USUBA'S PRINCIPLE UB_λ CAN FAIL AT SINGULAR CARDINALS

MOHAMMAD GOLSHANI AND SAHARON SHELAH

ABSTRACT. We answer a question of Usuba by showing that the combinatorial principle UB_λ can fail at a singular cardinal. Furthermore, λ can be taken to be $\aleph_\omega$.

§ 1. INTRODUCTION

In [4], Usuba introduced a new combinatorial principle, denoted UB_λ.[1] He showed that UB_λ holds for all regular uncountable cardinals and that for singular cardinals, some very weak assumptions like weak square or even ADS_λ imply it. It is known that ADS_λ can fail for singular cardinals, for example if κ is supercompact and $\lambda > \kappa$ is such that $\mathrm{cf}(\lambda) < \kappa$. Motivated by this results, Usuba asked the following question:

Question 1.1. ([4, Question 2.11]) Is it consistent that UB_λ fails for some singular cardinal λ?

In this paper we give a positive answer to the above question by showing that Chang's transfer principle $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ implies the failure of $\mathrm{UB}_{\aleph_\omega}$, see Theoem 3.1, where a stronger result is proved.

The paper is organized as follows. In Section 2, we present some preliminaries and results and then in Section 3, we prove our main result.

1991 *Mathematics Subject Classification.* Primary:03E05, 03E55 .

Key words and phrases. Usuba's question, singular cardinals, Chang's conjecture.

The first author's research has been supported by a grant from IPM (No. 1400030417). The second author's research has been partially supported by Israel Science Foundation (ISF) grant no: 1838/19. This is publication 1216 of second author.

[1]See Section 2 for the statement of the principle.

§ 2. Some preliminaries

In this section we proved some definitions and results that are needed for later section of this paper. Let us start by giving a definition of Usuba's principle.

Definition 2.1. Let λ be an uncountable cardinal. The principle UB_λ is the statement: there exists a function $f : [\lambda^+]^{<\omega} \to \lambda^+$ such that if $x, y \subseteq \lambda^+$ are closed under f, $x \cap \lambda = y \cap \lambda$ and $\sup(x \cap \lambda) = \lambda$, then $x \subseteq y$ or $y \subseteq x$.

It turned out this principle has many equivalent formulations. To state a few of it, let $S = \{x \subseteq \lambda : \sup(x) = \lambda\}$, $\theta > \lambda$ be large enough regular and let $\lhd$ be a well-ordering of $H(\theta)$. Then the following are equivalent (see [4]):

(1) UB_λ,
(2) If $M, N \prec (H(\theta), \in, \lhd, \lambda, S, \cdots)$ are such that $M \cap \lambda = N \cap \lambda \in S$, then either $M \cap \lambda^+ \subseteq N \cap \lambda^+$ or $N \cap \lambda^+ \subseteq M \cap \lambda^+$,
(3) If $M, N \prec (H(\theta), \in, \lhd, \lambda, S, \cdots)$ are such that $M \cap \lambda = N \cap \lambda \in S$, and $\sup(M \cap \lambda^+) \leq \sup(N \cap \lambda^+)$, then $M \cap \lambda^+$ is an initial segment of $N \cap \lambda^+$.

The principle UB_λ has many nice implications. Here we only consider its relation with Chang transfer principles which is also related to our work.

Definition 2.2. Suppose $\lambda > \mu$ are infinite cardinal. The Chang's transfer principle $(\lambda^+, \lambda) \twoheadrightarrow (\mu^+, \mu)$ is the statement: if $\mathcal{L}$ is a countable first order language which contains a unary predicate U, then for any $\mathcal{L}$ -structure $\mathcal{M} = (M, U^{\mathcal{M}}, \cdots)$ with $|M| = \lambda^+$ and $|U^{\mathcal{U}}| = \lambda$, there exists an elementary submodel $\mathcal{N} = (N, U^{\mathcal{N}}, \cdots)$ of $\mathcal{M}$ with $|N| = \mu^+$ and $|U^{\mathcal{N}}| = \mu$.

The next lemma shows the relation between $\mathrm{UB}_{\aleph_\omega}$ and Chang's transfer principles.

