The Slicing Axioms

Ziemowit Kostana[*], Saharon Shelah[†‡]

May 16, 2021

Abstract

We introduce a family of axioms Slice_κ, which claim the existence of nontrivial decompositions of the form

$$2^{<\kappa} = \bigcup_{\alpha<\kappa} 2^{<\kappa} \cap M_\alpha,$$

where $\{M_\alpha \mid \alpha < \kappa\}$ is a sequence of transitive models of set theory. We study compatibility of these axioms with versions of Martin's Axiom, and in particular show that $\mathrm{Slice}_{\omega_1}$ is compatible only with some very weak form of MA.

Keywords: Martin's Axiom, Suslin forcing, transitive models
MSC classification: 03E50 03E17 03E35

1 Introduction

1.1 How "compact" is the real line?

We introduce and study a family of axioms Slice_κ for cardinal numbers κ. The axiom Slice_κ basically claims that there exists an increasing sequence $\{M_\alpha \mid \alpha < \kappa\}$ of transitive models of ZFC, which decomposes $2^{<\kappa}$ into an increasing union

$$2^{<\kappa} = \bigcup_{\alpha<\kappa} 2^{<\kappa} \cap M_\alpha.$$

Our initial motivation was to find a single model of Martin's Axiom, which doesn't satisfy typical consequences of PFA. This was in turn motivated by the following intuition:

If the universe is sufficiently complete, in the sense that it has many generic filters, then any transitive submodel containing enough reals, contain all the reals.

This intuition is supported for example by the following result:

Theorem 1 (Thm. 8.6, [8]). *If MM holds, then any inner model with correct ω_2 contains all reals.*

The conclusion is quite strong, so it makes sense to ask what is left if we weaken MM to MA_{ω_1}. This motivated us to formulate the axiom $\mathrm{Slice}_{\omega_1}$, which turned out to be inconsistent with MA_{ω_1}. The main results of this paper are the following

Theorem (Thm. 2). $\mathrm{Slice}_{\omega_1} \implies \neg\,\mathrm{MA}_{\omega_1}(\sigma\text{-}centred)$.

Theorem (Thm. 5). *If κ is a regular cardinal such that $\kappa^\omega = \kappa$, then the following theory is consistent*

$$ZFC + MA(\textit{Suslin}) + \mathrm{Slice}_{\omega_1} + \text{"}2^\omega = \kappa\text{"}.$$

Theorem (Thm. 7). *Assume that $\omega < \kappa \leq \theta$ are regular cardinals, such that $\theta^{<\kappa} = \theta$. Then the following theory is consistent*

$$ZFC + MA_{<\kappa} + \mathrm{Slice}_\kappa + \text{"}2^\omega = \theta\text{"}.$$

[*]Research of Z. Kostana was supported by the GAČR project EXPRO 20-31529X and RVO: 67985840.
[†]Research of second author partially supported by Israel Science Foundation (ISF) grant no: 1838/19. Paper number 1210 on the publication list.
[‡]Research of both authors partially supported by NSF grant no: DMS 1833363.

The first of these results provides another argument in favor of the informal claim from the beginning. The class of Suslin forcings is a class of c.c.c. forcings, which admit simple (analytic) definitions (see Definition 2). This class is more extensively described in [2]. Martin's Axiom for this class is a considerable weakening of the full MA.

Theorem ([6])**.** *$MA(Suslin)$ implies each of the following:*

1. $\mathrm{Add}(\mathcal{N}) = 2^\omega$,
2. $\mathrm{Add}(\mathcal{SN}) = 2^\omega$,
3. 2^ω *is regular,*
4. *each MAD family of subsets of ω has size 2^ω.*

It follows from 1. that all cardinal characteristics in the Cichoń's diagram have value 2^ω. $\mathcal{SN}$ stands for the class of strong measure zero sets.

Theorem ([6])**.** *$MA(Suslin)$ does not imply any of the following:*

1. $\mathfrak{t} = 2^\omega$,
2. $\mathfrak{s} = 2^\omega$,
3. $\forall \kappa < 2^\omega \quad 2^\kappa = 2^\omega$,
4. *there is no Suslin tree.*

For an elaborated discussion of cardinal invariants of the continuum we refer the reader to [4].

1.2 Preliminaries

All non-standard notions are introduced in the subsequent sections. By *reals* we denote elements of the sets ω^ω, 2^ω, or seldom $\mathbb{R}$. We take the liberty to freely identify Borel functions with their Borel codes, so whenever we claim that

$$f \in M,$$

for some Borel function $f \subseteq 2^\omega \times 2^\omega$, and $M \models ZFC$, it should be understood that it is the Borel code of f that belongs to M (so we don't bother if, for instance, $\operatorname{dom} f \not\subseteq M$).
When we write $\mathbb{P} = \{\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha |\ \alpha < \theta\}$ for a finite-support iteration of forcings, we sometimes denote by $\mathbb{P}$ the final step of the iteration, that is $\mathbb{P} = \mathbb{P}_\theta$. When dealing with infinite iterations, we assume that $\mathbb{P}_0$ is the trivial forcing. A function $i : \mathbb{P}_0 \hookrightarrow \mathbb{P}_1$ is a *complete embedding* if the following assertions hold:

1. $\forall p, q \in \mathbb{P}_0 \quad p_0 \leq p_1 \implies i(p_0) \leq i(p_1)$,
2. $\forall p, q \in \mathbb{P}_0 \quad p_0 \perp p_1 \implies i(p_0) \perp i(p_1)$,
3. If $\mathcal{A} \subseteq \mathbb{P}_0$ is a maximal antichain, then $i[\mathcal{A}] \subseteq \mathbb{P}_1$ is a maximal antichain.

