PARTITION THEOREMS FOR EXPANDED TREES 1176

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Abstract. We look for partition theorems for large subtrees for suitable uncountable trees and colourings. The intension is to apply it to model theoretic problems.

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Anotated Content

§ 0. Introduction

§(0A) Background and Results

We continue two lines of research. One is set theoretic: pure partition relations on trees and the other is model theoretic-Hanf numbers and non-deniability of well ordering, in particular related to ω_1 . This relate to the existence of GEM (generalized Eherenfuecht-Mostowski) for suitable templates (see [Sheb]), and applications to descriptive set theory.

Halpern-Levy [HL71] had proved a milestone theorem on independence of versions of the axiom of choice: in ZF, AC is strictly stronger than the maximal prime ideal theorem (i.e. every Boolean algebra has a maximal ideal).

This work isolated a partition theorem[1] on the tree ${}^{\omega>}2$, sufficient for the proof. This partition theorem was then proved by Halpern-Lauchli [HL66] and was a major and early theorem in Ramsey theory, (so the proof above rely on it).

See more Laver [Lav71], [Lav73] and [She90, AP,§2] and Milliken [Mil79], [Mil81].

The [HL66] proof uses induction, later Harrington found a different proof using forcing: adding many Cohens and a name of a (non-principal) ultrafilter on $\mathbb{N}$. Earlier, (on using adding many reals and a partition theorem) see Silver proof of π^1_1-equivalence relations, in [Sil80]

Now [She92, §4] turn to uncountable trees, i.e. for some $\kappa > \aleph_0$, we consider trees $\mathscr{T}$ which are are sub-trees of $({}^{\kappa>}2, \triangleleft)$, such that (as in [HL66]) for every level $\varepsilon < \kappa$, either $(\forall \eta \in \mathscr{T} \cap {}^{\varepsilon}2)(\eta\hat{}\langle 0\rangle, \eta\hat{}\langle 1\rangle \in \mathscr{T})$ or $(\forall \eta \in \mathscr{T} \cap {}^{\varepsilon}2)(\exists! \iota < 2)[\eta\hat{}\langle \iota\rangle \in \mathscr{T}]$. But a new point is that we have to use a well ordering of $\mathscr{T} \cap {}^{\varepsilon}2$ for $\varepsilon < \kappa$ (and of course the second occurs unboundedly often). Naturally we add "is closed enough (that is under unions of increasing sequences of length $< \kappa$)". Also colouring with infinite number of colours, the proof uses "measurable κ which remains so when we add λ many κ-Cohens for appropriate λ"; it generalizes Harrington's proof. This was continued in several works, see Dobrinen-Hathaway [DH17] and references there.

We are here mainly interested in a weaker version which is enough for the model theoretic applications we have in mind, see more in §(1B). In this case the embedding does not preserve the equality of levels, also we may start with a large tree and get one of smaller cardinality, in a sense this is solving ?/(Erdös-Rado theorem) = [She92]/ (the partition relation of a weakly compact cardinal)= [HL66] / (Ramsey theorem). See Dzamonja-Shelah [DS04] where such indiscernibility is considered in model theoretic context.

Turning to model theory, a central direction in model theory in the sixties were two cardinal models. For infinite cardinals $\mu > \lambda$ let $K_{\mu,\lambda}$ be the class of models M such that M is of cardinality μ and P^M of cardinality λ. The main problems were transfer, compactness and completeness. For connection to partition theorems, Morley-s proof of Vaught far apart two cardinal theorem used Erdös-Rado theorem; generally see [She71b], [She71a],[She78]. Jensen celebrated gap n two cardinal theorem solve those problems for e.g $(\aleph_n, \aleph_0)$ <u>when</u> $\mathbf{V} = \mathbf{L}$. But can we get a nice picture in different universes?

Note that by [She89], [Sheb], consistently we have GEM (generalized Eherenfuecht-Mostowski) models for ordered graphs as index models.

[1]Using not splitting to 2 but other finite splitting make a minor difference; similarly here

On a different direction Douglas Ulrich has asked me (and it has descriptive set theoretic consequences, see [SU19]), and we intend to prove (in the sequel) that for $n < \omega$:

$(*)_n$ consistently

(a) if $\psi \in \mathbb{L}_{\aleph_1,\aleph_0}$ has a model M of cardinality $\beth_{n+1}$ with $(P^M, <^M)$ having order type ω_1 <u>then</u> ψ has a model N of cardinality $\beth_{n+1}$ and $(P^N, <^N)$ is not well ordered

(b) moreover, it is enough that M will have cardinality $\aleph_\delta.\delta \geq \beth_n^{++}$

(c) of course, preferably not using large cardinals

This requires consistency of many cases of partition relations on trees and more complicated structures, analysing GEM models. Much earlier we have intended to prove the parallel for first order logic; and $(\beth_n, \aleph_0)$, using measurables $\kappa_1 < \ldots \kappa_n$ as in [She92, §4] and forcing by blowing $2^{\aleph_0}$ to κ_1, 2^{κ_1} to κ_2 etc relying on [She92]; but have not carry that. In preparation are solutions to the two cardinal problems above and

$(*)$ (a) $\alpha_\bullet < \omega_1, \beth_{\alpha+1} = (\beth_\alpha)^{+\omega_1+1}$ for $\alpha < \alpha_\bullet$ and well ordering of ω_1 is not definable in $\{\mathrm{EC}_\psi(\beth_{\alpha_\bullet}) : \psi \in \mathbb{L}_{\aleph_1,\aleph_0}\}$ or at least

(b) as above but for $\beth_{\alpha+1} = \aleph_{\beth_\alpha^{++}}$.

(c) parallel results replacing $\aleph_0$ by μ

Contrary to the a priory expectation no large cardinal is used.

In a sequel we intend also to deal wih intermediate partition relationrelated to weakly compact cardinals. We thank Shimoni Garti for many helpful comments.

§ 0(A). Preliminaries.

This is a warm up for the main frame.

Definition 0.1. If $\mu = \mu^{<\kappa}$ then "for a (μ,κ)-club of $u \subseteq X$ we have $\varphi(u)$" means that: for some χ such that $\mu, u \in \mathscr{H}(\chi)$ and e.g. $\beth_3(\mu + |u|) < \chi$ and some $x \in \mathscr{H}(\chi)$ if $x \in \mathscr{B} \prec (\mathscr{H}(\chi), \in), \|\mathscr{B}\| = \mu, [\mathscr{B}]^{<\kappa} \subseteq \mathscr{B}$ and $\mu + 1 \subseteq \mathscr{B}$, <u>then</u> the set $u = \mathscr{B} \cap X$ satisfies $\varphi(u)$; there are other variants.

Definition 0.2. For κ regular (usually $\kappa = \kappa^{<\kappa}$) and ordinal γ the forcing $\mathbb{P} = \mathrm{Cohen}(\kappa,\gamma)$ of adding γ many many κ-Cohen is defined as follows:

(A) $p \in \mathbb{P}$ <u>iff</u>:

(a) p is a function with domain from $[\gamma]^{<\kappa}$

(b) if $\alpha \in \mathrm{dom}(p)$ then $p(\alpha) \in {}^{\kappa>}2$

(B) $\mathbb{P} \models p \leq q$ <u>iff</u>:

(a) $p, q \in \mathbb{P}$

(b) $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$

(c) if $\alpha \in \mathrm{dom}(p)$ then $p(\alpha) \trianglelefteq q(\alpha)$

(C) for $\alpha < \gamma$ let $\underset{\sim}{\eta}_\alpha = \cup\{p(\alpha) : p \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}}$ satisfies $\alpha \in \mathrm{dom}(p)\}$, so $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\eta}_\alpha \in {}^\kappa 2$"

(D) for $u \subseteq \lambda$ let $\mathbb{P}_u = \{p \in \mathbb{P} : \mathrm{dom}(p) \subseteq u\}$ so $\mathbb{P}_u \lessdot \mathbb{P}$ and $\bar{\eta}_u = \langle \underset{\sim}{\eta}_\alpha : \alpha \in u\rangle$ is generic for $\mathbb{P}_u$.

§ 1. Partition theorems

§ 1(A). The First Frame for Partition Theorems.

Definition 1.1. 1) Let $\mathbf{M}_1$ be the class of objects $\mathbf{m}$ consisting of (so $\kappa = \kappa_{\mathbf{m}}$, etc.):

(a) κ, a regular cardinal
(b) $\lambda \geq \kappa$
(c) $\mathbb{Q}$ a quasi-order, $(< \kappa)$-complete
(d) val, a function from $\mathbb{Q}$ into $[\lambda]^\lambda$
(e) (monotonicity) if $p \leq_{\mathbb{Q}} q$ then $\mathrm{val}(p) \supseteq \mathrm{val}(q)$
(f) (decidability) if $\mathrm{val}(p) = A \subseteq \lambda$ and $A \subseteq A_0 \cup A_1$, <u>then</u> for some $\ell \in \{0, 1\}$ and $q \in \mathbb{Q}$ above p we have $\mathrm{val}(q) \subseteq A_\ell$.

2) Let $\mathbf{M}_2$ be the class of $\mathbf{m} \in \mathbf{M}_1$ such that:

(g) (non-atomicity) if $p \in \mathbb{Q}$ <u>then</u> for some $q_1, q_2 \in \mathbb{Q}$ above p, the sets $\mathrm{val}(q_1), \mathrm{val}(q_2)$ are disjoint.

Claim 1.2. *Assume* $\mathbf{m} \in \mathbf{M}_1$.
1) If there is no $(< \kappa)$-complete uniform ultrafilter on λ <u>then</u> $\mathbf{m} \in \mathbf{M}_2$.
2) The cofinality of $\lambda_{\mathbf{m}}$ is $\geq \kappa$.
3) In Definition 1.1(1) we can weaken clause (f) demanding $A_0, A_1 \in [\lambda]^\lambda$ and $A = A_0 \cup A_1$.

Proof. 1) Toward contradiction assume that $\mathbf{m} \notin \mathbf{M}_2$, that is, clause (g) of 1.1(2) fails, let p_* witness this, so:

$(*)_1$ if $q_1, q_2 \in \mathbb{Q}$ are above p_*, <u>then</u> $\mathrm{val}(q_1) \cap \mathrm{val}(q_2) \neq \emptyset$.

Define

$(*)_2$ let $D = D_{p_*} = \{A \subseteq \lambda$: for some $q \in \mathbb{Q}$ above p_* we have $\mathrm{val}(q) \subseteq A\}$.

We shall prove that D is a κ-complete uniform ultrafilter on λ, thus arriving at the promised contradiction.

$(*)_3$ $D \subseteq [\lambda]^\lambda$ and $\lambda \in D$ and D is upward closed.

[Why? Check the definition of D in $(*)_2$ and 1.1(1)(d)].

$(*)_4$ D is a $(< \kappa)$-complete filter on λ.

[Why? Let $\zeta < \kappa$ and $A_\varepsilon \in D$ for $\varepsilon < \zeta$ and we shall prove that $\bigcap\limits_{\varepsilon<\zeta} A_\varepsilon \in D$. As $A_\varepsilon \in D$ we can choose $q_\varepsilon \in \mathbb{Q}$ above p_* such that $\mathrm{val}(q_\varepsilon) \subseteq A_\varepsilon$. Now choose p_ε by induction on $\varepsilon \leq \zeta$ such that:

$(*)_{4.1}$
(a) $p_\varepsilon \in \mathbb{Q}$ is above p_*
(b) if $\varepsilon(1) < \varepsilon$ then $p_{\varepsilon(1)} \leq_{\mathbb{Q}} p_\varepsilon$
(c) if $\varepsilon = \varepsilon(1) + 1$ then $\mathrm{val}(p_\varepsilon) \subseteq A_{\varepsilon(1)}$.
(d) if $\xi < \varepsilon$ then $\mathrm{val}(p_\varepsilon) \subseteq \mathrm{val}(p_{\xi+1}) \subseteq A_\xi$; this actually follows.

If we succeed, then $\mathrm{val}(p_\zeta) \subseteq \bigcap_{\varepsilon<\zeta} \mathrm{val}(p_{\varepsilon+1}) \subseteq \bigcap_{\varepsilon<\zeta} A_\varepsilon$ hence p_ζ witnesses $\bigcap_{\varepsilon<\zeta} A_\varepsilon \in D$ and we are done, that is, finish proving $(*)_4$

Why can we carry the induction? For $\varepsilon = 0$ let $p_\varepsilon = p_*$, for limit ε recall $\mathbb{Q}$ is $(< \kappa)$-complete and $\zeta < \kappa$ is assumed.

Lastly, for ε a successor ordinal, let $\varepsilon = \varepsilon(1) + 1$, now by clause (f) of 1.1(1) applied to the pair $(A_{\varepsilon(1)}, \lambda \backslash A_{\varepsilon(1)})$ there is $p_\varepsilon \in \mathbb{Q}$ above $p_{\varepsilon(1)}$ such that $\mathrm{val}(p_\varepsilon) \subseteq A_{\varepsilon(1)}$ or $\mathrm{val}(p_\varepsilon) \subseteq \lambda \backslash A_{\varepsilon(1)}$. But if $\mathrm{val}(p_\varepsilon) \subseteq \lambda \backslash A_{\varepsilon(1)}$ then, recalling q_ε was chosen before $(*)_{4.1}$, the pair $p_\varepsilon, q_\varepsilon$ contradict the choice of p_*.

Hence we can carry the induction in $(*)_{4.1}$. Together we are done proving $(*)_4$.]

