LARGE STRONGLY ANTI-URYSOHN SPACES EXIST

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Dedicated to the memory of our old friend Ken Kunen

ABSTRACT. As defined in [3], a Hausdorff space is *strongly anti-Urysohn* (in short: SAU) if it has at least two non-isolated points and any two *infinite* closed subsets of it intersect. Our main result answers the two main questions of [3] by providing a ZFC construction of a locally countable SAU space of cardinality $2^{\mathfrak{c}}$. The construction hinges on the existence of $2^{\mathfrak{c}}$ weak P-points in ω^*, a very deep result of Ken Kunen.

It remains open if SAU spaces of cardinality $> 2^{\mathfrak{c}}$ could exist, while it was shown in [3] that $2^{2^{\mathfrak{c}}}$ is an upper bound. Also, we do not know if *crowded* SAU spaces, i.e. ones without any isolated points, exist in ZFC but we obtained the following consistency results concerning such spaces.

(1) It is consistent that $\mathfrak{c}$ is as large as you wish and there is a locally countable and crowded SAU space of cardinality $\mathfrak{c}^+$.
(2) It is consistent that both $\mathfrak{c}$ and $2^{\mathfrak{c}}$ are as large as you wish and there is a crowded SAU space of cardinality $2^{\mathfrak{c}}$.
(3) For any uncountable cardinal κ the following statements are equivalent:
  (i) $\kappa = cof([\kappa]^{\omega}, \subseteq)$.
  (ii) There is a locally countable and crowded SAU space of size κ in the generic extension obtained by adding κ Cohen reals.
  (iii) There is a locally countable and countably compact T_1-space of size κ in some CCC generic extension.

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1. Introduction

Anti-Urysohn (AU) and strongly anti-Urysohn (SAU) spaces were introduced and studied in [3]. An AU space is a Hausdorff space in which any two non-empty *regular* closed sets intersect and a SAU space is a Hausdorff space that has at least two non-isolated points and in which any two *infinite* closed sets intersect. Note that a non-singleton AU space has no isolated points, i.e. is crowded and a crowded SAU space is AU, this explains the terminology. Also, the requirement of having at least two non-isolated points is needed to exclude the trivial case of one-point compactifications of discrete spaces.

All relevant questions concerning AU spaces were settled in [3], in particular it was shown that for every infinite cardinal κ there is an AU space of cardinality κ, but only inconclusive partial results were proved for SAU spaces. For instance, we could only construct consistent examples of SAU spaces, moreover all of them were of size $\le \mathfrak{c}$, while only $2^{2^{\mathfrak{c}}}$ was established as an upper bound for their cardinality.

This, of course, naturally led to the following two questions raised in [3]:

(1) Is there a SAU space in ZFC?

(2) Can there be a SAU space of cardinality greater than $\mathfrak{c}$?

Our main result answers affirmatively both of these questions, namely we shall present a ZFC example of a locally countable SAU space of cardinality $2^{\mathfrak{c}}$. It is easy to see that $2^{\mathfrak{c}}$ is an upper bound for the sizes of locally countable SAU spaces, so this result is sharp. However, it remains an open question if $2^{\mathfrak{c}}$ is an upper bound for the sizes of all SAU spaces. It was proved in [3] that $2^{2^{\mathfrak{c}}}$ is such an upper bound.

It was proved in [3, Theorem 3.7] that $\mathfrak{r} = \mathfrak{c}$ implies the existence of a locally countable and *crowded* SAU space of size $\le \mathfrak{c}$, moreover several other forcing constructions yielded *crowded* SAU spaces. This then led to the question if the existence of a SAU space is equivalent to the existence of crowded ones.

By Theorem 2.7 this question can now be reformulated as follows: Is there a crowded SAU space in ZFC? Now, the method of construction of Theorem 2.7 yields a SAU space that is right-separated, i.e. scattered, hence it may not help to answer this question that we could not answer. However, we could get several partial results about it. We could prove the consistency of the existence of a locally countable and *crowded* SAU space of size $\le \mathfrak{c}^+$, moreover we proved that the equality $\kappa = cof([\kappa]^\omega, \subseteq)$ is equivalent to the existence of a locally countable and *crowded* SAU space in the generic extension obtained by adding κ Cohen reals.

2. A large SAU space in ZFC

The aim of this section is to present a construction that yields, in ZFC, a locally countable SAU space of cardinality $2^{\mathfrak{c}}$.We actually present a general construction of right-separated spaces that uses functions with values that are free filters. We start by fixing some notation and terminology.

For any infinite set S we write $\Phi(S)$ to denote the collection of all free filters on S. As is customary, $S^* \subset \Phi(S)$ denotes the family of all free ultrafilters on S.

Definition 2.1. Let κ be an uncountable cardinal. We call a function φ with domain $\kappa \setminus \omega$ a *nice filter assignment* for κ if for all $\alpha \in \kappa \setminus \omega$ we have $\varphi(\alpha) \in \Phi(S_\alpha)$ for some infinite subset S_α of α. We shall denote by $\mathfrak{F}(\kappa)$ the family of all nice filter assignments for κ.

Note that S_α is simply the maximal member of $\varphi(\alpha)$.

Next we shall show how a nice filter assignment for κ defines a natural topology on κ.

Definition 2.2. Let φ be a nice filter assignment for $\kappa > \omega$. Then the topology τ_φ on κ is defined by the formula

$$\tau_\varphi = \{G \subset \kappa : \forall\, \alpha \in G \setminus \omega \left(G \cap S_\alpha \in \varphi(\alpha)\right)\}.$$

It is left to the reader to check that τ_φ is a T_1 topology on κ, this is where considering only free filters is essential.

Since SAU spaces are Hausdorff by definition, we shall need a condition that will imply in case of a nice filter assignment φ for κ that τ_φ is Hausdoff. To formulate this condition, we shall use the following terminology that was introduced in [2].

Definition 2.3. The indexed family of filters $\{F_i : i \in I\}$ is called *disjointly representable* if there are sets $\{A_i \in F_i : i \in I\}$ such that $A_i \cap A_j = \emptyset$ whenever $i, j \in I$ and $i \neq j$.

Lemma 2.4. *Let φ be a nice filter assignment for $\kappa > \omega$ such that*

(i) S_α is countable for all $\alpha \in \kappa \setminus \omega$,

(ii) $\{\varphi(i) : i \in I\}$ is disjointly representable for all countable $I \subset \kappa \setminus \omega$.

Then τ_φ is a Hausdoff topology.

Proof. Note first that all $n \in \omega$ are isolated in τ_φ. So, it suffices to show that distinct $\alpha, \beta \in \kappa \setminus \omega$ have disjoint neighborhoods.

To see that, we set $U_0 = \{\alpha\}$ and $V_0 = \{\beta\}$ and then define by recursion countable sets U_n and V_n with $U_n \cap V_n = \emptyset$ as follows.

Given U_n and V_n, we apply (ii) to the index set $I_n = U_n \cup V_n \setminus \omega$ to obtain pairwise disjoint sets $A_i \in \varphi(i)$ for all $i \in I_n$. Then we set $U_{n+1} = U_n \cup \bigcup\{A_i : i \in U_n\}$ and similarly $V_{n+1} = V_n \cup \bigcup\{A_i : i \in V_n\}$. Clearly, then both U_{n+1} and V_{n+1} are countable by (i), moreover we also have $U_{n+1} \cap V_{n+1} = \emptyset$. Now, it is obvious that $U = \bigcup\{U_n : n < \omega\} \in \tau_\varphi$ and $V = \bigcup\{V_n : n < \omega\} \in \tau_\varphi$ are disjoint open neighborhoods of α and β, completing the proof. □

In view of this result it is natural to look for conditions that imply disjoint representability of certain countable families of filters. Here is a very simple such condition about families of ultrafilters on ω.