We write $\mathbb{P}_0 \lessdot \mathbb{P}_1$ if $\mathbb{P}_0 \subseteq \mathbb{P}_1$ and the inclusion is a complete embedding. We will be frequently using the following observation

Proposition 1. *If V is a countable transitive model of ZFC, $\mathbb{P}_0, \mathbb{P}_1 \in V$, and $\mathbb{P}_0 \subseteq \mathbb{P}_1$ is an inclusion of partial orders, then the following conditions are equivalent:*

1. $\mathbb{P}_0 \lessdot \mathbb{P}_1$,
2. *If a filter $G \subseteq \mathbb{P}_1$ is $\mathbb{P}_1$-generic over V then $G \cap \mathbb{P}_0$ is $\mathbb{P}_0$-generic over V.*

2 The Slicing Axioms

Definition 1. Let κ be any uncountable cardinal. We will say that Slice_κ holds if there exists a sequence of transitive classes (not necessarily proper) $\{M_\alpha | \ \alpha < \kappa\}$, such that the following conditions are satisfied

- $\forall \alpha < \kappa \quad M_\alpha \models ZFC$,
- $\forall \alpha < \omega_1 \quad \omega_1^{M_\alpha} = \omega_1$,
- $2^{<\kappa} = \bigcup_{\alpha<\kappa} 2^{<\kappa} \cap M_\alpha$,
- $\forall \alpha < \beta < \kappa \quad 2^{<\kappa} \cap M_\alpha \subsetneq 2^{<\kappa} \cap M_\beta$.

We will say that the sequence $\{M_\alpha | \ \alpha < \kappa\}$ *preserves cardinals* if $\kappa \in M_0$ and for each cardinal $\lambda \in M_0$ and each $\alpha < \kappa$, λ^{M_α} is a cardinal.

The most important of the slicing axioms is perhaps Slice_{ω_1}, since it claims that the real line can be decomposed into an increasing union of ω_1 many sets, which belong to bigger and bigger models. The fact that MA_{ω_1} is inconsistent with Slice_{ω_1} shows, that the Martin's Axiom on ω_1 imposes certain compactness on the real line.

3 Slicing the real line

We begin with showing that Martin's Axiom on ω_1 is not compatible with Slice_{ω_1}.

Theorem 2. $\text{Slice}_{\omega_1} \implies \neg\, \text{MA}_{\omega_1}(\sigma\text{-}centred)$.

In the proof we will utilize the known result from [5]. Recall that a set $A \subseteq \mathbb{R}$ is a *Q-set*, if each subset of A is a relative F_σ.

Theorem 3 ([5]). $\text{MA}(\sigma\text{-}centred)$ *implies that each set of cardinality less than* 2^ω *is a Q-set.*

Proof of Theorem 2. Assume that MA_{ω_1} holds, and $(M_\alpha)_{\alpha<\omega_1}$ is a sequence of models witnessing Slice_{ω_1}. $M_0 \models$ "2^ω is uncountable" , so there exists a sequence of pairwise distinct reals $X = \{x_\alpha | \ \alpha < \omega_1\} \in M_0$ (note that this sequence is *really* of the length ω_1). Let $f : \omega_1 \hookrightarrow 2^\omega$ be a function such that $\forall \alpha < \omega_1 \ f(\alpha) \notin M_\alpha$. We will obtain a contradiction, by showing that there exists some $\eta < \omega_1$, for which $\text{rg}(f) \subseteq M_\eta$.

For every natural number m, let $A_m = \{x_\alpha | \ f(\alpha)(m) = 1\} = X \cap F_m$, where F_m is an F_σ subset of reals. Since the sequence $(F_m)_{m<\omega}$ can be coded by a real, clearly it belongs to some model M_η. It is enough to show that using this sequence we can give a definition of $\text{rg}(f)$. But

$$\text{rg}(f) = \{x \in 2^\omega | \ \exists \alpha < \omega_1 \ \forall m < \omega \quad x_\alpha \in F_m \iff x(m) = 1\}.$$

□

It is compatible with any value of 2^ω that Slice_{ω_1} holds and is witnessed by a cardinal preserving sequence.

Proposition 2. *Let* $\mathbb{P}$ *be any finite-support product of c.c.c. forcings adding reals, of the length at least* ω_1. *Then* $\mathbb{P} \Vdash \text{Slice}_{\omega_1}$, *and the corresponding sequence of models is cardinal preserving.*

Proof. Let us consider a finite-support product of c.c.c. forcings

$$\mathbb{P} = \prod_{i\in I} \mathbb{P}_i,$$

where each $\mathbb{P}_i$ adds some real number, and $|I| \geq \omega_1$. We can decompose I into a strictly increasing union $I = \bigcup_{\gamma<\omega_1} I_\gamma$. For each $\alpha < \omega_1$ the product $\prod_{i\in I_\alpha} \mathbb{P}_i$ can be identified with a complete suborder of $\mathbb{P}$.

If $G \subseteq \mathbb{P}$ is generic over some model V, then Slice_{ω_1} is witnessed by the sequence

$$M_\alpha = V[G \cap \prod_{i\in I_\alpha} \mathbb{P}_i].$$

□

The following was proved by Baumgartner in [3].

Theorem ([3]). *It is consistent with MA_{ω_1}, that all ω_1-dense subsets of reals are order-isomorphic. In particular, each ω_1-dense set of reals has a non-trivial order-automorphism.*

The natural question whether this assertion follows from MA_{ω_1} was resolved by Avraham and the second author in [1].

Theorem ([1]). *It is consistent with MA_{ω_1}, that there exists a rigid ω_1-dense real order type.*

This is also an easy consequence of $\mathrm{Slice}_{\omega_1}$.

Theorem 4. $\mathrm{Slice}_{\omega_1}$ *implies that there is an ω_1-dense rigid subset of the real line.*

Proof. Let $(M_\alpha)_{\alpha<\omega_1}$ be a sequence witnessing $\mathrm{Slice}_{\omega_1}$. For each α, we choose

$$x_\alpha \in \mathbb{R} \cap (M_\alpha \setminus \bigcup_{\beta<\alpha} M_\beta).$$

We can easily arrange the construction, so that we hit each open interval ω_1-many times. The set $X = \{x_\alpha |\ \alpha < \omega_1\}$ is ω_1-dense, and it remains to prove, that it is also rigid. Suppose that $f : X \to X$ is an order isomorphism. f extends uniquely to a continuous function $f' : \mathbb{R} \to \mathbb{R}$, and each such function can be coded by a real number. Therefore there is some $\eta < \omega_1$, such that $f' \in M_\eta$. Now, for any $\xi > \eta$, it is not possible that $f(x_\eta) = x_\xi$, because it would mean $x_\xi \in M_\eta$, contrary to the choice of x_ξ. But, likewise, it is not possible that $f^{-1}(x_\eta) = x_\xi$. The conclusion is that for all $\xi > \eta$, $f(x_\xi) = x_\xi$. But this means that f is identity on a dense set, and therefore everywhere. □

4 Slicing the real line while preserving MA(Suslin)

We are going to show that $\mathrm{Slice}_{\omega_1}$ is consistent with a version of Martin's Axiom which takes into account only partial orders representable as analytic sets (see [2], Ch. 3.6, or [6]).

