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SAHARON SHELAH

ABSTRACT. We find new “reasons” for a class of models for not having a universal model in a cardinal λ . This work, though has consequences in model theory, is really combinatorial. We concentrate on a prototypical class which is a simply defined class of models, of combinatorial character - models of T_{ceq} (essentially another representation of T_{feq} which was already considered but the proof with T_{ceq} is more transparent). Models of T_{ceq} consist essentially of an equivalence relation on one set and a family of choice functions for it. This class is not simple (in the model theoretic sense) but seems to be very low among the non-simple (first order complete countable) ones. We give sufficient conditions for the non-existence of a universal model for it in λ .

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Annotated Content

- §0 Introduction, (labels z,w), pg.3
- §1 For Mahlo cardinals, (label a), pg.6
- §2 On Successor Cardinals and Club Guessing, (label b), pg.10

§ 0. INTRODUCTION

On a recent survey on the universality spectrum see [She21], an earlier survey is [Dža05]; there have been several advances meanwhile (and this is one of the advances after [She21]). The problem for general first order theories is a model theoretic one, but specific examples are combinatorial set theoretic ones (and serve as proto-types for suitable families of theories); so combinatorialists may ignore model theoretic notions like “ T is simple, have the tree property, is TP_2 ”, and consider only the concrete universal theories considered; so ignore 1.4(1),(2) and their proof. Here we concentrate on the theory T_{ceq} , which we considered as a proto-typical “minimal” non-simple T , so are expecting it (under \leq_{univ}) to be low, so is it (like T_{feq} , see below, $NSOP_1$, see [She93b], [DS04b], [SU08], [CR16], [KR17a], [KR17b]). True, there were non-existence results near a strong limit singular cardinal (see on the T_{feq} in [She93b], generalizing it the oak property [DS06], [She17, §3]), but there were weak consistency results on existence (see [She93b], [DS04a]). We had considered T_{feq} , a prototypical example of such theories, now T_{ceq} is essentially equivalent to it for our aims, see 1.4(3),(4) but T_{ceq} seem more transparent; we intend to deal with “to what family of T ’s versions of our proof apply, in particular, NPT_2 and non-simple” elsewhere ([S⁺a]).

We have hoped/expected that for the $\lambda > \mu = \mu^{<\mu}$ but $\lambda = \mu^+ < 2^\mu$ we shall have consistency results for theories like T_{feq} and the class of triangle free graphs, see [S⁺b], [Shee].

We first give a case with stronger set theoretic assumptions, but more transparent proof in §1. In §2 we give such proof under reasonable set theoretic assumptions, (close to the so called club guessing) but then have to consider finer points in combinatorial set theory on guessing clubs. Elsewhere we hope to have relevant complimentary consistency (see [S⁺c], [Shee]) and families of theories (so e.g. T_{feq} fit in, see [S⁺d]).

A priori we think that T_{ifg} , the theory of triangle free graphs, is “more complicated” than T_{feq}, T_{ceq} , but now have doubts.

We thank Mark Poor for doing much to improve the paper.

- Question 0.1.* 1) Does §1 apply to more theories than in §2?
 2) Can we characterize the dividing line? Simple/non-simple in our context.
 3) Does it help to have:

(*) for some $\mu, \mu < \lambda < 2^\mu$ there is no $\mathcal{A} \subseteq [\lambda]^\lambda$ which is μ -AD of cardinality $> \lambda$?

This would justify the use of μ -AD family $\mathcal{A} \subseteq [\lambda]^\lambda$ in some consistency results, see [She90], [Shee], see below.

Discussion 0.2. Note that:

- ⊞ if $2 < n \leq \omega, \theta \leq \mu \leq \lambda < 2^\theta, \lambda \nrightarrow [\mu]_\theta^{<n}$ and we let T_n be the theory “ $\{P_k$ is a reflexive asymmetric k -place relation”: $k < n, k \geq 2$ ” and T_n has a universal model M_* in λ then there is a μ -disjoint $\mathcal{A} \subseteq [\lambda]^\lambda$ of cardinality 2^θ .

[Why? Let $\mathbf{c} : [\lambda]^{<n} \rightarrow \theta$ witness $\lambda \nrightarrow [\mu]_\theta^{<n}$ and for $u \subseteq \theta$ let $M_u = (\lambda, \dots, P_k^{M_u}, \dots)_{k \in [2, n]}$ where $P_k^{M_u} = \{\eta \in {}^k \lambda : \eta \text{ is with no repetitions and } \mathbf{c}(\text{Rang}(\eta)) \in u\}$. So there is an embedding f_u of M_u into M_* ; now $\langle A_u = \{\text{pr}(\alpha, f_u(\alpha)) : \alpha < \lambda\} : u \subseteq \theta \rangle$ is a family as promised when pr is a pairing function on λ . Why? if $B = A_{u_1} \cap A_{u_2}$ has cardinality $\geq \theta$

and $u_1 \neq u_2$ then without loss of generality $u_1 \not\subseteq u_2$ and $\mathfrak{c} \setminus B$ omit any member of $u_1 \setminus u_2$.
]

§ 0(A). **Preliminaries.**

Notation 0.3. 1) T is a theory with vocabulary $\tau_T = \tau(T)$ and is a first order, if not said otherwise.

2)

- (a) $EC_T = \{M : M \text{ a model of } T\}$,
- (b) $EC_T(\lambda) = \{M \in EC_T : M \text{ of cardinality } \lambda\}$,
- (c) $EC_T(\lambda!) = \{M \in EC_T : M \text{ has universe } \lambda\}$,
- (d) for a set A of ordinals and ordinal α let $\text{suc}_A(\alpha)$ be $\min\{\beta \in A : \beta > \alpha\}$.

Convention 0.4. 1)

- (A) If T is universal theory not complete (like $T_{\text{ceq}}^0, T_{\text{feq}}^0$), then embedding are the usual ones, (on EC_T) and \subseteq_T (on EC_T) means \subseteq and we assume EC_T has amalgamation and JEP.
- (B) If T is complete, then embeddings are elementary (on EC_T) and \subseteq_T means \prec on EC_T .
- (C) We say f is a T -embedding of M into N or $f : M \rightarrow_T N$ when M, N are models of T , f embed M into N and $f(M) \subseteq_T N$.

2) If $\Delta \subseteq \mathbb{L}(\tau_T)$ then $\text{univ}_{T,\Delta}(\lambda)$ is the minimal χ such that there is a sequence \vec{M} which is a (λ, T, Δ) -universal sequence which means:

- (a) $\vec{M} = \langle M_\alpha : \alpha < \chi \rangle$ is a sequence of models of T ,
- (b) each M_α is of cardinality λ ,
- (c) for every model M of T of cardinality λ there is a Δ -embedding of M into some M_α , see below.

3) For given T, Δ as above and models M, N of T , we say f is a Δ -embedding of M into N when:

- (a) f is a function from M into N ,
- (b) if $\varphi(x_0, \dots, x_{n-1}) \in \Delta$ and $a_0, \dots, a_{n-1} \in M$ and $M \models \varphi[a_0, \dots, a_{n-1}]$ then $N \models \varphi[f(a_0), \dots, f(a_{n-1})]$,
- (c) so f is one-to-one when $(x \neq y) \in \Delta$.

4) For T, Δ as above in part (2) we may omit Δ when:

- (a) T is complete, $\Delta = \mathbb{L}(\tau_T)$, all first order formulas,
- (b) T not complete, Δ the set of quantifier free formulas in $\mathbb{L}(\tau_T)$.

5) We may write at, ep for $\Delta_{\text{at}(T)} = \{\varphi \in \mathbb{L}(\tau_T) : \varphi \text{ is atomic}\}$, $\Delta_{\text{ep}(T)} = \{\varphi \in \mathbb{L}(\tau_T) : \varphi \text{ existential positive}\}$ respectively. We may write τ instead of T . We may write φ instead $\Delta = \{\varphi\}$ and $\pm\varphi$ instead $\Delta = \{\varphi, \neg\varphi\}$.

- Notation 0.5.* 1) Let $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \xi, i, j$ denote ordinals.
 2) Let $\kappa, \lambda, \mu, \chi, \partial, \theta, \Upsilon$ denote cardinals, infinite if not said otherwise.
 3) Let k, ℓ, m, n denote natural numbers.
 4) Let φ, ψ, ϑ denote formulas, f.o. if not said otherwise.

Definition 0.6. 1) $J_\theta^{\text{bd}} = \{A \subseteq \theta : \sup(A) < \theta\}$, bd stands for bounding, for θ a regular cardinal or just a limit ordinal.

1A) For θ regular uncountable let:

- $D_\theta^{\text{club}} = \{A \subseteq \theta : \text{there is a club (= closed unbounded subset) } E \text{ of } \theta \text{ such that } E \subseteq A\}$.
- NS_θ is the non-stationary ideal on θ .

2) For a regular θ let:

- (a) $\mathfrak{d}_\theta = \text{Min}\{|\mathcal{F}| : \mathcal{F} \subseteq {}^\theta\theta \text{ is } <_{J_\theta^{\text{bd}}}\text{-cofinal in } \mu_\mu\}$
 (b) $\mathfrak{b}_\theta = \text{Min}\{|\mathcal{F}| : \mathcal{F} \subseteq {}^\theta\theta \text{ has no } <_{J_\theta^{\text{bd}}}\text{-upper bound}\}$.

- 3) Let $\mathfrak{d}_\theta^{\text{club}}$ be defined similarly using $<_{\text{NS}_\theta}$ when θ is regular uncountable.
 4) For a model M and a set $u \subseteq M$ let $M \upharpoonright u$ is defined naturally, allowing a function symbol to be interpreted as a partial function (and so an individual constant to be not defined)

Definition 0.7. 1) For a regular uncountable cardinal λ let $\check{I}[\lambda] = \{S \subseteq \lambda : \text{some pair } (E, \bar{a}) \text{ witnesses } S \in \check{I}(\lambda), \text{ see below}\}$.

2) We say that (E, u) is a witness for $S \in \check{I}[\lambda]$ iff:

- (a) E is a club of the regular cardinal λ
 (b) $u = \langle u_\alpha : \alpha < \lambda \rangle, a_\alpha \subseteq \alpha$ and $\beta \in a_\alpha \Rightarrow a_\beta = \beta \cap a_\alpha$
 (c) for every $\delta \in E \cap S, u_\delta$ is an unbounded subset of δ of order-type $< \delta$ (and δ is a limit ordinal, necessarily δ is not a regular cardinal).

3) For $\kappa < \lambda = \text{cf}(\lambda)$ let $\check{I}_{\leq \kappa}[\lambda, \cdot]$ be the ideal $\{S \subseteq \lambda : S \subseteq S_{\leq \kappa}^\lambda, S \in \check{I}[\lambda]\}$

By ([She79], [Sheb] and better) [She93a] and [Shea]

Claim 0.8. *Let λ be regular uncountable.*

1) *If $S \in \check{I}[\lambda]$ then we can find a witness (E, \bar{a}) for $S \in \check{I}[\lambda]$ such that (clauses (a), (b), (c) from 0.7 and):*

- (d) $\delta \in S \cap E \Rightarrow \text{otp}(a_\delta) = \text{cf}(\delta)$
 (e) *if $\alpha \notin S$ then $\text{otp}(a_\alpha) < \text{cf}(\delta)$ for some $\delta \in S \cap E$.*

2) $S \in \check{I}[\lambda]$ iff there is a pair $(E, \bar{\mathcal{P}})$ such that:

- (a) E is a club of the regular uncountable λ
 (b) $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$, where $\mathcal{P}_\alpha \subseteq \{u : u \subseteq \alpha\}$ has cardinality $< \lambda$
 (c) *if $\alpha < \beta < \lambda$ and $\alpha \in u \in \mathcal{P}_\beta$ then $u \cap \alpha \in \mathcal{P}_\alpha$*
 (d) *if $\delta \in E \cap S$ then some $u \in \mathcal{P}_\delta$ is an unbounded subset of δ (and δ is a limit ordinal).*

§ 1. ON T_{ceq} FOR MAHLO CARDINALS

As here we consider T_{ceq} the simplest, non-simple theory, we may consider how much does it behave like the class of graphs (equivalently random graph)? We prove that not by a non-existence result, but with quite specific set theoretic assumptions.