Lemma 2.3. *([4, Corollary 4.2]) Suppose* $\mathrm{UB}_{\aleph_\omega}$ *holds. Then the Chang transfer principles* $(\aleph_{\omega+1,\aleph_\omega}) \twoheadrightarrow (\aleph_{n+1}, \aleph_n)$ *fail for all* $1 \leq n < \omega$.

Since the consistency of the transfer principle $(\aleph_{\omega+1,\aleph_\omega}) \twoheadrightarrow (\aleph_{n+1}, \aleph_n)$ is open for $n \geq 1$, one can not use the above result to get the consistent failure of $\mathrm{UB}_{\aleph_\omega}$.

In the next section we show that $\mathrm{UB}_{\aleph_\omega}$ implies the failure of $(\aleph_{\omega+1,\aleph_\omega}) \twoheadrightarrow (\aleph_1, \aleph_0)$ as well, and hence by the results of [3] (see also [1] and [2], where the consistency of $(\aleph_{\omega+1,\aleph_\omega}) \twoheadrightarrow (\aleph_1, \aleph_0)$ is proved using weaker large cardinal assumptions) $\mathrm{UB}_{\aleph_\omega}$ can fail. We also need the following notion.

Definition 2.4. An uncountable cardinal κ is said to be Jonsson, if for every function $f : [\kappa]^{<\omega} \to \kappa$ there exists a set $H \subseteq \kappa$ of order type κ such that for each n, $f''[H]^n \neq \kappa$.

§ 3. UB_λ CAN FAIL AT SINGULAR CARDINALS

In this section we prove the following theorem which answers Usuba's question 1.1.

Theorem 3.1. UB_λ *fails if at least one of the following hold:*

(a) $\lambda = \aleph_\omega$ *and the Chang's transfer principle* $(\lambda^+, \lambda) \twoheadrightarrow (\aleph_1, \aleph_0)$ *holds,*

(b) $\lambda > \mu \geq \mathrm{cf}(\mu)$ *are such that* $(\lambda^+, \lambda) \twoheadrightarrow (\mu^+, \mu)$ *holds,*

(c) $\lambda > \mu \geq \mathrm{cf}(\lambda)$ *and for every model* M *with universe* λ^+ *and vocabulary of cardinality* $\mathrm{cf}(\lambda)$*, we can find a sequence* $\vec{\alpha} = \langle \alpha_i : i < \mu^+ \rangle$ *of ordinals less than* λ *such that*

$$S^M_{\vec{\alpha}} = \{i < \mu^+ : cl(\{\alpha_i\}, M) \cap \lambda \subseteq cl(\{\alpha_j : j < i\}, M)\}$$

is stationary in μ^+

(d) *there exists* χ *with* $\lambda > \chi = \mathrm{cf}(\chi) > \mathrm{cf}(\lambda)$ *such that for every model* M *with universe* λ^+ *and vocabulary of cardinality* $\mathrm{cf}(\lambda)$*, we can find a sequence* $\vec{\alpha} = \langle \alpha_i : i < \chi \rangle$ *of ordinals less than* λ *such that*

$$S^M_{\vec{\alpha}} = \{i < \chi : cl(\{\alpha_i\}, M) \cap \lambda \subseteq cl(\{\alpha_j : j < i\}, M)\}$$

is stationary in χ

(e) *there is no sequence* $\vec{X} = \langle U_i : i < \lambda^+ \rangle$ *of cofinal subsets* $U_i \subseteq [\lambda]^{\mathrm{cf}(\lambda)}$ *such that for every* $i < \lambda^+$ *there is a sequence* $\vec{X}_i = \langle (\alpha_{i,j}, \beta_{i,j}) : j < i \rangle$ *such that:*

- $\vec{X}_i$ *has no repetition,*

- $\alpha_{i,j} \in U_i$,
- $\beta_{i,j} \in U_j$,

Furthermore, the statement (e) is equivalent to $\neg\mathrm{UB}_\lambda$, *provided that* $\mathrm{cf}(\lambda)$ *is not a Jonsson cardinal.*

Remark 3.2. Given a model M as in (c) and a subset A of M, by $cl(A, M)$ we mean the least substructure of M which includes A as a subset.

