Appendix: On stationary set

We represent the relevant facts from [Sh 6] (hopefully in a better way) and add slightly. This was written essentially by accident.

1. Definition: 1) For λ regular, a set $S \subseteq \lambda$ is called *good* if there is a sequence $\bar{a} = \langle a_i : i < \lambda \rangle$, a_i a subset of λ, such that for some closed unbounded $C \subseteq \lambda$:

$C \cap S \subseteq S^{*q}_{\lambda}[\bar{a}] = \{\gamma : (\exists a \subseteq \gamma)[\gamma = \sup a \wedge otp(a) < \gamma \wedge (\forall \alpha < \gamma)(\exists i < \gamma) a \cap \alpha = a_i] \ or \ \gamma = cf\,\gamma\}$

We say $\langle a_i : i < \lambda \rangle$ witness the goodness of S, and C exemplify this (p stands for positive, q for a variant n for negative.)

2) $I[\lambda]$ is the family of good subsets of α.

2. Lemma: 1) We can in 1.1 replace a_i by $\mathcal{P}_i$, $|\mathcal{P}_i| < \lambda$, $\mathcal{P}_i \subseteq \{a : a \subseteq \lambda$ is bounded, and "$a \cap \alpha = a_i$" by "$a \cap \alpha \in \mathcal{P}_i$" (and get an equivalent definition). [see 4) and 5) below]

2) we can demand in 1(1) that a has order type cf (γ) and $a_i \subseteq i$.

I.e. if for λ, $\bar{a}$ as in Definition 1(1) we let $S^{*p}_{\lambda}[\bar{a}] = \{\gamma < \lambda$: there is $a \subseteq \gamma$ of order type $cf\,\gamma$ such that $otp(a) = cf(\gamma)$, $\sup a = \gamma$ and $(\forall \alpha < \gamma)(\exists i < \gamma)[a \cap \alpha = a_i]\}$ we can use $S^{*q}_{\lambda}[\bar{a}]$ instead of $S^{*p}_{\lambda}[\bar{a}]$ in defining "a good set" (and hence $I[\lambda]$).

3) if $\langle a_i : i < \lambda \rangle$ witness the goodness of $S \subseteq \lambda$ and $\{a_i : i < \lambda\} \subseteq \{b_i : i < \lambda\} \subseteq \mathcal{P}(\lambda)$ *then* $\langle b_i : i < \lambda \rangle$ witness the goodness of S. In fact $S^{*p}_{\lambda}(\langle b_i : i < \lambda \rangle) \subseteq S^{*p}_{\lambda}(\langle a_i : i < \lambda \rangle)$ *mod* D_{λ}.

4) $\langle a_i : i < \lambda \rangle$ witness that $S \subseteq \lambda$ is good iff $\langle \{a_i\} : i < \lambda \rangle$ witness that S is good.

5) If $\overline{\mathcal{P}}^{\ell} = \langle \mathcal{P}^{\ell}_i : i < \lambda \rangle$ are as in 2(1) for $\ell = 1,2$ and $\bigcup_i \mathcal{P}^1_i \subseteq \bigcup_i \mathcal{P}^2_i$ and $\overline{\mathcal{P}}^1$ witness that $S \subseteq \lambda$ is good *then* also $\overline{\mathcal{P}}^2$ witnesses it.

6) For λ uncountable regular, $\{\delta < \lambda : \delta$ a (weakly) inaccessible cardinal$\}$ belongs to $I[\lambda]$.

Proof: Trivial, e.g.

2) Let $\langle a_i : i < \lambda \rangle$ witness $S \subseteq \lambda$ is good.

For every limit $\delta < \lambda$ choose a closed unbounded subset C_δ of δ of order type $cf\,\delta$; let for $i < \lambda$, $\delta < \lambda$, $a_{i,\delta} = \{j \in a_i :$ the order type of $a_i \cap j$ belongs to $C_\delta\}$.

Let $\{a_{i,\delta} : i < \lambda, \delta < \lambda\} \cup \{\{i : i < \alpha\} : \alpha < \lambda\} = \{b_i : i < \delta\}$, let C exemplify $\langle a_i : i < \lambda \rangle$ witness the goodness of S.

Let $C_0 = \{\alpha \in C :$ for every $i < \alpha$ and limit $\delta < \alpha$ there is $\zeta < \delta$ such that $b_\zeta = a_{i,\delta}$ (if defined and α is a limit ordinal)$\}$.

Clearly $C_0 \subseteq C$ is closed unbounded in λ. Now for any $\gamma \in C_0 \cap S$ we know there is a set $a \subseteq \gamma$ such that $\sup(a) = \gamma$, $otp(a) < \gamma$, $\alpha \cap a = a_{i(\alpha)}$ for $\alpha \in a$ and $i(\alpha)$ is an ordinal $< \gamma$. Let $a^* = \{i \in a : otp(a \cap \alpha) \in C_{otp(a)}\}$. Now a^* is as required.

3. Lemma: 1) $I[\lambda]$ is a normal ideal, which include all non-stationary subsets of λ.

2) If $\lambda = \lambda^{<\lambda}$, *then* for some $S^{*n}_\lambda \subseteq \lambda$:

$$I[\lambda] = \{S \subseteq \lambda : S \cap S_\lambda \text{ is not stationary}\} = \{S \subseteq \lambda : \langle a_i : i < \lambda \rangle \text{ witness } S \text{ is good}\}$$

for any $\langle a_i : i < \lambda \rangle$ enumerating $\{a \subseteq \lambda : |a| < \lambda\}$.