Proposition 2.5. *A countable subfamily of ω^* is disjointly representable iff it is a discrete subspace of ω^*, considered as the remainder of $\beta\omega$.*

Proof. Indeed, this follows from the fact that in a regular space the points in a countable discrete subspace have pairwise disjoint neighborhoods, moreover for members of ω^* this means that they have pairwise almost disjoint elements. □

Contrary to this proposition, the following result needed in our construction of SAU spaces, is highly non-trivial. But, luckily for us, it is an immediate consequence of a deep result of Kunen in [5]. We recall that an ultrafilter $u \in \omega^*$ is a *weak P-point* if u is not in the closure of any *countable* subset of its complement.

Theorem 2.6. *The family $\mathcal{U}$ of all weak P-point ultrafilters in ω^* has cardinality $2^{\mathfrak{c}}$, moreover all countable subfamilies of $\mathcal{U}$ are disjointly representable.*

Proof. We have $|\mathcal{U}| = 2^{\mathfrak{c}}$ by [5] and it is obvious that all countable subsets of $\mathcal{U}$ are discrete in ω^*, hence disjointly representable by Proposition 2.5. □

We are now ready to present our main result.

Theorem 2.7. *If $\kappa = \kappa^\omega \le 2^{\mathfrak{c}}$ then there is a locally countable SAU space of cardinality κ.*

Proof. To start with, we fix using Theorem 2.6 for every countably infinite set $S \subset \kappa$ the family $\mathcal{U}(S)$ of size $2^{\mathfrak{c}}$ of all weak P-point ultrafilters in S^*. Then all countable subfamilies of $\mathcal{U}(S)$ are disjointly representable. Also, if $T \in [S]^\omega$ then

$$\mathcal{U}(T) = \{u \upharpoonright T : u \in \mathcal{U}(S) \text{ and } T \in u\},$$

where $u \restriction T = \{A \cap T : A \in u\}$. In other words, $u \in T^*$ belongs to $\mathcal{U}(T)$ iff the ultrafilter $\widehat{u} \in S^*$ generated by u belongs to $\mathcal{U}(S)$. Indeed, this is because T^* is clopen in S^*.

Next, using $\kappa = \kappa^\omega$ we fix an enumeration $\{\langle A_\alpha, B_\alpha \rangle : \omega \leq \alpha < \kappa\}$ of $[\kappa]^\omega \times [\kappa]^\omega$ such that $S_\alpha = A_\alpha \cup B_\alpha \subset \alpha$ for all $\alpha \in \kappa \setminus \omega$. We then pick, by transfinite recursion on $\alpha \in \kappa \setminus \omega$, weak P-points $u_\alpha \in \mathcal{U}(S_\alpha)$ and $v_\alpha \in \mathcal{U}(S_\alpha)$ such that $A_\alpha \in u_\alpha$ and $B_\alpha \in v_\alpha$ as follows.

Assume that $\alpha \in \kappa \setminus \omega$ and u_β, v_β have been chosen for all $\omega \leq \beta < \alpha$ and then let us put

$$I = \{\beta \in \alpha \setminus \omega : A_\beta \cap S_\alpha \in u_\beta\} \text{ and } J = \{\beta \in \alpha \setminus \omega : B_\beta \cap S_\alpha \in v_\beta\}.$$

For each $\beta \in I$ then $u_\beta \restriction A_\alpha$ generates a weak P-point ultrafilter $\widehat{u}_\beta \in \mathcal{U}(S_\alpha)$ and, similarly, for each $\beta \in J$, $v_\beta \restriction B_\alpha$ generates a weak P-point ultrafilter $\widehat{v}_\beta \in \mathcal{U}(S_\alpha)$.

But we have on one hand $|I \cup J| < \kappa \leq 2^{\mathfrak{c}}$ and, on the other hand, $|\mathcal{U}(S_\alpha)| = |\mathcal{U}(A_\alpha)| = |\mathcal{U}(B_\alpha)| = 2^{\mathfrak{c}}$, so we may clearly choose distinct $u_\alpha, v_\alpha \in \mathcal{U}(S_\alpha) \setminus \{\widehat{u_\beta} : \beta \in I\} \cup \{\widehat{v_\beta} : \beta \in J\}$ such that $A_\alpha \in u_\alpha$ and $B_\alpha \in v_\alpha$.

After having completed the recursion, we let

$$\varphi(\alpha) = \{U \cup V : U \in u_\alpha \text{ and } V \in v_\alpha\}$$

for each $\alpha \in \kappa \setminus \omega$. Clearly, then $\varphi(\alpha) \in \Phi(S_\alpha)$, *hence* φ is a nice filter assignment for κ. It is also clear from the definitions that each $\alpha \in \kappa \setminus \omega$ is a common τ_φ-accumulation point of both A_α and B_α, hence τ_φ turns out to be a SAU topology on κ if we can prove that it is Hausdorff.

To see that, it suffices to show that for any $I \in [\kappa \setminus \omega]^\omega$ the family $\{u_\alpha, v_\alpha : \alpha \in I\}$ is disjointly representable. Indeed, if

$$\{U_\alpha : \alpha \in I\} \cup \{V_\alpha : \alpha \in I\}$$

are pairwise disjoint sets with $U_\alpha \in u_\alpha$ and $V_\alpha \in v_\alpha$ then $\{U_\alpha \cup V_\alpha : \alpha \in I\}$ are pairwise disjoint as well. But this means that $\{\varphi(\alpha) : \alpha \in I\}$ is disjointly representable for all countable $I \subset \kappa \setminus \omega$, hence τ_φ is Hausdorff by Lemma 2.4.

So, consider $I \in [\kappa \setminus \omega]^\omega$ and put $S = \bigcup\{S_\alpha : \alpha \in I\}$. For each $\alpha \in I$ then u_α generates a weak P-point $\widehat{u}_\alpha \in \mathcal{U}(S)$ and similarly v_α generates $\widehat{v}_\alpha \in \mathcal{U}(S)$, moreover by our recursive construction they are all distinct. Consequently, by Theorem 2.6 the family $\{\widehat{u}_\alpha, \widehat{v}_\alpha : \alpha \in I\}$ is disjointly representable, hence so is $\{u_\alpha, v_\alpha : \alpha \in I\}$, completing the proof. $\square$

It is worth to mention that for $\kappa \geq \mathfrak{c}$ the condition $\kappa = \kappa^\omega \leq 2^{\mathfrak{c}}$ in Theorem 2.7 is actually necessary to have a locally countable SAU space of cardinality κ. Since SAU spaces are countably compact, this follows immediately from the following result.

Theorem 2.8. *If X is a locally countable and countably compact T_1-space of cardinality $\kappa > \mathfrak{c}$ then $\kappa = \kappa^\omega$.*

Proof. Assume, on the contrary that $\kappa < \kappa^\omega$ and let λ be the smallest cardinal such that $\lambda^\omega > \kappa$. It is well-known that then $cf(\lambda) = \omega$. But then by a classical result of Tarski in [7], there is an almost disjoint family $\mathcal{A} \subset [\lambda]^\omega$ with $|\mathcal{A}| = \lambda^\omega > \kappa$.

Since $\lambda \le \kappa$, we may assume without any loss of generality that $\lambda \subset X$, hence every $A \in \mathcal{A}$ has an accumulation point $x_A \in X$. But $|\mathcal{A}| > \kappa$ then implies the existence of some $\mathcal{B} \subset \mathcal{A}$ and $x \in X$ such that $|\mathcal{B}| > \kappa$ and $x_A = x$ for all $A \in \mathcal{B}$. Let U be a countable neighbourhood of x, then $A \cap U$ is infinite for all $A \in \mathcal{B}$ which is impossible because $\mathcal{B}$ is almost disjoint with $|\mathcal{B}| > \kappa > \mathfrak{c}$. □

Corollary 2.9. *For $\mathfrak{c} \le \kappa \le 2^\mathfrak{c}$ there is a locally countable SAU space of cardinality κ iff $\kappa = \kappa^\omega$.*

As was mentioned in the introduction, any locally countable SAU space X has cardinality $\le 2^\mathfrak{c}$. Indeed, this is because it does have an infinite closed subset F of cardinality $\le 2^\mathfrak{c}$, namely the closure of any subset of size ω. But local countability then implies that F is covered by an open set U with $|U| = |F| \le 2^\mathfrak{c}$ and the SAU property implies that $X \setminus U$ is finite.

