ON κ-HOMOGENEOUS, BUT NOT κ-TRANSITIVE PERMUTATION GROUPS

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Abstract. A permutation group G on a set A is *κ-homogeneous* iff for all $X, Y \in [A]^\kappa$ with $|A \setminus X| = |A \setminus Y| = |A|$ there is a $g \in G$ with $g[X] = Y$. G is *κ-transitive* iff for any injective function f with $\operatorname{dom}(f) \cup \operatorname{ran}(f) \in [A]^{\le\kappa}$ and $|A \setminus \operatorname{dom}(f)| = |A \setminus \operatorname{ran}(f)| = |A|$ there is a $g \in G$ with $f \subset g$.

Giving a partial answer to a question of P. M. Neumann [6] we show that there is an ω-homogeneous but not ω-transitive permutation group on a cardinal λ provided

(i) $\lambda < \omega_\omega$, or
(ii) $2^\omega < \lambda$, and $\mu^\omega = \mu^+$ and $\square_\mu$ hold for each $\mu \le \lambda$ with $\omega = \operatorname{cf}(\mu) < \mu$, or
(iii) our model was obtained by adding $(2^\omega)^+$ many Cohen generic reals to some ground model.

For $\kappa > \omega$ we give a method to construct large κ-homogeneous, but not κ-transitive permutation groups. Using this method we show that there exist κ^+-homogeneous, but not κ^+-transitive permutation groups on κ^{+n} for each infinite cardinal κ and natural number $n \ge 1$ provided $V = L$.

1. Introduction

Denote by $\mathrm{S}(A)$ the group of all permutations of the set A. The subgroups of $\mathrm{S}(A)$ are called *permutation groups on A*.

Let A be a set and $\kappa \le |A|$ be a cardinal. We say that a permutation group G on A is *κ-homogeneous* iff for all $X, Y \in [A]^\kappa$ with $|A \setminus X| = |A \setminus Y| = |A|$ there is a $g \in G$ with $g[X] = Y$.

We say that a permutation group G on A is *κ-transitive* iff for any injective function f with $\operatorname{dom}(f) \cup \operatorname{ran}(f) \in [A]^{\le\kappa}$ and $|A \setminus \operatorname{dom}(f)| = |A \setminus \operatorname{ran}(f)| = |A|$ there is a $g \in G$ with $f \subset g$.

In this paper we give a partial answer to the following question which was raised by P.N. Neumann in [6, Question 3]:

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> *Suppose that $\kappa < \lambda$ are infinite cardinals. Does there exist a permutation group on λ that is κ-homogeneous, but not κ-transitive?*

In section 2 we show that there exist ω-homogeneous, but not ω-transitive permutation groups on $\lambda < \omega_\omega$ in ZFC, and on any infinite λ if $V = L$ (see Theorem 2.5).

In section 3 we develop a general method to obtain large κ-homogeneous, but not κ-transitive permutation groups for arbitrary $\kappa \geq \omega$ (see Theorem 3.2). Applying our method we show that if $\kappa^\omega = \kappa$, $\lambda = \kappa^{+n}$ for some $n < \omega$, and $\square_\nu$ holds for each $\kappa \leq \nu < \lambda$, then there is a κ-homogeneous, but not κ-transitive permutation group on λ (Corollary 3.12).

In section 4 first we show that if Martin's axiom holds for countable posets, then every subgroup of $\mathrm{S}_\omega(\omega_1)$ with cardinality $< 2^\omega$ can be extended to an ω-homogeneous, but not ω-transitive permutation group on ω_1. Based on this theorem we prove that after adding $(2^\omega)^+$ Cohen reals to any ground model in the generic extension for each infinite λ there exist ω-homogeneous, but not ω-transitive permutation groups on λ (Corollary 4.9).

Our notation is standard.

Definition 1.1. If λ is fixed and $f \in S(A)$ for some $A \subset \lambda$, we take

$$f^+ = f \cup (\mathrm{id} \restriction (\lambda \setminus A)) \in S(\lambda).$$

Given a family of functions, $\mathcal{G}$, we say that a function y is *$\mathcal{G}$-large* iff

$$|y \setminus \bigcup \mathcal{H}| = |y|$$

for each finite $\mathcal{H} \subset \mathcal{G}$.

We say that a permutation group on A is *κ-intransitive* iff there is a G-large injective function y with $\mathrm{dom}(y) \cup \mathrm{ran}(y) \in \big[A\big]^\kappa$ and $|A \setminus \mathrm{dom}(y)| = |A \setminus \mathrm{ran}(y)| = |A|$.

A κ-intransitive group is clearly not κ-transitive.

2. ω-HOMOGENEOUS BUT NOT ω-TRANSITIVE

Definition 2.1. Given a set A we say that a family $\mathcal{A} \subset \big[A\big]^\omega$ is *nice on A* iff $\mathcal{A}$ has an enumeration $\{A_\alpha : \alpha < \mu\}$ such that

(N1) $\mathcal{A}$ is cofinal in $\langle \big[A\big]^\omega, \subset \rangle$,

(N2) for each $\beta < \mu$ there is a countable set $I_\beta \in \big[\beta\big]^\omega$ such that for all $\alpha < \beta$ there is a finite set $J_{\alpha,\beta} \in \big[I_\beta\big]^{<\omega}$ such that

$$A_\alpha \cap A_\beta \subset \bigcup_{\zeta \in J_{\alpha,\beta}} A_\zeta.$$

Theorem 2.2. *Assume that λ is an infinite cardinal, and $\mathcal{A} \subset \big[\lambda\big]^\omega$ is a nice family on λ. Then for each $A \in \mathcal{A}$ there is an ordering $\leq_A$ on A such that*

(1) $tp(A, \leq_A) = \omega$ for each $A \in \mathcal{A}$,
(2) if $A, B \in \mathcal{A}$, then there is a partition $\{C_i : i < n\}$ of $A \cap B$ into finitely many subsets such that $\leq_A \restriction C_i = \leq_B \restriction C_i$ for all $i < n$.

Proof. Fix an enumeration $\{A_\beta : \beta < \mu\}$ of $\mathcal{A}$ witnessing that $\mathcal{A}$ is nice.

We will define $\leq_{A_\beta}$ by induction on $\beta < \mu$.

Assume that $\leq_{A_\alpha}$ is defined for $\alpha < \beta$.

By (N2) we can fix a countable set $I_\beta = \{\beta_i : i < \omega\} \in [\beta]^\omega$ such that for all $\alpha < \beta$ there is $n_\alpha < \omega$ such that

$$A_\alpha \cap A_\beta \subset \bigcup_{i<n_\alpha} A_{\beta_i}.$$

Choose an order $\leq_{A_\beta}$ on A_β such that

(i) for each $i < \omega$ writing $D_i = A_{\beta_i} \setminus \bigcup_{j<i} A_{\beta_j}$ we have

$$\leq_{A_\beta} \restriction (A_\beta \cap D_i) \quad = \quad \leq_{A_{\beta_i}} \restriction (A_\beta \cap D_i);$$

(ii) $tp(A_\beta, \leq_{A_\beta}) = \omega$.

By induction on β we show that (2) holds for A_α and A_β for each $\alpha < \beta$. Assume that this statement holds for each $\beta' < \beta$. To check for β fix $\alpha < \beta$.

To define $\leq_\beta$ we considered a set $I_\beta = \{\beta_i : i < \omega\} \in [\beta]^\omega$ such that we had $n_\alpha < \omega$ with

$$A_\alpha \cap A_\beta \subset \bigcup_{i<n_\alpha} A_{\beta_i}.$$

For $i < n_\alpha$ let $C'_i = A_\alpha \cap A_\beta \cap D_i$, where $D_i = A_{\beta_i} \setminus \bigcup_{j<i} A_{\beta_j}$. Then $\{C'_i : i < n_\alpha\}$ is a partition of $A_\alpha \cap A_\beta$ and

$$\leq_{A_\beta} \restriction C'_i = \leq_{A_{\beta_i}} \restriction C'_i$$

by (i). By the inductive hypothesis, $A_{\beta_i} \cap A_\alpha$ has a partition into finitely many pieces $\{C_{i,j} : j < k_i\}$ such that $\leq_{A_\alpha} \restriction C_{i,j} = \leq_{A_{\beta_i}} \restriction C_{i,j}$. Then the partition

$$\{C'_i \cap C_{i,j} : i < n, j < k_i\}$$

of $A_\alpha \cap A_\beta$ works for α and β. Indeed,

$$\leq_{A_\alpha} \restriction C'_i \cap C_{i,j} \quad = \quad \leq_{A_{\beta_i}} \restriction C'_i \cap C_{i,j} \quad = \quad \leq_{A_\beta} \restriction C'_i \cap C_{i,j}.$$

□

Theorem 2.3. *Assume that λ is an infinite cardinal, $\mathcal{A} \subset [\lambda]^\omega$ is a cofinal family and for each $A \in \mathcal{A}$ we have an ordering $\leq_A$ on A such that*

(1) $tp(A, \leq_A) = \omega$ for each $A \in \mathcal{A}$,
(2) if $A, B \in \mathcal{A}$, then there is a partition $\{C_i : i < n\}$ of $A \cap B$ into finitely many subsets such that $\leq_A \restriction C_i = \leq_B \restriction C_i$ for all $i < n$.