Definition 2. A partial order $(\mathbb{P}, \leq)$ has a *Suslin definition* if $\mathbb{P} \in \Sigma^1_1(\omega^\omega)$, and both ordering and incompatibility relations in $\mathbb{P}$ are analytic relations on ω^ω. $\mathbb{P}$ is *Suslin* if it has a Suslin definition and is c.c.c.

The following is the main result of this Section.

Theorem 5. *If κ is a regular cardinal such that $\kappa^\omega = \kappa$, then the following theory is consistent*

$$ZFC + MA(\textit{Suslin}) + \mathrm{Slice}_{\omega_1} + \text{"}2^\omega = \kappa\text{"}.$$

Let $\psi(-,-,-,-)$ be a universal analytic formula, i.e. a Σ^1_1 formula with the property that for each analytic set $P \subseteq \omega^\omega \times \omega^\omega \times \omega^\omega$ there exists $r \in \omega^\omega$ such that

$$P = \{x \in \omega^\omega \times \omega^\omega \times \omega^\omega |\ \psi(x, r)\}.$$

We want to use ψ to add generic filters to all possible Suslin forcings. We will say that $\psi(-,-,-,\dot{r}_\alpha)$ *defines* $\dot{\mathbb{Q}}_\alpha$ if $\dot{r}_\alpha$ is a $\mathbb{P}_\alpha$-name for a real and $\mathbb{P}_\alpha$ forces each of the following

$$\begin{gathered}\dot{\mathbb{Q}}_\alpha \text{ is a separative partial order with the greatest element } 0,\\ \psi(x,1,1,\dot{r}_\alpha) \iff x \in \dot{\mathbb{Q}}_\alpha,\\ \psi(x,y,2,\dot{r}_\alpha) \iff x \leq_{\dot{\mathbb{Q}}_\alpha} y,\\ \psi(x,y,3,\dot{r}_\alpha) \iff x \perp_{\dot{\mathbb{Q}}_\alpha} y.\end{gathered}$$

We will write $\psi^{\in}(x,z)$ for $\psi(x,1,1,z)$, $\psi^{\perp}(x,y,z)$ for $\psi(x,y,3,z)$, and $\psi^{\leq}(x,y,z)$ for $\psi(x,y,2,z)$.

We are going to iterate all Suslin forcings, each of them cofinally many times. More precisely, we define by induction a finite-support iteration $\{\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha |\ \alpha < \kappa\}$:

- $\mathbb{P}_0 = \{0\}$,
- $\mathbb{P}_\alpha \Vdash \text{"}\dot{\mathbb{Q}}_\alpha = \{x \in \omega^\omega |\ \psi(x, \dot{r}_\alpha)\}$ if this formula defines a Suslin forcing; else $\dot{\mathbb{Q}} = \{0\}\text{"}$,

The variable $\dot{r}_\alpha$ ranges over all reals, and all possible names for reals, each of them cofinally many times. In order to iterate through all possible parameters using a suitable bookkeeping, we introduce the class of *simple* conditions, following [2].

Definition 3. By induction on α we define *simple* conditions in $\mathbb{P}_\alpha$.

- $\alpha = 0$. $\mathbb{P}_0 = \{0\}$, and we declare 0 to be simple.
- $\alpha + 1$. $(p, \dot{q}) \in \mathbb{P}_{\alpha+1}$ is simple if $p \in \mathbb{P}_\alpha$ is simple and
$$\dot{q} = \{(m, n, p^m_n)|\ m, n < \omega,\ p^m_n \in \mathbb{P}_\alpha\},$$
where each p^m_n is a simple condition in $\mathbb{P}_\alpha$. (for each $m \in \omega$, the set $\{p^m_n|\ n < \omega\}$ is a maximal antichain deciding $\dot{q}(m)$, i.e. $p^m_n \Vdash \dot{q}(m) = n$)
- $\lim \alpha$. $p \in \mathbb{P}_\alpha$ is simple if for each $\beta < \alpha$, $p \restriction \beta \in \mathbb{P}_\beta$ is simple.

It is straightforward to check by induction, that the set of simple $\mathbb{P}_\alpha$-conditions is dense in $\mathbb{P}_\alpha$, and that each $\mathbb{P}_\alpha$ has at most κ many names for reals (if we restrict to names with simple conditions).

Proposition 3. *If $2^\omega \leq \kappa$ is an uncountable regular cardinal such that $\kappa^\omega = \kappa$, then*
$$\mathbb{P}_\kappa \Vdash MA(\textit{Suslin}) + "2^\omega = \kappa".$$

Proof. Let us denote by W_α the corresponding extensions of V by $\mathbb{P}_\alpha$. Let $(S, \leq)$ be a Suslin forcing in W_κ. Assume S is defined by the formula $\psi(-, r)$. We fix a family $\{A_\gamma|\ \gamma < \lambda\}$ of maximal antichains in S, where $\lambda < \kappa$. By the Löwenheim-Skolem theorem, we can find an elementary substructure of $(S, \leq, A_\gamma)_{\gamma<\lambda}$ of size λ. For simplicity of notation we can assume that S is this substructure, and so $|S| \leq \lambda$. Therefore $(S, A_\gamma, \leq)_{\gamma<\lambda} \in W_\delta$, for some $\delta < \kappa$, and we can enlarge δ so that
$$\mathbb{P}_\delta \Vdash \dot{r}_\delta = r.$$
By absoluteness of the formulae $\psi^\in(-, r)$, $\psi^\perp(-, r)$, and $\psi^\leq(-, r)$, the partial order defined by $\psi(-, r)$ in W_δ is a suborder of the one defined by this formula in W_κ (even a complete suborder, which is not relevant here). Therefore the generic filter added for $\dot{\mathbb{Q}}_\delta$ in W_δ will be a filter intersecting the sets A_γ in S. $\square$

If N is a transitive class containing κ, we can define by induction the relativized iteration $\mathbb{P}^N_\kappa \subseteq \mathbb{P}_\kappa$, taking into account only names from N.