Finally, we mention the following accidental consequence of Theorem 2.7. In this, $F(X)$ is the free set number and X_δ denotes the G_δ-modification of the space X, see [4].

Corollary 2.10. *The locally countable SAU space X of cardinality $2^\mathfrak{c}$ of Theorem 2.7 is an example of a Hausdorff space with $F(X) = \omega$ and $F(X_\delta) = 2^\mathfrak{c}$.*

Indeed, $F(X) = \omega$ because every free sequence in a SAU space has order type $< \omega + \omega$, while X_δ is discrete. It was shown in [4] that for any Hausdorff space X with $F(X) = \omega$ we have $F(X_\delta) \le 2^{2^\mathfrak{c}}$ and we do not know if this upper bound could be replaced by $2^\mathfrak{c}$, i.e. if Corollary 2.10 is sharp. It is curious that the same upper bound $2^{2^\mathfrak{c}}$ is known for the size of any SAU space and the same problem arises if this could be improved to $2^\mathfrak{c}$.

3. Forcing "large" crowded SAU spaces

All the *consistent* examples of SAU spaces constructed in [3] were crowded but of cardinality $\le \mathfrak{c}$. As mentioned above, we do not know if *crowded* SAU spaces exist in ZFC but the aim of this section is to produce consistent examples of crowded SAU spaces of size $> \mathfrak{c}$.

By Theorem 3.7 of [3], the assumption $\mathfrak{r} = \mathfrak{c}$ implies the existence of a locally countable and crowded SAU space of cardinality $\mathfrak{c}$. Our next result says that, under the same assumption $\mathfrak{r} = \mathfrak{c}$ together with $\mathfrak{c} = 2^{<\mathfrak{c}}$, a forcing construction yields a generic extension of the ground model in which there is a locally countable and crowded SAU space of cardinality $\mathfrak{c}^+$.

Theorem 3.1. *If $\mathfrak{r} = \mathfrak{c} = 2^{<\mathfrak{c}}$ then we have a $\mathfrak{c}$-closed and $\mathfrak{c}^+ - CC$ notion of forcing $\mathbb{P}$ such that, in the generic extension $V^{\mathbb{P}}$, there is a locally countable and crowded SAU space of cardinality $\mathfrak{c}^+$.*

Proof. Our aim is to obtain a function $U : \mathfrak{c}^+ \to [\mathfrak{c}^+]^\omega$ in $V^{\mathbb{P}}$ such that $\alpha \in U(\alpha)$ for each $\alpha \in \mathfrak{c}^+$ and $\{U(\alpha) : \alpha \in \mathfrak{c}^+\}$ generates a SAU topology on $\mathfrak{c}^+$. Our conditions then will be approximations to U of size $< \mathfrak{c}$ with some "side conditions" that will ensure that any two infinite subsets of $\mathfrak{c}^+$ have a common accumulation point. Hausdorffness of τ will follow from the assumption $\mathfrak{r} = \mathfrak{c}$ and genericity.

Now we define the notion of forcing $\mathbb{P} = \langle P, \leq \rangle$. The elements of P will be pairs of the form $p = \langle U_p, \mathcal{C}_p \rangle$, where U_p is a function with domain $A_p \in [\mathfrak{c}^+]^{<\mathfrak{c}}$ and values taken in $[A_p]^\omega$ such that $\omega \subset A_p$ and $U_p(n) = \omega$ for all $n \in \omega$, moreover $\alpha \in U(\alpha) \subset \alpha + 1$ if $\alpha \in A_p \setminus \omega$.

Then $\{U(\alpha) : \alpha \in A_p\}$ as a subbase generates a topology τ_p on A_p that is required to be crowded, i.e. we assume that all non-empty members of τ_p are infinite. To handle this, we define

$$B_{p,I} = \bigcap \{U_p(\alpha) : \alpha \in I\}$$

for any finite subset I of A_p. Then

$$\mathcal{B}_p = \{B_{p,I} : I \in [A_p]^{<\omega}\} \setminus \{\emptyset\}$$

is a base for τ_p, hence our assumption just means that all members of $\mathcal{B}_p$ are infinite.

For any $x \in A_p$ we shall denote by $ac_p(x)$ the family of all sets $C \in [A_p]^\omega$ such that x is a τ_p-accumulation point of C, i.e. every τ_p-neighborhood of x has infinite intersection with C. Now, the second coordinate $\mathcal{C}_p$ of the condition p is also a function with domain A_p but such that $\mathcal{C}_p(x) \in [ac_p(x)]^{<\mathfrak{c}}$ for $x \in A_p$.

Next we define the partial order $\leq$ on P by the following stipulations:

Definition 3.2. For $p, q \in P$ we have $p \leq q$, i.e. p is a stronger condition than q iff

(a) $U_p \supset U_q$ and

(b) $\mathcal{C}_p(x) \supset \mathcal{C}_q(x)$ for all $x \in A_q$.

We may reformulate item (a) above as follows: $A_p \supset A_q$ and $U_p(x) = U_q(x)$ for all $x \in A_q$. Note that this implies $B_{p,I} = B_{q,I}$ for all $I \in [A_q]^{<\omega}$. Also, it should be noted that in item (b) it is implicit that every $C \in \mathcal{C}_q(x)$ accumulates to x in the topology τ_p as well.

We next present several lemmas concerning the forcing $\mathbb{P}$ which together will yield the desired function $U : \mathfrak{c}^+ \to [\mathfrak{c}^+]^\omega$ in $V^{\mathbb{P}}$.

Lemma 3.3. *The forcing $\mathbb{P}$ is $\mathfrak{c}$-closed.*

Proof of 3.3. Assume that $\langle p_\xi : \xi < \varrho \rangle$ is a decreasing ϱ-sequence in $\mathbb{P}$, where $p_\xi = \langle U_\xi, \mathcal{C}_\xi \rangle$ for $\xi < \varrho$ and $\varrho < \mathfrak{c}$ is an infinite regular cardinal. (To enhance readability, we use the ξ's instead of the p_ξ's as indices.) We may then define a lower bound $p = \langle U_p, \mathcal{C}_p \rangle$ for the p_ξ's as follows.

(1) $U_p = \bigcup_{\xi<\varrho} U_\xi$, hence $A_p = \bigcup_{\xi<\varrho} A_\xi$, and
(2) $\mathcal{C}_p(x) = \bigcup \{\mathcal{C}_\xi(x) : x \in A_\xi\}$ for any $x \in A_p$.

To see that $p \in P$, first note that $\mathfrak{c} = 2^{<\mathfrak{c}}$ implies that $\mathfrak{c}$ is regular, hence we have both $|A_p| < \mathfrak{c}$ and $|\mathcal{C}_p(x)| < \mathfrak{c}$ for any $x \in A_p$. That τ_p is crowded follows from the fact that $\mathcal{B}_p = \bigcup_{\xi<\varrho} \mathcal{B}_\xi$. It remains to check that for any $C \in \mathcal{C}_\xi(x)$ we have $C \in ac_p(x)$, and that is clear because then $C \in ac_\eta(x)$ for all $\eta \in \varrho \setminus \xi$ and every $B \in \mathcal{B}_p$ eventually belongs to $\mathcal{B}_\eta$ as well. □

Our next lemma is an amalgamation result that will imply the $\mathfrak{c}^+ - CC$-property of $\mathbb{P}$. To formulate it we use the following notation: if M and N are non-empty sets of ordinals, we write $M < N$ iff $\alpha < \beta$ for each $\alpha \in M$ and $\beta \in N$.

Lemma 3.4. *Assume that $p, q \in P$ are such that $A_p \cap A_q < A_p \Delta A_q$, $otp(A_p) = otp(A_q)$, and for the unique order isomorphism $\pi : A_p \to A_q$ between them we have, for all $\alpha \in A_p$, that*

(i) $U_q(\pi(\alpha)) = U_p(\alpha)$, *and*
(ii) $\mathcal{C}_q(\pi(\alpha)) = \{\pi[C] : C \in \mathcal{C}_p(\alpha)\}$.

