Math. Log. Quart. **67,** No. 1, 18–42 (2021) / **DOI** 10.1002/malq.201900006

# Atomic saturation of reduced powers

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Received 24 January 2019, revised 26 April 2020, accepted 30 April 2020
Published online 20 May 2021

Our aim was to try to generalize some theorems about the saturation of ultrapowers to reduced powers. Naturally, we deal with saturation for types consisting of atomic formulas. We succeed to generalize "the theory of dense linear order (or $T$ with the strict order property) is maximal and so is any pair $(T, \Delta)$ which is $\mathrm{SOP}_3$", (where $\Delta$ consists of atomic or conjunction of atomic formulas). However, the theorem on "it is enough to deal with symmetric pre-cuts" (so the $\mathfrak{p} = \mathfrak{t}$ theorem) cannot be generalized in this case. Similarly the uniqueness of the dual cofinality fails in this context.



## 1 Introduction

### 1.1 Background, questions and answers

We know much on saturation of ultrapowers, see Keisler [3], [22, Ch. IV, ] and later mainly works of Malliaris and the author, e.g., [9], [10]. But we know considerably less on reduced powers. For transparency, let $T$ denote a first order complete countable theory with elimination of quantifiers and $M$ will denote a model of $T$. For $D$ a regular filter on $\lambda > \aleph_0$ we may ask: when is $M^\lambda/D$ $\lambda^+$-saturated? For $D$ an ultrafilter, Keisler [2] proves that this holds for every $T$ iff $D$ is $\lambda^+$-good iff this holds for $T =$ theory of Boolean algebras, such $T$ is called $\trianglelefteq_\lambda$-maximal.

By [21, Ch. VI, 2.6] the maximality holds for $T =$ theory of dense linear orders or just any $T$ with the strict order property and by [24], any $T$ with the 3-strong order property, $\mathrm{SOP}_3$ is $\trianglelefteq_\lambda$-maximal.

What about reduced powers for $\lambda$-regular filter $D$ on $\lambda$? By [19], $M^\lambda/D$ is $\lambda^+$-saturated for every $T$ (of cardinality $\leq \lambda$) iff $D$ is $\lambda^+$-good and $\mathscr{P}(\lambda)/D$ is a $\lambda^+$-saturated Boolean algebra. Parallel results hold when we replace $\lambda^+$-saturated by $(\lambda^+, \Sigma_{1+n}(\mathbb{L}_{\tau(T)}))$-saturated. We shall concentrate on ($\lambda^+$,atomic)-saturated and introduce the related partial order $\trianglelefteq_\lambda^{\mathrm{rp}}$, see definitions below.

Concerning ultrapowers, lately Malliaris and Shelah [9] proved that a regular ultrafilter $D$ on a cardinal $\lambda$ is $\lambda^+$-good iff for any linear order $M$ we have $M^\lambda/D$ has no symmetric pre-cut with cofinality $\leq \lambda$. This was proved together with the theorem $\mathfrak{p} = \mathfrak{t}$ and "for a f.o. complete countable $T$, being $\mathrm{SOP}_2$ suffices for $\triangleleft_\lambda$-maximality". In a later work [11], it is proved that at least for a relative $\triangleleft_\lambda^*$ (cf. [24]) this is "iff" assuming a case of G.C.H., relying also on works with Dzamonja [1], and with Usvyatsov [25]. Part of the proof is axiomatized by Malliaris and Shelah [8].

Note also that [17] deals with saturation but only for ultrapowers by $\vartheta$-complete ultrafilters for $\vartheta$ a compact cardinal; and also with $\omega$-ultra-limits.

Now what do we accomplish here?

First, in § 2 we axiomatize the proof of [21, Ch. VI, 2.6], i.e., we define when $\mathbf{r} = (M, \Delta)$ is a so called RSP and for it prove that the relevant model $N_{\mathbf{r}}$ is $(\min\{\mathfrak{p}_{\mathbf{r}}, \mathfrak{t}_{\mathbf{r}}\}, \Delta)$-saturated. Second, in § 3 we prove, of course, that [21, Ch. VI, 2.6] follows, but also we show that the axiomatization of RSP is by Horn sentences. Hence we can apply it to reduced powers. So $T$ is $\trianglelefteq_\lambda^{\mathrm{rp}}$-maximal if $T = \mathrm{Th}(\mathbb{Q}, <)$ and moreover for every $T$ having the $\mathrm{SOP}_3$; lastly we comment on models of Peano Arithmetic.

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In § 4 we try to sort out when for models of $T$ we get the relevant atomic saturation.

Can we generalize also results [9] to reduced powers? The main result of § 5 says that no. We also sort out the parallel of goodness, excellence and morality for filters and atomic saturation for reduced powers. In a hopeful continuation [15], we shall try to sort out the order $\trianglelefteq^{\rm rp}_\lambda$, and in particular consider non-maximality and parallel statements for infinitary logics (cf. [17]).

The reader can ignore Boolean ultrapowers (i.e., Definitions 1.12 and 1.13, Claim 1.14) for §§ 2, 3 and can in first reading deal only with first order logic (so $\vartheta = \aleph_0$), and the assumptions concerning the completeness of filters disappear.

Note that by Conclusion 3.10

**Conclusion 1.1** *If $(T, \Delta)$ has the* SOP$_3$*, then it is $\trianglelefteq^{\rm rp}_\lambda$-maximal.*

**Question 1.2** Do we have: if $D$ is $(\lambda_2, T)$-good and regular then $D$ is $(\lambda_1, T)$-good when $\lambda_1 < \lambda_2$ (or more)?

### 1.2 Further questions

**Convention 1.3** 1. *Let $T$ be a theory with elimination of quantifiers if not said otherwise. Let* $\mathrm{Mod}_T$ *be the class of models of $T$.*

2. *The main case is for $T$ a countable complete first order theory with elimination of quantifiers, moreover, with every formula equivalent to an atomic one.*

So it is natural to ask

**Conjecture 1.4** *The pair $(T, \Delta)$ is $\trianglelefteq_{\rm rp}$-maximal iff $(T, \Delta)$ has the* SOP$_3$.

So which $T$ (with elimination of quantifiers) are maximal under $\triangleleft^{\rm rp}_\lambda$? That is, when for every regular filter $D$ on $\lambda$, $M^\lambda/D$ is $(\lambda^+$, atomic)-saturated iff $D$ is $\lambda^+$-good? Is $T_{\rm feq}$ maximal? (cf. [23], it is a proto-typical non-simple $T$, but see more in [18]) As we have not proved this even for ultrafilters, the reasonable hope is that it will be easier to show non-maximality for $\triangleleft^{\rm rp}_\lambda$. Also in light of [10] for simple theories we like to prove non-maximality with no large cardinals. We may hope to use just NSOP$_2$, but still it would not settle the problem of characterizing the maximal ones as, e.g., SOP$_2 \equiv$ SOP$_3$ is open for such $T$; for pairs $(T, \varphi(\bar x, \bar y))$ they are different.

Note that for first order $T$, it makes sense to use $\mu^+$-saturated models and $D$ is $\mu^+$-complete.

Also the "$T$ stable" case should be resolved.

**Conjecture 1.5** *$M^\lambda/D$ is $(\aleph_0^\lambda/D$, atomic)-saturated when:*

*(a) $T$ a theory as in 1.3;*

*(b) $T$ is stable without the* fcp*;*

*(c) $D$ is a regular filter on $\lambda$.*

**Remark 1.6** Maybe given a 1-$\varphi$-type $p \subseteq \{\varphi(x, \bar a) : \bar a \in {}^m(M^I/D)\}$ of cardinality $\le \lambda$ in $M^I/D$, we try just to find a dense set of $A \in D^+$ such that in $M^I/(D + A)$ the 1-$\varphi$-type is realized. Then continue; opaque.

### 1.3 Preliminaries

**Notation 1.7** *1. $T$ denotes a f.o. theory, usually complete.*

*2. Let $\tau$ denote a vocabulary, $\tau_T = \tau(T)$ denotes the vocabulary of the theory $T$*

*3. We use $M, N$ to denote models, $\tau_M = \tau(M)$ is the vocabulary of $M$ and $P^M$, $F^M$ denote the interpretation of $P, F$ respectively.*

*4. let $\mathbb{L}(\tau)$ denote the f.o. language for the vocabulary $\tau$.*

*5. We allow function symbol $F \in \tau$ to be interpreted in a $\tau$-model $M$ as a partial function, but with domain $P^M_F$, with $P_F \in \tau$ a predicate with the same arity.*

 

**Notation 1.8** *1. Let $\mathfrak{B}$ denote a Boolean algebra,* $\mathrm{comp}(\mathfrak{B})$ *its completion,* $\mathfrak{B}^+ = \mathfrak{B}\backslash\{0_{\mathfrak{B}}\}$, $\mathrm{uf}(\mathfrak{B})$ *the set of ultrafilters on* $\mathfrak{B}$, $\mathrm{fil}(\mathfrak{B})$ *the set of filters on* $\mathfrak{B}$. *For* $\mathbf{a} \in \mathfrak{B}$ *let* $\mathbf{a}^{\mathrm{if(true)}} = \mathbf{a}^{\mathrm{if}(1)}$ *be* $\mathbf{a}$ *and let* $\mathbf{a}^{\mathrm{if(false)}} = \mathbf{a}^{\mathrm{if}(0)}$ *be* $1_{\mathfrak{B}} - \mathbf{a}$.

*1A. Let $\mathfrak{B}_1 \lessdot \mathfrak{B}_2$ mean that $\mathfrak{B}_1$ is a subalgebra of $\mathfrak{B}_2$, and moreover a complete one, which means that every maximal antichain of $\mathfrak{B}_1$ is a maximal antichain of $\mathfrak{B}_2$.*

2. *For a model $M$ let $\tau_M = \tau(M)$ be its vocabulary.*
3. *For a filter $D$ on a set $I$ let $D^+ = \{B \subseteq I : I \setminus B \notin D\}$.*

Now about cuts (they are closed to but different than gaps, see [12]).

**Definition 1.9** 1. For a partial order $\mathscr{T} = (\mathscr{T}, \leq_{\mathscr{T}})$, we say $(C_1, C_2)$ is pre-cut when (but we may in this paper omit the "pre"):

(a) $C_1 \cup C_2$ is a subset of $\mathscr{T}$ linearly ordered by $\leq_{\mathscr{T}}$;

(b) if $a_1 \in C_1, a_2 \in C_2$ then $a_1 \leq_{\mathscr{T}} a_2$;

(c) for no $c \in \mathscr{T}$ do we have $a_1 \in C_1 \Longrightarrow a_1 \leq_{\mathscr{T}} c$ and $a_2 \in C_2 \Longrightarrow c \leq_{\mathscr{T}} a_2$.

2. Above we say $(C_1, C_2)$ is a $(\kappa_1, \kappa_2)$-pre-cut when in addition:

(d) $C_1$ has cofinality $\kappa_1$;

(e) $C_2^*$, the inverse of $C_2$, has cofinality $\kappa_2$;

(f) so $\kappa_1, \kappa_2$ are regular cardinals (here we ignore the 0,1 if not said otherwise).

2A. Above we call $\kappa_1, \kappa_2$ the cofinalities of the pre-cut $(C_1, C_2)$. We say that the pre-cut is symmetric when $\kappa_1 = \kappa_2$ and then we may say $\kappa_1$ is its cofinality,

3. We may replace $C_\ell$ by a sequence $\bar{a}_\ell$, if not said otherwise such that $\bar{a}_1$ is $\leq_{\mathscr{T}}$-increasing and $\bar{a}_2$ is $\leq_{\mathscr{T}}$-decreasing.
4. We say $(C_1, C_2)$ is a $(\kappa_1, \kappa_2)$-linear-cut of $\mathscr{T}$ when it is a $(\kappa_1, \kappa_2)$-pre-cut and $C_1 \cup C_2$ is downward closed, so natural for $\mathscr{T}$ a tree.
5. We say $(C_1, C_2)$ is a weak pre-cut when (b),(c) of part (1) holds.

**Remark 1.10** 1. If $\mathscr{T}$ is a (model theoretic) tree, $\kappa_2 > 0$ and $(C_1, C_2)$ is a $(\kappa_1, \kappa_2)$-pre-cut then it induces one and only one $(\kappa_1, \kappa_2)$-linear-cut $(C_1', C_2')$, i.e., one satisfying $C_1 \subseteq C_1', C_2 \subseteq C_2'$ such that $C_1 \cup C_2$ is cofinal in $C_1' \cup C_2'$.

2. In 1.11 below, if $L = \mathbb{L}(\tau)$ then $\vartheta = \aleph_0, \sigma = 1$ suffice, but not so in more general cases.

**Definition 1.11** 1. We say $M$ is fully $(\lambda, \vartheta, \sigma, L)$-saturated (may omit the fully); where $L \subseteq \mathscr{L}(\tau_M)$ and $\mathscr{L}$ is a logic; we may write $\mathscr{L}$ if $L = \mathscr{L}(\tau_M)$, when:

> if $\Gamma$ is a set of $< \lambda$ formulas from $L$ with parameters from $M$ with $< 1 + \sigma$ free variables, and $\Gamma$ is $(< \vartheta)$-satisfiable in $M$, then $\Gamma$ is realized in $M$.

2. We say "locally" when using one $\varphi = \varphi(\bar{x}, \bar{y}) \in \mathscr{L}$ with $\ell g(\bar{x}) < 1 + \sigma$, i.e., all members of $\Gamma$ have the form[1] $\varphi(\bar{x}, \bar{b})$.
3. Saying "locally/fully $(\lambda, \mathscr{L})$-saturated" the default values (i.e., we may omit) of $\sigma$ is $\sigma = \vartheta$, of $(\sigma, \vartheta)$ is $\vartheta = \aleph_0 \wedge \sigma = \aleph_0$ and of $\mathscr{L}$ is $\mathbb{L}$ (first order logic) and of $L$ is $\mathscr{L}$. Omitting $\lambda$ means $\lambda = \|M\|$.
4. If $\varphi(\bar{x}, \bar{y}) \in \mathscr{L}(\tau_M)$ and $\bar{a} \in {}^{\ell g(\bar{y})}M$ then $\varphi(M, \bar{a}) := \{\bar{b} \in {}^{\ell g(\bar{x})}M : M \models \varphi[\bar{b}, \bar{a}]\}$.
5. Let $\bar{x}_{[u]} = \langle x_s : s \in u \rangle$.

In Definitions 1.12, 1.13, and 1.14 we shall deal with complete Boolean algebras and ultrapowers, and then we define an order between theories.

**Definition 1.12** Assume we are given a Boolean algebra $\mathfrak{B}$ usually complete and a model or a set $M$ and $D$ a filter on $\mathrm{comp}(\mathfrak{B})$, the completion of $\mathfrak{B}$.

[1] In [17] we use a $L \subseteq \mathbb{L}_{\vartheta,\vartheta}$, $\vartheta$ a compact cardinal and if $\sigma > \vartheta$ we use a slightly different version of the definition of local and of the default value of $\sigma$ was $\vartheta$.

 

1. Let $M^{\mathfrak{B}}$ be the set of partial functions $f$ from $\mathfrak{B}^+$ into $M$ such that for some maximal antichain $\langle a_i : i < i(*)\rangle$ of $\mathfrak{B}$, $\mathrm{dom}(f)$ includes $\{a_i : i < i(*)\}$ and is included in[2] $\{a \in \mathfrak{B}^+ : (\exists i)(a \leq a_i)\}$ and $f$ is a function into $M$ and $f\restriction\{a \in \mathrm{dom}(f) : a \leq a_i\}$ is constant for each $i$.
   1A. Naturally for $f_1, f_2 \in M^{\mathfrak{B}}$ we say $f_1, f_2$ are $D$-equivalent, or $f_1 = f_2 \mod D$ when for some $b \in D$ we have $a_1 \in \mathrm{dom}(f_1) \wedge a_2 \in \mathrm{dom}(f_2) \wedge a_1 \cap a_2 \cap b > 0_{\mathfrak{B}} \Longrightarrow f_1(a_1) = f_2(a_2)$.
2. We define $M^{\mathfrak{B}}/D$ naturally, as well as $\mathrm{TV}_M(\varphi(f_0, \ldots , f_{n-1})) \in \mathrm{comp}(\mathfrak{B})$ when $\varphi(x_0, \ldots , x_{n-1}) \in \mathbb{L}(\tau_M)$ and $f_0, \ldots , f_{n-1} \in M^{\mathfrak{B}}$ where
   (a) TV stands for truth value;
   (b) $\mathrm{TV}_M(\varphi(f_0, \ldots , f_{n-1})) = \sup\{a \in \mathfrak{B}^+ : a \cap \bigcap_{\ell<n} \mathrm{Dom}(f_\ell) : M \models (\varphi(f_0(a), \ldots , f_{n-1}(a)))\}$;
   (c) $M$ is defined by letting, for $\varphi$ an atomic formula, $M^{\mathfrak{B}}/D \models \varphi[f_0/D, \ldots , f_{n-1}/D]$ iff $\mathrm{TV}_M(\varphi(f_0, \ldots , f_{n-1})) \in D$.
   2A. Abusing notation, not only $M^{\mathfrak{B}_1} \subseteq M^{\mathfrak{B}_2}$ but $M^{\mathfrak{B}_1}/D_1 \subseteq M^{\mathfrak{B}_2}/D_2$ when $\mathfrak{B}_1 \lessdot \mathfrak{B}_2$, $D_\ell \in \mathrm{fil}(\mathfrak{B}_\ell)$ for $\ell = 1, 2$ and $D_1 = \mathfrak{B}_1 \cap D_2$. Also $[f_1, f_2 \in M^{\mathfrak{B}_1} \Longrightarrow f_1 = f_2 \mod D_1 \leftrightarrow f_1 = f_2 \mod D_2]$. So for $f \in M^{\mathfrak{B}_1}$ we identify $f/D_1$ and $f/D_2$.
3. For complete $\mathfrak{B}$, we say $\langle a_n : n < \omega\rangle$ represents $f \in \mathbb{N}^{\mathfrak{B}}$ when $\langle a_n : n < \omega\rangle$ is a maximal antichain of $\mathfrak{B}$ (so $a_n = 0_{\mathfrak{B}}$ is allowed) and for some $f' \in \mathbb{N}^{\mathfrak{B}}$ which is $D$-equivalent to $f$ (cf. Definition 1.12(1A)) we have $f'(a_n) = n$.
4. We say $\langle (a_n, k_n) : n < \omega\rangle$ represents $f \in \mathbb{N}^{\mathfrak{B}}$ when:
   (a) the $k_n$ are natural numbers with no repetition;
   (b) $\langle a_n : n < \omega\rangle$ is a maximal antichain;
   (c) $f(a_n) = k_n$.
5. If $\mathscr{I}$ is a maximal antichain of $\mathfrak{B}$ and $\bar{M} = \langle M_a : a \in \mathscr{I}\rangle$ is a sequence of $\tau$-models, then we define $\bar{M}^{\mathfrak{B}}$ be the set of partial functions $f$ from $\mathfrak{B}^+$ to $\cup\{M_a : a \in \mathscr{I}\}$ such that for some maximal antichain $\langle a_i : i < i(*)\rangle$ of $\mathfrak{B}$ refining $\mathscr{I}$ (i.e., $(\forall i < i(*))(\exists b \in \mathscr{I})(a_i \leq_{\mathfrak{B}} b)$) we have:
   (a) $\{a_i : i < i(*)\} \subseteq \mathrm{dom}(f) \subseteq \{b \in \mathfrak{B}^+ : b \leq_{\mathfrak{B}} a_i$ for some $i < i(*)\}$;
   (b) if $a \in \mathrm{dom}(f)$ and $a \leq a_i$ then $f(a) = f(a_i)$;
   (c) if $a_i \leq_{\mathfrak{B}} b, b \in \mathscr{I}$ then $f(a_i) \in M_b$.
6. For $\bar{M}, \mathfrak{B}, \mathscr{I}$ as above and a filter $D$ on $\mathfrak{B}$ we define $\bar{M}^{\mathfrak{B}}/D$ as in part (2) replacing $M^{\mathfrak{B}}$ there by $\bar{M}^{\mathfrak{B}}$ here, see part (7).
7. For $\bar{M}, \mathfrak{B}, \mathscr{I}$ as above, $\varphi = \varphi(\bar{x}) = \varphi(x_0, \ldots , x_{n-1}) \in \mathbb{L}(\tau_M)$ and $\bar{f} = \langle f_\ell : \ell < n\rangle$ where $f_0, \ldots , f_{n-1} \in \bar{M}^{\mathfrak{B}}$, let $\mathrm{TV}(\varphi[\bar{f}]) = \mathrm{TV}(\varphi[\bar{f}], \bar{M}^{\mathfrak{B}})$ be $\sup\{a \in \mathfrak{B}^+$: if $\ell < n$ then $a \in \mathrm{dom}(f_\ell)$ and $a \leq b \in \mathscr{I}$ then $M_b \models \varphi[f_0(b), \ldots , f_{n-1}(b)]\}$.
8. We say $\mathfrak{B}$ is $(< \sigma)$-distributive when it is $\vartheta$-distributive for every $\vartheta < \sigma$, where
   8A. $\mathfrak{B}$ is $\vartheta$-distributive when: if for $\alpha < \vartheta$, $\mathscr{I}_\alpha$ is a maximal antichain of $\mathfrak{B}$ then there is a maximal antichain of $\mathfrak{B}$ refining every $\mathscr{I}_\alpha (\alpha < \vartheta)$; this holds, e.g., when $\mathfrak{B} = \mathscr{P}(\lambda)$ or just there is a dense $Y \subseteq \mathfrak{B}^+$ closed under intersection of $\vartheta$.

