \subseteq u \in [\omega]^{<\omega}$,

then there is an $\ell<\omega$ and a quasi–embedding $\pi: u \longrightarrow w^{p_\ell}$ (into S) such that $(\pi\circ\pi_0)(\mathfrak{b}(p_k){\restriction} w) \Subset \mathfrak{b}(p_\ell){\restriction}(\pi\circ\pi_0[w])$.

For $m<\omega$ let $T_m=\bigcup\{t^{p_\ell}_m : \ell<\omega \wedge m<M^{p_\ell}\}$ and note that each T_m is a subtree of ${}^{\omega>}2$ without terminal nodes. Let $\bar{T}=\langle T_m : m<\omega\rangle$ and let $\bar{c}=\bigcup\limits_{\ell<\omega}\bar{d}^{p_\ell}$. One easily verifies that $\bar{T}$ has $(\bar{c},S)$–controlled amalgamation property.

Let ndrk_ι be the non-disjointness rank on $\mathbf{M}_{\bar{T},\iota}$ (see Definitions 2.8, 2.12). Note that $\mathcal{M}^{p_\ell}\subseteq \mathbf{M}_{\bar{T},\iota}$ for each $\ell<\omega$.

Claim 4.4.7. *Assume $N<\omega$, $w_0\subseteq w^{p_N}$, $\ell_0\le n^{p_N}$ and $\mathbf{n}_0=\mathbf{m}^{\mathfrak{b}(p_N)}(\ell_0,w_0)\in \mathcal{M}^{p_N}$. Then*

1. *$\bar{r}(w_0)\le \mathrm{ndrk}_\iota(\mathbf{n}_0)$.*
2. *If $|w_0|\ge 4$, then $\mathrm{ndrk}_\iota(\mathbf{n}_0)\le \omega\cdot\big(\bar{r}(w_0)+1\big)$ (ordinal product).*

Proof of the Claim. (1) By induction on α we show (for all $\mathbf{n}_0,N,\ell_0,w_0$) that $\alpha\le\bar{r}(w_0)$ implies $\alpha\le \mathrm{ndrk}_\iota(\mathbf{n}_0)$.

For the successor step, suppose that $\alpha+1\le\bar{r}(w_0)$. Assume $\nu\in u_{\mathbf{n}_0}$ and let $a\in w_0$ be such that $\nu=\eta^{p_N}_a{\restriction}\ell_0$. Let $L=|w_0|+1$, $\ell=|a\cap w_0|$ and let $\varphi_0: w_0\longrightarrow L\setminus\{\ell+1\}$ and $\varphi_1: w_0\longrightarrow L\setminus\{\ell\}$ be the increasing bijections. Note that

- $\varphi_0(x)=\varphi_1(x)$ whenever $x\in w_0\setminus\{a\}$, and
- $\varphi_0^{-1}[u]=\varphi_1^{-1}[u]$ whenever $u\subseteq L\setminus\{\ell,\ell+1\}$.

We define a $\mathcal{YZR}(\varepsilon)$–system $s=(X^s,\bar{r}^s,\bar{\jmath}^s,\bar{k}^s)$ as follows. We set $X^s=L$ and for $u\subseteq X^s$ we put

$(\circledcirc)_1$ if $\ell\notin u$, then $\bar{r}^s(u)=\bar{r}\big(\varphi_1^{-1}[u]\big)$, $\bar{\jmath}^s(u)=\bar{\jmath}\big(\varphi_1^{-1}[u]\big)$ and $\bar{k}^s(u)=\bar{k}\big(\varphi_1^{-1}[u]\big)$,

$(\circledcirc)_2$ if $\ell+1\notin u$, then $\bar{r}^s(u)=\bar{r}\big(\varphi_0^{-1}[u]\big)$, $\bar{\jmath}^s(u)=\bar{\jmath}\big(\varphi_0^{-1}[u]\big)$ and $\bar{k}^s(u)=\bar{k}\big(\varphi_0^{-1}[u]\big)$,

$(\circledcirc)_3$ if $\ell,\ell+1\in u$, then $\bar{r}^s(u)=\alpha$, $\bar{\jmath}^s(u)=\max\big(\bar{\jmath}(v):\emptyset\neq v\subseteq w_0\big)+1$, and $\bar{k}^s=|\ell\cap u|$.