Then p and q are compatible in $\mathbb{P}$.

Proof of 3.4. Let us note first that, as π is the identity on $A_p \cap A_q < A_p \Delta A_q$, we have $U_p(\alpha) = U_q(\alpha)$ for all $\alpha \in A_p \cap A_q$ by (i), hence $U_r = U_p \cup U_q$ is a well-defined map on $A_r = A_p \cup A_q$. We claim that if we set $\mathcal{C}_r(x) = \mathcal{C}_p(x) \cup \mathcal{C}_q(x)$ for $x \in A_p \cap A_q$, moreover $\mathcal{C}_r(x) = \mathcal{C}_p(x)$ for $x \in A_p \setminus A_q$ and similarly $\mathcal{C}_r(x) = \mathcal{C}_q(x)$ for $x \in A_q \setminus A_p$ then $r = \langle U_r, \mathcal{C}_r \rangle$ is a common extension of p and q in $\mathbb{P}$.

To see that $r \in P$, the only non-trivial thing to check is that $\mathcal{C}_r(x) \subset ac_r(x)$ for any $x \in A_p \cap A_q$. This, however, is clear because

$$U_r(x) = U_p(x) = U_q(x) \subset \omega \cup (x+1),$$

and hence $C \in ac_r(x)$ iff $C \cap (\omega \cup x) \in ac_r(x)$, moreover π is the identity on $A_p \cap (\omega \cup x) = A_q \cap (\omega \cup x)$. Now, that r extends both p and q is obvious. □

Using our assumption $\mathfrak{c} = 2^{<\mathfrak{c}}$, standard counting and delta-system arguments imply that every subset Q of P with $|Q| = \mathfrak{c}^+$ contains two elements $p, q \in Q$ that satisfy the conditions of Lemma 3.4, and hence are compatible. Consequently, $\mathbb{P}$ is indeed $\mathfrak{c}^+ - CC$.

It is an immediate consequence of the above results that we have $\mathfrak{c}^V = \mathfrak{c}^{V^{\mathbb{P}}}$ and $(\mathfrak{c}^+)^V = (\mathfrak{c}^+)^{V^{\mathbb{P}}}$.

Lemma 3.5. *For every condition $q \in P$ and $\alpha \in \mathfrak{c}^+ \setminus \omega$ there is $p \le q$ such that $\alpha \in A_p$.*

Proof of 3.5. Indeed, if $\alpha \notin A_q$ then let $A_p = A_q \cup \{\alpha\}$ and extend U_q to A_p by putting $U_p(\alpha) = \omega \cup \{\alpha\}$. Also, we extend $\mathcal{C}_q$ by letting $\mathcal{C}_p(\alpha) = \emptyset$. It is obvious then that $p = \langle U_p, \mathcal{C}_p \rangle$ is as required. □

It immediately follows that if $G \subset P$ is $\mathbb{P}$-generic over V then $U = \bigcup\{U_p : p \in G\}$ maps $\mathfrak{c}^+$ into $[\mathfrak{c}^+]^\omega$ and the (obviously locally countable) topology τ generated by the range of U is crowded because

$$\mathcal{B} = \bigcup\{\mathcal{B}_p : p \in G\} \subset [\mathfrak{c}^+]^\omega$$

forms a base for τ.

We still have to work to show that τ is SAU. The Hausdorff property of τ immediately follow from the following result.

Lemma 3.6. *For every condition $q \in P$ and distinct $\alpha, \beta \in A_q$ there is $p \le q$ such that for some $\gamma, \delta \in A_p$ we have $\alpha \in U_p(\gamma)$, $\beta \in U_p(\delta)$ and $U_p(\gamma) \cap U_p(\delta) = \emptyset$.*

Proof of 3.6. Start by fixing a countable τ_q-open set W containing both α and β, e.g. $W = U_q(\alpha) \cup U_q(\beta)$ will work. For every $x \in W$ consider the following two subfamilies of $[W]^\omega$:

$$\mathcal{B}_x = \{B \in \mathcal{B}_q : x \in B \subset W\},\ \mathcal{A}_x = \{B \cap C : B \in \mathcal{B}_x \text{ and } C \in \mathcal{C}_q(x)\}.$$

Then we have $|\mathcal{A}_x \cup \mathcal{B}_x| < \mathfrak{c}$ and hence for $\mathcal{A} = \bigcup\{\mathcal{A}_x : x \in W\}$ and $\mathcal{B} = \bigcup\{\mathcal{B}_x : x \in W\}$ we have $|\mathcal{A} \cup \mathcal{B}| < \mathfrak{c}$ as well.

We may thus apply our assumption $\mathfrak{r} = \mathfrak{c}$ to the family $\mathcal{A} \cup \mathcal{B} \subset [W]^\omega$ to obtain a partition $W = E \cup F$ of W such that $|E \cap A| = |F \cap A| = \omega$ for all $A \in \mathcal{A}$ and $|E \cap B| = |F \cap B| = \omega$ for all $B \in \mathcal{B}$. Without loss of generality, we may also assume that $\alpha \in E$ and $\beta \in F$.

Let us now choose $\gamma, \delta \in \mathfrak{c}^+$ such that $A_q < \gamma < \delta$, and extend U_q to $A_p = A_q \cup \{\gamma, \delta\}$ by putting $U_p(\gamma) = E \cup \{\gamma\}$ and $U_p(\delta) = F \cup \{\delta\}$. It is clear from our choice of E and F that τ_p is crowded and $\mathcal{C}_q(x) \subset ac_p(x)$ for all $x \in A_q$. Consequently, if we extend $\mathcal{C}_q(x)$ to A_p by putting $\mathcal{C}_p(\gamma) = \mathcal{C}_p(\delta) = \emptyset$ then $p = \langle U_p, \mathcal{C}_p \rangle \in P$ is as required. □

Now, it is trivial from Lemma 3.6 that the generic topology τ in $V[G]$ is Hausdorff. Our next lemma will imply the only missing thing required to show that τ is SAU, namely that any two infinite τ-closed sets intersect.

Lemma 3.7. *Given any condition $q \in P$ and two countably infinite sets $C, D \in [A_q]^\omega$, there is $p \le q$ such that both C and D belong to $\mathcal{C}_p(x)$ for some $x \in A_p$.*

Proof of 3.7. As in the previous proof, we start by fixing a countable τ_q-open set W such that $C \cup D \subset W$. Then we choose $\gamma \in \mathfrak{c}^+$ such that $A_q < \gamma$ and extend U_q to $A_p = A_q \cup \{\gamma\}$ by putting $U_p(\gamma) = W \cup \{\gamma\}$. We also extend $\mathcal{C}_q(x)$ to A_p by putting $\mathcal{C}_p(\gamma) = \{C, D\}$. It is trivial that then $p = \langle U_p, \mathcal{C}_p \rangle \in P$ and $p \le q$, completing the proof. □

As a corollary we have that the generic topology τ is SAU. Indeed, for any two sets $C, D \in [\mathfrak{c}^+]^\omega$, putting together Lemmas 3.3, 3.5, and 3.7 we may conclude that

$$\{p \in P : C \cup D \subset A_p \text{ and } \exists\, x \in A_p\, \{C, D\} \subset ac_p(x)\}$$

is dense in $\mathbb{P}$. But clearly if $\{C, D\} \subset ac_p(x)$ then p forces $\{C, D\} \subset ac_\tau(x)$ and so $cl_\tau(C) \cap cl_\tau(D) \neq \emptyset$ as well. Thus the proof of Theorem 3.1 is completed. □

We do not know if the above result is (consistently) sharp, i.e. if $\mathfrak{c}^+$ could be replaced by, say, $2^{\mathfrak{c}}$. Our next theorem shows that this is possible if we give up local countability. Also, the assumption $\mathfrak{r} = \mathfrak{c}$ in the ground model is strengthened to $\mathfrak{r}^* = \mathfrak{c}$, where $\mathfrak{r}^*$ is defined to be the smallest cardinal ϱ such that there is a family $\mathcal{A}$ of infinite sets that cannot be reaped (or bisected) by a single set. So, $\mathfrak{r}^* = \mathfrak{c}$ just says that any family $\mathcal{A}$ of infinite sets with $|\mathcal{A}| < \mathfrak{c}$ can be reaped.