3) Always there is $S^{*n}_\lambda \subseteq T_\lambda \stackrel{def}{=} \{\delta < \lambda : \lambda^{<cf\,\delta} = \lambda\}$ such that $S \in I[\lambda] \wedge S \subseteq T_\lambda \Leftrightarrow S \subseteq \lambda \wedge S \cap S^{*n}_\lambda$ not stationary.

Proof: Easy.

4. Lemma: 1) If λ is regular, $\kappa < \lambda$, $(\forall \alpha < \lambda) |\alpha|^{<\kappa} < \lambda$ (e.g., $\lambda = \mu^+$, $\mu = \mu^{<\kappa}$) *then* $\{\delta < \lambda : cf(\delta) \leq \kappa\} \in I[\lambda]$

2) Suppose $\lambda = \mu^+$, $cf(\mu) < \kappa < \mu$ and $(\forall \theta < \kappa)(\forall \chi < \mu)[\chi^\theta < \mu]$. Then there is $S \in I[\lambda]$ such that:

(*) *if* $\delta < \lambda, \theta < \kappa$, and $cf\,\delta = (2^\theta)^+$ or even just $(\forall \alpha < cf(\delta))\ [|\alpha|^\theta < cf(\delta)]$ *then* for some closed unbounded $C_\delta \subseteq \delta$, $(\forall \alpha)[\alpha \in C_\delta \wedge cf(\alpha) \leq \theta \rightarrow \alpha \in S]$.

3) For λ,μ,κ as in (2), there is a 2-place function c from λ to $cf\,\mu$ such that:

(a) for $\alpha < \beta < \gamma$, $c(\alpha,\gamma) \le Max\{c(\alpha,\beta), c(\beta,\gamma)\}$.

(b) $|\{\alpha < \beta : c(\alpha,\beta) = \gamma\}| < \mu$.

(c) $S^{*p}_{\lambda}[c] \stackrel{def}{=} \{\delta < \lambda$: δ has cofinality $\le \kappa$ and there is an unbounded $a \subseteq \delta$ such that $c \restriction a$ is bounded in $cf\,\mu$ (i.e. $(\exists \gamma < cf\,\mu)(\forall \alpha,\beta \in a)\,[\alpha < \beta \to c(\alpha,\beta) < \gamma]\}$ belongs to $I[\lambda]$.

Proof: Note that 4(1), is easy, and 4(2) follows from 8(1), 4(3). It is easy to satisfies (a), (b) of (4) and (c) follows [choose an increasing sequence $\langle \mu_i : i < cf\,\mu \rangle$ such that $\mu = \Sigma\{\mu_i : i < cf\,\mu\}$, and then define by induction on β, $\langle c(\alpha,\beta) : \alpha < \beta \rangle$such that (a) holds and

(b)$^+$ $\quad |\{\alpha < \beta : c(\alpha,\beta) = \gamma\}| = \mu_\gamma$.

Why (c) follows from (a) + (b)? Clearly for $\alpha < \lambda$, $i < cf(\mu)$, $\mathcal{P}_{\alpha,i} = \{a : a$ as a subset of $\{\beta : \beta < \alpha$ and $c(\beta,\alpha) < i\}$ of cardinality $< \kappa\}$ has cardinality $\le \mu$, so $\mathcal{P}_\alpha = \bigcup_{i<cf\,\mu} \mathcal{P}_{\alpha,i}$ has cardinality $\le \mu$. Now $S^{*p}_{\lambda}[\langle \mathcal{P}_\alpha : \alpha < \lambda \rangle]$ is a subset of $S^{*p}_{\lambda}[c]$.

There are no problems].

5. Remark: 1) In 4(2), 4(3) we can replace $\lambda = \mu^+$ by $\lambda = \mu^{+\alpha}$, as α increases we get less information. See [Sh 6] xx.

2) In (3) really (a) + (b) implies (c) and note (7) below.

6. Definition: 1) A two place function c from an ordinal ζ to an ordinal ξ is called subadditive if:

for $\alpha < \beta < \gamma < \zeta \quad c(\alpha,\gamma) \le Max\{c(\alpha,\beta), c(\beta,\gamma)\}$

and $c(\alpha,\beta) = c(\beta,\alpha)$, $c(\alpha,\alpha) = 0$

2)$\lambda \to_p (S)^2_\theta$ mean: (for λ,θ regular cardinals, $S \subseteq \lambda$.)

Suppose

(*) c is a two place function from λ to θ, c subadditive.

Then for some closed unbounded $C \subseteq \lambda$, for every $\delta \in S \cap C$ of cofinality $> \theta$,

$(**)_\delta$ there is $A \subseteq \delta$, such that $\sup A = \delta$, and $\sup\{c(\alpha,\beta) : \alpha < \beta, \alpha \in A, \beta \in A\} < \theta$.