Then there is a permutation group on λ that is ω-homogeneous and ω-intransitive.

Proof. For $A \in \mathcal{A}$ let

$$\mathcal{G}_A = \{f^+ \in \mathrm{S}(\lambda) : f \in \mathrm{S}(A) \wedge \text{there is a finite partition } \{C_i : i < n\} \text{ of } A \\ \text{such that } f \restriction C_i \text{ is } \leq_A\text{-order preserving}\}.$$

Let G be the permutation group on λ generated by

$$\bigcup \{\mathcal{G}_A : A \in \mathcal{A}\}.$$

Claim 2.3.1. *G is ω-homogeneous.*

Indeed, let $X, Y \in [\lambda]^\omega$ with $|\lambda \setminus X| = |\lambda \setminus Y| = \lambda$. Pick $A \in \mathcal{A}$ such that $X \cup Y \subset A$ and $|A \setminus X| = |A \setminus Y| = \omega$.

Let c be the unique $\leq_A$-monotone bijection between X and Y and d be the unique $\leq_A$-monotone bijection between $A \setminus X$ and $A \setminus Y$. Then taking $g = c \cup d$ we have $g^+ \in \mathcal{G}_A \subset G$ and $g^+[X] = Y$.

Claim 2.3.2. *G is ω-intransitive.*

Pick $A \in \mathcal{A}$ and choose $B \in [A]^\omega$ such that $|A \setminus B| = \omega$.

Let $b_0, b_1, \ldots$ be the $\leq_A$-increasing enumeration of B. Define a bijection $y : B \to \omega$ as follows: for $i < \omega$ and $j < 2^i$ let

$$y(b_{2^i+j}) = b_{2^{i+1}-j}.$$

Observe that if c is $\leq_A$-monotone then

$$|\{i < \omega : |\{j < 2^i : c(b_{2^i+j}) = r(b_{2^i+j})\}| \geq 2\}| \leq 1.$$

Indeed, if $|\{j < 2^i : c(b_{2^i+j}) = y(b_{2^i+j})\}| \geq 2$, then c should be $\leq_A$-decreasing, and if $|\{i : \{j < 2^i : c(b_{2^i+j}) = y(b_{2^i+j})\} \neq \emptyset\}| \geq 2$, then y should be $\leq_A$-increasing.

So y can not be covered by finitely many $\leq_A$-monotone functions. But for any $h \in G$, $h \cap (A \times A)$ can be covered by finitely many $\leq_A$-monotone functions by (2) and by the construction of G.

Thus y is G-large. □

To obtain nice families we recall some topological results. We say that a topological space X is *splendid* (see [2]) iff it is countably compact, locally compact, locally countable such that $|\overline{A}| = \omega$ for each $A \in [X]^\omega$.

We need the following theorem:

Theorem (Juhasz, Nagy, Weiss, [2])**.** *If*

(i) $\kappa < \omega_\omega$, *or*

(ii) $2^\omega < \kappa$, $\mathrm{cf}(\kappa) > \omega$ *and* $\mu^\omega = \mu^+$ *and* $\square_\mu$ *hold for each* $\mu < \kappa$ *with* $\omega = \mathrm{cf}(\mu) < \mu$,

then there is a splendid space X of size κ.

Remark. In [2, Theorem 11] the authors formulated a bit weaker result: *if $V = L$ and $\mathrm{cf}(\kappa) > \omega$ then there is a splendid space X of size κ.* However, to obtain that results they combined “Lemmas 7, 9 and 16 with the remark after Theorem 8" and their arguments used only the assumptions of the theorem above.

If $\mathcal{A}$ is a family of sets, and X is a set, write

$$\mathcal{A}\lceil X = \{A \cap X : A \in \mathcal{A}\}$$

and

$$\mathcal{A}\lceil^* X = \{\bigcap \mathcal{A}' \cap X : \mathcal{A}' \in \left[\mathcal{A}\right]^{<\omega}\}.$$

Lemma 2.4. *If X is a splendid space, $\mathcal{U}$ is the family of compact open subsets of X, and $Y \subset X$, then $\mathcal{U}\lceil Y$ is nice on Y.*

Proof. Let $A \in \left[Y\right]^{\omega}$. Then $\overline{A}$ is countable, so it is compact. Since a splendid space is zero-dimensional, A can be covered by finitely many compact open set, and so A can be covered by an element of $\mathcal{U}$. Thus $\mathcal{U}\lceil Y$ is cofinal in $\langle \left[Y\right]^{\omega}, \subset \rangle$.

To check (N2) observe that every $U \in \mathcal{U}$ is a countable compact space, so it is homeomorphic to a countable successor ordinal. Thus U has only countably many compact open subsets. Hence $\mathcal{U}\lceil U$ is countable which implies (N2) in the following stronger form:

(N2$^+$) for each $\beta < \mu$ there is a set $I_\beta \in \left[\beta\right]^{\omega}$ such that for all $\alpha < \beta$ there is $\zeta_\alpha \in I_\beta$ such that

$$A_\alpha \cap A_\beta = A_{\zeta_\alpha} \cap A_\beta.$$

□

Remark. By [3, Corollary 2.2], *if $(\omega_{\omega+1}, \omega_\omega) \to (\omega_1, \omega)$ holds, then the cardinality of a splendid space is less than ω_ω.* So we need some new ideas if we want to construct arbitrarily large nice families in ZFC.

Theorem 2.5. *If λ is an infinite cardinal, and*

(i) $\lambda < \omega_\omega$, or

(ii) $2^\omega < \lambda$, and $\mu^\omega = \mu^+$ and $\square_\mu$ hold for each $\mu \leq \lambda$ with $\omega = \mathrm{cf}(\mu) < \mu$.

then there is an ω-homogeneous and ω-intransitive permutation group on λ.

Proof. Applying the Juhasz-Nagy-Weiss theorem for $\kappa = \lambda$ if $\mathrm{cf}(\lambda) > \omega$, and for $\kappa = \lambda^+$ if $\lambda > \mathrm{cf}(\lambda) = \omega$, we obtain a splendid space on $\kappa \geq \lambda$. So, by Lemma 2.4, we obtain a nice family $\mathcal{A}$ on λ.

Thus, putting together Theorems 2.2 and 2.3 we obtained the desired permutation group on λ. □

3. κ-HOMOGENEOUS BUT NOT κ-TRANSITIVE FOR $\kappa > \omega$

Definition 3.1. Let $\kappa < \lambda$ be cardinals. We say that a cofinal family $\mathcal{A} \subset \left[\lambda\right]^{\kappa}$ is *locally small* iff $|\mathcal{A}\lceil A| \leq \kappa$ for all $A \in \mathcal{A}$.

Theorem 3.2. *Assume that $2^\kappa = \kappa^+$ and there is a cofinal, locally small family $\mathcal{A} \subset \left[\lambda\right]^\kappa$. Then there is a permutation group G on λ which is κ-homogeneous, but not κ-transitive.*

Before proving this theorem we need some preparation.

Definition 3.3. If X, Y are subsets of ordinals with the same order types, then let $\rho_{X,Y}$ be the unique order preserving bijection between X and Y.

Definition 3.4. If $\mathcal{F}$ is a set of functions, an $\mathcal{F} \cup \{x\}$-*term* t is a sequence $\langle h_0, \dots, h_{n-1}\rangle$, where $h_i = x$ or $h_i = x^{-1}$ or $h_i = f_i$ or $h_i = f_i^{-1}$ for some $f_i \in \mathcal{F}$. If g is function we use $t[g]$ to denote the function $h'_0 \circ h'_1 \circ \dots \circ h'_{n-1}$, where

$$h'_i = \begin{cases} f_i & \text{if } h_i = f_i, \\ f_i^{-1} & \text{if } h_i = f_i^{-1}, \\ g & \text{if } h_i = x, \\ g^{-1} & \text{if } h_i = x^{-1}. \end{cases}$$

If $\mathcal{H}$ is a set of $\mathcal{F} \cup \{x\}$-terms, then write

$$\mathcal{H}[g] = \{t[g] : t \in H\}.$$

We say that an $\mathcal{F} \cup \{x\}$-term t is an $\mathcal{F}$-*term* iff neither x nor x^{-1} appear in t. If t is a $\mathcal{F}$-term, then the function $t[g]$ does not depend on g, so we will write $t[\,]$ instead of $t[g]$ in that situation.