Clearly, we have $\omega < \mathfrak{r}^* \le \mathfrak{r} \le \mathfrak{c}$, moreover $\mathfrak{r}^* < \mathfrak{r} = \mathfrak{c}$ is consistent. Indeed, by [1] it is consistent that $\mathfrak{c}$ is as large as you wish, moreover both ♣ and $MA(countable)$ hold. But a ♣-sequence can not be reaped, hence $\mathfrak{r}^* = \omega_1$, while $MA(countable)$ implies $\mathfrak{r} = \mathfrak{c}$.

Theorem 3.8. *If $\mathfrak{r}^* = \mathfrak{c} = 2^{<\mathfrak{c}}$ then there is a $\mathfrak{c}$-closed and $\mathfrak{c}^+ - CC$ notion of forcing $\mathbb{P}$ such that, in the generic extension $V^{\mathbb{P}}$, there is a crowded SAU space of cardinality $(2^{\mathfrak{c}})^V = (2^{\mathfrak{c}})^{V^{\mathbb{P}}}$.*

Proof. To simplify notation, we shall write $\kappa = 2^{\mathfrak{c}}$ in what follows. Similarly as in the proof of Theorem 3.1, our aim is then to force a function $U : \kappa \to \mathcal{P}(\kappa)$ such that $\alpha \in U(\alpha)$ for all $\alpha \in \kappa$ and the topology τ generated by the range $\{U(\alpha) : \alpha \in \kappa\}$ of U on κ is a crowded SAU space. The notion of forcing $\mathbb{P} = \langle P, \le\rangle$ will also be quite similar to the one used there.

The elements of P will be pairs of the form $p = \langle U_p, \mathcal{C}_p\rangle$, where U_p is a function with domain $A_p \in [\kappa]^{<\mathfrak{c}}$ with values taken in $\mathcal{P}(A_p)$ such that $\alpha \in U(\alpha)$ for all $\alpha \in A_p$, moreover the second coordinate $\mathcal{C}_p$ of the condition p is also a function with domain A_p but such that $\mathcal{C}_p(x) \in [ac_p(x)]^{<\mathfrak{c}}$ for $x \in A_p$. Here we are freely using the analogs of the pieces of notation from the above proof of 3.1, so $ac_p(x)$ denotes the family of all countable subsets of A_p that accumulate to x in the topology τ_p generated by the range of U_p on A_p. Also, we shall use the notation $B_{p,I} = \bigcap\{U_p(\alpha) : \alpha \in I\}$ for $I \in [A_p]^{<\omega}$ to obtain the base

$$\mathcal{B}_p = \{B_{p,I} : I \in [A_p]^{<\omega}\} \setminus \{\emptyset\}$$

for τ_p.

Next we define the partial order $\le$ on P that, as far as the first coordinate is concerned, is quite different from the corresponding part of 3.2.

Definition 3.9. For $p, q \in P$ we have $p \le q$, i.e. p is a stronger condition than q iff

(a) $A_p \supset A_q$ and for every $\alpha \in A_q$ we have $A_q \cap U_p(\alpha) = U_q(\alpha)$;
(b) $U_q(\alpha) \cap U_q(\beta) = \emptyset$ implies $U_p(\alpha) \cap U_p(\beta) = \emptyset$ for any $\alpha, \beta \in A_q$;
(c) $\mathcal{C}_p(x) \supset \mathcal{C}_q(x)$ for all $x \in A_q$.

It is obvious that $\le$ is indeed a partial order on P.

We next present a series of lemmas that will help us prove the required properties of the forcing $\mathbb{P}$.

Lemma 3.10. *The forcing $\mathbb{P}$ is $\mathfrak{c}$-closed.*

Proof of 3.10. Assume that $\langle p_\xi : \xi < \varrho\rangle$ is a decreasing ϱ-sequence in $\mathbb{P}$, where $p_\xi = \langle U_\xi, \mathcal{C}_\xi\rangle$ for $\xi < \varrho$ and $\varrho < \mathfrak{c}$ is an infinite regular cardinal. We may then define a lower bound $p = \langle U_p, \mathcal{C}_p\rangle$ for the p_ξ's as follows.

(1) $A_p = \bigcup_{\xi<\varrho} A_\xi$, and $U_p(\alpha) = \bigcup\{U_\xi(\alpha) : \xi_\alpha \le \xi < \varrho\}$, where $\xi_\alpha = \min\{\xi : \alpha \in A_\xi\}$.
(2) $\mathcal{C}_p(x) = \bigcup\{\mathcal{C}_\xi(x) : x \in A_\xi\}$ for any $x \in A_p$.

Since $\mathfrak{c}$ is regular, we have both $|A_p| < \mathfrak{c}$ and $|\mathcal{C}_p| < \mathfrak{c}$. For any $\xi < \varrho$ and $C \in \mathcal{C}_\xi(x)$ we have $C \in ac_p(x)$ because then $C \in ac_\eta(x)$ for all $\eta \in \varrho \setminus \xi$ and every $B \in \mathcal{B}_p$ eventually belongs to $\mathcal{B}_\eta$ as well. Thus we have $p \in P$. The easy verification of $p \leq p_\xi$ for all $\xi < \varrho$ is left to the reader. □

As in the proof of 3.4, we next give an amalgamation result that will imply the $\mathfrak{c}^+ - CC$ property of $\mathbb{P}$.

Lemma 3.11. *Assume that $p, q \in P$ are* isomorphic *conditions, i.e. $otp(A_p) = otp(A_q)$, the unique order isomorphism $\pi : A_p \to A_q$ is the identity on $A = A_p \cap A_q$, moreover for all $\alpha \in A_p$ we have*

(i) $U_q(\pi(\alpha)) = U_p(\alpha)$, and

(ii) $\mathcal{C}_q(\pi(\alpha)) = \{\pi[C] : C \in \mathcal{C}_p(\alpha)\}$.

Then p and q are compatible in $\mathbb{P}$.

Proof of 3.11. We define the desired common extension $r = \langle U_r, \mathcal{C}_r \rangle$ of p and q by the following stipulations:

(a) $U_r(x) = U_r(\pi(x)) = U_p(x) \cup U_q(\pi(x))$ for $x \in A_p$;

(b) $\mathcal{C}_r(x) = \mathcal{C}_r(\pi(x)) = \mathcal{C}_p(x) \cup \mathcal{C}_q(\pi(x))$ for $x \in A_p$.

As π is the identity on A, both functions U_r and $\mathcal{C}_r$ are well-defined on $A_r = A_p \cup A_q$.

To see that $r \in P$, we have to check that $\mathcal{C}_r(x) \subset ac_r(x)$ for all $x \in A_r$. By symmetry, it suffices to do this for $x \in A_p$. In view of conditions (ii) and (b), what we have to show is that $x \in B \in \mathcal{B}_r$ implies $|B \cap C| = |B \cap \pi[C]| = \omega$ for any $C \in \mathcal{C}_p(x)$. Now, any member of $\mathcal{B}_r$ is of the form

$$B_{r,I\cup J} = \bigcap\{U_r(i) : i \in I\} \cap \bigcap\{U_r(j) : j \in J\}$$

with $I \in [A_p]^{<\omega}$ and $J \in [A_q]^{<\omega}$. It is easy to compute from condition (a) that we have

$$B_{r,I\cup J} = B_{p,I\cup\pi^{-1}[J]} \cup B_{q,\pi[I]\cup J},$$

moreover $B_{q,\pi[I]\cup J} = \pi[B_{p,I\cup\pi^{-1}[J]}]$.

Then $x \in B_{r,I\cup J}$ implies $x \in B_{p,I\cup\pi^{-1}[J]}$, hence $|C \cap B_{p,I\cup\pi^{-1}[J]}| = \omega$, consequently, then $|\pi[C] \cap B_{q,\pi[I]\cup J}| = \omega$ as well. But this clearly implies

$$|B_{r,I\cup J} \cap C| = |B_{r,I\cup J} \cap \pi[C]| = \omega,$$

just as we claimed.