- $\mathbb{P}^N_0 = \{0\}$,
- $\mathbb{P}^N_\alpha \Vdash "\dot{\mathbb{Q}}^N_\alpha = \{x \in \omega^\omega|\ \psi^\in(x, \dot{r}_\alpha)\}$ if this formula defines a Suslin forcing, $\dot{r}_\alpha \in N$, and $\dot{r}_\alpha$ is a $\mathbb{P}^N_\alpha$-name; else $\dot{\mathbb{Q}}^N_\alpha = \{0\}"$,
- $\mathbb{P}^N_{\alpha+1} = \mathbb{P}^N_\alpha * \dot{\mathbb{Q}}^N_\alpha$.

If we take direct limits in the limit step, it is clear that $\mathbb{P}^N_\alpha$ is really a subset of $\mathbb{P}_\alpha$. Note, that we do not define names $\dot{r}_\alpha$ inductively along the way, since they have already been defined in the construction of $\mathbb{P}_\kappa$, which we take as granted. This construction is inspired by the lemmas 1.4 and 1.5 from [6], and conceptually is very similar. In order for it to work as desired, we prove by induction some properties of $\mathbb{P}^N_\alpha$.

Theorem 6. *If N is a transitive class containing κ, then for all $\alpha \leq \kappa$*
$$\mathbb{P}^N_\alpha \lessdot \mathbb{P}_\alpha.$$

Specifically:

1. *If $p_0 \perp p_1$ in $\mathbb{P}^N_\alpha$, then $p_0 \perp p_1$ in $\mathbb{P}_\alpha$.*
2. *If $p_0 \leq p_1$ in $\mathbb{P}^N_\alpha$, then $p_0 \leq p_1$ in $\mathbb{P}_\alpha$.*
3. *If $G \subseteq \mathbb{P}_\alpha$ is a filter generic over V, then $G \cap \mathbb{P}^N_\alpha \subseteq \mathbb{P}^N_\alpha$ is also generic over V.*

Proof.

1.

- $\alpha = 0$. Clear.

- $\alpha + 1$. We can assume that $\dot{\mathbb{Q}}^N_\alpha$ is defined by the formula $\psi(-, \dot{r}_\alpha)$, for otherwise $\mathbb{P}^N_{\alpha+1} = \mathbb{P}^N_\alpha$, and we are done by the induction hypothesis. Fix two incomparable conditions $p_0, p_1 \in \mathbb{P}^N_{\alpha+1}$. Then $p_0 = (p'_0, \dot{q}_0)$, $p_1 = (p'_1, \dot{q}_1)$, where $p'_0, p'_1 \in \mathbb{P}^N_\alpha$, and

$$p'_0 \Vdash \psi^{\in}(\dot{q}_0, \dot{r}_\alpha),$$

$$p'_1 \Vdash \psi^{\in}(\dot{q}_1, \dot{r}_\alpha).$$

The forcing relation used above is a relation from $\mathbb{P}^N_\alpha$, however since $\dot{r}_\alpha$, $\dot{q}_0$ and $\dot{q}_1$ are $\mathbb{P}^N_\alpha$-names, this is the same relation as coming from $\mathbb{P}_\alpha$ (remember that $\mathbb{P}^N_\alpha \lessdot \mathbb{P}_\alpha$). We aim to show that $p_0 \perp p_1$ in $\mathbb{P}_{\alpha+1}$.

If $p'_0 \perp p'_1$ in $\mathbb{P}_\alpha$, then clearly $p_0 \perp p_1$ in $\mathbb{P}_{\alpha+1}$, so assume otherwise, and fix $p \leq p'_0, p'_1$ (in $\mathbb{P}_\alpha$). Let $p \in G \subseteq \mathbb{P}_\alpha$ be a filter generic over V. Conditions p_0 and p_1 were incomparable in $\mathbb{P}^N_{\alpha+1}$ and, by the induction hypothesis, $G \cap \mathbb{P}^N_\alpha \subseteq \mathbb{P}^N_\alpha$ is generic over V, therefore

$$V[G \cap \mathbb{P}^N_\alpha] \models \psi^{\perp}(\dot{q}_0[G], \dot{q}_1[G], \dot{r}_\alpha[G]).$$

By absoluteness

$$V[G] \models \psi^{\perp}(\dot{q}_0[G], \dot{q}_1[G], \dot{r}_\alpha[G]).$$

Since p was arbitrary, it follows that $p_0 \perp p_1$ in $\mathbb{P}_{\alpha+1}$.

- $\lim \alpha$. Follows from the induction hypothesis, since conditions have finite supports.

2.

- $\alpha = 0$. Clear.

- $\alpha + 1$. Again, we can assume that $\dot{\mathbb{Q}}^N_\alpha$ is defined by the formula $\psi(-, \dot{r}_\alpha)$. Fix two conditions $p_0 \leq p_1 \in \mathbb{P}^N_{\alpha+1}$. Then $p_0 = (p'_0, \dot{q}_0)$, $p_1 = (p'_1, \dot{q}_1)$, where $p'_0, p'_1 \in \mathbb{P}^N_\alpha$, and

$$p'_0 \Vdash \psi^{\in}(\dot{q}_0, \dot{r}_\alpha),$$

$$p'_1 \Vdash \psi^{\in}(\dot{q}_1, \dot{r}_\alpha).$$

By the induction hypothesis $p'_0 \leq p'_1$ in $\mathbb{P}_\alpha$. Moreover $\dot{r}_\alpha$, $\dot{q}_0$ and $\dot{q}_1$ are $\mathbb{P}^N_\alpha$-names, so the forcing relation

$$p'_0 \Vdash \dot{q}_0 \leq \dot{q}_1$$

holds in $\mathbb{P}^N_\alpha$ as well as in $\mathbb{P}_\alpha$.

- $\lim \alpha$. Follows from the induction hypothesis, since conditions have finite supports.

3.

- $\alpha = 0$. Clear.