One easily verifies $(*)_1$, $(*)_2$ and $(*)_4$ of Definition 3.1. Concerning 3.1$(*)_3$, we note that if $\emptyset\neq u\subseteq w\subseteq X^s$ and $\{\ell,\ell+1\}\nsubseteq w$, then $\bar{r}^s(u)\ge\bar{r}^s(w)$ by $(\circledcirc)_1+(\circledcirc)_2$ and the properties of $\bar{r}$. If $\{\ell,\ell+1\}\subseteq w$, $\emptyset\neq u\subseteq w$, then $\bar{r}^s(w)=\alpha$ and either $\bar{r}^s(u)=\alpha$ (when $\ell,\ell+1\in u$) or $\bar{r}^s(u)\ge\alpha+1$ (if $\{\ell,\ell+1\}\nsubseteq u$). (Since $\bar{r}(w_0)\ge\alpha+1$, for every nonempty $v\subseteq w_0$ we have $\bar{r}(v)\ge\alpha+1$.) Thus the only possibly unclear demand is 3.1$(*)_5$. So suppose that $\emptyset\neq u\subseteq X^s$ and $u=\{a_0,\dots,a_m\}$ is the increasing enumeration. We want to argue that there is no $b\in X^s\setminus u$ such that

$(\spadesuit)_b$ $|u\cap b|=\bar{k}^s(u)$, $\bar{\jmath}^s\big((u\setminus\{a_{\bar{k}^s(u)}\})\cup\{b\}\big)=\bar{\jmath}^s(u)$ and $\bar{r}^s\big(u\cup\{b\}\big)=\bar{r}^s(u)$.

Case 1: $\ell\notin u$

By $(\circledcirc)_1$, there is no $b\in X^s\setminus u\setminus\{\ell\}$ satisfying $(\spadesuit)_b$. If $\ell+1\in u$, then $\bar{r}^s\big(u\cup\{\ell\}\big)=\alpha<\alpha+1=\bar{r}^s(u)$. Therefore, $(\spadesuit)_\ell$ fails when $\ell+1\in u$. If $\ell+1\notin u$, then by $(\circledcirc)_2$ the statement $(\spadesuit)_\ell$ must fail too. Consequently, in the current case, there is no $b\in X^s\setminus u$ for which $(\spadesuit)_b$ holds true.

Case 2: $\ell+1\notin u$

Similar to Case 1, just interchanging ℓ and $\ell+1$.

Case 3: $\ell,\ell+1\in u$

Then, by $(\circledcirc)_3$, $a_{\bar{k}^s(u)}=\ell$ and for all $b\in X^s\setminus u$ we get $\bar{\jmath}^s\big((u\setminus\{\ell\})\cup\{b\}\big)<\bar{\jmath}^s(u)$. Thus $(\spadesuit)_b$ fails for all $b\in X^s\setminus u$.

Since S is cute, there is a quasi-embedding $\varphi : X^s \longrightarrow \omega$ of s into S such that $\max(w_0) < \min\big(\mathrm{rng}(\varphi)\big)$. Let $b = \varphi(\ell)$, $b' = \varphi(\ell+1)$, $u = \mathrm{rng}(\varphi) \setminus \{b, b'\}$ and $\pi_0 = \varphi \circ \varphi_0$, $\pi_1 = \varphi \circ \varphi_1$. Then

- $u \cup \{b, b'\} \subseteq \omega \setminus w_0$, $b \neq b'$, and $\bar{r}\big(u \cup \{b, b'\}\big) = \alpha$,
- $\pi_0 : w_0 \longrightarrow u \cup \{b\}$ and $\pi_1 : w_0 \longrightarrow u \cup \{b'\}$ are quasi–embeddings and $\pi_0(a) = b$ and $\pi_1(a) = b'$, and $\pi_0{\restriction}\big(w_0 \setminus \{a\}\big) = \pi_1{\restriction}\big(w_0 \setminus \{a\}\big)$.