$(*)_5$ D contains all co-bounded subsets of λ

[Why? Let $B \in [\lambda]^{<\lambda}$ and choose $A_0 = B, A_1 = \lambda \setminus B$ so by clause (f) of Definition 1.1(1), there is $p \in \mathbb{Q}$ above p_* such that $\mathrm{val}(p) \in [\lambda]^\lambda$ and $\mathrm{val}(p)$ is a subset of A_0 or of A_1. But in the former case $\mathrm{val}(p)$ is a subset of B hence has cardinality $\leq |B| < \lambda$ contradiction to clause (d) of 1.1(1), so $\mathrm{val}(p) \subseteq A_1 = \lambda \setminus B$ and it belongs to D, so we are done proving $(*)_5$].

$(*)_6$ if $A \subseteq \lambda$ then $A \in D$ or $\lambda \backslash A \in D$.

[Why? By clause (f) of Definition 1.1(1).]

By $(*)_2, (*)_3, (*)_4, (*)_5, (*)_6$ clearly D is a uniform κ-complete ultrafilter on λ, contradicting an assumption, so we are done proving 1.2(1).

2) Toward contradiction assume $\theta = \mathrm{cf}(\lambda) < \kappa$. Choose $\langle \alpha_i : i < \theta \rangle$ an increasing sequence of ordinals $< \lambda$ with limit λ. Now choose p_i by induction on $i \leq \theta$ such that:

- (*) $p_i \in \mathbb{Q}$
- (*) $j < i \Rightarrow p_j \leq_{\mathbb{Q}} p_i$
- (*) if $i = j + 1$ then $\mathrm{val}(p_i) \cap \alpha_i = \emptyset$.

For $i = 0$ choose any $p_i \in \mathbb{Q}$ and for i a limit ordinal let p_i be any upper bound of $\langle p_j : j < i \rangle$ recalling $\mathbb{Q}$ is $(< \kappa)$-complete. Lastly for $i = j + 1$ use the proof of $(*)_5$ above. Now p_θ is well defined but $\mathrm{val}(p_\theta) \cap \alpha_i = \emptyset$ for every $i < \theta$ hence $\mathrm{val}(P_\theta) = \emptyset$, a contradiction.

3) Easy.

$\square_{1.2}$

Claim 1.3. *If (A) then (B) <u>where</u>:*

(A) (a) $\mathbf{m} \in \mathbf{M}_1$ *and let* $(\lambda, \kappa) = (\lambda_{\mathbf{m}}, \kappa_{\mathbf{m}})$

(b) $\theta < \kappa$

(c) $\mathbf{c} : [\lambda]^2 \to \theta$

(d) $\mathscr{T}$ *satisfies:*

(α) $\mathscr{T} \subseteq {}^{\kappa>}2$ *is a sub-tree so is closed under initial segments*

(β) $\mathscr{T}$ *is of cardinality* κ, *and* $\alpha < \kappa \Rightarrow |\mathscr{T} \cap {}^{\alpha}2| < \kappa$

(γ) <u>notation</u>: $\mathrm{suc}(\mathscr{T}) = \{\eta \in \mathscr{T} : \ell g(\eta)$ *is a successor ordinal*$\}$

(δ) $\eta \in \mathscr{T} \Rightarrow \eta\hat{\ }\langle 0 \rangle, \eta\hat{\ }\langle 1 \rangle \in \mathscr{T}$

(ε) *for every* $\eta \in \mathscr{T}$ *and* $\zeta \in (\ell g(\eta), \kappa)$ *there is* $\nu \in \mathscr{T} \cap {}^{\zeta}2$ *such that* $\eta \triangleleft \nu$ *(needed for* $(*)_1(b)(c)$ *in the proof; it follows that: for every limit* $\delta < \kappa, \mathscr{T} \cap {}^{\delta}2$ *is infinite and even of cardinality* $\geq |\delta|$*)*

(B) *there are* u *and* g *such that:*
  (a) $u \subseteq \lambda$ *has order-type* κ
  (b) $\{\mathbf{c}\{\alpha, \beta\} : \alpha \neq \beta \in u\}$ *has at most two members*
  (c) g *is a one-to-one function from* u *onto* $\mathscr{T}$
  (d) *if* $\alpha \leq \beta$ *are from* u, *<u>then</u>* $\ell g(g(\alpha)) \leq \ell g(g(\beta))$
  (e) *for* $\alpha < \beta$ *from* $u, \mathbf{c}\{\alpha, \beta\}$ *depends only on:*
    (α) *the truth value of* $g(\alpha) \triangleleft g(\beta)$
    (β) *if the answer is no and we let* $\rho = g(\alpha) \cap g(\beta) \notin \{g(\alpha)\}$ *<u>then</u>* $\mathbf{c}\{\alpha, \beta\}$ *depends also on* $(g(\beta))(\ell g(\rho))$.

Remark 1.4. 0) We can deduce clause (e) of 1.3(B) by making $\mathbf{c}\{\alpha, \beta\}$ code such information when we start with a one-to-one function $g^* : \lambda \to {}^{\lambda>}\kappa$ or $g^* : \lambda \to {}^{\kappa}2$, see 1.6 below. There we try to comment on how to adapt the present proof for proving 1.6.
1) Note that if κ is strongly inaccessible <u>then</u> $\mathscr{T} = {}^{\kappa>}2$ is O.K. and even ${}^{\kappa>}\alpha$ for any $\alpha < \kappa$.
2) Is it worthwhile to allow $|\mathscr{T} \cap {}^{\alpha}2| \leq \kappa$ for $\alpha < \kappa$? It seems we shall not have a real gain.

Proof. Here $\mathbb{Q} = \mathbb{Q}_{\mathbf{m}}$. As a warm-up:

$(*)_1$ for $p \in \mathbb{Q}$ let:
  (a) $\mathrm{col}_{\mathbf{c}}(p) = \{\mathbf{c}\{\alpha, \beta\} : \alpha \neq \beta \in \mathrm{val}_{\mathbf{m}}(p)\}$
  (b) $p \in \mathbb{Q}$ is $\mathbf{c}$-minimal <u>when</u>: for every $q \in \mathbb{Q}$ above p we have $\mathrm{col}_{\mathbf{c}}(q) = \mathrm{col}_{\mathbf{c}}(p)$.

Now

$(*)_2$ (a) if $p \leq_{\mathbb{Q}} q$, <u>then</u> $\mathrm{col}_{\mathbf{c}}(p) \supseteq \mathrm{col}_{\mathbf{c}}(q) \neq \emptyset$
  (b) a dense open set of $p \in \mathbb{Q}$ is $\mathbf{c}$-minimal.

[Why? Clause (a) holds as $\leq_{\mathbb{Q}}$ is monotonic. Clause (b) holds as $|\mathrm{Rang}(\mathbf{c})| \leq \theta < \kappa$ and $\mathbb{Q}$ is $(< \kappa)$-complete.]

But we shall really use:

$(*)_3$ For $p_1, p_2 \in \mathbb{Q}$ let:
  (a) $\mathrm{col}_{\mathbf{c}}(p_1, p_2) = \{\xi < \theta$: for some $\alpha \in \mathrm{val}_{\mathbf{m}}(p_1)$ the set $\{\beta \in \mathrm{val}_{\mathbf{m}}(p_2) : \mathbf{c}\{\alpha, \beta\} = \xi\}$ includes $\mathrm{val}_{\mathbf{m}}(q_1)$ for some q_1 above $p_2\}$
  (b) the pair $(p_1, p_2) \in \mathbb{Q} \times \mathbb{Q}$ is $\mathbf{c}$-minimal <u>when</u>: for every q_1, q_2 above p_1, p_2 respectively we have $\mathrm{col}_{\mathbf{c}}(q_1, q_2) = \mathrm{col}_{\mathbf{c}}(p_1, p_2)$ and $\mathrm{col}_{\mathbf{c}}(q_2, q_1) = \mathrm{col}_{\mathbf{c}}(p_2, p_1)$
  (c) let $h\mathrm{col}_{\mathbf{c}}(p_1, p_2)$ be $\cap\{\mathrm{col}_{\mathbf{c}}(q_1, q_2) : q_1, q_2 \in \mathbb{Q}$ are above p_1, p_2 respectively$\}$.

So as above

$(*)_4$ (a) if $p_1 \leq_{\mathbb{Q}} q_1$ and $p_2 \leq_{\mathbb{Q}} q_2$ <u>then</u> $\mathrm{col}_{\mathbf{c}}(p_1, p_2) \supseteq \mathrm{col}_{\mathbf{c}}(q_1, q_2)$

(b) if $p_1, p_2 \in \mathbb{Q}$ <u>then</u> for some $q_1, q_2 \in \mathbb{Q}$ we have $p_1 \le_{\mathbb{Q}} q_1, p_2 \le_{\mathbb{Q}} q_2$ and (q_1, q_2) is $\mathbf{c}$-minimal

(c) (p_1, p_2) is $\mathbf{c}$-minimal <u>iff</u> $\mathrm{col}(p_1, p_2) = h\mathrm{col}(p_1, p_2)$ and $\mathrm{col}(p_2, p_1) = h\mathrm{col}(p_2, p_1)$

(d) if $p_1, p_2 \in \mathbb{Q}$ then $\mathrm{col}_{\mathbf{c}}(p_1, p_2) \neq \emptyset$.

Now

$(*)_5$ we say $p \in \mathbb{Q}$ is (ξ_1, ξ_2)-minimal <u>when</u> $\xi_1, \xi_2 < \theta$ and for every $q \in \mathbb{Q}$ above p there are q_1, q_2 above q such that $\xi_1 \in h\mathrm{col}_{\mathbf{c}}(q_1, q_2)$ and $\xi_2 \in h\mathrm{col}_{\mathbf{c}}(q_2, q_1)$

$(*)_6$ the set $\{p \in \mathbb{Q} : p$ is (ξ_1, ξ_2)-minimal for some $\xi_1, \xi_2 < \theta\}$ is a dense (and open) subset of $\mathbb{Q}$.

[Why? As above by $\mathbb{Q}$ being $(< \kappa)$-complete and $\theta < \kappa$.]

$(*)_7$ (a) fix $\xi_1, \xi_2 < \theta$ and $p_* \in \mathbb{Q}$ such that p_* is (ξ_1, ξ_2)-minimal

(b) fix $\bar\eta$ such that:

(α) $\bar\eta = \langle \eta_i : i < \kappa \rangle$ lists the elements of $\mathscr{T}$

(β) $\ell g(\eta_i) < \ell g(\eta_j) \Rightarrow i < j$

(γ) $\eta_i = \rho\hat{\ }\langle 0 \rangle \Rightarrow \eta_{i+1} = \rho\hat{\ }\langle 1 \rangle$ and then $(\exists \zeta)(i = 1 + 2\zeta)$

(δ) for $i < j < \kappa$ we have $\eta_i = \eta_j$ <u>iff</u> $(\exists \zeta)(i = 1 + 2\zeta \wedge j = i + 1)$ and $\ell g(\eta_i)$ is a limit ordinal

(ε) if $\zeta < \kappa$ <u>then</u> $\{i : \ell g(\eta_i) = \zeta\}$ is an interval, (follows by clause (δ) and if ζ is a limit ordinal then it is $[\zeta_1, \zeta_2)$ for some limit ordinals ζ_1, ζ_2.

[Why such (ξ_1, ξ_2) as in clause (a) exists? By $(*)_6$. Why $\bar\eta$ as in clause (b) exists? Note that by clause (A)(d) of the claim's assumption, for every limit ordinal $\varepsilon < \kappa, \mathscr{T} \cap {}^{\varepsilon}2$ has cardinality $\ge |\varepsilon|$.]

Toward our inductive construction:

$(*)_{7.1}$ For $\zeta < \kappa$ let:

(a) $I^{\mathrm{is}}_\zeta = \{\eta_\varepsilon : \varepsilon < 1 + (2\zeta)\}$, where "is" stands for "initial segment"

(b) $I^{\mathrm{fr}}_\zeta = \{\eta \in \mathscr{T} : \eta \in I^{\mathrm{is}}_\zeta \wedge [\eta\hat{\ }\langle 0 \rangle \notin I^{\mathrm{is}}_\zeta]$ or $\ell g(\eta)$ is a limit ordinal and $\eta \notin I^{\mathrm{is}}_\zeta \wedge (\forall i < \ell g(\eta))(\eta{\restriction}i \in I^{\mathrm{is}}_\zeta)\}$; where fr stands for "front"

(c) hence I^{fr}_ζ is a maximal $\triangleleft$-antichain of $\mathscr{T}$

(d) let $I^{\mathrm{ac}}_\zeta = \{\eta_\varepsilon : \varepsilon < 1 + 2\zeta$ and η_ε is of the form $\rho\hat{\ }\langle 0 \rangle\}$

(e) $I^{\mathrm{fs}}_\zeta = I^{\mathrm{is}}_\zeta \cap I^{\mathrm{fr}}_\zeta$.

Clearly

$(*)_{7.2}$ (a) if $\varepsilon, \zeta < \kappa$ then $\eta_\varepsilon\hat{\ }\langle 0 \rangle \in I^{\mathrm{is}}_\zeta \Leftrightarrow \eta_\varepsilon\hat{\ }\langle 1 \rangle \in I^{\mathrm{is}}_\zeta$

(b) $I = \bigcup\limits_{\zeta < \kappa} I^{\mathrm{is}}_\zeta$ and $I^{\mathrm{is}}_\varepsilon$ is $\subseteq$-increasing continuous with ε and if $[\zeta_1, \zeta_2) = \{\varepsilon < \kappa : \ell g(\eta_\varepsilon) = \delta\}, \delta < \kappa$ a limit ordinal then $\langle I^{\mathrm{fr}}_\zeta : 1 + 2\zeta \in [\zeta_1, \zeta_2) \rangle$ is constantly $\mathscr{T} \cap {}^{\delta}2$

(c) if $\varepsilon < \zeta$ and $\eta \in I^{\mathrm{fr}}_\zeta$ then $(\exists! \nu \in I^{\mathrm{fr}}_\varepsilon)[\nu \trianglelefteq \eta]$.