**Definition 1.13** 1. Let $\mathfrak{B}$ be a complete Boolean algebra and $D$ a filter on $\mathfrak{B}$. We say that $D$ is $(\mu, \vartheta)$-regular when for some $(\bar{\mathbf{c}}, \bar{u})$ we have:
   (a) $\bar{\mathbf{c}} = \langle \mathbf{c}_\alpha : \alpha < \alpha_*\rangle$ is a maximal antichain of $\mathfrak{B}$;
   (b) $\bar{u} = \langle u_\alpha : \alpha < \alpha_*\rangle$ with $u_\alpha \in [\mu]^{<\vartheta}$;
   (c) if $i < \mu$ then $\sup\{\mathbf{c}_\alpha : \alpha$ satisfies $i \in u_\alpha\} \in D$.
2. A filter $D$ is called $\lambda$-regular when it is $(\lambda, \aleph_0)$-regular; the filter $D$ on a set $I$ (that is the Boolean algebra $\mathscr{P}(I)$) is called regular when it is a filter on a set $I$ and it is $|I|$-regular.

**Claim 1.14** *Assume $\mathfrak{B}$ is a complete Boolean algebra which is $(< \lambda)$-distributive and $D$ a filter on $\mathfrak{B}$ and $\vartheta = \mathrm{cf}(\vartheta) \leq \lambda$.*

[2] For the $D_\ell \in \mathrm{uf}(\mathfrak{B}_\ell)$ ultra-product, without loss of generality $\mathfrak{B}$ is complete, then without loss of generality $f\restriction\{a_i : i < i(*)\}$ is one to one. But in general we allow $a_i = 0_{\mathfrak{B}}$, those are redundant but natural in Definition 1.12(3).

 

1. *Assume $D$ is a $\vartheta$-complete ultrafilter. The parallel of Łoś theorem holds for $\mathbb{L}_{\vartheta,\vartheta}$ and if $D$ is $\lambda$-complete even for $\mathbb{L}_{\lambda,\vartheta}$ which means: if $\bar{M} = \langle M_b : b \in \mathscr{I}\rangle$ is a sequence of $\tau$-models, $\mathscr{I}$ is a maximal antichain of the complete Boolean algebra $\mathfrak{B}$ and $\varepsilon < \vartheta$, $\varphi = \varphi(\bar{x}_{[\varepsilon]}) \in \mathbb{L}_{\lambda,\vartheta}(\tau)$ and $f_\zeta \in \bar{M}^{\mathfrak{B}}$ for $\zeta < \varepsilon$ then $M^{\mathfrak{B}}/D \models$ "$\varphi[\langle f_\zeta/D : \zeta < \varepsilon\rangle]$" iff $\mathrm{TV}_M(\varphi[\langle f_\zeta/D : \zeta < \varepsilon\rangle])$ belongs to $D$.*
2. *If in addition $D$ is $(\lambda, \vartheta)$-regular and $M, N$ are $\mathbb{L}_{\vartheta,\vartheta}$-equivalent then $M^{\mathfrak{B}}/D, N^{\mathfrak{B}}/D$ are $\mathbb{L}_{\lambda^+,\vartheta}$-equivalent.*

**Definition 1.15** 1. Assume $\Delta_\ell$ is a of set atomic formulas in $\mathbb{L}(\tau(T_\ell))$. Then we say $(T_1, \Delta_1) \trianglelefteq^{\mathrm{rp}}_{\lambda,\vartheta} (T_2, \Delta_2)$ when: if $D$ is a $(\lambda, \vartheta)$-regular filter on $\lambda$ and $M_\ell$ is a $\lambda^+$-saturated model of $T_\ell$ for $\ell = 1, 2$ and $M_2^\lambda/D$ is $(\lambda^+, \vartheta, \Delta_2)$-saturated then $M_1^\lambda/D$ is $(\lambda^+, \vartheta, \Delta_1)$-saturated.

2. For general $\Delta_1, \Delta_2$ we define $(T_1, \Delta_1) \trianglelefteq^{\mathrm{rp}}_{\lambda,\vartheta} (T_2, \Delta_2)$ as meaning $(T_1^+, \Delta_1^+) \trianglelefteq^{\mathrm{rp}}_{\lambda,\vartheta} (T_2^+, \Delta_2^+)$ where (as Morley [13] does):
   (a) $T_\ell^+ = T_\ell \cup \{(\forall \bar{x})(\varphi(\bar{x}) \equiv P_\varphi(\bar{x})) : \varphi(\bar{x}) \in \Delta_\ell\}$ with $\langle P^\ell_\varphi : \varphi \in \Delta_\ell\rangle$ new pairwise distinct predicates with suitable number of places;
   (b) $\Delta_\ell^+ = \{P^\ell_\varphi(\bar{x}_\varphi) : \varphi \in \Delta_\ell\}$.
3. In (2), $T_1 \trianglelefteq^{\mathrm{rp}}_{\lambda,\vartheta} T_2$ means $\Delta_\ell =$ the set of atomic $\mathbb{L}_{\vartheta,\vartheta}(\tau_{T_\ell})$-formulas.

**Observation 1.16** *Assume $\Delta \subseteq \mathbb{L}(\tau_T)$ is closed under $\exists$ and $\wedge$. A model $M$ of $T$ is $(\mu^+, \mu^+, \Delta)$-saturated iff it is $(\mu^+, 1, \Delta)$-saturated.*

**Question 1.17** 1. Under $\trianglelefteq_{\mathrm{rp}}$ characterize the minimal/maximal pairs $(T, \Delta)$
2. What about the parallel of $\trianglelefteq^{**}$ (cf. [11, 23])?

## 2 Axiomatizing [22, Ch. VI, 2.6]

Note that while the notation $\mathfrak{t}(\mathscr{T})$ is obviously natural the notation $\mathfrak{p}(\mathscr{T})$ is really justified just by the results here.

**Definition 2.1** 1. For a partial order $\mathscr{T} = (\mathscr{T}, \leq_{\mathscr{T}})$ let $\mathfrak{p}_{\mathscr{T}} = \mathfrak{p}(\mathscr{T})$ be $\min\{\kappa_1 + \kappa_2 : (\kappa_1, \kappa_2) \in \mathscr{C}_{\mathscr{T}}\}$ and $\mathfrak{p}_\vartheta(\mathscr{T}) = \min\{\kappa_1 + \kappa_2 : (\kappa_1, \kappa_2) \in \mathscr{C}_{\mathscr{T},\vartheta}\}$; where:
2. $\mathscr{C}_\vartheta(\mathscr{T}) = \{(\kappa_1, \kappa_2)$: the partial order $\mathscr{T}$ has a $(\kappa_1, \kappa_2)$-cut and $\kappa_1 \geq \vartheta, \kappa_2 \geq \aleph_0\}$. If $\vartheta = \aleph_0$ then we may omit $\vartheta$, (yes, when $\vartheta > \aleph_0$ this is not symmetric).
3. For a partial order $\mathscr{T}$ let $\mathfrak{t}_{\mathscr{T}} = \mathfrak{t}(\mathscr{T})$ be the minimal $\kappa \geq \aleph_0$ such that there is a $<_{\mathscr{T}}$-increasing sequence of length $\kappa$ with no $<_{\mathscr{T}}$-upper bound.
4. Let $\mathfrak{p}^*_{\mathscr{T}} = \mathfrak{p}^*(\mathscr{T})$ be $\min\{\mathfrak{t}_{\mathscr{T}}, \mathfrak{p}_{\mathscr{T}}\}$.
5. $\mathfrak{p}_{\vartheta-\mathrm{sym}}(\mathscr{T}) = \min\{\kappa : (\kappa, \kappa) \in \mathscr{C}_\vartheta(\mathscr{T})\}$. and if $\vartheta = \aleph_0$ we may write $\mathfrak{p}^*_{\mathrm{sym}}(\mathscr{T})$
6. In Definition 2.2 below let $\mathfrak{t}_{\mathbf{r}} = \mathfrak{t}_{\mathscr{T}_{\mathbf{r}}}, \mathfrak{p}_{\mathbf{r}} = \mathfrak{p}_{\vartheta_{\mathbf{r}}}(\mathscr{T}_{\mathbf{r}})$.

**Definition 2.2** For $\iota = 1, 2$ (the difference is only in closed (i)), we say $\mathbf{r}$ or $(M, \Delta)$ is a $(\vartheta, \iota)$-realization[3] spectrum problem, in short $(\vartheta, \iota)$-RSP or $(\vartheta, \iota) - 1$-RSP when $\mathbf{r}$ consists of (if $\iota = 2$ we may omit it, similarly if $\vartheta = \aleph_0$; we may omit $\Delta$ and write $M$ when $\Delta$ is the set of atomic formulas in $\mathbb{L}(\tau_{N_M})$, see below, so $M$ below $= M_{\mathbf{r}}$, etc.):

(a) $M$ a model;
(b) for the relations $\mathscr{T} = \mathscr{T}^M, \leq_{\mathscr{T}} = \leq^M_{\mathscr{T}}$ of $M$ (i.e., $\mathscr{T}, \leq_{\mathscr{T}}$ are predicates from $\tau_M$) we have $\mathscr{T} = (\mathscr{T}, \leq_{\mathscr{T}})$ a partial order (so definable in $M$) with root $c^M = \mathrm{rt}(\mathscr{T})$, so $c \in \tau_M$ is an individual constant and $t \in \mathscr{T} \Longrightarrow \mathrm{rt}(\mathscr{T}) \leq_{\mathscr{T}} t$; as in other cases we may write $\mathscr{T}_{\mathbf{r}}, \leq_{\mathbf{r}}$ for $\mathscr{T}, \leq_{\mathscr{T}}$; we do not require $\mathscr{T}$ to be a tree; but do require $t \in \mathscr{T} \Longrightarrow t \leq_{\mathscr{T}} t$;
(c) a model $N = N_{\mathbf{r}} = N_M$ with universe $P^M$, $\tau(N) \subseteq \tau(M)$ such that
   i. $Q \in \tau_N \Longrightarrow Q^M = Q^N$;
   ii. $F \in \tau_N \Longrightarrow F^N = F^M$, (we understand $F^M, F^N$ to be partial functions), so every $\varphi \in \mathbb{L}(\tau_N)$ can be interpreted as $\varphi^{[*]} \in \mathbb{L}(\tau_M)$, all variables varying on $P$ (include quantification); we may forget the $[*]$.

[3] When $P$ and $\tau_N$ (hence $N$) are understood from the context we may omit them

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(d) the cardinal $\vartheta$ and $\Delta \subseteq \{\varphi : \varphi = \varphi(x, \bar{y}) \in \mathbb{L}_{\vartheta,\vartheta}(\tau_N)\}$ which is closed under conjunctions meaning: if $\varphi_\ell(x, \bar{y}_\ell) \in \Delta$ for $\ell = 1, 2$ then $\varphi(x, \bar{y}'_1, \bar{y}''_2) = \varphi_1(x, \bar{y}'_1) \wedge \varphi_2(\bar{x}, \bar{y}''_2) \in \Delta$;

(e) $R^M \subseteq |N| \times \mathscr{T}^M$ so a two-place relation; and let $R^M_t = \{b : bR^M t\}$ for $t \in \mathscr{T}^M$;

(f) $|N| \times \{\mathrm{rt}_{\mathscr{T}}\} \subseteq R^M$, i.e., $R^M_{\mathrm{rt}(\mathscr{T})} = |N|$;

(g) if $s \leq_{\mathscr{T}} t$ then $a \in N \wedge aRt \Longrightarrow aRs$, i.e., $R^M_s \supseteq R^M_t$;

(h) $t \in \mathscr{T} \Longrightarrow R^M_t \neq \varnothing$;

(i) if $s \in \mathscr{T}, \varphi(x, \bar{a}) \in \Delta(N) := \{\varphi(x, \bar{a})) : \varphi(x, \bar{y}) \in \Delta$ and $\bar{a} \in {}^{\ell g(\bar{y})}N\}$ and for some $b \in R^M_s, N \models \varphi[b, \bar{a}]$ then there is $t \in \mathscr{T}$ such that $s \leq_{\mathscr{T}} t$ and $R^M_t = \{b \in R^M_s : N \models \varphi[b, \bar{a}]\}$;

(i)$^+$ if $\iota = 1$ like clause $(i)$ but[4] moreover $t = F^M_{\varphi,1}(s, \bar{a})$ where $F^M_{\varphi,1} : \mathscr{T}_{\mathbf{r}} \times {}^{\ell g(\bar{y})}(P^M) \to \mathscr{T}_{\mathbf{r}}$;

(j) if $t \in \mathscr{T}_{\mathbf{r}}$ and $\varphi(x, \bar{a}) \in \Delta(N)$ and $\varphi(N, \bar{a}) \neq \varnothing$ then

($\alpha$) $s = F^M_{\varphi,2}(t, \bar{a})$ is such that $R^M_s \cap \varphi(N, \bar{a}) \neq \varnothing$ and $s \leq_{\mathscr{T}} t$;

($\beta$) if $s = F^M_{\varphi,2}(t, \bar{a}), s_1 \leq_{\mathscr{T}} t$ and $R^M_{s_1} \cap \varphi(N, \bar{a}) \neq \varnothing$ then $s_1 \leq_{\mathscr{T}} s$.

(k) if $\vartheta > \aleph_0$ then in $(\mathscr{T}, \leq_{\mathscr{T}})$ any increasing chain of length $< \vartheta$ which has an upper bound has a $\leq_{\mathscr{T}}$-lub.

**Remark 2.3** We may consider adding: $S^M$ a being successor, (but this is not Horn), i.e.:

(l) if $\iota = 1$ we also have $S^M = \{(a, b) : B$ is a $\leq_{\mathscr{T}}$-successor of $a$ which means

($\alpha$) if $a \leq b \wedge a \neq b$ then for some $c$, $S(a, c) \wedge c \leq b$;

($\beta$) if $b \in \mathscr{T} \backslash \{\mathrm{rt}_{\mathscr{T}}\}$ then for some unique $a$ we have $S^M(a, b)$;

($\gamma$) $S(a, b) \Longrightarrow a \leq b$;

($\delta$) $S(a, b_1) \wedge S(a, b_2) \wedge b_1 \neq b_2 \Longrightarrow \neg(b_1 \leq b_2)$;

($\varepsilon$) in clause (j) we can add $S^M(s, t)$.}

**Remark 2.4** Presently, it may be that $a \leq_{\mathscr{T}} b \leq_{\mathscr{T}} a$ but $a \neq b$. Not a disaster to forbid but no reason.

How does this axiomatize realizations of types?

**Claim/Definition 2.5** Let $\iota = \{1, 2\}$, $\vartheta$ is $\aleph_0$ or just a regular cardinal.

1. For any model $N$ and $\Delta \subseteq \{\varphi : \varphi = \varphi(x, \bar{y}) \in \mathbb{L}_{\vartheta,\vartheta}(\tau_T)\}$ closed under conjunctions of $< \vartheta$, the canonical $(\vartheta, \iota)$-RSP, $\mathbf{r} = \mathbf{r}^\vartheta_{N,\Delta}$ defined below is indeed a $\vartheta$-RSP.
2. $\mathbf{r} = \mathbf{r}^\vartheta_{N,\Delta}$ (if $\vartheta = \aleph_0$ we may omit it) is defined by:
   (a) $\Delta_{\mathbf{r}} = \Delta$, $N_{\mathbf{r}} = N$ and $\vartheta_{\mathbf{r}} = \vartheta$;
   (b) $\mathscr{T}_{\mathbf{r}} = \{\langle \varphi_\varepsilon(x, \bar{a}_\varepsilon) : \varepsilon < \zeta \rangle : \zeta < \vartheta$ and for every $\varepsilon < \zeta$ we have $\varphi_\varepsilon(x, \bar{a}_\varepsilon) \in \Delta(N)$ and $N \models (\exists x)(\bigwedge_{\varepsilon<\zeta} \varphi_\varepsilon(x, \bar{a}_\varepsilon))\}$;
   (c) $\leq_{\mathbf{r}}$= being the initial segment relation on $\mathscr{T}_{\mathbf{r}}$;
   (d) $M = M_{\mathbf{r}}$ is the model with universe $\mathscr{T}_{\mathbf{r}} \cup |N|$; without loss of generality $\mathscr{T}_{\mathbf{r}} \cap |N| = \varnothing$, with the relations and functions of $N$, $\mathscr{T}_r$, $\leq_{\mathbf{r}}$ and
   1. $P^M = |N|$;
   2. $c^M = \langle\rangle \in \mathscr{T}_{\mathbf{r}}$;
   3. $R^M = \{(b, t) : b \in N, t = \langle \varphi_{t,\varepsilon}(x, \bar{a}_{t,\varepsilon}) : \varepsilon < \zeta_t \rangle \in \mathscr{T}$ and $N \models \varphi_{t,\ell}(b, \bar{a}_{t,\varepsilon})$ for every $\varepsilon < \zeta_t\}$;
   4. $F^M_{\varphi,2}$ as in Definition 2.2(j);
   5. if $\iota = 1$ then $F^M_{\varphi,1}$ is as in Definition 2.2(i)$^+$.

**Remark 2.6** If we adopt Remark 2.3 it is natural to add:

(e) for $\iota = 1$, $S^M = \{(\bar{\varphi}_1, \bar{\varphi}_2) : \bar{\varphi}_2 = \bar{\varphi}_1 \langle \varphi(x, \bar{a}) \rangle \in \mathscr{T}_{\mathbf{r}}$ for some $\varphi(x, \bar{a}) \in \Delta(N)\}$.

*Proof.* Obvious. □

[4] We may not add a function, maybe it matters when we try to build $\mathbf{r}$ with $\mathrm{Th}(M_{\mathbf{r}})$ nice first order

**Main Claim 2.7** *1. Assume* $\mathbf{r}$ *is an* RSP. *If* $\kappa = \min\{\mathfrak{t}_\mathbf{r}, \mathfrak{p}_\mathbf{r}\}$ *then the model* $N$ *is* $(\kappa, 1, \Delta_\mathbf{r})$*-saturated, i.e.,*

$\oplus$ *if* $p(x) \subseteq \Delta_\mathbf{r}(N_\mathbf{r})$ *is finitely satisfiable in* $N_\mathbf{r}$ *(= is a type in* $N_\mathbf{r}$*) of cardinality* $< \kappa$ *then* $p$ *is realized in* $N_\mathbf{r}$.

2. *If* $\vartheta > \aleph_0$ *and* $\mathbf{r}$ *is a* $\vartheta$*-RSP, then* $N_\mathbf{r}$ *is* $(\kappa, 1, \Delta_\mathbf{r})$*-saturated where* $\kappa = \min\{\mathfrak{t}_\mathbf{r}, \mathfrak{p}_\mathbf{r}\}$ *recalling Definition 2.1(6), i.e.,* $\mathfrak{p}_\mathbf{r} = \mathfrak{p}_{\mathscr{T}_\mathbf{r},\vartheta}$.