To see that r is a common extension of p and q, by symmetry again, it suffices to show $r \leq p$. Clearly, only condition (b) of 3.9 requires any checking for this. But as π is an isomorphism, $U_p(x) \cap U_p(y) = \emptyset$ implies $U_q(\pi(x)) \cap U_q(\pi(y)) = \emptyset$, moreover $U_p(x) \cap A = U_q(\pi(x)) \cap A$ and $U_p(y) \cap A = U_q(\pi(y)) \cap A$, we indeed have $U_r(x) \cap U_r(y) = \emptyset$. □

Since $\mathfrak{c} = 2^{<\mathfrak{c}}$, standard counting and delta-system arguments imply that every subset Q of P with $|Q| = \mathfrak{c}^+$ contains two isomorphic elements $p, q \in Q$ that satisfy the conditions of Lemma 3.11, and hence are compatible. Consequently, $\mathbb{P}$ is indeed $\mathfrak{c}^+ - CC$.

It is an immediate consequence of the above results that we have $\mathfrak{c}^V = \mathfrak{c}^{V^{\mathbb{P}}}$, moreover by counting nice names we also may conclude $(2^{\mathfrak{c}})^V = \kappa = (2^{\mathfrak{c}})^{V^{\mathbb{P}}}$.

We also have the following immediate consequence of Lemma 3.11 which will be used to show that the generic topology τ is (very) crowded.

Lemma 3.12. *For every $\alpha < \kappa$ the set of conditions*

$$D_\alpha = \{r \in P : \forall\, B \in \mathcal{B}_r\, (B \setminus \alpha \neq \emptyset)\}$$

is dense in $\mathbb{P}$.

Proof of 3.12. Since $cf(\kappa) > \mathfrak{c}$, the domain A_p of any condition $p \in P$ is bounded in κ. So we can pick $q \in P$ isomorphic to p such that $A_q > \{\alpha\} \cup A_p$. Now, let r be the common extension of p and q constructed in Lemma 3.11. Then, as we have seen,

$$\mathcal{B}_r = \{B \cup \pi[B] : B \in \mathcal{B}_p\},$$

where π is the unique order isomorphism from A_p to A_q. Thus we clearly have $p \geq r \in D_\alpha$. □

That the generic map U is defined on all of κ, follows from the following trivial lemma.

Lemma 3.13. *For every $\alpha < \kappa$ the set of conditions $E_\alpha = \{p \in P : \alpha \in A_p\}$ is dense in $\mathbb{P}$.*

Proof of 3.13. Indeed, if $q \in P$ and $\alpha \notin A_q$ then define $p = \langle U_p, \mathcal{C}_p \rangle$ as follows: Let $A_p = A_q \cup \{\alpha\}$, $U_p(x) = U_q(x)$ and $\mathcal{C}_p(x) = \mathcal{C}_q(x)$ for $x \in A_q$, moreover $U_p(\alpha) = A_p$ and $\mathcal{C}_p(\alpha) = \emptyset$. Then $p \leq q$ and $p \in E_\alpha$. □

It immediately follows from Lemma 3.13 that if G is $\mathbb{P}$-generic over V then putting $U(\alpha) = \bigcup\{U_p(\alpha) : p \in G \cap E_\alpha\}$ for all $\alpha < \kappa$, then $U : \kappa \to \mathcal{P}(\kappa)$ is well-defined in $V[G]$, hence so is the topology τ generated by the range of U. It is also clear that if $\mathcal{B}$ is the base of τ consisting of all non-empty finite intersections of the subbase $\{U(\alpha) : \alpha < \kappa\}$ then every member of $\mathcal{B}$ is cofinal in κ, hence τ is crowded. Indeed, fix $\alpha < \kappa$ and assume that $B_I = \bigcap\{U(i) : i \in I\} \neq \emptyset$ for some $I \in [\kappa]^{<\omega}$. By lemmas 3.13 and 3.12 there is $p \in G \cap D_\alpha$ such that $I \subset A_p$ and $B_{p,I} \in \mathcal{B}_p$. But then $B_{p,I} \subset B_I$ implies $B_I \setminus \alpha \neq \emptyset$, so B_I is indeed cofinal in κ.

Our next result is used to show that the generic topology τ is Hausdorff.

Lemma 3.14. *For distinct $\alpha, \beta \in \kappa$ the set of conditions $D(\alpha, \beta)$ consisting of all $p \in P$ such that for some $\gamma, \delta \in A_p$ we have $\alpha \in U_p(\gamma)$, $\beta \in U_p(\delta)$, and $U_p(\gamma) \cap U_p(\delta) = \emptyset$ is dense in $\mathbb{P}$.*

Proof of 3.14. By Lemma 3.13 it suffices to show that any $q \in P$ with $\alpha, \beta \in A_q$ has an extension in $D(\alpha, \beta)$. For every $x \in A_q$ consider the following family of infinite sets:

$$\mathcal{A}_x = \{B \cap C : x \in B \in \mathcal{B}_q \text{ and } C \in \mathcal{C}_q(x)\} \subset [A_q]^\omega.$$

Then we have $|\mathcal{A}_x| < \mathfrak{c}$ and hence, as $\mathfrak{c}$ is regular and $|A_q| < \mathfrak{c}$, $\mathcal{A} = \bigcup\{\mathcal{A}_x : x \in A_q\}$ has cardinality less than $\mathfrak{c}$ as well.

We may thus apply our assumption $\mathfrak{r}^* = \mathfrak{c}$ to $\mathcal{A} \subset [A_q]^\omega$ to obtain a partition $A_q = E \cup F$ of A_q such that $|E \cap A| = |F \cap A| = \omega$ for all $A \in \mathcal{A}$. Without loss of generality, we may also assume that $\alpha \in E$ and $\beta \in F$.

Let us now choose distinct $\gamma, \delta \in \kappa \setminus A_q$ and define $p \in P$ as follows. Put $A_p = A_q \cup \{\gamma, \delta\}$, for $x \in A_q$ set $U_p(x) = U_q(x)$ and $\mathcal{C}_p(x) = \mathcal{C}_q(x)$, moreover $U_p(\gamma) = E \cup \{\gamma\}$, $U_p(\delta) = F \cup \{\delta\}$ and, finally, $\mathcal{C}_p(\gamma) = \mathcal{C}_p(\delta) = \emptyset$. It is clear from our choice of E and F that $\mathcal{C}_p(x) \subset ac_p(x)$ for all $x \in A_q$. It trivially follows then that $p = \langle U_p, \mathcal{C}_p\rangle \in D(\alpha, \beta)$ and $p \le q$. □

It is obvious from condition (b) of Definition 3.9 that every $p \in D(\alpha, \beta)$ forces $U(\alpha) \cap U(\beta) = \emptyset$, hence τ is indeed Hausdorff. Our last result will finish our proof by implying that τ is SAU.

Lemma 3.15. *For any two sets $C, D \in [\kappa]^\omega$ the set of conditions*

$$E(C, D) = \{p \in P : C \cup D \subset A_p \text{ and } \exists\, x \in A_p \, \{C, D\} \subset ac_p(x)\}$$

is dense in $\mathbb{P}$.

Proof of 3.15. By Lemmas 3.10 and 3.13 it suffices to show that any $q \in P$ with $C \cup D \subset A_q$ has an extension in $E(C, D)$. To see that, pick $\gamma \in \kappa \setminus A_q$ and define $p \in P$ as follows. First, set $A_p = A_q \cup \{\gamma\}$, for $x \in A_q$ set $U_p(x) = U_q(x)$ and $\mathcal{C}_p(x) = \mathcal{C}_q(x)$. We also set $U_p(\gamma) = \{\gamma\} \cup A_q$ and $\mathcal{C}_p(\gamma) = \{C, D\}$. Then the only member of $\mathcal{B}_p$ containing γ is A_p, hence we trivially have $\mathcal{C}_p(\gamma) \subset ac_p(\gamma)$. We thus have $p = \langle U_p, \mathcal{C}_p\rangle \in E(C, D)$ and $p \le q$, completing the proof. □

Since every condition $p \in E(C, D)$ clearly forces $cl_\tau(C) \cap cl_\tau(D) \ne \emptyset$, it immediately follows that τ is indeed SAU. □

4. Crowded SAU spaces from Cohen reals

In the previous section, with considerable effort, we presented consistent examples of crowded SAU spaces of size $> \mathfrak{c}$. In this section we show that to obtain such examples of size $\le \mathfrak{c}$ is much easier, in fact, we can get them by simply adding Cohen reals.