We say that a term t' is a *subterm* of a term $t = \langle h_0, \dots, h_{n-1}\rangle$ iff $t' = \langle h_{i_0}, h_{i_1}, \dots, h_{i_k}\rangle$, where $i_0 < i_1 < \dots < i_k < n$.

The set of all $\mathcal{F} \cup \{x\}$-terms is denoted by $TERM(\mathcal{F} \cup \{x\})$.

The set of all $\mathcal{F}$-terms is denoted by $TERM(\mathcal{F})$.

Lemma 3.5. *Assume that*

(1) λ is a cardinal, $\mathcal{H}$ is a finite set of $S(\lambda) \cup \{x\}$-terms, and $\mathcal{H}$ is closed for subterms,
(2) g is an injective function, $\operatorname{dom}(g) \cup \operatorname{ran}(g) \subset \lambda$,
(3) $\alpha, \alpha^* \in \lambda$ such that

$$\langle \alpha, \alpha^*\rangle \notin \bigcup \mathcal{H}[g],$$

(4) $\zeta_0 \in \lambda \setminus \operatorname{dom}(g)$ and $\zeta_1 \in \lambda \setminus \operatorname{ran}(g)$,
(5) $\eta_0 \in \lambda \setminus \operatorname{ran}(g)$ and $\eta_1 \in \lambda \setminus \operatorname{dom}(g)$ such that

$$\eta_0, \eta_1 \notin \{t[g](\alpha), t[g]^{-1}(\alpha^*) : t \in \mathcal{H}\}.$$

Let $g_0 = g \cup \{\langle \zeta_0, \eta_0\rangle\}$ and $g_1 = g \cup \{\langle \eta_1, \zeta_1\rangle\}$. Then

$$\langle \alpha, \alpha^*\rangle \notin \mathcal{H}[g_0] \cup \mathcal{H}[g_1].$$

Proof. We prove only $\langle \alpha, \alpha^*\rangle \notin \mathcal{H}[g_0]$. The proof of the other statement is similar.

Assume on the contrary that $\langle \alpha, \alpha^*\rangle \in \mathcal{H}[g_0]$.

Pick the shortest term $t = \langle f_0, \dots, f_n \rangle$ from $\mathcal{H}$ such that $t[g_0](\alpha) = \alpha^*$.

Write $\alpha_{n+1} = \alpha$ and $\alpha_i = \langle f_i, \dots, f_n \rangle [g_0](\alpha)$ for $0 \le i \le n$. Hence $\alpha_0 = \alpha^*$.

Let i maximal such that α_i is ζ_0 or η_0. Since $t[g](\alpha)$ can not be α^* by (3), i is defined.

Since $\alpha_i = \langle f_i, \dots, f_n \rangle [g](\alpha)$, it follows that $\alpha_i \ne \eta_0$ by (5). So $\alpha_i = \zeta_0$.

Let j minimal such that α_j is ζ_0 or η_0. Since

$$\alpha_j = (\langle f_0, \dots, f_{j-1} \rangle [g])^{-1}(\alpha^*),$$

it follows that $\alpha_j \ne \eta_0$ by (5). So $\alpha_j = \zeta_0$ by (5). Thus $\alpha_i = \alpha_j = \zeta_0$, and so

$$\alpha^* = \langle f_0, \dots, f_{j-1}, f_i, \dots, f_n \rangle [g_0](\alpha).$$

Since $j < i$, the term $t' = \langle f_0, \dots, f_{j-1}, f_i, \dots, f_n \rangle$ is shorter than t and still $\alpha^* = t'[g_0](\alpha)$. So the length of t was not minimal. Contradiction. $\square$

Lemma 3.6. *Assume that*

(1) $y \in \mathrm{S}(\kappa)$,
(2) $A \in \left[\lambda\right]^{\kappa}$, *and* $B, C \in \left[A\right]^{\kappa}$ *such that* $|A \setminus B| = |A \setminus C| = \kappa$,
(3) $\mathcal{F} \in \left[\mathrm{S}(\lambda)\right]^{\kappa}$ *such that*

$$|y \setminus \bigcup \mathcal{H}[\ ]| = \kappa$$

whenever $\mathcal{H}$ *is a finite set of* $\mathcal{F}$*-terms.*

Then there is $g \in \mathrm{S}(A)$ *such that*

(i) $g[B] = C$,
(ii)

$$|y \setminus \mathcal{H}[g^+]| = \kappa$$

whenever $\mathcal{H}$ *is a finite set of* $\mathcal{F} \cup \{x\}$*-terms.*

Proof of Lemma 3.6. Write

$$\mathbb{TASK}_0 = A \times \{\mathrm{dom}, \mathrm{ran}\} \text{ and } \mathbb{TASK}_1 = \left[TERM(\mathcal{F} \cup \{x\})\right]^{<\omega} \times \kappa.$$

Let $\{I_0, I_1\} \in \left[\left[\kappa\right]^{\kappa}\right]^2$ be a partition of κ, and fix enumerations $\{T_i : i \in I_0\}$ of $\mathbb{TASK}_0$, and $\{T_i : i \in I_1\}$ of $\mathbb{TASK}_1$.

By transfinite induction, for $i < \kappa$ we will construct a function g_i and if $i = j+1$ for some $j \in K_1$ then we also pick an ordinal $\alpha_{j+1} \in \kappa$ such that

(a) g_i is an injective function, $\mathrm{dom}(g_i) \cup \mathrm{ran}(g_i) \subset A$,
(b) $g_i[B] \subset C$ and $g_i[A \setminus B] \subset A \setminus C$;
(c) $|g_i| \le i$;
(d) if $i = j+1$, $j \in I_0$ and $T_j = \langle \zeta, \mathrm{dom} \rangle$, then $\zeta \in \mathrm{dom}(g_i)$;
(e) if $i = j+1$, $j \in I_0$ and $T_j = \langle \zeta, \mathrm{ran} \rangle$, then $\zeta \in \mathrm{ran}(g_i)$;
(f) if $i = j+1$, $j \in I_1$ and $T_j = \langle \mathcal{H}_j, \chi_j \rangle$, then

(i) $\alpha_{j+1} \in \kappa \setminus \{\alpha_{j'+1} : j' \in I_1 \cap j\}$, and
(ii) $t[g_i \cup \mathrm{id}_{\lambda\setminus A}](\alpha_{j+1})$ is defined and $t[g_i \cup \mathrm{id}_{\lambda\setminus A}](\alpha_{j+1}) \neq y(\alpha_{j+1})$ for each $t \in \mathcal{H}_j$.

Let $g_0 = \emptyset$.

If i is limit, then let $g_i = \bigcup_{j<i} g_j$.

Assume that $i = j + 1$.

Claim 3.6.1.

$$|y \setminus \bigcup \mathcal{H}[g_j \cup \mathrm{id}_{\lambda\setminus A}]| = \kappa. \tag{†}$$

for each finite set $\mathcal{H}$ of $\mathcal{F} \cup \{x\}$-terms.

Proof of the Claim. Fix $\mathcal{H}$. We can assume that $\mathcal{H}$ is closed for subterms. By (3) we have $|y \setminus \bigcup \mathcal{H}[\,]| = \kappa$, and

$$y \cap \bigcup \mathcal{H}[\,] = y \cap \bigcup \mathcal{H}[\mathrm{id}_{\lambda\setminus A}] \tag{○}$$

because $\mathcal{H}$ is closed for subterms. Since $|g_j| < \kappa$, we have

$$|t[\mathrm{g_j} \cup \mathrm{id}_{\lambda\setminus A}] \setminus t[\mathrm{id}_{\lambda\setminus A}]| < \kappa. \tag{•}$$

for each $t \in \mathcal{H}$. Putting together $|y \setminus \bigcup \mathcal{H}[\,]| = \kappa$, (○) and (•) we obtain (†). □

Case 1. *$j \in I_0$ and so $T_j = \langle \zeta_j, x_j \rangle \in A \times \{\mathrm{dom}, \mathrm{ran}\}$.*

Assume first that $x_j = \mathrm{dom}$. If $\zeta_j \in \mathrm{dom}(g_j)$, let $g_i = g_j$. If $\zeta_j \notin \mathrm{dom}(g_j)$, then pick $\eta \in C$ if $\zeta_i \in B$, and pick $\eta \in A \setminus C$ if $\zeta_i \in A \setminus B$ such that and $\eta \notin \mathrm{ran}(g_j)$.

Let $g_i = g_j \cup \langle \zeta_i, \eta \rangle$. Then g_i satisfies (a)–(f).

The case $x_j = \mathrm{ran}$ is similar.

Case 2. *$j \in I_1$ and so $T_j = \langle \mathcal{H}_j, \chi_j \rangle \in \left[TERM(\mathcal{F} \cup \{x\})\right]^{<\omega} \times \kappa$.*

We can assume that $\mathcal{H}_j$ is closed for subterms.