T_{ceq} is very closed to (and equivalent for our purposes) the older T_{feq} which a prime example for a theory with the tree order property equivalently non-simple (even TP_2 but neither the strict order property nor even just the SOP_2). For it we get here parallel and better results than [She93b] where it is proved that there are limitations on the universality spectrum for T_{feq} and in [DS06], which generalize the results for any T with the so called oak property, see somewhat more in [She17, §3]. The results in those papers are meaningful when SCH fails, that is, consider a cardinal λ such that: for some strong limit singular $\mu, \mu^+ < \lambda < \mu$ if λ is regular then “usually” T_{feq} has no universal in λ .

Here we get further such non-existence results for Mahlo cardinals, but what about $\lambda \in [\mu, 2^\mu)$ when for transparency we assume $\mu = \mu^{<\mu}$? In §2, we do better but the Mahlo case may cover more classes, comes first and the proofs are more transparent. The proof here can be axiomatized as in §2 using:

- ⊞ $\text{PGC}(\lambda, S)$, S where S is a stationary set of regular cardinals $< \lambda$ mean ssh \mathbf{u} witness it where $\mathbf{U} = \{ \langle \omega(1 + \varepsilon) : \varepsilon < \theta \rangle : \theta \in S \}$ cardinals (so $\mathcal{U} = S$).

First, recall (the reader can concentrate on the universal versions, $T_{\text{feq}}^0, T_{\text{ceq}}^0$, on T_{feq} see [She93b, 2.1=Lb3, 3.1=Lc3]):

Definition 1.1. $T_{\text{feq}} = T_{\text{feq}}^1$ is the model completion of the following theory, T_{feq}^0 which is defined by:

- (A) $\tau = \tau(T_{\text{feq}}^0)$ consists of:
 (a) predicates P, Q (unary),
 (b) E (three place predicate written as $xE_z y$ instead $E(x, y, z)$),
 (B) a τ -model M is a model of T_{feq}^0 iff:
 (a) the universe of M is the disjoint union of P^M and Q^M ,
 (b) $xE_z y \rightarrow P(z) \wedge Q^M(x) \wedge Q^M(y)$,
 (c) for any fixed $z \in P^M, E_z^M$ is an equivalence relation on Q^M .

Observation 1.2. 0) T_{ceq} is well defined and $\text{univ}(\lambda T_{\text{ceq}}) = \text{univ}(\lambda, T_{\text{ceq}})$

1) So if $M \models T_{\text{feq}}$ then:

- (*) (a) in (B)(c), for each $x \in P, E_x$ is with infinitely many equivalence classes,
 (b) if $n < \omega, x_1, \dots, x_n \in P$ with no repetition and $y_1, \dots, y_n \in Q$ then for some $y \in Q, \bigwedge_{\ell=1}^n yE_{x_\ell} y_\ell$,
 (c) if $n < \omega$ and $y_1, \dots, y_n \in Q$ and e is an equivalence relation on $\{1, \dots, n\}$ then for some $x \in P$ we have $y_\ell E_x y_k \Leftrightarrow \ell e k$,
 (d) P^M, Q^M are infinite.

2) Hence T_{feq} has elimination of quantifiers.

We present a close relative, the main one we consider here (and, as proved below, equivalent to T_{feq} for our purpose).

Definition 1.3. We define $T_{\text{ceq}}^0, T_{\text{ceq}} = T_{\text{ceq}}^1$ as in Definition 1.1 replacing the subscript feq by ceq (or omitting it) and:

- (A) $\tau = \tau(T_{\text{ceq}}^0) = \tau(T_{\text{ceq}})$ consists of: P, Q unary predicates, E a binary predicate and F a binary function symbol,
- (B) a τ -model M is a model of universal theory. T_{ceq}^0 iff:
 - (a) P^M, Q^M is a partition of M ,
 - (b) E^M is an equivalence relation on Q^M ,
 - (c) F^M is a function from $Q^M \times P^M$ into Q^M such that for every $c \in P^M, a \mapsto F^M(a, c)$ is choosing a representative for the a/E^M -equivalence class, that is, we have:
 - (α) $a \in Q^M \Rightarrow F^M(a, c) \in a/E^M$,
 - (β) if $a, b \in Q^M$ are E^M -equivalent then $F^M(a, c) = F^M(b, c)$.

Concerning λ in the neighborhood of a strong limit singular we shall not give details as we can just quote.

Claim 1.4. 0) Concerning T_{ceq}^0

- (a) For a model M of T_{ceq}^0 and $A \subseteq M$ with n elements, the closure of A inside M has at most $n + n^2$ elements, (even at most $n + (n/2)^2$ elements),
- (b) T_{ceq}^0 has amalgamation and JEP,
- (c) T_{ceq}^0 has a model completion, that is T_{ceq} is well defined,
- (d) $\text{univ}(\lambda, T_{\text{ceq}} = \text{univ}(\lambda, T_{\text{feq}})$.

1) T_{ceq} is not simple, is NSOP₂ and even NSOP₁ and has the oak property, in fact, by qf (quantifier free) and even atomic formulas.

2) We have (A) \Rightarrow (B) where:

- (A) (a) $\theta < \mu < \lambda < \chi$,
- (b) $\text{cf}(\lambda) = \lambda, \theta = \text{cf}(\theta) = \text{cf}(\mu), \mu^+ < \lambda$,
- (c) $\chi := \text{pp}_{\Gamma(\theta)}(\mu) > \lambda + |i^*|$,
- (d) there is $\{(a_i, b_i) : i < i^*\}$, $a_i \in [\lambda]^{<\mu}, b_i \in [\lambda]^\theta$ and $|\{b_i : i < i^*\}| \leq \lambda$ such that: for every $f : \lambda \rightarrow \lambda$ for some $i, f(b_i) \subseteq a_i$,
- (B) (a) T_{ceq} equivalently T_{ceq}^0 has no universal model in λ ,
- (b) Moreover, $\text{univ}(\lambda, T_{\text{ceq}}) \geq \chi = \text{pp}_{\Gamma(\theta)}(\mu)$.

3) T_{feq} can be interpreted in T_{ceq} hence $\text{univ}_{T_{\text{feq}}}(\lambda) \leq \text{univ}_{T_{\text{ceq}}}(\lambda)$.

4) Also the inverse of part (3) holds.

Proof. 0) Easy.

1) By (1) quoting [DS06] where the oak property was introduced.

2) Follows from part (3) and [She93b, Claim 2.2].

3) For a model M of T_{ceq}^0 we define a model $N = N[M]$ of T_{feq}^0 as follows:

- (*)_{N,M} (a) $Q^N = P^M, P^N = Q^M/E^M$,
- (b) $E^N = \{(a, b, C) : C \in P^N = Q^M/E^M \text{ and } a, b \in Q^N \text{ and } (\forall c \in C)[F^M(c, a) = F^M(c, b)] \text{ equivalently, } (\exists c \in C)[F^M(c, a) = F^M(c, b)]\}$.

Now check that $N \models T_{\text{feq}}^1$ and $M \models T_{\text{ceq}} \Leftrightarrow N \models T_{\text{feq}}$.

4) For a model N of T_{feq}^0 we define a model $M = M[N]$ of T_{ceq}^0 as follows:

- (*)_{M,N} (a) $P^M = Q^N$ and $Q^M = \{(c, A) : c \in P^N \text{ and } A \in Q^N/E_c^N\}$
 (b) $E^M = \{(c_1, A_1), (c_2, A_2) : c_1 = c_2 \in P^M \text{ and } A_1, A_2 \in Q^N/E_{c_2}^N\}$
 (c) $F^M : Q^M \times P^M \rightarrow Q^M$ is defined by: If $d \in Q^M, b \in P^M$ hence for some $c \in P^N, A \in P^M = Q^N \setminus E_c^N$ we have $d = (c, A)$ then we let $F^M(d, b) = (c, b/E_c^M)$.

Now check that $N \models T_{\text{feq}}^1$ and $M \models T_{\text{ceq}} \Leftrightarrow N \models T_{\text{feq}}$. □_{1.4}

We now point out a new reason involved “large \mathfrak{d}_θ ’s” for not having a universal model in λ , even for many non-simple T ’s. In this section we deal with a case where the proof is simpler using T_{ceq} and λ a Mahlo cardinal.

Claim 1.5. 1) Assume λ is a (weakly inaccessible) Mahlo cardinal and $S = \{\theta < \lambda : \theta \text{ regular (weakly inaccessible) and } \mathfrak{d}_\theta > \lambda\}$ is stationary in λ and S has club guessing.

Then

- (a) $\text{univ}(\lambda, T_{\text{ceq}})$ is $> \lambda$
 (b) even, $\geq \sup\{\chi^+ : \text{the set } \{\theta \in S : \mathfrak{d}_\theta > \chi\} \text{ is stationary and has club guessing}\}$.
 2) We have $\chi < \text{univ}(\lambda, T_{\text{ceq}})$ when:
 (a) λ is a Mahlo weakly inaccessible cardinal,
 (b) $\lambda \leq \chi$,
 (c) $S \subseteq \{\theta < \lambda : \theta \text{ is weakly inaccessible cardinal}\}$ is stationary.
 (d) $\mathcal{P} = \langle \mathcal{P}_\theta : \theta \in S \rangle$,
 (e) if $\theta \in S$ then \mathcal{P}_θ is a set of $\leq \lambda$ clubs of θ ,
 (f) \mathcal{P} guess clubs of λ , that is, for every club E of λ for some $C \in \mathcal{P}_\theta, \theta \in S$ we have $C \subseteq E$,
 (g) $\mathfrak{d}_\theta > \chi$. for every $\theta \in S$.

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Proof. 1) Clearly

- (*)₀ it suffices to:
 (a) fix $\chi \geq \lambda$ such that $S_\chi = \{\theta \in S : \mathfrak{d}_\theta > \chi\}$ is stationary,
 (b) prove $\text{univ}_T(\lambda) > \chi$.

Let $T = T_{\text{ceq}}$ and let:

- (*)₁ $\langle C_\delta^* : \delta \in S \rangle$ witness “ S has club-guessing”;
 (*)₂ if (A) below holds, then we define some objects in (B) where:
 (A) (a) $M \in \text{EC}_T(\lambda!)$,
 (b) $|P^M| = \lambda$,
 (c) θ is regular and from the interval (\aleph_0, λ) , usually $\theta \in S$,
 (d) E a club of θ .
 (B) we define:
 (a) for $a \in P^M$ hence $a < \lambda$ let $g_a = g_{M, E, a}$ be the following partial function from θ to θ :

- for $\alpha < \theta$, $g_\alpha(\alpha)$ is the minimal $\beta \in E$ such that: $\beta \in E \setminus (\alpha + 1)$ and $(\beta_1 \in Q^M \cap \beta) \wedge (F^M(\beta_1, a) < \theta) \Rightarrow F^M(\beta_1, a) < \beta$,
- (b) $\mathcal{G}_{M,E}^0 = \{g_{M,E,a} : a \in P^M\}$, note that E determine θ ,
- (c) for $\theta \in S$; et $\mathcal{G}_{M,\theta}^* = \{g_{M,C_\theta,a} : a \in P^M\}$.

