TORSION-FREE ABELIAN GROUPS ARE BOREL COMPLETE

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ABSTRACT. We prove that the Borel space of torsion-free Abelian groups with domain ω is Borel complete, i.e., the isomorphism relation on this Borel space is as complicated as possible, as an isomorphism relation. This solves a long-standing open problem in descriptive set theory, which dates back to the seminal paper on Borel reducibility of Friedman and Stanley from 1989.

1. INTRODUCTION

Since the seminal paper of Friedman and Stanley on Borel complexity [3], descriptive set theory has proved itself to be a decisive tool in the analysis of complexity problems for classes of countable structures. A canonical example of this phenomenon is the famous result of Thomas from [14] which shows that the complexity of the isomorphism relation for torsion-free abelian groups of rank $1 \leqslant n < \omega$ (denoted as $\cong_n$) is strictly increasing with n, thus, on one hand, finally providing a satisfactory reason for the difficulties found by many eminent mathematicians in finding systems of invariants for torsion-free abelian groups of rank $2 \leqslant n < \omega$ which were as simple as the one provided by Baer for $n = 1$ (see [1]), and, on the other hand, showing that for no $1 \leqslant n < \omega$ the relation $\cong_n$ is universal among countable Borel equivalence relations. As a matter of facts, abelian group theory has been one of the most important fields of mathematics from which taking inspiration for forging the general theory of Borel complexity as well as for finding some of the most striking applications thereof. The present paper continues this tradition solving one of the most important problems in the area, a problem open since the above mentioned paper of Friedman and Stanley from 1989. In technical terms, we prove that the space of countable torsion-free abelian groups with domain ω is *Borel complete*.

As we will see in detail below, saying that a class of countable structures is Borel complete means that the isomorphism relation on this class is as complicated as possible, as an isomorphism relation. The Borel completeness of countable abelian group theory is particularly interesting from the perspective of model theory, as this class is model theoretically "low", i.e., stable (in the terminology of [12]). In fact, as already observed in [3], Borel reducibility can be thought of as a weak version of $\mathfrak{L}_{\omega_1,\omega}$-interpretability, and for other classes of countable structures such as groups or fields much stronger results than Borel completeness exist, as in such cases we can first-order interpret graph theory, but such classes are unstable, while abelian group theory is stable. Reference [8] starts a systematic study of the relations between Borel reducibility and classification theory in the context of $\aleph_0$-stable theories.

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Coming back to us, we now introduce the notions from descriptive set theory which are necessary to understand our results, and we try to make a complete historical account of the problems which we tackle in this paper. The starting point of the descriptive set theory of countable structures is the following fact:

Fact 1.1. *The set K^L_ω of structures with domain ω in a given countable language L is endowed with a standard Borel space structure $(\mathrm{K}^L_\omega, \mathcal{B})$. Every Borel subset of this space $(\mathrm{K}^L_\omega, \mathcal{B})$ is naturally endowed with the Borel structure induced by $(\mathrm{K}^L_\omega, \mathcal{B})$.*

For example, if take $L = \{e, \cdot, ()^{-1}\}$, and we let K' to be one of the following:
(a) the set of elements of K^L_ω which are groups;
(b) the set of elements of K^L_ω which are abelian groups;
(c) the set of elements of K^L_ω which are torsion-free abelian groups;
(d) the set of elements of K^L_ω which are n-nilpotent groups, for some $n < \omega$;
then we have that K' is a Borel subset of $(\mathrm{K}^L_\omega, \mathcal{B})$, and so Fact 1.1 applies.

Thus, given a class K' as in Fact 1.1, we can consider K' as a standard Borel space, and so we can analyze the complexity of certain subsets of this space or of certain relations on it (i.e., subsets of $K' \times K'$ with the product Borel space structure). Further, this technology allows us to compare pairs of classes of structures or, in another direction, pairs of relations defined on pairs of classes of structures.

Definition 1.2. *Let X_1 and X_2 be two standard Borel spaces, and let also $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$. We say that Y_1 is reducible to Y_2, denoted as $Y_1 \leqslant_R Y_2$, when there is a Borel map $\mathbf{B} : X_1 \to X_2$ such that for every $x \in X_1$ we have:*

$$x \in Y_1 \Leftrightarrow \mathbf{B}(x) \in Y_2.$$

Notice that Definition 1.2 covers in particular the case $X_1 = K' \times K'$ for K' as in Fact 1.1, and so for example Y_1 could be the isomorphism relation on K'. Also, given a Borel space X, we can ask if there are subsets of X which are $\leqslant_R$-maxima with respect to a fixed family of subsets of an arbitrary Borel space (e.g., Borel sets, analytic sets, co-analytic sets, etc). In particular we can define:

Definition 1.3. *Let X_1 be a Borel space and $Y_1 \subseteq X_1$. We say that Y_1 is* complete analytic *(resp. complete co-analytic) if for every Borel space X_2 and analytic subset (resp. co-analytic subset) Y_2 of X_2 we have that $Y_2 \leqslant_R Y_1$.*

We now introduce the notion of Borel reducibility among equivalence relations.

Definition 1.4. *Let X_1 and X_2 be two standard Borel spaces, and let also E_1 be an equivalence relation defined on X_1 and E_2 be an equivalence relation defined on X_2. We say that E_1 is Borel reducible to E_2, denoted as $E_1 \leqslant_B E_2$, when there is a Borel map $\mathbf{B} : X_1 \to X_2$ such that for every $x, y \in X_1$ we have:*

$$xE_1y \ \Leftrightarrow \mathbf{B}(x)E_2\mathbf{B}(y).$$

Remark 1.5. *Notice that in the context of Definitions 1.2 and 1.4, $E_1 \leqslant_R E_2$ and $E_1 \leqslant_B E_2$ have two different meaning, as in the first case the witnessing Borel function has domain $X \times X$, while in the second case it has domain X. Furthermore, notice that $E_1 \leqslant_B E_2$ implies $E_1 \leqslant_R E_2$ (but the converse need not hold, see 1.7).*

We now define *Borel completeness*, the notion at the heart of our paper.

Definition 1.6. *Let K_1 be a Borel class of structures with domain ω and let $\cong_1$ be the isomorphism relation on K_1. We say that K_1 is* Borel complete *(or, in more* *modern terminology, $\cong_1$ is S_∞-complete) if for every Borel class K_2 of structures with domain ω there is a Borel map $\mathbf{B} : \mathrm{K}_2 \to \mathrm{K}_1$ such that for every $A, B \in \mathrm{K}_2$:*

$$A \cong B \Leftrightarrow \mathbf{B}(A) \cong_1 \mathbf{B}(B),$$

that is, the isomorphism relation on the space K_2 is Borel reducible (in the sense of Definition 1.4) to the the isomorphism relation on the space K_1.

The following fact will be relevant for our subsequent historical account.

Fact 1.7 ([3])**.** *Let K be a Borel class of structures with domain ω. If K is Borel complete, then its isomorphism relation is a complete analytic subset of $\mathrm{K} \times \mathrm{K}$, but the converse need not hold, as for example abelian p-groups with domain ω have complete analytic isomorphism relation but they are* not *a Borel complete space.*

We now have all the ingredients necessary to be able to understand the problems that we solve in this paper and to introduce the state of the art concerning them. But first a useful piece of notation which we will use throughout the paper.

Notation 1.8. *(1) We denote by* Graph *the class of graphs.*
(2) We denote by Gp *the class of groups.*
(3) We denote by AB *the class of abelian groups.*
(4) We denote by TFAB *the class of torsion-free abelian groups.*
(5) Given a class K we denote by K_ω the set of structures in K with domain ω.

Convention 1.9. *To simplify statements, we use the following convention: when we say that a class K of countable structures is Borel complete we mean that K_ω is Borel complete. Similarly, when we say that a class K of countable groups is complete co-analytic we mean that K_ω is a complete co-analytic subset of Gp_ω. Finally, when we say that the isomorphism relation on a class of countable groups is analytic, we mean that restriction of the isomorphism relation on K to $\mathrm{K}_\omega \times \mathrm{K}_\omega$ is an analytic subset of the Borel space $\mathrm{Gp}_\omega \times \mathrm{Gp}_\omega$ (as a product space).*

In [3], together with the general notions just defined, the authors studied some Borel complexity problems for specific classes of countable structures of interest. Among other things they showed (we mention only the results relevant to us):

(i) countable graphs, linear orders and trees are Borel complete;
(ii) torsion abelian groups have complete analytic $\cong$ but are *not* Borel complete;
(iii) nilpotent groups of class 2 and exponent p (p a prime) are Borel complete[1];
(iv) the isomorphism relation on finite rank torsion-free abelian groups is Borel.

In [3] Friedman and Stanley state explicitly:

> There is, alas, a missing piece to the puzzle, namely our conjecture that torsion-free abelian groups are complete. [...] We have not even been able to show that the isomorphism relation on torsion-free abelian groups is complete analytic, nor, in another direction, that the class of all abelian groups is Borel complete. We consider these problems to be among the most important in the subject.

The challenge was taken by several mathematicians. The first to work on this problem was Hjorth, which in [6] proved that any Borel isomorphism relation is

[1]As already mentioned in [3], this result is actually a straightforward adaptation of a model theoretic construction due to Mekler [9].

Borel reducible (in the sense of Definition 1.4) to the isomorphism relation on countable torsion-free abelian groups, and that in particular the isomorphism relation on TFAB_ω is not Borel (as there is no such Borel equivalence relation), leaving though open the question whether TFAB_ω is a Borel complete class, or even whether the isomorphism relation on TFAB_ω is complete analytic (cf. Def. 1.3 and Fact 1.7).

The problem resisted further attempts of the time and the interest moved to another very interesting problem on torsion-free abelian groups: for $1 \leqslant n < m < \omega$, is the isomorphism relation $\cong_n$ on torsion-free abelian groups of rank n strictly less complex (in the sense of Definition 1.4) than the isomorphism relation on torsion-free abelian groups of rank m? As mentioned above, the isomorphism relation on torsion-free abelian groups of finite rank is Borel while, as just mentioned, the isomorphim relation on countable torsion-free abelian groups is not, and so the two problems are quite different, but obviously related. Also this problem proved to be very challenging, until Thomas finally gave a positive solution to the problem, in a series of two fundamental papers [13, 14], proving in particular that, for every $n < \omega$, $\cong_n$ is not universal among countable Borel equivalence relations.

The fundamental work of Thomas thus resolved completely the case of torsion-free abelian groups of finite rank, leaving open the problem for countable torsion-free abelian groups of arbitrary rank, i.e. the problem referred to as "among the most important in the subject" in [3]. The problem remained "dormant" for various years (at the best of our knowledge), until Downey and Montalbán [2] made some important progress showing that the isomorphism relation on countable torsion-free abelian groups is complete analytic, a necessary but not sufficient condition for Borel completeness, as recalled in Fact 1.7. This was of course possible evidence that the isomorphism relation was indeed Borel complete, as conjectured in [3]. Despite this advancement, the problem of Borel completeness of countable torsion-free abelian groups resisted for other 12 years, until this day, when we prove:

Main Theorem. *The space* TFAB_ω *is Borel complete, in fact there exists a continous map* $\mathbf{B} : \mathrm{Graph}_\omega \to \mathrm{TFAB}_\omega$ *such that for every* $H_1, H_2, \in \mathrm{Graph}_\omega$*:*

$$H_1 \cong H_2 \text{ if and only if } \mathbf{B}(H_1) \cong \mathbf{B}(H_2).$$

The techniques employed in the proof of our Main Theorem, lead us to the consideration of classification questions of co-Hopfian torsion-free abelian groups, where we recall that a group G is said to be co-Hopfian if G does not have proper subgroups H isomorphic to G, i.e., every injective endomorphism of G is surjective. As well-known (see e.g. [4, Proposition 2.2, pg. 130]), for $G \in \mathrm{TFAB}$, G is co-Hopfian iff G is divisible and of finite rank, i.e. G is a finitely dimensional vector space over $\mathbb{Q}$, and so clearly the co-Hopfian groups form a Borel subset of TFAB_ω. We wonder: what if replace the notion of surjective morphism with a notion of "almost-surjective" morphism which is appropriate for the class TFAB? Does the classification problem becomes intractable? In particular we might consider:

Definition 1.10. *(1) We define the collection* Emb_1 *of embeddings between elements of* TFAB *as* $f : G \to H \in \mathrm{Emb}_1$ *if and only if* $H/f[G]$ *is torsion.*

(2) We define the maps Emb_2 *on* TFAB *as those* $f : G \to H \in \mathrm{Emb}_1$ *such that* $f[G]$ *is* $H/f[G]$ *torsion and bounded (i.e., there is* $n \in \omega$ *such that* $n(H/f[G]) = 0$*).*

(3) We define Emb_3 *as those* $f \in \mathrm{Emb}_1$ *of the form* $g \mapsto mg$ *for some* $m \in \mathbb{Z} \setminus \{0\}$*.*

These three notions of "almost-surjective" morphism lead to three variations of the notion of co-Hopfian group (cf. Definition 2.7) and for them we are able to show:

Theorem 1.11. *For $\ell \in \{1,2,3\}$, the set of Emb_ℓ-co-Hopfian torsion-free abelian groups is a complete co-analytic subset of the Borel space space TFAB_ω.*

In a work in preparation [10] we extend the ideas behind Theorem 1.11 to a systematic investigation of various classification problems for various rigidity conditions on abelian and nilpotent groups from the perspective of descriptive set theory of countable structures. In another work in preparation [11] we study the question of existence (and absolute exitence) of uncountable (co-)Hopfian abelian groups.

2. Notations and Preliminaries

For the readers of various backgrounds we try to make the paper self-contained.

2.1. General notations

Definition 2.1. *(1) Given a set X we write $Y \subseteq_\omega X$ for $Y \subseteq X$ and $|Y| < \aleph_0$.*

(2) Given a set X and $\bar{x}, \bar{y} \in X^{<\omega}$ we write $\bar{y} \triangleleft \bar{x}$ to mean that $\lg(\bar{y}) < \lg(\bar{x})$ and $\bar{x} \restriction \lg(\bar{y}) = \bar{y}$, where $\bar{x}$ is naturally considered as a function $X^{\lg(\bar{x})} \to X$.

(3) Given a partial function $f : M \to M$, we denote by $\mathrm{dom}(f)$ and $\mathrm{ran}(f)$ the domain and the range of f, respectively.

(4) For $\bar{a} \in B^n$ we write $\bar{x} \subseteq B$ to mean that $\mathrm{ran}(\bar{x}) \subseteq B$, where, as usual, $\bar{a}$ is considered as a function $\{0, ..., n-1\} \to B$.

(5) Given a sequence $\bar{f} = (f_i : i \in I)$ we write $f \in \bar{f}$ to mean that there exists $j \in I$ such that $f = f_j$.

2.2. Groups

Notation 2.2. *Let G and H be groups.*

(1) $H \leqslant G$ means that H is a subgroup of G.

(2) We let $G^+ = G \setminus \{e_G\}$, where e_G is the neutral element of G.

(3) If G is abelian we might denote the neutral element e_G simply as $0_G = 0$.

Definition 2.3. *Let $H \leqslant G$ be groups, we say that H is pure in G, denoted by $H \leqslant_* G$, when if $k \in H$, $n < \omega$ and (in additive notation) $G \models ng = k$, then there is $h \in H$ such that $H \models nh = k$.*

Observation 2.4. *$H \leqslant_* G \in \mathrm{TFAB}$, $k \in H$, $0 < n < \omega$, $G \models ng = k \Rightarrow g \in H$.*

Observation 2.5. *Let $G \in \mathrm{TFAB}$, p a prime and let:*

$$G_p = \{a \in G : a \text{ is divisible by } p^m, \text{ for every } 0 < m < \omega\},$$

then G_p is a pure subgroup of G.

Proof. This is well-known, see e.g. the discussion in [5, pg. 386-387]. ■

Notation 2.6. *We denote by* Emb *the class of embeddings between (abelian) groups.*

Definition 2.7. *(1) Let K be a class of groups and suppose that $(\mathrm{K}, \mathrm{Map}_1)$ and $(\mathrm{K}, \mathrm{Map}_2)$ are categories. Then we say that $G \in \mathrm{K}$ is $(\mathrm{K}, \mathrm{Map}_1, \mathrm{Map}_2)$-Hopfian (or $(\mathrm{Map}_1, \mathrm{Map}_2)$-Hopfian) when $f \in \mathrm{Map}_1(G,G)$ implies $f \in \mathrm{Map}_2(G,G)$.*

(2) We say that $G \in \mathrm{K}$ is co-Hopfian (resp. Hopfian) when G is $(\mathrm{K}, \mathrm{Map}_1, \mathrm{Map}_2)$-Hopfian, where K is the class of groups, Map_1 is the class of embeddings (resp. onto homom.) and Map_2 is the class of onto homom. (resp. embeddings).

(3) More generally, when $\mathrm{Map}_1 = \mathrm{Emb}$ (cf. Not. 2.6), instead of $(\mathrm{K}, \mathrm{Map}_1, \mathrm{Map}_2)$-Hopfian we simply talk of Map_2-co-Hopfian groups (we do this in Theorem 1.11).

2.3. Graphs and Trees

Definition 2.8. *By a directed graph we mean a structure in the language* $L = \{R\}$, *where* R *is a binary predicate symbol. We say that the directed graph* M *is irreflexive when* $M \models \forall x(\neg R(x,x))$. *We say that the directed graph* M *is asymmetric when* $M \models \forall x \forall y(R(x,y) \to \neg R(y,x))$. *We say that the directed graph* M *has no cycles (or that it is acyclic) when there is no* $n < \omega$ *and* $x_0, ..., x_n \in M$ *such that:*

$$M \models x_0 = x_n, \; M \models R(x_n, x_0) \text{ and, for every } i < n, \; M \models R(x_i, x_{i+1}).$$

Definition 2.9. *By a graph we mean a structure* M *in the language* $L = \{R\}$, *where* R *is a binary predicate symbol, satisfying the following axioms:*

(i) $\forall x(\neg R(x,x))$ *(irreflexivity of* R*);*
(ii) $\forall x \forall y(R(x,y) \to R(y,x))$ *(symmetry of* R*).*

The graph M *has no cycles when there is no* $2 \leqslant n < \omega$ *and* $x_0, ..., x_n \in M$ *such that:*

$$M \models x_0 = x_n, \; M \models R(x_n, x_0) \text{ and, for every } i < n, \; M \models R(x_i, x_{i+1}).$$

Definition 2.10. *Given an* L*-structure* M *by a partial automorphism of* M *we mean a partial function* $f : M \to M$ *such that* $f : \langle \mathrm{dom}(f) \rangle_M \cong \langle \mathrm{ran}(f) \rangle_M$.