- $\lim \alpha$. Let $\{\overline{p}_n |\ n < \omega\}$ be a maximal antichain in $\mathbb{P}^N_\alpha$, and $\overline{p} \in \mathbb{P}_\alpha$. There is some $\gamma < \alpha$ such that $\overline{p} \in \mathbb{P}_\gamma$. $\{\overline{p}_n \restriction \gamma |\ n < \omega\}$ might not be an antichain in $\mathbb{P}^N_\gamma$, however each condition in $\mathbb{P}^N_\gamma$ is compatible with some $p_n \restriction \gamma$. We can refine $\{\overline{p}_n \restriction \gamma |\ n < \omega\}$ to an antichain in $\mathbb{P}^N_\gamma$, and this antichain will remain maximal in $\mathbb{P}_\gamma$ by the induction hypothesis. Therefore $\{\overline{p}_n \restriction \gamma |\ n < \omega\}$ intersects every condition in $\mathbb{P}_\gamma$, and in particular some $\overline{p}_n \restriction \gamma$ is compatible with $\overline{p}$ in $\mathbb{P}_\gamma$. But then $\overline{p}_n$ is compatible with $\overline{p}$ in $\mathbb{P}_\alpha$.

- $\alpha+1$. We aim to show that for any $G \subseteq \mathbb{P}_{\alpha+1}$ generic over V, $G \cap \mathbb{P}^N_{\alpha+1}$ is also generic over V.

Lemma 1. *If $G \subseteq \mathbb{P}_\alpha$ is generic over V, and $H \subseteq \dot{\mathbb{Q}}_\alpha[G]$ is generic over $V[G]$, then $H \cap \dot{\mathbb{Q}}^N_\alpha[G] \subseteq \dot{\mathbb{Q}}^N_\alpha[G]$ is generic over $V[G \cap \mathbb{P}^N_\alpha]$.*

Why is this sufficient? Let $\overline{G} \subseteq \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$ be a filter generic over V. Recalling the notation from [7],

$$\overline{G} = G * H = \{(p, \dot{q}) |\ p \in G,\ \dot{q}[G] \in H\},$$

where

$$G = \{p \in \mathbb{P}_\alpha |\ \exists \dot{q}\ \in \dot{\mathbb{Q}} \quad (p, \dot{q}) \in \overline{G}\},$$

and

$$H = \{\dot{q}[G] |\ \exists p \in G \quad (p, \dot{q})\ \in \overline{G}\}.$$

It is known that for any iteration $\mathbb{P} * \dot{\mathbb{Q}}$, if $G \subseteq \mathbb{P}$ is generic over V and $H \subseteq \dot{\mathbb{Q}}[G]$ is generic over $V[G]$, then $G * H$ is generic for $\mathbb{P} * \dot{\mathbb{Q}}$ over V (for details consult for example [7], Section 5, Chapter VIII). Let $G' = G \cap \mathbb{P}^N_\alpha$. It is generic for $\mathbb{P}^N_\alpha$ over V by the induction hypothesis. Now for filters G and H defined above

$$(G * H) \cap (\mathbb{P}^N_\alpha * \dot{\mathbb{Q}}^N_\alpha) = \{(p, \dot{q}) |\ p \in G',\ \dot{q}[G] \in H,\ \dot{q} \in \dot{\mathbb{Q}}^N_\alpha\} =$$
$$\{(p, \dot{q}) \in \mathbb{P}^N_\alpha * \dot{\mathbb{Q}}^N_\alpha |\ p \in G',\ \dot{q}[G'] \in H\} = G' * (H \cap \dot{\mathbb{Q}}^N_\alpha[G']).$$

But if the conclusion of Lemma 1 holds, this is a $\mathbb{P}^N_\alpha * \dot{\mathbb{Q}}^N_\alpha$-generic filter over V.

We turn to the proof of Lemma 1.

Proof. Fix a maximal antichain $\mathcal{A} \subseteq \dot{\mathbb{Q}}^N_\alpha[G] = \dot{\mathbb{Q}}^N_\alpha[G']$, belonging to $V[G']$. As $\mathcal{A}$ is a countable set of reals, it can be coded using a single real $z \in \omega^\omega$. Recall that $\dot{\mathbb{Q}}^N_\alpha[G']$ is defined in $V[G']$ by the formula ψ with the parameter $\dot{r}_\alpha[G'] = \dot{r}_\alpha[G]$. It is standard to check, that the following claim can be written as a Π^1_1 formula.

$\phi(x, y) =$ "x is a real coding a maximal antichain in the partial ordering defined by the formula $\psi(-, -, -, y)$".

Now

$$V[G'] \models \phi(z, \dot{r}_\alpha[G']),$$

and so by absoluteness

$$V[G] \models \phi(z, \dot{r}_\alpha[G]).$$

But $\psi(-, \dot{r}_\alpha[G])$ is the formula defining $\dot{\mathbb{Q}}_\alpha[G]$ in $V[G]$. Therefore $\mathcal{A}$ remains maximal in $\dot{\mathbb{Q}}_\alpha[G]$, and conclusion of the Lemma easily follows. □

This concludes the proof.

□

Let us note that even if N is an inner model of ZFC, usually $\mathbb{P}^N_\kappa \notin N$. Definition of $\mathbb{P}^N_\kappa$ makes use of a list of $\mathbb{P}^N_\alpha$-names, for all $\alpha < \kappa$, and although *some* such enumeration belongs to N (as it is a model of choice), this particular might not. In what sense is $\mathbb{P}^N_\kappa$ a *relativized* version of $\mathbb{P}_\kappa$, is explained by the next lemma.

Lemma 2. *For any transitive class N containing κ, for each $\alpha \leq \kappa$, $N \cap \mathbb{P}_\alpha \subseteq \mathbb{P}^N_\alpha$.*

Proof. We proceed by induction.

- $\alpha = 0$. Clear.
- $\lim \alpha$. If $r \in N \cap \mathbb{P}_\alpha$, we choose $\gamma < \alpha$ containing the support of r. Then $r \restriction \gamma \in \mathbb{P}_\gamma \cap N \subseteq \mathbb{P}^N_\gamma$. It is routine to verify by induction that for all $\gamma \leq \delta \leq \alpha$, $r \restriction \delta \in \mathbb{P}^N_\delta$.
- $\alpha + 1$. If $r = (p, \dot{q}) \in N \cap (\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha)$, then $p \in \mathbb{P}^N_\alpha$, $\dot{q} \in N$, and we need only to prove that $\dot{q}$ is a $\mathbb{P}^N_\alpha$-name. But note, that $\dot{q} = \{(m, n, s^m_n) |\ m, n < \omega\}$, where $\forall m, n < \omega \quad s^m_n \in \mathbb{P}_\alpha \cap N \subseteq \mathbb{P}^N_\alpha$.