By Claim 3.6.1, we have

$$|y \setminus \bigcup \mathcal{H}_j[g_j \cup id_{(\lambda\setminus A)}]| = \kappa.$$

So we can pick $\alpha_{j+1} \in \kappa \setminus \{\alpha_{j'+1} : j' \in I_1 \cap j\}$ such that

(∗) for each $t \in \mathcal{H}_j$ either $t[g_j \cup \mathrm{id}_{\lambda\setminus A}](\alpha_{j+1})$ is undefined or $t[g_j \cup \mathrm{id}_{\lambda\setminus A}](\alpha_{j+1}) \neq y(\alpha_{j+1})$.

Now in finitely many steps, using Lemma 3.5, we can extend the function g_j to a function g_i such that

(∗) $t[g_i \cup \mathrm{id}_{\lambda\setminus A}](\alpha_{j+1})$ is defined and $t[g_i \cup \mathrm{id}_{\lambda\setminus A}](\alpha_{j+1}) \neq y(\alpha_{j+1})$ for each $t \in \mathcal{H}_j$.

Indeed, if $t[g' \cup \mathrm{id}_{\lambda\setminus A}](\alpha_{j+1})$ is not defined, where $t = \langle t_0, \ldots, t_n \rangle$ then there is $i < n$ such that either

$\zeta_i = \langle t_{i+1}, \ldots, t_n \rangle [g' \cup \mathrm{id}_{\lambda\setminus A}](\alpha_{j+1})$ is defined, $t_i = x$ and $\zeta_i \in A \setminus \mathrm{dom}(g')$

or

$\zeta_i = \langle t_{i+1}, \dots, t_n \rangle \, [g' \cup \mathrm{id}_{\lambda \setminus A}](\alpha_{j+1})$ is defined, $t_i = x^{-1}$ and $\zeta_i \in A \setminus \mathrm{ran}(g')$.

In both cases, using Lemma 3.5, we can extend g' to g'' such that $\langle t_i, \dots, t_n \rangle \, [g'' \cup \mathrm{id}_{\lambda \setminus A}](\alpha_{j+1})$ is defined and $\langle \alpha_{j+1}, y(\alpha_{j+1}) \rangle \notin \bigcup \mathcal{H}_j[g'' \cup id_{\lambda \setminus A}]$.

After the inductive construction, the function $g = \bigcup_{i<\kappa} g_i$ meets the requirements. □

Lemma 3.7. *Assume that $2^\kappa = \kappa^+$ and there is a cofinal, locally small subfamily $\mathcal{C} \subset \left[\lambda\right]^\kappa$. Then there is a family $\mathcal{D} \subset \left[\lambda\right]^\kappa \times \left[\lambda\right]^\kappa$ such that*

(1) if $\langle A, B \rangle \in \mathcal{D}$, then $B \cup \kappa \subset A$ and $|A \setminus B| = \kappa$.

Moreover, writing $\mathcal{A} = \{A : \langle A, B \rangle \in \mathcal{D}\}$ and $\mathcal{B} = \{B : \langle A, B \rangle \in \mathcal{D}\}$

(2) $\mathcal{A}$ is a cofinal, locally small subfamily of $\left[\lambda\right]^\kappa$,

(3) $\mathcal{B}$ is cofinal in $\langle \left[\lambda\right]^\kappa, \subset \rangle$,

(4) $\{X \subset \kappa : |X| = |\kappa \setminus X| = \kappa\} \subset \mathcal{B}$.

Proof of Lemma 3.7. Fix a locally small, cofinal subfamily $\mathcal{C} \subset \left[\lambda\right]^\kappa$ such that $\mu = |\mathcal{C}|$ is minimal. Then $|\{C \in \mathcal{C} : D \subset C\}| = |\mathcal{C}|$ for all $D \in \left[\lambda\right]^\kappa$.

Write $\mathcal{C} = \{C_\alpha : \alpha < \mu\}$. Since $2^\kappa = \kappa^+ \le \lambda \le \mu$ there is a sequence $\langle B_\alpha : \alpha < \mu \rangle \subset [\lambda]^\kappa$ such that

(a) $\{B_\alpha : \alpha < \kappa^+\} \supset \{X \subset \kappa : |X| = |\kappa \setminus X| = \kappa\}$,

(b) $\{B_\alpha : \alpha < \mu\} \supset \mathcal{C}$.

Thus $\mathcal{B} = \{B_\alpha : \alpha < \mu\}$ is cofinal in $[\lambda]^\kappa$. Now, for each $\alpha < \mu$ pick $A_\alpha \in \mathcal{C}$ such that $A_\alpha \supset C_\alpha \cup B_\alpha \cup \kappa$ and $|A_\alpha \setminus B_\alpha| = \kappa$.

Then $\mathcal{D} = \{\langle A_\alpha, B_\alpha \rangle : \alpha < \mu\}$ satisfies the requirements. □

After that preparation we prove the main theorem of this section.

Proof of Theorem 3.2. Fix $\mathcal{D}$, $\mathcal{A}$ and $\mathcal{B}$ as in Lemma 3.7.

For $\langle A, B \rangle \in \mathcal{D}$ consider the structure

$$\mathcal{M}_{\langle A,B \rangle} = \langle A, <, B, \{A \cap X : A \in \mathcal{A}\} \rangle \,.$$

Fix $\mathcal{D}' \in \left[\mathcal{D}\right]^{\kappa^+}$ such that writing $\mathcal{A}' = \{A' : \langle A', B' \rangle \in \mathcal{D}'\}$ and $\mathcal{B}' = \{B' : \langle A', B' \rangle \in \mathcal{D}'\}$ we have

(a) $\forall \langle A, B \rangle \in \mathcal{D} \; \exists \langle A', B' \rangle \in \mathcal{D}'$ such that $\rho_{A,A'}$ is an isomorphism between $\mathcal{M}_{\langle A,B \rangle}$ and $\mathcal{M}_{\langle A',B' \rangle}$.

(b) $\{X \subset \kappa : |X| = |\kappa \setminus X| = \kappa\} \subset \mathcal{B}'$.

Pick $K \in \left[\kappa\right]^\kappa$ with $|\kappa \setminus K| = \kappa$. Choose $y \in S(\kappa)$ such that $y(\alpha) \neq \alpha$ for each $\alpha \in \kappa$.

Lemma 3.8 (Key lemma)**.** *There are functions $\mathcal{F} = \{f_{\langle A,B \rangle} : \langle A, B \rangle \in \mathcal{D}'\}$ such that*

(a) $f_{\langle A,B \rangle} \in \mathrm{S}(A)$,

(b) $f_{\langle A,B \rangle}[B] = K$,

moreover, taking

$$\mathcal{S} = \big\{\rho_{C_0,C_1} : \langle A_0, B_0\rangle, \langle A_1, B_1\rangle \in \mathcal{D}', C_0 \in \mathcal{A}\lceil^* A_0, C_1 \in \mathcal{A}\lceil^* A_1, \rho_{C_0,C_1}[\mathcal{A}\lceil C_0] = \mathcal{A}\lceil C_1\big\},$$

if $\mathcal{H}$ is a finite collection of $\mathcal{F}\cup\mathcal{S}$-terms, then

$$|y \setminus \bigcup \mathcal{H}[\ ]| = \kappa.$$

Before proving the Key lemma, we show how the Key Lemma completes the proof of Theorem 3.2.

So assume that the Key lemma holds.

For each $\langle A, B\rangle \in \mathcal{D}$ pick $\langle A', B'\rangle \in \mathcal{D}'$ such that $\rho_{A,A'}$ is an isomorphism between $\mathcal{M}_{\langle A,B\rangle}$ and $\mathcal{M}_{\langle A',B'\rangle}$. We assume that $\langle A', B'\rangle = \langle A, B\rangle$ for $\langle A, B\rangle \in \mathcal{D}'$.

Let

$$g_{\langle A,B\rangle} = \rho_{A',A} \circ f_{\langle A',B'\rangle} \circ \rho_{A,A'} \in S(A).$$

Let G be the permutation group on λ generated by

$$\mathcal{G} = \{{g_{\langle A,B\rangle}}^+ : \langle A, B\rangle \in \mathcal{D}\}.$$

Lemma 3.9. *G is κ-homogeneous.*

Proof of Lemma 3.9. It is enough to show that for each $X \in \big[\lambda\big]^\kappa$ there is $g \in G$ with $g[X] = K$.

So fix $X \in \big[\lambda\big]^\kappa$. Pick $\langle A, B\rangle \in \mathcal{D}$ such that $X \subset B$.