Now easily

- (*)₃ for a, M, θ, E as above, $g_{M,E,a}$ is a non-decreasing function from θ into θ , in fact, into $E \subseteq \theta$
- (*)₄ if $M, N \in EC_T(\lambda!)$ and f embeds M into N then for some club E^* of λ : if $\theta \in S$, $\theta = \sup(E^* \cap \theta)$, $E \subseteq E^* \cap \theta$ is a club of θ and $a \in P^M$ then $g_{M,E,a} \leq g_{N,E,f(a)}$.
DEAR MARK:do not understand, we say above $E \subseteq E^* \cap \theta$

So without loss of generality $\theta \in S \Rightarrow \mathfrak{d}_\theta > \chi \geq \lambda$ and we shall prove that $\text{univ}(\lambda, T) > \chi$; this suffices. So assume $\langle M_\alpha : \alpha < \chi \rangle$ is a sequence of members of $EC_T(\lambda!)$.

So for each $\theta \in S$ the set $\mathcal{G}_\theta = \cup \{\mathcal{G}_{M_\alpha, \theta}^* : \alpha < \chi, |Q^{M_\alpha} \cap \theta| = \theta\}$ has cardinality $\leq \chi$ recalling $\lambda \leq \chi$.

For $\theta \in S$, as $|\mathcal{G}_\theta| < \mathfrak{d}_\theta$, necessarily there is an increasing $g_\theta \in {}^\theta \theta$ such that $g \in \mathcal{G}_\theta \Rightarrow g_\theta \not\leq g \pmod{J_\theta^{\text{bd}}}$. Now we define a model $N \in EC_{T_{\text{ceq}}^0}(\lambda!)$ with $\tau_N = \tau(T_{\text{ceq}}) = \tau(T_{\text{ceq}}^0)$ as follows:

- (A) universe is λ ,
- (B) (a) Q^N is the set of odd ordinals $< \lambda$,
- (b) if $\alpha = 4\beta + 1 < \lambda$ then α/E^N is disjoint to α ,
- (c) E^N is an equivalence relation on Q^N such that for every $\alpha < \beta < \lambda$ such that β is divisible by $|\alpha|$, $\alpha \in Q^N$ we have $|\alpha/E^N \cap \beta| = |\beta|$,
- (d) if $\theta \in S$ and $\alpha < \theta$ then $\theta \geq F^M(4\alpha + 1, \theta) > g_\theta(4\alpha + 1)$.

This is easy to do

To show that \bar{M} does not witnesses $\text{univ}(\lambda, T) \leq \chi$ is suffice to show that N cannot be embedded in M_α for any $\alpha < \chi$. Toward contradiction assume that $\alpha < \chi$ and f is an embedding of N into M_α . Let $E = \{\delta < \lambda : \delta \text{ a limit ordinal such that } ((M_\alpha \upharpoonright \delta, N \upharpoonright \delta, f \upharpoonright \delta, < \upharpoonright \delta) \prec (M_\alpha, N, f, < \upharpoonright \lambda))\}$. Clearly E is a club of λ hence for some $\theta \in S_\chi$ we have $C_\theta \subseteq E$. Let $h \in {}^\theta \theta$ be $g_{M_\alpha, C_\theta, f(\theta)}$, so is well defined and belongs to $\mathcal{G}_{M_\alpha}^0$ hence to \mathcal{G}_θ hence $\mathcal{G}_{M_\alpha}^*$ hence $g_\theta \not\leq h \pmod{J_\theta^{\text{bd}}}$,

So for some $\alpha < \theta$ we have $h(\alpha) < g_\theta(\alpha)$. By the choice of h clearly $h(\alpha) \in C_\theta$ and by the choice of θ , the ordinal $h(\alpha)$ is closed under $f \upharpoonright \theta$; also as $h(\alpha) \in C_\theta \subseteq E$ and the choice of E there is as $\gamma = 4\gamma^* + 1 \in (\alpha, h(\alpha))$ so $\gamma \in Q^N$ so $f(\gamma) \in (\alpha, h(\alpha))$ and by the choice of N , $F^N(\gamma, \theta) \in [g_\theta(\alpha), \theta)$ hence is in the interval $(g_\theta(\alpha), \delta)$ hence $f(\gamma) < h(\alpha) < g_\theta(\gamma) < f(F^N(\gamma, \theta)) < \theta$ and easily we get a contradiction.

2) Similarly. □_{1.5}

Remark 1.6. We can similarly prove that: for every sequence $\langle (E_\xi, \mathcal{G}_{\xi, \theta}) : \xi < \chi, \theta \in S \rangle$ satisfying clause (A) below, there is a sequence $\langle g_{\theta, \alpha} : \theta \in S, \alpha < \lambda \rangle$ with $g_{\theta, \alpha} \in {}^\theta \theta$ satisfying clause (B) below, where:

- (A) E_ξ is a club of λ for $\xi < \chi$ and $\mathcal{G}_{\xi, \theta} \subseteq {}^\theta \theta$ has cardinality $\leq \lambda$ for $\xi < \lambda, \theta \in S$.
- (B) for every $\xi < \chi$ and club E of λ there are $\theta \in \text{acc}(E_\xi) \cap E \cap S$ and $\alpha < \lambda$ such that $g \in \mathcal{G}_{\xi, \theta} \Rightarrow g_{\theta, \alpha} \not\leq g \pmod{J_{E_\xi \cap E}^{\text{bd}}}$.

§ 2. ON SUCCESSOR CARDINALS AND CLUB GUESSING

We first introduce the relevant notions (in 2.1); (we could add clause 2.1(2)(b) into the definition of $\mathbf{U}_{\lambda,\theta}$ in 2.1(1), but so far it does not matter¹). We then investigate it and use it for sufficient conditions for “no universal”.

Definition 2.1. Assume $\lambda > \theta$ are regular and $D \subseteq \mathcal{P}(\theta)$ satisfying $D \subseteq [\theta]^\theta$ is non-empty; omitting D means $D = \{\theta\}$, and \mathfrak{B} is a model with universe λ and countable vocabulary. Saying “for D -most $\varepsilon < \theta$ will mean “for some $X \in D$ for every $\varepsilon \in X$ ”. The main case is $\theta > \aleph_0$, this is necessary for the “full” cases (see parts (2)), but not for the others; we may forget to assume $\theta > \aleph_0$.

1) Let $\mathbf{U}_{\lambda,\theta} = \{\bar{u} : \bar{u} = \langle u_i : i < \theta \rangle \text{ is } \subseteq\text{-increasing continuous, and } i < \theta \Rightarrow u_i \in [\lambda]^{<\theta} \text{ but } \cup\{u_i : i < \theta\} \in [\lambda]^\theta \text{ and } \bigwedge_{i < \theta} u_i \cap \theta \in \theta \text{ and } \bigcup_{i < \theta} u_i \supseteq \theta\}$.

1A) We shall say that $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ obeys \mathfrak{B} when every $\bar{u} \in \mathbf{U}$ does, which means that for every $\varepsilon < \theta$ we have $\mathfrak{B} \upharpoonright u_\varepsilon \subseteq \mathfrak{B}$, (if \mathfrak{B} has Skolem functions this is equivalent to $\mathfrak{B} \upharpoonright u_\varepsilon \prec \mathfrak{B}$) and $\varepsilon \subseteq u_\varepsilon$.

2) We say $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ fully D -guess clubs when ($\theta < \aleph_0$ and) for every model M with universe λ and countable vocabulary there is $\bar{u} \in \mathbf{U}$ which D -guesses M meaning²

(a) (α) if $\varepsilon < \theta$ then $\text{cl}(u_\varepsilon, M) \subseteq \text{sup}(u_\varepsilon)$, moreover (actually follows using an expansion of M) $M \upharpoonright \text{sup}(u_\varepsilon) \prec M$

(β) $(\exists \mathcal{X} \in D)(\forall \varepsilon)[\varepsilon \in \mathcal{X} \Rightarrow M \upharpoonright u_\varepsilon \subseteq M]$, i.e. for D -most $\varepsilon < \theta$ the set u_ε is closed under the functions of M , (in an equivalent definition $M_\varepsilon \upharpoonright u_\varepsilon \prec M$ as we can expand M by Skolem functions).

(b) the sequence $\text{ord}(\bar{u} \upharpoonright \text{sup}(u_\varepsilon) : \varepsilon < \theta)$ is strictly increasing.

3) We say $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ almost D -guess clubs when:

(a) for every model M with universe λ and countable vocabulary and $A \in [\lambda]^\lambda$ for some $\bar{u} \in \mathbf{U}$ we have:

(α) as in (a)(α) of part (2)

(β) for D -most $\varepsilon < \theta$ we have $A \cap u_{\varepsilon+1} \not\subseteq \text{sup}(u_\varepsilon)$,

(γ) $\text{cl}(\bigcup_{\varepsilon < \theta} u_\varepsilon, M) = \bigcup_{\varepsilon < \theta} u_\varepsilon$, that is, $M \upharpoonright (\bigcup_{\varepsilon < \theta} u_\varepsilon) \subseteq M$,

(b) if $\bar{u} \in \mathbf{U}$ then $\text{ord}(\bar{u}) = \langle \text{sup}(u_\varepsilon) : \varepsilon < \theta \rangle$ is strictly increasing.

3A) We say \mathbf{U} medium D -guess clubs when as in part (3) omitting clause (a)(γ).

3B) We say $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ semi- D -guess clubs when:

(a)' as (a) in part (3) but replacing (β) by:

(β)' for D -most $\varepsilon < \theta$ for some $\zeta \in [\varepsilon, \theta)$ and $\alpha \in A$ we have $\alpha \in (u_{\zeta+1} \setminus u_\zeta) \cap (\text{sup}(u_{\varepsilon+1}) \setminus \text{sup}(u_\varepsilon))$,

(b) as in part (3).

3C) We say $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ pseudo- D -guess clubs when:

(a)'' if M is as above and $A \in [\lambda]^\lambda$ then for some $\bar{u} \in \mathbf{U}$ we have:

(α) as is (3A) clause (a)(α)

¹note that it is relevant to “fully D -guess clubs” implies “almost guess clubs”, see 2.2

² we may omit in 2.1(2) the clauses (a)(α), (b) but then we have problems with “FGC \Rightarrow AGC and the gain is doubtful

(β) for D -most $\varepsilon < \theta$ for some $\zeta \in [\varepsilon, \theta)$ and $\alpha \in A$ we have $\alpha \in (u_{\zeta+1} \setminus u_\zeta) \cap (\sup(u_{\varepsilon+1}) \setminus \sup(u_\varepsilon))$,

(b) as above.

3D) We say \mathbf{U} is (λ, θ) -reasonable (or just reasonable when (λ, θ) are clear from the context) when $\mathbf{U} \subseteq \mathbf{U}_{\lambda, \theta}$ satisfies clause (3)(b).

4) We say \mathbf{U} does $X - D$ -guess clubs when:

- \mathbf{U} does D -fully guess clubs and $X = F$,
- \mathbf{U} almost D -guess clubs and $X = A$,
- \mathbf{U} semi-guess clubs and $X = S$,
- \mathbf{U} medium D -guess clubs and $X = M$,
- \mathbf{U} pseudo guess clubs and $X = P$.

5) Let $XGC_D(\lambda, \theta) = \min\{|\mathbf{U}| : \mathbf{U} \subseteq \mathbf{U}_{\lambda, \theta} \text{ and } \mathbf{U} \text{ does } X - D\text{-guess clubs}\}$ so if $X = S, P$ we may omit D .

5A) Similarly $XGC_D(\lambda, \theta, \mathfrak{B})$ when we restrict ourselves to \mathbf{U} obeying \mathfrak{B} .