Since $\bar{r}(w_0) \geq \alpha + 1 \geq 1$, we are sure that for every nonempty $v \subseteq w_0$ we have $\bar{r}(v) \neq 0$. Therefore assumption (b) of condition (iii) of the construction above is satisfied vacuously and we may use that condition to claim that there are $K < \omega$ and a quasi–embedding $\pi : u \cup \{b, b'\} \longrightarrow w^{p_K}$ such that

$$(\pi \circ \pi_0)(\mathfrak{b}(p_N){\restriction}w) \Subset \mathfrak{b}(p_K){\restriction}(\pi \circ \pi_0[w]) \ \text{ and } \ (\pi \circ \pi_1)(\mathfrak{b}(p_N){\restriction}w) \Subset \mathfrak{b}(p_K){\restriction}(\pi \circ \pi_1[w]).$$

Note that $\bar{r}\big(\pi[u \cup \{b, b'\}]\big) = \alpha$. Let $\mathbf{n}_1 = \mathbf{m}^{\mathfrak{b}(p_K)}(n^{p_K}, \pi[u \cup \{b, b'\}]) \in \mathcal{M}^{p_K} \subseteq \mathbf{M}_{\bar{T},\iota}$. Then $\mathbf{n}_0 \sqsubseteq \mathbf{n}_1$, $2 \leq |\{\eta \in u_{\mathbf{n}_1} : \nu \lhd \eta\}|$ and (by the inductive hypothesis) $\alpha \leq \mathrm{ndrk}_\iota(\mathbf{n}_1)$.

Now we may conclude that $\mathrm{ndrk}_\iota(\mathbf{n}_0) \geq \alpha + 1$. The rest is clear.

(2) By induction on α we argue that $\bar{r}(w_0) \leq \alpha$ implies $\mathrm{ndrk}_\iota(\mathbf{n}_0) \leq \omega \cdot (\alpha + 1)$ (for all $\mathbf{n}_0, N, \ell_0, w_0$).

Assume first $\bar{r}(w_0) = 0$ and let $a \in w_0$ be such that $|a \cap w_0| = \bar{k}(w_0)$. Suppose that there is $\mathbf{m} \in \mathbf{M}_{\bar{T},\iota}$ such that $\mathbf{m} \sqsupseteq \mathbf{n}_0$ and $2 \leq |\{\nu \in u_{\mathbf{m}} : \eta_a^{p_N}{\restriction}\ell_0 \lhd \nu\}|$. We may also demand that for some $K > N$ we have $\mathbf{m} \in \mathbf{M}^{n^{p_K}}_{\bar{t}^{p_K},\iota}$ and $\ell_{\mathbf{m}} = n^{p_K}$ and $|u_{\mathbf{m}}| = |u_{\mathbf{n}_0}| + 1 \geq 5$. Now use Claim 4.4.2 to find $\mathbf{n}_1 \in \mathcal{M}^{p_K}$ and ρ such that $\mathbf{n}_1 \doteqdot \mathbf{m} + \rho$. Let $\mathbf{n}_1 = \mathbf{m}^{\mathfrak{b}(p_K)}(n^{p_K}, w_1)$. Let $b, b' \in w_1$ be such that

($\clubsuit$) $(\eta_a^{p_N}{\restriction}\ell_0) + (\rho{\restriction}\ell_0) = \eta_b^{p_k}{\restriction}\ell_0 = \eta_{b'}^{p_k}{\restriction}\ell_0$, and $b \neq b'$.