3. *If* $\vartheta > \aleph_0$, $\mathbf{r}$ *is a* $\vartheta$*-RSP satisfying* $(k)^+$ *below then* $N_\mathbf{r}$ *is* $(\mathfrak{t}_\mathbf{r}, 1, \Delta_\mathbf{r})$*-saturated when:*

$(k)^+$ *in* $(\mathscr{T}, \leq_\mathscr{T})$ *any increasing chain which has an upper bound, has a* $\leq_\mathscr{T}$*-lub.*

Proof. This is an abstract version of [21, Ch. VI, 2.6] = [22, Ch. VI, 2.6]; recall that [21, Ch. VI, 2.7] translates trees to linear orders.

1. Let $N = N_\mathbf{r}$, $\Delta = \Delta_\mathbf{r}$, etc.

Let $p$ be a $(\Delta, 1)$-type in $N$ of cardinality $< \kappa$. Without loss of generality $p$ is infinite and closed under conjunctions.

So let

$(*)_1$ $\alpha_* < \kappa$, $p = \{\varphi_\alpha(x, \bar{a}_\alpha) : \alpha < \alpha_*\} \subseteq \Delta(N)$, $p$ is finitely satisfiable in $N$.

We shall try to choose $t_\alpha$ by induction on $\alpha \leq \alpha_*$ such that

$(*)_2$ (a) $t_\alpha \in \mathscr{T}$ and $\beta < \alpha \Longrightarrow t_\beta \leq_\mathscr{T} t_\alpha$

(b) if $\beta < \alpha_*$ then there is $b \in R^M_{t_\alpha}$ such that $N \models \varphi_\beta[b, \bar{a}_\beta]$

(c) if $\beta < \alpha$ then $b \in R^M_{t_\alpha} \Longrightarrow N \models \varphi_\beta[b, \bar{a}_\alpha]$.

If we succeed, this is enough because if $t = t_{\alpha_*}$ is well defined then $R^M_t \neq \varnothing$ by Definition 2.2(h) and any $b \in R^M_t$ realizes the type by $(*)_2(c)$ and Definition 2.2(h). Why can we carry the definition?

Case 1: $\alpha = 0$.

Let $t_\alpha = \mathrm{rt}_\mathscr{T}$, hence $R^M_{t_\alpha} = |N|$ by Definition 2.2(f). Now clause (a) of $(*)_2$ holds as $t_\alpha \in \mathscr{T}_\mathbf{r}$ and there is no $\beta < \alpha$. Also clause (b) of $(*)_2$ holds because $p$ is a type and $R^M_{\mathrm{rt}(\mathscr{T})} = |N_\mathbf{r}|$ by Definition 2.2(h).

Lastly, clause (c) of $(*)_2$ holds trivially.

Case 2: $\alpha = \beta + 1$.

If $\iota = 1$ let $t = F^M_{\varphi_\beta,1}(t_\beta, \bar{a}_\beta)$ and see clause (i)$^+$ of Definition 2.2. If $\iota = 2$ use clause (i) of the definition recalling $p$ is closed under conjunctions.

Case 3: $\alpha$ a limit ordinal.

As $\mathfrak{t}_{\mathscr{T}_\mathbf{r}} \geq \kappa > \alpha_*$ by the claim's assumption (on $\mathfrak{t}_{\mathscr{T}_\mathbf{r}}$, cf. Definition 2.1(2)) necessarily there is $s \in \mathscr{T}$ such that $\beta < \alpha \Longrightarrow t_\alpha \leq_\mathscr{T} s$. We now try to choose $s_i$ by induction on $i \leq \alpha_*$ such that

$(*)_{2.1}$ (a) $s_i \in \mathscr{T}$;

(b) $\beta < \alpha \Longrightarrow t_\beta \leq_\mathscr{T} s_i$;

(c) $j < i \Longrightarrow s_i \leq_\mathscr{T} s_j$;

(d) if $i = j + 1$ then $R^M_{s_i}$ is not disjoint to $\varphi_j(N, \bar{a}_j)$.

If we succeed, then $s_{\alpha_*}$ satisfies all the demands on $t_\alpha$ (e.g., $(*)_2(b)$ holds by Definition 2.2(g) and $(*)_{2.1}(d)$), so we have just to carry the induction for $\alpha$. Now if $i = 0$ clearly $s_0 = s$ it as required. If $i = j + 1$ let $s_i = F^M_{\varphi,2}(s_j, \bar{a}_j)$, by Definition 2.2(j) it is as required. For $i$ a limit ordinal use $\kappa \leq \mathfrak{p}_\mathscr{T}$ hence to carry the induction on $i$ so finish case 3.

So we succeed to carry the induction on $\alpha$ hence (as said after $(*)_2$) get the desired conclusion.

2. Similar, except concerning case 3. Note that without loss of generality $\vartheta > \aleph_0$ by part (1).

Case 3A: $\alpha$ is a limit ordinal of cofinality $\geq \vartheta$.

As in the proof of part (1).

Case 3B: $\alpha$ is a limit ordinal of cofinality $< \vartheta$.

Again there is an upper bound $s$ of $\{t_\beta : \beta < \alpha\}$. Now by clause (k) of Definition 2.2, without loss of generality $s$ is a $<_{\mathscr{T}}$-lub of $\{t_\beta : \beta < \alpha\}$. So easily for every $i < \alpha_*$, $F^N_{\varphi_i,2}(s, \bar{a}_i)$ is $\geq t_\beta$ for $\beta < \alpha$ hence is equal to $s$, so $s_\alpha := s$ is as required.

3. Similarly. □

**Discussion 2.8** 1. What about "$(\lambda^+, n, \Delta)$-saturation"? We can repeat the same analysis or we can change the models to code $n$-tuples. More generally, replacing $\varphi(\bar{x}_{[\varepsilon]}, \bar{y})$ by $\varphi(\langle F_\zeta(x) : \zeta < \varepsilon\rangle, \bar{y})$, using $F_\zeta \in \tau_M$ (though not necessarily $F_\zeta \in \tau_{N_{\mathbf{r}}}$), so we can allow infinite $\varepsilon$.

2. Hence the same is true for $(\lambda^+, \aleph_0, \Delta)$-saturation, e.g., $\lambda^+$-saturated by an assumption.

## 3 Applying the axiomatized frame

Consider a filter $D$ on a set $I$ and cardinals $\lambda \geq \mu$. We may ask for a model $M$ of cardinality $\geq \mu$, whether $M^I/D$ is $(\lambda^+$, atomic)-saturated, varying $M$.

We here apply § 1 to show that: when $D$ is an ultrafilter, the model $({}^{\omega>}\mu, \lhd)$ is the hardest, this is 2.1, we then (in 2.2) show that § 1 has axiomatization which is Horn theory. Hence we can prove results like Conclusion 3.1 below for filters $D$ (not just for ultrafilters),

**Conclusion 3.1** *1. If $D$ is an ultrafilter on a set $I, N$ a model, $\mu = \|N\| + \|\tau_N\|$ and $({}^{\omega>}\mu, \lhd)^I/D$ is $(\lambda^+$,atomic)-saturated then $N^I/D$ is $\lambda^+$-saturated.*

*2. Instead of "$({}^{\omega>}\mu, \unlhd)^I/D$ is $(\lambda^+, 1$, atomic)-saturated" we can demand "$J^I/D$ is $(\lambda^+, 1$,atomic)-saturated" where $J$ is the linear order with set of elements $\{-1, 1\} \times {}^{\omega>}\mu$ ordered by $(\iota_1, \eta_1) < (\iota_2, \eta_2)$ iff $\iota_1 < \iota_2$ or $\iota_1 = -1 = \iota_2 \wedge \eta_1 <_{\text{lex}} \eta_2$ or $\iota_1 = -1 = \iota_2 \wedge \eta_2 <_{\text{lex}} \eta_1$.*

P r o o f . 1. Let $N_1 = N$. As $D$ is an ultrafilter without loss of generality, $\text{Th}(N_1)$ has elimination of quantifiers and even every formula is equivalent to an atomic formula. Let $\Delta = \mathbb{L}(\tau_N)$, by Claim/Definition 2.5 $\mathbf{r}_1 := \mathbf{r}_{N_1,\Delta}$ is an RSP. Let $N_2 = N_1^I/D$ and let $M_1 = M_{\mathbf{r}_1}$, $M_2 = M_1^I/D$ and let $\mathbf{r}_2$ be the RSP$(M_2, \Delta)$. Clearly $\mathbf{r}_2$ is an RSP as the demands in Definition 2.2 are first order (see more in Claim 3.2).

Now

$(*)_1$ $\mathscr{T}_{\mathbf{r}_1} \cong ({}^{\omega>}\mu, \lhd)$.

[Why? Cf. Claim/Definition 2.5(2).]

$(*)_2$ $\mathscr{T}_{\mathbf{r}_2} = (\mathscr{T}_{\mathbf{r}_1})^I/D$ is $(\lambda^+$,atomic)-saturated.

[Why? By an assumption.]

$(*)_3$ $\mathfrak{t}(\mathscr{T}_{\mathbf{r}_1}), \mathfrak{p}(\mathscr{T}_{r_2}) \geq \lambda^+$.

[Why? Follows by $(*)_2$.]

Hence by the Main Claim 2.7, $N_2$ is $(\lambda^+, 1, 1, \Delta)$-saturated which means $N_2 = (N_1)^I/D$ is $\lambda^+$-saturated.

2. Easy (or cf. [21, Ch. VI, 2.7], or cf. [20]). □

To apply the criterion of the Main Claim 2.7 to reduced products we need:

**Claim 3.2** *If $\Delta$ is the set of conjunctions of atomic formulas (no negation!) in $\mathbb{L}(\tau_0)$ and $\tau = \{\mathscr{T}, \leq_{\mathscr{T}}, R, P, c\} \cup \{F_{\varphi,\ell} : \varphi \in \Delta$ and $\ell = 2$ or $\ell = 1$ if relevant$\} \cup \tau_0$ (disjoint union, recall $c$ is $\text{rt}_{\mathscr{T}}$), then there is a set $T$ of Horn sentences from $\mathbb{L}(\tau)$ such that for every $\tau$-model $M$: $(M, \Delta)$ is a* RSP *(i.e., 2-RSP) iff $M \models T$.*

P r o o f . Consider Definition 2.2. For each clause we consider the sentences expressing the demands there.
Clause (a): Obvious.
Clause (b): Clearly the following are Horn:

1. $x \leq_{\mathscr{T}} y \to \mathscr{T}(x), x \leq_{\mathscr{T}} y \to \mathscr{T}(y)$,
2. $x \leq_{\mathscr{T}} y \wedge y \leq_{\mathscr{T}} z \to x \leq_{\mathscr{T}} z$,
3. $\mathscr{T}(\text{rt}_{\mathscr{T}})$ and $\mathscr{T}(s) \to \text{rt}_{\mathscr{T}} \leq s$,

 

4. $\mathscr{T}(x) \rightarrow x \leq_{\mathscr{T}} x$.

Note that $(\mathscr{T}, \leq_{\mathscr{T}})$ being a tree is not a Horn sentence but is not required.

Clause (c):

1. $Q(x_0, \ldots, x_{n(Q)-1}) \rightarrow P(x_\ell)$ when $Q$ is an $n(Q)$-place predicate from $\tau(N)$ and $\ell < n(Q)$; clearly it is Horn;
2. for any $n$-place function symbol $F \in \tau_0$ the sentence: $P(x_0) \wedge \ldots \wedge P(x_{n-1}) \rightarrow P(F(x_0, \ldots, x_{n-1}))$ and $y = F(x_0, \ldots, x_{n-1}) \rightarrow P(x_\ell)$.

Clause (d): nothing to prove—see the present claim assumption on $\Delta$.

Recall that for $F \in \tau_N$, $F$ stand for a partial function symbol with domain $P_F$.

Clause (e): $yRs \rightarrow \mathscr{T}(s)$, $yRs \rightarrow P(y)$ are Horn.

Clause (f): $P(x) \rightarrow xR(\mathrm{rt}_{\mathscr{T}})$ is Horn.

Clause (g): $s \leq_{\mathscr{T}} t \wedge xRt \rightarrow xRs$ is Horn.

Clause (h): $(\forall t)(\exists x)(\mathscr{T}(t) \rightarrow xRt)$ is Horn.

Clause (i): Let $\varphi(x, \bar{y}) \in \Delta$.

First assume $\iota = 1$. Note the following are Horn: for any $\varphi(x, \bar{y}) \in \Delta$

1. $\mathscr{T}(s) \wedge xRs \wedge \varphi(x, \bar{y}) \wedge \bigwedge_{\ell < \ell g(\bar{y})} P(y_\ell) \wedge t = F_{\varphi,1}(s, \bar{y}) \rightarrow \mathscr{T}(t) \wedge s \leq_{\mathscr{T}} t$;
2. $\mathscr{T}(s) \wedge xRs \wedge \varphi(x, \bar{y}) \wedge \bigwedge_{\ell < \ell g(\bar{y})} P(y_\ell) \wedge t = F_{\varphi,1}(s, \bar{y}) \rightarrow xRt$;
3. $\mathscr{T}(s) \wedge x'Rs \wedge x'RF_{\varphi,1}(s, \bar{y}) \rightarrow \varphi(x', \bar{y})$.

This suffices. The proof when $\iota = 2$ is similar.

Clause (j): Similarly but we give details.

Let $\varphi = \varphi(x, \bar{y}) \in \Delta$, so the following are Horn:

1. $\varphi(x_1, \bar{y}) \wedge P(x_1) \wedge \bigwedge_{\ell < \ell g(\bar{y})} P(y_\ell) \wedge s = F_{\varphi,2}(t, \bar{y}) \rightarrow s \leq_{\mathscr{T}} t$;
2. $\varphi(x_1, \bar{y}) \wedge P(x_1) \wedge \bigwedge_{\ell < \ell g(\bar{y})} P(y_\ell) \wedge s = F_{\varphi,2}(t, \bar{y}) \rightarrow (\exists x)(xRs \wedge \varphi(x, \bar{y}))$;
3. $P(x) \wedge \bigwedge_{\ell < \ell g(\bar{y})} P(y_\ell) \wedge s = F_{\varphi,2}(t, \bar{y}) \wedge z \leq_{\mathscr{T}} t \wedge xRz \wedge \varphi(x, \bar{y}) \rightarrow z \leq_{\mathscr{T}} s$.

Clause (k): As $\vartheta = \aleph_0$ this is empty.

This suffices. □

**Claim 3.3** *Also for $\vartheta > \aleph_0$ (cf. Definition 2.2(2)) Claim 3.2 holds but some of the formulas are in $\mathbb{L}_{\vartheta,\vartheta}$.*

Proof. Clause ($k$): When $\vartheta > \aleph_0$.

Should be clear because for each limit ordinal $\delta < \kappa$, the sentence

$$\psi_\delta = (\forall x_0, \ldots, x_\alpha, \ldots, x_\delta)\big(\exists y\big)\big(\forall z\big) \ \left(\left(\bigwedge_{\alpha < \beta < \delta} x_\alpha \leq_{\mathscr{T}} x_B \leq_{\mathscr{T}} y \leq_{\mathscr{T}} x_\delta\right) \wedge \left(\bigwedge_{\alpha < \beta < \delta} x_\alpha \leq_{\mathscr{T}} x_\beta \leq_{\mathscr{T}} z \leq_{\mathscr{T}} y \leq_{\mathscr{T}} x_\delta \rightarrow y = z\right)\right)$$

is a Horn sentence and it expresses "any $\leq_{\mathscr{T}}$-increasing chain of length $\delta$ has a $\leq$-lub". □

**Conclusion 3.4** *1. Assume*

*(a) $D$ be a filter on $I$,*

*(b) $N$ a model, $\lambda = \|N\| + |\tau_N|$, $\Delta$ the set of atomic formulas (in $\mathbb{L}(\tau_N)$),*

*(c) $\mathscr{T} = (\mathscr{T}, \leq_{\mathscr{T}}) := ({}^{\omega >}\lambda, \trianglelefteq)^I/D$,*

*(d) $\kappa = \mathfrak{p}^*_{\mathscr{T}} = \min\{\mathfrak{t}_{\mathscr{T}}, \mathfrak{p}_\vartheta(\mathscr{T}_1)\}$; cf. Definition 2.1(6).*

*Then the reduced power $N^I/D$ is $(\kappa, 1, \Delta)$-saturated.*

2. *Assume*[5]
   (a) *$D$ is a $\vartheta$-complete filter on $I$, $\vartheta = \mathrm{cf}(\vartheta) > \aleph_0$,*
   (b) *$N$ is $(\vartheta, \Delta)$-saturated, $\Delta$ a set of atomic formulas,*
   (c) *$\mathscr{T}_1 := ({}^{\vartheta >}\lambda, \trianglelefteq)^I/D$,*
   (d) *$\kappa = \min\{\mathfrak{t}_{\mathscr{T}_1}, \mathfrak{p}_\vartheta(\mathscr{T}_1)\}$.*
   *Then $N^I/D$ is $(\kappa, \vartheta, 1, \Delta)$-saturated.*
3. *We can above replace $N^I/D$ by $N^{\mathfrak{B}}/D$ where $D$ is a filter on the complete Boolean algebra $\mathfrak{B}$ which has $(< \vartheta)$-distributivity when $\vartheta > \aleph_0$.*

P r o o f. 1. Let $\vartheta = \aleph_0$ and $\mathbf{r}_0 = (M_0, \Delta)$ be $\mathbf{r}^\vartheta_{N,\Delta}$ from Claim/Definition 2.5, so $\vartheta_{\mathbf{r}_0} = \vartheta$.
By Claim/Definition 2.5, $M_0$ is an RSP hence by Claim 3.2 also $M = M_0^I/D$ is an RSP. Now apply the Main Claim 2.7(1).
2. Similarly using the Main Claim 2.7(2).
3. Similarly. □

**Remark 3.5** 1. No harm in assuming $\Delta = \{Q(\bar{y}) : Q$ a predicate$\}$. Note that allowing bigger $\Delta$ is problematic except in trivial cases ($\varphi$ and $\neg\varphi$ are equivalent to Horn formulas), see proof of clauses (i), (j) of Definition 2.2.
2. Using Conclusion 3.4(1) above, if $D$ is an ultrafilter, not surprisingly we get [22, Ch. VI, 2.6], i.e., the theory of dense linear orders is $\trianglelefteq$-maximal (well, using the translation from dense linear orders to trees in Conclusion 3.1(2) equivalently [22, Ch. VI, 2.7]). The new point here is that Conclusion 3.4 does this also for reduced powers, i.e., for $D$ a filter.
3. So a natural question is: can we replace the strict property by $\mathrm{SOP}_2$? We shall show that for reduced power we have also non-peculiar cuts, see § 4.
4. Why is the reduced power of a tree not necessarily a tree? Let $M$ be the tree $({}^{\omega>}\omega, \triangleleft)$. Let $\eta_1 \triangleleft \eta_2 \triangleleft \eta_3 \in {}^{\omega>}\omega$ and let $A_1, A_2 \in D^+$ be disjoint and define $f_\ell : I \to {}^{\omega>}\omega$ for $\ell = 1, 2, 3$ by:
   (a) $f_3(s) = \eta_3$ for $s \in I$;
   (b) $f_2(s)$ is $\eta_2$ if $s \in A_2$ and $\eta_0$ otherwise;
   (c) $f_1(s)$ is $\eta_2$ if $s \in A_1$ and $\eta_0$ otherwise.
   Clearly if $N = M^I/D$ then in $N$ we have:
   (a) $f_1/D \triangleleft f_3/D$;
   (b) $f_2/D \triangleleft f_3/D$;
   (c) $\neg(f_1/D \triangleleft f_2/D)$;
   (d) $\neg(f_2/D \triangleleft f_1/D)$;
   (e) $\neg(f_1/D = f_2/D)$.