To fix our notation, we shall denote by $\mathbb{C}_I$ the standard Cohen forcing

$$\mathbb{C}_I = \langle Fn(I, 2), \supset \rangle,$$

using the notation of [6]. Before turning to this promised result, we formulate and prove the following technical lemma.

Lemma 4.1. *Assume that $cof([\kappa]^\omega, \subset) = \kappa$, moreover τ is a crowded and locally countable Hausdorff topology on a set X of cardinality κ with $X \cap \kappa = \emptyset$. Then there is a crowded and locally countable Hausdorff topology σ on $Z = X \cup \kappa$ in $V^{\mathbb{C}_\kappa}$ such that*

(1) $\tau \subset \sigma$;

(2) $X \cap cl_\sigma(A) = cl_\tau(A)$ for every $A \in V \cap [X]^\omega$;

(3) $cl_\sigma(C) \cap cl_\sigma(D) \ne \emptyset$ for any two $C, D \in V \cap [X]^\omega$.

Proof of 4.1. Since X is locally countable, every countable subset of X is included in a countable τ-open set, hence by $cof([\kappa]^\omega, \subset) = \kappa$ we can fix $\mathcal{U} = \{U_\alpha : \alpha \in \kappa\} \subset \tau \cap [X]^\omega$ that is cofinal in $[X]^\omega$. For each $\alpha \in \kappa$ we also fix an ω-type one-to-one enumeration $\{x_{\alpha,n} : n < \omega\}$ of U_α.

Let G be $\mathbb{C}_\kappa$-generic over V and $g = \cup G : \kappa \to 2$ be the corresponding Cohen generic map. For every $\alpha < \kappa$ we then define

$$W_\alpha = \{\alpha\} \cup \{x_{\alpha,n} : g(\omega \cdot \alpha + n) = 1\}.$$

We claim that the topology σ generated in $V[G]$ on Z by the subbase

$$\tau \cup \{W_\alpha : \alpha < \kappa\} \cup \{T_\alpha = Z \setminus W_\alpha : \alpha < \kappa\}$$

is as required.

That σ is locally countable follows because $W_\alpha \subset U_\alpha \cup \{\alpha\}$ for all $\alpha < \kappa$. To see that it is crowded, consider any non-empty basic open set of the form $B = U \cap W_I \cap T_J$ where $U \in \tau^+$, $W_I = \cap\{W_i : i \in I\}$ and $T_J = \cap\{T_j : j \in J\}$ with $I, J \in [\kappa]^{<\omega}$. Let $H = U \cap \bigcap\{U_i : i \in I\}$, then $\emptyset \ne B \subset H$ implies that H is infinite because X is crowded and H is a non-empty open set in X. To see that $|B| = \omega$, it is clearly enough to prove the following Claim.

Claim 4.1.1. *For any $p \in \mathbb{C}_\kappa$ and $F \in [H]^{<\omega}$ there are $q \le p$ and $x \in H \setminus F$ such that $q \Vdash x \in B$.*

Proof of the Claim. Let

$$E = \{x_{\alpha,n} \in H : \alpha \in I \cup J \ \text{ and } \omega \cdot \alpha + n \in \operatorname{dom}(p)\}.$$

Clearly $|E| \le |\operatorname{dom}(p)| < \omega$, hence we can pick $x \in H \setminus (F \cup E)$. Then $x \in H \subset W_\alpha$ for each $\alpha \in I$, so $x = x_{\alpha,n_\alpha}$ for some $n_\alpha < \omega$. Similarly, if $\beta \in J$ and $x \in W_\beta$ then there is $m_\beta \in \omega$ such that $x = x_{\beta,m_\beta}$. Let us put $J' = \{\beta \in J : x \in W_\beta\}$. Now, since $x \notin E$, we have $\omega \cdot \alpha + n_\alpha \notin \operatorname{dom}(p)$ for $\alpha \in I$ and $\omega \cdot \beta + m_\beta \notin \operatorname{dom}(p)$ for $\beta \in J'$, so we may define $q \in \mathbb{C}_\kappa$ with $q \le p$ as follows:

$$q(\omega \cdot \alpha + n_\alpha) = 1 \text{ for } \alpha \in I \text{ and } q(\omega \cdot \beta + m_\beta) = 0 \text{ for } \beta \in J'.$$

Clearly, then $q \Vdash x \in B$. □

To check that σ is Hausdorff, consider first $\alpha \ne \beta \in \kappa$. In this case W_α and $Z \setminus W_\alpha$ are clearly their disjoint neighborhoods. If $x \in X$ and $\alpha \in \kappa$ then $|\{\beta : x \in U_\beta\}| = \kappa$ implies that for every condition $q \in Fn(\kappa, 2)$ there is $\beta \ne \alpha$ with $x = x_{\beta,n} \in U_\beta$ and

$$[\omega \cdot \beta, \omega \cdot \beta + \omega) \cap \operatorname{dom}(q) = \emptyset.$$

Now, if p extends q so that $p(\omega \cdot \beta + n) = 1$ then p forces $x \in W_\beta$ and $\alpha \in Z \setminus W_\beta$. Finally, there is nothing to prove if $x \ne y \in X$ because $\tau \subset \sigma$ is Hausdorff.

It remains to prove (1), (2) and (3). Now, (1) holds by definition. To check (2), consider any $x \in cl_\tau(A)$ for some $A \in V \cap [X]^\omega$ and $x \in B = U \cap W_I \cap T_J$. Let $H = U \cap \bigcap\{U_i : i \in I \cup J\}$, then $x \in H \in \tau$. We may assume that $x \notin A$, hence $|H \cap A| = \omega$. So, given any condition $q \in Fn(\kappa, 2)$, there is $y \in H \cap A$ such that for every $i \in I \cup J$ with $y = x_{i,n_i}$ we have $\omega \cdot i + n_i \notin \operatorname{dom}(q)$. Consequently, we may extend q to $p \in Fn(\kappa, 2)$ so that $p(\omega \cdot i + n_i) = 1$ whenever $i \in I$ and $p(\omega \cdot i + n_i) = 0$ whenever $i \in J$. But then p clearly forces $y \in B$, hence $B \cap A \ne \emptyset$ as well.

Finally, since for any $C, D \in V \cap [X]^\omega$ there is $\alpha < \kappa$ with $C \cup D \subset U_\alpha$, to prove (3), it clearly suffices to show that $\alpha \in cl_\sigma(A)$ whenever $A \in V \cap [U_\alpha]^\omega$. To see this, note first that the sets of the form

$$B_{\alpha,I} = W_\alpha \setminus \bigcup_{i \in I} W_i$$

constitute a σ-neighborhood base at α, where $\alpha \notin I \in [\kappa]^{<\omega}$. For any condition $q \in Fn(\kappa, 2)$ there is $x = x_{\alpha,n} \in A$ such that $\omega \cdot \alpha + n \notin \operatorname{dom}(q)$ and for every $i \in I$ if $x = x_{i,n_i} \in U_i$ then $\omega \cdot i + n_i \notin \operatorname{dom}(q)$. So, we may extend q to $p \in Fn(\kappa, 2)$ so that $p(\omega \cdot \alpha + n) = 1$ and $p(\omega \cdot i + n_i) = 0$ whenever $x \in U_i$ for some $i \in I$. But then p forces $x \in B_{\alpha,I}$, hence $A \cap B_{\alpha,I} \ne \emptyset$, completing the proof. □

From Lemma 4.1 the following result is easily deduced.