Now we choose $(\bar{p}_\zeta, \bar{\alpha}_\zeta)$ by induction on $\zeta < \kappa$ such that:

$(*)_{7.3}$ (a) $\bar{p}_\zeta = \langle p_{\zeta,\eta} : \eta \in I^{\rm is}_\zeta \rangle$
(b) $p_{\zeta,\eta} \in \mathbb{Q}$
(c) $p_* \le_{\mathbb{Q}} p_{\zeta,\eta}$ recalling $(*)_7(a)$
(d) if $\varepsilon < \zeta$ and $\nu \in I^{\rm is}_\varepsilon, \nu \trianglelefteq \eta \in I^{\rm is}_\zeta$ <u>then</u>
  (α) $p_{\varepsilon,\nu} \le_{\mathbb{Q}} p_{\zeta,\eta}$
  (β) if $\nu \notin I^{\rm fr}_\varepsilon$ then $p_{\varepsilon,\nu} = p_{\zeta,\eta}$
(e) $\bar{\alpha}_\zeta = \langle \alpha_\eta = \alpha(\eta) : \eta \in I^{\rm ac}_\zeta \rangle$ where $\alpha_\eta < \lambda$ so $\varepsilon < \zeta \Rightarrow \bar{\alpha}_\varepsilon = \bar{\alpha}_\zeta {\restriction} I^{\rm ac}_\varepsilon$ and if $\eta \triangleleft \nu$ are from $I^{\rm is}_\zeta$ and $\ell g(\eta) \le i < \ell g(\nu) \Rightarrow \nu(i) = 1$, <u>then</u> $\alpha_\eta = \alpha_\nu$
(f) if $\varepsilon_1 < \varepsilon_2 < 1 + 2\zeta$ and $\eta_{\varepsilon_1}, \eta_{\varepsilon_2} \in I^{\rm ac}_\zeta$, <u>then</u>
  $\bullet_1$ $\mathbf{c}\{\alpha(\eta_{\varepsilon_1}), \alpha(\eta_{\varepsilon_2})\} \in \{\xi_1, \xi_2\}$
  $\bullet_2$ in fact, $\mathbf{c}\{\alpha(\eta_{\varepsilon_1}), \alpha(\eta_{\varepsilon_2})\} = \xi_1$ <u>iff</u> letting $\rho = \eta_{\varepsilon_1} \cap \eta_{\varepsilon_2}$ we have $\rho\hat{\ }\langle 0 \rangle \trianglelefteq \eta_{\varepsilon_1}$
(g) if $\rho\hat{\ }\langle 0 \rangle, \rho\hat{\ }\langle 1 \rangle \in I^{\rm fr}_\zeta$, <u>then</u> $\xi_1 \in {\rm hcol}(p_{\rho\hat{\ }\langle 0 \rangle}, p_{\rho\hat{\ }\langle 1 \rangle})$ and $\xi_2 \in {\rm hcol}(p_{\rho\hat{\ }\langle 1 \rangle}, p_{\rho\hat{\ }\langle 0 \rangle})$.

Before carrying the induction:

$(*)_{7.4}$ it suffice to carry the induction

[Why? Let

- $u = \{\alpha_{\nu\hat{\ }\langle 0 \rangle} : \nu \in \mathscr{T}\}$
- $g(\alpha) = \nu$ iff $\alpha_{\nu\hat{\ }\langle 0 \rangle} = \alpha$

Now clearly u is well defined as $\nu \in \mathscr{T} \Rightarrow \nu\hat{\ }\langle 0 \rangle \in \mathscr{T} \Rightarrow (\alpha_{\nu\hat{\ }\langle 0 \rangle}$ is well defined). Also g is a function from u onto $\mathscr{T}$, it is onto again because for every $\nu \in \mathscr{T}$ also $\nu\hat{\ }\langle 0 \rangle \in \mathscr{T}$. Lastly, if $\eta_\varepsilon \ne \eta_\zeta \in \mathscr{T}, \varepsilon < \zeta$ then $\mathbf{c}\{\alpha_{\eta_\varepsilon\hat{\ }\langle 0 \rangle}, \alpha_{\eta_\zeta\hat{\ }\langle 0 \rangle}\}$ belongs to ξ_1, ξ_2 by $(*)_{7.3}(f)\bullet_2$.]

Now we turn to carrying the induction in $(*)_{7.3}$.

<u>Case 1</u>: $\zeta = 0$ Let $p_{\langle\rangle} = p_*$, easy to check that we have nothing more to choose and all relevant clauses hold

<u>Case 2</u>: $\zeta = \varepsilon + 1, \eta_{1+2\varepsilon} = \rho\hat{\ }\langle 0 \rangle$

So $\rho \in I^{\rm fr}_\varepsilon$ and let $\langle \nu_{\varepsilon,i} : i < i_\varepsilon = i(\varepsilon) \rangle$ list with no repetitions the set $I^{\rm fs}_\varepsilon \backslash \{\rho\}$. Now we choose $q_{\varepsilon,i}$ by induction on $i \le i_\varepsilon$ and if $i = j + 1$ also $r_{\varepsilon,j}$ (so $r_{\varepsilon,j}$ will be well defined for every $j < i_\varepsilon$) such that:

(a) $q_{\varepsilon,i} \in \mathbb{Q}$ is above $q_{\varepsilon,j}$ for $j < i$
(b) if $i = 0$ then $q_{\varepsilon,i} = p_{\varepsilon,\rho}$
(c) if $i = j + 1$ then
  (α) $p_{\varepsilon,\nu_{\varepsilon,j}} \le r_{\varepsilon,j}$ and
  (β) if $\alpha \in {\rm val}(q_{\varepsilon,i})$ <u>then</u> for some $q \ge r_{\varepsilon,j}$ we have: $\beta \in {\rm val}(q) \Rightarrow \mathbf{c}\{\alpha, \beta\} \in \{\xi_1, \xi_2\}$; moreover, the one which is as promised in $(*)_{7.3}(f)\bullet_3$

There is no problem to carry the induction. Then choose q', q'' above $q_{\varepsilon,i(\varepsilon)}$ such that $\xi_1 \in {\rm hcol}(q', q''), \xi_2 \in {\rm hcol}(q'', q')$ and $\alpha \in {\rm val}(q') \Rightarrow (\exists q)(q'' \le q \wedge (\forall \beta \in {\rm val}(q))(\mathbf{c}\{\alpha, \beta\} = \xi_2)$.

Lastly, we choose our objects for ζ:

(∗) (a) let $\alpha_{\rho\hat{}\langle 0\rangle}$ be any member of $\mathrm{val}(q')$, this is the only new case of an $\alpha_\rho, \rho \in I^{\mathrm{ac}}_\zeta$

(b) let $p_{\zeta,\nu_{\varepsilon,i}}$ be above $r_{\varepsilon,i}$ such that $\beta \in \mathrm{val}(p_{\zeta,\nu_{\varepsilon,i}}) \Rightarrow \mathbf{c}\{\alpha_{\rho\hat{}\langle 0\rangle}, \beta\} \in \{\xi_1, \xi_2\}$; moreover, one which is as promised

(c) let $(p_{\zeta,\rho\hat{}\langle 0\rangle}, p_{\zeta,\rho\hat{}\langle 1\rangle})$ be a pair of members of $\mathbb{Q}$ above q'' such that $\xi_1 \in h\mathrm{col}(p_{\zeta,\rho\hat{}\langle 0\rangle}, p_{\zeta,\rho\hat{}\langle 1\rangle})$ and $\xi_2 \in h\mathrm{col}(p_{\zeta,\rho\hat{}\langle 1\rangle}, p_{\zeta,\rho\hat{}\langle 0\rangle})$.

Case 3: $\zeta = \varepsilon + 1$ and $\ell g(\eta_{1+2\varepsilon})$ is a limit ordinal.

As $\eta_{1+2\varepsilon} = \eta_{1+2\varepsilon+1}$, we deal only with $\eta_{1+2\varepsilon}$. Let us choose $p'_{\eta_{1+2\varepsilon}} \in \mathbb{Q}_{\mathbf{m}}$ above $p_{\xi,\eta_{1+2\varepsilon}\restriction i}$ for every $i < \ell g(\eta_{1+2\varepsilon}), \xi \leq \varepsilon$ such that $\eta_{1+2\varepsilon}\restriction i \in I^{\mathrm{is}}_\xi$.

$\square_{1.3}$

Discussion 1.5. We shall later turn to "k-place colourings" and "end extension k-uniformity" as in [She92, §4].

Claim 1.6. *In 1.3 if we add (A)(e),(f) to (A) then we can add (B)(f) to (B) where:*

(A) (e) *g' is a one-to-one function from λ into $\lim_\kappa(\mathscr{T})$ so necessarily $\lambda \leq 2^\kappa$*

(f) *$\bar{<} = \langle <_\varepsilon : \varepsilon < \kappa\rangle$ where $<_\varepsilon$ is a well ordering of $\mathscr{T} \cap {}^\varepsilon 2$*

(B) (f) *there is an increasing function h from κ to κ such that: assuming $\alpha \neq \beta \in u$ and $g(\alpha) = \eta, g'(\beta) = \nu$ we have:*

(∗) *if $\rho, \eta, \nu \in \mathscr{T}$ and $(\rho\hat{}\langle 0\rangle \trianglelefteq \eta)$ and $(\rho = \nu) \vee \rho\hat{}\langle 1\rangle \trianglelefteq \nu$ then:*
- *if $\ell g(\eta) < \ell g(\nu)$ then $\mathbf{c}\{\alpha, \beta\} = \xi_2$*
- *if $\ell g(\eta) > \ell g(\nu)$ then $\mathbf{c}\{\alpha, \beta\} = \xi_1$*
- *if $\ell g(\eta) = \ell g(\nu) \leq \varepsilon$ and $\eta <_\varepsilon \nu$ then $\mathbf{c}\{\alpha, \beta\} = \xi_2$ and*
- *if $\ell g(\eta) = \ell g(\nu) = \varepsilon$ and $\nu <_\varepsilon \eta$ then $\mathbf{c}\{\alpha, \beta\} = \xi_1$.*

Proof. Similarly to 1.3. $\square_{1.6}$

§ 1(B). Expanded Trees and Second Frame for partition Theorem.

Discussion 1.7. We may like to replace ${}^{\kappa>}2$ by ${}^{\kappa>}I$, and even use creature tree forcing.; but (in second thought maybe see the paper with Zapletal [SZ11]), so for $\kappa > \aleph_0$ in each node we have a forcing notion which is quite complete but of cardinality $< \kappa =$ set of levels. So we do not have a tree but a sequence of creatures, $\langle \mathfrak{c}_\varepsilon : \varepsilon < \mathrm{ht}(\mathscr{T})\rangle$ such that for a colouring we like to find $\mathfrak{d}_\varepsilon \in \Sigma(\mathfrak{c}_\varepsilon)$ for $\varepsilon < \kappa$ which induce a sub-tree in which the colouring is 1-end-homogeneous. Alternatively we have $\langle \mathfrak{c}_\eta : \eta \in \mathscr{T}\rangle$ where $\mathfrak{c}_\eta$ is a creature with set of possible values being $\mathrm{suc}_{\mathscr{T}}(\eta)$.

So sometimes we need κ to be weakly compact

Clearly the answer is that we can, but it is not clear how interesting it is.

Here we consider partition on trees. Now in Halpern-Lauchli [HL66] (and [She92] and [DH17]) the embedding of the trees preserves level (which is a plus), for uncountable trees we find the need to consider a well ordering of each level, still preserving equality of level. But for the model theoretic applications we have in mind it is enough to consider embeddings where levels are not preserved, see Dzamonja-Shelah [DS04], (in the web version). Also we may waive completeness of the tree but usually still like to have many branches.

Here we denote by $\rightarrow_5 = \rightarrow_{HL}$ the older notion but we concentrate on the weaker one $\rightarrow = \rightarrow_1$, where the level is not preserved in the embedding. We intend to deal with an intermidiate one (for weakly compact cardinals) in a sequel.

Convention 1.8. the main case her eis $\iota = 1$, so we may omit it.