**Conclusion 3.6** *$N^I/D$ is $(\kappa, 1, \Delta_1)$-saturated and $\kappa \geq \vartheta$ when:*

$(*)$ *(a) $D$ is a $\vartheta$-complete filter on $I$;*
   *(b) $\Delta \subseteq \{\varphi : \varphi(x, \bar{y}) \in \mathbb{L}_{\vartheta,\vartheta}(\tau_N)$ is atomic (hence $\in \mathbb{L}(\tau_N))\}$;*
   *(c) $\Delta_1 = c\ell_{<\vartheta}(\Delta) =$ the closure of $\Delta$ under conjunction of $< \vartheta$ formulas;*
   *(d) $N$ is $(\vartheta, \Delta)$-saturated, i.e., if $p(x) \subseteq \Delta(N) = \{\varphi(x, \bar{a}) : \varphi(x, \bar{y}) \in \Delta, \bar{a} \in {}^{\lg(\bar{y})}M\}$ has cardinality $< \vartheta$ and is finitely satisfiable in $N$ then $p$ is realized in $N$;*
   *(e) $\kappa = \min\{\mathfrak{p}_{\mathscr{T}}, \mathfrak{t}_\vartheta(\mathscr{T})\}$ where $\mathscr{T} = ({}^{\vartheta >}\lambda, \trianglelefteq)^I/D$ and $\lambda = {}^{\vartheta >}(\|N\| + |\Delta|)$.*

P r o o f. Let $\mathbf{r} = \mathbf{r}^\vartheta_{N,\Delta_1}$ recalling Claim/Definition 2.5 and $M_0 = M_{\mathbf{r}}$.
Now apply the Main Claim 2.7(2) noting that:

[5] Note that $\kappa$ here may be bigger than in part (1)

 

$(*)_1$ $N_1 = N_0^I/D$ satisfies: every set of $< \vartheta$ formulas from $\Delta(N)$ which is finitely satisfiable in $N_1$ is realized in $N_1$.

[Why? Let $\langle \varphi_\alpha(x, f_{\alpha,0}/D, \ldots, f_{\alpha,n(\alpha)-1}/D) : \alpha < \alpha_* \rangle$ be finitely satisfiable in $N_1$ and $\alpha_* < \vartheta, \alpha < \alpha_* \implies \varphi_\alpha \in \Delta$. For every finite $u \subseteq \alpha_*$ we have $N_1 \models (\exists x)(\bigwedge_{\alpha \in u} \varphi_\alpha(x, f_{\alpha,0}/D, \ldots, f_{\alpha,n(\alpha)-1}/D))$ hence the set

$$I_u := \{s \in I : N_1 \models (\exists x) \bigwedge_{\alpha \in u} \varphi_\alpha(x, f_{\alpha,0}(s), \ldots, f_{\alpha,n(\alpha)-1}(s))\}$$

belongs to $D$. But $D$ is $\vartheta$-complete, hence $I_* = \cap\{I_u : u \subseteq \alpha_* \text{ is finite}\}$ belongs to $D$. Now for each $s \in I_*$, the set $p_s := \{\varphi_\alpha(x, f_{\alpha,0}(s), \ldots, f_{\alpha,n(\alpha)-1}(s)) : \alpha < \alpha_*\}$ is finitely satisfiable in $N$, hence is realized by some $a_s \in N$. Let $g \in {}^I N$ be such that $s \in I_* \implies g(s) = a_s$; clearly $g/D$ realizes $p$, so we are done.]

Similarly

$(*)_2$ in $\mathscr{T} = ({}^{\vartheta >}\lambda, \trianglelefteq)^I/D$ we have

(a) every increasing sequence of length $< \vartheta$ has an upper bound;

(b) any increasing sequence of length $< \vartheta$ with an upper bound has a lub;

(c) there is no infinite decreasing sequence so $(\kappa_1, \kappa_2) \in \mathscr{C}(\mathscr{T}) \implies \kappa_2 = 1$.

[Why? For clause (a) note that $(\forall x_0, \ldots, x_\alpha, \ldots)_{\alpha<\delta}(\exists y)(\bigwedge_{\alpha<\beta<\delta} x_\alpha \leq_{\mathscr{T}} x_\beta \to \bigwedge_{\alpha<\delta} x_\alpha \leq_{\mathscr{T}} y)$ is a Horn sentence. For clause (b) cf Claim 3.3, i.e., proof of clause (k) in Claim 3.3.]

$(*)_3$ $M_1 = M_{\mathbf{r}}^I/D$ is a $\vartheta$-RSP.

[Why? See above recalling Claims 3.2 and 3.3.]

$(*)_4$ if $\vartheta \geq \aleph_1$ then $\mathbf{r}$ satisfies $(k)^+$ from the Main Claim 2.7(3).

[Why? Easily as $D$ is a $\aleph_1$-complete ultrafilter.]

So we are done by the Main Claim 2.7(3). □

It is natural to wonder

**Question 3.7** Assume $\lambda \geq \vartheta = \mathrm{cf}(\vartheta) > \aleph_0$.

1. Is there a $\vartheta$-complete $(\lambda, \vartheta)$-regular ultrafilter $D$ on $\lambda$ such that $\lambda < \mathfrak{t}(({}^{\vartheta >}\vartheta, \trianglelefteq)^\lambda/D)$?
2. Similarly for filters.
3. Use $\leq_{\mathscr{T}} = \trianglelefteq$ or $<_{\mathscr{T}} = \triangleleft$?
4. If $\lambda = \lambda^{<\vartheta}$, $D$ a fine normal ultrafilter on $I = [\lambda]^{<\vartheta}$, we get $\lambda \leq \mathfrak{t}({}^{\vartheta >}\vartheta, \trianglelefteq)/D$.

**Remark 3.8** Now [10, § 5] answers Question 3.7(1) positively for $\vartheta$ a supercompact cardinal.

**Conclusion 3.9** *Let $\mathfrak{B}$ be a complete Boolean algebra and $D$ a filter on $\mathfrak{B}$.*

1. *For every model $N$, letting $\lambda = \|N\| + |\tau_N|$, we have $N^{\mathfrak{B}}/D$ is $(\mu^+$,atomic)-saturated if*

$$\mu^+ \leq \min\{\mathfrak{p}(({}^{\omega >}\lambda, \trianglelefteq)^{\mathfrak{B}}/D)), \mathfrak{t}(({}^{\omega >}\lambda, \trianglelefteq)^{\mathfrak{B}}/D)\}.$$

2. *Assume $\mathfrak{B}$ is $(< \vartheta)$-distributive (e.g., for some dense $Y \subseteq \mathfrak{B}^+$, for every decreasing sequence*[6] *in $\mathfrak{B}$ of elements from $Y$ of length $< \vartheta$ has a positive lower bound), and $D$ is a $\vartheta$-complete filter on $\mathfrak{B}$. If $N$ is $(\mu^+$, atomic)-saturated then $N^{\mathfrak{B}}/D$ is $\mathfrak{t}(({}^{\vartheta >}\lambda, \trianglelefteq)^{\mathfrak{B}}/D)$-atomic saturated.*

Proof. As, e.g., in Conclusion 3.6 above or in Conclusion 3.13 below. □

**Conclusion 3.10** *Assume $(T, \varphi(\bar{x}, \bar{y}))$ has* SOP$_3$. *Then, recalling Definition 1.15, $T$ is $\trianglelefteq_\lambda^{\mathrm{rp}}$-maximal for every $\lambda$ and even $(T, \{\varphi(\bar{x}, \bar{y})\})$ is.*

Proof. Should be clear. □

On the connection to Peano arithmetic and to Pabion [14], cf. Malliaris and Shelah [11]. We repeat some results of [12] in the present context; but first recalling:

[6] One can weaken the demand.

 

**Definition 3.11** 1. PA, Peano arithmetic, is the f.o. theory consisting of:

(a) the obvious axioms on $0, 1, x < y, x + y, xy$

(b) all the cases of the induction scheme, i.e., for every f.o. $\varphi$:
"if $\{x : \varphi(x, \bar{y})\}$ is not empty then is has a first member",

2. BPA, the bounded Peano arithmetic, is defined similarly, but in clause (b), the formulas $\varphi$ is bounded, i.e., all the quantifications inside it are of the form $(\forall x < y)$ or $(\exists x < y)$.

**Definition 3.12** 1. $N \models$ BPA is boundedly $\kappa$-saturated up to $(c_1, c_2)$ where $c_1, c_2 \in N$ when: if $p(x) \cup \{x < c_1\}$ is a type in $N$ (= finitely satisfiable) of cardinality $< \kappa$ consisting of bounded formulas but with parameters $\leq c_2$, then $p(x) \cup \{x < c_1\}$ is realized in $N$.

2. If above $c_1 = c = c_2$ we may write $c$ instead of $(c_1, c_2)$. We say $N$ is strongly boundedly $\kappa$-saturated up to $c$ when it holds for $(c, c_2)$, $c_2 = \infty$, i.e., we do not bound the parameters.

3. Omitting "up to $c$" in part (3) means for every $c \in N$.

**Conclusion 3.13** *Assume $N$ be a model of* BPA.

1. *Assume $a_* \in N$ is non-standard and the power in the $N$-sense $c^{a_*}$ exists for every $c \in N$.*
*For any uncountable cardinal $\kappa$ the following conditions are equivalent:*
   - *(a) $N$ is boundedly $\kappa$-saturated up to $c$ for any $c \in N$*
   - *(b) if $(C_1, C_2)$ is a cut of $N$ of cofinality $(\kappa_1, \kappa_2)$ and $\kappa_1, \kappa_2$ are infinite (so $C_1, C_2 \neq 0$) then $\kappa_1 + \kappa_2 \geq \kappa$.*
   - *(c) like clause (b) but $\kappa_1 = \kappa_2$, that is restricting ourselves to symmetric cuts.*
2. *We can weaken the assumption of part (1) by fixing $c$, as well as $N$, $a_*$. That is, assume $N \models$ "$n < a_*$ and $c_n = c^{(a_*)^n}$ exist" for every standard $n$ from $N$. For every uncountable cardinal $\kappa$ the following are equivalent:*
   - *(a)′ $N$ is boundedly $\kappa$-saturated up to $c_n$ for each $n$*
   - *(b)′ if $(C_1, C_2)$ is a cut of $N$ of cofinality $(\kappa_1, \kappa_2)$ with $\kappa_1, \kappa_1$ infinite such that $c_n \in C_2$ for some $n$ then $\kappa_1 + \kappa_2 \geq \kappa$*
   - *(c)′ like clause (b)' but $\kappa_1 = \kappa_2$.*
3. *Moreover we can add in part (2):*
   - *(c) $N$ is strongly boundedly $\kappa$-saturated up to $c$.*

Proof. 1. By (2).

2. (a)′ $\Longrightarrow$ (b)′: Trivial.
(b)′ $\Longrightarrow$ (a)′: Without loss of generality $c$ is not standard (in $N$) and $n = 0$. Let $N^+ = (N, c, a_*)$ and $\tau^+ = \tau(N^+) = \tau(N) \cup \{c, a_*\}$ and $\Delta = \{\varphi(x, \bar{y}) \wedge x < c \wedge \bigwedge_\ell y_\ell < c : \varphi(x, \bar{y}) \in \mathbb{L}(\tau_N)$ is a bounded formula$\}$.
We define $\mathbf{r}$ naturally - the tree of sequences of length $< a_*$ of members of $\Delta(N_{\leq c})$ possibly non-standard but of length $< a_*$. Now apply the Main Claim 2.7.
(b)′ $\Longrightarrow$ (c)′: Obvious.
(c)′ $\Longrightarrow$ (b)′: By [9].

3. We just repeat the proof of the Main Claim 2.7, or cf. Claim 3.16 below. □

**Question 3.14** Is $a_*$ necessary in Conclusion 3.13(1)? We conjecture that yes.

A partial answer:

**Fact 3.15** *If $N$ is a model of* PA*, then $N$ is $\kappa$-saturated iff $\mathrm{cf}(|N|, <^N) \geq \kappa$ and $N$ is boundedly $\kappa$-saturated.*

**Claim 3.16** *If (A) then (B) where:*

A. *(a) $\mathbf{r}_\alpha$ is an* RSP *for $\alpha < \delta$;*
   *(b) $\Delta_{\mathbf{r}_\alpha} = \Delta$ is a set of quantifier free formulas;*
   *(c) $\mathscr{T}_{\mathbf{r}_\alpha} = \mathscr{T}_{\mathbf{r}_0}$ and $N_{\mathbf{r}_\alpha}$ is increasing with $\alpha$;*
   *(d) $Q \in \tau(N_{\mathbf{r}_\alpha})$ and $Q^{N_{\mathbf{r}_\alpha}} = Q^{N_{r_0}}$;*

 

*(e) if* $\varphi(x, \bar{y}) \in \Delta_{\mathbf{r}_\alpha}$ *and* $\bar{b} \in {}^{\ell g(\bar{y})}(N_{\mathbf{r}_\alpha})$ *then* $\varphi(N_{\mathbf{r}_\alpha}, \bar{b}) \subseteq Q^{N_{\mathbf{r}_\alpha}}$*;*

*(f)* $\kappa = \min\{\mathfrak{p}_\alpha(\mathscr{T}_{\mathbf{r}_0}), \mathfrak{t}(\mathscr{T}_{\mathbf{r}_0})\}$*;*

*B. the model* $\cup\{N_{\mathbf{r}_\alpha} : \alpha < \delta\}$ *is* $(\kappa, 1, \Delta)$*-saturated.*

Proof. As in the Main Claim 2.7. □

## 4 Criterion for atomic saturation of reduced powers

Malliaris and Shelah [9] have dealt with such problems for ultrafilters (on sets). The main case here is $\vartheta = \aleph_0$.

**Definition 4.1** Assume $D$ is a filter on the complete Boolean algebra $\mathfrak{B}$, $T$ an $\mathbb{L}_{\vartheta,\vartheta}(\tau_T)$-theory, $\Delta \subseteq \mathbb{L}(\tau_T)$ and $\mu \geq |\Delta|$. We say $D$ is a $(\mu, \vartheta, \varepsilon!, \Delta, T)$-moral filter on $\mathfrak{B}$ (writing $\varepsilon$ instead $\varepsilon!$ means for every $\varepsilon' < 1 + \varepsilon$; if $\mathfrak{B} = \mathscr{P}(\lambda)$ we may say good instead of moral): when for every $D$-$(\mu, \vartheta, \varepsilon!, \Delta, T)$-problem there is a $D$-$(\mu, \vartheta)$-solution where:

(a) $\bar{\mathbf{a}}$ is a $D$-$(\mu, \vartheta, \varepsilon!, \Delta, T)$-(moral)-problem when:

($\alpha$) $\bar{\mathbf{a}} = \langle \mathbf{a}_u : u \in [\mu]^{<\vartheta} \rangle$;

($\beta$) $\mathbf{a}_u \in D$ (hence $\in \mathfrak{B}^+$);

($\gamma$) $\bar{\mathbf{a}}$ is $\subseteq$-decreasing, that is $u \subseteq v \in [\mu]^{<\vartheta} \Longrightarrow \mathbf{a}_v \leq \mathbf{a}_u$ and $\mathbf{a}_\varnothing = 1_{\mathfrak{B}}$;

($\delta$) for some sequence $\langle \varphi_\alpha(\bar{x}_{[\varepsilon]}, \bar{y}_\alpha) : \alpha < \mu \rangle$ of formulas from $\Delta$ for every $\mathbf{a} \in \mathfrak{B}^+$ and $u \subseteq \mu$ of cardinality $< \vartheta$ we can find $M \models T$ and $\bar{b}_\alpha \in {}^{\ell g(\bar{y}_\alpha)}M$ for $\alpha \in u$ such that:

$$(*) \quad \text{for every } v \subseteq u \text{ we have } \mathbf{a} \leq \mathbf{a}_v \Longrightarrow M \models \text{“}(\exists \bar{x}_{[\varepsilon]}) \bigwedge_{\alpha \in v} \varphi_\alpha(\bar{x}_{[\varepsilon]}, \bar{b}_\alpha)\text{” and}$$

$$\mathbf{a} \leq 1 - \mathbf{a}_v \Longrightarrow M \models \text{“}\neg(\exists \bar{x}_{[\varepsilon]}) \bigwedge_{\alpha \in v} \varphi_\alpha(\bar{x}_{[\varepsilon]}, \bar{b}_\alpha)\text{”};$$

(b) $\bar{\mathbf{b}}$ is a $D$-$(\mu, \vartheta)$-(moral)-solution of the $D$-$(\mu, \vartheta, \varepsilon!, \Delta, T)$-(moral)-problem $\bar{\mathbf{a}}$ when

($\alpha$) $\bar{\mathbf{b}} = \langle \mathbf{b}_u : u \in [\mu]^{<\vartheta} \rangle$;

($\beta$) $\mathbf{b}_u \in D$ and $\mathbf{b}_\varnothing = 1_{\mathfrak{B}}$;

($\gamma$) $\mathbf{b}_u \leq \mathbf{a}_u$;

($\delta$) $\bar{\mathbf{b}}$ is multiplicative, i.e., $\bar{\mathbf{b}}_u = \cap\{\mathbf{b}_{\{\alpha\}} : \alpha \in u\}$ and $\mathbf{b}_\varnothing = 1_{\mathfrak{B}}$.

**Remark 4.2** 1. The $\vartheta$ here means “a type is $(< \vartheta)$-satisfiable”.

2. The use of “$\varepsilon!$” is to conform with Definition 1.11.

Recall (from Definition 1.11)

**Definition 4.3** Let $\tau$ be a vocabulary and $\Delta \subseteq \{\varphi \in \mathbb{L}(\tau) : \varphi = \varphi(\bar{x}, \bar{y})\}$ but $\varphi(\bar{x}, \bar{y}) \in \Delta$ means we can add to $\bar{x}$ dummy variables. Let $\lambda > \vartheta$ (dull otherwise).

A $\tau$-model $M$ is $(\lambda, \vartheta, \varepsilon!, \Delta)$ -saturated when: if $p \subseteq \{\varphi(\bar{x}_{[\varepsilon]}, \bar{a}) : \varphi(\bar{x}_{[\varepsilon]}, \bar{y}) \in \Delta, \bar{a} \in {}^{\ell g(\bar{y})}M\}$ has cardinality $< \lambda$ and is $(< \vartheta)$ satisfiable in $M$, then $p$ is realized in $M$.

**Claim 4.4** *1. For a* $(\mu, \vartheta)$*-regular* $\vartheta$*-complete ultrafilter* $D$ *on a set* $I$ *and* $\vartheta$*-saturated or just* $(\vartheta, \aleph_0, \varepsilon!, \Delta)$*-saturated model* $M$*, a cardinal* $\mu$ *and* $\Delta = \mathbb{L}_{\vartheta,\vartheta}(\tau_M)$*, the following conditions are equivalent:*

*(a)* $D$ *is* $(\mu, \vartheta, \varepsilon!, \Delta, T)$*-moral ultrafilter on the Boolean algebra* $\mathscr{P}(I)$*;*

*(b) if* $M \in \mathrm{Mod}_T$*, then* $M^I/D$ *is* $(\mu, \vartheta, \varepsilon!, \Delta)$*-saturated.*

*2. Similarly for* $D$ *a ultrafilter on a* $(< \vartheta)$*-distributive (cf. Definition 1.12(8)) complete Boolean algebra* $\mathfrak{B}$*.*

Proof. Similar to Claim 4.5, it actually follows from it because as $D$ is an ultrafilter, we can start with $M \models T$, expand it to $M^+$ by adding a predicate to any definable relation and apply Claim 4.5 to $T^+ = \mathrm{Th}(M^+)$. □

**Claim 4.5** *1. If* $(A)$*, then* $(B) \iff (C)$ *where:*

*A. (a)* $\mathfrak{B} = \mathscr{P}(I)$*;*

 

*(b) $D$ is a $\vartheta$-complete $(\mu, \vartheta)$-regular filter on $\mathfrak{B}$;*
*(c) $\vartheta > \varepsilon$ or just $\mu^+ > \varepsilon$;*
*(d) $T$ is an $\mathbb{L}_{\vartheta,\vartheta}(\tau)$-theory;*
*(e) $\Delta$ is a set of conjunctions of $< \vartheta$ atomic formulas from $\mathbb{L}_{\vartheta,\vartheta}(\tau)$;*

B. *$D$ is a $(\mu, \vartheta, \varepsilon!, \Delta, T)$-moral filter on $\mathfrak{B}$;*

C. *if $M_s$ is a model of $T$ for $s \in I$ then $\prod_{s\in I} M_s/D$ is $(\mu^+, \vartheta, \varepsilon!, \Delta)$-saturated.*

2. *If (A′), then (B′) $\iff$ (C′) where*

A′. *(a) $\mathfrak{B}$ is a $(< \vartheta)$-distributive (cf. Definition 1.12(8)) complete Boolean algebra;*
*(b)-(e) as above (on regularity cf. Definition 1.13);*
*(d)$^+$ $T$ is a complete $\mathbb{L}_{\vartheta,\vartheta}(\tau)$-theory;*

B′. *as (B) above;*

C′. *(a) if $M$ is a model of $T$, then $M^{\mathfrak{B}}/D$ is $(\mu^+, \vartheta, \varepsilon!, \Delta)$-saturated;*
*(b) if $\mathscr{I}$ is a maximal antichain of $\mathfrak{B}$ and $\bar{M} = \langle M_b : b \in \mathscr{I}\rangle$ is a sequence of $\tau$-models, then $\bar{M}^{\mathfrak{B}}/D$ is $(\mu^+, \vartheta, \varepsilon!, \Delta)$-saturated.*

Proof. 1. Proving $(B) \Longrightarrow (C)$: Let $N = \prod_{s\in I} M_s/D$ let $\bar{x} = \bar{x}_{[\varepsilon]}$, $\varphi_\alpha = \varphi_\alpha(\bar{x}, \bar{y}_\alpha)$ and assume that $p(\bar{x}) = \{\varphi_\alpha(\bar{x}, \bar{b}_\alpha) : \alpha < \alpha_*\}$ is $(< \vartheta)$-satisfiable in $N$ and $|\alpha_*| \leq \mu$, so without loss of generality $\alpha_* = \mu$; without loss of generality let $\varphi_\alpha = \varphi_\alpha(\bar{x}, \bar{y}_{[\xi_\alpha]})$ so $\bar{b}_\alpha \in {}^{\xi_\alpha}(\prod_{s\in I} M_s)$.