Then

$$\begin{aligned} Z = g_{\langle A,B\rangle}[X] \subset g_{\langle A,B\rangle}[B] =& (\rho_{A',A} \circ f_{\langle A',B'\rangle} \circ \rho_{A,A'})[B] \\ =& (\rho_{A',A} \circ f_{\langle A',B'\rangle})[B'] = \rho_{A',A}[K] = K. \end{aligned}$$

Since $|Z| = |\kappa \setminus Z| = \kappa$, there is C such that $\langle C, Z\rangle \in \mathcal{D}'$. Then $f_{\langle C,Z\rangle}[Z] = K$. Thus ${g_{\langle C,Z\rangle}}^+[Z] = K$ because $\langle C', Z'\rangle = \langle C, Z\rangle$ and so $f_{\langle C,Z\rangle} = g_{\langle C,Z\rangle}$.

Thus $K = ({g_{\langle C,Z\rangle}}^+ \circ {g_{\langle A,B\rangle}}^+)[X]$. □

Lemma 3.10. *G is not κ-transitive.*

Proof of Lemma 3.10. We prove that $y \not\subset h$ for any $h \in G$.

Assume that

$$h = (g_0^+)^{\ell_0} \circ (g_1^+)^{\ell_1} \circ \cdots \circ (g_{n-1}^+)^{\ell_{n-1}},$$

where $g_i = g_{\langle A_i,B_i\rangle} = \rho_{A_i',A_i} \circ f_{A_i',B_i'} \circ \rho_{A_i,A_i'}$ and $\ell_i \in \{-1, 1\}$ for $i < n$.

Since $g_i^+ \setminus g_i$ is the identity function on $\lambda \setminus A_i$, we have

$$h \subset \bigcup\{(g_{i_0})^{\ell_{i_0}} \circ (g_{i_1})^{\ell_{i_1}} \circ \cdots \circ (g_{i_{k-1}})^{\ell_{i_{k-1}}} : k < n, i_0 < i_1 < \cdots < i_{k-1} < n\}.$$

Fix $k \le n$ and $i_0 < i_1 < \cdots < i_{k-1} < n$.

Observe that if $\ell_i = -1$ then

$$(g_i)^{\ell_i} = (\rho_{A_i',A_i} \circ f_{A_i',B_i'} \circ \rho_{A_i,A_i'})^{-1} = \rho_{A_i',A_i} \circ (f_{A_i',B_i'})^{-1} \circ \rho_{A_i,A_i'}.$$

So

$$(g_{i_0})^{\ell_{i_0}} \circ (g_{i_1})^{\ell_{i_1}} \circ \cdots \circ (g_{i_{k-1}})^{\ell_{i_{k-1}}} = \rho_{A'_{i_0},A_{i_0}} \circ (f_{A'_{i_0},B'_{i_0}})^{\ell_{i_0}} \circ \rho_{A_{i_0},A'_{i_0}} \circ \rho_{A'_{i_1},A_{i_1}} \circ (f_{A'_{i_1},B'_{i_1}})^{\ell_{i_1}} \circ \rho_{A_{i_1},A'_{i_1}} \circ$$

For $j < k$ let

$$\rho^*_j = \rho_{A_{i_j},A'_{i_j}} \circ \rho_{A'_{i_{j+1}},A_{i_{j+1}}}.$$

Observe that writing

$$C_{j+1} = \rho_{A_{i_{j+1}},A'_{i_{j+1}}}[A_{i_j} \cap A_{i_{j+1}}] \text{ and } C_j = \rho_{A_{i_j},A'_{i_j}}[A_{i_j} \cap A_{i_{j+1}}]$$

we have

$$\rho^*_j = \rho_{C_{j+1},C_j} \in \mathcal{S}$$

(see Figure 1).

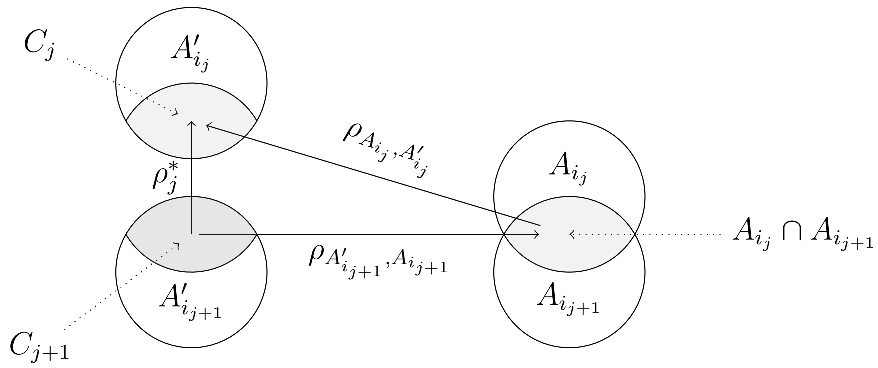

FIGURE 1. The function ρ^*_j

Thus

$$(g_{i_0})^{\ell_{i_0}} \circ (g_{i_1})^{\ell_{i_1}} \circ \cdots \circ (g_{i_{k-1}})^{\ell_{i_{k-1}}} = \rho_{A_{i_0},A'_{i_0}} \circ (f_{A'_{i_0},B'_{i_0}})^{\ell_0} \circ \rho^*_0 \circ (f_{A'_{i_1},B'_{i_1}})^{\ell_1} \circ \rho^*_1 \circ \dots \circ (f_{A'_{i_{k-1}},B'_{i_{k-1}}})^{\ell_{i_{k-1}}} \circ \rho_{A'_{i_{k-1}},A_{i_{k-1}}}.$$

Since $\rho_{A_\ell,A'_\ell} \restriction \kappa = \mathrm{id} \restriction \kappa$, we have

$$\big((g_{i_0})^{\ell_{i_0}} \circ (g_{i_1})^{\ell_{i_1}} \circ \cdots \circ (g_{i_{k-1}})^{\ell_{i_{k-1}}}\big) \cap \kappa \times \kappa \subset (f_{A'_{i_0},B'_{i_0}})^{\ell_0} \circ \rho^*_0 \circ (f_{A'_{i_1},B'_{i_1}})^{\ell_1} \circ \rho^*_1 \circ \dots \circ (f_{A'_{i_{k-1}},B'_{i_{k-1}}})^{\ell_{i_{k-1}}}$$

But $(f_{A'_{i_0},B'_{i_0}})^{\ell_0} \circ \rho^*_0 \circ (f_{A'_{i_1},B'_{i_1}})^{\ell_1} \circ \rho^*_1 \circ \cdots \circ (f_{A'_{i_{k-1}},B'_{i_{k-1}}})^{\ell_{i_{k-1}}} = t[]$ for the $\mathcal{F} \cup \mathcal{S}$-term $t = \Big\langle (f_{A'_{i_0},B'_{i_0}})^{\ell_0}, \rho^*_0, (f_{A'_{i_1},B'_{i_1}})^{\ell_1}, \rho^*_1, \dots, (f_{A'_{i_{k-1}},B'_{i_{k-1}}})^{\ell_{i_{k-1}}} \Big\rangle$.

Since there are only finitely many sequences $i_0 < \cdots < i_{k-1} < n$, we obtain that $h \cap \kappa \times \kappa$ is covered by the union of finitely many $\mathcal{F} \cup \mathcal{S}$-terms.

But y is not covered by the union of finitely many $\mathcal{F} \cup \mathcal{S}$-terms. So y witnesses that G is not κ-transitive. □

Proof of the Key Lemma 3.8. Write $\mathcal{D}' = \{\langle A_\alpha, B_\alpha \rangle : \alpha < \kappa^+\}$.

By transfinite induction, we define functions $\{f_\alpha : \alpha < \kappa^+\}$ such that taking

$$\mathcal{F}_{<\beta} = \{f_\gamma : \gamma < \beta\}$$

and

$$\mathcal{S}_{<\beta} = \{\rho_{C_0,C_1} : \delta, \gamma < \beta, C_0 \in \mathcal{A}\lceil^* A_\delta, C_1 \in \mathcal{A}\lceil^* A_\gamma, \\ \rho_{C_0,C_1}[\mathcal{A}\lceil C_0] = \mathcal{A}\lceil C_1\},$$

we have

(i) $f_\alpha \in \mathrm{S}(A_\alpha)$,
(ii) $f_\alpha[B_\alpha] = K$,
(iii) if $\mathcal{H}$ is a finite collection of $\mathcal{F}_{<\alpha+1} \cup \mathcal{S}_{<\alpha+1}$-terms, then

$$|y \setminus \mathcal{H}[\ ]| = \kappa.$$

Assume that we have constructed f_β for $\beta < \alpha$. Then we have:

if $\mathcal{H}$ is a finite collection of $\mathcal{F}_{<\alpha} \cup \mathcal{S}_{<\alpha}$-terms, then $|y \setminus \mathcal{H}[\ ]| = \kappa$. (∗)

To continue the construction we need a bit more.