Proof. We prove the theorem is a sequence of claims. First note that (a) is a special case of (b) and that (c) implies (d).

Claim 3.3. *(b) implies (c)*

Proof. Let M be a model with universe λ^+ and vocabulary of cardinality at most $\mathrm{cf}(\lambda)$. By (b), there exists an elementary submodel $N \prec M$ such that $||N|| = \mu^+$ and $|N \cap \lambda| = \mu$. Let $\vec{\alpha} = \langle \alpha_i : i < \mu^+ \rangle$ list in increasing order the first μ^+ elements of N. So for $i < \mu^+$ we have

$$cl(\{\alpha_i\}, M) \cap \lambda \subseteq N \cap \lambda,$$

and since $N \cap \lambda$ has size μ, we can find some $i(*) < \mu^+$ such that

$$\forall i < \mu^+,\ cl(\{\alpha_i\}, M) \cap \lambda \subseteq \bigcup_{j < i(*)} cl(\{\alpha_i\}, N).$$

Hence the set $S^M_{\vec{\alpha}}$ includes $[i(*), \mu^+)$ and so is stationary in μ^+, as requested. □

Claim 3.4. *(d) implies (e).*

Proof. Suppose towards a contradiction that (d) holds but (e) fails. As (e) fails, we can find sequences $\vec{X} = \langle U_i : i < \lambda^+ \rangle$ and $\vec{X}_i = \langle (\alpha_{i,j}, \beta_{i,j}) : j < i \rangle$ as in clause (e). Let M be a model in a vocabulary $\mathcal{L}$ such that:

(1) $|\mathcal{L}| = \mathrm{cf}(\lambda)$,
(2) M has universe λ^+,
(3) $M = (\lambda^+, \langle \tau_i^M : i < \mathrm{cf}(\lambda) \rangle, F^M, G^M, H^M)$, where
  (a) $\tau_i^M = i$,

(b) F^M is a unary function which maps $\mathrm{cf}(\lambda)$ to an unbounded subset of λ,

(c) G^M is a 3-place function such that for all $j < i$, $G^M(i, \alpha_{j,i}, \beta_{j,i}) = j$,

(d) H^M is a 2-place function such that for all i, $U_i = \langle H^M(i, \alpha) : \alpha < \mathrm{cf}(\lambda)\rangle$.

Now by (d) applied to the model M, we can find a sequence $\vec{\zeta} = \langle \zeta_i : i < \chi\rangle$ of ordinals less than λ such that the set $S^M_{\vec{\zeta}}$ is stationary in χ. Let $\zeta = \sup\limits_{i<\chi} \zeta_i$. Consider the sequence $\vec{X}_\zeta = \langle (\alpha_{\zeta,\xi}, \beta_{\zeta,\xi}) : \xi < \zeta\rangle$.

As $\chi = \mathrm{cf}(\chi) > \mathrm{cf}(\lambda) \geq |cl(\{\zeta\}, M)|$, for some β_* and some stationary set $U \subseteq S^M_{\vec{\zeta}}$,

$$i \in U \implies \beta_{\zeta_i, \zeta} = \beta_*.$$

For $i < \chi$ let

$$W_i = \{\lambda \cap cl(\{\zeta_j\}, M) : j < i\}.$$

So $\langle W_i : i < \chi$ is a $\subseteq$-increasing and continuous sequence of sets each of cardinality $< \chi$. We get some $i_i < i_2$ in U such that $\alpha_{\zeta_{i_1}, \zeta} = \alpha_{\zeta_{i_2}, \zeta}$, and we get a contradiction to the choice of $\vec{X}_\zeta$. □

Claim 3.5. *(e) implies* $\neg\mathrm{UB}_\lambda$.