Let $\bar{b}_\alpha = \langle f_{\alpha,\xi}/D : \xi < \xi_\alpha\rangle$ where $f_{\alpha,\xi} \in \prod_{s\in I} M_s$ and for $s \in I$ let $\bar{b}_{\alpha,s} = \langle f_{\alpha,\xi}(s) : \xi < \xi_\alpha\rangle$; now for $u \in [\mu]^{<\vartheta}$ we let

$(*)_0$ $\mathbf{a}_u := \{s \in I : M_s \models (\exists \bar{x}) \bigwedge_{\alpha\in u} \varphi(\bar{x}, \bar{b}_{\alpha,s})\}$.

Now

$(*)_1$ $\bar{\mathbf{a}} = \langle \mathbf{a}_u : u \in [\mu]^{<\vartheta}\rangle$ is a $D$-$(\mu, \vartheta, \varepsilon!, \Delta, T)$-problem.

[Why? We should check Definition 4.1, clause (a): now (a)($\alpha$) is trivial; also $\mathbf{a}_u \subseteq I$ holds by the choice of $\mathbf{a}_\alpha$. Toward clause (a)($\beta$) fix a set $u \in [\mu]^{<\vartheta}$; some $\bar{c} \in {}^{\varepsilon}N$ realizes the type $p_u(\bar{x}_{[\varepsilon]}) = \{\varphi_\alpha(\bar{x}, \bar{b}_\alpha) : \alpha \in u\}$ in $N$ because $p(\bar{x})$ is $(< \vartheta)$-satisfiable in $N$, cf. Definition 4.3, so let $\bar{c} = \langle g_\zeta/D : \zeta < \varepsilon\rangle$ for some $g_\zeta \in \prod_{s\in I} M_s$ for $\zeta < \varepsilon$ and let $\bar{c}_s = \langle g_\zeta(s) : \zeta < \varepsilon\rangle \in {}^{\varepsilon}(M_s)$. So $\mathbf{a}'_{\{\alpha\}} = \{s \in I : M \models \varphi_\alpha[\bar{c}_s, \bar{b}_s]\}$ belong to $D$ because $N \models \varphi_\alpha[\bar{c}, \bar{b}_\alpha]$ by the definition of $N$ if $\varphi_\alpha$ is atomic, but recalling $D$ is $\vartheta$-complete also for our $\varphi_\alpha$, remembering clause (A)(e) of Claim 4.5(1). As $D$ is $\vartheta$-complete clearly, $\mathbf{a}'_u = \cap\{\mathbf{a}'_{\{\alpha\}} : \alpha \in u\}$ belongs to $D$ and by our choices, $\mathbf{a}'_u \leq_{\mathfrak{B}} \mathbf{a}_u$, hence $\mathbf{a}_u \in D$ so subclause (a)($\beta$) of Definition 4.1 holds indeed.

By the choice of $\mathbf{a}_u$, $\bar{\mathbf{a}}$ is $\subseteq$-decreasing so subclause (a)($\gamma$) of Definition 4.1 holds.

Lastly, subclause (a)($\delta$) of Definition 4.1 holds by the definition of $\mathbf{a}_u$'s recalling $p(\bar{x})$ is $(< \vartheta)$-satisfiable (and $\varnothing \notin D$).]

$(*)_2$ there is $\bar{\mathbf{b}}$, a $D$-$(\mu, \vartheta)$-solution of $\mathbf{a}$ in $\mathfrak{B}$.

[Why? Because we are presently assuming clause (B) of Claim 4.5 which says that $D$ is $(\mu, \vartheta, \varepsilon!, \Delta, T)$-good, cf. Definition 4.1.]

$(*)_3$ without loss of generality $s \in I \Longrightarrow \{\alpha < \mu : s \in \mathbf{b}_{\{\alpha\}}\}$ has cardinality $< \vartheta$.

[Why? As $D$ is $(\mu, \vartheta)$-regular.]

Next for $s \in I$ let $u_s = \{\alpha < \mu : s \in \mathbf{b}_{\{\alpha\}}\}$ but $\bar{\mathbf{b}}$ is multiplicative (cf. Definition 4.1(b)($\delta$)) so $\mathbf{b}_{u_s} = \cap\{\mathbf{b}_{\{\alpha\}} : \alpha \in u_s\} = \cap\{\mathbf{b}_\alpha$: the ordinal $\alpha$ satisfies $s \in \mathbf{b}_{\{\alpha\}}\}$ hence $s \in \mathbf{b}_{u_s}$ hence (cf. Definition 4.1(b) recalling that $|u_\alpha| < \vartheta$ by $(*)_2$) we have $s \in \mathbf{a}_{u_s}$ hence (by the choice of $\mathbf{a}_{u_s}$) there is $\bar{a}_s \in {}^{\varepsilon}(M_s)$ realizing $\{\varphi(\bar{x}_{[\varepsilon]}, \langle f_{\alpha,\varepsilon_\zeta}(s) : \zeta < \varepsilon\rangle) : \alpha \in u_s\}$.

Let $\bar{a}_s = \langle a_{s,\zeta} : \zeta < \varepsilon \rangle$. Now for $\zeta < \varepsilon = \ell g(\bar{x})$ let $g_\zeta \in \prod_{s \in I} M_s$ be defined by $g_\zeta(s) = a_{s,\zeta} \in M_s$ and let $\bar{a} = \langle g_\zeta/D : \zeta < \varepsilon \rangle$ noting $g_\zeta/D \in \prod_{s \in I} M_s/D = N$. Hence for every $\alpha < \mu$, $\{s \in I : M_s \models \varphi_\alpha(\langle g_\zeta(s) : \zeta < \varepsilon \rangle, \bar{b}_{\alpha,s})\} \supseteq \mathbf{b}_{\{\alpha\}} \in D$ so $N \models \varphi[\bar{a}, \bar{b}_\alpha]$.

Hence $\bar{a}$ realizes $p(\bar{x})$ in $N$ as promised.

Proving $(C) \Longrightarrow (B)$: To prove clause (B), let $\bar{\mathbf{a}}$ be a $D$-$(\mu, \vartheta, \varepsilon!, \Delta, T)$-problem and let $\bar{\varphi} = \langle \varphi_\alpha(\bar{x}_{[\varepsilon]}, \bar{y}_\alpha) : \alpha < \mu \rangle$ be a sequence of formulas from $\Delta$ as in clause (a)($\delta$) of Definition 4.1.

As $D$ is $(\lambda, \vartheta)$-regular, we can choose $\bar{w} = \langle w_s : s \in I \rangle$ a sequence of subsets of $\mu$ each of cardinality $< \vartheta$ such that $\alpha < \mu \Longrightarrow \{s \in I : \alpha \in w_s\} \in D$. For $u \in [\mu]^{<\vartheta}$ let $\mathbf{c}_u = \{s \in I : u \subseteq w_s\}$, so clearly $\mathbf{c}_u \in D$ and $\langle \mathbf{c}_u : u \in [\lambda]^{<\vartheta} \rangle$ is multiplicative.

For each $s \in I$ applying Definition 4.1(a)($\delta$) to $\mathbf{a} = \{s\}$ and $u = w_s$ we can find a model $M_s$ of $T$ and $\bar{b}_{s,\alpha} \in {}^{\ell g(\bar{y}_\ell)}(M_s)$ for $\alpha \in w_s$ satisfying $(*)$ there.

Now choose $\bar{b}_{s,\alpha}$ also for $s \in I, \alpha \in \mu \backslash w_s$, as any sequence of members of $M_s$ of length $\ell g(\bar{y}_\alpha)$. Now for every $\alpha < \mu$ and $j < \ell g(\bar{y}_\alpha)$ we define $g_{\alpha,j} \in \prod_{s \in I} M_s$ by $g_{\alpha,j}(s) = (\bar{b}_{s,\alpha})_j$.

Hence $g_{\alpha,\zeta}/D \in \prod_{s \in I} M_s/D = N$ and $\bar{b}_\alpha = \langle g_{\alpha,\zeta}/D : \zeta < \ell g(\bar{y}_\alpha) \rangle \in {}^{\ell g(\bar{y}_\alpha)}N$ and consider the set $p = \{\varphi_\alpha(\bar{x}, \bar{b}_\alpha) : \alpha < \mu\}$. Is $p$ a $(< \vartheta)$-satisfiable type in $N$? We shall prove that Yes, so let $u \in [\mu]^{<\vartheta}$, then recall $\mathbf{c}_u = \{s \in I : u \subseteq W_s\} \in D$ and $s \in \mathbf{c}_u \cap \mathbf{a}_u \Longrightarrow \{\varphi_\alpha(\bar{x}_{[\varepsilon]}, \bar{b}_{s,\alpha}) : \alpha \in u\}$ is realized in $M_s$, [why? by the choice of $\langle \bar{b}_{s,\alpha} : \alpha \in w_s \rangle$.]

So let the type $\{\varphi_\alpha(\bar{x}_{[\varepsilon]}, \bar{b}_{s,\alpha}) : \alpha \in w_s\}$ be realized $\bar{a}_s = \langle a_{s,\zeta} : \zeta < \varepsilon \rangle$; for $s \in I$ and let $f_{\alpha,\zeta} \in \prod_{s \in I} M_s$ be $f_{\alpha,\zeta}(s) = a_{s,j}$. Easily $\langle f_{\alpha,j}/D : \zeta < \varepsilon \rangle$ realizes $\{\varphi_\alpha(\bar{x}_{[\varepsilon]}) : \alpha \in u\}$ because $\mathbf{a}_u \cap \mathbf{c}_u \in D$. Hence $p(\bar{x}_{[\varepsilon]})$ is $(< \vartheta)$-satisfiable indeed.

Next, we apply clause (C) we are assuming hence $p(\bar{x}_{[\varepsilon]})$ is realized in $N$. So let $\bar{a} = \langle a_\zeta : \zeta < \varepsilon \rangle \in {}^\varepsilon N$ realize $p$ and let $a_\zeta = h_\zeta/D$ where $h_\zeta \in \prod_{s \in I} M_s$ and lastly let

$$\mathbf{b}_u = \{s \in I : M_s \models \varphi_\alpha[\langle h_\zeta(s) : \zeta < \varepsilon \rangle, \bar{b}_{s,\alpha}] \text{for every } \alpha \in u \text{and } s \in \mathbf{c}_u\}.$$

Now check that $\langle \mathbf{b}_u : u \in [\lambda]^{<\vartheta} \rangle$ is as required, recalling $\langle \mathbf{c}_u : u \in [\lambda]^{<\vartheta} \rangle$ is multiplicative. So the desired conclusion of Definition 4.1(B) holds indeed so we are done proving $(C) \Longrightarrow (B)$.

2. Similarly; e.g., for clause (a) let $p(\bar{x})$ be as there but

$f_{\alpha,\xi} \in M^{\mathfrak{B}}$ is supported by the maximal antichain $\langle \mathbf{c}_{\alpha,\xi,i} : i < i(\alpha, \xi) \rangle$

$(*)_0$ $\mathbf{a}_u = \sup\{\mathbf{c}$: we have $\alpha \in u \wedge \xi < \xi_\alpha \Longrightarrow (\exists \mathbf{d})(\mathbf{d} \in \mathrm{dom}(f_{\alpha,\xi}) \wedge \mathbf{c} \leq \mathbf{d})$ and $M \models (\exists \bar{x}_{[\varepsilon]}) \bigwedge_{\alpha \in u} \varphi(\bar{x}_{[\varepsilon]}, \langle f_{\alpha,\xi}(\mathbf{c}) : \xi < \xi_\alpha \rangle)\}$;

$(*)_1$ $\bar{\mathbf{a}} = \langle \mathbf{a}_u : u \in [\mu]^{<\vartheta} \rangle$ is a $D$-$(\mu, \vartheta, \varepsilon!, \Delta, T)$-problem.

[Why? As there.]

$(*)_2$ let $\bar{\mathbf{b}}$ be a $D$-$(\mu, \vartheta)$-solution.

[Why does $\bar{\mathbf{b}}$ exist? By $(B)'$ recalling Definition 4.1.]

Also the rest is as above. □

**Remark 4.6** If $\mathscr{S} \subseteq [\mu]^{<\vartheta}$ is cofinal, $u \in [\mu]^{<\vartheta} \Longrightarrow |\mathscr{P}(u) \cap \mathscr{S}| < \vartheta_1$ we may consistently replace $[\mu]^{<\vartheta}$ by $\mathscr{S}$ and $2^{\vartheta_1}$ by $\vartheta_1$.

**Definition 4.7** 1. A filter $D$ on a complete Boolean algebra $\mathfrak{B}$ is $(\mu, \vartheta)$-excellent when: if $\bar{\mathbf{a}} = \langle \mathbf{a}_u : u \in [\mu]^{<\vartheta} \rangle$ is a sequence of members of $\mathfrak{B}$, (yes! not necessarily from $D$), then we can find $\bar{\mathbf{b}}$ which is a multiplicative refinement of $\bar{\mathbf{a}}$ for $D$, meaning:

(a) $\bar{\mathbf{b}} = \langle \mathbf{b}_u : u \in [\mu]^{<\vartheta} \rangle$;

(b) $\mathbf{b}_u \leq \mathbf{a}_u$ and $\mathbf{b}_u = \mathbf{a}_u \mod D$;

(c) if $\mathbf{a}_{u_1} \cap \mathbf{a}_{u_2} = \mathbf{a}_{u_1 \cap u_2} \mod D$, then $\mathbf{b}_{u_1} \cap \mathbf{b}_{u_2} = \mathbf{b}_{u_1 \cap u_2}$.

2. For a Boolean algebra $\mathfrak{B}$ and filter $D$ on $\mathfrak{B}$ we say $\bar{\mathbf{a}}$ is a $D$-$(\mu, \vartheta)$-problem (or a $D - (\mu, \vartheta)$-moral problem) when clauses (a)($\alpha$), ($\beta$), ($\gamma$) of Definition 4.1 holds.

 

3. A filter $D$ on a complete Boolean algebra $\mathfrak{B}$ is $(\mu, \vartheta)$-good when every $D - (\mu, \vartheta)$-problem has a $D - (\mu, \vartheta)$-solution

**Claim 4.8** *1. Assuming $(*)$ below, the filter $D$ on $I$ (i.e., on the Boolean algebra $\mathscr{P}(I)$) is $(\mu, \vartheta, \varepsilon!, \Delta, T)$-moral iff the filter $D_1$ on $\mathfrak{B}_1$ is $(\mu, \vartheta, \varepsilon!, \Delta, T)$-moral where:*

$(*)$ *(a) $\mathfrak{B}_1$ is a complete Boolean algebra;*
*(b) $\mathbf{j}$ is a homomorphism from $\mathscr{P}(I)$ onto $\mathfrak{B}_1$;*
*(c) $D_0 = \{A \subseteq I : \mathbf{j}(A) = 1_{\mathfrak{B}_1}\}$ is a $(\mu, \vartheta)$-excellent filter on $I$;*
*(d) $D_1$ is a filter on $\mathfrak{B}_1$;*
*(e) $D = \{A \subseteq I : \mathbf{j}(A) \in D_1\}$. is a filter on $I$*

*2. We can replace $\mathscr{P}(I)$ by a complete Boolean algebra $\mathfrak{B}_2$.*

Proof. 1. The "if" direction: We assume $D_1$ is $(\mu, \vartheta, \varepsilon!, \Delta, T)$-moral and should prove it for $D$. So let $\bar{A} = \langle A_u : u \in [\mu]^{<\vartheta}\rangle$ be a $D$-$(\mu, \vartheta, \varepsilon!, \Delta, T)$-problem and we should find a $D$-$(\mu, \vartheta)$-solution $\bar{B}$ of it.

Clearly $\mathbf{a}_u := \mathbf{j}(A_u) \in \mathfrak{B}^+$ and $\bar{\mathbf{a}} = \langle \mathbf{a}_u : u \in [\mu]^{<\vartheta}\rangle = \langle \mathbf{j}(A_u) : u \in [\mu]^{<\vartheta}\rangle$ is a $D_1 - (\mu, \vartheta, \varepsilon!, \Delta, T)$-problem.

Hence by our present assumption ($D_1$ is $(\mu, \vartheta, \varepsilon!, \Delta, T)$-moral) there is a $D_1 - (\mu, \vartheta)$-solution $\bar{\mathbf{b}}$ of $\bar{\mathbf{a}}$, let $\bar{\mathbf{b}} = \langle \mathbf{b}_u : u \in [\mu]^{<\vartheta}\rangle$ so in particular $u \in [\mu]^{<\vartheta} \Longrightarrow \mathbf{b}_u \in D_1$. For $u \in [\mu]^{<\vartheta}$ choose $B^1_u \subseteq I$ such that $\mathbf{j}(B^1_u) = \mathbf{b}_u$, possible because $\mathbf{j}$ is a homomorphism from $\mathscr{P}(I)$ onto $\mathfrak{B}_1$. So $\bar{B}^1 = \langle B^1_u : u \in [\mu]^{<\vartheta}\rangle$ is a multiplicative modulo $D_0$, i.e., $\langle B^1_u/D_0 : u \in [\mu]^{<\vartheta}\rangle$ is a multiplicative sequence of members of $\mathscr{P}(I)/D_0$.

Let $B^2_u = B^1_u \cap A_u$, let

i. $B^1_u \subseteq A_u \mod D_0$.

[Note that we have written $B^1_u$ and not $B^2_u$. So why this statement holds? As $\mathbf{j}(B^1_u) = \mathbf{b}_u \le \mathbf{a}_u = \mathbf{j}(A_u)$.]

i. $B^2_u \subseteq B^1_u$ and $B^2_u \subseteq A_u \mod D_0$;
ii. $B^2_u \in D$;
iii. $\langle B^2_u : u \in [\mu]^{<\vartheta}\rangle$ is multiplicative modulo $D_0$ (cf. Definition 4.7).

By Definition 4.7(1) applied to $\langle B^2_u : u \in [\mu]^{<\vartheta}\rangle$ recalling clause (c) of the assumption of the claim, we can find $\bar{B} = \langle B_u : u \in [\mu]^{<\vartheta}\rangle$ which is a multiplicative refinement of $\bar{B}^2$ and is multiplicative, and $B_u \in D$ because $B_n = B^2_n$ modulo $D_0 \subseteq D$ and $B^2_u \in D$.

So we are done for the "if" direction.

The "only if" direction: So we are assuming $D$ is a $(\mu, \vartheta, \varepsilon!, \Delta, T)$-good filter on $I$ and we have to prove $D_1$ is $(\mu, \vartheta, \varepsilon!, \Delta, T)$-moral.