Claim 3.10.1. *If $\mathcal{H}$ is a finite collection of $\mathcal{F}_{<\alpha} \cup \mathcal{S}_{<\alpha+1}$-terms, then*

$$|y \setminus \mathcal{H}[\ ]| = \kappa.$$

Proof. First observe that if $\rho_i = \rho_{A_i,A_i^*}$ for $i < 2$, then

$$\rho_1 \circ \rho_0 = \rho_{\rho_0^{-1}[A_0^* \cap A_1], \rho_1[A_0^* \cap A_1]}. \tag{‡}$$

Let

$$t = \langle t_0, t_1, \ldots, t_n \rangle$$

be an element of $\mathcal{H}$. Since $\rho_{C_0,C_1} \restriction \kappa = \mathrm{id} \restriction \kappa$, if $t_0 \in \mathcal{S}_{<\alpha+1}$, then $t[\ ] \cap \kappa \times \kappa = \langle t_1, \ldots t_n \rangle [\ ] \cap \kappa \times \kappa$. So we can assume that $t_0 \in \mathcal{F}_{<\alpha}$. Similar argument give that we can assume that $t_n \in \mathcal{F}_{<\alpha}$.

Now assume that

$$\langle t_i, \ldots, t_j \rangle = \left\langle f_{\alpha_i}, \rho_{C_{i+1},D_{i+1}}, \rho_{C_{i+2},D_{i+2}}, \ldots, \rho_{C_{j-1},D_{j-1}}, f_{\alpha_j} \right\rangle$$

Then, by (‡)

$$\rho_{C_{i+1},D_{i+1}} \circ \rho_{C_{i+2},D_{i+2}} \circ \cdots \circ \rho_{C_{j-1},D_{j-1}} = \rho_{E_i,E_j}.$$

for some $E_i \in \mathcal{A}\lceil C_{i+1}$ and $E_j \in \mathcal{A}\lceil D_{j-1}$.

Thus we can assume that $j = i+2$ and

$$\langle t_i, t_{i+1}, t_{i+2} \rangle = \langle f_{\alpha_0}, \rho_{E_0,E_1}, f_{\alpha_1} \rangle.$$

Now

$$f_{\alpha_0} \circ \rho_{E_0,E_1} \circ f_{\alpha_1} = f_{\alpha_0} \circ \rho_{A_{\alpha_0} \cap E_0, A_{\alpha_1} \cap E_1} \circ f_{\alpha_1}$$

and $\rho_{A_{\alpha_0} \cap E_0, A_{\alpha_1} \cap E_1} \in \mathcal{S}_{<\alpha}$.

Thus there is a $\mathcal{F}_{<\alpha} \cup \mathcal{S}_{<\alpha}$-term s_t such that

$$t[\,] \cap (\kappa \times \kappa) = s_t[\,] \cap (\kappa \times \kappa).$$

Since $|y \setminus \bigcup\{s_t[\,] : t \in \mathcal{H}\}| = \kappa$ by $(*)$, the Claim holds. $\square$

Since the claim holds, we can apply Lemma 3.6 for the family $\mathcal{F} = \mathcal{F}_{<\alpha} \cup \mathcal{S}_{<\alpha+1}$ to obtain f_α as g.

So we proved the Key Lemma 3.8. $\square$

So we proved theorem 3.2 $\square$

The following theorem is hidden in [5]:

Theorem 3.11. *If $\kappa^\omega = \kappa$, $\lambda = \kappa^{+n}$ for some $n < \omega$, and $\square_\nu$ holds for each $\kappa \le \nu < \lambda$, then there is a cofinal, locally small family in $\left[\lambda\right]^\kappa$.*

Indeed, in subsection 2.4 of [5] the author defines the *weakly rounded* subsets of $\lambda = \kappa^{+n}$, in Lemma 2.4.1 he shows that the family of weakly rounded sets is cofinal, finally on page 52 he proves a Claim which clearly implies that the family of weakly rounded sets is locally small.

Putting together Theorems 3.2 and 3.11 we obtain the following corollary.

Corollary 3.12. *If $\kappa^\omega = \kappa$, $\lambda = \kappa^{+n}$ for some $n < \omega$, and $\square_\nu$ holds for each $\kappa \le \nu < \lambda$, then there is a κ-homogeneous, but not κ-transitive permutation group on λ.*

4. ω-homogeneous but not ω-transitive permutation groups in the Cohen model

Let $MA(countable)$ denote the Martin's Axiom restricted to countable partial orderings.

For $f \in \mathrm{S}(\lambda)$ let $\mathrm{supp}(f) = \{\alpha : f(\alpha) \ne \alpha\}$. Write

$$\mathrm{S}_\omega(\lambda) = \{f \in \mathrm{S}(\lambda) : |\,\mathrm{supp}(f)| \le \omega\}.$$

Theorem 4.1. *If MA(countable) holds and $H \le \mathrm{S}_\omega(\omega_1)$ is a permutation group with $|H| < 2^\omega$, then there is an ω-homogeneous, but ω-intransitive permutation group $H^* \le \mathrm{S}_\omega(\omega_1)$ with $H^* \supset H$.*

Proof of Theorem 4.1. If $\mathcal{F}$ is a set of functions, let

$$\langle \mathcal{F} \rangle_{gen} = \{f_0 \circ \dots \circ f_{n-1} : n \in \omega, f_i \in \mathcal{F} \text{ or } f_i^{-1} \in \mathcal{F} \text{ for } i < n\}.$$

Lemma 4.2. *If $\mathcal{H}$ is a family of functions with $|\mathcal{H}| < 2^\omega$ then some $r \in S(\omega)$ is $\mathcal{H}$-large.*

Proof. Fix a family $\{r_\alpha : \alpha < 2^\omega\} \subset \mathrm{S}(\omega)$ such that $r_\alpha \cap r_\beta$ is finite for each $\{\alpha, \beta\} \in [2^\omega]^2$.

Assume on the contrary that for each $\alpha < 2^\omega$ the permutation r_α is not $\mathcal{H}$-large, i.e. there is $\mathcal{H}_\alpha \in [\mathcal{H}]^{<\omega}$ such that $r_\alpha \setminus \bigcup \mathcal{H}_\alpha$ is finite.

Let $\mathcal{U}$ be a non-principal ultrafilter on ω. Then for each $\alpha < 2^\omega$ there is $h(\alpha) \in \mathcal{H}_\alpha$ such that $U_\alpha = \{n \in \omega : r_\alpha(n) = h(\alpha)(n)\} \in \mathcal{U}$.

Since $|\mathcal{H}| < 2^\omega$, there are $\alpha \neq \beta$ such that $h(\alpha) = h(\beta)$. Thus for each $n \in U_\alpha \cap U_\beta$ we have $r_\alpha(n) = h(\alpha)(n) = h(\beta)(n) = r_\beta(n)$. Thus $r_\alpha \cap r_\beta$ is infinite. Contradiction. $\square$

Using Lemma 4.2 fix an H-large $r \in S(\omega)$. Enumerate $[\omega_1]^\omega \times [\omega_1]^\omega$ as $\{\langle A_\alpha, B_\alpha\rangle : \alpha < 2^\omega\}$. By transfinite recursion on $\alpha < 2^\omega$, we will construct permutations $f_\alpha \in \mathrm{S}_\omega(\omega_1)$ such that $f_\alpha[A_\alpha] = B_\alpha$ and writing

$$\mathcal{F}_\delta = \{t[] : t \text{ is a } H \cup \{f_\zeta : \zeta < \delta\}\text{-term}\} = \langle H \cup \{f_\zeta : \zeta < \delta\}\rangle_{gen},$$

the permutation r is $\mathcal{F}_{\alpha+1}$-large.

Since $\mathcal{F}_0 = H$, we know that $r \in S(\omega)$ is $\mathcal{F}_0$-large.

Assume that we have constructed $\langle f_\zeta : \zeta < \alpha\rangle$ such that the function r is $\mathcal{F}_{\zeta+1}$-large for $\zeta < \alpha$. Then r is $\mathcal{F}_\alpha$-large. Next we should construct $f_\alpha \in S(\omega_1)$ such that $f_\alpha[A_\alpha] = B_\alpha$ and r is $\mathcal{F}_{\alpha+1}$-large. We want to apply MA(countable) to construct f_α, but to do so we need some technical lemmas.

Fix first $C_\alpha \in [\omega_1]^\omega$ such that $A_\alpha \cup B_\alpha \subset C_\alpha$ and $C_\alpha \setminus (A_\alpha \cup B_\alpha) = \omega$.

Definition 4.3. Given sets X and Y let us denote by $\mathrm{Bij_p}(X,Y)$ the set of all finite bijections between subsets of X and Y.

For $A, B, C \in [\omega_1]^\omega$ define the poset $\mathcal{P}_{C,A,B} = \langle P_{C,A,B}, \leq\rangle$ as follows. Let

$$P_{C,A,B} = \{p \in \mathrm{Bij_p}(C,C) : p[A] \subset B, p[C \setminus A] \subset C \setminus B\}.$$

Write $p \leq q$ iff $p \supseteq q$.