3) We say λ is θ-sawc (sub-additively weakly compact) *if:* for every subadditive two place function d from λ to θ, *there is* an unbounded subset A of λ such that $\sup\{c(\alpha,\beta) : \alpha < \beta,\ \alpha \in A,\ \beta \in A\} < \theta$.

7. Fact: In Definition 6(2) the following demand on $\delta \in S \cap C$ is equivalent to $(**)_\delta$ when $cf(\delta) > \theta$:

$(**)'_\delta$ there are $\alpha_i, \beta_i < \delta$ for $i < cf(\delta)$, $\delta = \bigcup_i \alpha_i = \bigcup_i \beta_i$ and $\sup\{c(\alpha_i,\beta_j) : i < j < cf(\delta)\} < \theta$.

Proof: If A is as in $(**)_\delta$ choose $\alpha_i, \beta_i \in A$ s.t. $\delta = \bigcup\{\alpha_i : i < cf\,\delta\}$, $\sup\{\beta_j : j < i\} < \alpha_i < \beta_i$, they are as required.

If $\alpha_i, \beta_i (i < cf(\delta))$ are as in $(**)'_\delta$, w.l.o.g. $[j < i \Rightarrow \alpha_i < \beta_i < \alpha_j < \beta_j]$, so as $cf(\delta) > \theta$ for some $\gamma_1 < \theta$

$$B \overset{def}{=} \{i : c(\beta_i, \alpha_{i+1}) = \gamma_1\}$$

is unbounded below $cf(\delta)$. Let

$$\gamma_0 = \sup\{c(\alpha_i,\beta_j) : i < j < cf(\delta)\} < \theta.$$

Now $A = \{\beta_i : i \in B\}$ is as required: for $j < i$ in B

$$c(\beta_j,\beta_i) \le Max\{c(\beta_j,\alpha_{j+1}), c(\alpha_{j+1},\beta_i)\} \le$$

$$Max\ \{\gamma_1,\gamma_0\}$$

8. Lemma: 1) Suppose λ, μ, κ are as in 4(2) (so 4(3)) and $\lambda \to_p (S)^2_{cf(\mu)}$, $S \subseteq \{\delta < \lambda : cf\,\delta < \kappa\}$ *then* $S \in I[\lambda]$.

2) If $(\forall \sigma)[\sigma^+ < \mu \to 2^\sigma < \lambda]$, $S \subseteq \{\delta < \lambda : cf\,\delta < \mu\}$, $S \in I[\lambda]$ and λ, θ are regular *then*

$\lambda \to_P (S)^2_\theta$.

3) Suppose λ, μ, κ are as in 4(2), c as in 4(3) (a),(b). Then for any $\delta < \lambda$ and $S \subseteq cf\,\delta$, there is an increasing continuous function $h : cf(\delta) \to \delta$, $\delta = \sup\{h(i) : i < cf(\delta)\}$, and a club $c \subseteq \delta$ such that

$$[cf\,\delta \to_P (S)^2_{cf\,\mu} \Rightarrow c \cap h''(S) \subseteq S^{*P}_\chi[c]]$$

9. Remark: Particularly assuming G.C.H. 4(3), 8(1), 8(2), 8(3) fits nicely.

Proof of 8: 1) Let c be a two place function satisfying 4(3) (a) + (b). By Definition 6, there is a closed unbounded $C \subseteq \lambda$ such that for $\delta \in C \cap S$ of cofinality $> cf(\mu)$, $(**)_\delta$ hold. Now $\{\delta < \lambda : cf\,\delta \leq cf(\mu)\} \subseteq T_\lambda$ [by 4(2) as $\mu^{<cf\,\mu} \leq \sum_{\substack{\chi<\mu \\ \theta<cf\,\mu}} \chi^\theta \leq \sum_{\chi<\mu} \chi^{<\kappa}$ hence $\lambda^{<cf\,\mu} = (\mu^+)^{<cf\,\mu} = \mu^+ = \lambda$] so we can assume $cf(\delta) > cf(\mu)$. Now $(**)_\delta$ implies $\delta \in S^{*P}_\chi[c]$ (see 4(3)(c)), so by 4(3) we finish.