So let $\bar{\mathbf{a}}$ be a $D_1 - (\mu, \vartheta, \varepsilon!, \Delta, T)$-moral problem (on $\mathfrak{B}_1$), we have to find a solution. For $u \in [\mu]^{<\vartheta}$ choose $A^1_u \subseteq I$ such that $\mathbf{j}(A^1_u) = \mathbf{a}_u$, so $A^1_u \in D$ (by clause (e)) and $u \subseteq v \in [\mu]^{<\vartheta} \to A^1_u \subseteq A^1_v$ modulo $D_0$. Now by Definition 4.7, i.e., clause (b) of the assumption of the claim there is $\bar{A}^2 = \langle A^2_u : u \in [\mu]^{<\vartheta}\rangle$ such that $A^2_u \subseteq A^1_u$, $A^2_u = A^1_u$ mod $D_0$ hence $A^2_u \in D$ and $\bar{A}^2$ is $\subseteq$-decreasing [Why? Because $\bar{A}^1$ is $\subseteq$-decreasing modulo $D_0$ as $\bar{\mathbf{a}}$ is decreasing hence $\bar{A}^2$ is $\subseteq$-decreasing.]

As $D$ is $(\mu, \vartheta, \varepsilon!, \Delta, T)$-good filter on $I$ there is a $D$-multiplicative refinement $\langle B^2_u : u \in [u]^{<\vartheta}\rangle$ of $\langle A^2_u : u \in [\mu]^{<\vartheta}\rangle$. Let $\mathbf{b}_u = \mathbf{j}(B^2_u)$, now $\langle \mathbf{b}_u : u \in [\mu]^{<\vartheta}\rangle$ is as required.

2. Similarly. □

**Claim 4.9** *Let $D$ be a filter on $I$.*

*1. $D$ is $(\mu, \vartheta)$-excellent implies $D$ is $(\mu, \vartheta)$-good, cf. Definition 4.7(3).*
*2. $D$ is $(\mu, \vartheta)$-good implies $D$ is $(\mu, \vartheta, \varepsilon, \Delta, T)$-moral.*

Proof. 1. So let $\bar{\mathbf{a}} = \langle \mathbf{a}_u : u \in [\mu]^{<\vartheta}\rangle$ be a $D$-problem and we should find a $D$-$(\mu, \vartheta)$-solution $\bar{\mathbf{b}}$ below $\bar{\mathbf{a}}$. As $D$ is $(\mu, \vartheta)$-excellent we apply this to $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ as in Definition 4.7(2). Easily it is as required.

2. Just read the definitions: there are fewer problems. □

**Remark 4.10** We may wonder, e.g., in Claim 4.5(1): can we remove the regularity demand on the filter $D$ from clause (A) to clause (B)? The answer is yes for most $T$'s.

 

**Claim 4.11** *The filter $D$ is $(\mu, \vartheta)$-regular when:*

*A. (a) $\mathfrak{B} = \mathscr{P}(I)$;*
*(b) $D$ is a $\vartheta$-complete ultrafilter on $\mathfrak{B}$;*
*(c) $\vartheta > \varepsilon$, is natural but not actually required;*
*(d) $T$ is a complete $\mathbb{L}_{\vartheta,\vartheta}(\tau)$-theory, e.g., $T = \mathrm{Th}_{\mathbb{L}_{\vartheta,\vartheta}}(M)$, $M$ a $\vartheta$-saturated model (note that $T = T_0^{[\vartheta]}$ where $T_0 = \mathrm{Th}_{\mathbb{L}_{\aleph_0,\aleph_0}}(M)$, i.e., $T$ is determined by $T_0$ and $\vartheta$);*

*B. $T$ has a model $M$ and $p = \{\varphi_\alpha(\bar{x}_{[\varepsilon]}, \bar{b}_\alpha) : \alpha < \mu\}$, $\varphi_\alpha(\bar{x}_{[\varepsilon]}, \bar{y}_\alpha) \in \mathbb{L}_{\vartheta,\vartheta}$, $\bar{b}_\alpha \in {}^{\ell g(\bar{y}_\alpha)}M$ such that: for every $q \subseteq p$, $q$ is realized in $M$ iff $|q| < \vartheta$;*

*C. if $M_s$ is a model of $T$ for $s \in I$, then $\prod_s M_s/D$ is $(\mu^+, \vartheta, \varepsilon!, \Delta)$- saturated.*

P r o o f . Should be clear. □

## 5 A counter example

In § 3 we generalize [22, Ch. VI, 2.6] to filters, using the class of relevant RSP's **r** being closed under reduced powers (being a Horn class; cf. Claim 3.2). Can we generalize the result of Malliaris and Shelah [9]? Here we give a counter-example.

For this we have to find

$(*)_1$ $D$ a filter of $\lambda$ such that the partial order $N_1 = (\mathbb{Q}, <)^\lambda/D$ satisfies $\mathfrak{p}^*(N_1) = \kappa_1 + \kappa_2 < \mu^+ \leq \mathfrak{p}^*_{\mathrm{sym}}(N_1)$, $\kappa_1 \neq \kappa_2$, $(\kappa_1, \kappa_2) \in \mathscr{C}(N_1)$, so in fact $N_1$ has no $(\vartheta_1, \vartheta_2)$-cut when $\vartheta_1 = \mathrm{cf}(\vartheta_1) = \vartheta_2 \leq \mu$ and when $\vartheta_\ell \geq \mu^+ \wedge \vartheta_{3-\ell} \in \{0, 1\}$;

$(*)_2$ preferably: $\lambda = \mu$;

$(*)_3$ or at least for some dense linear order $M_0$ there is a complete Boolean algebra $\mathfrak{B}$ and a filter $D$ on $\mathfrak{B}$ such that $N_0 = M_0^{\mathfrak{B}}/D$ is as above.

We presently deal with the (main) case $\vartheta = \aleph_0$ and carry this out. It seems reasonable that we can prove, e.g., $T_{\mathrm{ceq}} \ntrianglelefteq_{\mathrm{rp}} T_{\mathrm{ord}}$ but we have not arrived to it; cf. [18] on $T_{\mathrm{ceq}}$ and [23] on the closely related $T_{\mathrm{feq}}$. Later we hope to say more. Clearly we can control the set of non-symmetric pre-cuts.

**Convention 5.1** $T_{\mathrm{ord}}$ is the first order theory of $(\mathbb{Q}, \leq)$; cf. Definition 5.4(1)(d).

**Definition 5.2** Let $\kappa$ be a regular cardinal.

1. Let $K^{\mathrm{ba}}_\kappa$ be the class of $\mathbf{m}$ such that:
   (a) $\mathbf{m} = (\mathfrak{B}, D) = (\mathfrak{B}_{\mathbf{m}}, D_{\mathbf{m}})$;
   (b) $\mathfrak{B}$ is a complete Boolean algebra satisfying the $\kappa$-c.c.;
   (c) $D$ is a filter on $\mathfrak{B}$.
2. Let $\leq^{\mathrm{ba}}_\kappa$ be the following two-place relation on $K^{\mathrm{ba}}_\kappa$ : $\mathbf{m} \leq^{\mathrm{ba}}_\kappa \mathbf{n}$ iff
   (a) $\mathbf{m}, \mathbf{n} \in K^{\mathrm{ba}}_\kappa$;
   (b) $\mathfrak{B}_{\mathbf{m}} \lessdot \mathfrak{B}_{\mathbf{n}}$;
   (c) $D_{\mathbf{m}} = D_{\mathbf{n}} \cap \mathfrak{B}_{\mathbf{m}}$.
3. Let $S^{\mathrm{ba}}_\kappa$ be the class of $\leq^{\mathrm{ba}}_\kappa$-increasing continuous sequences $\bar{\mathbf{m}}$ which means:
   (a) $\bar{\mathbf{m}} = \langle \mathbf{m}_\alpha : \alpha < \ell g(\bar{\mathbf{m}}) \rangle$;
   (b) $\mathbf{m}_\alpha \in K^{\mathrm{ba}}_\kappa$;
   (c) if $\alpha < \beta < \ell g(\bar{\mathbf{m}})$, then $\mathbf{m}_\alpha \leq^{\mathrm{ba}}_\kappa \mathbf{m}_\beta$;
   (d) if $\beta < \ell g(\bar{\mathbf{m}})$ is a limit ordinal, then:
   $(\alpha)\mathfrak{B}_{\mathbf{m}_\beta}$ is the completion of $\cup\{\mathfrak{B}_{\mathbf{m}_\alpha} : \alpha < \beta\}$;
   $(\beta)D_{\mathbf{m}_\beta}$ is generated (as a filter) by $\cup\{D_{\mathbf{m}_\alpha} : \alpha < \beta\}$.
4. If $\kappa = \aleph_1$ we may write $K^1_{\mathrm{ba}}, \leq^1_{\mathrm{ba}}, S^1_{\mathrm{ba}}$, and if $\kappa = \infty$ we may write $K^2_{\mathrm{ba}}, \leq^2_{\mathrm{ba}}, S^2_{\mathrm{ba}}$ or $K^{\mathrm{ba}}_\infty, \leq^{\mathrm{ba}}_\infty, S^{\mathrm{ba}}_\infty$,
5. We say $\mathbf{m}$ is of cardinality $\lambda$ when $\mathfrak{B}_{\mathbf{m}}$ is of cardinality $\lambda$.

**Claim 5.3** *1. For every $\lambda$ there is $\mathbf{m} \in K_\kappa^{\rm ba}$ of cardinality $\lambda^{<\kappa}$.*

*2. $\leq_\kappa^{\rm ba}$ is a partial order on $K_\kappa^{\rm ba}$.*

*3. If $\bar{\mathbf{m}} = \langle \mathbf{m}_\alpha : \alpha < \delta\rangle$ is a $\leq_\kappa^{\rm ba}$-increasing continuous sequence, then for some $\mathbf{m}_\delta$, the sequence $\bar{\mathbf{m}}\hat{\ }\mathbf{m}_\delta$ is $\leq_\kappa^{\rm ba}$-increasing continuous.*

P r o o f.

1. E.g., $\mathfrak{B}_{\mathbf{m}}$ is the completion of a free Boolean algebra generated by $\lambda^{<\kappa}$ elements.
2. Easy.
3. If $\mathrm{cf}(\delta) \geq \kappa$, then $\mathfrak{B}_{\mathbf{m}_\delta} = \bigcup_{\alpha<\delta} \mathfrak{B}_{\mathbf{m}_\alpha}$, if $\mathrm{cf}(\delta) < \kappa$ it is the (pendantically $a$) completion of the union. $D_{\mathbf{m}_\delta}$ is the filter generated by $\cup\{D_{\mathbf{m}_\alpha} : \alpha < \delta\}$. Classically $\kappa$-c.c. is preserved. □

**Definition 5.4** Let $\mathbf{m} \in K^2_{\rm ba}$ and $\kappa_1, \kappa_2$ are (infinite) regular cardinals.

1. We say $\bar{\mathbf{a}}$ is a $T_{\rm ord}$-$(\kappa_1, \kappa_2)$-moral problem in $\mathbf{m}$ when:
   (a) $\mathbf{m} \in K^2_{\rm ba}$, (actually already assumed).
   (b) $I = I(\kappa_1, \kappa_2)$ is the linear order $I_1 + I_2$ where
   1.$I_1 = I_1(\kappa_1) = (\{1\} \times \kappa_1)$,
   2.$I_2 = I_2(\kappa_2) = (\{2\} \times \kappa_2^*)$;
   (c) $\bar{\mathbf{a}} = \langle \mathbf{a}_{s,t} : s <_{I(\kappa_1,\kappa_2)} t\rangle$ is a sequence of members of $D_{\mathbf{m}}$;
   (d) if $u \subseteq I$ is finite, $\mathbf{t} : u \times u \to \{0, 1\}$ and $\cap\{\mathbf{a}_{s,t}^{\mathrm{if}(\mathbf{t}(s,t))} : s, t \in u\} > 0_{\mathbf{m}}$, then there is a function $f : u \to \{0, \dots, |u| - 1\}$ such that:
   1.if $s, t \in u$, then $\mathbf{t}(s, t) = 1$ iff $f(s) \leq f(t)$;
   (e) hence $s_1 <_I s_2 <_I s_2 \Longrightarrow \mathbf{a}_{s_1,s_2} \cap \mathbf{a}_{s_2,s_3} \leq \mathbf{a}_{s_1,s_3}$ and we stipulate $\mathbf{a}_{s,s} = 1_{\mathfrak{B}_{\mathbf{m}}}$, $\mathbf{a}_{t,s} = \mathbf{a}_{s,t}$ when $s <_I t$.
2. We say $\bar{\mathbf{b}}$ is a solution of $\bar{\mathbf{a}}$ in $\mathbf{m}$ where $\bar{\mathbf{a}}$ is as above when:
   (a) $\bar{\mathbf{b}} = \langle \mathbf{b}_s : s \in I\rangle$;
   (b) $\mathbf{b}_s \in D_{\mathbf{m}}$;
   (c) if $s_1 \in I_1$, $s_2 \in I_2$, then $\mathbf{b}_{s_1} \cap \mathbf{b}_{s_2} \leq \mathbf{a}_{s_1,s_2}$.

**Definition 5.5** 1. For $\iota = 1, 2$ let $\mathbf{S}_\iota$ be the class of tuples $\mathbf{s} = (I, D_0, \mathbf{j}, \mathfrak{B}, D_1, D)$ such that:
   (a) $\mathbf{j}$ is a homomorphism from $\mathscr{P}(I)$ onto the complete Boolean algebra $\mathfrak{B}$;
   (b) $D_1$ is a filter on $\mathfrak{B}$;
   (c) $D_0 = \{A \subseteq I : \mathbf{j}(A) = 1_{\mathfrak{B}}\}$ (or see § 4);
   (d) $D = \{A \subseteq I : \mathbf{j}(A) \in D_1\}$;
   (e) the pair $(\mathfrak{B}, D)$ belongs to $K^\iota_{\rm ba}$;
2. For $\mathbf{s} \in \mathbf{S}$ let $\mathbf{m}_\mathbf{s} = (\mathfrak{B}_\mathbf{s}, D_\mathbf{s})$.
3. We say $\mathbf{s} \in \mathbf{S}$ is $(\mu, \vartheta)$-excellent (if $\vartheta = \aleph_0$ may omit) when $D_0$ is an excellent filter on $I$, cf. Definition 4.7(2).
4. We say $\mathbf{s} \in \mathbf{S}$ is $(\mu, \vartheta)$-regular (if $\vartheta = \aleph_0$ we may omit $\vartheta$) when $D_0$ is a $(\mu, \vartheta)$-regular filter.
5. Let $\mathbf{S}^\iota_{\mu,\vartheta}$ be the class of $(\mu, \vartheta)$-excellent $(\mu, \vartheta)$-regular $\mathbf{s} \in \mathbf{S}_\iota$; we may omit $\vartheta$ if $\vartheta = \aleph_0$.
6. Let $\mathbf{S}_{\mu,\vartheta,\kappa}$ be the class of $\mathbf{s} \in \mathbf{S}^2_{\mu,\vartheta}$ such that $\mathfrak{B}_\mathbf{s}$ satisfies the $\kappa$-c.c.

**Claim 5.6** *1. Assume $\mathbf{m} = (\mathfrak{B}, D) \in K_{\rm ba}$ and $\kappa_1, \kappa_2$ are infinite and regular cardinals. Then for some $M \in \mathrm{Mod}_{T_{\rm ord}}$, $M^{\mathfrak{B}}/D$ has a $(\kappa_1, \kappa_2)$-pre-cut iff some $T_{\rm ord}$-$(\kappa_1, \kappa_2)$-moral problem in $\mathbf{m}$ has no solution.*

*2. Let $\mu \geq \aleph_0 = \vartheta$. If $\mathbf{s} \in \mathbf{S}_{\mu,\vartheta}$ so is $\mu$-excellent and $\mu$-regular and $\kappa_1, \kappa_2 \geq \aleph_0$ are regular and $\kappa_1 + \kappa_2 \leq \mu$, then the following conditions are equivalent:*
   *(a) for some linear order $M$, $M^{I(\mathbf{s})}/D_\mathbf{s}$ has a $(\kappa_1, \kappa_2)$-pre-cut;*
   *(b) for every infinite linear order, $M^{I(\mathbf{s})}/D_\mathbf{s}$ has a $(\kappa_1, \kappa_2)$-pre-cut;*
   *(c) not every $T_{\rm ord}$-$(\kappa_1, \kappa_2)$-moral problem in $\mathbf{m}_\mathbf{s}$ has a solution.*

P r o o f. As in in the proof of Claim 4.5(1), relying on Definition 5.4 instead of Definition 4.1; recalling

 

⊞ if $M^\iota_s$ for $s \in I, \iota \in \{1,2\}$ are $\tau$-models, $|\tau| \le \mu$, $D$ a $\mu$-regular filter on $I$ and $M^1_s, M^2_s$ are elementarily equivalent, then $N_1 = \prod_{s\in I} M^1_s/D$, $N_2 = \prod_{s\in I} M^2_s/D$ are $\mathbb{L}_{\mu^+,\mu^+}$-equivalent (and more, cf. Kennedy and Shelah [4], [5] and Kennedy, Shelah, and Väänänen [6] on the subject). □

**Observation 5.7** *Assume $\mathbf{m} \in K^2_{\rm ba}$ and $\bar{\mathbf{a}}$ is a $T_{\rm ord}$-$(\kappa_1,\kappa_1)$-moral problem for $\mathbf{m}$ so (cf. Definition 5.5(5)) $I_\ell = I_\ell(\kappa_\ell)$ for $\ell = 1, 2$.*

1. *If $I'_1 \subseteq I_1$ is cofinal in $I_1$ and $I'_2 \subseteq I_2$ is co-initial in $I_2$, then $\bar{\mathbf{a}}$ has a solution in $\mathbf{m}$ iff $\bar{\mathbf{a}}' = \bar{\mathbf{a}}{\restriction}(I'_1 + I'_2) = \langle \mathbf{a}_{s,t} : s <_I t \text{ and } s,t \in I'_1 + I'_2\rangle$ has a solution in $\mathbf{m}$.*
   *1A. Also, above, if $\bar{\mathbf{b}}$ is a solution of $\bar{\mathbf{a}}$ in $\mathbf{m}$, then $\bar{\mathbf{b}}{\restriction}(I'_1 + I'_2)$ is a solution of $\bar{\mathbf{a}}'$.*
   *1B. Also above, if $\bar{\mathbf{b}}'$ is a solution of $\bar{\mathbf{a}}'$, then $\bar{\mathbf{b}}$ is a solution of $\bar{\mathbf{a}}$ when:*
   *(a) if $s \in I_1$ and $t \in I'_1$ is minimal such that $s \le_I t$, then $\mathbf{b}_s = \mathbf{b}'_t \cap \mathbf{a}_{s,t}$ if $s <_I t$ and $\mathbf{b}_s = \mathbf{b}'_t$ if $s = t$;*
   *(b) like (a) replacing $I_1, I'_1, s <_I t, \mathbf{a}_{s,t}$ by $I_2, I'_2, t \le_I s, \mathbf{a}_{t,s}$.*
2. *If $\bar{\mathbf{b}}$ is a solution of $\bar{\mathbf{a}}$ in $\mathbf{m}$ and $\mathbf{b}'_s \in D \wedge \mathbf{b}'_s \le \mathbf{b}_s$ for $s \in I_1 + I_2$, then $\langle \mathbf{b}'_s : s \in I\rangle$ is a solution of $\bar{\mathbf{a}}$ for $\mathbf{m}$.*

P r o o f . 1. Easy using the proofs of Claims 4.5 and 5.6 or using (1A), (1B).
2. Check. □

A key point in the inductive construction is:.

**Claim 5.8** *There is no solution to $\bar{\mathbf{a}}$ in $\mathbf{m}_\delta$ when:*

*(a)* $\bar{\mathbf{m}} = \langle \mathbf{m}_\alpha : \alpha \le \delta\rangle \in S^2_{\rm ba}$*;*
*(b)* $\bar{\mathbf{a}}$ *is a* $T_{\rm ord}$-$(\kappa_1,\kappa_2)$*-moral problem in* $\mathbf{m}_0$*;*
*(c) if* $\alpha < \delta$*, then* $\bar{\mathbf{a}}$ *has no solution in* $\mathbf{m}_\alpha$*;*
*(d)* ${\rm cf}(\delta) \ne \kappa_1$ *or* ${\rm cf}(\delta) \ne \kappa_2$.

P r o o f . Let $\mathbf{m}_\gamma = (\mathfrak{B}_\gamma, D_\gamma)$ for $\gamma \le \delta$; by symmetry without loss of generality ${\rm cf}(\delta) \ne \kappa_1$ and toward contradiction assume $\bar{\mathbf{b}} = \langle \mathbf{b}_s : s \in I_1 + I_2\rangle$ is a solution of $\bar{\mathbf{a}}$ in $\mathbf{m}_\delta$.