Definition 2.11. *Let* $(T, <_T)$ *be a strict partial order.*

(1) $(T, <_T)$ *is a* tree *when, for all* $t \in T$, $\{s \in T : s <_T t\}$ *is well-ordered by the relation* $<_T$. *Notice that according to our definition a tree* $(T, <_T)$ *might have more than one root, i.e. more than one* $<_T$*-minimal element. We say that the tree* $(T, <_T)$ *is rooted when it has only one* $<_T$*-minimal element (its root).*
(2) A branch *of the tree* $(T, <_T)$ *is a maximal chain of the partial order* $(T, <_T)$.
(3) A tree $(T, <_T)$ *is said to be* well-founded *if it has only finite branches.*
(4) Given a tree $(T, <_T)$ *and* $t \in T$ *we let the level of* t *in* $(T, <_T)$, *denoted as* $\mathrm{lev}(t)$, *to be the order type of* $\{s \in T : s <_T t\}$ *(recall item (1)).*

Concerning Def. 2.11(4), we will only consider trees $(T, <_T)$ such that, for every $t \in T$, $\{s \in T : s <_T t\}$ is finite, so for us $\mathrm{lev}(t)$ will always be a natural number.

Fact 2.12. *Let* M *be a graph,* $\mathcal{U} \neq \mathcal{V} \subseteq M$ *and assume that* $|\mathcal{U}| = |M| = |\mathcal{V}| = \aleph_0$. *Then the following are equivalent:*

(A) h *is an isomorphism from* $M \restriction \mathcal{U}$ *onto* $M \restriction \mathcal{V}$*;*
(B) there is $\bar{g} = (g_k : k < \omega)$ *such that:*
  (a) for every $k < \omega$, g_k *is a finite partial automorphism of* M*;*
  (b) for every $k < \omega$, $g_k \subsetneq g_{k+1}$*;*
  (c) for every $k < \omega$, $g_k \neq g_k^{-1}$*;*
  (d) $\bigcup_{k<\omega} g_k = h$.

3. Borel Completeness of Torsion-Free Abelian Groups

3.1. The Frame

Hypothesis 3.1. *(1)* M *is (a copy of) the universal homogeneous graph of size* $\aleph_0$ *(a.k.a. the countable random graph) and* M *has set of nodes* $\subseteq \omega$*;*
(2) $\mathcal{G}$ *is the set of finite partial automorphisms* g *of* M *such that either* $\mathrm{dom}(g) = \emptyset$ *or* $g \neq g^{-1}$. *Notice that in particular* $\mathcal{G}$ *is closed under* $g \mapsto g^{-1}$*;*
(3) for $m < \omega$, $\mathcal{G}^m_* = \{(g_0, ..., g_{m-1}) \in \mathcal{G}^m : g_0 \subsetneq \cdots \subsetneq g_{m-1}\}$.

Notation 3.2. *(1) We use* $s, t, ...$ *to denote finite subsets of* M *and* $\mathcal{U}, \mathcal{V}, ...$ *to denote arbitrary subsets of* M.

(2) For $\bar{g} = (g_0, ..., g_{\lg(\bar{g})-1}) \in \mathcal{G}_*^{\lg(\bar{g})}$ *and* $s, t \subseteq_\omega M$*, we let:*
  (a) $\bar{g}[s] = t$ *mean that* $g_{\lg(\bar{g})-1}[s] = t$*;*
  (b) $\mathrm{dom}(\bar{g}) = \mathrm{dom}(g_{\lg(\bar{g})-1})$*;*
  (c) $\mathrm{ran}(\bar{g}) = \mathrm{ran}(g_{\lg(\bar{g})-1})$*;*
  (d) $\bar{g}^{-1} = (g_i^{-1} : i < \lg(\bar{g}))$

Definition 3.3. *In the context of Hypothesis 3.1, let* $\mathrm{K}_1^{\mathrm{bo}}(M)$ *be the class of objects* $\mathfrak{m}(M) = \mathfrak{m} = (X^{\mathfrak{m}}, \bar{X}^{\mathfrak{m}}, I^{\mathfrak{m}}, \bar{I}^{\mathfrak{m}}, \bar{f}^{\mathfrak{m}}, \bar{E}^{\mathfrak{m}}, \bar{p}^{\mathfrak{m}}, S^{\mathfrak{m}}, <_{\mathfrak{m}}) = (X, \bar{X}, I, \bar{I}, \bar{f}, \bar{E}, \bar{p}, S, <)$ *such that the following conditions are satisfied:*

(1) X *is a non-empty countable set and* $X \subseteq \omega$*;*

(2) (a) $(X'_s : s \subseteq_\omega M)$ *is a partition of* X *into infinite sets;*
  (b) for $s \subseteq_\omega M$*, let* $X_s = \bigcup_{t \subseteq s} X'_t$*;*
  (c) $\bar{X} = (X_s : s \subseteq_\omega M)$ *and so* $s \subseteq t \subseteq_\omega M$ *implies* $X_s \subseteq X_t$*;*

(3) for $\mathcal{U} \subseteq M$ *let* $X_{\mathcal{U}} = \bigcup\{X_s : s \subseteq_\omega \mathcal{U}\}$ *and so* $X = X_M = \bigcup\{X_s : s \subseteq_\omega M\}$*;*

(4) (a) $\bar{I} = (I_n : n < \omega) = (I_n^{\mathfrak{m}} : n < \omega)$ *are pairwise disjoint;*
  (b) $\bar{g} \in I_n$ *implies* $\bar{g} \in \mathcal{G}_*^m$ *for some* $m \leqslant n$*;*
  (c) I_n *is finite;*

(5) if $\bar{g}' \triangleleft \bar{g} \in I_n$*, then* $\bar{g}' \in I_{<n} := \bigcup_{\ell < n} I_\ell$*;*

(6) $I = I^{\mathfrak{m}} = \bigcup_{n<\omega} I_n$ *and* $<_I$ *is the order of being an initial segment;*

(7) $\bar{f} = (f_{\bar{g}} : \bar{g} \in I)$ *and:*
  (a) $f_{\bar{g}}$ *is a finite partial permutation of* X *with no cycles, i.e. there are no* $x_0, ..., x_n \in \mathrm{dom}(f_{\bar{g}})$ *such that* $x_0 = x_n$*,* $f_{\bar{g}}(x_n) = x_0$*, and, for every* $i < n$*,* $f_{\bar{g}}(x_i) = x_{i+1}$*, so in particular* $f_{\bar{g}}(x) \neq x$ *and* $f_{\bar{g}}(x) = y$ *implies* $f_{\bar{g}}(y) \neq x$*;*
  (b) $\mathrm{dom}(f_{\bar{g}}) \subseteq X_{\mathrm{dom}(\bar{g})}$ *and* $\mathrm{ran}(f_{\bar{g}}) \subseteq X_{\mathrm{ran}(\bar{g})}$ *(cf. Notation 3.2(2b)(2c));*
  (c) for $s, t \subseteq_\omega M$ *and* $\bar{g}[s] = t$ *we have:*

$$f_{\bar{g}}(x) = y \text{ implies } (x \in X'_s \text{ iff } y \in X'_t).$$

  (d) for $s, t \subseteq_\omega M$*,* $(f_{\bar{g}}(x) = y, x \in X'_s, y \in X'_t)$ *implies* $(\bar{g}[s] = t)$*;*
  (e) if $\bar{g} \in I_n$*, then* $\bar{g}^{-1} \in I_n$ *and* $f_{\bar{g}^{-1}} = f_{\bar{g}}^{-1}$*;*

(8) $\bar{g} \triangleleft \bar{g}' \Rightarrow f_{\bar{g}} \subsetneq f_{\bar{g}'}$*;*

(9) for $Z \subseteq X$*, we let* $\mathrm{seq}(Z) = \bigcup_{0<n<\omega} \mathrm{seq}_n(Z)$*, where, for* $0 < n < \omega$*, we let:*

$$\mathrm{seq}_n(Z) = \{\bar{x} \in Z^n : \bar{x} \text{ injective}\};$$

(10) we define the graph $(\mathrm{seq}_n(X), R_n^{\mathfrak{m}})$ *as* $(\bar{x}, \bar{y}) \in R_n^{\mathfrak{m}} = R_n$ *when* $\bar{x} \neq \bar{y}$ *and:*

$$\text{for some } \bar{g} \in I \text{ we have } f_{\bar{g}}(\bar{x}) = \bar{y},$$

  notice that $f_{\bar{g}}(\bar{x}) = \bar{y}$ *implies* $f_{\bar{g}}^{-1}(\bar{y}) = \bar{x}$ *and* $f_{\bar{g}}^{-1} = f_{\bar{g}^{-1}} \in \bar{f}$*, as* $\bar{g}^{-1} \in I$*.*

(11) the graph $(\mathrm{seq}_n(X), R_n)$ *has no cycles (cf. Definition 2.9);*

(12) (a) $\bar{E}^{\mathfrak{m}} = \bar{E} = (E_n : n < \omega) = (E_n^{\mathfrak{m}} : n < \omega)$*, and, for* $n < \omega$*,* E_n *is the equivalence relation corresponding to the partition of* $\mathrm{seq}_n(X)$ *given by the connected components of the graph* $(\mathrm{seq}_n(X), R_n)$*;*
  (b) $Y = Y_{\mathfrak{m}} = \{x \in X : \text{ for some } \bar{g} \in I,\, x \in \mathrm{dom}(f_{\bar{g}})\}$*, and:*

$$\mathrm{seq}_k(\mathfrak{m}) = \{\bar{x} \in \mathrm{seq}_k(X) : \text{ for some } \bar{g} \in I,\, \bar{x} \subseteq \mathrm{dom}(f_{\bar{g}})\},$$

  notice that $\mathrm{seq}_k(\mathfrak{m}) \subseteq \mathrm{seq}_k(Y_{\mathfrak{m}})$ *but the converse need not hold;*

(13) $\bar{p}$ *is a sequence of prime numbers without repetitions such that:*

$$\bar{p} = (p_{(e,\bar{q})} : e \in \mathrm{seq}_n(X)/E_n \text{ for some } 0 < n < \omega \text{ and } \bar{q} \in (\mathbb{Z} \setminus \{0\})^n);$$

(14) $S^{\mathfrak{m}} \subseteq I^{\mathfrak{m}}$ *satisfies the following conditions:*
  (a) if $\bar{g} \in I^{\mathfrak{m}}$ *is trivial, i.e.* $\mathrm{dom}(\bar{g}) = \emptyset$*, then* $\bar{g} \in S^{\mathfrak{m}}$*;*

(b) *for non-trivial* $\bar{g} \in I^{\mathfrak{m}}$, $\bar{g} \in S^{\mathfrak{m}}$ *iff* $\bar{g}^{-1} \notin S^{\mathfrak{m}}$;

(c) *if* $\bar{g}^\frown(g) \in I^{\mathfrak{m}}$ *and* $\bar{g}$ *is non-trivial, then* $\bar{g}^\frown(g) \in S^{\mathfrak{m}} \Leftrightarrow \bar{g} \in S^{\mathfrak{m}}$;

(15) *if* $n < \omega$, $\bar{g} \in S^{\mathfrak{m}} \subseteq I^{\mathfrak{m}}$, $\bar{g} = (g_0, ..., g_n)$ *and* $s \subseteq \mathrm{dom}(g_n)$, *then we have:*
  (a) n *even implies* $X_s \cap \{0, ..., n-1\} \subseteq \mathrm{dom}(f_{\bar{g}})$;
  (b) n *odd implies* $X_{g_n[s]} \cap \{0, ..., n-1\} \subseteq \mathrm{ran}(f_{\bar{g}})$;

(16) *if* $2 \leqslant i_* < \omega$ *and* $\bar{x}^0, ..., \bar{x}^{i_*-1} \in \bar{x}/E_n$ *are pairwise distinct, then there are* $i_1 \neq i_2 < i_*$ *and* $\ell_1, \ell_2 < n$ *such that the following holds:*
  (α) $x^{i_1}_{\ell_1} \notin \{x^i_\ell : \ell < n, i < i_*, (i, \ell) \neq (i_1, \ell_1)\}$;
  (β) $x^{i_2}_{\ell_2} \notin \{x^i_\ell : \ell < n, i < i_*, (i, \ell) \neq (i_2, \ell_2)\}$;

(17) *if* $k \geqslant 1$, $\bar{y} \in \bar{x}/E_k$ *and for some* $i, j < \mathrm{lg}(\bar{x}) = \mathrm{lg}(\bar{y})$, $y_i = x_j$, *then* $i = j$.

(18) $<_{\mathfrak{m}}$ *is a linear order of* $Y_{\mathfrak{m}}$ *of order type* $\leqslant \omega$;

(19) *if* $\bar{g} \in S^{\mathfrak{m}}$, $\bar{x} \in \mathrm{seq}_k(\mathrm{dom}(f_{\bar{g}}))$, $k \geqslant 1$, *then* $\max_{<_{\mathfrak{m}}}(\bar{x}) <_{\mathfrak{m}} \max_{<_{\mathfrak{m}}}(f_{\bar{g}}(\bar{x}))$, *further if* $\ell < k$, $\bar{g} = (g_i : i \leqslant n)$ *and* $x_\ell \notin \mathrm{dom}(f_{\bar{g}\restriction n})$, *then* $\max_{<_{\mathfrak{m}}}(\bar{x}) <_{\mathfrak{m}} f_{\bar{g}}(x_\ell)$;

(20) *for* $k \geqslant 1$, $\bar{x} \in \mathrm{seq}_k(Y_{\mathfrak{m}})$, *let* $\mathrm{suc}^{\mathfrak{m}}_k(\bar{x}) = \{f_{\bar{g}}(\bar{x}) : \bar{g} \in S^{\mathfrak{m}}, \bar{x} \in \mathrm{seq}_k(\mathrm{dom}(f_{\bar{g}}))\}$;

(21) (a) *the directed graph* $(\mathrm{seq}_k(Y_{\mathfrak{m}}), P^{\mathfrak{m}}_k)$ *has no cycles (cf. Def. 2.8), where:*

$$P^{\mathfrak{m}}_k = \{(\bar{x}, \bar{y}) : \bar{x} \in \mathrm{seq}_k(Y_{\mathfrak{m}}), \bar{y} \in \mathrm{suc}^{\mathfrak{m}}_k(\bar{x})\};$$

  (b) *the transitive closure of* $\{(\bar{x}, \bar{y}) : \bar{x} \in \mathrm{seq}_k(Y_{\mathfrak{m}}), \bar{y} \in \mathrm{suc}^{\mathfrak{m}}_k(\bar{x})\}$, *denoted as* $<^{\mathfrak{m}}_k$, *is a strict partial order on the set* $\mathrm{seq}_k(Y_{\mathfrak{m}})$;

(22) $(\mathrm{seq}_k(Y_{\mathfrak{m}}), <^{\mathfrak{m}}_k)$ *is a tree (possibly with more than one root);*

(23) *if* $\bar{x}, \bar{y} \in \mathrm{seq}_k(Y_{\mathfrak{m}})$ *are* E_k*-equivalent, then:*
  (a) *clearly, by (22), and recalling that* $\bar{x}, \bar{y} \in \mathrm{seq}_k(Y_{\mathfrak{m}})$ *are* E_k*-equivalent, we have that* $\bar{x}, \bar{y}$ *have a greatest* $<^{\mathfrak{m}}_k$*-lower bound, denoted as* $\bar{x} \wedge^{\mathfrak{m}}_k \bar{y}$;
  (b) *if* $\bar{z} = (\bar{x} \wedge^{\mathfrak{m}}_k \bar{y}) <^{\mathfrak{m}}_k \bar{y}$, $i < k$ *and* $x_i = y_i$, *then* y_i *is not* $<_{\mathfrak{m}}$*-maximal in* $\mathrm{ran}(\bar{y})$;
  (c) *if* $\bar{x} <^{\mathfrak{m}}_k \bar{y}$, *then* $\mathrm{ran}(\bar{x}) \cap \mathrm{ran}(\bar{y}) = \emptyset$;

(24) *if* $s, t \subseteq_\omega M$ *and* $X'_s \cap Y_{\mathfrak{m}} \neq \emptyset \neq X'_t \cap Y_{\mathfrak{m}}$, *then for some* $x \in X'_s$ *and* $y \in X'_t$ *we have that* $x \neq y \in Y_{\mathfrak{m}}$ *and* (x, y) *is* $<^{\mathfrak{m}}_2$*-minimal in* $(x, y)/E_2$.

Remark 3.4. *Let* $\mathfrak{m} \in \mathrm{K}^{\mathrm{bo}}_1(M)$. *Notice that conditions (7a) and (17) of Definition 3.3 imply that for every* $f_{\bar{g}} \in \bar{f}^{\mathfrak{m}}$ *we have that* $\mathrm{dom}(f_{\bar{g}}) \cap \mathrm{ran}(f_{\bar{g}}) = \emptyset$. *Why? Suppose there is* $x_1 \in \mathrm{dom}(f_{\bar{g}}) \cap \mathrm{ran}(f_{\bar{g}})$, *and let* $x_0 \in X$ *be such that* $f_{\bar{g}}(x_0) = x_1$ *and* $x_2 := f_{\bar{g}}(x_1)$. *Then* $(x_0, x_1)E_2(x_1, x_2)$, *contradicting Definition 3.3(17).*

Observation 3.5. *Let* $\mathfrak{m} \in \mathrm{K}^{\mathrm{bo}}_1(M)$. *The set of conditions (1)-(23) from Definition 3.3 is not minimal. In particular clauses (18)-(23) imply (16) and also:*

($\cdot$) *if* $\bar{x}^0, ..., \bar{x}^{i_*-1} \in \bar{x}/E_n$ *are pairwise distinct, then there exists* $j < i_*$ *and* $\ell < n$ *such that* x^j_ℓ *is* $<_{\mathfrak{m}}$*-maximal in* $\{x^j_0, ..., x^j_{n-1}\}$ *and the following holds:*

$$x^j_\ell \notin \{x^i_m : i < i_*, m < n, (i, m) \neq (j, \ell)\}.$$

Proof. For $n = 1$, both (16) and ($\cdot$) are trivial. Let then $n \geqslant 2$.