6) We say $\mathbf{U} \subseteq \mathbf{U}_{\lambda, \theta}$ is bounded when there is an F witnessing it which means: F is a function from $\{\bar{u} \upharpoonright \zeta : \bar{u} \in \mathbf{U}, \zeta < \theta\}$ into λ such that $F(\bar{u}_1 \upharpoonright \zeta_1) = F(\bar{u}_2 \upharpoonright \zeta_2) \Rightarrow \bar{u}_1 \upharpoonright \zeta_1 = \bar{u}_2 \upharpoonright \zeta_2$ and $F(\bar{u} \upharpoonright (\zeta + 1)) < \sup(u_{\zeta+1})$.

7) We say “strongly bounded” when in addition $F(\bar{u} \upharpoonright (\zeta + 1)) \in u_{\zeta+1}$ for every $\zeta < \theta$.

8) We say $\mathbf{U} \subseteq \mathbf{U}_{\lambda, \theta}$ is weakly bounded, when there is a function F witnessing it which means:

- (a) $\text{Dom}(F) = \{\text{ord}(\bar{u}) \upharpoonright \zeta : \bar{u} \in \mathbf{U} \text{ and } \zeta < \theta \text{ is a successor ordinal}\}$ recalling $\text{ord}(\bar{u}) = \langle \sup(u_\varepsilon) : \varepsilon < \theta \rangle$,
- (b) $\text{Rang}(F) \subseteq \lambda$ and $F(\text{ord}(\bar{u}) \upharpoonright (\zeta + 1)) < \sup(u_{\zeta+1})$ for $\bar{u} \in \mathbf{U}$ and $\zeta < \theta$,
- (c) if $\zeta_1, \zeta_2 < \theta$ are successor of successor ordinals and $\bar{u}_1, \bar{u}_2 \in \mathbf{U}$ and $F(\text{ord}(\bar{u}_1) \upharpoonright \zeta_1) = F(\text{ord}(\bar{u}_2) \upharpoonright \zeta_2)$ then $\text{ord}(\bar{u}_1) \upharpoonright \zeta_1 = \text{ord}(\bar{u}_2) \upharpoonright \zeta_2$.

9) Notation:

- (a) if $\bar{u} \in \mathbf{U}_{\lambda, \theta}$ and $f : \theta \rightarrow \theta$ is \leq -increasing continuous with limit θ then $\bar{u}^{[f]} = \bar{u}[f] := \langle u_{f(\varepsilon)} : \varepsilon < \theta \rangle$,
- (b) if $\mathbf{U} \subseteq \mathbf{U}_{\lambda, \theta}$ and $f : \theta \rightarrow \theta$ is \leq -increasing continuous with limit θ then $\mathbf{U}^{[f]} := \mathbf{U}[f] = \{\bar{u}[f] : \bar{u} \in \mathbf{U}\}$,
- (c) if $\mathbf{U} \subseteq \mathbf{U}_{\lambda, \theta}$ and \mathcal{F} is a set of \leq -increasing continuous function from θ into θ with limit θ then $\mathbf{U}[\mathcal{F}] = \{\bar{u}[f] : \bar{u} \in \mathbf{U}, f \in \mathcal{F}\}$,
- (d) if $w \in [\theta]^\theta$ then $f_w = f[w]$ is the $g : \theta \rightarrow \theta$ such that $g(\varepsilon) = \min\{\zeta : \text{otp}(w \cap \zeta) = \varepsilon\}$, so is \leq -increasing continuous with limit θ .

10) In (a),(b) of part (9) above we may write $\bar{u}[w], \mathbf{U}[w]$ for $w \in [\theta]^\theta$ means $\bar{u}[f], \mathbf{U}[f]$ where $f = f_w$, writing $\mathbf{U}[W], W \subseteq [\theta]^\theta$ mean $\cup\{\mathbf{U}[w] : w \in W\}$.

11) Now for $X \in \{F, A, S, M, P\}$ we let (naturally and we can add \mathfrak{B} as in part (5A)):

- (a) $BXGC_D(\lambda, \theta) = \text{Min}\{|\mathbf{U}| : \mathbf{U} \subseteq \mathbf{U}_{\lambda, \theta} \text{ does } X - D\text{-guess clubs and is strongly bounded}\}$,
- (b) $CXGC_D(\lambda, \theta)$ is defined as in (a) but \mathbf{U} is just bounded,
- (c) $WXGC_D(\lambda, \theta)$ is defined as in clause (a) but \mathbf{U} is weakly bounded.

Some of the obvious implications are:

Observation 2.2. 1) If \mathbf{U} is fully D -guess clubs, then \mathbf{U} almost D -guess clubs,
 2) If \mathbf{U} almost D -guess clubs then \mathbf{U} semi-guess-club and medium D -guess clubs.
 3) If \mathbf{U} semi-guess-clubs or medium D -guess clubs for D as above, then \mathbf{U} does pseudo guess clubs.

4) If $D_1 \subseteq D_2 \subseteq [\theta]^\theta$ then “ \mathbf{U} does $X - D_1$ -guess clubs” implies “ \mathbf{U} does $X - D_2$ -guess clubs” for $X \in \{F, A, M, S, P\}$, we may write {full, almost, medium, semi, pseudo}.

5) Assume $\mathbf{U} \subseteq \mathbf{U}_{\lambda, \theta}$ and \mathfrak{B} is as in 2.2. Then there is \mathbf{U}' such that:

- (a) $\mathbf{U}' \subseteq \mathbf{U}_{\lambda, \theta}$
- (b) $|\mathbf{U}'| \leq |\mathbf{U}|$
- (c) if \mathbf{U} does $X - D$ -guess clubs, for $X \in \{F, A, M, S, P\}$ as in part (4) then so does \mathbf{U}' ,
- (d) \mathbf{U} obeys \mathfrak{B} .

6) In 2.1(11) the number is $\geq \lambda$.

Proof. E.g. **DEAR MARK: I DO NOT UNDERSTAND YOUR COMPLAIN**

5) Let $\mathbf{U}' = \{\bar{u}' \in \mathbf{U} : \text{for some } \bar{u} \in \mathbf{U} \text{ for every } \varepsilon < \theta \text{ we have } \text{cl}_{\mathfrak{B}}(u_\varepsilon) = u'_\varepsilon \subseteq \text{sup}(u_\varepsilon) \text{ and } M \upharpoonright \text{sup}(u_\varepsilon) \prec M\}$ it suffice to prove that \mathbf{U}' is as required. The main point is to verify the appropriate version of clause (a) in Def 2.1. So let M be a model with universe λ and countable vocabulary, we have to find a suitable $\bar{u} \in \mathbf{U}'$. By renaming, without loss of generality the vocabulary of M is disjoint to the one of \mathbf{G} and let M' be a common expansion of M and \mathfrak{B} with $\tau(M') = \tau(M) \cup \tau(\mathfrak{B})$. Let $E = \{\delta : M; \upharpoonright \delta[eM]\}$. So $(M, E), < \upharpoonright \lambda$ is as required in clause (a) for \mathbf{U} hence there is a suitable $\bar{u} \in \mathbf{U}$. Obviously $\bar{u} \in \mathbf{U}$ is as required here, so we are done.

□_{2.2}

Definition 2.3. 1) For the model theory: for a model $M \in \text{EC}_T(\lambda!), \Delta \subseteq \mathbb{L}(\tau_T)$ and $u \subseteq \lambda, A \subseteq M$ let $M^{[A]} \upharpoonright_\Delta u$ be the model $M \upharpoonright u$ expanded by all the restriction to u of all relations definable by a Δ -formula with parameters from A .

1A) If $\Delta = \mathbb{L}_{\text{qf}}(\tau_M)$ then we may omit Δ ; writing \bar{a} instead A means $\text{Rang}(\bar{a})$.

2) For $M \in \text{EC}_T(\lambda!), \bar{u} \in \mathbf{U}_{\lambda, \theta}$ and $\bar{a} \in {}^{>\omega}M$ let $g_{\bar{a}, \bar{u}, M}$ be the function from θ to θ such that for $\zeta < \theta, g_{\bar{a}, \bar{u}, M}(\zeta)$ is the minimal $\varepsilon \in (\zeta, \theta)$ such that $M^{[\bar{a}]} \upharpoonright_{u_\varepsilon} \prec M^{[\bar{a}]} \upharpoonright \bigcup_{\xi < \theta} u_\xi$.

Claim 2.4. We assume that \mathfrak{B} is a model with universe λ and countable vocabulary.

1) We have

- (A) $\text{CSGC}(\lambda, \theta) = \lambda$, moreover $\lambda = \text{CSGC}(\lambda, \theta, \mathfrak{B})$ provided that:
 - $\lambda = \text{cf}(\lambda) = \theta^{++}$ and $\theta = \text{cf}(\theta)$
- (B) $\text{BSGC}(\lambda, \theta) = \lambda$ provided³ that
 - $\lambda = \theta^+, \theta = \text{cf}(\theta)$
 - there is a stationary $S \subseteq S_\theta^{\theta^+}$ from $\check{I}_\theta[\theta^+]$,
- (C) $\text{BFGC}(\lambda, \theta) = \lambda$ even with a reasonable witness. provided that:
 - $\lambda = \lambda^\theta > \aleph_0$ and $\theta = \text{cf}(\theta)$
- (D) $\text{MGC}_D(\lambda, \theta) = \lambda$ when:
 - (*) $\theta = \text{cf}(\theta) < \lambda$ and there is \mathcal{S} such that:
 - (a) $\mathcal{S} \subseteq \{w : w \subseteq \lambda, \text{otp}(w) = \theta\}$,

³We can weaken the demand: if we weaken the demand in Definition 2.1(5) to “for stationary many $\varepsilon < \theta$ ” and $\theta \geq \aleph_2$.

- (b) \mathcal{S} has cardinality λ ,
(c) if $A \in [\lambda]^\lambda$ then for some $w \in \mathcal{S}$ the set $w \cap A$ has cardinality θ ,
(d) $D = [\theta]^\theta$.
(E) $\text{AGC}(\lambda, \theta) = \lambda$ when
(*) we have
(a),(b),(c) as above
(d) the cofinality of $([\lambda]^\theta, \subseteq) = \lambda$
- 2) For regular $\lambda > \theta = \text{cf}(\theta)$
(A) if $\text{SGC}(\lambda, \theta) = \lambda$ and $\mathfrak{b}_\theta \leq \lambda$ then $\text{AGC}_D(\lambda, \theta) = \lambda$, recalling that the default value of D is $\{\theta\}$.
(B) if $\text{SGC}(\lambda, \theta) = \lambda$ and $\mathfrak{d}_\theta \leq \lambda$ then $\text{AGC}(\lambda, \theta) = \lambda$.
- 3) For $\lambda > \theta = \text{cf}(\theta)$ such that⁴ $\lambda > \theta^+$ we have $\text{SGC}(\lambda, \theta) = \lambda$ provided that (e.g. $\lambda = \theta^{+n}$ for some n):
 $\boxplus_{\lambda, \theta}^3 \text{cf}([\lambda]^\theta, \subseteq) = \lambda$.
- 4) If $\mathbf{U}_1 \subseteq \mathbf{U}_{\lambda, \theta}$ medium guesses clubs, then there is $\mathbf{U} \subseteq \mathbf{U}_{\lambda, \theta}$ which medium guesses clubs of cardinality $\leq |\mathbf{U}_1|$ and for $\bar{u} \in \mathbf{U}$ we have:
(a) if $u = \cup\{u_i : i < \theta\}$ then $u \subseteq \delta = \sup(u)$ for some $\delta < \lambda$ of cofinality θ ; (this actually follows by 2.1(3)(b)),
(b) if $\mathfrak{b}_\theta \leq \lambda$ then $u = \cup\{u_i : i < \theta\}$ and $\bar{u} \in \mathbf{U}_{\lambda, \theta}$ then $\text{otp}(u \setminus \theta) = \theta$,
- 5) DEAR MARK:intend to OMIT If $\lambda \geq \theta^+$ and $\theta = \text{cf}(\theta) > \aleph_0$ and $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \theta\}$ is stationary and some $\bar{C} = \langle C_\delta : \delta \in S \rangle$ guesses clubs, then $\text{PGC}(\lambda, \theta) = \lambda$.
6) If $\text{cf}([\lambda]^\theta, \subseteq) = \delta$ and $\mathfrak{d}_\theta \leq \lambda$ then $\text{FGC}(\lambda, \theta) = \lambda$, moreover $\text{BFGC}(\lambda, \theta) = \lambda$, (in fact, looking at [She93a] we get strongly bounding).