2) Let c be a two place function from λ to θ, subadditive. Let χ be regular large enough, and w.l.o.g. let $\langle a_i : i < \lambda\rangle$ exemplify $S \in I[\lambda]$ witnessed by C_0 with $otp(a_i) < \mu$ (see 2(2) above). Let $\langle N_i : i < \lambda\rangle$ be increasing continuous such that $N_i \prec (H(\chi),\in)$, $\langle a_i : i < \lambda\rangle \in N_0$, $N_i \cap \lambda$ is an ordinal, $\|N_i\| < \lambda$, and $\langle N_j : j \leq i\rangle \in N_{i+1}$. Let $C = \{i < \lambda : N_i \cap \lambda = i$ and $i \in C_0\}$ (it is closed unbounded). Suppose $\delta \in C \cap S$, $cf(\delta) > \theta$, then there is $a \subseteq \delta = \sup \delta$ such that $(\forall \alpha \in a)[a \cap \alpha \in \{a_j : j < \delta\}]$ hence $(\forall \alpha \in a)$ $[a \cap \alpha \in N_\delta]$, and we also know $otp(a) = cf\,(\delta)$; and let $\{\alpha_i : i < cf(\delta)\}$ be an increasing enumeration of a. So there are $\alpha_i < \delta$, $[i < j \Rightarrow \alpha_i < \alpha_j]$, $\delta = \bigcup\{\alpha_i : i < cf\,\delta\}$ and for $i < cf(\delta)$, $\{\alpha_j : j < i\} \in N_{\beta_i}$ for some $\beta_i < \delta$. As for $i < cf\,\delta$, $|\{\alpha_j : j < i\}| < cf\,\delta < \mu$, so $2^{|\{\alpha_j\ j<i\}|} < \lambda$ hence $\{\zeta : \zeta < 2^{|\{\alpha_j : j<i\}|}\} \subseteq N_\delta$, so every subset of $\{\alpha_j : j < i\}$ belongs to $N_{\beta_{i+1}} \prec N_\delta$. As $cf(\delta) > \theta$ for some $\gamma < \theta$, $A = \{i < cf(\delta)$: $c(\alpha_i, \delta) < \gamma\}$ is unbounded below $cf(\delta)$, so by the previous sentence w.l.o.g. $A = cf(\delta)$. So $N_{\delta+1} \models (\exists x)(\forall y \in \{\alpha_j : j < i\})$ $[c(y, x) \leq \gamma \wedge \alpha_i < x]$ (as δ witness the $\exists x$) so there is such x in N_{β_i+1} call is β_i. So α_i, β_i are as required in $(**)'_\delta$ of Fact 7, so by 7 we finish.

3) Follows.

10. Lemma: 1) If $S \in I[\lambda]$ is stationary, and $(\forall \delta \in S)[cf(\delta) < \mu]$, and P is a μ-complete forcing notion ($\mu > \aleph_0$) then "$\Vdash_P$"S is a stationary subset of the ordinal λ"

2) If $S \subsetneq \{\delta < \lambda : cf(\delta) < \mu\}$ is stationary but included in S^{*n}_{λ} (see 3(3)), μ regular and $\lambda = \lambda^{<\mu}$ *then* for some μ-complete forcing notion P, $\Vdash_P$"S is not stationary" (in fact $P =$ Levi (μ, λ) is O.K.)

Remark: As for 10(2), it repeats Theorem 21, p. 366 of [Sh 6], Donder and Ben David note a defect: in the case $\lambda = \lambda^{<\lambda}$ (really $\lambda = \lambda^{<\mu}$) in the definition of the forcing P $(= \{\langle \alpha_i : i \leq \zeta \rangle$: α_i increasing continuous $B_{\alpha_{i+1}} = \{\alpha_j : j \leq i\}\}$) we forget to demand $\zeta < \mu$. [Note however that automatically $\zeta \leq \mu$ as each B_i has cardinality $< \mu$, so we should just omit the maximal elements of P which make P totally trivial].

For the general case $(\lambda < \lambda^{<\mu})$ note that if some weak form of it fails, our definition of the set S^{*n}_{λ} make it empty. I.e. by Definition 8, p. 36 of [Sh 6], S^{*n}_{λ} make it empty. I.e. by Definition 2(1),2(1), p. 359 of [Sh 6] relaying on Definition 1, p. 358 of [Sh 6]. This demand "$S^{*n}_{\lambda} \subseteq gcf(x)$" is reasonable, as otherwise we cannot prove there is such a set. See here later. [18,19]

Proof: 1) Use $(\forall s)(s \in I[\lambda] \Rightarrow s \in I^+[\lambda])$ from 16(2) (see Definition 15)

2) Let $\langle a_i : i < \lambda \rangle$ list the subsets of λ of cardinality $< \mu$, each appearing λ times. If $P =$ Levi (μ, λ), in V^P λ has cofinality μ, so let $\langle \alpha_i : i < \mu \rangle$ be increasing, $\alpha_i < \lambda$, $\bigcup_{i<\kappa} \alpha_i = \lambda$. But forcing with P add no sequences of ordinals of length $<\mu$, so we can find inductively $j(i) < \lambda$, $j(i) > \bigcup \{j(\xi), \alpha_\xi : \xi < i\}$, $a_{j(i)} = \{\alpha_\xi : \xi < i\}$. Now $\{\delta < \lambda$: the set $\{j(i) : i < \mu\} \cap \delta$ is unbounded in $\delta\}$ is a club of λ in V^P, included in a good subset of λ from V.

10A Remark: It is natural to force with $Q(\langle a_i : i < \lambda \rangle) = \{\langle i_\zeta : \zeta < \xi^* \rangle : \zeta^* < \kappa, i_\zeta < \lambda,$ $[\zeta(1) < \zeta(2) \Rightarrow i_{\zeta(1)} < i_{\zeta(2)}]$, and $a_{i_\zeta} = \langle i_\xi : \xi < \zeta \}$.

* * *

In [Sh 6] we define S^{*n}_{λ} inside a larger set than $\{\delta < \lambda : \lambda^{<cf\,\delta} = \lambda\}$ (see 3(3)). We will present this addition, improved, i.e. $Gcf\,[\lambda]$, $gcf(\lambda)$ are bigger sets here than in [Sh 6, Definition 2].

11 Definition: 1) For a family F of subsets of θ let

$$tr(F) = \{A \cap \alpha : A \in F, \alpha < \theta\}$$

2) For θ regular uncountable let $club_{tr}(\theta) = Min\{|tr(F)| : F$ is a family of closed unbounded subsets of θ such that: every closed unbounded subset of θ contains some members of $F\}$.