Hence $\mathbf{b}_s \in D$. Now $D_\delta$ is not necessarily equal to $\bigcup_{\gamma<\delta} D_\delta$ but recalling Definition 5.2(3)(d)($\beta$) and $\langle D_\gamma : \gamma < \delta\rangle$ being increasing, clearly every member of $D_\delta$ is above some member of $\bigcup_{\gamma<\delta} D_\gamma$.

So by Observation 5.7(2) without loss of generality $s \in I_1 + I_2 \Longrightarrow \mathbf{b}_s \in \bigcup_{\gamma<\delta} D_\gamma \subseteq \bigcup_{\gamma<\delta} \mathfrak{B}_\gamma$.

As ${\rm cf}(\delta) \ne \kappa_1$, for some $\gamma < \delta$ we have $\kappa_1 = \sup\{\alpha < \kappa_1 : \mathbf{b}_{(1,\alpha)} \in \mathfrak{B}_\gamma\}$, i.e., $\{s \in I_1 : \mathbf{b}_s \in \mathfrak{B}_\gamma\}$ is co-final in $I_1$. So by Observation 5.7(1) without loss of generality

(a) $s \in I_1 \Longrightarrow \mathbf{b}_s \in \mathfrak{B}_\gamma$.

As $D_\gamma = D_\delta \cap \mathfrak{B}_\gamma$ by Definition 5.2(2)(c) clearly

(b) $s \in I_1 \Longrightarrow \mathbf{b}_s \in D_\gamma$.

For $t \in I_2$ let $\mathbf{b}'_t = \min\{\mathbf{b} \in \mathfrak{B}_\gamma : \mathfrak{B}_\delta \models \mathbf{b}_t \le \mathbf{b}\}$, well defined because $\mathfrak{B}_\gamma$ is complete.
Now

(c) $\mathbf{b}'_t \in D_\gamma$ for $t \in I_2$.

[Why? Clearly $\mathbf{b}_t \in \mathfrak{B}_\delta$ as $\bar{\mathbf{b}}$ is a solution of $\bar{\mathbf{a}}$ in $\mathbf{m}_\delta$ and $\mathbf{b}_t \le \mathbf{b}'_t$, $\mathbf{b}'_t \in \mathfrak{B}_\gamma$ by its choice. Also $\mathbf{b}'_t \in D_\delta$ because $\mathbf{b}_t \le \mathbf{b}'_t \wedge \mathbf{b}_t \in D_\delta$ and $D_\delta$ is a filter on $\mathfrak{B}_\delta$ and lastly $\mathbf{b}'_t \in D_\gamma$ as $D_\gamma = D_\delta \cap \mathfrak{B}_\gamma$.]

(d) if $s \in I_1, t \in I_2$, then $\mathbf{b}_s \cap \mathbf{b}'_t \le \mathbf{a}_{s,t}$.

[Why? Note $\mathfrak{B}_\delta \models$ "$\mathbf{b}_s \cap \mathbf{b}_t \le \mathbf{a}''_{s,t}$ because $\bar{\mathbf{b}}$ is a solution of $\mathbf{a}$ in $\mathfrak{B}_\delta$ hence $\mathbf{b}_t \le \mathbf{a}_{s,t} \cup (1 - \mathbf{b}_s)$ and the later $\in \mathfrak{B}_\gamma$. So by the choice of $\mathbf{b}'_t$, $\mathbf{b}'_t \le \mathbf{a}_{s,t} \cup (1 - \mathbf{b}_s)$ hence $\mathbf{b}_s \cap \mathbf{b}'_t \le \mathbf{a}_{s,t}$.]

(e) $\langle \mathbf{b}_s : s \in I_1\rangle^\frown\langle \mathbf{b}'_t : t \in I_2\rangle$ solves $\bar{\mathbf{a}}$ in $\mathfrak{B}_\gamma$.

[Why? By (a) + (b) + (c) + (d).]

But this contradicts an assumption. □

**Definition 5.9** Assume $\mathbf{m} \in K^2_{\mathrm{ba}}$ and $\bar{\mathbf{a}}$ is a $(\kappa_1, \kappa_2)$-moral problem in $\mathbf{m}$. We say $\mathbf{n}$ is a simple $\bar{\mathbf{a}}$-solving extension of $\mathbf{m}$ when:

(a) $\mathfrak{B}_{\mathbf{n}}$ is the completion of $\mathfrak{B}^o_{\mathbf{n}}$ where

(b) $\mathfrak{B}^o_{\mathbf{n}}$ is the Boolean algebra generated by $\mathfrak{B}_{\mathbf{m}} \cup \{y_s : s \in I(\kappa_1, \kappa_2)\}$ freely except the equations which holds in $\mathfrak{B}_{\mathbf{m}}$ and $\Gamma_{\bar{\mathbf{a}}} = \{y_{s_1} \cap y_{s_2} \le \mathbf{a}_{s_1,s_2} : s_1 \in I_1(\kappa_1) \text{ and } s_2 \in I_2(\kappa_2)\}$;

(c) $D_{\mathbf{n}}$ is the filter on $\mathfrak{B}_{\mathbf{n}}$ generated by $D_{\mathbf{m}} \cup \{y_s : s \in I(\kappa_1, \kappa_2)\}$.

**Claim 5.10** *Assume $\bar{\mathbf{a}}$ is a $T_{\mathrm{ord}}$-$(\kappa_1, \kappa_2)$-moral problem in $\mathbf{m} \in K^{\mathrm{ba}}_\kappa$ and*[7] $\kappa = \mathrm{cf}(\kappa) > \kappa_1 + \kappa_2$.

1. *There is $\mathbf{n} \in K^{\mathrm{ba}}_\kappa$ which is a simple $\bar{\mathbf{a}}$-solving extension of $\mathbf{m}$, unique up to isomorphism over $\mathfrak{B}_{\mathbf{m}}$.*
2. *Above $\mathbf{m} \le^{\mathrm{ba}}_\kappa \mathbf{n}$ (so $\mathbf{n} \in K^{\mathrm{ba}}_\kappa$).*
3. *If $\bar{\mathbf{a}}^*$ is a $T_{\mathrm{ord}}$-$(\vartheta_1, \vartheta_2)$-moral problem of $\mathbf{m}$ with no solution in $\mathbf{m}$ and $\vartheta_1 \notin \{\kappa_1, \kappa_2\}$ or $\vartheta_2 \notin \{\kappa_1, \kappa_2\}$, then $\bar{\mathbf{a}}^*$ has no solution in $\mathbf{n}$.*

Proof. 1. As above let $I_\ell = I_\ell(\kappa_\ell)$ for $\ell = 1, 2$ and $I = I_1 + I_2$.
First

$(*)_1$ the set of equations $\Gamma_{\bar{\mathbf{a}}}$ is finitely satisfiable in $\mathfrak{B}_{\mathbf{m}}$.
Why? We shall prove two stronger statements (each implying $(*)_1$).

$(*)_{1.1}$ if $t_1 \in I_1$, then we can find $\langle \mathbf{b}'_s : s \in I \rangle \in {}^I\mathfrak{B}$ such that:
(a) $\mathbf{b}'_s \in D_{\mathbf{m}} \subseteq \mathfrak{B}_m$ if $(s \le_{I_1} t_1) \vee (s \in I_2)$;
(b) if $s_1 \in I_1, s_2 \in I_2$, then $\mathbf{b}'_{s_1} \cap \mathbf{b}'_{s_2} \le \mathbf{a}_{s_1,s_2}$.
[Why? Let $\mathbf{b}'_s$ be:
1. $\mathbf{a}_{s,t_1}$ if $s \le_I t_1$ (so $s \in I_1$)
2. $\mathbf{a}_{t_1,s}$ if $s \in I_2$
3. $0_{\mathfrak{B}}$ if $t_1 <_I s \in I_1$.
Now clause (a) is obvious (recalling $\mathbf{a}_{t_1,t_1} = 1_{\mathfrak{B}_{\mathbf{m}}}$ and as for clause (b), let $s_1 \in I_1, s_2 \in I_2$, now if $t_1 \le_I s_1 \in I_1$, then $\mathbf{b}'_{s_1} \cap \mathbf{b}'_{s_2} = 0_{\mathfrak{B}_{\mathbf{m}}} \cap \mathbf{b}'_{s_2} = 0_{\mathfrak{B}_{\mathbf{m}}} \le \mathbf{a}_{s_1,s_2}$ and if $s_1 <_I t_1$, then $\mathbf{b}'_{s_1} \cap \mathbf{b}'_{s_2} = \mathbf{a}_{s_1,t_1} \cap \mathbf{a}_{t_1,s_2}$ which is $\le \mathbf{a}_{s_1,s_2}$ by Definition 5.4(1)(d),(e).]

$(*)_{1.2}$ if $t_2 \in I_2$, then we can find $\langle \mathbf{b}'_s : s \in I \rangle \in {}^I\mathfrak{B}$ such that
(a) $\mathbf{b}'_s \in D_{\mathbf{m}} \subseteq \mathfrak{B}_{\mathbf{m}}$ if $s \in I_1$ or $t_2 \le_{I_2} s$;
(b) if $s_1 \in I_2, s_2 \in I_2$, then $\mathbf{b}'_{s_1} \cap \mathbf{b}'_{s_2} \le \mathbf{a}_{s_1,s_2}$.
[Why? Similarly.]
Now $(*)_1$ is easy: if $\Gamma' \subseteq \Gamma_{\bar{\mathbf{a}}}$ is finite let $t_* \in I_1$ be such that: if $t \in I_1$ and $y_t$ appears in $\Gamma'$, then $t \le_I t_*$. Choose $\langle b'_s : s \in I \rangle$ as in $(*)_{1.1}$ for $t_*$ and let $h$ be the function $y_s \mapsto \mathbf{b}'_s$ for $s \in I$. Now think, so $(*)_1$ holds indeed.
Clearly it follows by $(*)_1$ that

$(*)_2$ (a) there is a Boolean algebra $\mathfrak{B}^o_{\mathbf{n}}$ extending $\mathfrak{B}_{\mathbf{m}}$ as described in clause (b) of Definition 5.9;
(b) there is a Boolean algebra $\mathfrak{B}_{\mathbf{n}}$ as described in (a) of Definition 5.9: the completion of $\mathfrak{B}^0_{\mathbf{n}}$;
(c) $D_{\mathbf{n}}$ is chosen as the filter on $\mathfrak{B}_{\mathbf{n}}$ generated by $D_{\mathbf{m}} \cup \{y_s : s \in I\}$; satisfies $D_{\mathbf{m}} = D_{\mathbf{n}} \cap \mathfrak{B}_{\mathbf{m}}$, in particular $0_{\mathfrak{B}_{\mathbf{m}}} \notin D_{\mathbf{n}}$;
(d) $\mathfrak{B}_{\mathbf{n}}$ satisfies the $\kappa$-c.c.;
(e) $D_{\mathbf{n}}$ is generated (as a filter) by $D_{\mathbf{n}} \cap \mathfrak{B}^o_{\mathbf{n}}$.
[Why? Clauses (a),(b) follows by $(*)_1$ and for clauses (c),(d) see $(*)_4$ and $(*)_5$ in the proof of (2), respectively; in particular $0_{\mathfrak{B}\mathbf{m}} \notin D_{\mathbf{n}}$.]
Together we have $\mathbf{n} = (\mathfrak{B}_{\mathbf{n}}, D_{\mathbf{n}}) \in K^2_{\mathrm{ba}}$, as for $\mathbf{m} \le_{\mathrm{ba}} \mathbf{n}$, see part (2).

2. Now (by part (1) we have $\mathfrak{B}_{\mathbf{m}} \subseteq \mathfrak{B}_{\mathbf{n}}$, but we shall show that moreover)

[7] It seems that $\min\{\kappa_1, \kappa_2\} < \kappa$ suffice; the only difference in the proof is in proving $(*)_5$.

 

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$(*)_3$ $\mathfrak{B}_{\mathbf{m}} \lessdot \mathfrak{B}_{\mathbf{n}}$.

[Why? If not, then some $\mathbf{d} \in \mathfrak{B}^+_{\mathbf{n}}$ is disjoint to $\mathbf{b}$ for a dense subset of $\mathbf{b} \in \mathfrak{B}^+_{\mathbf{m}}$. Let $\mathbf{d} = \sigma(y_{s_0}, \dots, y_{s_{n-1}}, \bar{c})$ where $\sigma$ is a Boolean term, $s_0 <_I \dots <_I s_{n-1}$ and $\bar{c}$ is from $\mathfrak{B}_{\mathbf{m}}$. We may replace $\mathbf{d}$ by any $\mathbf{d}' \in \mathfrak{B}^+_{\mathbf{n}}$ satisfying $\mathbf{d}' \leq_{\mathfrak{B}} \mathbf{d}$. Hence without loss of generality $\mathbf{d} = \cap\{y^{\text{if}(\eta(\ell))}_{s_\ell} : \ell < n\} \cap c > 0_{\mathbf{n}}$ where $c \in \mathfrak{B}_{\mathbf{m}}$, $\eta(\ell) \in \{0, 1\}$ for $\ell < n$; also without loss of generality for every $\ell, k < n$ we have $s_\ell \in I_1 \wedge s_k \in I_2 \Longrightarrow (c \leq \mathbf{a}_{s_\ell, s_k}) \vee (c \cap \mathbf{a}_{s_\ell, s_k} = 0_{\mathfrak{B}_{\mathbf{n}}})$.

We now define a function $h$ from $\{y_s : s \in I\}$ into $\mathfrak{B}_{\mathbf{m}}$ as follows: $h(y_s)$is:

i. $c$ if $s = s_\ell \wedge \eta(\ell) = 1$;

ii. $0_{\mathfrak{B}_{\mathbf{m}}}$ if otherwise.

Now

iii. if $t_1 \in I_1, t_2 \in I_2$, then $\mathfrak{B}_{\mathbf{m}} \models$ "$h(y_{t_1}) \cap h(y_{t_2}) \leq \mathbf{a}_{t_1, t_2}$".

[Why? If $h(t_1) = 0_{\mathfrak{B}_{\mathbf{m}}} \vee h(t_2) = 0_{\mathfrak{B}_{\mathbf{m}}}$ this is obvious, otherwise for some $\ell(1) < \ell(2) < n$ we have $t_1 = s_{\ell(1)}, t_2 = s_{\ell(2)}$ and $\eta(\ell(1)) = 1 = \eta(\ell(2))$. So it suffice to prove $c = c \cap c \leq \mathbf{a}_{t_1, t_2}$ but otherwise by the choice of $c$, $c \cap \mathbf{a}_{t_1, t_2} = 0$, hence recalling Definition 5.9(b) we have $\mathfrak{B}_{\mathbf{n}} \models$ "$y_{s_1} \cap y_{s_2} \cap c = 0$" contradiction to our current assumption $\mathfrak{B}_{\mathbf{n}} \models$ "$d > 0$"; so (iii) holds indeed.]

By the choice of $\Gamma_{\bar{\mathbf{a}}}$ and of $\mathfrak{B}_{\mathbf{n}}$ recalling $\mathfrak{B}_{\mathbf{m}}$ is complete, by the choice of $h$ and (iii) there is a projection $\hat{h}$ from $\mathfrak{B}_{\mathbf{n}}$ onto $\mathfrak{B}_{\mathbf{m}}$ extending $h$, so clearly $\hat{h}(d) = c$ and this implies $c_1 \in \mathfrak{B}_{\mathbf{m}} \wedge 0 < c_1 \leq c \Longrightarrow \mathfrak{B}_{\mathbf{n}} \models$ "$c_1 \cap \mathbf{d} \geq 0''_{\mathfrak{B}_{\mathbf{n}}}$ contradicting the choice of $\mathbf{d}$. So indeed $(*)_3$ holds.]

$(*)_4$ $D_{\mathbf{m}} = D_{\mathbf{n}} \cap \mathfrak{B}_{\mathbf{m}}$.

[Why? Otherwise there are $c_1 \in D_{\mathbf{m}}$, $c_2 \in \mathfrak{B}_{\mathbf{m}} \backslash D_{\mathbf{m}}$ and $s_0 <_I \dots <_I s_{n-1}$ such that $\mathfrak{B}_{\mathbf{n}} \models$ "$\bigcap\limits_{\ell<n} y_{s_\ell} \cap c_1 \leq c_2$". As $\mathbf{a}_{t_1, t_2} \in D_{\mathbf{m}}$ for $t_1 <_I t_2$, without loss of generality $c_1 \leq \mathbf{a}_{s_\ell, s_k}$ for $\ell < k < n$, $s_\ell \in I_1$, $s_k \in I_2$.

Now letting $c = c_1 - c_2$ we continue as in the proof of $(*)_3$ defining $h, \hat{h}$ and apply the projection $\hat{h}$ to "$\bigcap\limits_{\ell<n} y_{s_\ell} \cap c_1 \leq c_2$".]

$(*)_5$ $\mathfrak{B}_{\mathbf{n}}$ satisfies the $\kappa$-c.c.

[Why? If not, then there are pairwise disjoint, positive $d_i \in \mathfrak{B}_{\mathbf{n}}$ for $i < \kappa$. So as in the proof of $(*)_3$, without loss of generality $d_i = \cap\{y^{\text{if}(\eta(i,\ell))}_{s(i,\ell)} : \ell < n(i)\} \cap c_i$ where $c_i \in \mathfrak{B}_{\mathbf{m}}$, $\eta(i, \ell) \in \{0, 1\}$ and $s(i, 0) <_I s(i, 1) <_I \dots <_I s(i, n(i) - 1)$. Let $m(i) \leq n(i)$ be such that for every $\ell < n(i)$ we have $s_\ell \in I_1$ iff $\ell < m(i)$.

Again as there, without loss of generality for every $\ell < m(i) \leq k < n(i)$ we have $(\mathbf{a}_{s(i,\ell),s(i,k)} \leq c_i) \vee (\mathbf{a}_{s(i,\ell),s(i,k)} \cap c_i = 0)$ so $\eta(i, \ell) = 1 = \eta(i, k) \wedge \ell < m(i) \leq k < n(i) \Longrightarrow c_i \leq \mathbf{a}_{s(i,\ell),s(i,k)}$.

As $\kappa = \text{cf}(\kappa) > \kappa_1 + \kappa_2$ by an assumption of Claim 5.10without loss of generality $n(i) = n, m(i) = m\eta(i, \ell) = \eta(\ell)$ and $s(i, \ell) = s_\ell$ for $i < \kappa, \ell < n$ and as $\mathfrak{B}_{\mathbf{m}}$ satisfies the $\kappa$-c.c. we can find $i < j < \kappa$ such that $\mathfrak{B}_{\mathbf{m}} \models$ "$0 < c_i \cap c''_j$ and let $c = c_i \cap c_j$ so we can continue as before.]

So together by $(*)_3$, $(*)_4$, $(*)_5$ we have $\mathbf{m} \leq^{\text{ba}}_\kappa \mathbf{n} \in K^{\text{ba}}_\kappa$ as promised.

3. Let $I^* = I(\vartheta_1, \vartheta_2)$, $I^*_1 = I_1(\vartheta_1)$, $I^*_2 = I_2(\vartheta_2)$ and recall $\bar{\mathbf{a}}^* = \langle \mathbf{a}^*_{s,t} : s <_{I^*} t \rangle$ is a $T_{\text{ord}}$-$(\vartheta_1, \vartheta_2)$-moral problem in $\mathbf{m}$. Toward a contradiction assume that the sequence $\bar{\mathbf{b}} = \langle \mathbf{b}_t : t \in I^* \rangle$ solve the problem $\bar{\mathbf{a}}^*$ in $\mathbf{n}$ so $\mathbf{b}_t \in D_{\mathbf{n}}$ and let $\mathbf{b}_t = \sigma_t(y_{s(t,0)}, \dots, y_{s(t,n(t)-1)}, c_{t,0}, \dots, c_{t,m(t)-1})$ with $c_{t,k} \in \mathfrak{B}_{\mathbf{m}}$, $s(t, \ell) \in I$ and without loss of generality $s(t, \ell) <_I s(t, \ell + 1)$ for $\ell < n(t) - 1$ so $s(t, k) \in I$ for $k < n(t)$.