Discussion 2.5. DEAR MARK:NEED TO REREAD

- 1) In 2.4 we have ZFC results, we may get stronger results (on the full and almost versions) in some forcing extensions see 2.14. and [S⁺e].
2) We can look at the cases of Definition 2.1 for singular λ , replacing $(u_\zeta \setminus \sup(u_\varepsilon))$ by $u_\zeta \setminus u_\varepsilon$, but we have not arrive to it.
3) When we have clause (a)(γ) of the Definition 2.1(3) there is less need of clause (a)(α). E.g. in 2.4(1)(C) we do not need “ λ regular”.
4) In clauses (D), (E) of 2.1(1) we may add bounded/weakly bounded under natural assumption.

Proof. Without loss of generality \mathfrak{B} has a pairing function $\text{pr}^{\mathfrak{B}}$.

- 1) Clause (A): First, choose S, S^+, \bar{C} such that (partial square guessing clubs):

- (*)₁ (a) $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \theta \text{ and } \delta > \theta^+\}$ is stationary,
(b) $S \subseteq S^+ \subseteq \{\delta < \lambda : \text{cf}(\delta) \leq \theta \text{ and } \delta > \theta^+\}$, moreover if $\delta \in S$ then $\delta = \sup(S^+ \cap \delta)$,
(c) $\bar{C} = \langle C_\alpha : \alpha \in S^+ \rangle$,
(d) C_α is a closed subset of α of order type $\leq \theta$, and $\text{otp}(C_\alpha)$ is a limit ordinal iff $\alpha = \sup(C_\alpha)$,

⁴see footnote to part (2)

- (e) for $\alpha \in S^+$ we have $\alpha \in S \Leftrightarrow \text{otp}(C_\alpha) = \theta$,
- (f) if $\alpha \in C_\beta$ then $\alpha \in S^+$ and $C_\alpha = C_\beta \cap \alpha$,
- (g) $\bar{C} \upharpoonright S$ guess clubs, i.e.: if E is a club of λ then for stationarily many $\delta \in S$ we have $C_\delta \subseteq E$,
- (h) if $\alpha \in S^+$ then $\alpha > \theta^+$ and α is closed under \mathfrak{B} , that is $\mathfrak{B} \upharpoonright \alpha \subseteq \mathfrak{B}$

[Why do they exist (provably in ZFC)? see [Shea, 1.3=L1.3(b)], but we elaborate (for the case $\theta > \aleph_0$); by [She91, 4.4,pg.47] there are S, S^+, \bar{C} satisfying $(*)_1$ except possibly clauses (g) and (h) which we can add as follows: choose $\langle C_\alpha : \alpha \in S^+ \rangle$ satisfying the rest. For any club E of λ define the sequence $\bar{C}[E] = \langle C_\alpha[E] : \alpha \in S^+ \rangle$ as follows.

First let $S_E = \{\delta \in S : \delta \in E, \delta = \sup(\delta \cap E)\}$. Second let S_E^+ be $\{\alpha \in S^+ : \alpha \in S_E \text{ or } \alpha \in S^+ \cap E \setminus S\}$. Third for $\alpha \in S_E^+$ let $C_\alpha[E] = C_\alpha \cap E$.

Now $S_E, S_E^+, \bar{C}[E]$ satisfies clauses (a)-(f). Next the set $E_{\mathfrak{B}} = \{\delta < \lambda : \mathfrak{B} \upharpoonright \delta \subseteq \mathfrak{B}\}$ is a club of λ and for any club $E \subseteq E_{\mathfrak{B}}$ clearly $\bar{C}[E]$ satisfies clause (h). Lastly for satisfying clause (g), just we shall try θ^+ times as in [She94, Ch.III,2.3(2)], but we elaborate.

We choose $E_{<\varepsilon}, E_\varepsilon^1, E_\varepsilon^2$ by induction on $\varepsilon \leq \theta^+$. First $E_{<\varepsilon} = \bigcap_{\zeta < \varepsilon} (E_\zeta^1 \cap E_\zeta^2) \cap E_{\mathfrak{B}}$. Second, E_ε^1 is a club of λ included in $E_{<\varepsilon}$ which if possible witnesses that $\bar{C}[E_{<\varepsilon}] := \langle C_\delta \cap E_{<\varepsilon} : \delta \in S_{E_{<\varepsilon}} \rangle$ fails clause (g), that is, the set $\{\delta \in S_{E_{<\varepsilon}} : C_\delta[E_{<\varepsilon}] \subseteq E_\varepsilon^1\}$ is not stationary (in λ).

Lastly E_ε^2 is a club of λ , disjoint, if possible, to $\{\delta \in S_{E_{<\varepsilon}} : C_\delta[E_{<\varepsilon}] \subseteq E_\varepsilon^1\}$.

If for some $\varepsilon < \theta^+$, $\bar{C}[E_{<\varepsilon}]$ satisfies clause (g) then we are done. Otherwise as S is stationary and $\theta = \text{cf}(\theta) > \aleph_0$, clearly for some $\delta \in S$ we have $\delta = \sup(E_{<\theta^+} \cap \delta)$, hence $\langle C_\delta \cap E_{<\varepsilon} : \varepsilon < \theta^+ \rangle$ is a strictly decreasing sequence of closed unbounded subsets of C_δ , but $|C_\delta| = \theta$, contradiction, so $(*)_1$ holds indeed].

$(*)_2$ For $\delta \in S$ let $\langle \gamma_{\delta,\varepsilon}^\bullet : \varepsilon < \theta \rangle$ list C_δ in increasing order.

Second, fix \bar{f}, \bar{g} such that:

- $(*)_3$ (a) $\bar{f} \equiv \langle f_\alpha : \alpha \in [\theta^+, \lambda) \rangle$,
- (b) f_α is a one-to-one function from θ^+ onto α ,
- (c) $\bar{g} = \langle g_\xi : \xi \in [\theta, \theta^+) \rangle$,
- (d) g_ξ is a one-to-one function from θ onto ξ .

Third,

- $(*)_4$ (a) for $\delta \in S$ let $e_\delta = \{\xi < \theta^+ : \text{if } \alpha \in C_\delta \text{ then } \text{Rang}(f_\alpha \upharpoonright \xi) = \alpha \cap \text{Rang}(f_\delta \upharpoonright \xi) \text{ and this set include } C_\delta \cap \alpha \text{ and has cardinality } \theta\}$
- (b) e_δ is a club of θ^+ .

[Why clause (b) holds? As $\text{otp}(C_\delta) = \theta$ and $\alpha \in C_\delta \cup \{\delta\} \Rightarrow |\alpha| = \theta^+$, this should be clear.]

$(*)_5$ for $\delta \in S$ and $\xi \in e_\delta$ let:

- (a) $u_{\delta,\xi} = \text{Rang}(f_\delta \upharpoonright \xi)$, it belongs to $[\delta]^\theta$ and it includes C_δ
- (b) we choose $\bar{u}_{\delta,\xi} = \langle u_{\delta,\xi,\varepsilon} : \varepsilon < \theta \rangle$ by $u_{\delta,\xi,\varepsilon} = \text{cl}_{\mathfrak{B}}(\{f_{\gamma_{\delta,v}^\bullet}(g_\xi(\zeta)) : v < \omega(1+\varepsilon) \text{ and } \zeta < \omega(1+\varepsilon)\} \cup \{\gamma_{\delta,v}^\bullet : v < \omega(1+\varepsilon)\})$
- (c) for $w \in [\theta]^\theta$ let $\bar{u}_{\delta,\xi}^{[w]}$ be $\langle u_{\delta,\xi,\varepsilon}^{[w]} : \varepsilon < \theta \rangle$ where $u_{\delta,\xi}^{[w]} = \bigcup u_{\delta,\xi,\iota}$ where: $\iota \in w$ satisfies $\text{otp}(w \cap \iota) = \varepsilon$, this fit 2.1(9)(d).

Note that (recalling $(*)_2$)

- $(*)_6$ For $\delta \in S, \xi \in e_\delta$ we have:
- (a) $\bar{u}_{\delta, \xi}$ is a \subseteq -increasing continuous sequence of subsets of $u_{\delta, \xi}$,
 - (b) each $u_{\delta, \xi, \varepsilon}$ include $C_{\gamma_{\delta, \omega(1+\varepsilon)}^\bullet}$ and is an unbounded subset of $\gamma_{\delta, \omega(1+\varepsilon)}^\bullet$ and it is of cardinality $< \theta$,
 - (c) $\cup\{u_{\delta, \xi, \varepsilon} : \varepsilon < \theta\}$ is equal to $u_{\delta, \xi}$
 - (d) $u_{\delta, \xi, \varepsilon}$ is computable from $\text{pr}^{\text{BS}}(\gamma_{\delta, \varepsilon}^\bullet, \xi)$ recalling that pr^{BS} is a pairing function, well, using as parameters \bar{f}, \bar{g} which were fixed in $(*)_2$.

[Why? should be clear]

Lastly,

- $(*)_7$ let
- (a) $\mathbf{U} = \{\bar{u}_{\delta, \xi} : \delta \in S, \xi \in C_\delta\}$
 - (b) $\mathbf{U}_w = \{\bar{u}_{\delta, \xi}^{[w]} : \delta \in S \text{ and } \xi \in e_\delta\}$ for $w \in [\theta]^\theta$.

Now we shall deal which is of the five clauses of 2.1(1) separately.

Clause (A) We shall prove that (why the w ? for the use in the proof of part (4) of the claim):

- $(*)_8$ if $w \in [\theta]^\theta$ then \mathbf{U}_w witness $\text{WSGC}(\lambda, \theta) \leq \lambda$.

Fix w now and we shall deal with all the demands:

- $(*)_{8.1}$ \mathbf{U}_w has cardinality $\leq \lambda$; in fact is equal to λ .

[Why? As $|\mathbf{U}_w| \leq |\{(\delta, \xi) : \delta \in S, \xi \in e_\delta \subseteq \theta\}| \leq \lambda + \theta = \lambda$. The other inequality is also easy as $\cup\{u_{\delta, \xi} : \delta \in S, \xi \in e_\delta\} = \lambda$ and each $u_{\delta, \xi}$ has cardinality $\theta < \lambda$.]

- $(*)_{8.2}$ $\mathbf{U}_w \subseteq \mathbf{U}_{\lambda, \theta}$ is reasonable.

[Why? By the choices above.]

- $(*)_{8.3}$ \mathbf{U}_w semi-guess clubs.

[Why? Let M and A be as in Definition 2.1(3B)(a)', let M^+ be the expansion of M by the relation $<^{M^+}$, the order of the ordinals $< \lambda$ and $P^{M^+} = A$, and let $E := \{\delta < \lambda : M^+ \upharpoonright \delta \prec M^+\}$, clearly E is a club of λ . By the choice of \bar{C} there is $\delta \in S$ such that $C_\delta \subseteq E$ (hence $\delta \in S$). Note that if $\alpha \in C_\delta$ then $A \cap \alpha$ is unbounded in α .