□

Lemma 3. *For each $\alpha \leq \kappa$, if $p \in \mathbb{P}_\alpha$ is simple then p is definable (in the language of set theory) with a parameter from κ^ω.*

Proof.

- $\alpha = 0$. Clear, since each real is definable with a real parameter.
- $\alpha + 1$. Let $r = (p, \dot{q})$ be simple. We can write

  $$\dot{q} = \{(m, n, p^m_n) |\ m, n \in \omega,\ p^m_n \in \mathbb{P}_\alpha\},$$

  where each p^m_n is simple. By the induction hypothesis each p^m_n definable with a parameter from κ^ω, and so is p. Clearly r can be defined from them, and so r is definable with countably many parameters from κ^ω. We can easily code them as a single parameter.
- $\lim \alpha$. Fix $r \in \mathbb{P}_\alpha$. r has finite support, so there exists $\beta < \alpha$ containing the support of r. By the induction hypothesis $p \restriction \beta$ is definable with a parameter from κ^ω, and p is definable with parameters $p \restriction \beta$, β, and α.

$\square$

Proof of Theorem 5. We start with a model $V \models \text{Slice}_{\omega_1} +$"$2^\omega = \kappa$", and we assume moreover that the sequence $\{M_\alpha |\ \alpha < \omega_1\}$ witnessing Slice_{ω_1} satisfies the following stronger property:

$$\kappa^\omega = \bigcup_{\alpha < \omega_1} \kappa^\omega \cap M_\alpha.$$

Such model is easy to get, for example by adding κ many Cohen reals to a model of CH, and proceeding like in the proof of Proposition 2.

Let $\mathbb{P} = \{\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha |\ \alpha < \kappa\}$ be the iteration described above, which forces

$$MA(\text{Suslin}) + \text{"}2^\omega = \kappa\text{"}.$$

We claim that if $G \subseteq \mathbb{P}$ is generic over V, then the sequence $V[G \cap \mathbb{P}^{M_\alpha}]$ witnesses Slice_{ω_1} in $V[G]$. For this we need to show two things

1. If $r \in \omega^\omega$, then $r \in V[G \cap \mathbb{P}^{M_\alpha}]$ for some $\alpha < \kappa$.
2. None of the models $V[G \cap \mathbb{P}^{M_\alpha}]$ contains all reals.

Concerning 1. assume that $\mathbb{P}_\kappa \Vdash \dot{r} \in \omega^\omega$. We can assume that

$$\dot{r} = \{(m, n, p^m_n) |\ m, n < \omega\},$$

and all conditions p^m_n are simple. By Lemma 3 each condition p^m_n is definable with a parameter from κ^ω. It follows, that $\dot{r}$ is definable with a parameter from κ^ω, and by our assumption this parameter belongs to some model M_α. Therefore $\dot{r}$ is a $\mathbb{P}^{M_\alpha}$-name, and so

$$\dot{r}[G] = \dot{r}[G \cap \mathbb{P}^{M_\alpha}] \in V[G \cap \mathbb{P}^{M_\alpha}].$$

Concerning 2. fix a real $r \in \omega^\omega \setminus M_\alpha$. There exists a representation of the Cohen forcing as a Borel subset of ω^ω, from which the real r is definable. For concreteness, let us put

$$\mathbb{C}_r = \omega^{<\omega} \cup \{r\} \subseteq \omega^\omega,$$

where $\omega^{<\omega}$ is ordered by the end-extension and

$$\forall s \in \omega^{<\omega}\ s \perp r.$$

Since $r \notin M_\alpha$, it follows that for some $\gamma < \kappa$

$$\mathbb{P}^{M_\alpha}_\gamma \Vdash \dot{\mathbb{Q}}^{M_\alpha}_\gamma = \{0\},$$

and

$$\mathbb{P}_\gamma \Vdash \dot{\mathbb{Q}}_\gamma = \mathbb{C}_r.$$

Therefore we can find a complete embedding of $\mathbb{C}_r$ into the quotient forcing

$$\mathbb{C}_r \hookrightarrow \mathbb{P}_\kappa / (\mathbb{P}^{M_\alpha}_\kappa \cap G),$$

given by the formula

$$x \mapsto 1_{\mathbb{P}_\gamma} {}^\frown (1_{\mathbb{P}_\gamma}, x) {}^\frown 1_{\mathbb{P}_{\kappa \setminus (\gamma+1)}}.$$

This shows that $V[G]$ contains a Cohen real over $V[\mathbb{P}^{M_\alpha} \cap G]$. $\square$

5 Slicing $2^{<\kappa}$

Although MA_{ω_1} is inconsistent with Slice_{ω_1}, it is consistent with Slice_κ for any $\kappa > \omega_1$. The idea of the proof is very much like that of Theorem 5, and actually even simpler, because we iterate forcings directly, instead of coding them as Borel sets.

Theorem 7. *Assume that $\omega < \kappa \leq \theta$ are regular cardinals, such that $\theta^{<\kappa} = \theta$. Then the following theory is consistent*

$$ZFC + MA_{<\kappa} + \text{Slice}_\kappa + "2^\omega = \theta".$$

We are going to apply a finite-support iteration of the form

$$\mathbb{P} = \{\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha |\ \alpha < \theta\},$$

where for each $\alpha < \theta$

$$\mathbb{P}_\alpha \Vdash \dot{\mathbb{Q}}_\alpha = (\lambda_\alpha, \dot{\leq}_\alpha),$$

for $\lambda_\alpha < \kappa$. We also assume that $0 \in \lambda_\alpha$ is always the largest element in $\dot{\mathbb{Q}}_\alpha$. We can arrange the iteration so that each partial order of size $< \kappa$ will appear cofinally many times (see the proof of Proposition 3).

Definition 4. By induction on α, we define the class of *simple* $\mathbb{P}_\alpha$-conditions.

- $\alpha = 0$. $\mathbb{P}_0 = \{0\}$, and we declare 0 to be simple.
- $\alpha + 1$. $(p, \dot{q}) \in \mathbb{P}_{\alpha+1}$ is simple if $p \in \mathbb{P}_\alpha$ is simple and $\dot{q} = \{(\gamma_n, p_n)|\ n < \omega\}$, where conditions p_n are simple.
- $\lim \alpha$. $p \in \mathbb{P}_\alpha$ is simple if for each $\beta < \alpha$, $p \restriction \beta \in \mathbb{P}_\beta$ is simple.