Definition 1.9. 1) Let $\mathbf{T}$ be the class of structures $\mathscr{T}$ (letting $\mathbf{T} = \mathbf{T}_{\rm a}$) such that:

(a) $\mathscr{T} = (u, <_*, E, <, \cap, S, R_0, R_1) = (u_{\mathscr{T}}, <^*_{\mathscr{T}}, E_{\mathscr{T}}, <_{\mathscr{T}}, \cap_{\mathscr{T}}, S_{\mathscr{T}}, R^0_{\mathscr{T}}, R^1_{\mathscr{T}})$ but we may write $s \in \mathscr{T}$ instead of $s \in u$

(b) $(u, <_*)$ is a well ordering, linear, u non-empty

(c) $<_{\mathscr{T}}$ is a partial order included in $<_*$

(d) $(u, <_{\mathscr{T}})$ is a tree, i.e. if $t \in \mathscr{T}$ then $\{s : s <_{\mathscr{T}} t\}$ is linearly ordered by $<_{\mathscr{T}}$; the tree is with $\mathrm{ht}(\mathscr{T})$ levels

(e) E is an equivalence relation on u, convex under $<_*$

(f) (α) each E-equivalence class is the set of $t \in \mathscr{T}$ of level ε for some ε
 (β) we denote the ε-th equivalence class by $\mathscr{T}_{[\varepsilon]}$
 (γ) E has no last E-equivalence class if not said otherwise
 (δ) let $\mathrm{lev}_{\mathscr{T}}(s) = \mathrm{lev}(s, \mathscr{T})$ be ε when $s \in \mathscr{T}_{[\varepsilon]}$, equivalently $\{t : t <_{\mathscr{T}} s\}$ has order type ε under the order $\leq_{\mathscr{T}}$
 (ε) let $\mathrm{ht}(\mathscr{T})$ be $\cup\{\mathrm{lev}(s) + 1 : s \in \mathscr{T}\}$
 (ζ) if $s \in u, \mathrm{lev}_{\mathscr{T}}(s) < \zeta < \mathrm{ht}(\mathscr{T})$ then there is $t \in \mathscr{T}_{[\zeta]}$ which is $<_{\mathscr{T}}$-above s

(g) for each $s \in u$ there is $t \in (\mathscr{T})$ above (i.e. $\leq_{\mathscr{T}}$-above) s

(h) each $s \in \mathscr{T}$ has exactly two immediate successors by $<_{\mathscr{T}}$

(i) for $s \in \mathscr{T}$
 $\bullet_1$ $\mathscr{T}_{\geq s} = \{t \in \mathscr{T} : s \leq_{\mathscr{T}} t\}$
 $\bullet_2$ $\mathrm{suc}_{\mathscr{T}}(s) = \{t : t \in \mathscr{T}_{[\mathrm{lev}(s)+1]}$ satisfies $s <_{\mathscr{T}} t\}$

(j) let $s = t{\restriction}\varepsilon$ means that $\mathrm{lev}_{\mathscr{T}}(s) = \varepsilon \leq \mathrm{lev}_{\mathscr{T}}(t)$

(k) for $t_1, t_2 \in \mathscr{T}, t_1 \cap_{\mathscr{T}} t_2$ is the maximal common lower bound of t_1, t_2 so we demand it always exists, i.e. $(\mathscr{T}, <)$ is normal

(l) for $\ell = 0, 1$ we have $R_\ell \subseteq \{(s,t) : s \in (\mathscr{T})$ and $s <_{\mathscr{T}} t\}$ and if $s \in (\mathscr{T})$ then for some $t_0 \neq t_1$ we have $\mathrm{suc}_{\mathscr{T}}(s) = \{t_0, t_1\}$ and $\ell < 2 \Rightarrow (\forall t)(sR_\ell t$ iff $t_\ell \leq_{\mathscr{T}} t)$; so $sR_\ell t$ is the parallel to $\eta\hat{\ }\langle \ell \rangle \trianglelefteq \nu$; we may think of it as a division to the left side and the right side of the set of t-s above s.

2) Let $\mathbf{T}_{\theta,\kappa} = \{\mathscr{T} \in \mathbf{T}$: the tree $\mathscr{T}$ has δ levels, for some ordinal δ of cofinality κ and for every $\varepsilon < \delta$ we have $\theta > |\{s \in \mathscr{T} : s$ of level $\leq \varepsilon\}|\}$.
3) For $\iota = 1$ let $\mathscr{T}_1 \subseteq_\iota \mathscr{T}_2$ mean:

(a) $\mathscr{T}_1, \mathscr{T}_2 \in \mathbf{T}_\frown$

(b) $<_{\mathscr{T}_1} = <_{\mathscr{T}_2} {\restriction} u_{\mathscr{T}_1}$

(c) $\bullet$ if $\mathscr{T}_1 \models$ "$\eta \cap \nu = \rho$" then $\mathscr{T}_2 \models$ "$\eta \cap \nu = \rho$"

- $R_{\mathscr{T}_1,\ell} = R_{\mathscr{T}_2,\ell} \restriction u_{\mathscr{T}_1}$ for $\ell = 0, 1$

(d) $(\mathscr{T}_1) \subseteq (\mathscr{T}_2) \cap u_{\mathscr{T}_1}$
(e) $<^*_{\mathscr{T}_1} = <^*_{\mathscr{T}_2} \restriction u_{\mathscr{T}_1}$.
(f) if $\iota = 5$ then $E_{\mathscr{T}_1} = E_{\mathscr{T}_2} \restriction u_{\mathscr{T}_1}$ and $(\mathscr{T}_1) = (\mathscr{T}_2) \cap u_{\mathscr{T}_1}$

4) For $s \in \mathscr{T}$ and $\ell \in \{0, 1\}$, let $\mathrm{suc}_{\mathscr{T},\ell}(s)$ be the unique immediate successor of s in $\mathscr{T}$ such that $(s, t) \in R^{\mathscr{T}}_\ell$.
5)We say $\mathscr{T}_1, \mathscr{T}_2 \in \mathbf{T}$ are neighbors <u>when</u> they are equal except that on each equivalence class we can change the order $<_*$.

Definition 1.10. 1) We say f is a $\subseteq_\iota$-embedding of $\mathscr{T}_1 \in \mathbf{T}$ into $\mathscr{T}_2 \in \mathbf{T}$ <u>when</u>: f is an isomorphism from $\mathscr{T}_1$ onto $\mathscr{T}'_1$ where $\mathscr{T}'_1 \subseteq_\iota \mathscr{T}_2$
2) For any ordinal α and sequence $\bar{<} = \langle <_\beta : \beta < \alpha \rangle, <_\beta$ a well ordering of ${}^\beta 2$ we define $\mathscr{T} = \mathscr{T}_{1.5,\alpha,\bar{<}}$ as follows (omitting $\bar{<}$ means "for some"):

(a) universe ${}^{\alpha>}2$
(b) $<_{\mathscr{T}}$ is $\triangleleft \restriction {}^{\alpha>}2$
(c) $E_{\mathscr{T}} = \{(\eta, \nu) : \eta, \nu \in {}^\beta 2$ for some $\beta < \alpha\}$
(d) $<^*_{\mathscr{T}} = \{(\eta, \nu) : \eta, \nu \in {}^{\alpha>}2$ and $\ell g(\eta) < \ell g(\nu)$ or $(\exists \beta < \alpha)(\ell g(\eta) = \beta = \ell g(\nu) \wedge \eta <_\beta \nu)\}$
(e) $S_{\mathscr{T}} = {}^{\alpha>}2$.

Claim 1.11. *If $\theta = \sup\{(2^{|\alpha|})^+ : \alpha < \kappa\}$ and $\bar{<} = \langle <_\beta : \beta < \kappa \rangle$ as above, <u>then</u> $\mathscr{T}^\bullet_{,\kappa,\bar{<}}$ is well defined and belongs to $\mathbf{T}_{\theta,\kappa}$;*

Proof. Is clear. $\square_{1.11}$

Definition 1.12. 1) For $\mathscr{T} \in \mathbf{T}$ let $\mathrm{eseq}^\iota_n(\mathscr{T})$ be the set of sequences $\bar{a}$ such that:
(a) $\bar{a}$ is an $<^*_{\mathscr{T}}$-increasing sequence of length n of members of $(\mathscr{T})$
(b) if $\iota = 1, 5$ then $k < \ell < n \Rightarrow a_k \cap a_\ell \in \{a_m : m < n\}$
(c) if $\iota = 5$ then $k, \ell < n \wedge \mathrm{lev}(a_k) \le \mathrm{lev}(a_\ell) \Rightarrow a_\ell \restriction \mathrm{lev}(a_k) \in \{a_m : m < n\}$

1A) For $\mathscr{U} \subseteq \mathscr{T}$ we let $\mathrm{eseq}^\iota_n(\mathscr{U}, \mathscr{T})$ be $\mathrm{eseq}^\iota_n(\mathscr{T}) \cap ({}^n\mathscr{U})$, similarly for part (2).
2) Let $\mathrm{eseq}_\iota(\mathscr{T}) = \mathrm{eseq}^\iota_{<\omega} = \cup\{\mathrm{eseq}^\iota_n(\mathscr{T}) : n < \omega\}$.
2A) for finite $A \subseteq \mathscr{T}$ we define the sequence $\bar{b} = \mathrm{cl}_\iota(A) = \mathrm{cl}^\iota_{\mathscr{T}}(A) = \mathrm{cl}_\iota(A, \mathscr{T})$ as the unique $\bar{b}$ and we also define $\mathrm{pos}(A)$ such that:
(a) $\bar{b} \in \mathrm{eseq}_\iota(\mathscr{T})$
(b) $A \subseteq \mathrm{Rang}(\bar{b})$
(c) $\mathrm{Range}(\bar{b})$ is minimal under those restrictions
(d) $\mathrm{pos}_\iota(A) = \mathrm{pos}_\iota(A, \mathscr{T})$ is the unique function h from A into $\lg(\bar{b})$ such that $b_i = h(a)$ <u>iff</u> $b_i = a$

2B) We may replace above A by a finite sequence $\bar{a}$, and let $\mathrm{pos}^\iota_{\mathscr{T}}(\bar{a})$ be the natural function f such that $a_i = b_{f(\mathrm{pos}(,\bar{a},\mathscr{T})(i))}$
2C) For a function F from $\mathrm{eseq}^\iota_{<\omega}(\mathscr{T})$ to $\mathrm{ht}(\mathscr{T})$ let $\mathrm{eseq}^\iota_{<\omega}(\mathscr{T}, F)$ be the set of $\bar{a} \in \mathrm{eseq}^\iota_{<\omega}(\mathscr{T})$ such that $k < \lg(\bar{a}) \Rightarrow F(\bar{a} \restriction k) < \mathrm{lev}_{\mathscr{T}}(a_k)$
3) We say $\bar{a}, \bar{b} \in \mathrm{eseq}_\iota(\mathscr{T})$ are $\mathscr{T} - \iota$-similar or $\bar{a} \sim_{\mathscr{T},\iota} \bar{b}$ <u>when</u> for some n we have:

(a) $\bar{a}, \bar{b} \in \mathrm{eseq}^\iota_n(\mathscr{T})$
(b) for any $k, \ell, m < n$ we have:
  - $\bullet_1$ $a_k \le_{\mathscr{T}} a_\ell$ <u>iff</u> $b_k \le_{\mathscr{T}} b_\ell$

- $\bullet_2$ if $\iota = 1, 1.5, 5$ then $(a_k, a_i) \in R^{\mathscr{T}}_\ell$ <u>iff</u> $(b_k, b_i) \in R^{\mathscr{T}}_\ell$ for $\ell = 0, 1$
- $\bullet_3$ assuming $\iota = 0, \ell = 0, 1$ for $k, m, i < n$ we have: $(a_k \cap a_m) R_{\mathscr{T},\ell} a_i$ iff $(b_k \cap b_m) R_{\mathscr{T},\ell} b_i$
- $\bullet_4$ $\mathrm{lev}_{\mathscr{T}}(a_k) \le \mathrm{lev}_{\mathscr{T}}(a_\ell)$ <u>iff</u> $\mathrm{lev}_{\mathscr{T}}(b_k) \le_{\mathscr{T}} \mathrm{lev}_{\mathscr{T}}(b_\ell)$
- $\bullet_5$ $a_k \cap_{\mathscr{T}} a_\ell = a_m$ <u>iff</u> $b_k \cap_{\mathscr{T}} b_\ell = b_m$ when $\iota = \iota = 1, 1.5, 5$
- $\bullet_6$ $a_\ell \in S_{\mathscr{T}}$ <u>iff</u> $b_\ell \in S_{\mathscr{T}}$ when $\iota = \iota = 1, 1.5, 5$
- $\bullet_7$ if $\iota = 5$ and $k < \ell < n$ we have $a_k = a_\ell \restriction \mathrm{lev}(a_k)$ iff $b_k = b_\ell \restriction \mathrm{lev}(b_k)$
- $\bullet_8$ note that $a_k <^*_{\mathscr{T}} a_\ell \Leftrightarrow b_k <^*_{\mathscr{T}} b_\ell$ follows by part (1).