Then $\mathbf{m}_0 \stackrel{\mathrm{def}}{=} \mathbf{m}^{\mathfrak{b}(p_K)}(\ell_0, w_1 \setminus \{b\}) \in \mathcal{M}^{p_K}$ and $\mathbf{n}_0 + (\rho{\restriction}\ell_0) \doteqdot \mathbf{m}_0$. By condition 3.6($\boxplus$)$_6$ for p_K the order isomorphism $\pi : w_0 \longrightarrow w_1 \setminus \{b\}$ is a quasi–embedding and $(\eta_c^{p_K}{\restriction}\ell_0) + (\rho{\restriction}\ell_0) = \eta_{\pi(c)}^{p_K}{\restriction}\ell_0$. Therefore $\bar{r}(w_1 \setminus \{b\}) = 0$ and $\bar{k}(w_1 \setminus \{b\}) = \bar{k}(w_0) = |a \cap w_0| = |b' \cap w_1|$. But then $\mathbf{m}_0 \sqsubseteq \mathbf{n}$, $b, b' \in w_1$ and ($\clubsuit$) contradict condition 3.6($\boxplus$)$_7$. Consequently, $\mathrm{ndrk}_\iota(\mathbf{n}_0) = 0$.

Assume now $0 < \bar{r}(w_0) \leq \alpha$ (and the statement is true for ranks below α). Let $a \in w_0$ be such that $|a \cap w_0| = \bar{k}(w_0)$ and suppose that $\mathbf{n}^* \in \mathbf{M}_{\bar{T},\iota}$ satisfies

$$\mathbf{n}_0 \sqsubseteq \mathbf{n}^* \ \text{ and } \ |u_{\mathbf{n}^*}| = |u_{\mathbf{n}_0}| + 1 \geq 5 \ \text{ and } \ 2 \leq |\{\nu \in u_{\mathbf{n}^*} : \eta_a^{p_N}{\restriction}\ell_0 \lhd \nu\}|.$$

We will argue that $\mathrm{ndrk}_\iota(\mathbf{n}^*) < \omega \cdot (\alpha + 1)$. So suppose this is not the case and $\mathrm{ndrk}_\iota(\mathbf{n}^*) \geq \omega \cdot \alpha + \omega$. Let L be such that $c_{h_i^{\mathbf{n}^*}(\eta,\nu)} \leq L$ for all $(\eta, \nu) \in \big(u_{\mathbf{n}^*}\big)^{\langle 2\rangle}$ and $i < \iota$. Using Lemma 2.13(8) and Observation 2.14 we may find $\mathbf{n}^+ \in \mathbf{M}_{\bar{T},\iota}$ such that

$$\mathbf{n}^* \sqsubseteq \mathbf{n}^+ \ \text{ and } \ |u_{\mathbf{n}^*}| = |u_{\mathbf{n}^+}| \geq 5 \ \text{ and } \ \ell_{\mathbf{n}^+} > L \ \text{ and } \ \mathrm{ndrk}_\iota(\mathbf{n}^+) \geq \omega \cdot \alpha + 1170.$$

Take $K > N + L$ such that $\mathbf{n}^+ \in \mathbf{M}^{n^{p_K}}_{\bar{t}^{p_K},\iota}$. Now we may find $\mathbf{n}_1 \in \mathcal{M}^{p_K}$ which is essentially the same as a translation of $\mathbf{n}^+$ (exists by Claim 4.4.2), say $\mathbf{n}_1 = \mathbf{m}^{\mathfrak{b}(p_K)}(\ell_{\mathbf{n}^+}, w_1)$. Then for some distinct $b, b' \in w_1$ we have