Theorem 4.2. *Assume that W is a model of ZFC, $\langle V_\alpha : \alpha < \omega_1\rangle$ is an increasing sequence of ZFC submodels in W, $(\omega_1)^{V_0} = (\omega_1)^W$, and $\kappa \in V_0$ is a cardinal such that*

$$([\kappa]^\omega)^W = \bigcup_{\alpha<\omega_1} ([\kappa]^\omega)^{V_\alpha}$$

and $cof([\kappa]^\omega, \subset) = \kappa$ holds in all the V_α's. Assume also that $V_{\alpha+1}$ contains a $\mathbb{C}_\kappa$-generic filter G_α over V_α for every $\alpha < \omega_1$. Then there is a crowded locally countable SAU space of cardinality κ in W.

Proof. To start with, we fix a crowded locally countable space $\langle X_0, \tau_0\rangle \in V_0$ of cardinality κ such that $X_0 \cap (\omega_1 \times \kappa) = \emptyset$. By transfinite recursion on $\alpha \le \omega_1$ we then define crowded locally countable topologies $\tau_\alpha \in V_\alpha$ on $X_\alpha = X_0 \cup (\alpha \times \kappa)$ as follows.

To obtain $\tau_{\alpha+1}$ from τ_α, we apply Lemma 4.1 to get a crowded locally countable topology $\sigma_\alpha \in V[G_\alpha] \subset V_{\alpha+1}$ on $X_{\alpha+1} = X_\alpha \cup (\{\alpha\} \times \kappa)$ with properties (1) - (3) applied to X_α and $\{\alpha\} \times \kappa$ instead of X and κ. Then $\tau_{\alpha+1}$ is the topology generated by σ_α on $X_{\alpha+1}$ in $V_{\alpha+1}$. For α limit we simply let τ_α be the topology on X_α generated by $\bigcup_{\beta<\alpha} \tau_\beta$ on X_α in $V_{\alpha+1}$.

Now, it is straightforward to check that τ_{ω_1} is a crowded locally countable topology on X_{ω_1} in the final model W. The topology is SAU because if $A, B \in [X_{\omega_1}]^\omega \cap W$ then there is $\alpha < \omega_1$ with $A, B \in [X_{\omega_\alpha}]^\omega \cap V_\alpha$, and so A and B have a common accumulation point in $X_{\alpha+1}$. Then x is a common accumulation point in X_{ω_1} as well. □

While it is an immediate corollary of Theorem 4.2 that the equality $cof([\kappa]^\omega, \subset) = \kappa$ implies the existence of a crowded locally countable SAU space of cardinality κ in the generic extension obtained by adding $\kappa > \omega$ Cohen reals, it may be somewhat surprising that the two statements are actually equivalent.

Theorem 4.3. *For any uncountable cardinal κ TFAE:*

(i) $cof([\kappa]^\omega, \subset) = \kappa$.

(ii) There is a crowded locally countable SAU space in $V^{\mathbb{C}_\kappa}$.

(iii) There is a locally countable and countably compact T_1-space in some CCC generic extension W of V.

Proof. Since (i) $\Rightarrow$ (ii) is implied by Theorem 4.2 and (ii) $\Rightarrow$ (iii) is trivial, it suffices to show that (iii) $\Rightarrow$ (i).

We first note that if X is a locally countable and countably compact T_1-space of cardinality $\kappa > \omega$ then, choosing a countable neighbourhood U_x of every non-isolated point $x \in X'$, the family

$$\mathcal{U} = \{U_x : x \in X'\} \subset [X]^\omega$$

has the property that for every $A \in [X]^\omega$ there is $U_x \in \mathcal{U}$ with $|A \cap U_x| = \omega$, i.e. $\mathcal{U}$ is ω-hitting.

Now, it is well-known that if $\kappa > \omega$ then the existence of an ω-hitting family $\mathcal{H}$ of size λ in $[\kappa]^\omega$ implies that of a cofinal family of size λ in $[\kappa]^\omega$. Indeed, we may then take $\mathcal{H}$ with $|\mathcal{H}| = \lambda$ that is ω-hitting in $[\kappa]^{<\omega}$ and then $\{\cup H : H \in \mathcal{H}\}$ is cofinal in $[\kappa]^\omega$. This is because if

$$A = \{\alpha_i : i < \omega\} \in [\kappa]^\omega \text{ and } S_A = \big\{\{\alpha_i : i < n\} : n < \omega\big\},$$

then $|H \cap S_A| = \omega$ implies $A \subset \cup H$.

Thus (iii) implies that $cof([\kappa]^\omega, \subset) = \kappa$ holds in W, but this, in turn, implies the same in V because for any set $A \in W$ with $A \subset V$ there is $B \in V$ such that $A \subset B$ and $|A| = |B|$. □

It is easy to see that $\mathfrak{r} \ge \mathfrak{r}^* \ge \kappa$ holds in $V^{\mathbb{C}_\kappa}$. Indeed, if $\mathcal{G}$ is a $\mathbb{C}_\kappa$-generic filter over V, and $\mathcal{A} \subset [\lambda]^\omega$ with $|\mathcal{A}| = \lambda < \kappa$, then we can assume that $\mathcal{A} \in W = V[\mathcal{G} \restriction \mathbb{C}_\lambda]$. Then $\mathcal{H} = \mathcal{G} \restriction \mathbb{C}_{\kappa\setminus\lambda}$ is a $\mathbb{C}_{\kappa\setminus\lambda}$-generic filter over W. So if we take $h = \bigcup \mathcal{H}$ and define $X \subset \lambda$ by the formula $\alpha \in X$ iff $h(\lambda + \alpha) = 1$, then X reaps $\mathcal{A}$, because $\mathcal{A} \in W$.

Moreover Theorem 3.7. of [3] as well as Theorem 3.1 above used the assumption $\mathfrak{r} = \mathfrak{c}$ to obtain ‘"large" crowded locally countable SAU spaces. This lead us to raise the question if one could get such spaces when $\mathfrak{r}$ is small. Our last result gives an affirmative answer to this question.

Theorem 4.4. *There are models of ZFC containing crowded locally countable SAU spaces of cardinality* $\mathfrak{c}$ *in which* $\mathfrak{r} = \omega_1$ *but* $\mathfrak{c}$ *is arbitrarily large.*

Proof. To get such a model we first fix a cardinal $\kappa = \kappa^\omega$ in the ground model and then will do a finite support iteration $\langle \mathbb{P}_\alpha : \alpha \le \omega_1 \rangle$ of length ω_1 of CCC forcings where $\mathbb{P}_{\alpha+1} = \mathbb{C}_\kappa * \mathbb{Q}_\alpha$ for any $\alpha < \omega_1$. Then, independently of the choice of the $\mathbb{Q}_\alpha$'s, we get from Theorem 4.2 that a crowded locally countable SAU space of cardinality $\mathfrak{c} = \kappa$ exists in the final model $W = V^{\mathbb{P}_{\omega_1}}$.

The posets $\mathbb{Q}_\alpha$ will be obtained together with ultrafilters $u_\alpha \in \omega^*$ in $V^{\mathbb{P}_\alpha}$ by recursion so that $\beta < \alpha < \omega_1$ implies $u_\beta \subset u_\alpha$. Our u_0 is an arbitrary free ultrafilter on ω in the ground model and for α limit we take $u_\alpha \supset \bigcup\{u_\beta : \beta < \alpha\}$. Once we have u_α, we let $\mathbb{Q}_\alpha$ be the standard CCC, in fact even σ-centered, partial order that adds an infinite pseudo-intersection S_α of u_α.

Then $u = \bigcup\{u_\alpha : \alpha < \omega_1\}$ is a free ultrafilter on ω in the final model $W = V^{\mathbb{P}_{\omega_1}}$ that is generated by the family

$$\{S_\alpha : \alpha < \omega_1\} \cup \{\omega \setminus a : a \in [\omega]^{<\omega}\}.$$

Thus W actually satisfies $\mathfrak{u} = \omega_1$ (see [6, V.4.27] for more details). However the base of a free ultrafilter on ω obviously cannot be reaped, hence $\mathfrak{u} = \omega_1$ implies $\mathfrak{r} = \omega_1$. □

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