(+) Let $j < i_*$ be such that $\bar{x}^j$ is locally $<^{\mathfrak{m}}_n$-maximal (i.e., $i < i_*$ implies $\bar{x}^j \not<^{\mathfrak{m}}_n \bar{x}^i$). Let $\ell < n$ be such that x^j_ℓ is $<_{\mathfrak{m}}$-maximal in $\bar{x}^j$ (recall that $\bar{x}^j$ is with no repetitions). We claim that (j, ℓ) is as required in ($\cdot$).

We prove (+). Let (i, m) be a counterexample, i.e. $i < i_*$, $m < n$, $(i, m) \neq (j, \ell)$ and $x^i_m = x^j_\ell$. By Definition 3.3(17), $m = \ell$, so necessarily $i \neq j$ and let $\bar{y} := \bar{x}^i \wedge^{\mathfrak{m}}_n \bar{x}^j$. If $\bar{y} \neq \bar{x}^j$, then, noticing firstly that $\bar{y} <^{\mathfrak{m}}_n \bar{x}^j$ we may apply Definition 3.3(23b) with $\bar{x}^i, \bar{x}^j, \bar{y}$ here standing for $\bar{x}, \bar{y}, \bar{z}$ there, and so we have that x^j_ℓ is not $<_{\mathfrak{m}}$-maximal in $\bar{x}^j$, a contradiction. If on the other hand $\bar{y} = \bar{x}^j$, then $\bar{x}^j = \bar{y} <^{\mathfrak{m}}_n \bar{x}^i$, as $\bar{x}^j = \bar{y} = \bar{x}^i$ cannot happen, but this contradicts the fact that $i < i_*$ implies $\bar{x}^j \not<^{\mathfrak{m}}_n \bar{x}^i$.

We are then left with proving (16). Now, if $\{\bar{x}^i : i < i_*\}$ has two $<^{\mathfrak{m}}_n$-incomparable elements, then we are done by (+). Indeed, let $i(1), i(2) < i(*)$ be such that $\bar{x}^{i(1)}, \bar{x}^{i(2)}$ are $<^{\mathfrak{m}}_n$-incomparable. Now, for $\ell \in \{1,2\}$ there is $j(\ell) < i(*)$ such that $\bar{x}^{i(\ell)} \leqslant^{\mathfrak{m}}_n \bar{x}^{j(\ell)}$ and $\bar{x}^{j(\ell)}$ is locally $<^{\mathfrak{m}}_n$-maximal among $\{\bar{x}^i : i < i(*)\}$. We can then choose $m(\ell) < n$ such that $x^{j(\ell)}_{m(\ell)}$ is $<_{\mathfrak{m}}$-maximal in $\bar{x}^{i(\ell)}$. By (+) we know that $(j(\ell), m(\ell))$ are as required for $(\cdot)$. But then, by the choice of $i(1), i(2)$ and $j(1), j(2)$, also $\bar{x}^{j(1)}, \bar{x}^{j(2)}$ are $<^{\mathfrak{m}}_n$-incomparable (by Definition 3.3(22)), hence $j(1) \neq j(2)$. It follows that $(j(1), m(1))$, $(j(2), m(2))$ are as required for (16), and so in this case we are done. So we are left with the case in which $\{\bar{x}^i : i < i_*\}$ is $<^{\mathfrak{m}}_n$-linearly ordered. W.l.o.g. we have the following situation:

$$\bar{x}^0 <^{\mathfrak{m}}_n \bar{x}^1 <^{\mathfrak{m}}_n \cdots <^{\mathfrak{m}}_n \bar{x}^{i_*-1}.$$

By Def. 3.3(23c) the sets $(\mathrm{ran}(\bar{x}^i) : i < i_*)$ are pairwise distinct, so we are done. ■

Definition 3.6. *For $\mathfrak{m} \in \mathrm{K}^{\mathrm{bo}}_1(M)$, we say that $\mathfrak{m} \in \mathrm{K}^{\mathrm{bo}}_2(M)$ when:*

$(*)_0$ $X^{\mathfrak{m}} = Y_{\mathfrak{m}}$;

$(*)_1$ *if $s \subseteq_\omega M$, then for some $x \neq y \in X'_s$ we have $\neg((x)E_1(y))$ (this condition actually follows by Definition 3.3(24) but we chose to include it for clarity);*

$(*)_2$ $I = \bigcup_{n<\omega} I_n = \bigcup_{m<\omega} \mathcal{G}^m_*$ *(cf. Hypothesis 3.1(3));*

$(*)_3$ *if, for every $n < \omega$, $g_n \in \mathcal{G}$ and $g_n \subsetneq g_{n+1}$, and $\mathcal{U} = \bigcup_{n<\omega} \mathrm{dom}(g_n) \subseteq M$, then $\bigcup_{n<\omega} \mathrm{dom}(f_{(g_\ell : \ell<n)}) = \bigcup\{X_s : s \subseteq_\omega \mathcal{U}\}$ (this condition actually follows by Definition 3.3(15) but we chose to include it for clarity).*

Definition 3.7. *(1) $\mathrm{K}^{\mathrm{bo}}_0(M)$ is the class of $\mathfrak{m} \in \mathrm{K}^{\mathrm{bo}}_1(M)$ such that for some $n < \omega$ we have that for every $m \geqslant n$, $I_m = \emptyset$, in this case we let $n = n(\mathfrak{m})$ to be the minimal such $n < \omega$. Note that if $\mathfrak{m} \in \mathrm{K}^{\mathrm{bo}}_0(M)$, then $Y_{\mathfrak{m}}$ is finite.*

(2) We say that $\mathfrak{n} \in \mathrm{suc}(\mathfrak{m})$ when:

- *(a) $\mathfrak{n}, \mathfrak{m} \in \mathrm{K}^{\mathrm{bo}}_0(M)$;*
- *(b) $n(\mathfrak{n}) = n + 1$, where $n(\mathfrak{m}) = n$;*
- *(c) if $\ell < n(\mathfrak{m})$, then $I^{\mathfrak{m}}_\ell = I^{\mathfrak{n}}_\ell$ and $\bigwedge_{\bar{g} \in I^{\mathfrak{m}}_\ell} f^{\mathfrak{m}}_{\bar{g}} = f^{\mathfrak{n}}_{\bar{g}}$;*
- *(d) $I^{\mathfrak{n}}_n = \{\bar{g}, \bar{g}^{-1}\}$, $lg(\bar{g}) \leqslant n$, and $\ell < \mathrm{lg}(\bar{g})$ implies:*

$$\bar{g} \restriction \ell \in \bigcup_{\ell<n} I^{\mathfrak{m}}_\ell,$$

  notice that $\bar{g} \notin \bigcup_{\ell<n} I^{\mathfrak{m}}_\ell$ (by Definition 3.3(4a)) and the symmetric condition $\bar{g}^{-1} \restriction \ell \in \bigcup_{\ell<n} I^{\mathfrak{m}}_\ell$ follows from Definition 3.3(7e);
- *(e) if $\bar{x}E^{\mathfrak{n}}\bar{y}$ and $\neg(\bar{x}E^{\mathfrak{m}}\bar{y})$, then $\bar{x} \notin \mathrm{seq}_k(\mathfrak{m})$ or $\bar{y} \notin \mathrm{seq}_k(\mathfrak{m})$, hence we have:*

$$(\bar{x}/E^{\mathfrak{m}}_k \textit{ is a singleton}) \;\textit{ or }\; (\bar{y}/E^{\mathfrak{m}}_k \textit{ is a singleton});$$

- *(f) if $\bar{g}$ is as in (d) and $s \subseteq \mathrm{dom}(\bar{g}) \cup \mathrm{dom}(\bar{g}^{-1})$, then for some $x, y \in Y_{\mathfrak{n}} \cap X'_s$:*

$$\neg((x)E^{\mathfrak{n}}_1(y));$$

- *(g) if $s \subseteq \mathrm{dom}(\bar{g})$, then $Y_{\mathfrak{n}} \cap X_s \subseteq \mathrm{dom}(f_{\bar{g}})$;*
- *(h) $<_{\mathfrak{m}} \subseteq <_{\mathfrak{n}}$ (cf. Definition 3.3(18)), $x \in Y_{\mathfrak{m}}$ and $y \in Y_{\mathfrak{n}} \setminus Y_{\mathfrak{m}}$ implies $x <_{\mathfrak{n}} y$;*
- *(i) $S^{\mathfrak{m}} \subseteq S^{\mathfrak{n}}$, so $S^{\mathfrak{m}} = S^{\mathfrak{n}} \cap I^{\mathfrak{m}}$;*

(3) $<_{\mathrm{suc}}$ on $\mathrm{K}^{\mathrm{bo}}_0(M)$ is the transitive closure of the relation $\mathfrak{n} \in \mathrm{suc}(\mathfrak{m})$.

Claim 3.8. *For M as in Hypothesis 3.1, $\mathrm{K}^{\mathrm{bo}}_2(M) \neq \emptyset$.*

Proof. $(*)_1$ $\mathrm{K}_0^{\mathrm{bo}}(M) \neq \emptyset$.
[Why? Let $\mathfrak{m}$ be such that:

(a) $|X| = \aleph_0$, and $X \subseteq \omega$;
(b) $(X'_s : s \subseteq_\omega M)$ is a partition of X into infinite sets;
(c) for $s \subseteq_\omega M$, $X_s = \bigcup_{t \subseteq s} X'_t$;
(d) $\bar{X} = (X_s : s \subseteq_\omega M)$;
(e) $I_0^{\mathfrak{m}} = \{()\}$, $f_{()}$ is the empty function, $\bar{f} = (f_{()})$ and $I_{1+n} = \emptyset$, for every $n < \omega$;
(f) $S^{\mathfrak{m}} = \{()\}$.

Notice that $()$ denotes the empty sequence and under this choice of $\mathfrak{m}$, $n(\mathfrak{m}) = 1$, where we recall that the notation $n(\mathfrak{m})$ was introduced in Definition 3.7(1).]

$(*)_2$ If $\mathfrak{m} \in \mathrm{K}_0^{\mathrm{bo}}(M)$, $n = n(\mathfrak{m}) > 0$, $\bar{g} = (g_0, ..., g_{m-1}) \in I_{<n}^{\mathfrak{m}}$ (so $n > m$) and:
 (i) $g \in \mathcal{G}$;
 (ii) $\bigcup_{\ell < m} g_\ell \subsetneq g$;
 (iii) $\bar{g}^\frown(g) \notin I^{\mathfrak{m}}$;
 (iv) if m is even, then $\bar{g}$ is trivial or $\bar{g} \in S^{\mathfrak{m}}$;
 (v) if m is odd then $\bar{g} \notin S^{\mathfrak{m}}$;

 then there is $\mathfrak{n} \in \mathrm{K}_0^{\mathrm{bo}}(M)$ such that (recall that$f_{\bar{g}} = f_{\bar{g}}^{\mathfrak{m}}$):
 (a) $\mathfrak{n} \in \mathrm{suc}(\mathfrak{m})$;
 (b) $\bar{g}^\frown(g) \in I_n^{\mathfrak{n}}$;
 (c) $\mathrm{dom}(f_{\bar{g}^\frown(g)})$ contains $\{0, ..., m-1\} \cap X_{\mathrm{dom}(g)}$;
 (d) if m is even, then $S^{\mathfrak{n}} \cap I_n^{\mathfrak{n}} = \{\bar{g}^\frown(g)\}$;
 (e) if m is odd, then $S^{\mathfrak{n}} \cap I_n^{\mathfrak{n}} = \{(\bar{g}^{-1})^\frown(g)^{-1}\}$, so $n(\mathfrak{n}) = n(\mathfrak{m}) + 1$.

We prove $(*)_2$. To this extent:

$(*)_{2.0}$ Let $s_* = \mathrm{dom}(g) \subseteq_\omega M$, hence $\mathrm{dom}(\bar{g}) \subsetneq s_*$.

$(*)_{2.1}$ Let u_1, u_2 be such that:
 (a) u_1 is a finite initial segment of X;
 (b) u_1 includes $(X \cap \{0, ..., m-1\}) \cup Y_{\mathfrak{m}}$;
 (c) for every $s \subseteq \mathrm{dom}(g) \cup \mathrm{dom}(g^{-1})$ the set $(u_1 \cap X'_s) \setminus Y_{\mathfrak{m}}$ has $\geqslant 2$ elements;
 (d) $u_2 = u_1 \cap X_{s_*}$.

$(*)_{2.2}$ For $\ell \in u_2$, let $y_\ell = \ell$ and $z_\ell = f_*(y_\ell)$, where we let f_* be such that:

(a) f_* is a finite permutation of X obeying Def. 3.3(7a)-(7d) with $\mathrm{dom}(f_*) = u_2$;
(b) f_* extends $f_{\bar{g}}$;
(c) if $x \in \mathrm{dom}(f_*) \setminus \mathrm{dom}(f_{\bar{g}})$, then $f_*(x) \notin (u_2 \cup Y_{\mathfrak{m}})$;
(d) if $t \subseteq \mathrm{ran}(\bar{g})$, then the set $(X'_t \cap (\mathrm{ran}(f_*)) \setminus \mathrm{ran}(f_{\bar{g}}^{\mathfrak{m}})$ is an initial segment of the set $X'_t \setminus Y_{\mathfrak{m}}$ with respect to the standard order on the natural numbers (recall that by Definition 3.3(1) we have that $X \subseteq \omega$);.

It follows that:

$(*)_{2.2.1}$ $(\cdot_1)$ if $s \subseteq \mathrm{dom}(g)$, then $((X'_s \cap u_2) \setminus Y_{\mathfrak{m}}) \setminus f_*[Y_{\mathfrak{m}}]$ has at least two elements;
 $(\cdot_2)$ if $s \subseteq \mathrm{ran}(g)$, then $((X'_s \cap f_*[u_2]) \setminus Y_{\mathfrak{m}}) \setminus f_*[Y_{\mathfrak{m}}]$ has at least two elements.

[Why $(*)_{2.2.1}$? E.g., for $(\cdot_2)$, let $t = g^{-1}[s]$, so $(X'_t \cap \mathrm{dom}(f_*)) \setminus Y_{\mathfrak{m}}$ has at least two elements, and thus f_* maps them into distinct members of $X'_s \setminus Y_{\mathfrak{m}}$.]
We now define $\mathfrak{n}$, as required in $(*)_2$.

$(*)_{2.3}$ (a) $n(\mathfrak{n}) = n + 1$;
 (b) $f^{\mathfrak{n}}_{\bar{g}^\frown(g)} = f_*$, $f^{\mathfrak{n}}_{(\bar{g}^{-1})^\frown(g^{-1})} = f_*^{-1}$ and $I_n^{\mathfrak{n}} = \{\bar{g}^\frown(g), (\bar{g}^{-1})^\frown(g^{-1})\}$;
 (c) $<_{\mathfrak{m}} \subseteq <_{\mathfrak{n}}$ and $x \in Y_{\mathfrak{m}}$ and $y \in Y_{\mathfrak{n}} \setminus Y_{\mathfrak{m}}$ implies $x <_{\mathfrak{n}} y$;
 (d) $Y_{\mathfrak{n}} = Y_{\mathfrak{m}} \cup \mathrm{dom}(f_*) \cup \mathrm{ran}(f_*)$;

(e) for $\ell, m \in Y_{\mathfrak{n}}$ we let $\ell <_{\mathfrak{n}} m$ iff one of the following three exclusive conditions is verified:
  (i) $\ell <_{\mathfrak{m}} m$;
  (ii) $\ell \in Y_{\mathfrak{m}}$ and $m \in Y_{\mathfrak{n}} \setminus Y_{\mathfrak{m}}$;
  (iii) $m, \ell \in Y_{\mathfrak{n}} \setminus Y_{\mathfrak{m}}$ and ℓ is smaller than m as natural numbers (recall that by Definition 3.3(1) we have that $X \subseteq \omega$);

(f) if n is even, then $S^{\mathfrak{n}} = S^{\mathfrak{m}} \cup \{\bar{g}^\frown(g)\}$;

(g) if n is odd, then $S^{\mathfrak{n}} = S^{\mathfrak{m}} \cup \{(\bar{g}^{-1})^\frown(g)^{-1}\}$.

We have to check that $\mathfrak{n} \in \mathrm{K}_1^{\mathrm{bo}}$, in fact it being in $\mathrm{suc}(\mathfrak{m})$ (and so in $\mathrm{K}_0^{\mathrm{bo}}$) is then obvious, and (b) and (c) of $(*)_2$ are obvious too. Comparing the graphs $(\mathrm{seq}_k(X), R_k^{\mathfrak{n}})$ and $(\mathrm{seq}_k(X), R_k^{\mathfrak{m}})$ the set of new edges are the following:

$$\{(\bar{x}, \bar{y}) : (\bar{x}, \bar{y}) \in Z_1^k \cup Z_{-1}^k\},$$

where we let:

$(*)_{2.4}$

$$Z_1^k = \{(\bar{x}, \bar{y}) : \bar{x} \in \mathrm{seq}_k(u_2), f_*(\bar{x}) = \bar{y}, \bar{y} \notin \mathrm{seq}_k(\mathfrak{m})\},$$
$$Z_{-1}^k = \{(\bar{x}, \bar{y}) : (\bar{y}, \bar{x}) \in Z_1^k\},$$

Notice that possibly $\bar{x} \subseteq \mathrm{dom}(f_*) \wedge \bar{x} \notin \mathrm{seq}_k(\mathfrak{m})$, and possibly $\bar{x} \subseteq \mathrm{dom}(f_*) \wedge \bar{x} \not\subseteq \mathrm{dom}(f_{\bar{g}}^{\mathfrak{m}}) \wedge \bar{x} \in \mathrm{seq}_k(\mathfrak{m})$ (as witnessed by some $\bar{g}' \in I_{<n}^{\mathfrak{m}}$), anyhow the union $Z_1^k \cup Z_{-1}^k$ is disjoint. Notice:

$(*)_{2.4.1}$ if $\bar{x} \subseteq u_2$, $\mathrm{lg}(\bar{x}) = k$ and $\bar{y} = f_*(\bar{x})$, then:
  $(\cdot_1)$ $\bar{x} \subseteq \mathrm{dom}(f_{\bar{g}}) \Leftrightarrow \bar{y} \subseteq \mathrm{ran}(f_{\bar{g}}) \Rightarrow \bar{x} \in \mathrm{seq}_k(\mathfrak{m})$;
  $(\cdot_2)$ $\bar{x} \subseteq \mathrm{dom}(f_*) \cap Y_{\mathfrak{m}} \not\Rightarrow \bar{x} \in \mathrm{seq}_k(\mathfrak{m})$, in general;
  $(\cdot_3)$ there might be $s \subseteq \mathrm{dom}(\bar{g})$ and $x \in X'_s$ such that $x \in Y_{\mathfrak{m}}$ but $x \notin \mathrm{dom}(f_{\bar{g}})$, but then $x \in u_2$ and so $x \in \mathrm{dom}(f_{\bar{g}^\frown(g)}) \setminus \mathrm{dom}(f_{\bar{g}})$.