- $\eta_b^{p_K}{\restriction}\ell_0 = \eta_{b'}^{p_K}{\restriction}\ell_0$, $\mathbf{m}^{\mathfrak{b}(p_K)}(\ell_0, w_1 \setminus \{b\}), \mathbf{m}^{\mathfrak{b}(p_K)}(\ell_0, w_1 \setminus \{b'\}) \in \mathcal{M}^{p_K}$ and
- $\mathbf{m}^{\mathfrak{b}(p_K)}(\ell_0, w_1 \setminus \{b\}) \doteqdot \mathbf{m}^{\mathfrak{b}(p_K)}(\ell_0, w_1 \setminus \{b'\})$, and they are essentially the same as a translation of $\mathbf{n}_0$, and
- the translation above maps $\eta_a^{p_K}{\restriction}\ell_0 \in u_{\mathbf{n}_0}$ to $\eta_{b'}^{p_K}{\restriction}\ell_0$ ($\eta_b^{p_K}{\restriction}\ell_0$, respectively).

By condition 3.6($\boxplus$)$_6$ for p_K we know that the order isomorphism from w_0 onto $w_1\setminus\{b\}$ ($w_1\setminus\{b'\}$, respectively) is a quasi–embedding mapping a onto b' (b, respectively). Therefore

- $\bar{r}(w_0)=\bar{r}(w_1\setminus\{b\})=\bar{r}(w_1\setminus\{b'\})$, and
- $\bar{k}(w_0)=\bar{k}(w_1\setminus\{b\})=\bar{k}(w_1\setminus\{b'\})$, and
- $|w_1\cap b|=|w_1\cap b'|=|w_0\cap a|=\bar{k}(w_0)$.

Since $\bar{r}(w_0)>0$, also $\bar{\jmath}(w_0)=\bar{\jmath}(w_1\setminus\{b\})=\bar{\jmath}(w_1\setminus\{b'\})$. Therefore, by 3.1$(*)_5$, $\bar{r}(w_1)<\bar{r}(w_1\setminus\{b\})\leq\alpha$ and by the inductive hypothesis we get

$$\mathrm{ndrk}_\iota(\mathbf{n}^+)=\mathrm{ndrk}_\iota(\mathbf{n}_1)\leq\omega\cdot(\bar{r}(w_1)+1)\leq\omega\cdot\alpha,$$

contradicting the choice of $\mathbf{n}^+$.

Now we may conclude that $\mathrm{ndrk}_\iota(\mathbf{n}_0)\leq\omega\cdot\alpha+1<\omega\cdot(\alpha+1)$. □

Claim 4.4.8. $\varepsilon\leq\mathrm{ndrk}_\iota(\bar{T})\leq\omega\cdot(\varepsilon+2)+2$.

Proof of the Claim. By the cuteness of S, there are $w\in[\omega]^{<\omega}$ with $\bar{r}(w)=\varepsilon$. Therefore Claim 4.4.7(1) immediately implies the first inequality.

For the second inequality, suppose towards contradiction that $\mathbf{m}\in\mathbf{M}_{\bar{T},\iota}$ is such that $\mathrm{ndrk}_\iota(\mathbf{m})\geq\omega\cdot(\varepsilon+2)+3$. Then we may pick $\mathbf{n}\in\mathbf{M}_{\bar{T},\iota}$ such that

$$\mathbf{m}\sqsubseteq\mathbf{n},\quad 5\leq|u_{\mathbf{n}}|,\quad\text{ and }\quad\mathrm{ndrk}_\iota(\mathbf{n})\geq\omega\cdot(\varepsilon+2).$$

Let L be such that $c_{h^{\mathbf{n}}_i(\eta,\nu)}\leq L$ for all $(\eta,\nu)\in\big(u_{\mathbf{n}}\big)^{\langle 2\rangle}$ and $i<\iota$. Like in the previous Claim, use Observation 2.14 to find $\mathbf{n}^+\in\mathbf{M}_{\bar{T},\iota}$ such that

$$\mathbf{n}\sqsubseteq\mathbf{n}^+\ \text{ and }\ |u_{\mathbf{n}}|=|u_{\mathbf{n}^+}|\geq 5\ \text{ and }\ \ell_{\mathbf{n}^+}>L\ \text{ and }\ \mathrm{ndrk}_\iota(\mathbf{n}^+)\geq\omega\cdot(\varepsilon+1)+1170.$$