Like in the previous section, is is easy to check that the set of simple conditions is always dense.

Lemma 4. *For each $\alpha \leq \kappa$, if $p \in \mathbb{P}_\alpha$ is simple then p is definable (in the language of set theory) with a parameter from κ^ω.*

Proof.

- $\alpha = 0$. Clear.
- $\alpha+1$. Let $r = (p, \dot{q})$ be simple. We can write $\dot{q} = \{(\gamma_n, p_n)|\ n < \omega\}$, where conditions p_n are simple. By the induction hypothesis each p_n is definable with a parameter from κ^ω, and so is p. Clearly r can be defined from them, and so r is definable with countably many parameters, which we can code as one.
- $\lim \alpha$. Fix $r \in \mathbb{P}_\alpha$. r has finite support, so there exists $\beta < \alpha$ containing the support of r. By the induction hypothesis $p \restriction \beta$ is definable with a parameter from κ^ω, and p is definable with parameters $p \restriction \beta$, β, and α.

$\square$

Like in the previous Section, we can define by induction the relativized forcings $\mathbb{P}^N_\kappa \subseteq \mathbb{P}_\kappa$, taking into account only names from N.

- $\mathbb{P}^N_0 = \{0\}$,
- Assume $\mathbb{P}^N_\alpha$ is defined. We define a $\mathbb{P}^N_\alpha$-name $\dot{\mathbb{Q}}^N_\alpha$ as follows
  - $\dot{\mathbb{Q}}^N_\alpha = \dot{\mathbb{Q}}_\alpha$ if $\dot{\mathbb{Q}}_\alpha \in N$, and $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}^N_\alpha$-name,
  - $\dot{\mathbb{Q}}^N_\alpha = \{0\}$ otherwise.
- $\mathbb{P}^N_{\alpha+1} = \mathbb{P}^N_\alpha * \dot{\mathbb{Q}}^N_\alpha$.

In limit steps we take direct limits, so $\mathbb{P}^N_\kappa \subseteq \mathbb{P}_\kappa$. Repeating the proof of Lemma 2, we obtain

Lemma 5. *If N is a transitive class containing κ, $\alpha \leq \kappa$, then $\mathbb{P}_\alpha \cap N \subseteq \mathbb{P}^N_\alpha$.*

Lemma 6. *If N is a transitive class, then for all $\alpha \leq \kappa$*

$$\mathbb{P}^N_\alpha \lessdot \mathbb{P}_\alpha.$$

Specifically:

1. *If $p_0 \perp p_1$ in $\mathbb{P}^N_\alpha$, then $p_0 \perp p_1$ in $\mathbb{P}_\alpha$.*
2. *If $p_0 \leq p_1$ in $\mathbb{P}^N_\alpha$, then $p_0 \leq p_1$ in $\mathbb{P}_\alpha$.*
3. *If $\mathcal{A} \subseteq \mathbb{P}^N_\alpha$ is a maximal antichain, then $\mathcal{A}$ is maximal in $\mathbb{P}_\alpha$.*

Proof. We proceed by induction on α.

1.

- $\alpha = 0$. Clear.
- $\alpha + 1$. Assume $(p_0, \dot{q}_0) \perp (p_1, \dot{q}_1)$ in $\mathbb{P}^N_{\alpha+1}$. If $p_0 \perp p_1$ in $\mathbb{P}^N_\alpha$, then by the induction hypothesis $p_0 \perp p_1$ in $\mathbb{P}_\alpha$ and we are done. Suppose otherwise, and fix a condition $p \leq p_0, p_1$ from $\mathbb{P}_\alpha$. Let $G \subseteq \mathbb{P}_\alpha$ be any filter generic over V, containing p. $p_0, p_1 \in G \cap \mathbb{P}^N_\alpha$, so
$$\dot{q}_0[G \cap \mathbb{P}^N_\alpha] \perp \dot{q}_1[G \cap \mathbb{P}^N_\alpha]$$
in model $V[G \cap \mathbb{P}^N_\alpha]$, and in $V[G]$ as well. Since p and G were arbitrary, it follows that $(p_0, \dot{q}_0) \perp (p_1, \dot{q}_1)$ in $\mathbb{P}_{\alpha+1}$.
- $\lim \alpha$. Follows from the induction hypothesis, since the supports are finite.

2.

- $\alpha = 0$. Clear.
- $\alpha + 1$. Assume $(p_0, \dot{q}_0) \leq (p_1, \dot{q}_1)$ in $\mathbb{P}^N_{\alpha+1}$. From the induction hypothesis we know, that $p_0 \leq p_1$ in $\mathbb{P}_\alpha$, and $p_0 \Vdash \dot{q}_0 \leq \dot{q}_1$ in the sense of $\mathbb{P}^N_\alpha$. We must show that the assertion
$$p_0 \Vdash \dot{q}_0 \leq \dot{q}_1$$
holds also in the sense of $\mathbb{P}_\alpha$. If $\dot{\mathbb{Q}}^N_\alpha = \{0\}$ it is trivial. Otherwise $\dot{\mathbb{Q}}^N_\alpha = \dot{\mathbb{Q}}_\alpha$. In that case $\dot{q}_0$ and $\dot{q}_1$ are $\mathbb{P}^N_\alpha$-names, and the $\Vdash$ relation for them is the same in $\mathbb{P}^N_\alpha$ as in $\mathbb{P}_\alpha$.
- $\lim \alpha$. Follows from the induction hypothesis, since the supports are finite.

3.