Now, we have:

$(*_{2.5})$ (a) if $(\bar{x}, \bar{y}) \in Z_1^k$, then:
  (α) $\bar{x} \in \mathrm{seq}_k(u_2)$, $\bar{x} \not\subseteq \mathrm{dom}(f_{\bar{g}})$ and $\bar{x} \cap f_*(Y_{\mathfrak{m}}) \subseteq \mathrm{dom}(f_{\bar{g}})$;
  (β) $\bar{y} \subseteq f_*(u_2)$, $\bar{y} \not\subseteq \mathrm{dom}(f_{\bar{g}})$, $\bar{y} \not\subseteq \mathrm{ran}(f_{\bar{g}})$ and $\bar{y} \cap Y_{\mathfrak{m}} \subseteq \mathrm{ran}(f_{\bar{g}})$;
 (b) the dual of item (a) for $(\bar{x}, \bar{y}) \in Z_{-1}^k$;
 (c) if $\bar{y} \in \mathrm{seq}_k(\mathfrak{n}) \setminus \mathrm{seq}_k(\mathfrak{m})$, then $\bar{y}$ occurs in exactly one edge of $R_k^{\mathfrak{n}}$.

Notice now that:

$(*)_{2.6}$ in the graph $(\mathrm{seq}_k(X), R_k^{\mathfrak{n}})$ we have:
  (i) all the new edges have at least one node in $\mathrm{seq}_k(u_2) \setminus \mathrm{seq}_k(\mathrm{dom}(f_{\bar{g}}))$ and one in $\mathrm{seq}_k(f_*[u_2]) \setminus \mathrm{seq}_k(\mathrm{ran}(f_{\bar{g}}))$;
  (ii) every node in $\mathrm{seq}_k(\mathfrak{n}) \setminus \mathrm{seq}_k(Y_{\mathfrak{m}})$ has valency 1.

Notice also that:

$(*)_{2.6.1}$ (a) if $\bar{x}_0, ..., \bar{x}_m$ is a path in $(\mathrm{seq}_k(\mathfrak{n}), R_k^{\mathfrak{n}})$ with no repetitions and $0 < \ell < m$, then $\bar{x}_\ell \in \mathrm{seq}_k(\mathfrak{m})$;
 (b) $E_k^{\mathfrak{n}} \restriction \mathrm{seq}_k(\mathfrak{m}) = E_k^{\mathfrak{m}} \restriction \mathrm{seq}_k(\mathfrak{m})$.

Thus, by $(*)_{2.6}$ and $(*)_{2.6.1}$, $\mathfrak{n}$ satisfies Definition 3.3, in fact we have:

$(*)_{2.7}$ Definition 3.3(1)-(3) are obvious as they are the same for $\mathfrak{m}$;

$(*)_{2.8}$ Definition 3.3(4)-(15) and (18)-(24) are easy to check by our choices.

$(*)_{2.9}$ As noticed in Observation 3.5, Definition 3.3(16) follows from Definition 3.3(18)-(23), and Definition 3.3(17) is easy.

So we finished proving $(*)_2$.

$(*)_3$ We can choose an $<_{\rm suc}$-increasing sequence $(\mathfrak{m}_\ell : \ell < \omega)$ in $\mathrm{K}^{\rm bo}_0(M)$ whose limit $\mathfrak{m}$ is as wanted, i.e. $\mathfrak{m} \in \mathrm{K}^{\rm bo}_2(M)$.

We show this. We can find a list $(\bar{g}^\ell : \ell < \omega)$ of $\bigcup_{m<\omega} \mathcal{G}^m_*$ such that:

(i) $\lg(\bar{g}^\ell) \leqslant \ell$;
(ii) if $\bar{g}^\ell \triangleleft \bar{g}^k$, then $\ell < k$;
(iii) $\lg(\bar{g}^\ell) = 0$ iff $\ell = 0$.

Now, by induction on $\ell < \omega$, we choose $\mathfrak{m}_\ell \in \mathrm{K}^{\rm bo}_0$ such that $n(\mathfrak{m}_\ell) \leqslant \ell + 1$ and $\mathfrak{m}_{\ell+1} \in \mathrm{suc}(\mathfrak{m}_\ell)$. We proceed as follows:

$(\ell = 0)$ use $(*)_1$;

$(\ell = k+1)$ $(\cdot_1)$ if $\bar{g}^{k+1} \in I^{\mathfrak{m}_k}$, then $\mathfrak{m}_\ell = \mathfrak{m}_k$;
$(\cdot_2)$ if $\bar{g}^{k+1} \notin I^{\mathfrak{m}_k}$, let $m_k = \lg(\bar{g}^{k+1}) - 1$, so $\bar{g}^{k+1} \restriction m \in I^{\mathfrak{m}_k}$, and:
$(\cdot_{2.1})$ if m_k is even, use $(*)_2$ with the pair $n(\mathfrak{m}_k)$, $\bar{g}^{k+1}$ here standing for the pair $n, \bar{g}^\frown(g)$ there;
$(\cdot_{2.2})$ if m_k is odd, use $(*)_2$ with the pair $n(\mathfrak{m}_k)$, $\bar{g}^{k+1}$ here standing for the pair $n, \bar{g}^\frown(g)$ there.

Clearly $\mathfrak{m} = \lim_{\ell<\omega}(\mathfrak{m}_\ell)$ is as promised by the choice of $(\bar{g}^\ell : \ell < \omega)$, e.g. Def. 3.6 $(*_1)$ holds by Def. 3.7(2f), which in turn holds by $(*_{2.1})(c)$ of the present proof. Concerning $<_\mathfrak{m}$, which is needed for Definition 3.3(18), let $<_\mathfrak{m} = \bigcup\{<_{\mathfrak{m}_\ell} : \ell < \omega\}$. As, by the first half of Definition 3.7(2h), $(Y_{\mathfrak{m}_\ell}, <_{\mathfrak{m}_\ell})$ is an increasing sequence of linear orders, clearly $(Y_\mathfrak{m}, <_\mathfrak{m})$ is a linear order, and it is easy to see that it is of order type $\leqslant \omega$, by the second half of Definition 3.7(2h). Finally, we show that $Y_\mathfrak{m} = X$. It suffices to prove that, for any $s \subseteq_\omega M$, $X'_s \subseteq Y_\mathfrak{m}$. For this it suffices to prove that for any $m < \omega$, $\{0, ..., m-1\} \cap X'_s \subseteq Y_\mathfrak{m}$. But clearly for some $\bar{g} \in \mathcal{G}^*_m$ we have that $s \subseteq \mathrm{dom}(\bar{g})$, and so for some $\ell > 0$ we have that $\bar{g}^\ell = \bar{g}$, and also for some $i > \ell$ and $g', g'' \in \mathcal{G}$ we have that $\bar{g}^i = \bar{g}^\frown(g', g'')$. Thus by Definition 3.3(15) we have that $\{0, ..., m-1\} \cap X'_s \subseteq \mathrm{dom}(f^{\mathfrak{m}(i+1)}_{\bar{g}}) \subseteq Y_\mathfrak{m}$. This concludes the proof. ∎

Definition 3.9. *Let $\mathfrak{m} \in \mathrm{K}^{\rm bo}_1(M)$.*

(1) Let $G_2 = G_2[\mathfrak{m}]$ be $\bigoplus\{\mathbb{Q}x : x \in X\}$.
(2) Let $G_0 = G_0[\mathfrak{m}]$ be the subgroup of G_2 generated by X, i.e. $\bigoplus\{\mathbb{Z}x : x \in X\}$.
(3) Let $G_1 = G_1[\mathfrak{m}]$ be the subgroup of G_2 generated by:
  (a) G_0;
  (b) $p^{-m}(\sum_{\ell<n} q_\ell x_\ell)$, where:
    (i) $0 < m < \omega$;
    (ii) $\bar{x} = (x_\ell : \ell < n) \in \mathrm{seq}_n(X)$, $e = \bar{x}/E_n$;
    (iii) $\bar{q} = (q_\ell : \ell < n) \in (\mathbb{Z} \setminus \{0\})^n$;
    (iv) $p = p_{(e,\bar{q})}$ (so a prime).
(4) For a prime p, let $G_{(1,p)} = \{a \in G_1 : a$ is divisible by p^m, for every $0 < m < \omega\}$ (notice that, by Observation 2.5, $G_{(1,p)}$ is always a pure subgroup of G_1).
(5) For $\mathcal{U} \subseteq M$, we let:

$$G_{(1,\mathcal{U})}[\mathfrak{m}] = G_{(1,\mathcal{U})}[\mathfrak{m}(M)] = G_{(1,\mathcal{U})} = \langle y : y \in X_u, u \subseteq_\omega \mathcal{U} \rangle^*_{G_1} = \langle X_\mathcal{U} \rangle^*_{G_1}.$$

Notice that the notation $\mathfrak{m}(M)$ was introduced in the second line of Def. 3.3.
(6) For $f_{\bar{g}} \in \bar{f}$ (cf. Definition 3.3(7)), let $\hat{f}^2_{\bar{g}}$ be the unique partial automorphism of G_2 which is induced by $f_{\bar{g}}$, explicitly: if $k < \omega$ and for every $\ell < k$ we have

that $y^1_\ell \in \mathrm{dom}(f_{\bar g})$, $y^2_\ell = f_{\bar g}(y^1_\ell)$, $q_\ell \in \mathbb{Q}$ and $a = \sum_{\ell<k} q_\ell y^1_\ell \in G_2$, then:

$$\hat f^2_{\bar g}(a) = \sum_{\ell<k} q_\ell y^2_\ell.$$

Notice that if $\sum_{\ell<k} q_\ell y^1_\ell \in G_1$, then also $\sum_{\ell<k} q_\ell y^2_\ell \in G_1$, by Definition 3.9(3) recalling Definition 3.3(7a) and (12a), this is relevant for Lemma 3.10(2).

(7) *For $\ell \in \{0,1\}$ we let $\hat f^2_{\bar g} \restriction G_\ell = \hat f^\ell_{\bar g}$, a partial automorphism of G_ℓ, and $\hat f_{\bar g} = \hat f^1_{\bar g}$.*

(8) *For $i \in \{0,1,2\}$, $a = \sum_{\ell<m} q_\ell x_\ell \in G_i$, with $x_\ell \in X$ and $q_\ell \in \mathbb{Q} \setminus \{0\}$, let $\mathrm{supp}(a) = \{x_\ell : \ell < m\}$, i.e., $\mathrm{supp}(a) \subseteq_\omega X$ is the smallest subset of X s.t.:*

$$a \in \langle \mathrm{supp}(a) \rangle^*_{G_i}.$$

Lemma 3.10. *Let $\mathfrak{m} \in \mathrm{K}^{\mathrm{bo}}_2$ and $\ell \in \{0,1,2\}$.*

(1) *$G_\ell[\mathfrak{m}] \in \mathrm{TFAB}$ and $|G_\ell[\mathfrak{m}]| = \aleph_0$.*

(2) *Recalling $\hat f_{\bar g} = \hat f^1_{\bar g} := \hat f^2_{\bar g} \restriction G_{(1,\mathrm{dom}(\bar g))}$ (cf. Definition 3.9(5)(7)), we have that the map $\hat f_{\bar g}$ is a well-defined partial automorphism of G_1, and $\mathrm{dom}(\hat f_{\bar g})$ is a pure subgroup of $G_1[\mathfrak{m}]$, in fact $\mathrm{dom}(\hat f_{\bar g})$ is the pure closure in G_1 of $\mathrm{dom}(\hat f^0_{\bar g})$.*

(3) *If $p = p_{(e,\bar q)}$, $e \in \mathrm{seq}_n(X)/E_n$, $\bar q = (q_\ell : \ell < n) \in (\mathbb{Z} \setminus \{0\})^n$ and $n \geqslant 1$, then:*

$$G_{(1,p)} = \langle \sum \{\mathbb{Z}(\sum_{\ell<n} q_\ell y_\ell) : \bar y \in e\} \rangle^*_{G_1}.$$

(4) *For e as in (3), assume that we have $i_* \geqslant 2$ and $\bar y^i \in e$, for $i < i_*$, which are pairwise distinct, $q^i \in \mathbb{Q} \setminus \{0\}$, for $i < i_*$, and $a = \sum_{i<i_*} q^i(\sum q_\ell y^i_\ell) \in G_1$. Then $\mathrm{supp}(a)$ has at least two elements, where $\mathrm{supp}(a)$ is as in Definition 3.9(8).*

Proof. Items (1), (2) are clear. We elaborate on items (3), (4). Concerning item (3), if $\bar y \in e$ and $0 < m < \omega$, then $p^{-m} \sum_{\ell<k} q_\ell y_\ell$ is one of the generators of G_1, as this holds for every $0 < m < \omega$ it follows that $\sum_{\ell<k} q_\ell y_\ell \in G_{(1,p)}$, by the definition of $G_{(1,p)}$. As $G_{(1,p)}$ is a subgroup of G_1, for every $\bar y \in e$ we have that $\mathbb{Z}(\sum_{\ell<n} q_\ell y_\ell) \subseteq G_{(1,p)} \leqslant G_1$. But then we have that $Z = \{\mathbb{Z}(\sum_{\ell<n} q_\ell y_\ell) : \bar y \in e\} \subseteq G_{(1,p)}$, and so $\langle Z \rangle_{G_1} \leqslant G_{(1,p)}$. Lastly, $\langle Z \rangle^*_{G_1} \leqslant G_{(1,p)}$, because by Definition 3.9(4) we have that:

$(*_1)$ $G_{(1,p)}$ is a pure subgroup of G_1, as $G_1 \in \mathrm{TFAB}$.

So we are done with one inclusion. Concerning the other inclusion, toward contradiction assume that $g \in G_{(1,p)} \setminus \langle Z \rangle^*_{G_1}$, where Z is as above. Now, we have:

$(*_2)$ $Z_1 := \langle Z \rangle^*_{G_1}$ is a pure subgroup of G_1 and $G_1/Z_1 \in \mathrm{TFAB}$;

$(*_3)$ each non-zero element of G_1/Z_1 is not divisible by p^m for some $m < \omega$.

[Why? By the choice of G_1 and Z.]

Hence we reach the following contradicting statement:

$(*_4)$ $g \in G_{(1,p)} \subseteq G_1$ is not divisible by p^m for some $m < \omega$.

Concerning item (4), by Definition 3.3(16) applied to $\bar y^0, ..., \bar y^{i_*-1} \in e$ there are $i_1 \neq i_2 < i_*$ and $\ell_1, \ell_2 < n$ such that the following holds:

(α) $y^{i_1}_{\ell_1} \notin \{y^i_\ell : \ell < n, i < i_*, (i,\ell) \neq (i_1,\ell_1)\}$;

(β) $y^{i_2}_{\ell_2} \notin \{y^i_\ell : \ell < n, i < i_*, (i,\ell) \neq (i_2,\ell_2)\}$.

But then, by (α), $y^{i_1}_{\ell_1}$ appears exactly once in the sum $\sum_{i<i_*} q^i(\sum q_\ell y^i_\ell) \in G_1$, and similarly, by (β), $y^{i_2}_{\ell_2}$ appears exactly once in the sum $\sum_{i<i_*} q^i(\sum q_\ell y^i_\ell) \in G_1$. Hence, as $a = \sum_{i<i_*} q^i(\sum q_\ell y^i_\ell)$, we have that $|\mathrm{supp}(a)| \geqslant |\{y^{i_1}_{\ell_1}, y^{i_2}_{\ell_2}\}| = 2$. ■

Claim 3.11. *Assume that* $\mathfrak{m} \in \mathrm{K}_2^{\mathrm{bo}}(M)$, $\mathcal{U} \neq \mathcal{V} \subseteq M$ *and* $|\mathcal{U}| = |\mathcal{V}| = \aleph_0$. *Then:*

$$M \restriction \mathcal{U} \cong M \restriction \mathcal{V} \ \Rightarrow \ G_{(1,\mathcal{U})}[\mathfrak{m}] \cong G_{(1,\mathcal{V})}[\mathfrak{m}].$$

Proof. Let h be an isomorphism from $M \restriction \mathcal{U}$ onto $M \restriction \mathcal{V}$. Let $(r_\ell : \ell < \omega)$ list $\mathcal{U}$ with no repetitions (recall $|\mathcal{U}| = |\mathcal{V}| = \aleph_0$) in a such a way that:

(i) for $k < \omega$, $g_k = h \restriction \{r_\ell : \ell \leqslant k\}$;
(ii) $(g_k : k < \omega)$ is as in Fact 2.12 with respect to h;
(iii) for $k < \omega$, $\bar{g}_k = (g_\ell : \ell \leqslant k)$, so $\bar{g}_k \in \mathcal{G}_*^{k+1}$ (cf. Hypothesis 3.1(3));
(iv) $s_k = \{r_\ell : \ell \leqslant k\} = \mathrm{dom}(g_k)$ and $t_k = \{h(r_\ell) : \ell \leqslant k\} = \mathrm{ran}(g_k)$;
(v) by Definition 3.6$(*)_2$, for every $k < \omega$ we have that $\bar{g}_k \in I^{\mathfrak{m}}$ and so $f_{\bar{g}_k} \in \bar{f}^{\mathfrak{m}}$.