Take an N such that $\mathbf{n}^+\in\mathbf{M}^{n^{p_N}}_{\bar{t}^{p_N},\iota}$. By Claim 4.4.2 there are $\rho\in{}^{\ell_{\mathbf{n}^+}}2$ and $\mathbf{n}^*\in\mathcal{M}^{p_N}$ such that $(\mathbf{n}^++\rho)\doteqdot\mathbf{n}^*$. But now by Claim 4.4.7(2) and Lemma 2.13(4) we have $\mathrm{ndrk}_\iota(\mathbf{n}^+)=\mathrm{ndrk}_\iota(\mathbf{n}^*)\leq\omega\cdot(\varepsilon+1)$, a contradiction. □

□

5. Conclusions and Questions

For a countable ordinal $\varepsilon>0$ and $2\leq\iota<\omega$ let $\bar{T}^{\varepsilon,\iota}=\langle T^{\varepsilon,\iota}_m:m<\omega\rangle$ be the sequence of trees given by Theorem 4.4 (for some S and $\bar{c}$ as there). Let $B_{\varepsilon,\iota}=\bigcup\limits_{m<\omega}\lim(T^{\varepsilon,\iota}_m)$.

Corollary 5.1. *If λ is a cardinal such that $\mathrm{NPr}^\varepsilon(\lambda)$ holds true, then there exist a ccc forcing notion $\mathbb{P}$ such that*

> $\Vdash_{\mathbb{P}}$ *" there is a sequence $\langle\rho_\alpha:\alpha<\lambda\rangle$ of distinct elements of ${}^\omega 2$ such that*
> *$\big|(\rho_\alpha+B_{\varepsilon,\iota})\cap(\rho_\beta+B_{\varepsilon,\iota})\big|\geq 2\iota$ for all $\alpha,\beta<\lambda$*
> *but there is no perfect set of such ρ's. "*

Corollary 5.2. *Assume MA and $\aleph_\varepsilon<\mathfrak{c}$. Then*

- *there is a sequence $\langle\rho_\alpha:\alpha<\aleph_\varepsilon\rangle$ of distinct elements of ${}^\omega 2$ such that for $\alpha,\beta<\aleph_\varepsilon$*
$$\big|(\rho_\alpha+B_{\varepsilon,\iota})\cap(\rho_\beta+B_{\varepsilon,\iota})\big|\geq 2\iota,$$
- *for every perfect set $P\subseteq{}^\omega 2$ there are $\eta,\nu\in P$ such that*
$$\big|(\eta+B_{\varepsilon,\iota})\cap(\nu+B_{\varepsilon,\iota})\big|<2\iota.$$

Corollary 5.3. *There exists a sequence $\langle \eta_\alpha : \alpha < \omega_1 \rangle$ of distinct elements of ${}^\omega 2$ such that $\big|(\rho_\alpha + B_{\varepsilon,\iota}) \cap (\rho_\beta + B_{\varepsilon,\iota})\big| \geq 2\iota$ for all $\alpha, \beta < \omega_1$, but there is no perfect set of such η's.*

Proof. Since $\mathrm{ndrk}_\iota\big(\bar{T}^{\varepsilon,\iota}\big) < \omega_1$ we know that there is no perfect set $P \subseteq {}^\omega 2$ with the property that $\big|(\rho_0 + B_{\varepsilon,\iota}) \cap (\rho_1 + B_{\varepsilon,\iota})\big| \geq 2\iota$ for all $\rho_0, \rho_1 \in P$. On the other hand, by Theorem 3.10, there is a ccc forcing notion forcing that

($\circledast$) "there is a sequence $\langle \eta_\alpha : \alpha < \omega_1 \rangle$ of distinct elements of ${}^\omega 2$ such that $\big|(\rho_\alpha + B_{\varepsilon,\iota}) \cap (\rho_\beta + B_{\varepsilon,\iota})\big| \geq 2\iota$ for all $\alpha, \beta < \omega_1$".