Now recall that $Mp1\delta \prec M$, $\langle u_{\delta, \xi} : \xi \in e_\delta \rangle$ is \subseteq -increasing continuous with union δ , each $u_{\delta, \xi}$ is of cardinality $\leq \theta$ and e_δ is a club of θ^+ hence $e = \{\xi \in e_\delta : M^+ \upharpoonright u_{\delta, \xi} \prec M^+\}$ is a club of θ^+ . So if $\xi \in e_\delta$ then $A \cap u_{\delta, \xi}$ is unbounded in $u_{\delta, \xi}$. Now choose $\xi \in e$, so $\bar{u} = \bar{u}_{\delta, \xi}$ is as required.]

- $(*)_{8.4}$ \mathbf{U} is weakly bounded.

[Why? Just think, recalling $(*)_1$ and Definition 2.1(7), that is, note that $\langle C_\delta \cap \alpha : \delta \in S^+ \rangle$ has cardinality $\leq \theta^+$ for each $\alpha < \lambda$ because $\beta \in C_{\delta_1} \cap C_{\delta_2} \Rightarrow C_{\delta_1} \cap \beta = C_{\delta_2} \cap \beta$, anyhow below we shall get more.]

- $(*)_9$ \mathbf{U} is bounded hence $\text{CSGC}(\lambda, \theta)$ holds, in fact:

- (a) if $u_1 = u_{\delta_1, \xi_1, \varepsilon_1}, u_2 = u_{\delta_2, \xi_2, \varepsilon_2}$ and $\text{pr}(\gamma_{\delta_1, \varepsilon_1}^\bullet, \xi_1) = \text{pr}(\gamma_{\delta_2, \varepsilon_2}^\bullet, \xi_2)$ then:

- (α) $\langle \gamma_{\delta_1, \varepsilon}^\bullet : \varepsilon \leq \varepsilon_1 \rangle = \langle \gamma_{\delta_2, \varepsilon}^\bullet : \varepsilon \leq \varepsilon_2 \rangle$
 (β) $u_1 = u_2$
 (b) $\text{pr}(\gamma_{\delta, \varepsilon}^\bullet, \xi_1) < \gamma_{\delta, \varepsilon+1}^\bullet$.

[Why? Clause (a) holds by $(*)_6(d)$ and clause (b) by $(*)_1(h)$.]

We have finished proving $\lambda = \text{CSGC}(\lambda, \theta)$, and even $\text{CSGC}(\lambda, \theta, \mathfrak{B})$, that is clause (A) of part (1).

Clause (B): Fix a stationary $S \subseteq S_\theta^{\theta^+}$ which belongs to $\check{I}_\theta[\theta^+]$, see Def 0.7. By 0.8 we can choose $\zeta_{\xi, \varepsilon}$ for $\xi \in S$ such that for any such $\xi \in S$, $\langle \zeta_{\xi, \varepsilon} : \varepsilon < \theta \rangle$ is increasing continuous with limit ξ and $(\zeta_{\xi_1, \varepsilon+1} = \zeta_{\xi_1, \varepsilon+1}) \wedge (v \leq \varepsilon) \Rightarrow \zeta_{\xi_1, v} = \zeta_{\xi_2, v}$.

Now in the proof of part (1) we can restrict ourselves to $\xi \in S$ such that $u_{\delta, \xi}$ is closed under pr. Then we can restrict ourselves to (ω, δ, ξ) such that $\varepsilon_1 < \varepsilon_2 \in w \Rightarrow \text{pr}(\gamma_{\delta, \varepsilon_1}^\bullet, \zeta_{\xi, \varepsilon_1}) \in u_{\delta, \xi, \varepsilon_2}$.

Clause (C): Easy but we elaborate.

We are assuming $\lambda = \lambda^\theta$, $\theta = \text{cf}(\theta)$; so $\mathbf{U} = \mathbf{U}_{\lambda, \theta}$ is trivially a subset of $\mathbf{U}_{\lambda, \theta}$ of cardinality λ . Let M be a model with universe λ and $A \in [\lambda]^\lambda$ and we have to find \bar{u} as promised. Toward this we choose u_ε by induction on ε as follow:

- (a) u_ε is a subset of λ of cardinality $< \theta$,
- (b) $u_\varepsilon = \text{cl}(u_\varepsilon, M)$ and has no last member,
- (c) if $\varepsilon = \zeta + 1$ then for some $\alpha \in A \setminus \text{sup}(u_\zeta)$ belongs to u_ε ,
- (d) if ε is a limit ordinal then $u_\varepsilon = \cup\{u_\zeta : \zeta < \varepsilon\}$

There is no problem to carry the induction and $\langle u_\varepsilon : \varepsilon < \theta \rangle$ is as required.

Clause (D) Recall that $\mathcal{S} \subseteq \{w \subseteq \lambda : \text{otp}(w) = \theta\}$ and more by our assumption. For each $w \in \mathcal{S}$ let \bar{u}_w be $\langle u_{w, \varepsilon} : \varepsilon < \theta \rangle$ were $u_{w, \varepsilon} = \{\alpha \in w : \text{otp}(w \cap \alpha) < \omega(1 + \varepsilon)\} \cup \varepsilon$. Now let $\mathbf{U} = \{\bar{u}_w : w \in \mathcal{S}\}$, it suffice to prove that \mathbf{U} witness $\text{MGC}(\lambda, \theta) = \lambda$.

Clearly most demands hold: \mathcal{U} is a subset of $\mathbf{U}_{\lambda, \theta}$ of cardinality λ , and for each $\bar{u} \in \mathbf{U}$ the sequence $\langle \text{sup}(u_\varepsilon) : \varepsilon < \theta \rangle$ is increasing. The main point is, to be given M, A as in clause (a) of Def 2.1(3) and to prove that sub-clauses (α), (β) there hold.

Let M^+ be an expansion of M by the order $<^{M^+}$ of the ordinals $< \lambda$, $R^{M^+} = A$ and let $E = \{\delta < \lambda : M^+ \upharpoonright \delta \prec M^+\}$, clearly it is a club of λ . Now there is no harm in replacing A by a smaller sub-set so let $A' = \{\alpha \in A : \alpha = \min(A \cap \beta) \text{ for some } \beta \in E\}$. Clearly $A' \in [\lambda]^\lambda$ so by the choice of \mathcal{S} there is $w \in \mathcal{S}$ such that $w \cap A$ has cardinality θ .

Now $\bar{u}_w \in \mathbf{U}$ is as required.

Clause E

By [She93a] there is a stationary $\mathcal{A} \subseteq [\lambda]^\theta$ of cardinality λ , see detail in the proof of part (3). Now for each $w \in \mathcal{S}$ let $\mathcal{S}_w = v \in \mathcal{A} : w \subseteq v$, so it is non-empty. Now for each $w \in \mathcal{S}$ let $\langle \alpha_{w, \varepsilon} : \varepsilon < \theta \rangle$ list the members of w in increasing order. Also for each such pair (w, v) let $\bar{u}_{w, v} = \langle u_{w, v, \varepsilon} : \varepsilon < \theta \rangle$ be such that:

- (a) $u_{w, v, \varepsilon}$ is a subset of v of cardinality $< \theta$,
- (b) $u_{w, v, \varepsilon}$ is increasing continuous with ε ,
- (c) $u_{w, v, \varepsilon}$ include $\{\alpha_{w, \zeta} : \zeta < \omega(1 + \varepsilon)\}$
- (d) $u_{w, v, \varepsilon}$ is included in $\cup\{\alpha_{w, \zeta} : \zeta < \omega(1 + \varepsilon)\}$
- (e) $\cup\{u_{w, v, \varepsilon} : \sigma < \theta\} = v \cap \cup\{\alpha_{w, \varepsilon} : \varepsilon < \theta\}$

Lastly we define \mathbf{U} as the set $\{\bar{u}_{w, v} : w \in \mathcal{S}, v \in \mathcal{A}_w\}$; so it suffice to prove that \mathbf{U} witness $\text{MGC}_D(\lambda, \theta) = \lambda$; this is an in previous cases.

2) Easy. DEAR MARK:: to give details?

2) DEAR MARK:THIS INTEND TO OMIT By the assumption there is \mathcal{S} such that:

- (*)₁ (a) $\mathcal{S} \subseteq [\lambda]^\theta$ has cardinality λ ,
- (b) if $A \in [\lambda]^\theta$ then for some $w \in \mathcal{S}$ we have $|A \cap w| = \theta$.
- (*)₂ let S, \bar{e}, \bar{C} be such that:
 - (a) $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \theta\}$ is stationary,
 - (b) $\bar{C} = \langle C_\delta : \delta \in S \rangle$ with C_δ a club of δ of order type θ ,
 - (c) \bar{C} guesses clubs, we may add $\alpha \in \text{nacc}^9 C_{\delta_1} \cap \text{nacc}(C_{\delta_2}) \Rightarrow C_{\delta_1} \cap \alpha = C_{\delta_2} \cap \alpha$,
 - (d) $S \in \check{I}[\lambda]$ as witnessed by \bar{e} , (see Def 0.7), not used but will be helpful⁵.

[Why they exist? see [She05], [Shea, §0, 0.6,]]. Without loss of generality

- (*)₃ if $w \in \mathcal{S}$ then $w \subseteq \delta = \text{sup}(w)$ for some δ of cofinality θ .

[Why? Let $\mathcal{S}' = \{w \cap \delta : \delta < \lambda \text{ has cofinality } \theta \text{ and } \delta = \text{sup}(w \cap \delta)\}$, clearly it is as required.]

Now for every $w \in \mathcal{S}$, we choose $\bar{u} = \bar{u}_w$ such that:

- $\bar{u} = \langle u_\varepsilon : \varepsilon < \theta \rangle$
- $u_\varepsilon \in [w]^{<\theta}$ is increasing continuous with ε
- $w = \cup \{u_\varepsilon : \varepsilon < \theta\}$
- u_ε has no last element, $\langle \text{sup}(u_\varepsilon) : \varepsilon < \theta \rangle$ is increasing continuous,
- $u_\varepsilon \subseteq \alpha$ when α is the ε -th member of $C_{\text{sup}(w)}$

Let $\mathbf{U} = \{\bar{u}_w : w \in \mathcal{S}\}$, it is as required. Obviously $\mathbf{U} \subseteq [\lambda]^\theta$ has the right cardinality; so let Now we shall show that the relevant clauses 2.1(3A)(a)(α), (β) hold. $M_*, A \in [\lambda]^\lambda$ be as there, and we have to find a suitable \bar{u} . Let E be the following club of λ : $E = \{\delta : \delta \text{ is a limit ordinal } < \lambda \text{ and } \upharpoonright \delta \prec \}$ and $\delta = \text{sup}(A \cap \delta)\}$, we can ask for more. Now choose $\delta \in E \cap S$ such that $C_\delta \subseteq E$, possible by (*)₂. Choose $B \subseteq A \cap \delta$ of order type θ such that $\alpha \in C_\delta \Rightarrow 1 = \|B \cap [\alpha, \text{succ}_{C_\delta}(\alpha)]\|$, that is the intersection is a singleton and $B \cap \min(C_\delta) = \emptyset$. Next by (*)₁(b) choose $w \in \mathcal{S}$ such that $w \cap B$ has cardinality θ , by (*)₃ without loss of generality we have $\text{sup}(w) = \delta$. Now let $\bar{u} = \bar{u}_w \in \mathbf{U}$ be as

Why \bar{u} is as required? Clause 2.1(3)(a)(α), that is $\varepsilon < \theta \rightarrow \text{cl}(u_\varepsilon, M) \subseteq \text{sup}(U)^\varepsilon$, hold because of the choice of δ such that $C_\delta \subseteq E$. Clause 2.1(3)(a)(β)' holds by the choice of B .