Notice now that for $k < \omega$ we have:

$(\star_1)$ (a) $\mathrm{dom}(f_{\bar{g}_k}) \subseteq X_{s_k}$, $\mathrm{ran}(f_{\bar{g}_k}) \subseteq X_{t_k}$;
(b) $\{0, ..., k-1\} \cap X_{s_k} \subseteq \mathrm{dom}(f_{\bar{g}_{k+1}})$;
(c) $\{0, ..., k-1\} \cap X_{t_k} \subseteq \mathrm{dom}(f^{-1}_{\bar{g}_{k+1}}) = \mathrm{ran}(f_{\bar{g}_{k+1}})$.

[Why? (a) is by Def. 3.3(7b). (b) and (c) are by Definition 3.3(14)(15).]
Notice also that:

$(\star_2)$ (d) $\bigcup_{k<\omega} \mathrm{dom}(f_{\bar{g}_k}) = \bigcup_{k<\omega} X_{s_k} = X_{\mathcal{U}}$;
(e) $\bigcup_{k<\omega} \mathrm{dom}(f^{-1}_{\bar{g}_k}) = \bigcup_{k<\omega} X_{h[s_k]} = X_{\mathcal{V}}$.

[Why? (d) is by $(\star_1)$(a)-(b) and the fact that $X \subseteq \omega$. (e) is by $(\star_1)$(a)-(b), the fact that $X \subseteq \omega$ and that h is from $\mathcal{U}$ onto $\mathcal{V}$.]
Hence, we have:

$(\star_3)$ $\bigcup_{k<\omega} \hat{f}_{\bar{g}_k}$ is an isomorphism from $G_{(1,\mathcal{U})}$ onto $G_{(1,\mathcal{V})}$ (cf. Def. 3.9(7)).
[Why? By Def. 3.9(5)(6)(7).] ■

3.2. **Analyzing Isomorphism**

Hypothesis 3.12. *Throughout this subsection the following hypothesis holds:*

(1) $\mathfrak{m} \in \mathrm{K}_2^{\mathrm{bo}}(M)$;
(2) $\mathcal{U} \neq \mathcal{V} \subseteq M$;
(3) $|\mathcal{U}| = \aleph_0 = |\mathcal{V}|$;
(4) π *is an isomorphism from* $G_{(1,\mathcal{U})}[\mathfrak{m}]$ *onto* $G_{(1,\mathcal{V})}[\mathfrak{m}]$.

Lemma 3.13. *Let* $a \in G_{(1,\mathcal{U})}[\mathfrak{m}] \setminus \{0\}$ *and let* $b = \pi(a)$.

(1) For a prime p, $a \in G_{(1,p)} \Leftrightarrow b \in G_{(1,p)}$;
(2) if $a = qx$, *for some* $q \in \mathbb{Q} \setminus \{0\}$ *and* $x \in X_{\mathcal{U}}$, *then for some* $y \in X_{\mathcal{V}}$:
(a) $(x)E_1(y)$;
(b) $b \in \mathbb{Q}y$, *i.e. there exist* $m_1, m_2 \in \mathbb{Z} \setminus \{0\}$ *such that* $m_1 b = m_2 y$.

Proof. Item (1) is obvious by Hypothesis 3.12(4). Concerning item (2), let $n < \omega$, $\bar{y} \in \mathrm{seq}_n(X)$ and $\bar{q} \in (\mathbb{Q} \setminus \{0\})^n$ be such that $b = \sum\{q_\ell y_\ell : \ell < n\}$. It suffices to prove (2)(b), as if $b = \frac{m_2}{m_1} y$ let $p' = p_{((x)/E_1, \bar{q})}$, then $x \in G_{(1,p')}$ and so, by (1), $y \in G_{(1,p')}$ and thus by Lemma 3.10(3) we are done. Trivially, $n > 0$, we shall show that $n = 1$, so toward contradiction we assume that $n > 1$. Let $q_* \in \omega \setminus \{0\}$ be such that $b_1 := q_* b \in G_0[\mathfrak{m}]$. Let $e = \bar{y}/E_n$, $q'_\ell = q_* q_\ell$ and $\bar{q}' = (q'_\ell : \ell < n)$, so that $q_* q_\ell y_\ell = q'_\ell y_\ell$ and $q'_\ell \in \mathbb{Z} \setminus \{0\}$. Let $p = p_{(e, \bar{q}')}$. Then we have:

$(*_1)$ (i) $b \in G_{(1,p)}$;
(ii) $a \in G_{(1,p)}$.

[Why (i)? By the choice of p we have that $b_1 \in G_{(1,p)}$ (cf. Def. 3.9(3)(4)) and so, as $G_{(1,p)}$ is pure in G_1 (cf. Observation 2.5), $b_1 = q_1 b$ and $q_1 \in \mathbb{Z}$, we have $b \in G_{(1,p)}$ (cf. Observation 2.4). Why (ii)? By (1) and $(*_1)$(i), recalling Hyp. 3.12(4).]
By Lemma 3.10(3), there are $k < \omega$, and, for $i < k$, $\bar{y}^i \in \bar{y}/E_n$ and $q^i \in \mathbb{Q} \setminus \{0\}$ s.t.:

$(*)_2$ $a = \sum_{i<k} q^i(\sum_{\ell<n} q'_\ell y^i_\ell) = \sum_{i<k}(\sum_{\ell<n} q^i q'_\ell y^i_\ell)$.

Notice that by the assumption in item (2) of the present lemma we have that $a = qx$, for some $x \in X$ and $q \in \mathbb{Q} \setminus \{0\}$, and so we have that:

$(*)_3$ (a) $qx = a = \sum_{\ell<n}(\sum_{i<k} q^i q'_\ell y^i_\ell)$;
(b) $x \in \{y^i_\ell : i < k, \ell < n\}$.

By Lemma 3.10(4) $k = 1$, so $qx = a = q^0 \sum_{\ell<n} q'_\ell y^0_\ell$, hence (recalling that $q \neq 0$ and that $\bar{y}^0 \in \mathrm{seq}_n(X)$, as E_n is an equivalence relation on $\mathrm{seq}_n(X)$) necessarily $n = 1$, as wanted. Finally, as $\pi(a) = b \in \mathbb{Q}x$ and $\pi(a) \in G_1[\mathcal{V}]$ necessarily $y \in X_{\mathcal{V}}$. ■

Conclusion 3.14. *(1) There is a sequence $(q^1_x : x \in X_{\mathcal{U}})$ of non-zero rationals and a function $\pi_1 : X_{\mathcal{U}} \to X_{\mathcal{V}}$ such that for every $x \in X_{\mathcal{U}}$ we have that $\pi(x) = q^1_x(\pi_1(x))$, moreover the function π_1 is 1-to-1.*
(2) There is a sequence $(q^2_x : x \in X_{\mathcal{V}})$ of non-zero rationals and a function $\pi_2 : X_{\mathcal{V}} \to X_{\mathcal{U}}$ such that $\pi^{-1}(x) = q^2_x(\pi_2(x))$, moreover π_2 is 1-to-1.
(3) (i) $\pi_2 \circ \pi_1 : X_{\mathcal{U}} \to X_{\mathcal{U}} = id_{\mathcal{U}}$;
(ii) $\pi_1 \circ \pi_2 : X_{\mathcal{V}} \to X_{\mathcal{V}} = id_{\mathcal{V}}$;
(iii) $\pi_1 : X_{\mathcal{U}} \to X_{\mathcal{V}}$ is a bijection.

Proof. (1) by Lemma 3.13. (2) by Lemma 3.13 applied to π^{-1}. (3) by (1) and (2). ■

Claim 3.15. *In the context of Conc. 3.14 and letting $(q^1_x : x \in X_{\mathcal{U}}) = (q_x : x \in X_{\mathcal{U}})$.*
(1) For some $q_* \in \mathbb{Q} \setminus \{0\}$ we have that, for every $x \in X_{\mathcal{U}}$, $q_x = q_*$.
(2) There is an isomorphism $\sigma_1 : M \restriction \mathcal{U} \cong M \restriction \mathcal{V}$.

Proof. Let $x \neq y \in X_{\mathcal{U}}$ be such that $(x)/E^{\mathtt{m}}_1 \neq (y)/E^{\mathtt{m}}_1$ (cf. Definition 3.3(24)). Let then $e = (x, y)/E_2$, $\bar{q} = (1, 1)$ and $p = p_{(e,\bar{q})}$. Now, by the choice of p, we have that $x + y \in G_{(1,p)}$ and so $q_x\pi_1(x) + q_y\pi_1(y) = \pi(x) + \pi(y) = \pi(x + y) \in G_{(1,p)}$. So, by Lemma 3.10(3), there are $(x_i, y_i) \in (x, y)/E_2$ and $q^i \in \mathbb{Q} \setminus \{0\}$, for $i < k$, s.t.:

$(\star_1)$ $q_x\pi_1(x) + q_y\pi_1(y) = \sum_{i<k} q^i(x_i + y_i) = (\sum_{i<k} q^i x_i) + (\sum_{i<k} q^i y_i)$.

Let $(x_k, y_k) = (\pi_1(x), \pi_1(y))$. Now, $(\pi_1(x), \pi_1(y)) \in \mathrm{seq}_2(X)$ by Conclusion 3.14. Let $e_1 = (x)/E^{\mathtt{m}}_1$, $e_2 = (y)/E^{\mathtt{m}}_1$ and, for $\ell = 1, 2$, $p_\ell = p_{(e_\ell,1)}$. So:

$(\star_{1.1})$ for $z \in X$, $z \in G_{(1,p_\ell)} \Leftrightarrow z \in e_\ell$.

Hence, $x_k = \pi_1(x) \in e_1$ and $y_k = \pi_1(y) \in e_2$, since π is an isomorphism and (by Lemma 3.14(1)) π_1 is such that for every $x \in X_{\mathcal{U}}$ we have $\pi(x) = q^1_x(\pi_1(x))$. Also, $x_i \in e_1$ and $y_i \in e_2$, for every $i < k$, as $(x_i, y_i) \in (x, y)/E_2$ and so $x_i \in x/E_2$ and $y_i \in y/E_2$. Lastly, clearly $e_1 \cap e_2 = \emptyset$, by the choice of (x, y). Hence, the sets $\{x_i : i \leqslant k\}$ and $\{y_i : i \leqslant k\}$ are disjoint. But this means that:

$(\star_{1.2})$ the sets $\{x_i : i < k\} \cup \{\pi_1(x)\}$ and $\{y_i : i < k\} \cup \{\pi_1(y)\}$ are disjoint.

Thus, by $(\star_1)$ and $(\star_{1.2})$ we have:

$(\star_2)$ (a) $q_x\pi_1(x) = \sum\{q^i x_i : i < k\}$;
(b) $q_y\pi_1(y) = \sum\{q^i y_i : i < k\}$.

Let $E_x = \{(i,j) : i,j < k, x_i = x_j\}$, so E_x is an equivalence relation on k. By $(\star_2)$(a), $\pi_1(x) \in \{x_i : i < k\}$, and so for some $i_* < k$ we have that $x_{i_*} = \pi_1(x)$. Hence, again by $(\star_2)$(a), we have:

$(\star_{2.5})$ (a) $q_x = \sum\{q^i : i \in i_*/E_x\}$;
(b) if $i < k$ and $i \notin i_*/E_x$, then $\sum\{q^j : j \in i/E_x\} = 0$.

Together we have:

$(\star)_3$ $q_x = \sum\{q^i : i < k\}$.

Similarly using $(\star)_2$(b) we get that:

$(\star)_4$ $q_y = \sum\{q^i : i < k\}$.

Thus, $q_x = q_y$. So $x, y \in X_{\mathcal{U}}$ and $x/E_1^{\mathfrak{m}} \neq y/E_1^{\mathfrak{m}}$ implies $q_x = q_y$. Hence, for every $x, y \in X_{\mathcal{U}}$ we have $q_x = q_y$. Why? If $x/E_1^{\mathfrak{m}} \neq y/E_1^{\mathfrak{m}}$ see above, if $x/E_1^{\mathfrak{m}} = y/E_1^{\mathfrak{m}}$, by Definition 3.6$(*)_1$, we can find $z \in X \setminus x/E_1^{\mathfrak{m}} = X \setminus y/E_1^{\mathfrak{m}}$, so $q_x = q_z = q_y$.

Moving to item (2), we shall show:

$(\star_5)$ if $k = 2$, $\bar{z} \in \mathrm{seq}_k(X_{\mathcal{U}})$ and $\bar{z}$ is $<_k^{\mathfrak{m}}$-minimal, then there is a sequence $(\bar{g}^\ell : \ell < n)$ of members of $I^{\mathfrak{m}}$ and $\bar{z}_\ell \in \mathrm{seq}_k(X)$, for $\ell \leqslant n$, such that:
(a) $\bar{z}_0 = \bar{z}$;
(b) $\bar{z}_n = \pi_1(\bar{z})$;
(c) if $\ell < n$, then $f_{\bar{g}^\ell}(\bar{z}_\ell) = \bar{z}_{\ell+1}$.

We shall now prove $(\star_5)$. Let $a = \sum\{z_\ell : \ell < k\} \in G_1$ and $b = \pi(a)$, so:

$$b = \pi(\sum_{\ell<k} z_\ell) = \sum_{\ell<k} \pi(z_\ell) = \sum_{\ell<k} q_* \pi_1(z_\ell) = q_* \sum_{\ell<k} \pi_1(z_\ell).$$

Let $e = \bar{z}/E_k$, $\bar{q} = \underbrace{(1, ..., 1)}_{k}$ and $p = p_{(e,\bar{q})}$, so $a \in G_{(1,p)}$ and thus $b = \pi(a) \in G_{(1,p)}$, hence there are $i(*)$ and, for $i < i(*)$, $\bar{y}^i \in \bar{z}/E_k$ and $q^i \in \mathbb{Q} \setminus \{0\}$ such that:

(i) $(\bar{y}^i : i < i(*))$ is with no repetitions;
(ii) $b = \sum_{i<i(*)} q^i (\sum_{\ell<k} y^i_\ell)$.

Hence we have:

$$(+_1) \qquad q_* \sum_{\ell<k} \pi_1(z_\ell) = \sum_{i<i(*)} q^i (\sum_{\ell<k} y^i_\ell).$$

Now, $k = 2$, and so we have:

$$(+_2) \qquad q_* \pi_1(z_0) + q_* \pi_1(z_1) = (\sum_{i<i(*)} q^i y^i_0) + (\sum_{i<i(*)} q^i y^i_1).$$

Recall that by Definition 3.3(17) we have:

$$(+_3) \qquad \{y^i_0 : i < i(*)\} \cap \{y^i_1 : i < i(*)\} = \emptyset.$$

We now distinguish three cases.

Case 1. $i_* = 1$.

Recall that $\bar{y}^i \in \bar{z}/E_k^{\mathfrak{m}}$ and $\bar{z}$ is $<_k^{\mathfrak{m}}$-minimal, so the tree $(\bar{z}/E_k^{\mathfrak{m}}, <_k^{\mathfrak{m}} \restriction \bar{z}/E^{\mathfrak{m}})$ has only one root and this root is $\bar{z}$, hence $i < i(*)$ implies $\bar{z} \leqslant_k^{\mathfrak{m}} \bar{y}^i$. By the definition of $<_k^{\mathfrak{m}}$, as $\bar{z} \leqslant_k^{\mathfrak{m}} \bar{y}^0$, clearly there are $n < \omega$, $(\bar{z}_\ell : \ell \leqslant n)$ and $(\bar{g}^\ell : \ell < n)$ such that $\bar{z}_0 = \bar{z}$, $\bar{z}_n = \bar{y}^0$ and, for $\ell < n$, $f_{\bar{g}^\ell}(\bar{z}_\ell) = \bar{z}_{\ell+1}$. Thus, as by assumption $i_* = 1$, we have that $(\star)_5$ holds, noticing that $\bar{z}_n = \bar{y}^0 = \pi_1(\bar{z})$.

Case 2. $i_* \geqslant 2$.

As $(\bar{y}^i : i < i(*))$ is with no repetitions, there is $i < i(*)$ such that $\bar{y}^i \neq \bar{z}$. Choose

$i < i(*)$ such that $\bar{y}^i$ is (locally) $<^{\mathfrak{m}}_k$-maximal among $\{\bar{y}^i : i < i(*)\}$, hence $\bar{y}^i \neq \bar{z}$, and let $\ell < k$ be such that y^i_ℓ is $<_{\mathfrak{m}}$-maximal in $\bar{y}^i$, then, by Definition 3.3(23c), y^i_ℓ appears only once in $(\bar{y}^j : j < i(*))$, and it does not appear in $\bar{z}$ (as $\bar{z} <^{\mathfrak{m}}_k \bar{y}^i$), contradicting the equality $(+_2)$. Hence, this case is not possible.

Case 3. $i_* = 0$.

Contrary to the assumption that π is an isomorphism from $G_{(1,\mathcal{U})}[\mathfrak{m}]$ onto $G_{(1,\mathcal{V})}[\mathfrak{m}]$, as then an element $a \neq 0$ is mapped to 0. Hence, this case is not possible.

As only Case 1 is possible and in that case $(\star_5)$ holds, we are done proving $(\star_5)$.

$(\star_6)$ If $s \subseteq_\omega \mathcal{U}$, then for some unique $t = t_s \subseteq_\omega \mathcal{V}$ we have:
  - (a) π_1 maps X'_s into X'_t;
  - (b) in (a) we have $|t_s| = |s|$;
  - (c) $s_1 \neq s_2$ implies $t_{s_1} \neq t_{s_2}$;
  - (d) for every $t \subseteq_\omega \mathcal{V}$ there exists $s \subseteq_\omega \mathcal{U}$ such that $t_s = t$.