3A) We say that $\bar{a}, \bar{b} \subseteq \mathscr{T}$ are $\mathscr{T} - \iota -$similar when $\mathrm{cl}_\iota(\bar{a}), \mathrm{cl}_\iota(\bar{b})$ are and $\mathrm{pos}_\iota(\bar{a}, \mathscr{T}) = \mathrm{pos}_\iota(\bar{b}, \mathscr{Y})$; so $\lg(\bar{a}) = \lg(\bar{b})$
4) For $\bar{a} \in {}^n\mathscr{T}$ let $\mathrm{Lev}(\bar{a})$ be the set $\{\mathrm{lev}_{\mathscr{T}}(a_\ell) : \ell < \lg(\bar{a})\}$
5) We say that $\mathscr{T} \in \mathbf{T}_\iota$ is weakly $\aleph_0$-saturated <u>when</u>:

- (*) for every $\varepsilon < \zeta < \mathrm{ht}(\mathscr{T})$ and $s_0, \dots, s_{n-1}$ from $\mathscr{T}_{[\varepsilon]}$ there are $t_0 <^*_{\mathscr{T}} \cdots <^*_{\mathscr{T}} t_{n-1}$ from $\mathscr{T}_{[\zeta]}$ satisfying $k < n \Rightarrow s_k <_{\mathscr{T}} t_k$

6) For $\mathscr{T} \in \mathbf{T}_\iota$ let:

- (a) $\mathrm{incr}_n(\mathscr{T})$ be the set on $<^*_{\mathscr{T}}$-increasing $\bar{a} \in {}^n\mathscr{T}$ and let $\mathrm{seq}(\mathscr{T}) = {}^n\mathscr{T}$ that is the set sequences of length n from $\mathscr{T}$
- (b) $\mathrm{incr}(\mathscr{T}) = \cup\{\mathrm{incr}_n(\mathscr{T} : n < \omega)\}$ and $\mathrm{seq}(A) = \cup\{\mathrm{seq}_n(A) : n < \omega\}$

7) For $\mathscr{T} \in \mathbf{T}_\iota$:

- (a) for $\bar{t} \in \mathrm{incr}(\mathscr{T})$ or just $\bar{t} \in {}^n\mathscr{T}$ let $\mathrm{sim} - \mathrm{tp}_\iota(\bar{t}, \mathscr{T})$ be the similarity type of $\mathrm{cl}_\iota(\bar{a}, \mathscr{T})$ and $\mathrm{pos}(\bar{a}, \mathscr{T})$, that is all the information from part (3) (of 1.12) (including pos).
- (b) if in addition, $\mathscr{U} \subseteq \mathscr{T}$ then we let $\mathrm{sim} - \mathrm{tp}_\iota(\bar{t}, \mathscr{U}, \mathscr{Y}))$ be the function mapping $\bar{s} \in {}^{\omega >}\mathscr{U}$ to $\mathrm{sim} - \mathrm{tp}_\iota(\bar{t}\hat{\ }\bar{s}, \mathscr{T})$.

8) Let $\mathbb{S}^n_\iota$ be the set of ι-similarity types of sequences of length n in some $\mathscr{T} \in \mathbf{T}^n_\iota$, so the sequences are not necessarily increasing.
9) Naturally $\mathbb{S}_\iota = \cup\{\mathbb{S}^n_\iota : n < \omega\}$ (and of course omitting ι means $\iota = 1$)¿

Now comes the main property.

Definition 1.13. 1) $\mathscr{T}_1, \mathscr{T}_2 \in \mathbf{T}_\iota$ and $n < \omega$ and a cardinal σ let $\mathscr{T}_1 \to_\iota (\mathscr{T}_2)^n_\sigma$ mean:

- ($*$) if $\mathbf{c} : \mathrm{eseq}^\iota_n(\mathscr{T}_1) \to \sigma$, <u>then</u> there is a ι-embedding g of $\mathscr{T}_2$ into $\mathscr{T}_1$ such that the colouring $\mathbf{c} \circ g$ is homogeneous for $\mathscr{T}_2$ which means:
  - if $\bar{a}, \bar{b} \in \mathrm{eseq}^\iota_n(\mathscr{T}_2)$ are $\mathscr{T}_2 - \iota$-similar then $\mathbf{c}(g(\bar{a})) = \mathbf{c}(g(\bar{b}))$.

2) For $\mathscr{T}_1, \mathscr{T}_2 \in \mathbf{T}$ and n and σ let $\mathscr{T}_1 \to_\iota (T_2)^{\mathrm{end}(k)}_\sigma$ mean that:

- ($*$) $\mathscr{T}_1, \mathscr{T}_2 \in \mathbf{T}$ and: if $\mathbf{c} : \mathrm{eseq}_\iota(\mathscr{T}_1) \to \sigma$ <u>then</u> there is an $\subseteq_\iota$-embedding g of $\mathscr{T}_2$ into $\mathscr{T}_1$ such that the colouring $\mathbf{c}' = \mathbf{c} \circ g$ satisfies $\mathbf{c}'(\bar{\eta})$ does not depend on the last k levels, that is,
  - if $\bar{a}, \bar{b} \in \mathrm{eseq}^\iota_n(\mathscr{T}_2)$ are $\mathscr{T}_2 - \iota$-similar and $\ell < n \wedge (k \le |\mathrm{Lev}(\bar{a}) \setminus \mathrm{lev}(a_\ell)| \Rightarrow b_\ell = a_\ell$ then $\mathbf{c}'(\bar{a}) = \mathbf{c}(\bar{b})$

3) We define $\mathscr{T}_1 \to^+_\iota (\mathscr{T}_2)^n_\sigma$ as in part (1) but $\bar{a}, \bar{b} \in \mathrm{seq}_n(\mathscr{T}_2)$
4) Let $\to'_\iota$ be defined similarly but "for $\bar{a}, \bar{b} \in \mathrm{eseq}^\iota_x(\mathscr{T}_2)$" is replaced by "for some function F from $\mathrm{eseq}^\iota_{<\omega}(\mathscr{T}_2)$ into $\mathrm{ht}(\mathscr{T})$ for every $\bar{a}, \bar{b} \in \mathrm{eseq}^\iota_x(\mathscr{T}_2, F)$"; of interest particularly when $\mathrm{ht}(\mathscr{T}_2) = \mathrm{ht}(\mathscr{T}_2)$ is not inaccessible.

We may mention some implications among the $\to_\iota$,

Claim 1.14. *Let* $\mathscr{T} \in \mathbf{T}$.
1) If $A \subseteq \mathscr{T}$ *is finite non-empty with* m *elements* <u>*then*</u>*:*

- (a) *if* $\iota = 1$ *and* $A \in [\mathscr{T}]^{\leq m}$ *then for some* $n \leq (2m-1)$ *and* $\bar{a} \in \mathrm{ceseq}_n(\mathscr{T})$ *we have* $A \subseteq \mathrm{Rang}(\bar{a})$*; moreover if* $\eta \in \mathrm{Rang}(\bar{a}), \eta$ *is* $<_{\mathscr{T}}$*-maximal in* $\mathrm{Rang}(\bar{a})$ *then* $\eta \in A$*. Also if* A *is a set of pairwise* $<_{\mathscr{T}}$*-incomparables then we get iff.*
- (b) *if* $\iota = 5$ *then for some* $n \leq (2m-1)m$ *and* $\bar{a} \in \mathrm{eseq}^\iota_n(\mathscr{T})$ *we have* $A \subseteq \mathrm{Rang}(\bar{a})$*; moreover* $\max\{\mathrm{lev}_{\mathscr{T}}(a) : a \in A\} = \max\{\mathrm{lev}_{\mathscr{T}}(a_\ell) : \ell < n\}$
- (c) *for* $\mathscr{T} \in \mathbf{T}_\iota$ *and finite* $A \subseteq \mathscr{T}$ *then* $\mathrm{cl}_\iota(A, \mathscr{T}), \mathrm{pos}_\iota(A, \mathscr{T})$ *is well defined*

2) The number of quantifier free complete n*-types realized by some* $\mathscr{T} \in \mathbf{T}_\iota$ *and* $\bar{a} \in \mathrm{eseq}^\iota_n(\mathscr{T})$ *is, e.g.* $\leq 2^{2n^2+n}$ *but* $\geq n$*.*
3) Assume $\iota = 1$ *If* $\mathscr{T} \in \mathbf{T}$ *is weakly* $\aleph_0$*-saturated* <u>*then*</u> $\mathscr{T}$ *realizes all possible such types, i.e. one realized in some* $\mathscr{T}' \in \mathbf{T}$
4) Assume $\iota = 1$ *and* $\mathscr{T} \in \mathbf{T}$ *and* $\mathscr{U}$ *is a subset of* $\mathscr{T}$ *closed under* $<_{\mathscr{T}}$*, (that is* $s <_{\mathscr{T}} t \in \mathscr{U} \Rightarrow s \in \mathscr{U}$*).*

Let $\mathrm{lev}(\mathscr{U}, \mathscr{T}) = \sup\{\mathrm{lev}(s, \mathscr{T}) + 1 : s \in \mathscr{U}\}$*.*

If $\mathrm{lev}(\mathscr{U}, \mathscr{T}) \leq \mathrm{lev}(t, \mathscr{T})$ *and* $A \subseteq \mathscr{U}$ *is finite* <u>*then*</u> $\bar{b} = \mathrm{cl}_\iota(A \cup \{t\})$ *has the form* $\bar{c}\hat{\ }\langle t\rangle$ *with* $\bar{c} \in \mathrm{eseq}_\iota(\mathscr{T}) \cap {}^{\omega>}\mathscr{U}$*; we shall use this freely.*

Proof. 1) <u>Clause (a)</u>: Let $B_1 = \{\eta \cap_{\mathscr{T}} \nu : \eta, \nu \in A\}$ and note that $\eta \in A \Rightarrow \eta = \eta \cap \eta \in B_1$. Now by induction on $|A|$ easily $|B_1| \leq 2m - 1$.

Also the two additional statements are easy too.
<u>Clause (b)</u> Let $w = \{\mathrm{lev}(_{\mathscr{T}}(a)\} : a \in B_1\}$; and let $\zeta = \max(w)$. Now we shall choose $\bar{b}$ such that:

- (a) $\bar{b} = \langle b_a : a \in A\rangle$
- (b) $b_a \in \mathscr{T}_{[\zeta]}$
- (c) $a \leq_{\mathscr{T}} b_a$
- (d) if $a_1 \leq_{\mathscr{T}} a_2$ then $b_{a_1} = b_{a_2}$

Clearly possible. Lastly let
$B_2 = \{b_a{\restriction}\varepsilon : a \in A, \varepsilon \in w\}$.

Now clearly $|w| \leq |B_1|$ and $|B_2| \leq |w| \times |A| \leq |B_1| \times |A| \leq (2m-1)m$ as promised. (We may improve the boud but this does not matter here; similarly below)

Also clause (c) is clear.
2) Considering the class of such pairs $(\bar{a}, \mathscr{T})$ (fixing n) the number of $E_{\bar{a}} = \{(k, \ell) : a_k E_{\mathscr{T}} a_\ell\}$ is $\leq 2^{n^2}$ and the number of $<_{\bar{a}} = \{(k, \ell) : a_k <_{\mathscr{T}} a_\ell\}$ is $\leq 2^{n^2}$ and the number of $\{(a_k, a_\ell) : (a_k, a_\ell) \in R^{\mathscr{T}}_1$ and for no $i, a_k <_{\mathscr{T}} a_i <_{\mathscr{T}} a_\ell\}$ is $\leq 2^n$.

Lastly, from those we can compute $\{(a_k, a_\ell) : (a_k, a_\ell) \in R^{\mathscr{T}}_0\}$ as $\{(a_k, a_\ell, a_m) : a_k \cap_{\mathscr{T}} a_\ell = a_m\}$, so together the number is $\leq 2^{2n^2+n}$.

Clearly we can get a better bound, e.g. letting $m^\bullet_n(\mathscr{T}) = |\{\mathrm{tp}_{\mathrm{qf}}(\bar{a}{\restriction}n, \emptyset, \mathscr{T}) : \bar{a} \in \mathrm{eseq}(\mathscr{T})$ has length $\geq n\}|$ then:

- $m^\bullet_n(\mathscr{T}) = 1$ for $n = 0, 1$
- $m^\bullet_{n+1}(\mathscr{T}) \leq 2n(m^\bullet_n(\mathscr{T}))$
- hence $m^\bullet_n(\mathscr{T}) \leq 2^{n-1}(n-1)!$

3), 4), 5) Clear. $\square_{1.14}$

Next we define $\mathbf{N}_\iota$ parallel to $\mathbf{M}_1, \mathbf{M}_2$ in §(1A) using $\mathscr{T}$ from Definition 1.9, that is:

Definition 1.15. For $\iota = 1$ let $\mathbf{N} = \mathbf{N}_\iota$ be the class of objects $\mathbf{m}$ consisting of (so $\kappa = \kappa_{\mathbf{m}}$, etc.):

- (a) κ, a regular cardinal
- (b) $(*)_\alpha$ $\mathscr{T} \in \mathbf{T}$
  $(*)_\beta$ $\mathbb{B}$ a Boolean algebra of subsets of $\mathrm{dom}(\mathbb{B})$ which is $(\mathscr{T})$; (see 1.9(1)(f)(η)),
- (c) $\mathbb{Q}$ is a quasi-order, $(< \kappa)$-complete
- (d) (α) val, a function from $\mathbb{Q}$ into $\mathbb{B}$
  (β) if $p \in \mathbb{Q}$ then $\{\mathrm{lev}_{\mathscr{T}}(\eta) : \eta \in \mathrm{val}(p)\}$ is unbounded in $\mathrm{ht}(\mathscr{T})$
  (γ) if $\varepsilon < \mathrm{ht}(\mathscr{T})$ and $p \in \mathbb{Q}$ <u>then</u> for some s, t, p_0, p_1 we have: $s \in \mathrm{val}_{\mathscr{T}}(p), \mathrm{lev}_{\mathscr{T}}(s) \geq \varepsilon, s \leq_{\mathscr{T}} t \in (\mathscr{T})$ and for $\ell = 0, 1$ we have $p \leq_{\mathbb{Q}} p_\ell$ and $(\forall r \in \mathrm{val}(p_\ell))(tR_{\mathscr{T},\ell} r)$
  (δ) above, if $\iota > 0$ then we add $s = t$
- (e) (monotonicity) if $p \leq q$ then $\mathrm{val}(p) \supseteq \mathrm{val}(q)$
- (f) (decidability) if $\mathrm{val}(p) = A \subseteq \lambda$ and $A = A_0 \cup A_1$ and $A_0, A_1 \in \mathbb{B}$ <u>then</u> for some $\ell \in \{0, 1\}$ and $q \in \mathbb{Q}$ above p we have $\mathrm{val}(q) \subseteq A_\ell$.
- (g) (compatibility) if $t \in \mathscr{T}$ and $p \in \mathbb{Q}$ then for some pair (q, s) we have
  (α) $p \leq q$ in $\mathbb{Q}$
  (β) $s \leq t$ in $\mathscr{T}$
  (γ) if $r \in \mathrm{val}(q)$ then $t \cap r = s$
- (h) (extension existence) if $\iota = 5$ and $\gamma < \kappa$ and $p_\alpha \in \mathbb{Q}$ for $\alpha < \gamma$ <u>then</u> $\bigcap\limits_{\alpha<\gamma} \{\ell g(\eta) : \eta \in \mathrm{val}(p_\alpha)\}$ is unbounded in $\mathrm{ht}(\mathscr{T})$