Also clause 2.1(3)(b) is easy.

DEAR MARK:END OF WHAT I INTEND TO OMIT

3) By [She96] there is \mathcal{S} :

- (*)₁ (a) $\mathcal{S} \subseteq [\lambda]^\theta$ has cardinality λ
- (b) \mathcal{S} is stationary, i.e. for every model M_* with universe λ and vocabulary $\leq \theta$ there is $w \in \mathcal{S}$ such that $M_* \upharpoonright w \prec M_*$.

Now as we can increase \mathcal{S} , without loss of generality:

- (*)₂ $\mathcal{S} \cap [\alpha]^\theta$ is a stationary subset of $[\alpha]^\theta$ for every $\alpha \leq \lambda$.

Also without loss of generality

⁵ by this without loss of generality for every $\beta < \lambda$ the set $\mathcal{C}_\beta = \{C_\delta \cap \beta : \beta \in \text{nacc}(C_\delta)\}$ has cardinality $< \lambda$

- (*)₃ (a) if $u \in \mathcal{S}$ then $u \subseteq \delta = \sup(u)$ for some δ of cofinality θ
 (b) if $v < \lambda$ and $\text{cf}(v) = \theta$ and $\mathfrak{B} \upharpoonright (\delta \cap v) \subseteq \mathfrak{B}$ then $v \cap u \in \mathcal{S}$.

[Why? Let $\mathcal{S}' = \{u \cap \delta : \delta < \lambda \text{ has cofinality } \theta, u \in \mathcal{S}, \text{ or } v \subseteq \mathfrak{B} \text{ and } \delta = \sup(u \cap \delta)\}$.

We continue as in the proof of part (1). DEAR MARK:TO SAY MORE?

4) Let $\mathbf{U} \subseteq \mathbf{U}_{\lambda, \theta}$ medium guess clubs,

Now clause (a) follows by 2.1(3)(b).

For clause (b), $\mathbf{U} = \{\bar{u}' : \bar{u} \in \mathbf{U}\}$ where for $\bar{u} \in \mathbf{U}_1$ we let $\bar{u}' = \langle u'_\varepsilon : \varepsilon < \theta \rangle$ where $u'_\varepsilon = \cup \{u_{\zeta+1} \setminus \sup(u_\zeta) : \zeta < \varepsilon\} \cup \varepsilon$, Now we can check that \mathbf{U}' is as required.

5) DEAR MARK:HAJMET??

As in part (2).

6) Combine things above. □_{2.4}

Discussion 2.6. Assume $\lambda > \theta \geq \sigma = \text{cf}(\sigma)$, ($2^\sigma > \lambda$ in the interesting case). Let $\mathcal{U}_{\lambda, \theta, \sigma} = \{\bar{u} : \bar{u} = \langle u_\varepsilon : \varepsilon < \sigma \rangle \text{ is } \subseteq\text{-increasing and } u_\varepsilon \in [\lambda]^\theta\}$ and repeat the definition. Of doubtful help, otherwise $(\theta^{++}, \theta^+, \theta)$ would have helped.

Theorem 2.7. 1) Assume $\lambda = \text{cf}(\lambda) \geq \theta = \text{cf}(\theta)$, $D = [\theta]^\theta$ and $\text{AGC}_D(\lambda, \theta) = \lambda$ or just $\text{MGC}_D(\lambda, \theta) = \lambda$ and $\mathfrak{b}_\theta > \lambda$. Then $\lambda \notin \text{Univ}(T_{\text{ceq}})$; moreover, $\text{univ}_{T_{\text{ceq}}}(\lambda) \geq \mathfrak{b}_\theta$.

2) If $\lambda = \text{cf}(\lambda) > \theta = \text{cf}(\theta)$ and $\text{FGC}(\lambda, \theta) = \lambda$ and $\chi = \mathfrak{d}_\theta > \lambda$ or just $\text{cf}(\theta, \leq D) \geq \xi > \lambda$, then $\text{univ}_{T_{\text{ceq}}}(\lambda) \geq \chi$.

3) If D is a uniform filter on θ , $({}^\theta \theta, <_D)$ is $(< \chi)$ -directed and $\chi > \lambda$, $\text{FGC}_D(\lambda, \theta) = \lambda$ then $\text{univ}_{T_{\text{ceq}}}(\lambda) \geq \chi$.

Remark 2.8. 1) The Claim 2.11 below shows that we cannot weaken the assumption on T too much.

2) Note that the above work also for $\theta = \aleph_0$,

Proof. 1) So let $(T = T_{\text{ceq}})$ and

(*)₁ assume $\alpha_* < \mathfrak{b}_\theta$ and $M_\alpha^* \in \text{EC}_T(\lambda!)$ for $\alpha < \alpha_*$; it suffices to find $N \in \text{EC}_{T_{\text{ceq}}}(\lambda!)$ not embeddable into M_α^* for every $\alpha < \alpha_*$

(*)₂ let $\mathbf{U} \subseteq \mathbf{U}_{\lambda, \theta}$ witness $\text{AGC}(\lambda, \theta) = \lambda$

(*)₃ for $\bar{u} \in \mathbf{U}$, $\alpha < \alpha_*$ and $d \in P^{M_\alpha^*}$ we define the set $E_{\bar{u}, d, \alpha}$; clearly is a club of θ , as follows: newline

$E_{\bar{u}, d, \alpha} = \{\varepsilon < \theta : \varepsilon \text{ is a limit ordinal such that } u_\varepsilon \text{ is closed inside } \cup \{u_\zeta : \zeta < \theta\}$
 under the functions of M_α^* and the function $F^{M_\alpha^*}(-, d)$, i.e. if $a \in u_\varepsilon, b \in \bigcup_{\zeta < \theta} u_\zeta$
 and $M_\alpha^* \models "F(a, d) = b"$ then $b \in u_\varepsilon\}$.

So $\mathcal{E} = \{E_{\bar{u}, d, \alpha} : \bar{u} \in \mathbf{U}, \alpha < \alpha_* \text{ and } d \in P^{M_\alpha^*}\}$ is a set of clubs of θ of cardinality $\leq |\mathbf{U}| + |\alpha_*| + |P^{M_\alpha^*}| < \mathfrak{b}_\theta$. Hence there is an increasing function $g : \theta \rightarrow \theta$ such that $(\forall E \in \mathcal{E})(\forall^\infty \varepsilon < \theta)(g(\varepsilon) > \text{succ}_E(\varepsilon))$.

Now we can construct $N = N_g \in \text{EC}_T(\lambda!)$ such that:

- (*)₄ (a) $P^N = \{3\beta : \beta < \lambda\}$ hence $Q^N = \{3\beta + 1, 3\beta + 2 : \beta < \lambda\}$
 (b) if $\alpha = 3\beta + 1 < \lambda$ (hence $\alpha \in Q^N$) then $\alpha = \min(\alpha/E^N)$

⁶ recall that his mean that $D = \{\theta\}$

- (c) if $\bar{u} \in \mathbf{U}$ then for some $\alpha(\bar{u}) = \alpha_{\bar{u},g} \in P^N$ we have: if $\beta \in (u_{\varepsilon+1} \setminus u_\varepsilon) \cap Q^N$ and $(\beta/E^N) \cap u_\varepsilon = \emptyset$ but $v < \theta \Rightarrow (\beta/E^N) \cap \bigcup_{\zeta < \theta} u_\zeta \not\subseteq \text{sup}(u_v)$ then
- $$F^N(\beta, \alpha(\bar{u})) \in \bigcup_{\zeta < \theta} u_\zeta \setminus \text{sup}(u_{g(\varepsilon+1)}).$$

Now toward contradiction assume that:

- (*)₅ f embeds N_g into M_α^* and $\alpha < \alpha_*$.

Let $N_g^+ = (N_g, <)$ and let M_* be a model with universe λ expanding M_α^* and (a renaming of) N_g ; (that is $\tau(M_\alpha^*)$ contains also a disjoint copy τ' of $\tau(N_g^+)$ such that the restriction of M_* to τ' is the suitable copy of N_g^+).

Also we have $f = G^{M_*}$ for some unary function symbol $G \in \tau(M_*)$ and $<^{M_*} = \{(\alpha, \beta) : \alpha < \beta < \lambda\}$ and M_* has Skolem functions and $\tau(M_*)$ is countable.

Let $A = \{f(3\beta + 1) : \beta < \lambda\}$. By the choice of \mathbf{U} there is \bar{u} such that:

- (*)₆ (a) $\bar{u} \in \mathbf{U}$
 (b) $M_* \upharpoonright \bigcup_{\varepsilon < \theta} u_\varepsilon \prec M_*$, not used when we assume only $\text{MGC}(\lambda, \theta) = \lambda$,
 (c) $c\ell(u_\varepsilon, M_*) \subseteq \text{sup}(u_\varepsilon)$, and (essentially follows) $M_* \upharpoonright \text{sup}(u_\varepsilon) \prec M_*$,
 (d) the set $v = \{\varepsilon < \theta : A \cap u_{\varepsilon+1} \setminus \text{sup}(u_\varepsilon) \neq \emptyset\}$ has cardinality θ .

Now let $d = f(\alpha_{\bar{u},g})$, so it is a member of $P^{M_\alpha^*}$, and

- (*)₇ if $\varepsilon \in v$ then for some $a = a_\varepsilon \in u_{\varepsilon+1} \setminus \text{sup}(u_\varepsilon)$ we have $a_\varepsilon \in A$,

Note that the set $B = \{a \in Q^{N_g} : a \text{ is minimal (ordinal) in } a/E^{N_g}\}$ is definable in the model M_* , and it include the set $\{3\beta + 1 : \beta < \lambda\}$.

Also in the model M_* we can define the function h from B into $Q^{M_\alpha^*}$ by: $h(a)$ is the minimal ordinal in $f(a)/E^{M_\alpha^*}$. Clearly

- (*)₈ B, h are indeed definable and h is a one to one function.

Hence:

- (*)_{*} if $\delta < \lambda$ is a limit ordinal and $M_* \upharpoonright \delta \prec M_*$ then for every $a \in B$ we have $a < \delta \Leftrightarrow h(a) < \delta$

Hence

- (*)₍ $b_\varepsilon = f(a_\varepsilon)$ satisfies:
 (a) $b_\varepsilon \in u_{\varepsilon+1}$,
 (b) b_ε does not belong to $M_* \upharpoonright \text{sup}(u_\varepsilon)$
 (c) $f(a_\varepsilon)$ and $\min(f(a_\varepsilon)/E^{M_\alpha^*}) = b_\varepsilon$ both belong to $u_{\varepsilon+1} \setminus (\text{sup}(u_\varepsilon))$

By the choice of $E_{\bar{u},d,\alpha}$ and g we get a contradiction.

2),3) Similarly. DEAR MARK:ADD DETAILS

□_{2.7}

Conjecture 2.9. 1) Assume T (is countable complete first order) with the PT_2 . If $\lambda > \theta > \aleph_0$ are regular, $\mathfrak{d}_\theta > \lambda$ and $\mathfrak{d}_\kappa > \text{FGC}(\lambda, \theta)$ (maybe θ inaccessible), then $\text{univ}_T(\lambda) \geq \mathfrak{d}_\kappa$.
 2) Assume T (is countable complete first order) non-simple. If $\lambda > \theta > \aleph_0$ are regular, $\mathfrak{d}_\theta > \lambda$ and $\mathfrak{d}_\kappa > \text{CFGC}(\lambda, \theta)$, then $\text{univ}_T(\lambda) \geq \mathfrak{d}_\kappa$.