We prove $(\star_6)$. Concerning item (a), if it fails, then there is $\bar{z} = (z_1, z_2) \in \mathrm{seq}_2(X'_s)$ such that $\bigwedge_{\ell=1,2} \pi_1(z_\ell) \in X'_{t_\ell}$ and $t_1 \neq t_2$ and $\bar{z}/E^{\mathfrak{m}}_2$ is $<^{\mathfrak{m}}_k$-minimal (see Definition 3.3(24)). Now, applying $(\star_5)$ to $\bar{z}$, let $(\bar{g}^\ell : \ell < n)$ be as there. For each $\ell < n$ and $s_1 \subseteq_\omega M$, either $X'_{s_1} \cap \mathrm{dom}(f_{\bar{g}_\ell}) = \emptyset$ or $f_{\bar{g}_\ell}$ maps $X'_{s_1} \cap \mathrm{dom}(f_{\bar{g}_\ell}) \neq \emptyset$ into X'_{s_2}, for some $s_2 \subseteq_\omega M$, by Definition 3.3(7d), so the contradiction follows, and clause (a) holds. Concerning item (b), by the proof of clause (a) we can choose $s_0, ..., s_n$ such that $s_0 = s$ and $\bar{g}_\ell(s_\ell) = s_{\ell+1}$, for $\ell < n$, and $t = s_n$. Now, by induction on $\ell \leqslant n$ we can prove that $|s_\ell| = |s_0|$, where the case $\ell = 0$ is trivial, and the case $\ell+1$ is by Definition 3.3(7c)(7d), recalling that partial automorphisms are 1-to-1 maps. Finally, concerning items (c) and (d), we can apply the above replacing $(\mathcal{U}, \mathcal{V}, \pi)$ with $(\mathcal{V}, \mathcal{U}, \pi^{-1})$. So for every $t \subseteq_\omega \mathcal{V}$ there is $s_t \subseteq_\omega \mathcal{U}$ such that $|t| = |s_t|$ and π^{-1} maps $\langle X'_t \rangle^*_{G_1}$ into $\langle X'_{s_t} \rangle^*_{G_1}$, but then it is easy to conclude. For example, concerning item (c), toward contradiction, assume that $s_1 \neq s_2$ but $t_{s_1} = t_{s_2} := t$. Choose then $x_1 \in X'_{s_1}$ and $x_2 \in X'_{s_2}$, so by (a) we have:

$$y_1 = \pi_1(x_1) \in X_{t_{s_1}} \text{ and } y_2 = \pi_1(x_2) \in X_{t_{s_2}}.$$

Hence, $y_1, y_2 \in X'_t = X_{t_{s_1}} = X_{t_{s_2}}$, but then:

$$x_1 = \pi_1^{-1}(y_1) \in X'_{s_1} \text{ and } x_2 = \pi_1^{-1}(y_2) \in X'_{s_2}.$$

By clause (a) for $(\mathcal{V}, \mathcal{U}, \pi^{-1})$, we get a contradiction, since $s_1 \neq s_2$, recalling that by Definition 3.3(2a) we have that $X'_{s_1} \cap X'_{s_2} = \emptyset$. All together $(\star_6)$ holds.

We shall now show:

$(\star_7)$
  - (a) the map $\mathbf{h} : \mathcal{U} \to \mathcal{V}$ such that $r \in \mathcal{U}$ implies $t_{\{r\}} = \{\mathbf{h}(r)\}$ is well-defined;
  - (b) $\mathbf{h} : \mathcal{U} \to \mathcal{V}$ is one-to-one and onto;
  - (c) $\mathbf{h}$ is an isomorphism from $M \restriction \mathcal{U}$ onto $M \restriction \mathcal{V}$.

We prove $(\star_7)$. Clause (a) is by $(\star_6)$(a)(b). Clause (b) is by $(\star_6)$(c)(d). Clause (c) holds by $(\star_5)$, but we elaborate. Let $r_0 \neq r_1 \in \mathcal{U}$ and let $r'_0 = \mathbf{h}(r_0)$ and $r'_1 = \mathbf{h}(r_1)$. Since $\mathbf{h}$ is one-to-one from $\mathcal{U}$ onto $\mathcal{V}$ it suffices to prove:

$(*)$ (r_0, r_1) is an edge of M iff (r'_0, r'_1) is an edge of M.

Now, apply $(\star_5)$ to $\bar{z} = (z_0, z_1)$ for $z_0 \in X'_{\{r_0\}}$ and $z_1 \in X'_{\{r_1\}}$ such that (z_0, z_1) is $<^{\mathfrak{m}}_2$-minimal (cf. Definition 3.3(24)), and let $(\bar{g}^\ell : \ell < n)$ and $(\bar{z}^\ell : \ell \leqslant n)$ be a witness of this, so $\bar{z}^n = \pi_1(\bar{z})$. For $\ell < n$ and $i \in \{0, 1\}$, let:

(1) $z^0_i = z_i$;
(2) $z^{\ell+1}_i = f_{\bar{g}_\ell}(z^\ell_i)$;

(3) $r^0_i = r_i$;
(4) $r^{\ell+1}_i = \mathbf{h}(r^\ell_i)$.

Notice that by the choice above we have that, for $i \in \{0,1\}$, $r^n_i = r'_i$. Furthermore, by Definition 3.3(7b)-(7e) we have that, for $\ell < n$, $\bar{g}_\ell$ maps (r^ℓ_0, r^ℓ_1) to $(r^{\ell+1}_0, r^{\ell+1}_1)$. Recall now that, for $\ell < n$, $\bar{g}_\ell$ is a partial automorphism of M, and so it is immediate to prove by induction on $\ell \leqslant n$ that the following holds:

$$M \models R(r_0, r_1) \leftrightarrow R(r^\ell_0, r^\ell_1),$$

and so we are done, since, as already noticed, $r^n_i = r'_i$, for $i \in \{0,1\}$. ■

Claim 3.16. *In the context of Claim 3.15, q_* is an integer.*

Proof. If not, then $q_* = \frac{m}{k}$, for $m \in \mathbb{Z} \setminus \{0\}$ and $k \in \omega \setminus \{0,1\}$. Let p be a prime dividing k. Let $x_1 \in X_{\mathcal{U}}$. If in G_1 we have that x_1 is not divisible by p, then we are done (since then $\pi(x_1)$ cannot be $q_* x_1$). Thus, by Lemma 3.10(3)(4), it must be the case that $p = p_{(x_1/E_1,(q))}$, for some $q \in \mathbb{Q} \setminus \{0\}$ such that $qx_1 \in G_0$, but by Definition 3.6$(*)_1$ we can find $x_2 \in X_{\mathcal{U}}$ such that $(x_2) \notin (x_1)/E_1$, and so, by Lemma 3.10(3), also in this case we reach a contradiction. Thus, $q_* \in \mathbb{Z}$. ■

Claim 3.17. *In the context of Claim 3.15, $q_* \in \{1,-1\}$.*

Proof. If not, then we contradict Claim 3.16 when applied to π^{-1}. ■

3.3. The Proof of the Main Theorem

Notice that in this subsection Hypothesis 3.12 is no longer assumed.

Conclusion 3.18. *Let $\mathfrak{m}[M] \in \mathrm{K}^{\mathrm{bo}}_2$, $\mathcal{U}, \mathcal{V} \subseteq M$ and $|\mathcal{U}| = |\mathcal{V}| = \aleph_0$. Then:*

$$(\star) \qquad M \restriction \mathcal{U} \cong M \restriction \mathcal{V} \Leftrightarrow G_{(1,\mathcal{U})}[\mathfrak{m}] \cong G_{(1,\mathcal{V})}[\mathfrak{m}].$$

Proof. First assume that $\mathcal{U} = \mathcal{V}$, then clearly both the left-hand-side (LHS) and the right-hand-side (RHS) of $(\star)$ holds. Assume then that $\mathcal{U} \neq \mathcal{V}$. If the LHS of $(\star)$ holds, then by Claim 3.11 also the RHS of $(\star)$ holds. On the other hand, if the RHS of $(\star)$ holds, then the assumptions in Hyp. 3.12 are fulfilled and thus 3.13-3.17 holds, so in particular Claim 3.15(2) holds, and thus the LHS of $(\star)$ holds. ■

Convention 3.19. *In Fact 1.1 and Definition 1.8(5) instead of considering structures with domain ω we could have considered structures with domain an infinite subset of ω. We take the liberty of not distinguishing between these two variants. This happens most notably in the Proof of Main Theorem right below.*

Proof of Main Theorem. Let M be as in Hypothesis 3.1. Fix $\mathfrak{m} \in \mathrm{K}^{\mathrm{bo}}_2(M)$ (cf. Claim 3.8) and assume without loss of generality that $G_1[\mathfrak{m}]$ has set of elements ω. For every graph H with domain ω we define $F[H] : H \to M$ by defining $F[H](n)$ by induction on $n < \omega$ as the minimal $k < \omega$ such that $\{(\ell, F[H](\ell)) : \ell < n\} \cup \{(n,k)\}$ is a graph isomorphism from $H \restriction (n+1)$ onto $M \restriction (\{F[H](\ell) : \ell < n\} \cup \{k\})$. The map $H \mapsto M \restriction \{F[H](n) : n < \omega\}$ is clearly continuous. We will show that the map $F' : M \restriction \mathcal{U} \mapsto G_{(1,\mathcal{U})}[\mathfrak{m}]$, for $\mathcal{U} \subseteq M$ infinite, is also continuous (recall Convention 3.19), thus concluding that the map $\mathbf{B} := F' \circ F : H \mapsto G_{(1,\{F[H](n)\, :\, n<\omega\})}[\mathfrak{m}]$ is a continuous map from Graph_ω into TFAB_ω (Convention 3.19) and so, by Conclusion 3.18, the map $\mathbf{B}$ is as wanted.

In order to show that F' is continuous, first recall that $\mathfrak{m}$ as well as $\bar{p}$ from 3.3(13) are fixed. Now, given $a \in G_1[\mathfrak{m}]$, we have to compute from $\mathcal{U}$ whether $a \in G_{(1,\mathcal{U})}[\mathfrak{m}]$

or not. To this extent, let $a = \sum\{q^a_\ell x^a_\ell : \ell < n\}$ with the x_ℓ's pairwise distinct and $q_\ell \in \mathbb{Q} \setminus \{0\}$. Now, as by 3.3(3), $X_{\mathcal{U}} = \bigcup\{X_s : s \subseteq_\omega \mathcal{U}\} = \bigcup\{X'_s : s \subseteq_\omega \mathcal{U}\}$ and the latter is a partition of X, for every $\ell < n$, there is a unique finite $s^a_\ell \subseteq M$ s.t.:

$$a \in G_{(1,\mathcal{U})}[\mathfrak{m}] \Leftrightarrow \bigwedge_{\ell<n} s^a_\ell \subseteq \mathcal{U}.$$

This suffices to show continuity of F', thus concluding the proof of the theorem. ■

Remark 3.20. *We observe that in the context of the Proof of Main Theorem we can choose both M and $\mathfrak{m}$ to be computable stuctures, in the sense of computable model theory, i.e., all the relations and functions of the structure are computable.*

4. The Co-Hopfian Problem for Torsion-Free Abelian Groups

Fact 4.1 ([4, Proposition 2.2, pg. 130])**.** *For $G \in$ TFAB, G is co-Hopfian iff G is divisible and of finite rank, i.e., G is a finitely dimensional vector space over $\mathbb{Q}$.*

Conclusion 4.2. *The co-Hopfian groups in TFAB_ω form a Borel subset of TFAB_ω.*

On the other hand, we will show below that there are variations on the notion of co-hopfianity (cf. Definition 2.7) which give a completely different answer.

Hypothesis 4.3. *Throughout this section the following hypothesis stands:*

(1) $T = (T, <_T)$ is a rooted tree with ω levels and we denote by $\mathrm{lev}(t)$ the level of t;
(2) $T = \bigcup_{n<\omega} T_n$, $T_n \subseteq T_{n+1}$, and $t \in T_n$ implies that $\mathrm{lev}(t) \leqslant n$;
(3) $T_0 = \emptyset$, T_n is finite, and we let $T_{<n} = \bigcup_{\ell<n} T_\ell$ (so $T_{<(n+1)} = T_n$);
(4) if $s <_T t \in T_n$, then $s \in T_n$.

Definition 4.4. *Let $\mathrm{K}^{\mathrm{co}}_1(T)$ be the class of objects:*

$$\mathfrak{m}(T) = \mathfrak{m} = (X^T_n, \bar{f}^T_n : n < \omega) = (X_n, \bar{f}_n : n < \omega)$$

satisfying the following requirements:

(a) $X_0 \neq \emptyset$, X_n is finite and strictly increasing with n, and $X_{<n} = \bigcup_{\ell<n} X_\ell$;
(b) $\bar{f}_n = (f_t : t \in T_n)$, so if $s \in T_m$ for some $m < n$, then f_s is determined by $\bar{f}_m$;
(c) if $n > 0$ and $t \in T_n \setminus T_{<n}$, then f_t is a one-to-one function from X_{n-1} into X_n;
(d) for every $t \in T$, $X_0 \cap \mathrm{ran}(f_t) = \emptyset$;
(e) if $s <_T t \in T_n$, then $f_s \subseteq f_t$;
(f) if $t \in T_n \setminus T_{<n}$, $f_t(x) = y$ and $y \in X_{n-1}$, then for some $s <_T t$, $x \in \mathrm{dom}(f_s)$;
(g) if $s, t \in T_n$ and $x \in \mathrm{ran}(f_s) \cap \mathrm{ran}(f_t)$, then for some $r \in T_n$ such that $r \leqslant_T s, t$ we have that $x \in \mathrm{ran}(f_r)$, equivalently, $\mathrm{ran}(f_s) \cap \mathrm{ran}(f_t) = \mathrm{ran}(f_r)$, for $r = s \wedge t$, where $\wedge$ is the natural semi-lattice operation taken in the tree $(T, <_T)$;
(h) $X_{n+1} = \bigcup\{\mathrm{ran}(f_t) : t \in T_{n+1} \setminus T_n\} \cup X_n$;
(i) we let $X = X^{\mathfrak{m}} = \bigcup_{n<\omega} X_n$.

Convention 4.5. $\mathfrak{m} = (X_n, \bar{f}_n : n < \omega) \in \mathrm{K}^{\mathrm{co}}_1(T)$ *(cf. Definition 4.4).*

Observation 4.6. *In the context of Definition 4.4, we have:*

(1) If $m < n < \omega$, $t \in T_n \setminus T_{<n}$ and for every $s <_T t$ we have $s \in T_m$, then:

$$(X_{n-1} \setminus X_m) \cap \mathrm{ran}(f_t) = \emptyset.$$

(2) Let $t \in T$, then for every $x \in \mathrm{dom}(f_t)$ we have that $x \neq f_t(x)$, moreover there is unique $0 < n < \omega$ such that $x \in X_{n-1}$ and $f_t(x) \in X_n \setminus X_{n-1}$.

Proof. We prove (1), by Definition 4.4(c) we know that f_t is one-to-one from X_{n-1} into X_n. If $n = 1$, then $m = 0$ and so $X_{n-1} = X_0 = X_m$, thus the conclusion is trivial. Suppose then that $n > 1$ and let $y \in (X_{n-1} \setminus X_m) \cap \mathrm{ran}(f_t)$, and let $x \in \mathrm{dom}(f_t)$ be such that $f_t(x) = y$. Then, by Definition 4.4(f) there exists $s <_T t$ such that $x \in \mathrm{dom}(f_s)$. But then, using the assumption in (1), we have that $s \in T_m$ (so $m = 0$ is impossible by Definition 4.3(3)). Hence, by Definition 4.4(c), $\mathrm{ran}(f_s) \subseteq X_m$, so $y = f(x) \in X_m$, contradicting the fact that $y \in (X_{n-1} \setminus X_m)$.

We prove (2). Assume that x, t, and thus also f_t, are fixed and $x \in \mathrm{dom}(f_t)$. Let $s \leqslant_T t$ be $\leqslant_T$-minimal such that $f_s(x)$ is well-defined, and let $n < \omega$ be such that $s \in T_n \setminus T_{<n}$ (notice that $n \geqslant 1$ since $T_0 = \emptyset$). Clearly, there is unique $m < \omega$ such that $x \in X_m \setminus X_{<m}$. As $x \in \mathrm{dom}(f_s)$ and $s \in T_n \setminus T_{<n}$ necessarily $m < n$, so $x \in X_{<n}$. But by the choice of s we have that $r <_T s$ implies $x \notin \mathrm{dom}(f_r)$. By the last two sentences and Def. 4.4(f) we have $f_s(x) \in X_n \setminus X_{<n}$, but $f_t(x) = f_s(x)$. ■

Claim 4.7. *For T as in Hypothesis 4.3, $\mathrm{K}_1^{\mathrm{co}}(T) \neq \emptyset$ (cf. Definition 4.4).*

Proof. Straightforward. ■

Definition 4.8. *On X (cf. Convention 4.5) we define:*

(1) for $x \in X$, $\mathrm{suc}(x) = \{f_t(x) : t \in T, x \in \mathrm{dom}(f_t)\}$;
(2) for $x, y \in X$, we let $x <_X y$ if and only if for some $0 < n < \omega$ and $x_0, ..., x_n \in X$ we have that $\bigwedge_{\ell<n} x_{\ell+1} \in \mathrm{suc}(x_\ell)$, $x = x_0$ and $y = x_n$;
(3) $\mathrm{seq}_k(X) = \{\bar{x} \in \mathrm{seq}_k(X) : \bar{x}$ is injective$\}$;
(4) we say that $\bar{x} \in \mathrm{seq}_k(X)$ is reasonable when the following happens:

$$n_1 < n_2,\ x_{i(1)} \in X_{n(1)} \setminus X_{<n(1)},\ x_{i(2)} \in X_{n(2)} \setminus X_{<n(2)} \Rightarrow i(1) < i(2);$$

(5) $<^k_X$ is the partial order on $\mathrm{seq}_k(X)$ defined as $\bar{x}^1 <^k_X \bar{x}^2$ if and only if $\bar{x}^1, \bar{x}^2 \in \mathrm{seq}_k(X)$ and there are $0 < n < \omega$, $\bar{y}^0, ..., \bar{y}^n \in \mathrm{seq}_k(X)$ and $t_0, ..., t_{n-1} \in T_{\mathfrak{m}}$ such that for every $\ell < n$ we have that $f_{t_\ell}(\bar{y}^\ell) = \bar{y}^{\ell+1}$, and $(\bar{x}^1, \bar{x}^2) = (\bar{y}^0, \bar{y}^n)$;
(6) notice that for $k = 1$ we have that $<^k_X = <_X$, where $<_X$ is as in (2).