Let $club_{tr}[\aleph_0] = \aleph_0$ and let F_θ exemplify $club_{tr}(\theta) = |F_\theta|$.

3) $Gcf[\lambda] = \{\theta : \theta$ is regular $\geq \aleph_0$ and, $\lambda = \lambda^{<\theta}$ or $club_{tr}(\theta) < \lambda\}$

4) $gcf[\lambda] = \{\delta < \kappa : cf\, \delta \in gcf[\lambda],\ cf(\delta) < \delta\} \cup \{\delta < \lambda : \delta$ a (weakly) inaccessible cardinal$\}$ $\cup \{\alpha < \lambda : \alpha = 0$,, or α successor ordinal$\}$

4) $gcf_{ac}[\lambda] = \{\delta \in gcf[\lambda] : cf\, \delta < \delta\}$

12 Fact: 1) If GCH, $\lambda > \aleph_0$ regular *then* $Gcf[\lambda] = \{\theta : \theta$ regular $< \lambda\}$, $gcf[\lambda] = \lambda$.

2) For regular uncountable θ, $\theta < club_{tr}(\theta) \leq 2^{<\theta} \leq 2^{\theta}$.

3) If $2^{<\theta} \leq \lambda$, (θ, λ regular) *then* $\theta \in Gcf(\lambda)$ [as this implies either $\lambda = 2^{<\theta}$ hence $\lambda = \lambda^{<\theta}$ or $\lambda > 2^{<\theta}$ hence $\lambda > club_{tr}(\theta)$].

13 Definition: 1) We call $\bar{a}$ an *enumeration* for λ if $\bar{a} = \langle a_i : i < \lambda \rangle$, each a_i a bounded subset of λ.

2) We call $\bar{a}$ a *rich enumeration* for λ if:

(i) $\bar{a}$ is an enumeration for λ

(ii) if $\lambda = \lambda^{\theta}$, (hence $\theta < \lambda$) *then* every subset of λ of cardinality $\leq \theta$ appears in a

(iii) if θ is an uncountable regular cardinal, and $club_{tr}(\theta) \leq \lambda$ *then* letting F_θ exemplify $club_{tr}(\theta) \leq \lambda$, for every limit ordinal $\delta < \lambda$ of cofinality θ, there is a closed unbounded subset $\{\beta_i^\delta : i < \theta\rangle$ of δ (β_i^δ increasing continuous) such that

(*) for every $A \in F_\theta$ and $\zeta < \theta$, $\{\beta_i^\delta : i \in A \cap \zeta\}$ appear in $\bar{a}$.

3) In (1) (and (2)) we replace "enumeration" by ($<\mu$)-enumeration if we restrict ourselves to subsets of λ of cardinality $<\mu$ i.e. in (ii) $\theta \leq \mu$, in (iii) $\theta \leq \mu$.

4) For an unbounded subset S of λ, we say $\bar{a}$ is a rich enumeration for $(S, \theta)^{\ell}$ *if:*

(i) $\bar{a}$ is an enumeration for λ

(ii) if $\ell = 1$, $\lambda = \lambda^{<\theta}$ and every $b \subseteq \lambda$, $|b| < \theta$ appear in $\bar{a}$

(iii) if $\ell = 2$ $club_{tr}(\theta) \leq \lambda$, then for every $\delta \in S$ of cofinality θ the condition in (2) (iii) above holds.

14 Fact: 1) For every regular uncountable λ there is a rich enumeration;

2) For every $\lambda = cf\,\lambda > \mu$, λ has a rich μ-enumeration.

15 Definition: For λ regular uncountable

$I^+[\lambda] = \{S \subseteq \lambda$: for every cardinal $\chi > \lambda$ and $x \in H(\chi)$, for some closed $C \subseteq \lambda$, for every $\delta \in C \cap S$ there are a limit $\gamma < \delta$, and $N_i \prec (H(\chi), \in, x, \lambda)$, for $i < \gamma$, such that $\langle N_j : j \leq i \rangle \in N_i$, $N_i \cap \lambda$ is an ordinal $\alpha_i < \delta$ and $\delta = \bigcup_{i<\gamma} \alpha_i\}$.

16 Fact: 1) $I^+[\lambda]$ is a normal ideal on λ and in its definition w.l.o.g. $\gamma = cf\,\delta$,

2) $I[\lambda] \subseteq I^+[\lambda]$

3) If $S \subseteq gcf[\lambda]$ then: $S \in I[\lambda] \Leftrightarrow S \in I^+[\lambda]$

4) There is $S^{*n}_{\lambda} \subseteq gcf[\lambda]$, such that for every rich enumeration $\bar{a}$ for λ and $S \subseteq Gcf[\lambda]$: $S \in I[\lambda]$ *if and only if* $S \in I^+[\lambda]$ *if and only if* $S \cap S^{*n}_{\lambda} = \emptyset \mod D_{\lambda}$ *if and only if* $S \subseteq S^{*p}_{\lambda}[\bar{a}] \mod D_{\lambda}$. We let $S^{*n}_{\lambda,\theta} = \{\delta < \lambda : cf(\delta) = \theta,\ \delta \in S^{*}_{\lambda}\}$ (this replace 3(3)) and

$$S^{*n}_{\lambda,<\theta} \overset{def}{=} \{\delta < \lambda : cf\,\delta < \theta,\ \delta \in S^{*n}_{\lambda}\}$$

5) for every rich enumeration $\bar{a}$ for λ, $gcd[\lambda] - S^{*n}_{\lambda}[\bar{a}] \equiv S^{*q}_{\lambda} \mod D_{\lambda}$.