Remark 2.10. See hopefully [Shed], [S⁺f], [S⁺d].

Claim 2.11. Assume $\mu = \mu^{<\mu} \leq \theta = \text{cf}(\theta) < \lambda = \text{cf}(\lambda) < \chi = \chi^\lambda$, λ is strongly inaccessible Mahlo and for transparency GCH holds in the interval $[\mu, \chi)$. For some \mathbb{P} :

- (a) \mathbb{P} is a $(< \mu)$ -complete forcing of cardinality χ neither collapsing any cardinal, nor changing cofinalities
- (b) $(2^\mu)^{\mathbb{V}^{\mathbb{P}}} = \chi$
- (c) in $\mathbb{V}^{\mathbb{P}}$ we have $\mathfrak{d}_\partial = \chi$ for every inaccessible $\partial \in [\mu, \lambda)$,
- (d) in $\mathbb{V}^{\mathbb{P}}$ there is \mathcal{P} as in 1.5(2),
- (e) T_{ceq} has no universal member in λ moreover $\text{univ}(\lambda, T_{\text{ceq}}) \geq \xi$.
- (f) the results of [She90] holds, i.e. there is a universal random graph in λ , and see [She21]

Remark 2.12. Note that here the case “ $\cup_{\varepsilon < \theta} u_\varepsilon \cap \text{sup}(u_\zeta)$ has cardinality θ ” does not arise.

Proof. DEAR MARK: This is changed

Essentially as in [She90],.

That is, we choose:

- (*) $\mathbb{P} = \mathbb{P}_3$, where $\langle \mathbb{P}_k, \mathbb{Q}_\ell : k \leq 3, \ell < 3 \rangle$ is an iteration and:
 - (A) \mathbb{Q}_0 is adding χ λ -Cohen, so it satisfies:
 - ₁ \mathbb{Q}_0 is a $(< \lambda)$ -complete forcing notion of cardinality χ ,
 - ₂ \mathbb{Q}_0 neither collapse some cardinal nor change any cofinality (in fact is λ^+ -cc),
 - ₃ in $\mathbb{V}^{\mathbb{Q}_0}$ there is a family \mathcal{A}_0 of χ -many subsets of λ each of cardinality λ , the intersection of any two having cardinality $< \lambda$,
 - (B) in $\mathbb{V}^{\mathbb{Q}_0} = \mathbb{V}^{\mathbb{P}_1}$ the forcing notion \mathbb{Q}_1 satisfies:
 - ₁ \mathbb{Q}_1 is a $(< \mu)$ -complete λ -cc forcing notion of cardinality χ ,
 - ₂ \mathbb{Q}_1 is neither collapses some cardinal nor changes any cofinality, (yes, λ -cc not λ^+ -cc),
 - ₃ in $\mathbb{V}^{\mathbb{P}_2}$ there is a family \mathcal{A}_1 of χ -many subsets of λ , each of cardinality λ , the intersection of any two having cardinality $< \mu$,
 - ₄ in $\mathbb{V}^{\mathbb{P}_2}$ we have $\mathfrak{d}_\partial = \chi$ for every weakly inaccessible $\partial \in (\mu, \lambda)$,
 - (C) in $\mathbb{V}^{\mathbb{P}_2}$ we have \mathbb{Q}_2 which is $(< \mu)$ -complete μ^+ -cc forcing notion forcing that there is a universal graph of cardinality λ .

Now, why are there such \mathbb{Q}_ℓ -s? For clause (A) use the forcing of adding χ λ -Cohens.

For clause (B) we use \mathbb{Q}_1 such that where:

- (*) \mathbb{Q}'_1 we quote [S⁺e] which continue [Bau76, Th.6.1] as in [She90, 1.2]. starting with $(\mu, \lambda, \chi, \mathcal{A}_0)$ so in particular we force a refinement \mathcal{A}_1 of \mathcal{A}_0 ,

Now why clause (B)•₁, •₂, •₃ holds? As said above by [S⁺e] while [Bau76] prove that it holds for \mathbb{Q}'_1 , the proof works also for $\mathbb{Q}'_1 \times \mathbb{Q}''_1$.

Lastly for clause (C) we apply [She90].

Having constructed $\mathbb{P} = \mathbb{P}_3$ we have to check that is is as required.

Now being $(< \mu)$ -complete, of cardinality χ , pedantically of density χ , is obvious by the properties of the \mathbb{Q}_ℓ -s. Similarly concerning “no cardinal is collapsed and no cofinality changed”, so clause (a) of Claim 2.11 holds. Also forcing the existence of a universal graph of cardinality λ holds by the choice of \mathbb{Q}_2 , so clause (f) of 2.11 holds.

Next, clause (c) there saying $\mathfrak{d}_\partial = \chi$ holds because it obviously holds in $\mathbb{V}^{\mathbb{P}_2}$ by the choice of \mathbb{Q}''_1 and the later forcing preserve it because it satisfies the μ^+ -cc.. Now, lastly, why clause (d) of 2.11 saying holds? First, in $\mathbb{V}^{\mathbb{P}_1}$ we have GCH in the interval $[\mu, \lambda)$ so

there is such $\bar{\mathcal{D}}$, and $\mathbb{P}_3/\mathbb{P}_1$ satisfies the λ -cc so the old clubs of λ are dense. and this continue to holds in $\mathbf{V}^{\mathbb{P}}$. Hence the non-existence of a universal model of T_{ceq} in λ , (holds by 1.5(2)). $\square_{2.11}$

Question 2.13. 1) Can we for theories T satisfying $\text{NSOP}_1 + \text{PT}_2$ get similar results?
2) Is T_{ceq} in some sense minimal non-simple in a suitable family of theories?

Claim 2.14. Assume $\lambda > \theta = \text{cf}(\theta) > \aleph_0$ and $\lambda = \lambda^\theta$ and \mathbb{P} is a θ -cc forcing notion

1) In $\mathbf{V}^{\mathbb{P}}$ there is a reasonable strongly bounding $\mathbf{U} \subseteq \mathbf{U}_{\lambda, \theta}$ of cardinality λ witnessing $\lambda = \text{FGC}(\lambda, \theta, \mathfrak{B})$

2) Assume $\mathbf{U} \subseteq \mathbf{U}_{\lambda, \theta}$ fully/almost guess clubs then in $\mathbf{V}^{\mathbb{P}}$, \mathbf{U} still fully/almost (resp) guess clubs.

3) In part (2), if \mathbf{U} is reasonable/bounding/strongly bounding/weakly bounding in \mathbf{V} , then so it is in $\mathbf{V}^{\mathbb{P}}$.

Remark 2.15. 1) This will help in consistency results, see $[\text{S}^+ \text{b}]$.

2) Similarly for the other versions of guessing clubs from 2.1, but take care of what is \mathcal{D} .

Proof. DEAR MARK:I DID NOT READ Part (1) follows by parts (2),(3) because in \mathbf{V} there is such \mathbf{U} by 2.4(1)(C). The point is:

(*) (A) \Rightarrow (B) where:

(A) if $\chi > \lambda$ and $\{\mathfrak{B}, \mathbb{P}, \lambda\} \cup \{\varepsilon : \varepsilon \leq \theta\} \subseteq N \prec (\mathcal{H}(\chi), \in)$ and $\|N\| < \lambda$ and \mathbb{P} satisfies the θ^+ -cc where $\Vdash_{\mathbb{P}} \mathfrak{B}$ a model with universe λ and vocabulary of cardinality $< \theta$

(B) $\Vdash_{\mathbb{P}} \mathfrak{N} \cap \lambda = \text{cl}\{|N|, \mathfrak{B}\}$ and $\mathfrak{B} \upharpoonright |N|$ is an elementary submodel of \mathfrak{B} .

$\square_{2.14}$

Definition 2.16. Assuming $\theta = \text{cf}(\theta) \leq \lambda$ we let $\mathfrak{d}_{\theta\lambda}^\dagger$ be the cofinality of the partial order $(\mathcal{F}_{\theta,\lambda}^\dagger, \leq_{\theta,\lambda}^\dagger)$ where:

- (*) $\mathcal{F}_{\theta,\lambda}^\dagger$ is the family of subsets of ${}^\theta\theta$ of cardinality $\leq \lambda$
- (*) let $\leq_{\theta,\lambda}^\dagger$ is the following partial order on $\mathcal{F}_{\theta,\lambda}^\dagger$:
 $F_1 \leq F_2$ if $(\forall f_1 \in F_1)(\exists f_2 \in F_2)[f_1 \leq f_2]$

The following was part of 2.7, maybe we shall return it.

Claim 2.17. 1) If $\lambda > \theta > \aleph_0$ are regular, $D = [\theta]^\theta$, $\text{FGC}(\lambda, \theta) = \lambda$ and $\mathfrak{d}_\theta > \lambda$ then $\text{univ}(\lambda, T_{\text{ceq}}) \geq \mathfrak{d}_\theta$.

2) Above we can replace \mathfrak{d}_θ by $\mathfrak{d}_{\theta,\lambda}^\dagger$

Proof. 1) Like the proof of part (1) of 2.7(1) so we mainly note the changes.

- (*)₁ as above.
- (*)₂ $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ witness $\text{FGC}(\lambda, \theta) = \lambda$.
- (*)₃ we let:
 - (a) M_α^+ is an expansion of M_α^* by a paring function and Skolem functions,
 - (b) $\mathcal{Z} = \{(\alpha, \bar{u}, d) : \alpha < \alpha_*, \cup_{\varepsilon < \theta} u_\varepsilon \text{ is closed under the function } F^{M_\alpha^+} \text{ and each } u_\varepsilon \text{ is closed under the functions of } M_\alpha^+\}$,
 - (c) for $(\alpha, \bar{u}, d) \in \mathcal{Z}$ let $E_{\alpha,\bar{u},d} = \{\varepsilon < \theta : u_\varepsilon \text{ is closed under } F^{M_\alpha^*}(-, d)\}$,
 - (d) above let $g_{\alpha,\bar{u},d} \in {}^\theta\theta$ be such that $g_{\alpha,\bar{u},d}(\varepsilon) = \min(E_{\alpha,\bar{u},d} \setminus (\varepsilon + 1))$ $g \in {}^\theta\theta$

Next choose $g \in {}^\theta\theta$ no bounded by any well defined $g_{\alpha,\bar{u},d}$ Now we choose $N = N_g$ as follows:

- (*)₄ we let
 - (a),(b) as above,
 - (c) for every $\bar{u} \in \mathbf{U}$ for some $\alpha(\bar{u}) = \alpha_{\bar{u},v} \in P^N$ for every $\varepsilon < \theta$ we have:
 - $F^N(-, d)$ maps $\cup_{\varepsilon < \theta} u_\varepsilon$ into itself
 - for $\varepsilon < \theta$ we have: $\varepsilon \in v$ iff there is $a \in u_{\varepsilon+1} \setminus u_\varepsilon$ such that $F_{\varepsilon+1}^N(a, d) \notin u_{\varepsilon+1}$.

The rest is as in the proof of part (1) of 2.7. □_{2.17}

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EINSTEIN INSTITUTE OF MATHEMATICS, EDMOND J. SAFRA CAMPUS, GIVAT RAM, THE HEBREW UNIVERSITY OF JERUSALEM, JERUSALEM, 9190401, ISRAEL, AND, DEPARTMENT OF MATHEMATICS, HILL CENTER - BUSCH CAMPUS, RUTGERS, THE STATE UNIVERSITY OF NEW JERSEY, 110 FRELINGHUYSEN ROAD, PISCATAWAY, NJ 08854-8019 USA

Email address: shelah@math.huji.ac.il

URL: <http://shelah.logic.at>