Observation 4.9. *(1) $(X, <_X)$ is a tree with ω levels;*
(2) $(X, <_X)$ is well-founded iff $(T, <_T)$ is well-founded;
(3) every $z \in X_0$ is a root of the tree $(X, <_X)$;
(4) if $y \in X_{n+1} \setminus X_n$, then for one and only one $x \in X_n$ we have $y \in \mathrm{suc}(x)$;
(5) if $y \in \mathrm{suc}(x)$, then $\{t \in T : f_t(x) = y\}$ is a cone of T;
(6) if $\bar{x} \in \mathrm{seq}_k(X)$, then some permutation of $\bar{x}$ is reasonable (cf. Definition 4.8(4));
(7) if $f_t(\bar{x}) = \bar{y}$ and $\bar{x}$ is reasonable, then so is $\bar{y}$;
(8) for every $k \geqslant 1$, $(\mathrm{seq}_k(X), <^k_X)$ is a tree;
(9) if $\bar{x} <^k_X \bar{y}$ and $\bar{x}$ is reasonable, then $\bar{y}$ is also reasonable;
(10) if $\bar{x} \in \mathrm{seq}_k(X)$ is reasonable, $\bar{x} \leqslant^k_X \bar{y}^1 = (y^1_0, ..., y^1_{k-1})$, $\bar{x} \leqslant^k_X \bar{y}^2 = (y^2_0, ..., y^2_{k-1})$ and $y^1_{k-1} = y^2_{k-1}$, then $\bar{y}^1 = \bar{y}^2$.

Proof. Items (1)-(3) and (8)(9) are clear. We prove (4). By Definition 4.4(h) we have at least one $x \in X_n$ such that $y \in \mathrm{suc}(x)$, but by Definition 4.4(f)-(g) we have at most one $x \in X_n$ such that $y \in \mathrm{suc}(x)$, putting everything together (4) holds. Items (5) and (6) are also easy. Item (7) can be proved for $t \in T_n \setminus T_{<n}$ by induction on $n < \omega$. Finally, concerning item (10), w.l.o.g. $n[\bar{y}^1] \leqslant n[\bar{y}^2]$, where for $\bar{y} \in \mathrm{seq}_k(X)$ we let $n[\bar{y}] = \min\{m < \omega : \bar{y} \subseteq X_m\}$, and we can prove item (10) by induction on $n[\bar{y}^2]$. Item (5) is not used but we retain it to give the picture. ■

Definition 4.10. *Let $\mathfrak{m} \in \mathrm{K}_1^{\mathrm{co}}(T)$ (i.e. as in Convention 4.5).*

(1) Let $G_2 = G_2[\mathfrak{m}]$ be $\bigoplus\{\mathbb{Q}x : x \in X\}$.
(2) Let $G_0 = G_0[\mathfrak{m}]$ be the subgroup of G_2 generated by X, i.e. $\bigoplus\{\mathbb{Z}x : x \in X\}$.
(3) For $t \in T$, let:
   (a) $H_{(2,t)} = \bigoplus\{\mathbb{Q}x : x \in \mathrm{dom}(f_t)\}$;
   (b) $I_{(2,t)} = \bigoplus\{\mathbb{Q}x : x \in \mathrm{ran}(f_t)\}$;
   (c) $\hat{f}^2_t$ is the (unique) isomorphism from $H_{(2,t)}$ onto $I_{(2,t)}$ such that $x \in \mathrm{dom}(f_t)$ implies that $\hat{f}^2_t(x) = f_t(x)$ (cf. Definition 4.4(c)).
(4) For $t \in T$, we define $H_{(0,t)} := H_{(2,t)} \cap G_0$ and $I_{(0,t)} := I_{(2,t)} \cap G_0$;
(5) For $\hat{f}^2_t$ as above, we have that $\hat{f}^2_t[H_{(0,t)}] = I_{(0,t)}$. We define $\hat{f}^0_t$ as $\hat{f}^2_t \restriction H_{(0,t)}$.
(6) We define the partial order $<_*$ on $G_0^+ := G_0 \setminus \{0\}$ by letting $a <_* b$ if and only if $a \neq b \in G_0^+$ and, for some $0 < n < \omega$, $a_0, ..., a_n \in G_0, a_0 = a, a_n = b$ and:

$$\ell < n \Rightarrow \exists t \in T(\hat{f}_t(a_\ell) = a_{\ell+1}).$$

(7) For $a = \sum_{\ell<m} q_\ell x_\ell$, with $x_\ell \in X$ and $q_\ell \in \mathbb{Q} \setminus \{0\}$, let $\mathrm{supp}(a) = \{x_\ell : \ell < m\}$.
(8) For $a \in G_2^+$, let $n[a]$ be the minimal $n < \omega$ such that $a \in \langle X_{<n}\rangle^*_{G_2}$.

Lemma 4.11. *(1) If $\{t \in T : \hat{f}^2_t(a) = b\} \neq \emptyset$, then it is a cone of T.*
(2) $<_* \restriction X = <_X$ (where $<_X$ is as in Definition 4.8(2)).
(3) $(G_0, <_*)$ is a tree with ω levels (recall Hypothesis 4.3(1)).
(4) If $s \leqslant_T t$, then $\hat{f}^\ell_s \subseteq \hat{f}^\ell_t$, for $\ell \in \{0, 2\}$.
(5) If $t \in T$, $\hat{f}^2_t(a) = b$ and $a \in G_0^+$, then $n[a] < n[b]$ (cf. Definition 4.10(8)).
(6) If $a <_* b$ (so $a, b \in G_0^+$), then the sequence $(a_\ell : \ell \leqslant n)$ from 4.10(6) is unique.
(7) If $a <_* b$, and, for $\ell \in \{0,1\}$, $a_\ell = \sum_{i<k} q_i x^\ell_i$, $q_i \in \mathbb{Q} \setminus \{0\}$, $\bar{x}^\ell = (x^\ell_i : i < k) \in \mathrm{seq}_k(X)$, then maybe after replacing $\bar{x}^1$ with a permutation of it, $\bar{x}^0 \leqslant^k_X \bar{x}^1$.

Proof. Unraveling definitions, e.g. for item (3) use Definition 4.4(e), we elaborate only on item (5). Concerning item (5), as $a \neq 0$, let $a = \sum_{i \leqslant n} q_i x_i$, $x_i \in X$ with no repetitions, $q_i \in \mathbb{Q} \setminus \{0\}$. Let $x_i \in X_{k(i)} \setminus X_{<k(i)}$ and w.l.o.g. $k(i) \leqslant k(i+1)$, for $i < n$ (cf. Observation 4.9(6)). Clearly $a \in \langle X_{k(n)}\rangle^*_{G_2}$ but $a \notin \langle X_{<k(n)}\rangle^*_{G_2}$. As $\hat{f}^2_t(a)$ is well-defined, clearly $\{x_i : i \leqslant n\} \subseteq \mathrm{dom}(f_t)$ and $b = \hat{f}^2_t(a) = \sum_{i \leqslant n} q_i f_t(x_i)$ and, as f_t is 1-to-1, the sequence $(f_t(x_i) : i \leqslant n)$ is with no repetitions. By Observation 4.6(2) applied with n there as $k(n)$ here, $f_t(x_n) \notin \langle X_{k(n)}\rangle^*_{G_2}$, hence we have that $n[b] \geqslant n(f_t(x_n)) + 1 > k(n) + 1 = n[a]$, so (5) holds. ■

Claim 4.12. *If (A), then (B), where:*

(A) (a) $a, b_\ell \in G_0$, for $\ell < \ell_*$;
   (b) $a \leqslant_* b_\ell$ and the b_ℓ's are with no repetitions;
   (c) $a = \sum\{q_i x_i : i < j\}$;
   (d) $\bar{x} = (x_i : i < j) \in X^j$ is injective and reasonable;
   (e) $q_i \in \mathbb{Z} \setminus \{0\}$;
(B) there are ℓ_* and for $\ell < \ell_*$, $\bar{y}^\ell = (y_{(\ell,i)} : i < j)$ such that:
   (a) $y_{(\ell,i)} =: y^\ell_i \in X$ and $\bar{x} \leqslant^j_* \bar{y}^\ell$ (cf. Definition 4.8(5));
   (b) $b_\ell = \sum\{q_i y_{(\ell,i)} : i < j\}$, and so the $\bar{y}^\ell$ are pairwise distinct;
   (c) $(y_{(\ell,i)} : i < j)$ is injective and reasonable;
   (d) if $j > 1$ and $\ell_* > 1$, then there are at least two $y \in X$ such that:

$$|\{(\ell, i) : \ell < \ell_*,\ i < j \text{ and } y_{(\ell,i)} = y\}| = 1;$$

(e) if $j > 1$ and $\ell_* > 1$, then there are $\ell_1 \neq \ell_1 < \ell_*$ and $i_1, i_2 < j$ such that:
(i) if $\ell < \ell_*$, $i < j$ and $y_{(\ell,i)} = y_{(\ell_1,i_1)}$, then $(\ell, i) = (\ell_1, i_1)$;
(ii) if $\ell < \ell_*$, $i < j$ and $y_{(\ell,i)} = y_{(\ell_2,i_2)}$, then $(\ell, i) = (\ell_2, i_2)$.
(f) $(y_{(\ell,j-1)} : \ell < \ell_*)$ is without repetitions and none of $\{y_{(\ell,i)} : \ell < \ell_*, i < j-1\}$ appears in it (recall that $\bar{x}, \bar{y}^0, ..., \bar{y}^{\ell_*-1}$ are reasonable).

Proof. By the definition of $\leqslant_*$ there are $(y_{(\ell,i)} : i < j,\ \ell < \ell_*)$ satisfying clauses (a)-(c) of (B) as in the proof of Lemma 4.11(7). Recall that $(\{\bar{y} : \bar{x} \leqslant^j_X \bar{y}\}, \leqslant^j_X)$ is a tree. We now imitate the proof of Observation 3.5.

<u>Case 1</u>. $\{\bar{y}^\ell : \ell < \ell_*\}$ is not linearly ordered by $\leqslant^j_X$.

Then there are $\ell(1) \neq \ell(2) < \ell_*$ such that $\bar{y}^{\ell(1)}, \bar{y}^{\ell(2)}$ are locally $\leqslant^j_X$-maximal. So as in the analogous case in the proof of Obs. 3.5 we can choose $i_1, i_2 < j$ s.t.:

$$x^{\ell_1}_{i_1} \in X_{n[b_{\ell_1}]} \setminus X_{<n[b_{\ell_1}]} \text{ and } x^{\ell_2}_{i_2} \in X_{n[b_{\ell_2}]} \setminus X_{<n[b_{\ell_2}]},$$

notice that by the assumption that the sequences are reasonable we can choose $i_1 = j - 1 = i_2$, see Lemma 4.11(5). Hence, $y^{\ell(1)}_{i_1}, y^{\ell(2)}_{i_2}$ are as required.

<u>Case 2</u>. Not Case 1.

So w.l.o.g. we have that, for every $\ell < \ell_* - 1$, $\bar{y}^\ell <^j_X \bar{y}^{\ell+1}$. Now, for $\ell < \ell_*$ and $i < j$, let $n(\ell, i) < \omega$ be such that $y^\ell_i \in X_{n(\ell,i)} \setminus X_{<n(\ell,i)}$. Let:

$(\cdot_1)$ $i(1) < j$ be such that $i < j$ implies $n(0, i) \geqslant n(0, i(1))$;

$(\cdot_2)$ $i(2) < j$ be such that $i < j$ implies $n(\ell_* - 1, i) \leqslant n(\ell_* - 1, i(2))$.

Then $(0, i(1))$, $(\ell_* - 1, i(2))$ are as required. As, for $\ell < \ell_*$, $\bar{y}^\ell$ is reasonable we can actually choose $i(1), i(2)$ such that $i(1) = 0$ and $i(2) = j_* - 1$. ■

Definition 4.13. *Let $(p_a : a \in G^+_0)$ be a sequence of pairwise distinct primes.*

(1) For $a \in G^+_0$, let:

$$\mathbb{P}^{\leqslant_*}_a = \{p_b : b \in G^+_0, b \leqslant_* a\} \text{ and } \mathbb{P}^{\geqslant_*}_a = \{p_b : b \in G^+_0, a \leqslant_* b\}.$$

(2) Let $G_1 = G_1[\mathfrak{m}] = G_1[\mathfrak{m}(T)] = G_1[T]$ be the subgroup of G_2 generated by:

$\{m^{-1}a : a \in G^+_0,\ m \in \omega\backslash\{0\}$ *a product of primes from* $\mathbb{P}^{\leqslant^*}_a$, *poss. with repetitions*$\}$.

(3) For a prime p, let $G_{(1,p)} = \{a \in G_1 : a$ is divisible by p^m, for every $0 < m < \omega\}$ (notice that, by Observation 2.5, $G_{(1,p)}$ is always a pure subgroup of G_1).

(4) For $b \in G^+_1$, let $\mathbb{P}_b = \{p_a : a \in G^+_0, G_1 \models \bigwedge_{m<\omega} p^m_a \,|\, b\}$.

Remark 4.14. *(1) If $a, b \in G^+_1$ and $\mathbb{Q}a = \mathbb{Q}b \subseteq G_2$, then $\mathbb{P}_a = \mathbb{P}_b$.*
(2) If $b \in G^+_1$, then $\mathbb{P}_b$ is infinite.

Proof. Concerning (1), let $q^*_1 a = q^*_2 b$, where $q^*_1, q^*_2 \in \mathbb{Q}\backslash\{0\}$. W.l.o.g. $q^*_1, q^*_2 \in \mathbb{Z}\backslash\{0\}$ and so $q^*_1 a = q^*_2 b \in G_1$. Let now p be an arbitrary prime, and, for $\ell \in \{1, 2\}$, let $m(\ell) < \omega$ be such that that $q_\ell = p^{m(\ell)} q^*_\ell$, $p \nmid q^*_\ell$ and $(q^*_\ell, p) = 1$. By transitivity of equality, w.l.o.g. $a \in G^+_0$. Now, let $m \in \mathbb{Z}$ with $m > 0$. Then we have:

(a) In G_1, $p^m |\, a$ iff $p^m |\, q^*_1 a$.

[Why (a)? First assume that $G_1 \models p^m |\, a$, then there is $a_1 \in G_1$ such that $G_1 \models p^m a_1 = a$. Let $a_2 = q^*_1 a_1$, then $G_1 \models q^*_1 a = q^*_1(p^m a_1) = p^m(q^*_1 a_1) = p^m a_2$, so $G_1 \models p^m |\, q^*_1 a$. Assume now that $G_1 \models p^m |\, q^*_1 a$, and let $q^*_1 a = p^m a_3$, with $a_3 \in G_1$. By the choice of p we know that $(p, q^*_1) = 1$ and so also $(p^m, q^*_1) = 1$. It follows that 1 belongs to the ideal of $\mathbb{Z}$ that p^m and q^*_1 generates, hence:

$$\text{for some } m_1, m_2 \in \mathbb{Z}, \text{ we have } m_1 p^m + m_2 q^*_1 = 1.$$

But then:

$$\begin{aligned} a &= 1 \cdot a \\ &= (m_1 p^m + m_2 q_1^*) a \\ &= m_1 p^m a + m_2 q_1^* a \\ &= p^m (m_1 a) + (m_2 p^m a_3) \\ &= p^m (m_1 a) + p^m (m_2 a_3) \\ &= p^m (m_1 a + m_2 a_3), \end{aligned}$$

and so $p^m \,|\, a$, and thus we are done proving item (a).]

(b) In G_1, $p^m \,|\, q_1^* a$ iff $p^{m+m(1)} \,|\, p^{m(1)} q_1^* a$.

[Why (b)? Similar to (a).]

(c) In G_1, $p^{m+m(1)} \,|\, p^{m(1)} q_1^* a$ iff $p^{m+m(1)} \,|\, p^{m(2)} q_2^* b$.

[Why? Since by our assumptions, $p^{m(1)} q_1^* a = p^{m(2)} q_2^* b$.]

(d) In G_1, $p^{m+m(1)} \,|\, p^{m(2)} q_2^* b$ iff $p^{m+m(1)-m(2)} \,|\, q_2^* b$.

[Why? Like (b).]

(e) In G_1, $p^{m+m(1)-m(2)} \,|\, q_2^* b$ iff $p^{m+m(1)-m(2)} \,|\, b$.

[Why? Like (a).]

Thus, putting everything together we have:

(f) In G_1, $p^m \,|\, a$ iff $p^{m+m(1)-m(2)} \,|\, b$.

As for $n < \omega$ we have $p^{n+1} \,|\, c$ implies $p^n \,|\, c$, clearly:

$$\bigwedge_{n<\omega} p^m \,|\, a \Leftrightarrow \bigwedge_{n<\omega} p^m \,|\, b.$$

As p was an arbitrary prime, this concludes the proof of (1). Also, item (2) follows from (1) considering the distinct primes $p_b, p_{2b}, p_{3b}, \ldots$. ■

Lemma 4.15. *(1) If $p = p_a$, $a \in G_0^+$, then:*

$$G_{(1,p)} = \langle b \in G_0^+ : a \leqslant_* b \rangle_{G_1}^*.$$

(2) For $t \in T$, $H_{(1,t)} := H_{(2,t)} \cap G_1$ and $I_{(1,t)} := I_{(2,t)} \cap G_1$ are pure in G_1.

(3) For $\hat{f}_t^{(i,2)}$ as in Definition 4.10(3c), $\hat{f}_t^{(i,2)}[H_{(1,t)}] \subseteq I_{(1,t)}$. We define $\hat{f}_t^{(i,1)}$ as $\hat{f}_t^{(i,2)} \restriction H_{(1,t)}$.

(4) $\hat{f}_t^2 \restriction H_{(1,t)} = I_{(1,t)}$.

Proof. Item (1) is clear by Claim 4.13(1)(2). Concerning item (2), simply notice:

$$H_{(1,t)} = \langle \mathbb{Z}x : x \in \mathrm{dom}(f_t) \rangle_{G_1}^*,$$

$$I_{(2,t)} = \langle \mathbb{Z}x : x \in \mathrm{ran}(f_t) \rangle_{G_2}^*.$$

Item (3) is by item (2) and the following observation, if $f_t(x) = y$, then we have $x \leqslant_* y$ (recall Lemma 4.11(2)), and so $\mathbb{P}_x \subseteq \mathbb{P}_y$ (cf. Definition 4.13(1)). Finally, concerning item (4), assume that $0 < n < \omega$ and $t \in T_n \setminus T_{<n}$, then there is $x \in X_{n-1}$ such that $y = f_t(x) \in X_n \setminus X_{<n}$ (cf. Observation 4.6), notice that in particular $x <_* y$. So p_y is well-defined, since $y \in G_0^+$, and we have the following:

(a) $G_1 \models p_y \nmid x$, and so $H_{(1,t)} \models p_y \nmid x$ (as $H_{(1,t)}$ is pure in G_1, cf. item (2));
(b) $G_1 \models \bigwedge_{m<\omega} p_y^m \,|\, y$.