- $\alpha = 0$. Clear.
- $\alpha + 1$. The proof is exactly the same, as in the paragraph after Lemma 1, so we need to prove the conclusion of Lemma 1 in the current setting. But this is trivial, once we recall that
$$\mathbb{P}^N_\alpha \Vdash \dot{\mathbb{Q}}^N_\alpha = \{0\},$$
or
$$\mathbb{P}^N_\alpha \Vdash \dot{\mathbb{Q}}^N_\alpha = \dot{\mathbb{Q}}_\alpha.$$
- $\lim \alpha$. Let $\{\overline{p}_n |\ n < \omega\}$ be a maximal antichain in $\mathbb{P}^N_\alpha$, and $\overline{p} \in \mathbb{P}_\alpha$. There is some $\gamma < \alpha$ such that $\overline{p} \in \mathbb{P}_\gamma$. $\{\overline{p}_n \restriction \gamma |\ n < \omega\}$ might not be an antichain in $\mathbb{P}^N_\gamma$, however each condition in $\mathbb{P}^N_\gamma$ is compatible with some $p_n \restriction \gamma$. We can refine $\{\overline{p}_n \restriction \gamma |\ n < \omega\}$ to an antichain in $\mathbb{P}^N_\gamma$, and this antichain will remain maximal in $\mathbb{P}_\gamma$ by the induction hypothesis. Therefore $\{\overline{p}_n \restriction \gamma |\ n < \omega\}$ intersects every condition in $\mathbb{P}_\gamma$, and in particular some $\overline{p}_n \restriction \gamma$ is compatible with $\overline{p}$ in $\mathbb{P}_\gamma$. But then $\overline{p}_n$ is compatible with $\overline{p}$ in $\mathbb{P}_\alpha$.

□

Proof of Theorem 7. Let
$$V \models ZFC + GCH + \text{Slice}_\kappa$$
and let $\mathbb{P}$ be the forcing defined in the beginning of the Section. Suppose that $\{M_\alpha |\ \alpha < \kappa\}$ witnesses Slice_κ in V, and $G \subseteq \mathbb{P}$ is generic over V. We aim to shows that the sequence $V[G \cap \mathbb{P}^{M_\alpha}]$ witnesses Slice_κ in $V[G]$. For this we need to show two things

1. If $F \in 2^{<\kappa}$, then $F \in V[G \cap \mathbb{P}^{M_\alpha}]$ for some $\alpha < \kappa$.
2. None of the models $V[G \cap \mathbb{P}^{M_\alpha}]$ contains all sequences from $2^{<\kappa}$ (here we use the regularity of κ, so we can later pass to a strictly increasing subsequence).

Concerning 1. assume that $\mathbb{P}_\kappa \Vdash \dot{F} \in 2^\delta$, for some ordinal $\delta < \kappa$. We can assume that $\dot{F} = \{(\alpha, \alpha_n, p_n^\alpha) | \ \alpha < \delta, \ n < \omega\}$, and all conditions p_n^α are simple. By Lemma 4 each condition p_n^α is definable with a parameter $E_n^\alpha \in \kappa^\omega$. The set $\{E_n^\alpha | \ \alpha < \delta, n < \omega\}$ is definable from a sequence of the length $< \kappa$, so it belongs to some model M_α, and so $\dot{F} \in M_\alpha$. Therefore $\dot{F}$ is a $\mathbb{P}_\kappa^{M_\alpha}$-name, and it follows that

$$\dot{F}[G] = \dot{F}[G \cap \mathbb{P}^{M_\alpha}] \in V[G \cap \mathbb{P}^{M_\alpha}].$$

Concerning 2. fix a sequence $F \in 2^{<\kappa} \setminus M_\alpha$. There exists a representation of the Cohen forcing, say $\mathbb{C}_F$, from which the sequence F is definable and $|\mathbb{C}_F| < \kappa$. Since $F \notin M_\alpha$, it follows that for some $\gamma < \kappa$

$$\mathbb{P}_\gamma^{M_\alpha} \Vdash \dot{\mathbb{Q}}_\gamma^{M_\alpha} = \{0\},$$

and

$$\mathbb{P}_\gamma \Vdash \dot{\mathbb{Q}}_\gamma = \mathbb{C}_F.$$

Therefore we can find a complete embedding of $\mathbb{C}_F$ into the quotient forcing

$$\mathbb{C}_F \hookrightarrow \mathbb{P}_\kappa / (\mathbb{P}_\kappa^{M_\alpha} \cap G),$$

given by the formula

$$x \mapsto 1_{\mathbb{P}_\gamma} {}^\frown \ (1_{\mathbb{P}_\gamma}, x) \ {}^\frown 1_{\mathbb{P}_{\kappa \setminus (\gamma+1)}}.$$

This shows that $V[G]$ contains a Cohen real over $V[\mathbb{P}^{M_\alpha} \cap G]$.

$\square$

Corollary 1. *The following theories are consistent*

$$ZFC + MA_{\omega_1} + \text{Slice}_{\omega_2} + \text{"}2^\omega = \omega_2\text{"},$$
$$ZFC + MA_{\omega_1} + \text{Slice}_{\omega_2} + \text{"}2^\omega = \omega_3\text{"},$$
$$ZFC + MA_{\omega_2} + \text{Slice}_{\omega_3} + \text{"}2^\omega = \omega_{29}\text{"}.$$

6 Final comments

It is easy to see that all sequences of models witnessing Slice_κ that we built, are cardinal preserving. Moreover, we proved that MA_{ω_1} and Slice_{ω_1} are not compatible. It looks reasonably to expect that for any regular cardinal κ

$$MA_\kappa \implies \neg \, \text{Slice}_\kappa \, .$$

What about singular κ?

References

[1] U. Avraham, S. Shelah, *Martin's Axiom does not imply that every two $\aleph_1$-dense sets of reals are isomorphic*, Israel Journal of Mathematics, vol. 38, Nos. 1-2 (1981)

[2] T. Bartoszyński, H. Judah, *Set Theory: On The Structure of The Real Line*, A.K. Peters Ltd. 1995

[3] J.E. Baumgartner, *All $\aleph_1$-dense sets of reals can be isomorphic*, Fund. Math. 79 (1973), 101-106

[4] A. Blass, *Combinatorial cardinal characteristics of the continuum*, Handbook of set theory. Springer, Dordrecht, 2010, 395-489

[5] W. G. Fleissner, A.W. Miller, *On Q-sets*. Proc. Am. Math. Soc. vol. 78, n. 2 (1980)

[6] J. Ihoda, S. Shelah, *Souslin Forcing*, The Journal of Symbolic Logic, vol. 53, No. 4 (1988)

[7] K. Kunen, *Set Theory. Introduction to Independence Proofs*, 1980 Elsevier Science B.V.

[8] J. T. Moore, *What makes the continuum $\aleph_2$*, Foundations of mathematics, essays in honor of W. Hugh Woodin's 60th birthday, Harvard University (Caicedo et al., eds.) (2017), 259-287