We want to apply MA(countable) for the countable poset

$$\mathcal{P} = \mathcal{P}_{C_\alpha, A_\alpha, B_\alpha}.$$

Our plan is to define a family $\mathbb{D}$ of dense subsets in P with $|\mathbb{D}| < 2^\omega$ such that if $\mathcal{K}$ is a $\mathbb{D}$-generic filter in P, then $(\bigcup \mathcal{K}) \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}$ works as f_α.

Lemma 4.4. *For $i \in C_\alpha$ the sets $D_i = \{p \in P_{C,A,B} : i \in \mathrm{dom}(p)\}$ and $R_i = \{p \in P_{C,A,B} : i \in \mathrm{ran}(p)\}$ are dense in P.*

Proof. Straightforward. $\square$

Lemma 4.5. *If $M \in \omega$ and $\mathcal{H}$ is a finite set of $\mathcal{F}_\alpha \cup \{x\}$-terms then*

$$E_{\mathcal{H},M} = \{p \in P : \exists m \in \omega \setminus M$$
$$t[p](m) \textit{ is defined, but } t[p](m) \neq r(m) \textit{ for each } t \in \mathcal{H}\}$$

is dense in P.

Proof of the lemma. Fix $q \in P$. We can assume that $\mathcal{H}$ is closed for subterms.

We know that $|r \setminus \bigcup \mathcal{H}[\,]| = \omega$ because r is $\mathcal{F}_\alpha$-large.

Since $\mathcal{H}$ is closed for subterms,

$$r \cap \bigcup \mathcal{H}[\,] = r \cap \bigcup \mathcal{H}[\mathrm{id}_{\omega_1 \setminus C_\alpha}].$$

Since $|q| < \omega$, we have

$$|r \setminus \bigcup \mathcal{H}[q \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}]| = \omega.$$

So we can pick $m \in \omega \setminus M$ such that

($*$) for each $t \in \mathcal{H}$ either $t[q \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}](m)$ is undefined or $t[q \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}](m) \neq r(m)$.

Since $\mathcal{H}$ is finite, we can find $p \leq q$ such that

($\star$) for each $t \in \mathcal{H}$ either $t[p \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}](m)$ is undefined or $t[p \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}](m) \neq r(m)$,

($\bullet$) the cardinality of the finite set

$$\{t \in \mathcal{H} \ : \ t[p \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}](m) \text{ is undefined}\}$$

is minimal.

To show that $p \in E_{\mathcal{H},M}$ we prove that

($\circ$) there is no $t \in \mathcal{H}$ such that $t[p \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}](m)$ is undefined.

Assume on the contrary that this statement is not true.

Fix $t \in \mathcal{H}$ such that $t[p \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}](m)$ is not defined, where $t = \langle t_0, \ldots, t_n \rangle$. Thus there is $i < n$ such that

(1) $\langle t_{i+1}, \ldots, t_n \rangle \, [p \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}](m)$ is defined, but
(2) $\langle t_i, \ldots, t_n \rangle \, [p \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}](m)$ is not defined.

Then $t' = \langle t_i, \ldots, t_n \rangle \in \mathcal{H}$. Let $\zeta_i = \langle t_{i+1}, \ldots, t_n \rangle \, [p \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}](m)$. Then either $t_i = x$ and $\zeta_i \notin \mathrm{dom}(p)$ or $t_i = x^{-1}$ and $\zeta_i \notin \mathrm{ran}(p)$.

In both cases, using Lemma 3.5, we can extend p to p' such that $\langle t_i, \ldots, t_n \rangle \, [p' \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}](m)$ is defined and $\langle m, r(m) \rangle \notin \mathcal{H}[p' \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}]$. Thus $p' \leq q$ and

$$\{t \in \mathcal{H} \ : \ t[p' \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}](m) \text{ is undefined}\} \subsetneq \{t \in \mathcal{H} \ : \ t[p \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}](m) \text{ is undefined}\}$$

which contradicts ($\bullet$).

So we proved Lemma 4.5. $\square$

Let

$$\mathbb{D} = \{D_i, R_i : i \in C_\alpha\} \cup \{E_{\mathcal{F},M} : M \in \omega, \ \mathcal{F} \text{ is a finite set of } \mathcal{F}_\alpha \cup \{x\}\text{-terms.}\}$$

Then $\mathbb{D}$ is a family of dense sets in $P_{C_\alpha, A_\alpha, B_\alpha}$ with cardinality $< 2^\omega$. So, by MA(countable), there is a $\mathbb{D}$-generic filter $\mathcal{K}$. Let $f_\alpha = (\bigcup \mathcal{K}) \cup id_{\omega_1 \setminus C_\alpha}$

The assumption $\{D_i, R_j : i \in C_\alpha\} \subset \mathbb{D}$ yields $C_\alpha = \mathrm{dom}(\bigcup \mathcal{K}) = \mathrm{ran}(\bigcup \mathcal{K})$. Since $f_\alpha[A_\alpha] \subset B_\alpha$ and $f_\alpha[C_\alpha \setminus A_\alpha] \subset C_\alpha \setminus B_\alpha$ by the construction of $P_{C_\alpha, A_\alpha, B_\alpha}$ we have $f_\alpha[A_\alpha] = B_\alpha$.

If $\mathcal{F}$ is a finite subset of $\mathcal{F}_{\alpha+1}$, then there is a finite set $\mathcal{H}$ of $\mathcal{F}_\alpha \cup \{x\}$-terms such that

$$\mathcal{F} = \{t[f_\alpha] : t \in \mathcal{H}\}.$$

Then $E_{\mathcal{H},M} \cap \mathcal{K} \neq \emptyset$ implies that there is $m > M$ such that $r(m) \notin \{t[f_\alpha](m) : t \in \mathcal{H}\} = \{f(m) : f \in \mathcal{F}\}$. Thus r is $\mathcal{F}_{\alpha+1}$-large. Hence f_α satisfies the requirements.

So we carried out the inductive construction, and so we have constructed $\langle f_\alpha : \alpha < 2^\omega \rangle$ such that r is $\mathcal{F}_{2^\omega}$-large. So the group $H^* = \mathcal{F}_{2^\omega}$ satisfies the requirements. This completes the proof of Theorem 4.1. $\square$

Next we need a "stepping-up" theorem.

Theorem 4.6. *Assume that $\lambda \geq \omega_1$ is a cardinal, $G \leq \mathrm{S}(\lambda)$ and $H^* \leq \mathrm{S}(\omega_1)$ are permutation groups such that*

(i) H^* is ω-homogeneous, but ω-intransitive,

(ii) $\forall g \in G\ \forall \delta < \omega_1\ \exists h \in H^*\ g \cap (\delta \times \delta) \subset h$.

(iii) $\{g[\omega] : g \in G\}$ is cofinal in $\langle [\lambda]^\omega, \subset \rangle$.

Then $G^* = \langle G \cup \{h^+ : h \in H\} \rangle_{gen} \leq \mathrm{S}(\lambda)$ is ω-homogeneous, but ω-intransitive.

Proof of Theorem 4.6. First we show that G^* is ω-homogeneous.

Let $X, Y \in [\lambda]^\omega$ be arbitrary. First, by (iii) we can pick $f, g \in G$ such that $f[\omega] \supset X$ and $g[\omega] \supset Y$. Since H^* is ω-homogeneous, there is $h \in H^*$ such that

$$h[f^{-1}(X)] = g^{-1}(Y).$$

Then $g \circ h^+ \circ f^{-1} \in G^*$ and $(g \circ h^+ \circ f^{-1})[X] = Y$.

Next we show that G^* is ω-intransitive. Fix a countable injective function function r with $\mathrm{dom}(r) \cup \mathrm{ran}(r) \in [\omega_1]^\omega$ which is H^*-large. Without loss of generality we can assume that $r \in S(\gamma)$ for some $\gamma < \omega_1$. We will verify that

$$r \text{ is } G^*\text{-large}$$

as well. It is enough to show that

Lemma 4.7. *For each $g \in G^*$ there is a finite subset H_g of H^* such that*

$$g \cap (\gamma \times \gamma) \subset \bigcup H_g.$$

Proof of the Lemma. Since $G^* = \langle G \cup H^+ \rangle_{gen}$, where $H^+ = \{h^+ : h \in H^*\}$ and both G and H^+ are subgroups, we can assume that

$$g = e_0 \circ g_0 \circ \cdots \circ e_n \circ g_n$$

where $g_i \in G$ and $e_i \in H^+$.

For $e \in H^+$, write $e^- = e \restriction \omega_1 \in H^*$.

By finite induction, define countable subsets $A_{n+1}, B_n, A_n, \ldots, B_0, A_0$ of λ as follows: let $A_{n+1} = \gamma$ and $B_i = g_i[A_{i+1}]$ and $A_i = e_i[B_i]$ for $i = n, n-1, \ldots, 0$.