6) for any $\theta < \lambda$, suppose (A) $\lambda = \lambda^{<\theta}$, $\{b \subseteq \lambda : |b| < \theta\} \subseteq \{a_i : i < \lambda\}$ (like 11(2)(ii) *or* (B) $club_{tr}(\theta) < \lambda$, F_{θ} exemplify it and $\bar{a}$ satisfies 11(2)(iii) for every $\delta \in S \subseteq \{\delta < \lambda : cf\,\delta > cf\,\theta\}$. *Then* $S \cap S^{*n}_{\lambda,\theta} = S - S^{*p}_{\lambda}[\bar{a}]$.

Proof: 1) The normality is easy, the "w.l.o.g. $\gamma = cf\ \delta$" is proved as in 2(2).

2) Let $S \in I[\lambda]$, so for some enumeration $\bar{a} = \langle a_i : i < \lambda \rangle$ for λ, $C \cap S \subseteq S^{*q}_{\lambda}[\bar{a}]$ for some closed unbounded $C \subseteq \lambda$. Let $\chi > \lambda$, $x \in H(\chi)$. We can find $\langle N_\zeta : \zeta < \lambda \rangle$ increasing continuous, $N_\zeta \prec (H(\chi), \in, x)$, such that $N_\zeta \cap \lambda$ is an ordinal $\|N_\zeta\| < \lambda$, $\langle N_j : j \leq \zeta \rangle \in N_{\zeta+1}$ and C, $\bar{a} \in N_0$. So $C' \stackrel{def}{=} \{\delta < \lambda : \delta \in C \text{ and } N_\delta \cap \lambda = \delta\}$ is a closed unbounded subset of λ.

Now for every $\delta \in C' \cap S$, there is a_i from $\bar{a}$, $otp(a_i) < \sup(a_i) = \delta$ and for $\alpha \in a_i$, $\alpha \cap a_i \in \{a_j : j < \delta\}$. As $\bar{a} \in N_0$ clearly $\{a_j : i < \delta\} \subseteq \bigcup_{\zeta<\delta} N_\zeta$. Let $a_i = \{\gamma_\varepsilon : \varepsilon < otp\ a_i\}$. Now we try to define by induction on $\varepsilon < otp(a_i)$ an ordinal $\zeta_\varepsilon < \delta$:

for $\varepsilon = 0$: $\zeta_\varepsilon = 0$

for ε *limit:* $\zeta_\varepsilon = \bigcup_{\beta<\varepsilon} \zeta_\beta$,

for ε *successor:* ζ_ε is the first ordinal ζ satisfying ζ is bigger than γ_ε and $\langle N_{\zeta_\beta} : \beta < \varepsilon \rangle$ belongs to N_{ζ_ε}.

The only reason for stopping is: ε limit $\bigcup_{\beta<\varepsilon} \zeta_\beta = \delta$; once this occurs at ε_0, $\langle N_{\zeta_\varepsilon} : \varepsilon < \varepsilon_0 \rangle$ is as required [otherwise for limit and for zero there is no problem, and for ε successor, $\zeta_{\varepsilon-1}$ is defined and $< \delta$, so for some β, $\zeta_{\varepsilon-1} < \gamma_\beta < \delta$ [where $a_i = \{\gamma_\beta : \beta < otp\ a_i\}$) now $\langle \zeta_\beta : \beta < \varepsilon \rangle$ is definable inside the model $(H(\chi), \in)$ from the parameters $\langle N_j : j < \gamma_\beta \rangle$, $\langle \gamma_j : j < \beta \rangle$ only; as both are in $\bigcup_{j<\delta} N_j$, is $\langle \zeta_\beta : \beta < \alpha \rangle$, and similarly so is ζ_ε].

3) Fix $S \subseteq gcf[\lambda]$; by 16(2) it is enough to assume $S \in I^+[\lambda]$ and prove $S \in I[\lambda]$, we prove more in 16A below.

4) S^{*n}_{λ} is $gcf[\lambda] - S^{*p}_{\lambda}[\bar{a}]$ for any rich enumeration $\bar{a}$ for λ.

5), 6) Should be clear.

16A Subfact: If $S \subseteq gcf[\lambda]$, (λ regular uncountable) S belongs to $I^+[\lambda]$ and $\bar{a} = \langle a_i : i < \lambda \rangle$ is a rich enumeration for λ, *then* $S \subseteq S^{*p}_{\lambda}[\bar{a}]$ *mod* D_λ.

Proof of 16A: Let $x = \bar{a}$, $\chi = (2^\lambda)^+$, so as $S \in I^+[\lambda]$ (see Definition 15), there is a closed unbounded $C \subseteq \lambda$ such that (see 16(1)):

(*) for every $\delta \in C \cap \delta$ there is $\bar{N} = \langle N_i : i < cf(\delta) \rangle$ as in Definition 15.