Proof. Suppose not. Thus we can assume that both (e) and UB_λ hold. Let $f : [\lambda^+]^{<\omega} \to \lambda^+$ witness UB_λ. Choose a vocabulary $\mathcal{L}$ of size $\mathrm{cf}(\lambda)$ and an $\mathcal{L}$-model M such that:

(1) M has universe λ^+,

(2) $M = (\lambda^+, \langle \tau^M_i : i < \mathrm{cf}(\lambda)\rangle, F^M, \langle F^M_n : n < \omega\rangle, p^M)$

(a) $\tau^M_i = i$,

(b) F^M is a unary function which maps $\mathrm{cf}(\lambda)$ to an unbounded subset of λ,

(c) F^M_n is an n-ary function such that

$$F^M_n(\alpha_0, \cdots, \alpha_{n-1}) = f(\{\alpha_0, \cdots, \alpha_{n-1}\}),$$

(d) p^M is a pairing function on λ^+, For $\alpha < \lambda$, set $N_\alpha = cl(\{\alpha\}, M)$.

$(*)_1$ N_α belongs to $[\lambda^+]^{\mathrm{cf}(\lambda)}$ and it contains an unbounded subset of λ.

Proof. As $\mathcal{L}$ has size $\mathrm{cf}(\lambda)$, so $|N_\alpha| \leq \mathrm{cf}(\lambda)$. On the other hand by clause (2)(a), $\mathrm{cf}(\lambda) \subseteq N_\alpha$ and hence N_α belongs to $[\lambda^+]^{\mathrm{cf}(\lambda)}$. As $N_\alpha \supseteq \mathrm{cf}(\lambda)$, $\mathrm{range}(F^M) \subseteq N_\alpha$, and thus by clause (2)(b), N_α contains an unbounded subset of λ. □

Let

$$E = \{\delta \in (\lambda, \lambda^+) : \delta = cl(\delta, M)\}.$$

E is clearly a club of λ^+ and $E \cap \lambda = \emptyset$.

$(*)_2$ Suppose $\xi < \zeta$ are in E. Then

$$\xi \in cl(\{\xi\} \cup (N_\xi \cap \lambda) \cup (N_\zeta \cap \lambda), M).$$

Proof. Let $V = (N_\xi \cap \lambda) \cup (N_\zeta \cap \lambda)$, $X = cl(\{\xi\} \cup V, M)$ and $Y = cl(\{\zeta\} \cup V, M)$. Thus X and Y are f-closed subsets of $\lambda^+, \xi \in X$ and $\zeta \in Y$. Furthermore, by the choice of $\zeta \in E$, we have $\zeta \notin X$. By UB_λ it follows that $X \subseteq Y$, and hence $\xi \in Y$, which gives the result. □

Let $\langle \sigma_i(x_0, \cdots, x_{n(i)-1}) : i < \mathrm{cf}(\lambda)\rangle$ list all terms of $\mathcal{L}$. By $(*)_2$, for each $\xi < \zeta$ from E, we can choose some $i(\xi, \zeta) < \mathrm{cf}(\lambda)$ together with sequences $\vec{a}_{\xi,\zeta} \in (N_\xi \cap \lambda)^{<\omega}$ and $\vec{b}_{\xi,\zeta} \in (N_\zeta \cap \lambda)^{<\omega}$ such that

$$\xi = \sigma_{i(\xi,\zeta)}(\xi, \vec{a}_{\xi,\zeta}, \vec{b}_{\xi,\zeta}).$$

For $\xi \in E$ set $u_\xi = N_\xi \cap \min(E)$. It follows that $u_\xi = cl(u_\xi, M)$. For $\xi < \zeta$ use the pairing function p^M to find $\alpha_{\xi,\zeta}$ and $\beta_{\xi,\zeta}$ such that $\alpha_{\xi,\zeta}$ codes $\langle i(\xi, \zeta)\rangle^\frown \vec{a}_{\xi,\zeta}$ and $\beta_{\xi,\zeta}$ codes $\langle i(\xi, \zeta)\rangle^\frown \vec{b}_{\xi,\zeta}$.