Pick $\delta < \omega_1$ with

$$\bigcup \{A_i, B_i : 0 \leq i \leq n+1\} \cap \omega_1 \subset \delta.$$

For $0 \le k < m \le n$ let

$$g_{k,m} = g_k \circ \cdots \circ g_{m-1}.$$

By (ii) we can pick $h_{k,m} \in H^*$ such that $h_{k,m} \supset g_{k,m} \cap (\delta \times \delta)$. Let

$$\mathcal{H}_g = \{e^-_{i_0} \circ h_{i_0,i_1} \circ e^-_{i_1} \circ h_{i_1,i_2} \circ \cdots \circ e^-_{i_\ell} \circ h_{i_\ell,i_{\ell+1}} : 0 \le i_0 < \cdots < i_\ell < i_{\ell+1} = n\}.$$

Claim 4.7.1. $g \cap (\gamma \times \gamma) \subset \bigcup \mathcal{H}_g$.

Proof of the Claim. Let $\alpha \in \gamma$ be arbitrary with $g(\alpha) \in \gamma$. Write $\alpha_{n+1} = \alpha$, $\beta_i = g_i(\alpha_{i+1})$ and $\alpha_i = e_i(\beta_i)$ for $i = n, n-1, \dots, 0$. So $\alpha_0 = g(\alpha) \in \gamma$.

Let $i_0 = 0 < \cdots < i_s = n+1$ be the enumeration of the set $I = \{i \le n+1 : \alpha_i \in \omega_1\} = \{i \le n+1 : \alpha_i \in \delta\}$.

Fix $\ell < s$, and write $k = i_\ell$ and $m = i_{\ell+1}$.

If $k+1 = m$, then $\alpha_k, \beta_k, \alpha_m \in \delta$ and so then

$$\alpha_k = e_k(\beta_k) = e_k(g_k(\alpha_m)) = (e^-_k \circ h_{k,m})(\alpha_m).$$

If $k+1 < m$, then

(i) $\alpha_k \in \delta$, $\beta_m \in \delta$, but
(ii) $\alpha_i, \beta_i \in \lambda \setminus \omega_1$ and so $\alpha_i = \beta_i$ for $k < i < m$,

(see Figure 2).

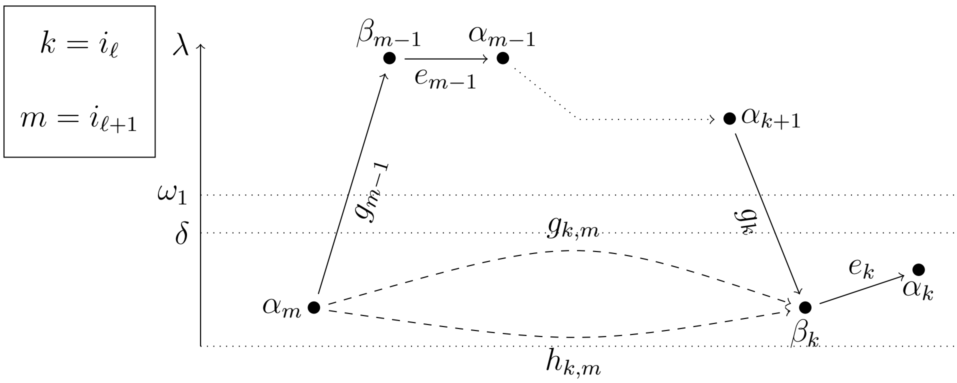

FIGURE 2. The function $h_{k,m}$

Thus

$$\beta_k = (g_k \circ e_k \circ g_{k+1} \cdots \circ e_{m-1} \circ g_{m-1})(\alpha_m) = = (g_k \circ g_{k+1} \circ \cdots \circ g_{m-1})(\alpha_m) = g_{k,m}(\alpha_m) = h_{k,m}(\alpha_m),$$

and so

$$\alpha_k = e_k(\beta_k) = e_k(h_{k,m}(\alpha_m)) = (e^-_k \circ h_{k,m})(\alpha_m).$$

Hence

$$g(\alpha) = (e_0 \circ g_0 \circ \cdots \circ e_n \circ g_n)(\alpha) = (e_{i_0}^- \circ h_{i_0,i_1} \circ \cdots \circ e_{i_\ell}^- \circ h_{i_{s-1},i_s})(\alpha)$$

and $(e_{i_0}^- \circ h_{i_0,i_1} \circ \cdots \circ e_{i_\ell}^- \circ h_{i_{s-1},i_s}) \in \mathcal{H}_g$. □

So we proved the Claim which completes the proof of the Lemma. □

As we observed, the previous lemma implies that r is G^*-large, and so G^* is ω-intransitive which completes the proof of Theorem 4.6. □

Putting together Theorems 4.1 and 4.6 we can get the following result.

Theorem 4.8. *Assume that λ is an uncountable cardinal and there is a permutation group $G \leq \mathrm{S}_\omega(\lambda)$ such that*

(1) $|\{g \cap (\omega_1 \times \omega_1) : g \in G\}| < 2^\omega$.
(2) $\{g[\omega] : g \in G\}$ *is cofinal in* $\langle [\lambda]^\omega, \subset \rangle$.

If MA(countable) holds, then there is an ω-homogeneous but not ω-transitive permutation group $G^* \leq \mathrm{S}_\omega(\lambda)$ with $G^* \supset G$.

Proof of Theorem 4.8. First observe that (2) implies that $|\{g \cap (\omega_1 \times \omega_1) : g \in G\}| \geq \omega_1$, and so $2^\omega > \omega_1$ by (1).

For each countable injective function f with $\mathrm{dom}(f) \cup \mathrm{ran}(f) \subset \omega_1$ pick a permutation $h(f) \in \mathrm{S}_\omega(\omega_1)$ with $h(f) \supset f$.

Let

$$H = \langle \{h(g \cap (\alpha \times \alpha)) : g \in G, \alpha < \omega_1\} \rangle_{gen}.$$

Since $2^\omega > \omega_1$, we have

(3) $|H| \leq |\{g \cap (\omega_1 \times \omega_1) : g \in G\}| \cdot \omega_1 < 2^\omega$, and
(4) $\forall g \in G\ \forall \alpha < \omega_1\ \exists h \in H$ such that $g \cap (\alpha \times \alpha) \subset h$.

By (3) we can apply Theorem 4.1 and so there is an ω-homogeneous, but ω-intransitive permutation group $H^* \leq \mathrm{S}_\omega(\omega_1)$ with $H^* \supset H$.

By (2) and (4) we can apply Theorem 4.6 for G and H^* to show that the permutation group $G^* = \langle G \cup \{h^+ : h \in H^+\} \rangle_{gen} \leq \mathrm{S}_\omega(\lambda)$ is ω-homogeneous, but ω-intransitive. □

Given sets X and Y let us denote by $\mathrm{Fin}(X, Y)$ the following poset: its underlying set is the set of all finite functions mapping a finite subset of X into Y, and $p \leq_{\mathrm{Fin}(X,Y)} q$ iff $p \supseteq q$. In particular, $\emptyset$ is the greatest element of $\mathrm{Fin}(X, 2)$.

Corollary 4.9. *If $P = \mathrm{Fin}((2^\omega)^+, 2)$ then*

$$V^P \models \text{“for each } \lambda \geq \omega_1 \text{ there is an } \omega\text{-homogeneous, but not } \omega\text{-transitive permutation group on } \lambda.\text{”}$$

Remark. In section 2 we showed that if there is a splendid space of cardinality at least λ, then there is a ω-homogeneous but not ω-transitive permutation group on λ. However, it was proved in [3] that it is consistent (modulo some large cardinal assumption), that there is no splendid space of size at least $\aleph_{\omega+1}$ in any c.c.c. generic extension of a certain ZFC model.

Proof of Corollary 4.9 from Theorem 4.8. We work in V^P. Let $G = \mathrm{S}_\omega(\lambda)^V$. Then

$$|\{g \cap \omega_1 \times \omega_1 : g \in G\}| = |\,\mathrm{S}_\omega(\omega_1)^V| = (2^\omega)^V < ((2^\omega)^+)^V = (2^\omega)^{V^P}.$$

So (1) holds. Since P is c.c.c., $\{g[\omega] : g \in G\} = [\lambda]^\omega \cap V$ is cofinal in $\langle [\lambda]^\omega, \subset \rangle$. Hence (2) also holds.

So we can apply Theorem 4.8 because it is known that MA(countable) holds after adding $(2^\omega)^+$-many Cohen reals to a ground model, (e.g. $cov(\mathcal{M}) = 2^\omega$ in the Cohen model by [1, Table 4], and $cov(\mathcal{M}) = 2^\omega$ implies MA(countable) by [4, Theorem 1]). $\square$

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