Fix $\delta \in C \cap S$, and $\bar{N}$ and let $\theta = cf\,\delta$, $\alpha_i = N_i \cap \lambda$. Remember that $N_i \cap \{a_j : j < \lambda\} = \{a_j : j < \alpha_i\}$. We shall show that $\delta \in S^{*p}_{\chi}[\bar{a}]$, thus finishing.

Case 1: $\lambda^{<\theta} = \lambda$ (e.g. $cf\,\delta \leq \aleph_0$).

In this case for each $i(*) < \theta$, $\{\alpha_i : i < i(*)\}$ belongs to $\{a_j : j < \lambda\}$ (as $\bar{a}$ is rich) and to $N_{i(*)+1}$ (as $\langle N_i : i < i(*) \rangle \in N_{i(*)+1}$, and $\lambda \in N_{i(*)+1}$); hence $\{\alpha_i : i < i(*)\}$ belongs to their intersection which is $\{a_j : j < \alpha_i\}$. So $\langle a_i : i < i(*) \rangle$ exemplify $\delta \in S^{*p}_{\chi}(\bar{a})$, as required.

Case 2: $cf\,\delta < \delta$, $club_{tr}(\theta) < \lambda$ where $\theta = cf\,\delta > \aleph_0$).

Let $F_{cf\,\delta}$ exemplify $club_{tr}(\theta) = |tr(F_\theta)|$, and let $\{\beta^\delta_i : i < \theta\}$ be as in Definition 13(1) (iii). So $A_0 = \{i < \theta : \beta^\delta_i = \alpha_i\}$ is a club of θ, hence for some club $A \in F_\theta$, $A \subseteq A_0$. By 13(1) (iii) for every $i(*) < \theta$, $\{\beta^\delta_i : i \in A,\ i < i(*)\}$ belongs to $\{a_i : i < \lambda\}$, but $A \cap i(*) \in \bigcup_{i<\theta} N_i$ [as $\theta < \delta$, hence w.l.o.g. $F_\theta \in \bigcup_{i<\theta} N_i$ hence $tr(F_\theta) \in \bigcup_{i<\theta} N_i$, but $|tr(F_\theta)| < \lambda$ hence $tr(F_\theta) \subseteq \bigcup_{i<\theta} N_i$]. Hence $\{\alpha_i : i \in A \cap i(*)\} \in \bigcup_{i<\theta} N_i$, so we finish.

Case 3: $\delta = cf\,\delta$.

Trivial.

17 Lemma: Suppose in V, $\lambda > \aleph_0$ is regular, $\theta \in Gcf[\lambda]$, so $S^{*n}_{\lambda,\theta}$ is defined.

Suppose further V^1 is an extension of the universe V (say same ordinals), $V^1 \models$ "$\lambda > \aleph_0$ is regular", and

$(*)_1$ $V^1 \models$ "every subset of λ of cardinality $< \theta$ belongs to V", $V \models$ "$\lambda = \lambda^{<\theta}$", or

$(*)_2$ $V^1 \models$ "F^V_θ satisfies for every club C of θ, there is $A \in F^V_\theta$, $A \subseteq C$" (but maybe $V^1 \models$ "θ not a cardinal") and $V \models$ "$|F^V_\theta| = club_{tr}(\theta) < \lambda$"

Then

(i) $V^1 \models$ "$cf\,\theta \in Gcf[\lambda]$ or $cf\,\theta = \aleph_0$"; and

(ii) $V^1 \models$ "$S^{*n}_{\lambda,cf\theta} \cap \{\delta : V \models cf\,\delta = \theta\} \equiv (S^{*n}_{\lambda,\theta})^V \bmod D_\lambda$" or equivalently: in V^1, $(S^{*n}_{\lambda,\theta})^V/D_\lambda$ is disjoint to every S/D_λ, $S \in I[\lambda]$.

Proof: Let $\bar{a}$ be a rich enumeration for λ in V.

By (*), $\bar{a}$ is still a rich enumeration in V^1 for $S = \{\delta < \lambda : V \models cf\,\delta = \theta\}$. By 16(6) we finish.

18 Lemma: If $\lambda > \aleph_0$ is regular, $S \in I^+[\lambda]$, $S \subseteq \{\delta < \lambda : cf\,\delta < \mu\}$, P is a μ-complete forcing notion *then*

$\Vdash_P$ "S is a stationary subset of λ (as an ordinal, λ may or may not be a cardinal)"

Complementary to 18 is

19 Lemma: Suppose $\theta \in Gcf(\lambda)$, $\aleph_0 < \theta \le \mu = cf\,\mu < \lambda$ so $S^{*n}_{\lambda,\theta}$ is well defined.

1) If $\mu = \theta$, $\lambda = \lambda^{<\theta}$, $\Vdash_{Levi(\mu,\lambda)}$ "$(S^{*n}_{\lambda,\theta})^V$ is not stationary (as a subset of the ordinal λ, (remember $Levi(\theta,\lambda) = \{f : f$ a function from some $\alpha < \theta$ to $\lambda\}$, it is θ-complete).