Definition 1.16. If $\mathbf{m} \in \mathbf{N}_\iota$ <u>then</u> we shall say $\mathbf{c}$ is an $\mathbf{m}$-colouring <u>when</u>:

- (a) $\mathbf{c}$ is a function with domain $\mathrm{eseq}_\iota(\mathscr{T}_{\mathbf{m}})$
- (b) $\mathbf{c}$ is a function into some $\sigma < \kappa_{\mathbf{m}}$
- (c) if $\bar{a} \in \mathrm{eseq}_\iota$ and $j < \sigma$ then the set $\{b \in \mathscr{T}_{\mathbf{m}} : \bar{a}\char`^\langle b\rangle \in \mathrm{eseq}_\iota(\mathscr{T}), \mathbf{c}(\bar{a}\char`^\langle b\rangle) = j\}$ belongs to $\mathbb{B}_{\mathbf{m}}$

Crucial Claim 1.17. *Assume $\mathbf{m} \in \mathbf{N}$ so $\kappa = \kappa_{\mathbf{m}}$, etc. and $\sigma < \kappa, \mathscr{S} \in \mathbf{T}^\iota_{\theta,\partial}$ and $\partial \leq \kappa, \theta \leq \kappa$ and $\alpha < \theta \Rightarrow \sigma^{|\alpha|} < \kappa$. <u>Then</u> $\mathscr{T}_{\mathbf{m}} \to_{\mathbf{m}} (\mathscr{S})^{\mathrm{end}(1)}_\sigma$ where $\to_{\mathbf{m}}$ means that we restrict ourselves to $\mathbf{m}$-colourings, that is: for every $\mathbf{m}$-colouring $\mathbf{c} : \mathrm{eseq}^\iota_{<\omega}(\mathscr{T}) \to \sigma$ there is an ι-embedding g of $\mathscr{S}$ into $\mathscr{T}_{\mathbf{m}}$ such that $\mathbf{c}' = \mathbf{c} \circ g$ which has domain $\mathrm{eseq}^\iota_{<\omega}(\mathscr{S})$ is 1-end homogeneous, recalling 1.13(2).*

Remark 1.18. 1) Note that our embedding (in 1.17) preserves $<^*$ but not necessarily equality of levels, i.e. E.
2) A delicate point to check: consider sequence $\bar{\eta}$ in ${}^{n+1}\mathscr{T}, <^*_{\mathscr{T}}$-increasing, consider a colouring $\mathbf{c}_1$ such that $\mathbf{c}_1(\langle \eta_\ell : \ell \leq n\rangle) = \mathbf{c}_2(\langle \eta_\ell : \ell < n\rangle\char`^\langle \eta_n \restriction \mathrm{lev}_{\mathscr{T}}(\eta_{n-1})\rangle)$. See colouring below.
3) Recall κ regular, it is natural to have $\mathrm{ht}(\mathscr{T}) = \kappa$ but $\mathrm{cf}(\mathrm{ht}(\mathscr{T})) \geq \kappa$ is enough.

Proof. So we fix $\mathbf{m} \in \mathbf{T}_\iota, \mathscr{T} = \mathscr{T}_{\mathbf{m}}$ etc, $\sigma < \kappa$ and $\mathbf{m}$-colouring $\mathbf{c} : \mathrm{eseq}^\iota_{<\omega}(\mathscr{T}) \to \sigma$.

$(*)_0$ For $\mathscr{U} \subseteq \mathscr{T}$ we shall say that $p \in \mathbb{Q}$ is similarity fixed over $\mathscr{U}$ when:
if $s, t \in \mathrm{val}(p)$ then $\mathrm{sim}-\mathrm{tp}(s, \mathscr{U}, \mathscr{T}) = \mathrm{sim}-\mathrm{tp}_\iota(t, \mathscr{U}\mathscr{T})$

Now:

$(*)_1$ for any $\mathscr{U} \subseteq \mathscr{T}$ which has cardinality $< \theta$ and similarity fixed $p \in \mathbb{Q}$ <u>then</u> we let:

(a) $\mathrm{Col}_{\mathscr{U}}(p)$ is the set of partial functions $\mathbf{d}$ from $\mathrm{eseq}^\iota_{<\omega}(\mathscr{U}, \mathscr{T})$ into σ such that for some $s \in \mathrm{val}(p) \backslash \mathscr{U}$ for every $\bar{a} \in \mathrm{eseq}^\iota_{<\omega}(\mathscr{U}, \mathscr{T})$ we have:

- $\bar{a}{}^\frown\langle s\rangle \in \mathrm{eseq}^\iota_{<\omega}(\mathscr{T}) \Leftrightarrow \bar{a} \in \mathrm{Dom}(\mathbf{d})$
- if this holds then $\mathbf{c}(\bar{a}{}^\frown\langle s\rangle) = \mathbf{d}(\bar{a})$

(b) $\mathrm{col}_{\mathscr{U}}(p) = \mathrm{col}(p, \mathscr{U})$ is the function $\mathbf{d}$ from $\mathrm{eseq}_{<\omega}(\mathscr{U}, \mathscr{T})$ into $\sigma + 1$ such that (if there is no such function, then $\mathrm{col}_{\mathscr{U}}(p)$ is not defined) for every $s \in \mathrm{val}(p)$ and $\bar{a} \in \mathrm{eseq}_{<\omega}(\mathscr{U}, \mathscr{T})$ we have
  (α) $\sup\{\mathrm{lev}_{\mathscr{T}}(t) + 1 : t \in \mathscr{U}\} \le \mathrm{lev}_{\mathscr{T}}(s)$ and
  (β) if $\bar{a} \in \mathrm{eseq}(\mathscr{U}, \mathscr{T})$ and $i < \sigma$ then $\mathbf{d}(\bar{a}{}^\frown\langle s\rangle) = i$ iff $\bar{a}{}^\frown\langle s\rangle \in \mathrm{eseq}(\mathscr{T})$ is well defined and $\mathbf{c}(\bar{a}{}^\frown\langle s\rangle) = i$.

Next

$(*)_2$ if $\mathscr{U} \subseteq \mathscr{T}, |\mathscr{U}| < \kappa_{\mathbf{m}}$ and $p \in \mathbb{Q}$ then for some pair (q, g)
(a) $p \le q$ in $\mathbb{Q}$
(b) g is a function with domain $\mathscr{U}$
(c) if $t \in \mathscr{U}$ then $g(t) \le_{\mathscr{T}} t$
(d) if $s \in \mathrm{val}(q)$ and $t \in \mathscr{U}$ then $s \cap t = g(s)$

[Why? By clause (g) of Definition 1.15; now we can note that if $\mathrm{ht}(\mathscr{T}) \le \kappa$ (so really equal) then clause (g) in this case follows because $\mathbb{Q}$ is $(< \kappa)$-complete, $\mathrm{ht}(\mathscr{T}) = \kappa$, and $\mathbf{m}$ has decidability.]

Now

$(*)_3$ if $\mathscr{U} \subseteq \mathscr{T}, p \in \mathbb{Q}$ and $|\mathscr{U}| < \theta$, <u>then</u> for some q we have:

- $p \le_{\mathbb{Q}} q$
- $\mathrm{col}_{\mathscr{U}}(q)$ is well defined, hence
- q is similarity fixed over $\mathscr{U}$ and
- if $r \in \mathbb{Q}$ is above q then $\mathrm{col}_{\mathscr{U}}(r) = \mathrm{col}_{\mathscr{U}}(q)$ so both are well defined.

[Why? Without loss of generality for some g, the triple $(p, g, \mathscr{U})$ is as in $(*)_2$. Let $\langle(\bar{a}_i, c_i) : i < |\mathrm{eseq}_{<\omega}(\mathscr{U}, \mathscr{T}) \times \sigma|\rangle$ list the pairs $(\bar{a}, c) \in \mathrm{eseq}^\iota_{<\omega}(\mathscr{U}, \mathscr{T}) \times \sigma$. Now choose p_i by induction on $i \le |\mathrm{eseq}_{<\omega}(\mathscr{U}, \mathscr{T}) \times \sigma|$ such that:

- $\bullet_1$ $p_0 = p$
- $\bullet_2$ $j < i \Rightarrow p_j \le p_i$
- $\bullet_3$ if $i = j+1$ then $\mathrm{val}(p_i)$ is included in $\{s \in (\mathscr{T}) : \bar{a}_i{}^\frown\langle s\rangle \in \mathrm{eseq}_(\mathscr{T})$ and $\mathbf{c}(\bar{a}_i{}^\frown\langle s\rangle) = c_j\}$ or is disjoint to it.

Recall that by the properties of val, we can ensure $s \in \mathrm{val}(p_i) \Rightarrow (\forall t \in \mathscr{U})(t <_{\mathscr{T}} s)$, see 1.15(1)(e),(f).

This suffices (note that by our assumptions: $(\alpha < \theta \Rightarrow \sigma^{|\alpha|} < \kappa)$).]

$(*)_4$ fix $\bar{\eta}$ such that:

(a) $\bar{\eta} = \langle \eta_i : i < i(*) \rangle$ lists the elements of $\mathscr{S}$ where $i(*) \leq \theta$
(b) $i < j \Rightarrow \eta_i <^*_{\mathscr{S}} \eta_j$
(c) if $\eta_i = \mathrm{suc}_{\mathscr{S},0}(\rho)$ then $\eta_{i+1} = \mathrm{suc}_{\mathscr{S},1}(\rho)$ and $i \in \{1+2j : 1+2j < i(*)\}$

[Why exists? Just think.]

Toward our inductive construction:

$(*)_5$ for $\zeta < \kappa$ let:
- (a) $I^{\mathrm{is}}_\zeta = \{\eta_\varepsilon : \varepsilon < 1 + (2\zeta)\}$, where "is" stands for "initial segment"
- (b) I^{fr}_ζ, where "fr" stands for front, is the set of η satisfying one of the following
  - (α) $\eta \in \mathscr{S}, \eta \in I^{\mathrm{is}}_\zeta$ but $\neg(\exists \nu)(\eta <_{\mathscr{T}} \nu \in I^{\mathrm{is}}_\zeta)$;
  - (β) $\eta \in \mathscr{S}, \eta \notin I^{is}_\zeta$ and $\lg(\eta)$ is a limit ordinal but $(\forall \nu)[\nu <_{\mathscr{T}} \eta \Rightarrow \nu \in I^{\mathrm{is}}_\zeta]$
- (c) hence I^{fr}_ζ is a maximal $\triangleleft$-antichain of $\mathscr{T}$.

Now we shall choose $(\bar{p}_\zeta, \bar{s}_\zeta, \bar{t}_\zeta, \bar{\mathbf{d}}_\zeta, \mathscr{U}_\zeta)$ by induction on $\zeta < \theta$ such that:

$(*)_6$
- (a) $\bar{p}_\zeta = \langle p_{\zeta,\eta} : \eta \in I^{\mathrm{is}}_\zeta \cup I^{\mathrm{fr}}_\zeta \rangle$
- (b) $p_{\zeta,\eta} \in \mathbb{Q}$
- (c) $p \leq_{\mathbb{Q}} p_{\zeta,\eta}$ for $\eta \in I^{\mathrm{is}}_\zeta \cup I^{\mathrm{fr}}_\zeta$
- (d) if $\varepsilon < \zeta$ and $\nu \in I^{\mathrm{is}}_\varepsilon, \nu \trianglelefteq \eta$ then
  - (α) $p_{\varepsilon,\nu} \leq_{\mathbb{Q}} p_{\zeta,\eta}$
  - $\beta)$ if $\nu \notin I^{\mathrm{fr}}_\zeta$ then $p_{\varepsilon,\nu} = p_{\zeta,\nu}$
- (e)
  - (α) $\bar{s}_\zeta = \langle s_\eta = s(\eta) : \eta \in I^{\mathrm{is}}_\zeta \setminus I^{\mathrm{fr}}_\zeta \rangle$ so $\varepsilon < \zeta \Rightarrow \bar{s}_\varepsilon = \bar{s}_\zeta {\restriction} (I^{\mathrm{is}}_\varepsilon \setminus I^{\mathrm{fr}}_\zeta$
  - (β) if $\varepsilon < \xi$ and $\eta_\varepsilon, \eta_\xi \in I^{\mathrm{is}} \setminus I^{\mathrm{fr}}_\zeta$ then $s_{\eta_\varepsilon} <^*_{\mathscr{T}} s_{\eta_\xi}$
  - (γ) if $\nu <_{\mathscr{T}} \eta \in I^{\mathrm{is}}_\zeta \setminus I^{\mathrm{fr}}_\zeta$ then $s_\eta \in \mathrm{val}(p_{\zeta,\nu})$
- (f)
  - (α) $\bar{t}_\zeta = \langle t_\eta : \eta \in I^{\mathrm{is}}_\zeta \backslash I^{\mathrm{fr}}_\zeta \rangle$
  - (β) $s_\eta \leq_{\mathscr{T}} t_\eta \in \mathscr{T}$ for $\eta \in I^{\mathrm{is}}_\zeta \backslash I^{\mathrm{fr}}_\zeta$
  - (γ) $\nu \triangleleft \eta \in I^{\mathrm{is}}_\zeta \backslash I^{\mathrm{fr}}_\zeta \Rightarrow t_\nu \leq_{\mathscr{T}} s_\eta$
  - (δ) if $\iota = 1, 1.5$ then $s_\eta = t_\eta$
- (g) if $\eta \in I^{\mathrm{is}}_\zeta \backslash I^{\mathrm{fr}}_\zeta, \ell < 2$ and $\nu = \mathrm{suc}_{\mathscr{S},\ell}(\eta) \in I^{\mathrm{is}}_\zeta \cup I^{\mathrm{fr}}_\zeta$ then $(\forall t)[t \in \mathrm{val}(p_{\zeta,\nu}) \to t_\eta R_{\mathscr{T},\ell} t]$
- (h) $\bar{\mathbf{d}}_\zeta = \langle \mathbf{d}_{\zeta,\eta} : \eta \in I^{\mathrm{is}}_\zeta \setminus I^{\mathrm{fr}}_\zeta \rangle$ but $\mathbf{d}_{\zeta,\eta} = \mathbf{d}_\eta$, so it does not depend on ζ
- (i) $\mathbf{d}_\eta = \mathrm{col}_{\mathscr{U}_\zeta}(p_{\zeta,\eta}, \mathscr{U}_\zeta)$ is well defined, where:
- (j) $\mathscr{U}_\zeta = \{s_\eta, t_\eta : \eta \in I^{\mathrm{is}_\zeta} \setminus I^{\mathrm{fr}}_\zeta\}$.