Now the sequences $\vec{X} = \langle U_x i : \xi \in E\rangle$ and $\langle\langle(\alpha_{\xi,\zeta}, \beta_{\xi,\zeta}) : \xi \in \zeta \cap E\rangle : \zeta \in E$ witness the failure of (e). We get a contradiction and the claim follows. □

Thus so far we have shown that

$$(a) \implies (b) \implies (c) \implies (d) \implies (e) \implies \neg\mathrm{UB}_\lambda.$$

Now suppose that in (e), $\mathrm{cf}(\lambda)$ is not a Jonsson cardinal and we show that $\neg\mathrm{UB}_\lambda$ also implies (e). Thus suppose towards a contradiction that (e) fails and let $\vec{X}$

and $\langle \vec{X}_i : i < \lambda^+ \rangle$ as in clause (e) witness this failure. Let $\langle \lambda_i : i < \mathrm{cf}(\lambda) \rangle$ be an increasing sequence cofinal in λ and define the function $c : \lambda \to \mathrm{cf}(\lambda)$ as

$$c(\alpha) = \min\{i < \mathrm{cf}(\lambda) : \alpha < \kappa_i\}.$$

For $\xi < \lambda^+$ let $\langle \gamma_{\xi,i} : i < \mathrm{cf}(\lambda) \rangle$ enumerate U_ξ. Let $f : [\lambda^+]^{<\omega} \to \lambda^+$ be such that:

(1) if $\xi < \zeta < \lambda^+$, then

$$f(\alpha_{\xi,\zeta}, \beta_{\xi,\zeta}, \xi) = \xi,$$

(2) if $\zeta < \lambda^+$ and $\xi < \mathrm{cf}(\lambda)$, then for arbitrary large $j < \mathrm{cf}(\lambda)$, we have

$$\sup_{i<j} \lambda_i < \alpha \implies f(\alpha, \zeta) = \gamma_{\zeta,j}.$$

(3) if $A \in [\mathrm{cf}(\lambda)]^{\mathrm{cf}(\lambda)}$, $c(\alpha_i) = i$ for $i \in A$ and $j < \mathrm{cf}(\lambda)$, then for some n and some sequence $\vec{\xi} = \langle \xi_0, \cdots, \xi_{n-1} \rangle \in A^n$, we have

$$j = c(f(\alpha_{\xi_0}, \cdots, \alpha_{\xi_{n-1}})).$$

Since $\mathrm{cf}(\lambda)$ is not a Jonsson cardinal, we can define such a function f[2]. Then f witnesses that UB_λ holds, which contradicts our assumption. This completes the proof of the theorem. □

Remark 3.6. The above proof shows that the following are equivalent:

(1) clause (e) of Theorem 3.1,

(2) for each model M with universe λ^+ and vocabulary of cardinality $\mathrm{cf}(\lambda)$, there are substructures N_0, N_1 of M such that $N_0 \cap \lambda = N_1 \cap \lambda$, $N_0 \nsubseteq N_1$ and $N_1 \nsubseteq N_0$.

As we noticed earlier, it is consistent relative to the existence of large cardinals that Chang's transfer principle $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ holds. Hence by our main theorem, we have the following corollary.

Corollary 3.7. *It is consistent, relative to the existence of large cardinals, that* $\mathrm{UB}_{\aleph_\omega}$ *fails.*

[2]this assumption is used to guarantee clause (3) in definition of f holds.

References

[1] Eskew, Monroe; Hayut, Yair; On the consistency of local and global versions of Chang's conjecture. Trans. Amer. Math. Soc. 370 (2018), no. 4, 2879-2905. Erratum: Trans. Amer. Math. Soc. 374 (2021), no. 1, 753.

[2] Hayut, Yair; Magidor-Malitz reflection. Arch. Math. Logic 56 (2017), no. 3-4, 253-272.

[3] Levinski, Jean-Pierre; Magidor, Menachem; Shelah, Saharon; Chang's conjecture for $\aleph_\omega$. Israel J. Math. 69 (1990), no. 2, 161-172.

[4] Usuba, Toshimichi; New combinatorial principle on singular cardinals and normal ideals. MLQ Math. Log. Q. 64 (2018), no. 4-5, 395-408.

Mohammad Golshani, School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395–5746, Tehran, Iran.

E-mail address: golshani.m@gmail.com

Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem, Jerusalem, 91904, Israel, and, Department of Mathematics, Hill Center - Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019 USA

E-mail address: shelah@math.huji.ac.il

URL: http://shelah.logic.at