The reader may wonder: we have to prove that there is no solution in $\mathfrak{B}_{\mathbf{n}}$ , not just in $\mathfrak{B}^o_{\mathbf{n}}$ , so how can we use finitary terms? The point is that though $\mathfrak{B}_{\mathbf{n}}$ is the completion of $\mathfrak{B}^o_{\mathbf{n}}$, the filter $D_{\mathbf{n}}$ is generated (as a filter) by $\mathfrak{B}^o_{\mathbf{n}} \cap D_{\mathbf{n}}$.

By symmetry without loss of generality

$(*)_6$ $\vartheta_1 \notin \{\kappa_1, \kappa_2\}$.

Recalling Observation 5.7, we can replace $\mathbf{b}_t$ by any $\mathbf{b}'_t \leq \mathbf{b}_t$ which is from $D_{\mathbf{n}}$, so as $\bigwedge\limits_{\ell} y_{s(t,\ell)} \in D_{\mathbf{n}}$, without loss of generality $\ell < n(t) \Longrightarrow \mathbf{b}_t \leq y_{s(t,\ell)}$, so without loss of generality

$(*)_7$ $\mathbf{b}_t = \cap\{y_{s(t,\ell)} : \ell < n(t)\} \cap c_t$ for some $c_t \in D_{\mathbf{m}}$ recalling $D_{\mathbf{m}} = D_{\mathbf{m}} \cap \mathfrak{B}_{\mathbf{m}}$.

By the $\Delta$-system lemma (recalling Observation 5.7(1)) without loss of generality

$\oplus$ if $\vartheta_1 > \aleph_0$, then

 

(a) $t \in I_1^* \Longrightarrow n(t) = n(*)$;

(b) if $t \in I_1^*$, then $s(t, \ell) \in I_1^* \iff \ell < \ell(*)$;

(c) $\langle\langle s(t, \ell) : \ell < n(*)\rangle : t \in I_1^*\rangle$ is an indiscernible sequence in the linear order $I = I(\kappa_1, \kappa_2)$, for quantifier free formulas.

But we shall not use $\oplus$. As $\vartheta_1 \neq \kappa_1, \kappa_2$, by Observation 5.7(1),(1A) it follows that without loss of generality for some $s_1^\circ, s_2^\circ$ we have:

$(*)_8$ $s_1^\circ \in I_1, s_2^\circ \in I_2$ and $s(t, \ell) \notin [s_1^\circ, s_2^\circ]_I$ for every $t \in I_1^*, \ell < n(t)$.

Again by Observation 5.7(2) without loss of generality

$(*)_9$ if $t \in I_2^*$, then $\mathbf{b}_t \leq y_{s_1^\circ} \cap y_{s_2^\circ}$.

We now define a function $h$ from $\{y_s : s \in I\}$ into $\mathfrak{B}_\mathbf{n}$, (yes! not $\mathfrak{B}_\mathbf{m}$) by:

$(*)_{10}$ $h(y_s)$ is:

(a) $\mathbf{a}_{s,s_1^\circ} \cap \mathbf{a}_{s_1^\circ,s_2^\circ}$ if $s <_I s_1^\circ$;

(b) $\mathbf{a}_{s_1^\circ,s} \cap y_s \cap \mathbf{a}_{s,s_2^\circ}$ if $s \in I$, $s_1^\circ \leq_I s \leq_I s_2^\circ$;

(c) $\mathbf{a}_{s_1^\circ,s_2^\circ} \cap \mathbf{a}_{s_2^\circ,s}$ if $s_2^\circ <_I s$.

Note

$(*)_{11}$ $h(y_s) \in D_\mathbf{n}$ for $s \in I$.

[Why? Because $\mathbf{a}_{s,t} \in D_\mathbf{n}$ for $S \in I_1, t \in I)2$ and $y_s \in D_\mathbf{n}$ for $s \in I$.]

$(*)_{12}$ $h(y_{s_1}) \cap h(y_{s_2}) \leq \mathbf{a}_{s_1,s_2}$ for $s_1 \in I_1, s_2 \in I_2$.

[Why? If $s_1, s_2 \in [s_1^\circ, s_2^\circ]_I$ this holds by the definition of $\mathfrak{B}_\mathbf{n}$, i.e., as $h(y_{s_1}) \leq y_{s_1}$, $h(y_{s_2}) \leq y_{s_2}$ and $\mathfrak{B}_\mathbf{n} \models$ "$y_{s_1} \cap y_{s_2} \leq \mathbf{a}''_{s_1,s_2}$.

If $s_1 <_{I^*} s_1^\circ \wedge s_2^\circ <_{I^*} s_2$, then $(*)_{11}$ says: $\mathbf{a}_{s_1,s_1^\circ} \cap \mathbf{a}_{s_1^\circ,s_2^\circ} \cap \mathbf{a}_{s_2^\circ,s_2} \leq \mathbf{a}_{s_1,s_2}$ which obviously holds (as $\bar{\mathbf{a}}$ is a $T_{\mathrm{ord}}$-$(\kappa_1, \kappa_2)$-problem in $\mathbf{m}$).

If $s_1 <_{I^*} s_1^\circ \wedge s_2 \in [s_1^\circ, s_2^\circ]_{I^*}$ then this means: $(\mathbf{a}_{s_1,s_1^\circ} \cap \mathbf{a}_{s_1^\circ,s_2^\circ}) \cap (\mathbf{a}_{s_1^\circ,s_2} \cap y_{s_2} \cap \mathbf{a}_{s,s_2^\circ}) \leq \mathbf{a}_{s_1,s_2}$; but as we have $\mathbf{a}_{s_1,s_1^\circ} \cap \mathbf{a}_{s_1^\circ,s_2} \leq \mathbf{a}_{s_1,s_2}$ this holds.

If $s_1 \in [s_1^\circ, s_2^\circ]_{I^*}$ and $s_2^\circ <_{I^*} s_2$ this means $(\mathbf{a}_{s_1^\circ,s_1} \cap y_{s_1} \cap \mathbf{a}_{s_1,s_2}) \cap (\mathbf{a}_{s_1^\circ,s_2^\circ} \cap \mathbf{a}_{s_2^\circ,s}) \leq \mathbf{a}_{s_1,s_2}$ which holds for similar reasons. So $(*)_{12}$ holds indeed.]

By the choice of $\mathfrak{B}_\mathbf{n}^\circ$ and $\mathfrak{B}_\mathbf{n}$ there is a homomorphism $\hat{h}$ from $\mathfrak{B}_\mathbf{n}$ into $\mathfrak{B}_\mathbf{n}$, extending $\mathrm{id}_{\mathfrak{B}_\mathbf{m}}$ and extending $h$. Now easily $\hat{h}(\mathbf{b}_t) \in D$ for $t \in I^*$ because $\mathbf{b}_t = \cap\{y_{s(t,\ell)} : \ell < n(t)\} \cap c_t$, $c_t \in D_\mathbf{m}$ hence $\hat{h}(c_t) = c_t \in D_\mathbf{m}$ and by $(*)_{10}$ we have $\hat{h}(y_{s(\ell,t)}) \in D_\mathbf{m}$.

Now $\langle \hat{h}(\mathbf{b}_t) : t \in I^*\rangle$ still form a solution of $\bar{\mathbf{a}}^*$ and by $(*)_7 + (*)_8 + (*)_{10}$ we have $t \in I_1^* \Longrightarrow h(\mathbf{b}_t) \in \mathfrak{B}_\mathbf{m}$ hence without loss of generality:

$(*)_{13}$ $t \in I_1^* \Longrightarrow \mathbf{b}_t \in \mathfrak{B}_\mathbf{m}$.

Now define $\mathbf{b}'_t$ for $t \in I^*$ by: $\mathbf{b}'_t$ is:

(a) $\mathbf{b}_t$ if $t \in I_1^*$

(b) $c_t$ if $t \in I_2^*$.

It suffices to prove that $\langle \mathbf{b}'_t : t \in I^*\rangle$ solves $\bar{\mathbf{a}}^*$ in $\mathbf{m}$. Clearly $t \in I^* \Longrightarrow \mathbf{b}'_t \in D_\mathbf{m}$, so let $t_1 \in I_1^*, t_2 \in I_2^*$. We have to prove that $\mathbf{b}'_{t_1} \cap \mathbf{b}'_{t_2} \leq \mathbf{a}_{t_1,t_2}$ but we know only that $\mathbf{b}_{t_1} \cap \mathbf{b}_{t_2} \leq \mathbf{a}_{t_1,t_2}$ which means $\mathbf{a}_{t_1,t_2} \geq \mathbf{b}'_{t_1} \cap (\bigcap_{\ell < n(t_2)} y_{s(t_2,\ell)} \cap c_{t_2}) = (\mathbf{b}'_{t_1} \cap \mathbf{b}'_{t_2}) \cap \bigcap\{y_{s(t_2,\ell)} : \ell < n(t_2)\}$.

Let $h_{t_2}$ be a projection from $\mathfrak{B}_\mathbf{n}$ onto $\mathfrak{B}_\mathbf{m}$ such that $h_{t_2}(y_{s(t_2,\ell)}) = c_t$ if $\ell < n(t)$ and $h_{t_2}(y_s) = 0_{\mathfrak{B}_\mathbf{m}}$ if $s \in I \backslash \{s(t_2, \ell) : \ell < n(t_2)\}$, as earlier it exists and applying it we get the desired inequality. □

**Theorem 5.11** *For any $\lambda$ and regular $\vartheta_1, \vartheta_2 \leq \lambda$ such that $\vartheta_1 + \vartheta_2 > \aleph_0$ there is a regular filter $D$ on $\lambda$ such that:*

*(a) for every dense linear order $M$, in $M^\lambda/D$ there is a $(\vartheta_1, \vartheta_2)$-pre-cut but no $(\kappa_1, \kappa_2)$-pre-cut when $\kappa_1, \kappa_2$ are regular $\leq \lambda$ and $\{\vartheta_1, \vartheta_2\} \nsubseteq \{\kappa_1, \kappa_2\}$*

*(b) if $M$ is $({}^{\omega>}2, \triangleleft)^\lambda/D$, then $\mathfrak{t}(M) \geq \lambda^+$.*

**Remark 5.12** 1. Why do we need $\vartheta_1 + \vartheta_2 > \aleph_0$? To prove $(*)_1$.

2. In fact, this demand is necessary; cf. Observation 5.14 below.

 

P r o o f . We prove clause (a), which is the main result, clause (b) holds by 5.15. Let $\kappa = \lambda^+$.

$(*)_1$ there are $\mathbf{m}_0$, $\mathbf{a}$ such that:
(a) $\mathbf{m}_0 \in K^{\rm ba}_\kappa$
(b) $\mathbf{a}$ is a $T_{\rm ord}$-$(\vartheta_1, \vartheta_2)$-moral problem in $\mathbf{m}_0$ not solved in it.

[Why? By [22, Ch. VI, § 3] there is an ultrafilter $D$ on $\lambda$ such that in $(\mathbb{Q} <)^\lambda/D$ there is a $(\vartheta_1, \vartheta_2)$-cut. Define $\mathbf{m}$ by $\mathfrak{B}_\mathbf{m} = \mathscr{P}(\lambda)$, $D_\mathbf{m} = D$, now check. E.g., as $\kappa = \lambda^+$ , easily the Boolean algebra $\mathfrak{B}_\mathbf{m}$ satisfies the $\kappa$-c.c.; alternatively let $\mathfrak{B}_\mathbf{n}$ be generated by $\{\mathbf{a}_{s,t} : s \in I_1, t \in I_2\}$ freely; and let $D_\mathbf{n}$ be the ultrafilter on $\mathfrak{B}_\mathbf{n}$ generated by $\{\mathbf{a}_{s,t} : s \in I_1, t \in I_2\}$. Now check.]

Let $\langle W_\alpha : \alpha < 2^\lambda\rangle$ be a partition of $2^\lambda$ to sets each of cardinality $2^\lambda$ such that $W_\alpha \cap \alpha = \varnothing$.

$(*)_2$ we can choose $\mathbf{m}_\alpha$ and $\langle \bar{\mathbf{a}}_\gamma : \gamma \in W_\alpha\rangle$ by induction on $\alpha \le 2^\lambda$ such that:
(a) $\mathbf{m}_\alpha \in K^{\rm ba}_\kappa$ has cardinality $\le 2^\lambda$;
(b) $\langle \mathbf{m}_\beta : \beta \le \alpha\rangle \in S^{\rm ba}_\kappa$;
(c) $\mathbf{m}_0$ is as in $(*)_1$;
(d) $\langle \bar{\mathbf{a}}_\gamma : \gamma \in W_\alpha\rangle$ be such that $\bar{\mathbf{a}}_\gamma$ is a $T_{\rm ord}$-$(\kappa_{\gamma,1}, \kappa_{\gamma,2})$ problem in $\mathbf{m}_\alpha$ and $\kappa_{\gamma,1}, \kappa_{\gamma,2}$ are regular $\le \lambda$ and $\{\vartheta_1, \vartheta_2\} \nsubseteq \{\kappa_{\gamma,1}, \kappa_{\gamma,2}\}$ and any such $\bar{\mathbf{a}}$ appears in the sequence;
(e) if $\alpha = \gamma + 1$, then necessarily $\gamma \in W_\beta$ for some $\beta \le \alpha$ and in $\mathbf{m}_\alpha$ there is a solution for $\bar{\mathbf{a}}_\gamma$;
(f) in $\mathbf{m}_\alpha$ there is no solution to $\bar{\mathbf{a}}^*$.

[Why can we carry the induction?
Now for $\alpha = 0$ use $(*)_1$, for $\alpha$ limit use Claim 5.8 and for $\alpha$ successor use Claim 5.10.]

$(*)_3$ letting $\mathbf{m} = \mathbf{m}_{2^\lambda}$ we have $\mathfrak{B}_\mathbf{m} = \cup\{\mathfrak{B}_{\mathbf{m}_\alpha} : \alpha < 2^\lambda\}$ and $D_\mathbf{m} = \cup\{D_{\mathbf{m}_\alpha} : \alpha < 2^\lambda\}$.

[Why? Because $\langle \mathbf{m}_\alpha : \alpha \le 2^\lambda\rangle \in S^{\rm ba}_\kappa$ and $\mathrm{cf}(2^\lambda) \ge \kappa$.]

$(*)_4$ there is a regular excellent filter $D_0$ on $\lambda$ and homomorphism $\mathbf{j}$ from $\mathscr{P}(\lambda)$ onto $\mathfrak{B}_\mathbf{m}$.

[Why? Cf. [7].]

$(*)_5$ let $D = \mathbf{j}^{-1}(D_\mathbf{m})$.

So $D$ is a filter on $\lambda$, and by 4.8 for $\vartheta = \aleph_0$ (or Malliaris and Shelah [7]) we are done. □

**Conclusion 5.13** *If $\lambda \ge \aleph_2$ the results of Malliaris and Shelah [9] cannot be generalized to reduced powers (atomic types, of course),that is (clause (A) is in contrast to [9, Th.10.25 (b) $\Longrightarrow$ (d)]; clause (B) is in contrast to [9, Th.10.1], and clause (C) is in contrast to [9, Th.3.1])*

A. *If $\lambda \ge \aleph_1$, then for some regular filter $D$ on $\lambda$ we have: in ultrapowers of infinite linear orders we have a pre-cut with small cofinalities, but no symmetric pre-cut, that is:*
 *(a) in the ultrapower $(\mathbb{Q}, <)^\lambda/D$ there is a $(\aleph_1, \aleph_0)$-pre-cut;*
 *(b) in this ultrapower, there is no symmetric pre-cut of cofinality $\sigma$ for $\sigma \le \lambda$;*
B. *treetops: we can add above above that in $({}^{\omega>}\omega, \lhd)^\lambda/D$ every increasing sequence of length $\le \lambda$ has an upper bound;*
C. *if $\lambda \le \aleph_2$ then we can add in part (A), there are two pre-cuts with the same small left cofinality but different small right cofinalities, e.g., $\aleph_1$ from the left and $\aleph_2, \aleph_0$ from the right.*

P r o o f . For clause (A) we apply clause (a) of Theorem 5.11 choosing the pair $(\vartheta_1, \vartheta_2)$ as $(\aleph_1, \aleph_0)$.
For clause (B) apply clause (b) of Theorem 5.11.
For clause (C) we repeat the proof of Theorem 5.11 but starting (with $\kappa = \lambda^+$ as there) and choose as there $\mathbf{m}_0 \in K_\kappa$ of cardinality $\le 2^\lambda$ such that some $(\aleph_1, \aleph_0)$-moral problem and $(\aleph_1, \aleph_2)$-moral problem in $\mathbf{m}_0$ are not solve. Then continue as there. □

**Observation 5.14** *If $\mathbf{m} \in K^{\rm ba}_\kappa$, then any $T_{\rm ord}$-$(\aleph_0, \aleph_0)$-problem $\bar{\mathbf{a}}$ in $\mathbf{m}$ has a solution.*

 

P r o o f . Let $\mathbf{b}_{(1,n)} = \mathbf{b}_{(2,n)} = \mathbf{b}_n := \cap\{\mathbf{a}_{(1,\ell),(2,k)} : \ell, k \le n\}$, clearly $s \in I(\aleph_0, \aleph_0) \Longrightarrow \mathbf{b}_s \in D$ and $(s,t) \in I(1,\aleph_0) \times I(2,\aleph_0) \Longrightarrow \mathbf{b}_s \cap \mathbf{b}_t \le \mathbf{a}_{s,t}$. □

**Claim 5.15** *In $M_*^{\mathfrak{B}}/D$, any increasing sequence of length $< \kappa^+$ has an upper bound when (A) or (B) holds, where:*

A. (a) $M_* = ({}^{\omega>}\mu, \trianglelefteq)$
   (b) $\mathfrak{B}$ *is a complete Boolean algebra which is* $(< \vartheta)$*-distributive*
   (c) $D$ *is a* $(\mu, \vartheta)$*-regular,* $\vartheta$*-complete filter on* $\mathfrak{B}$
   (d) $(\mathbb{Q}, <)^{\mathfrak{B}}/D$ *has no* $(\sigma, \sigma)$*-pre-cut for any regular* $\sigma \le \kappa$
   (e) $\mathbf{m} = (\mathfrak{B}, D)$

B. *(a)-(c) as above.*
   (d) *every* $T_{\rm tr}$-$(\sigma, \sigma)$*-moral problem in* $\mathbf{m}$ *has a* $T_{\rm tr}$-$(\sigma, \sigma)$*-moral solution in* $\mathbf{m}$ *where:*
   ($\alpha$) $\bar{\mathbf{a}}$ *is a* $T_{\rm tr}$*-moral problem when:*

i. $\bar{\mathbf{a}} = \langle \mathbf{a}_{\alpha,\beta} : \alpha < \beta < \sigma \rangle$;

ii. $\mathbf{a}_{\alpha,\beta} \in D$;

iii. *if* $u \subseteq \sigma$ *is finite and* $\mathbf{c} \in \mathfrak{B}^+$*, then for some* $\bar{\eta} = \langle \eta_\alpha : \alpha \in u \rangle$ *we have* $\eta_\alpha \in {}^{|u|>}|u|$ *for* $\alpha \in u$ *and* $\mathbf{c} \le \mathbf{a}_{\alpha,\beta} \Longrightarrow \eta_\alpha \trianglelefteq \eta_\beta$ *and* $\mathbf{c} \cap \mathbf{a}_{\alpha,\beta} = 0_{\mathfrak{B}} \Longrightarrow \neg(\eta_\alpha \trianglelefteq \eta_\beta)$ *for* $\alpha < \beta$ *from* $u$;

($\beta$) $\bar{\mathbf{b}} = \langle \mathbf{b}_\alpha : \alpha < \sigma \rangle$ *is a* $T_{\rm tr}$-$(\sigma\sigma)$*-solution of* $\bar{\mathbf{a}}$ *when* $\mathbf{b}_\alpha \in D$ *and* $\mathbf{b}_\alpha \cap \mathbf{b}_\beta \le \mathbf{a}_{\alpha,\beta}$ *for* $\alpha < \beta < \sigma$.

P r o o f . If clause (A), as in [21, Ch. VI, 2.7] or [9].
If clause (B), as above. □

**Acknowledgements** The research of the author was partially supported by European Research Council, Grant No. 338821. The author thanks Alice Leonhardt for the beautiful typing. First typed February 26, 2014. References like [16, Th0.2=Ly5] means the label of Theorem 0.2 is y5. The reader should note that the version in my website is usually more updated than the one in the mathematical archive.
We thank the referee for many helpful comments.
Paper no. Sh:1064.

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