Clearly it is enough to carry the induction, because then the mapping $\eta \mapsto s_\eta$ is a $\subseteq_\iota$-embedding as required.

Case 1: $\zeta = 0$

Let $p_{\langle\rangle} \in \mathbb{Q}$ be above p and such that $\mathbf{d}_{0,\emptyset} = \mathrm{col}_\emptyset(p_{\langle\rangle})$ is well defined. Let $\alpha_{\langle\rangle} \in \mathrm{val}(p_{\langle\rangle})$.

Case 2: $\zeta = \varepsilon + 1, \eta_{1+2\varepsilon} = \mathrm{suc}_{\mathscr{S},0}(\rho)$

So $\rho \in I^{\rm is}_\varepsilon \cap I^{\rm fr}_\varepsilon, I^{\rm fr}_\zeta = I^{\rm is}_\varepsilon \cup \{\rho\hat{}\langle 0\rangle, \rho\hat{}\langle 1\rangle\} = I^{\rm is}_\varepsilon \cup \{\eta_{1+2\varepsilon}, \eta_{1+2\varepsilon+1}\}$ and $I^{\rm fr}_\zeta = I^{\rm fr}_\varepsilon \cup \{\rho\hat{}\langle 0\rangle, \rho\hat{}\langle 1\rangle\} \setminus \{\rho\}$.

Now we choose a quadruple $(s_\rho, t_\rho, p^0_\rho, p^1_\rho)$ such that:

$(*)_7$ (a) $s_\rho \in {\rm val}(p_{\varepsilon,\rho})$ and ${\rm lev}_{\mathscr{T}_{\mathbf{m}}}(s_\rho) > {\rm lev}(\mathscr{U}_\varepsilon, \mathscr{T}_{\mathbf{m}})$
(b) $s_\rho \le_{\mathscr{T}} t_\rho$
(c) p^0_ρ, p^1_ρ are above $p_{\varepsilon,\rho}$ in $\mathbb{Q}$
(d) for $\ell < 2$ for every $r \in {\rm val}(p^\ell_\rho)$ we have $t_\rho R_{\mathscr{T},\ell} r$
(e) if $\iota = 1.5$ then $s_\rho = t_\rho$

[Why such a quadruple exists? by 1.15(1)(d)(γ)]

$(*)_8$ Let $\mathscr{U}_\zeta = \mathscr{U}_\varepsilon \cup \{s_\rho, t_\rho\}$.

Next we choose $p_{\zeta,\eta}$ for $\eta \in I^{\rm is}_\zeta \cup I^{\rm fr}_\zeta$ as follows

$(*)_9$ (a) if $\eta \notin I^{\rm fr}_\zeta$ then $p_{\zeta,\eta} = p_{\varepsilon,\eta}$
(b) if $\eta \in I^{\rm fr}_\varepsilon \setminus \{\rho\} = I^{\rm fr}_\zeta \setminus \{\rho\hat{}\langle 0\rangle, \rho\hat{}\langle 1\rangle\}$ then we choose $p_{\zeta,\eta} \in \mathbb{Q}$ above $p_{\varepsilon,\eta}$ such that ${\rm col}_{\mathscr{U}_\zeta}(p_{\zeta,\eta}, \mathscr{U}_\zeta)$ is well defined.
(c) if $\ell = 0, 1$ and $\eta = \rho\hat{}\langle \ell\rangle$ <u>then</u> we choose $p_{\zeta,\eta} \in \mathbb{Q}$ above $p^\ell_{\rho,\eta}$ such that ${\rm col}_{\mathscr{U}_\zeta}(p_{\zeta,\eta}\mathscr{U}_\zeta)$ is well defined.

Clearly we can do this by $(*)_3$. Lastly

$(*)_{10}$ We choose $\bar{\mathbf{d}}_\zeta$ by $(*)_9(b),(c)$ in the new cases and $(*)_9(a)$ in the old ones.

Now check.

<u>Case 3: $\zeta = \varepsilon + 1$ and $\eta_{1+2\varepsilon} = \rho = \eta_{1+2\varepsilon+1}$</u>

Necessarily ${\rm lg}(\rho)$ is a limit ordinal and $\rho \in I^{\rm fr}_\varepsilon$. So $\{\rho\} \cup I^{\rm is}_\varepsilon$ is equal to $I^{\rm is}_\zeta$ and $I^{\rm fr}_\zeta = I^{\rm fr}_\zeta$ so $I^{\rm is}_\varepsilon \cup I^{\rm fr}_\varepsilon = I^{\rm is}_\zeta \cup I^{\rm fr}_\zeta$. So $\bar{s}_\zeta = \bar{s}_\varepsilon$ and $\bar{t}_\zeta = \bar{t}_\varepsilon$ and we can choose $p_{\zeta,\eta} = p_{\varepsilon,\eta}$ for $\eta \in I^{\rm is}_\varepsilon \cup I^{\rm fr}_\varepsilon \setminus \{\rho\} = I^{\rm is}_\zeta \cup I^{\rm fr}_\zeta \setminus \{\rho\}$. Also $\mathscr{U}_\zeta = \mathscr{U}_\varepsilon$ is well defined.

So we still are left with choosing $(p_{\zeta,\rho}, \mathbf{d}_{\zeta,\rho})$. Now $p_{\zeta,\rho} \ge p_{\varepsilon,\rho}$ such that $\mathbf{d}_{\zeta,\rho}$ is as required can be chosen by $(*)_3$.

<u>Case 4: ζ is a limit ordinal</u>

This is easy, recalling $\mathbb{Q}$ is $(< \kappa)$-complete.

$\square_{1.17}$

Now we derive the obvious conclusion

Claim 1.19. *Assume $\iota = 1$*
1) If $\mathscr{T} \in \mathbf{T}$ and $\mathscr{T} \to (\mathscr{T})^{{\rm end}(1)}_\sigma$ then $\mathscr{T} \to_\iota (\mathscr{T})^{{\rm end}(k)}_\sigma$ for every k.
2) If $\mathscr{T} \in \mathbf{T}$ and $\mathscr{T} \to (\mathscr{T})^{{\rm end}(1)}_\sigma$ then $\mathscr{T} \to (\mathscr{T})^n_\sigma$ for every n.
3) If $k \ge 1$ and $\mathscr{T}_\ell \in \mathbf{T}$ for $\ell = 0, \dots, k$ and $\mathscr{T}_{\ell+1} \to_\iota (T_\ell)^{{\rm end}(1)}_\sigma$ for $\ell < k$, <u>then</u> $\mathscr{T}_k \to (\mathscr{T}_0)^{{\rm end}(k)}_\sigma$ hence $\mathscr{T}_k \to (\mathscr{T}_0)^n_\sigma$.

Proof. Should be clear. $\square_{1.19}$

§ 2. Examples

§ 2(A). Consistency with no Large Cardinal.

Recall that $\mathscr{T} \to_\iota (\mathscr{T}_2)^{\text{end}(1)}_\sigma$ is the following statement. If $\mathbf{c}$ is a function from $\text{eseq}_\iota(\mathscr{T}_1)$ into σ then there is a $\subseteq_\iota$-embedding g of $\mathscr{T}_2$ into $\mathscr{T}_1$ such that the colouring $\mathbf{c}' = \mathbf{c} \circ g$ does not depend on the last level. That is, if if $\bar{a}, \bar{b} \in \text{eseq}^\iota_{n+1}(\mathscr{T})_2$ are so called similar and $\bar{a}{\restriction}n = \bar{b}{\restriction}n$ then $\mathbf{c}'(\bar{a}) = \mathbf{c}'(\bar{b})$.

Claim 2.1. *Assume $\iota = 1$ and (A) then (B) where:*

(A) (a) *$\kappa = \kappa^{<\kappa} < \mu = \kappa^+$ (can get it by a preliminary forcing for κ regular by first Levy collapsing $2^{<\kappa}$ to κ and then Levy collapsing 2^κ to μ^+).*
  (b) *$\lambda > \mu_1 = \mu(1) = 2^{<\mu} = 2^\kappa$*
  (c) *$\mathbb{P}$ is the forcing $\text{Cohen}(\kappa, \lambda)$, adding λ many κ-Cohens*
  (d) *the $\mathbb{P}$-name $\underset{\sim}{\mathscr{T}}_1 \in \mathbf{T}$ expands $({}^{\mu>}2, \triangleleft)$ as in 1.11*
  (e) *$\mathscr{T}_2 \in \mathbf{T}$ expands $({}^{\kappa>}2, \triangleleft)$ as in 1.11 so can be chosen in $\mathbf{V}$*

(B) *in $\mathbf{V}^{\mathbb{P}}$, for every $\sigma < \kappa$ we have $\mathscr{T}_1 \to (\mathscr{T}_2)^{\text{end}(1)}_\sigma$*

Proof. First

$\boxplus_1$ so $\mathbb{P}$ is $\text{Cohen}(\kappa, \lambda)$ and $\underset{\sim}{\eta}_\alpha$ for $\alpha < \lambda$ are as in 0.2

For the rest of the proof we assume:

$\boxplus_2$ $\Vdash_{\mathbb{P}}$: $\underset{\sim}{\mathbf{c}} : \text{eseq}_\iota(\underset{\sim}{\mathscr{T}}_1) \to \sigma$ and σ is a cardinal $< \kappa$

Next, in $\mathbf{V}$, we choose:

$(*)_1$ (a) let $\chi > \lambda$ and $<^*_\chi$ a well ordering of $\mathscr{H}(\chi)$
  (b) let $\mathfrak{B} \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ be of cardinality κ such that $[\mathfrak{B}]^{<\kappa} \subseteq \mathfrak{B}$ and $\lambda, \kappa, \mu, \sigma, \mathscr{T}_1, \mathscr{T}_2, \underset{\sim}{\mathbf{c}} \in \mathfrak{B}$
  (c) let $u_* = \mathfrak{B} \cap \lambda \in [\lambda]^\kappa$
  (d) let $\mathbf{G}_{u_*} \subseteq \mathbb{P}_{u_*}$ be generic over $\mathbf{V}_0 = \mathbf{V}$
  (e) let $\bar{\eta}_{u_*} = \langle \underset{\sim}{\eta}_\alpha[\mathbf{G}_{u_*}] : \alpha \in u_* \rangle$
  (f) let $\mathbf{V}_1 = \mathbf{V}_0[\mathbf{G}_{u_*}] = \mathbf{V}_0[\bar{\eta}_{u_*}]$
  (g) Let $\mathbf{V}_2 = \mathbf{V}_0[\bar{\eta}_\lambda] = \mathbf{V}_1[\bar{\eta}_{\lambda \setminus u_*}]$

$(*)_2$ (a) let $\underset{\sim}{\mathscr{T}}_0$ be the sub-structure of $\underset{\sim}{\mathscr{T}}_1$ with set of elements $\{\underset{\sim}{\eta} : \underset{\sim}{\eta}$ is a canonical $\mathbb{P}$-name of a member of $\underset{\sim}{\mathscr{T}}_1$ and this name belongs to $\mathfrak{B}\}$; ("canonical" mean it is defined by κ maximal anti-chains of $\mathbb{P}$ each of cardinality κ etc)
  (b) let $\delta_* = \delta(*)$ be $\min(\lambda \setminus u_*) = \min(\mu \setminus u_*) = \mu \cap u_*$

Clearly

$(*)_3$ (a) $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathscr{T}}_0 \subseteq_\iota \underset{\sim}{\mathscr{T}}_1$ expands $({}^{\mu>}2, \triangleleft)^{\mathbf{V}[\mathbb{P}]}$, $u_{\mathscr{Y}_0}$ is closed under initial segments, is of cardinality κ and has δ_* levels and is closed under unions of increasing chains of length $< \kappa$ and $\nu \in \underset{\sim}{\mathscr{T}}_0 \Rightarrow \nu\hat{\ }\langle 0 \rangle, \nu\hat{\ }\langle 1 \rangle \in \underset{\sim}{\mathscr{T}}_0$ so $\underset{\sim}{\mathscr{T}}_0 \in \mathbf{T}_5$"
  (b) $\underset{\sim}{\mathscr{T}}_0$ is actually a $\mathbb{P}_{u_*}$-name and we can use $\delta_* = \mathfrak{B} \cap \mu$ is its set of levels,

$(*)_4$ (a) let $\mathscr{T}_0 = \underset{\sim}{\mathscr{T}}_0[\mathbf{G}_{u_*}]$, so is from $\mathbf{V}_1$
(b) let $\mathbb{P}_* = \mathbb{P}/\mathbf{G}_{u_*} = \mathbb{P}_{\lambda \backslash u_*}$

$(*)_5$ in $\mathbf{V}_1$ there are
(a) a $\mathbb{P}_*$-name $\underset{\sim}{\eta}_\bullet$ of a branch of $\mathscr{T}_0$ generic over $\mathbf{V}_1$, i.e. for the forcing notion $(u_{\mathscr{T}_0}, \triangleleft)$.
(b) hence $\underset{\sim}{\eta}_\bullet \in {}^{\delta(*)}2 \subseteq \underset{\sim}{\mathscr{T}}_1$ and $\varepsilon < \delta_* \Rightarrow \Vdash_{\mathbb{P}_{u_*}}$ "$\underset{\sim}{\eta}_\bullet \restriction \varepsilon \in \underset{\sim}{\mathscr{T}}_1$"

[Why? By the character of $\mathscr{T}_0$, see $(*)_1(b) + (*)_2(a) + (*)_3$ being of cardinality κ.]