[Why (b)? By the definition of G_1 recalling that $x <_* y$.]
But then, since by item (3), $\hat{f}_t \restriction H_{(1,t)}$ is an embedding of $H_{(1,t)}$ into $I_{(1,t)}$ we have that $\hat{f}_t[H_{(1,t)}] \models p_y \not| f(x) \wedge f(x) = y$. On the other hand, since $I_{(1,t)}$ is pure in G_1 (cf. item (2) of the present lemma) we have that for every $m < \omega$, $p_y^{-m} y \in I_{(t,1)}$ (cf. Observation 2.4), and so we are done. Finally, the fact that $I_{(1,t)}/\hat{f}^1_t[H_{(1,t)}]$ is torsion is easy (and it is not used in the proof of Theorem 4.16). ■

Recall Definition 1.10 for the definition of Emb_ℓ-co-Hopfianity, for $\ell \in \{1, 2, 3\}$.

Theorem 4.16. *Let $\mathfrak{m}(T) \in \mathrm{K}_1^{\mathrm{co}}(T)$.*

1. *We can modify the construction so that $G_1[\mathfrak{m}(T)] = G_1[T]$ has domain ω and the function $T \mapsto G_1[T]$ is Borel (for T a tree with domain ω).*
2. *T has an infinite branch iff $G_1[T]$ is not Emb_1-co-Hopfian.*
3. *T has an infinite branch iff $G_1[T]$ is not Emb_2-co-Hopfian.*
4. *T has an infinite branch iff $G_1[T]$ is not Emb_3-co-Hopfian.*

Proof. Item (1) is easy. We prove items (2)-(4) with a single proof. Concerning the "left-to-right" direction of items (2)-(4), let $(t_n : n < \omega)$ be an infinite branch of T. By Lemma 4.11(4), $(\hat{f}_{t_n} : n < \omega)$ is increasing, by Definition 4.10(3c), $\hat{f}^2_{t_n}$ embeds $H_{(2,t_n)}$ into $I_{(2,t_n)}$, thus $\hat{f}^2 = \bigcup_{n<\omega} \hat{f}^2_{t_n}$ is an embedding of G_2 into G_2, since $G_2 = \bigcup_{n<\omega} H_{(2,t_n)}$, where $(H_{(2,t_n)} : n < \omega)$ is a chain of pure subgroups of G_2 with limit G_2, because, recalling 4.4(e), we have that:

$$H_{(2,t_n)} \supseteq \mathrm{dom}(f_{t_n}) \subseteq \mathrm{dom}(f_{t_{n+1}}) \subseteq H_{(2,t_{n+1})}$$

and by 4.4(c) we have that $\bigcup_{n<\omega} H_{(2,t_n)} = G_2$. Thus $\hat{f}^1 := \hat{f} \restriction G_1 = \bigcup_{n<\omega} \hat{f}^1_{t_n} = \bigcup_{n<\omega} \hat{f}^2_{t_n} \restriction H_{(1,t_n)}$ is an embedding of G_1 into G_1 (cf. Lemma 4.15(3)), in fact we have that $\mathrm{dom}(\hat{f}_{t_n}) = H_{(1,t_n)}$ (cf. Lemma 4.15(3)) and $G_1 = \bigcup_{n<\omega} H_{(1,t_n)}$, where $(H_{(1,t_n)} : n < \omega)$ is chain of pure subgroups of G_1 with limit G_1. Clearly $\hat{f}^1$ is not of the form $g \mapsto mg$ for some $m \in \mathbb{Z} \setminus \{0\}$, since for every $x \in \mathrm{dom}(f_t)$ we have $x \neq f_t(x)$ (cf. Obs. 4.6), this is enough for the "left-to-right" direction of item (4).

We claim that $G_1/\hat{f}^1[G_1]$ is not torsion. To this extent, first of all notice that $X_0 \neq \emptyset$ (by Definition 4.4(a)) and $X_0 \cap \mathrm{ran}(f_{t_n}) = \emptyset$ (by Definition 4.4(d)). Thus:

$$\mathrm{ran}(\hat{f}^1) \subseteq G^2_{X \setminus X_0} := \sum \{\mathbb{Q}x : x \in X \setminus X_0\} = \langle X \setminus X_0 \rangle^*_{G_2}.$$

Now, let $x \in X_0$, then $x \in G_1 \setminus \mathrm{ran}(\hat{f}^1)$, moreover, for $q \in \mathbb{Q} \setminus \{0\}$:

$$qx \notin G^2_{X \setminus X_0} \text{ and so } qx \notin \mathrm{ran}(\hat{f}^1),$$

and so in particular, for every $0 < n < \omega$ we have that $nx \notin \mathrm{ran}(\hat{f}^1)$, hence $n(x/(\mathrm{ran}(\hat{f}^1)) \neq 0$. This is enough for the "left-to-right" of items (2) and (3).

We now prove the "right-to-left" direction of item (2). To this extent, suppose that $(T, <_T)$ is well-founded and, for the sake of contradiction, suppose that there exists $f \in \mathrm{End}(G_1)$ one-to-one such that $G_1/f[G_1]$ is not torsion. Let $G^*_0 = G_1$ and $G^*_{n+1} = f(G^*_n)$ and notice that the sequence $(G^*_n : n < \omega)$ is strictly $\subseteq$-decreasing. Let now $c^*_0 \in G^*_0$ be such that $c^*_0/f[G^*_0]$ is not torsion in $G^*_0/f[G^*_0]$, and let then, for $0 < n < \omega$, $c^*_n = f^n(c^*_0)$, where $f^0 = f$ and $f^{n+1} = f^n \circ f$. Notice that then for every $n < \omega$, $c^*_n \in G^*_n$ and c^*_n/G^*_{n+1} is not torsion in G^*_n/G^*_{n+1}. Thus, for every $n < \omega$, $c^*_n \in G^*_n \subseteq G^*_0 = G_1 \subseteq G_2$, and so we have that:

$$(*_0) \qquad c^*_n = \sum\{q_{(n,x)}x : x \in w_n\},$$

where $w_n \subseteq X$ is finite and non-empty, and $q_{(n,x)} \in \mathbb{Q} \setminus \{0\}$. Notice that:

$(*_1)$ (i) for $n \neq m < \omega$ we have that $\mathbb{Q}c^*_n \neq \mathbb{Q}c^*_m$ (in G_2);
(ii) $\bigcup_{n<\omega} w_n$ is infinite.

[Why? Because $(c^*_n : n < \omega)$ is linearly independent in the $\mathbb{Q}$-vector space G_2. To see this, toward contradiction, suppose there is $\ell_* < \omega$ and $n(0) < \cdots < n(\ell_*)$ and $q_0, ..., q_{\ell_*} \in \mathbb{Q} \setminus \{0\}$ such that $G_2 \models \sum_{\ell \leqslant \ell_*} q_\ell c^*_{n(\ell)} = 0$. Then:

$$\sum_{0<\ell\leqslant\ell_*} q_\ell c^*_{n(\ell)} \in G^*_{n(0)+1},$$

and so $q_0 c^*_{n(0)} \in G^*_{n(0)+1}$, contradicting that $c^*_{n(0)}/G^*_{n(0)+1}$ is not torsion, as letting $q_0 = n_0/m_0$, with $n_0, m_0 \in \omega \setminus \{0\}$, $m_0 q_0 \in \mathbb{Z} \setminus \{0\}$ and $(m_0 q_0)c^*_{n(0)} \in G^*_{n(0)+1}$.]
Notice now that:

$(*_2)$ for every $n < \omega$, there is $h_n : w_{n+1} \to w_n$ such that $y \in w_{n+1} \Rightarrow h_n(y) \leqslant_X y$.

[Why? Fix $n < \omega$, then, by the definition of $G_{(1,p)}$ (Definition 4.13(3)) and the choice of $(f^m : m < \omega)$ and $(c^*_m : m < \omega)$, for every prime p, we have:

$$(*_{2.1}) \qquad c^*_n \in G_{(1,p)} \Rightarrow c^*_{n+1} \in G_{(1,p)}.$$

Let $m_n \in \omega \setminus \{0\}$ be such that $m_n c^*_n := c^+_n \in G_0$ and let $p' = p_{c^+_n}$, then $c^+_n \in G_{(1,p')}$, and so, since $G_{(1,p')}$ is pure in G_1 (cf. Observation 2.5), $m_n c^*_n = c^+_n$ and $m_n \in \mathbb{Z} \setminus \{0\}$, we have $c^*_n \in G_{(1,p')}$ (cf. Observation 2.4). Thus, by $(*_3)$ and Lemma 4.15(1), there is $k \in \mathbb{Z} \setminus \{0\}$ such that the following holds:

$$(*_{2.2}) \qquad kc^*_{n+1} \in \sum\{\mathbb{Z}b : c^*_n \leqslant_* b\}.$$

Hence, there are $j < \omega$, $c^*_n \leqslant_* b_0, ..., b_{j-1} \in G^+_0$, and $k, k_0, ..., k_{j-1} \in \mathbb{Z} \setminus \{0\}$ s.t.:

$$(*_{2.3}) \qquad G_1 \models kc^*_{n+1} = \sum_{i<j} k_i b_i.$$

Notice that by the definition of $<_*$ (cf. Def. 4.10(6)), for every $i < j$, we can find $\bar{f}_i = (\hat{f}_{t_{(i,\ell)}} : \ell < \ell(i))$ such that $\hat{f}_{t_{(i,\ell(i)-1)}} \circ \cdots \circ \hat{f}_{t_{(i,0)}} := f_i \in \mathrm{End}(G_1)$ satisfies:

$$(*_{2.4}) \qquad b_i = f_i(c^*_n) = \sum\{q_{(n,x)} f_i(x) : x \in w_n\}, \;\; q_{(n,x)} \in \mathbb{Q} \setminus \{0\},$$

and $x \leqslant_X f_i(x) := y_{(i,x)}$, for all $x \in w_n$ (cf. $(*_1)$). Thus, by $(*_{2.2})$, we have:

$$(*_{2.5}) \qquad c^*_{n+1} \in \sum\{\mathbb{Q}y_{(i,x)} : i < j, x \in w_n\}.$$

Hence, $w_{n+1} \subseteq \{y_{(i,x)} : i < j, x \in w_n\}$. Let $h_n : w_{n+1} \to w_n$ be such that if $y \in w_{n+1}$, then for some $i < j$, $y = y_{(i,x)}$ and $h_n(y) = x \leqslant_X y_{(i,x)} = y$.]

Let $\bar{y}_n = (y_{(n,\ell)} : \ell < \ell_n)$ be a reasonable sequence listing w_n. As in $(*_{2.2})$, for each $n < \omega$ there are $i_n < \omega$ such that for every $i < i_n$ and $\ell < \ell_n$ there are $q^1_{(n,i)}, q^0_{(n,\ell)} \in \mathbb{Q} \setminus \{0\}$ and $y^i_{(n,\ell)}$ such that the following holds:

$(*_{2.6})$ $c^*_{n+1} = \sum_{i<i_n} q^1_{(n,i)} \sum_{\ell<\ell_n} q^0_{(n,\ell)} y^i_{(n,\ell)}$, where $\bar{y}_n \leqslant^{\ell_n}_X \bar{y}^i_n = (y^i_{(n,\ell)} : \ell < \ell_n)$.

Now, firstly, we have:

$(*_{2.7})$ For no $n < \omega$ do we have $\bar{y}_n \in \{\bar{y}^i_n : i < i_n\}$.

[Why? As otherwise c^*_n and c^*_{n+1} would be linearly dependent, a contradiction.] Secondly, we have:

$(*_{2.8})$ For every $\ell < \ell_n$ we have:
- (a) $y_{(n,\ell)} \notin \mathrm{ran}(\bar{y}_{n-1})$, if $n > 0$;
- (b) there is $\ell' < \ell_{n+1}$ such that $h_n(y_{(n+1,\ell')}) = y_{(n,\ell)}$.

[Why? By $(*_{2.6})$ and $(*_{2.7})$ (or for more details see below between $(\star_{1.5})$ and $(\star_2)$.] So we can choose $\ell^2_n < \ell_n$ by induction on $n < \omega$ such that we have:

$$h_n(y_{(n+1,\ell^2_{n+1})}) = y_{(n,\ell^2_n)} \text{ and } y_{(n,\ell^2_n)} <_X y_{(n+1,\ell^2_{n+1})}.$$

This gives an infinite branch of $(X, <_X)$, contradicting the fact that $(X, <_X)$ is a well-founded tree (cf. Observation 4.9(2) recalling that T is well-founded). Thus, we have finished proving item (2) of the present theorem.

We now prove the "right-to-left" direction of items (3)-(4). To this extent, relying on the "right-to-left" direction of item (2), it suffices to show that if $f \in \mathrm{End}(G_1)$ is one-to-one and $G_1/f[G_1]$ is torsion, then:

(a) $G_1/f[G_1]$ is bounded;
(b) for some $m \in \mathbb{Z} \setminus \{0\}$ we have that $f(a) = ma$, for all $a \in G_1$.

Since $G_1/f[G_1]$ is torsion, for each $x \in X$, there is $m_x \in \mathbb{Z} \setminus \{0\}$ such that $m_x x \in \mathrm{ran}(f)$. Fix now $x \in X$. Then we can find $a \in G^+_1$ such that $f(a) = m_x x$, further, as we can replace the pair (a, m_1) by the pair (ma, mm_x) for any $m \in \mathbb{Z} \setminus \{0\}$, we can assume w.l.o.g. that $a \in G^+_0$. We claim that:

$(\star_1)$ $a \in \mathbb{Q}x$.

To this extent, let $p = p_a$. Then $a \in G_{(1,p)}$, and so $f(a) = m_x x \in G_{(1,p)}$. Thus, since $G_{(1,p)}$ is pure in G_1 (cf. Lemma 4.15(1)), we have that $x \in G_{(1,p)}$ (cf. Observation 2.4). But then, again by Lemma 4.15(1), we can find $n < \omega$ and $m_2, m_{(2,0)}, ..., m_{(2,n-1)} \in \mathbb{Z} \setminus \{0\}$, and $b_0, ..., b_{n-1} \in G^+_0$ such that:

$(\star_{1.5})$
- (i) $m_2 x = \sum\{m_{(2,\ell)} b_\ell : \ell < n\} \in G_1$;
- (ii) $a \leqslant_* b_\ell$, for every $\ell < n$;
- (iii) the b_ℓ's are pairwise distinct (w.l.o.g.).

Suppose now (as $a \in G^+_0$) that $a = \sum\{q^j y_j : j < j_*\}$ for some $j_* < \omega$, $q_0, ..., q_{j_*-1} \in \mathbb{Z} \setminus \{0\}$, and $y_j \in X$, with $(y_1, ..., y_{j-1})$ without repetitions. Clearly $j_* > 0$.

$(\star_{1.6})$ We claim that $j_* = 1$.

For the sake of contradiction suppose that $j_* > 1$. Now, as for every $\ell < n$, $a \leqslant_* b_\ell$, there are $(\hat{f}_{t_{(\ell,i)}} : i < i(\ell))$ such that $f_\ell = \hat{f}_{t_{(\ell,i(\ell)-1)}} \circ \cdots \circ \hat{f}_{t_{(\ell,0)}}$ and $f_\ell(a) = b_\ell$. Thus, by $(\star_{1.5})$(i), we have:

$$(\star_{1.7}) \qquad m_2 x = \sum_{\ell < n} m_{(2,\ell)} \sum_{j < j_*} q^j f_\ell(y_j).$$

If $n = 0$ we get a contradiction (i.e. $m_x x = 0$). If $n = 1$, then $m_2 x = \sum_{j<j_*} q^\ell f_0(y_j)$, but as we are assuming that $(y_j : j < j_*)$ is without repetitions, necessarily $j_* = 1$, as desired. So assume $n \geqslant 2$. Then, w.l.o.g. $(y_j : j < j_*)$ is reasonable (cf. Lemma 4.9(6)), and so using Claim 4.12 (with $(y_j : j < j_*)$ here as $(x_i : i < j)$ there) we immediately get a contradiction, since the support of the right hand side of $(\star_{1.7})$ has at least 2 members by Claim 4.12(B)(d)-(e), while the support of the left hand side of $(\star_{1.7})$ has exactly one member. Thus, $(\star_{1.6})$ holds, as wanted.

Let $\hat{f}$ be the extension of f to an embedding of G_2 into G_2 (which exists as f embeds G_1 into G_1 and $G_1 \subseteq G_2 = \langle G_1 \rangle^*_{G_2}$). Hence, by $(\star_1)$ we have:

$(\star_2)$ for every $x \in X$ there is $q_x \in \mathbb{Q}$ such that $G_2 \models \hat{f}(q_x x) = x$.

Furthermore, we have:

$(\star_3)$ the sequence $(q_x : x \in X)$ is constant, call it q_*.

[Why? Toward contradiction, suppose that $x_1 \neq x_2 \in X$ and $q_{x_1} \neq q_{x_2}$. Then we have $x_1 + x_2 \in G^+_0$ and $f(x_1 + x_2) = \frac{1}{q_{x_1}} x_1 + \frac{1}{q_{x_2}} x_2$. Let now $p = p_{x_1+x_2}$, then $x_1 + x_2 \in G_{(1,p)}$ and so $\frac{1}{q_{x_1}} x_1 + \frac{1}{q_{x_2}} x_2 \in G_{(1,p)}$, but then arguing as in the proof of Claim 3.15 and using Claim 4.12 we reach a contradiction.]

$(\star_4)$ $q_* \in \mathbb{Z} \setminus \{0\}$.

[Why? Clearly $q_* \neq 0$, and if $q_* \notin \mathbb{Z}$, then G_1 is divisible by some prime number p, but clearly this is false. Hence $q_* \in \mathbb{Z} \setminus \{0\}$, as wanted.] ∎

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