By Keisler's completeness theorem for logic with the quantifier "there exists uncountably many", the assertion in ($\circledast$) is absolute between $\mathbf{V}$ and it ccc forcing extensions. To see this, one may use the Completeness Theorem for $\mathcal{L}_{\omega_1\omega}(Q)$ [7, Theorem 4.10] and repeat the argument in the proof of Kubiś and Shelah [8, Proposition 3.2]. Alternatively, one may use the Completeness Theorem for $\mathcal{L}^\omega(Q)$ [7, Corollary 3.10] and follow the arguments given in the proof of Kubiś and Vejnar [9, Theorem 3.1, pp 4364–4366]. $\square$

The results presented in this paper leave several natural questions open. First of all,

Problem 5.4. What is the value of $\mathrm{ndrk}_\iota\big(\bar{T}^{\varepsilon,\iota}\big)$?

A natural question is if we can replace the amalgamation property in Theorem 3.10 with a requirement on the rank $\mathrm{ndrk}_\iota(\bar{T})$. In the strongest form this would be the following question.

Problem 5.5. Suppose that

(a) $T_m \subseteq {}^{\omega>}2$ (for $m < \omega$) are trees with no maximal nodes, $\bar{T} = \langle T_m : m < \omega \rangle$, and
(b) ndrk_ι is the non-disjointness rank on $\mathbf{M}_{\bar{T},\iota}$, $2 \leq \iota < \omega$,
(c) $\varepsilon \leq \mathrm{ndrk}_\iota(\bar{T})$, and λ is a cardinal such that $\mathrm{NPr}^\varepsilon(\lambda)$ holds true,
(d) $B = \bigcup\limits_{m<\omega} \lim(T_m)$.

Does there exist a ccc forcing notion $\mathbb{P}$ of size λ such that

$\Vdash_{\mathbb{P}}$ " there is a sequence $\langle \eta_\alpha : \alpha < \lambda \rangle$ of distinct elements of ${}^\omega 2$ such that $\big|(\rho_\alpha + B) \cap (\rho_\beta + B)\big| \geq 2\iota$ for all $\alpha, \beta < \lambda$ " ?

The relevance of ι is yet to be discovered:

Problem 5.6. Does there exist a sequence $\bar{T} = \langle T_m : m < \omega \rangle$ of trees $T_m \subseteq {}^{\omega>}2$ (for $m < \omega$) such that for some $2 \leq \iota < \iota' < \omega$ we have $\mathrm{ndrk}_\iota(\bar{T}) \neq \mathrm{ndrk}_{\iota'}(\bar{T})$?

Of course, the next steps could be to investigate stnd_ω and stnd_{ω_1}:

Problem 5.7. Is is consistent to have a Borel set $B \subseteq {}^\omega 2$ such that

- for some uncountable set H, $(B + x) \cap (B + y)$ is uncountable for every $x, y \in H$, but
- for every perfect set P there are $x, y \in P$ with $(B+x) \cap (B+y)$ countable?

Similarly if "uncountable / countable" are replaced with "infinite / finite", respectively.

As mentioned before, our arguments relay on the algebraic properties of ${}^\omega 2$. So, one should ask for the following. (The constructions in [12] might be relevant here.)

Problem 5.8. Generalize the results of this paper (Theorems 3.10 and 4.4) to the case of Polish groups (not just ${}^{\omega}2$).

Hopefully, the investigations of stnd will shed some light on the dual case of std_κ. In particular:

Problem 5.9. Is it consistent to have a Borel set $B \subseteq {}^{\omega}2$ such that

- B has uncountably many pairwise disjoint translations, but
- there is no perfect set of pairwise disjoint translations of B ?

Finally, let us recall the big question concerning the "cutting point" in this considerations.

Problem 5.10. Is $\lambda_{\omega_1} = \aleph_{\omega_1}$?

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