$(*)_6$ (a) let $\mathscr{A}$ be the set of objects from $\mathfrak{B}$ which are $\mathbb{P}$-names of a subset of $\underset{\sim}{\mathscr{T}}_1$.
(b) if $\underset{\sim}{A}$ belongs to $\mathscr{A}$ then we let $\check{\underset{\sim}{A}}$ be the $\mathbb{P}_{u_*}$-name $\underset{\sim}{A} \cap \underset{\sim}{\mathscr{T}}_0$
(c) $\check{\underset{\sim}{A}}$ is actually a $\mathbb{P}_{u_*}$-name
(d) in $\mathbf{V}_1$ let $\mathbb{B} = \{\check{\underset{\sim}{A}}[\mathbf{G}_{u_*}] : \underset{\sim}{A} \in \mathscr{A}\}$ and so $\underset{\sim}{\mathbb{B}} = \{\check{\underset{\sim}{A}} : \underset{\sim}{A} \in \mathscr{A}\}$ is a $\mathbb{P}_{u_*}$-name such that $\underset{\sim}{\mathbb{B}}[\mathbf{G}_{u_*}] = \mathbb{B}$
(e) for $p \in \mathbb{P}_*$ we let (in $\mathbf{V}_1$) $D_p = \{\check{\underset{\sim}{A}}[\mathbf{G}_{u_*}] : p \Vdash_{\mathbb{P}_2}$ "$\underset{\sim}{\eta}_\bullet \in \underset{\sim}{A}\}$

Now comes the main point - find appropriate $\mathbf{m}$ in the universe $\mathbf{V}_1$

$(*)_7$ we define $\mathbf{m}$ as follows:
(a) $\mathscr{T}_{\mathbf{m}} = \underset{\sim}{\mathscr{T}}_0[\mathbf{G}_{u_*}]$
(b) the forcing notion $\mathbb{Q}_{\mathbf{m}}$ is defined by
- the set of elements is $\{(p, \underset{\sim}{A}) : p \in \mathbb{P}_*$ and $\underset{\sim}{A} \in \mathscr{A}$ are such that $p \Vdash$ "$\underset{\sim}{\eta}_\bullet \in \underset{\sim}{A}\}$
- the order is $(p_1, \underset{\sim}{A}_1) \leq_{\mathbb{Q}_{\mathbf{m}}} (p_2, \underset{\sim}{A}_2)$ iff (both are from $\mathbb{Q}$ and) $p_1 \leq_{\mathbb{Q}} p_2$ and $p_2 \Vdash$ "$\underset{\sim}{A}_2 \subseteq \underset{\sim}{A}_1$"

(c) $\mathbb{B}_{\mathbf{m}}$ is the family $\mathbb{B}$ defined above except that we intersect it with $\mathscr{P}(\mathscr{T}_0)$
(d) $\mathrm{val}_{\mathbf{m}}((p, \underset{\sim}{A})) = \check{\underset{\sim}{A}}[\mathbf{G}_{u_*}]$

$(*)_8$ if $p \in \mathbb{P}_*$ and $\varepsilon < \delta_*$ then for some $p \in \mathbb{P}_*$ above p and $\nu \in \mathscr{T}_0 \cap {}^{\varepsilon}(\delta_*) \cap \mathfrak{B}[\mathbf{G}_{u_*}]$ we have $q \Vdash_{\mathbb{P}_*}$ "$\nu \triangleleft \underset{\sim}{\eta}_\bullet$"
[Why? because for every $\eta \in \mathscr{T}_0$ there is $\nu \in \mathscr{T}_)$ such that $\eta \triangleleft \nu \wedge \lg(\nu) \geq \varepsilon$ and $(*)_5(a)$.]

$(*)_9$ if $p \in \mathbb{P}_*$ then for some $\varepsilon < \delta_*$ and $\nu_1 \neq \nu_2 \in \mathscr{T}_)$ of length ε and $q_1, q_2 \in \mathbb{P}_*$ above p we have $q_\ell \Vdash_{\mathbb{P}_*}$ "$\nu_\ell =$ '$\underset{\sim}{\eta}_\bullet \restriction \varepsilon$"

[Why? As $\Vdash_{\mathbb{P}_*}$ "$\underset{\sim}{\eta}_\bullet \notin \mathbf{V}_1$" beig generic for $\mathscr{T}_0$ ove $\mathbf{V}_1$]
Now

$(*)_{10}$ $\mathbf{m} \in \mathbf{N}_\iota$

Why? We should check all the clauses in Definition 1.15 in order to prove that $\mathbf{m} \in \mathbf{N}_\iota$ (in the universe $\mathbf{V}_1$ indeed).

Clause (a): κ is a regular cardinal

Recall clause (A)(a) of 2.1(1) and $\mathbb{P}_{u_*}$ being $(< \kappa)$-complete.

Clause (b): $\mathscr{T}_{\mathbf{m}} \in \mathbf{T}_\iota$ and $\mathbb{B}$ being as there.

On $\mathscr{T}_{\mathbf{m}}$ recall clause (A)(d) of 2.1(1) and $(*)_2 + (*)_3$. On $\mathbb{B}_{\mathbf{m}}$, being a Boolean algebra of sub-sets of $\mathscr{T}_1$, see its definition in $(*)_7(c), (*)_6(d)$.

Clause (c): $\mathbb{Q}_{\mathbf{m}}$ is a quasi order, $(< \kappa)$-complete

For being a quasi-order, see the choice of $\mathbb{Q}_{\mathbf{m}}$ in $(*)_7(b)$; and $\mathbb{P}_*$ being a quasi-order and $(\underset{\sim}{D}_p, \supseteq)$ is forced to be. As for being $(<\kappa)$-complete recall $\mathbb{P}_*$ is $(<\kappa)$-complete by clause (A)(b) of 2.1(1) and $\underset{\sim}{D}_p$ is forced to be $(<\kappa)$-complete (recalling the role of $\underset{\sim}{\eta}_\bullet$) and $\mathbf{G}_{u_*}$ is $(<\kappa)$-directed

Clause (d)(α): $\mathrm{val} = \mathrm{val}_{\mathbf{m}}$ being a function from $\mathbb{Q}$ into $\mathscr{P}((\mathscr{T}))$

See the choice of val in $(*)_7(d)$ above. In particular, why the value belongs to $\mathbb{B}_{\mathbf{m}}$? see $(*)_7(c)$.

Clause (d)(β): if $p \in \mathbb{Q}$ then $\{\mathrm{lev}_{\mathscr{T}}(\eta) : \eta \in \mathrm{val}(p)\}$ is unbounded in $\mathrm{ht}(\mathscr{T})$

Why? as it is forced that $\underset{\sim}{\eta}_\bullet$ is a cofinal branch of $\underset{\sim}{\mathscr{T}}_0$.

Clause (d)(γ) if $\varepsilon < \mathrm{ht}(\mathscr{T})$ and $p \in \mathbb{Q}$ then for some s, t, p_0, p_1 we have: $s \in \mathrm{val}(p), \mathrm{lev}_{\mathscr{T}}(s) \geq \varepsilon, s \leq t \in (\mathscr{T})$ and for $\ell = 0, 1$ we have $p \leq_{\mathbb{Q}} p_\ell$ and $(\forall r \in \mathrm{val}(p_\ell))(tR_{\mathscr{T},\ell} r)$

The reason is that it is forced (for $\mathbb{P}_*$)) that $\underset{\sim}{\eta}_\bullet$ is not from $\mathbf{V}_1$ that is $(*)_8$.

Clause (d)(δ): in (d)(γ) above, if $\iota = 1$ then we can add $s = t$

This is easy.

Clause (e): monotonicity

Just check the definition of the order of $\mathbb{Q}$.

Clause (f): decidability

Why? This is the point where the choice of $\mathbb{B}_{\mathbf{m}}$ help us.

So we are given $(p, \underset{\sim}{A}) \in \mathbb{Q}_{\mathbf{m}}$ and $A_0, A_1 \in \mathbb{B}$ such that $\underset{\sim}{A} = \underset{\sim}{A}_0 \cup \underset{\sim}{A}_1$ is forced and let $A = \underset{\sim}{A}[\mathbf{G}_{u_*}]$. Recalling $(p, \underset{\sim}{A}) \in \mathbb{Q}_{\mathbf{m}}$ clearly $p \cup r_1 \Vdash$ "$\underset{\sim}{\eta}_\bullet \in \underset{\sim}{A}$" for some $r_1 \in \mathbf{G}_{u_*}$. Also $A \subseteq A_0 \cup A_1$ hence some $r_2 \in \mathbf{G}_{u_*}$ forces (for $\mathbb{P}_{u_*}$) that $\underset{\sim}{A} \subseteq \underset{\sim}{A}_0 \cup \underset{\sim}{A}_1$, and let $r \in \mathbf{G}_{u_*}$ be a common upper bound of r_1, r_2. Hence $p \cup r$ forces (for $\mathbb{P}$) that $\underset{\sim}{\eta}_\bullet \in \underset{\sim}{A}_0$ or $\underset{\sim}{\eta}_\bullet \in \underset{\sim}{A}_1$. Hence for some q, r', ℓ we have $\ell \in \{0,1\}$ and $\mathbb{P}_2 \models$ "$p \leq q$" and $r' \in \mathbf{G}_{u_*}$ is above r and $q \cup r' \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\eta}_\bullet \in \underset{\sim}{A}_\ell$", So $\ell, (q, \underset{\sim}{A}_\ell)$ are as required.

Clause (g): compatibility

So we are given $(p, \underset{\sim}{A}) \in \mathbb{Q} = \mathbb{P}/\mathbf{G}_{u_*} \times \mathscr{A}$ and $t \in \mathscr{T}_{\mathbf{m}} = \mathscr{T}_0$.

So $t \cap \underset{\sim}{\eta}_\bullet$ is a $\mathbb{P}_*$-name of an initial segment of t hence $\mathrm{lg}(t \cap \underset{\sim}{\eta}_\bullet)$ is a $\mathbb{P}_*$-name of an ordinal $\leq \mathrm{lg}(t) < \delta_*$. It follows that for some $q \in \mathbb{P}_*$ above p and some ordinal ζ, we have $q \Vdash_{\mathbb{P}_*}$ "$\zeta = \mathrm{lg}(t \cap \underset{\sim}{\eta}_\bullet)$" hence $q \Vdash_{\mathbb{P}_*}$ "$t \restriction \zeta = t \cap \underset{\sim}{\eta}_\bullet$". But as $\mathrm{lg}(t) < \delta_* < \mu$ clearly $\zeta \in \mu \cap \mathfrak{B} = \delta_* \subseteq \mathfrak{B}$ hence $s = t \restriction \zeta \in \mathfrak{B}[\mathbf{G}_{u_*}$. So $s, (q, \underset{\sim}{A})$ is as required in clause (g).

Clause (h): extension existence

Problematic but not needed.

So we are done proving $(*)_8 10$.

$(*)_{11}$ in $\mathbf{V}_1$

(a) let $\mathbf{c}$ be the function from $\mathrm{eseq}_\iota(\mathscr{T}_0)$ to σ such that $\mathbf{c}(\bar{a}) = j$ iff for some $r \in \mathbf{G}_{u_*}$ we have $r \Vdash$ "$\underset{\sim}{\mathbf{c}}(\bar{a}) = j$"

(b) $\mathbf{c}$ is indeed a function from $\mathrm{eseq}_\iota(\mathscr{T}_0)$ to σ (and belong to $\mathbf{V}_1$)

$(*)_{12}$ there is an $\subseteq_\iota$-embedding g of $\mathscr{T}_2$ into $\mathscr{T}_0$ and $\mathbf{c} \circ g$ is an end(1)-extension homogeneous colouring of $\mathrm{eseq}(\mathscr{T}_2')$.

[Why? By §(1B), that is 1.17].

Now g witness our desired conclusion.

2), 3), 4) Proved similarly. $\square_{2.1}$

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