2) If $S^{*n}_{\mu,\theta} = \emptyset$, $\lambda = \lambda^{<\theta}$, $\Vdash_{Levi(\mu,\lambda)}$ "$(S^{*n}_{\lambda,\theta})^V$ is not stationary".

3) In (1) and (2) we can replace Levi (θ,λ), by any forcing notion P which adds to λ no new subset of power $< \mu$ and $\Vdash_P$ "$cf\,\lambda = \mu$".

4) In (1),(2) we can replace "$\lambda = \lambda^{<\theta}$" by $club_{tr}(\theta) < \lambda$, if we replace Levi (μ,λ) by $Levi(\lambda,\lambda^{<\theta}) * \underset{\sim}{Levi}(\mu,\lambda)$.

Remark: A more general forcing is as follows: Let $\theta < \lambda$, $\kappa \le \theta$, $\bar{b} = \langle b_i : i < \lambda \rangle$ exemplify that $S_\theta \in I[\theta]$ and $[\delta < \theta \wedge \delta \in S_\theta \Rightarrow cf\,\delta < \kappa]$ or just for some $\sigma = cf\,\sigma < \kappa$, $S_\theta = \{\delta < \theta : cf\,\delta = \sigma\}$, $S \subseteq \{\delta < \lambda : cf\,\delta = \sigma\}$ and $Q^{\bar{b},\theta} = \{\langle i_\zeta : \zeta < \zeta^* \rangle : \zeta^* < \theta,\ i_\zeta < \lambda,$

$[\zeta(1) < \zeta(2) \Rightarrow i_{\zeta(1)} < i_{\zeta(2)}]$, $a_{i_\zeta} = \{i_\xi : \xi \in b_\zeta\}\}$.

20 Claim: Suppose $\theta = cf\,\theta > \aleph_0$ for the regular cardinals λ and μ, $\lambda > \mu$ and $club_{tr}(\theta) < \mu$.

1) Given S^{*n}_λ, S^{*n}_μ, there is a club $C \subseteq \lambda$ such that: *for every* $\delta \in C$ of cofinality μ, *there is* an increasing continuous sequence $\langle \alpha_i : i < \mu \rangle$, $\bigcup_{i<\mu} \alpha_i = \delta$ and a club c of μ satisfying $[i \in c \wedge cf\, i = \theta \wedge i \notin S^{*n}_\mu \Rightarrow i \notin S^{*n}_\lambda]$.

2) If $\bar{a}$ is a rich enumeration for (λ,θ), then $\{\delta < \lambda : cf\,\delta = \mu$ implies that for some α_i, β_i $(i < \mu)$: $\langle \alpha_i : i < \mu \rangle$ is increasing continuous with limit δ, $\beta_i < \mu$ and defining for $i < \mu$, $b_i = \{j : a_j \in a_{\beta_i}\}$, $\langle b_i : i < \mu \rangle$ is a rich enumeration for $(\mu,\theta)\} \in D_\lambda$.

21. Lemma: 1) If κ is supercompact and e.g. $\lambda > \kappa > cf\,\lambda$, *then* $I[\lambda^+]$ is a proper ideal: $\lambda^+ \notin I[\lambda^+]$.

3) After suitable collapses, e.g. $cf\,\lambda = \aleph_0 < \lambda$ but still $\lambda^+ \notin I[\lambda^+]$.

22. Problem: 1) Is G.C.H. + $\{\delta < \aleph_{\omega+1} : cf\,\delta > \aleph_1\} \notin I[\aleph_{\omega+1}]$ consistent with ZFC.

2) Is

(*)$2^{\aleph_0} > \aleph_{\omega+1}$ + "there is no stationary $S \in I[\aleph_{\omega+1}]$"

consistent with ZFC?

3) Is

(*) $2^{\aleph_0} > \aleph_{\omega+1}$ + for no ultrafilter D on ω, $cf(\pi_n(\aleph_n, <)/D) = \aleph_{\omega+1}$

consistent with ZFC.

Remark: "$\aleph_{\omega+1}$ is a Jonson cardinal" implies (*) of (3) (see [Sh 9] which implies (*) of (2) (see [Sh]).

Having F cause slight inconvenience.

We define by induction on $\alpha < \lambda^+$, (M_α, N_α, a), and N^*_γ, N^*_γ, $M^*_\gamma, f_\gamma, g_\gamma$: for suitable γ's such that

(A) $M_\alpha, N_\alpha, M^*_\alpha, N^*_\alpha$ are isomorphic to M^*.

(B) $M_\beta (\beta \leq \alpha)$ is $\prec_K$- increasing continuous and similarly N_β, M^*_β, N^*_β.

(C) $F(M^*_{i+1}) = M_{i+2}$

(D) $F(N^*_{i+1}) = N^*_{i+2}$

(E) $(M_\beta, N_\beta, a) < (M_\alpha, N_\alpha, a)$ for $\beta < \alpha$.

(F) for γ limit or zero f_γ is an isomorphism from M^*_γ onto M_γ, g_γ is an isomorphism from N^*_γ onto N_γ.

(G) for γ limit or zero, $n > 0$: f_γ is an isomorphism from $N^*_{\gamma+n}$ onto $N_{\gamma+2n}$, g_γ is an isomorphism from $M^*_{\gamma+n}$ onto $M_{\gamma+2n-1}$.