Pacific Journal of Mathematics

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Volume 311 No. 2 April 2021

PACIFIC JOURNAL OF MATHEMATICS
Vol. 311, No. 2, 2021
https://doi.org/10.2140/pjm.2021.311.423

CHARACTERIZING THE SPECTRA OF CARDINALITIES OF BRANCHES OF KUREPA TREES

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We give a complete characterization of the sets of cardinals that in a suitable forcing extension can be the Kurepa spectrum, that is, the set of cardinalities of branches of Kurepa trees. This answers a question of Poór.

1. Introduction

A tree is a Kurepa tree if it is of height ω_1, each of its levels is countable, and it has more than ω_1-many cofinal (that is of order type ω_1) branches. In this paper we study the possible values of the branch spectrum of Kurepa trees, i.e., the set

$$\mathrm{Sp}_{\omega_1} = \{\lambda : \text{there exists a Kurepa tree } T \text{ such that } |\mathcal{B}(T)| = \lambda\} \subseteq [\omega_2, 2^{\omega_1}]$$

(where $\mathcal{B}(T)$ stands for the set of cofinal branches of T).

The spectrum is related to the model theoretical spectrum of maximal models of $\mathcal{L}_{\omega_1,\omega}$-sentences [Sinapova and Souldatos 2020]. Also canonical topological and combinatorial structures are associated with branches of Kurepa trees possessing a remarkably wide range of nonreflecting properties [Koszmider 2005]. For higher Kurepa trees (of weakly compact height) the consistency strength of certain types of the branch spectrum was studied in [Hayut and Müller 2019].

It was first shown by Silver [1971] that the Kurepa hypothesis (i.e., the existence of a Kurepa tree) is independent (also see [Kunen 1983, Chapter VIII, §3]). Moreover the nonexistence of Kurepa trees is equiconsistent with the existence of an inaccessible cardinal [Kunen 1983, Chapter VII, Example B8].

Questions about the possible values of the spectrum were addressed by Jin and Shelah [1992]. They proved (assuming an inaccessible cardinal) that consistently there are only Kurepa trees with ω_3-many cofinal branches while $2^{\omega_1} = \omega_4$.

Poór was supported by the National Research, Development and Innovation Office — NKFIH, grant nos. 124749, 129211. Shelah was partially supported by the European Research Council (ERC) grant 338821, and by the Israel Science Foundation grant 1838/19. Paper 1189 on Shelah's list. Supported by the ÚNKP-19-3 New National Excellence Program of the Ministry of Human Capacities. Poór is the corresponding author.

MSC2010: primary 03E35; secondary 03E05, 03E45.

Keywords: Kurepa tree, constructible universe, cardinal spectra.

Building on ideas of Jin and Shelah, Poór [2017] provided a sufficient condition for a set to be equal to Sp_{ω_1} in a forcing extension. Formally, it was shown that if $\boldsymbol{GCH}$ holds, and $0, 1 \notin S$ is a set of ordinals such that S satisfies either Case A:

(i) $2 \in S$,

(ii) $\{\sup C : C \in [S]^{\leq\omega_1}\} \subseteq S$,

(iii) (for all $\alpha \in S$) : $(\omega \leq \mathrm{cf}(\alpha) < \omega_2) \to (\alpha + 1 \in S)$,

or Case B:

(i) there exists an inaccessible κ,

(ii) $\{\sup C : C \in [S]^{<\kappa}\} \subseteq S$,

(iii) (for all $\alpha \in S$) : $(\omega \leq \mathrm{cf}(\alpha) < \kappa) \to (\alpha + 1 \in S)$,

then in a forcing extension we have $\{\alpha : \aleph_\alpha \in \mathrm{Sp}_{\omega_1}\} = S$ (cardinals are only collapsed in Case B, from (ω_1, κ)). It can be easily seen that if $\mathrm{cf}(\mu) = \omega$ and $(\mathrm{Sp}_{\omega_1} \cap \mu)$ is cofinal in μ, then there exists a Kurepa tree with μ-many branches, as the union of countably many Kurepa trees is a Kurepa tree, and it is not difficult to see that the same holds if $\mathrm{cf}(\mu) = \omega_1$, therefore Case A(ii) and Case B(ii) are in fact necessary. However, it remained a question whether the last clauses can be dropped.

In this paper as the main result we prove that assuming $\boldsymbol{CH} + (2^{\omega_1} = \omega_2)$ conditions (i), (ii) (in both cases) are in fact sufficient by forcing a model of $\{\alpha : \aleph_\alpha \in \mathrm{Sp}_{\omega_1}\} = S$. Also, we can arbitrarily prescribe 2^{ω_1} to be any cardinal $\lambda \geq \sup(\mathrm{Sp}_{\omega_1})$ if in Case A the equality $\lambda^{<\omega_2} = \lambda$ holds, or in Case B $\lambda^{<\kappa} = \lambda$ holds too.

Moreover, when we do not want Kurepa trees with ω_2-many cofinal branches, we prove that the inaccessible is necessary by verifying that if ω_2 is a successor in L, then there exists a Kurepa tree with only ω_2-many cofinal branches in V. It was known that these assumptions imply that there exists a Kurepa tree even in $L[A]$ for some $A \subseteq \omega_1$ [Kunen 1983, Chapter VII, Example B8] (possibly having more than ω_2-many cofinal branches in V). Our proof not only utilizes countable elementary submodels of initial segments of $L[A]$, but the nodes of the tree are such elementary submodels, and each cofinal branch uniquely corresponds to an initial segment of $L[A]$.

2. Preliminaries and notations

Under ordinals we always mean Neumann ordinals. For a fixed cardinal χ we will use the notation $\mathcal{H}(\chi)$ for the collection of sets of hereditary size less than χ, i.e.,

$$\mathcal{H}(\chi) = \{x : |\mathrm{trcl}(x)| < \chi\},$$

where $\mathrm{trcl}(x)$ stands for the transitive closure of x. In terms of forcing we will use the notations of [Kunen 2011], e.g., $p \le q$ means that p is the stronger. If it is clear from the context and won't make any confusion we will identify the set x in the ground model with its canonical name $\check{x}$. For a set A the symbol $\mathcal{P}(A)$ denotes the powerset of A, and $[A]^\lambda$ stands for $\{X \in \mathcal{P}(A) : |X| = \lambda\}$. For a function $s = \{\langle \beta, s(\beta)\rangle : \beta \in \mathrm{dom}(s)\}$ we will also use the following notation and refer to s as

$$\langle s_\beta : \beta \in \mathrm{dom}(s)\rangle.$$

Under a sequence we mean a function defined on a set of ordinals. For sequences s, t the relation $s = t \restriction \mathrm{dom}(s)$ (or equivalently $s \subseteq t$) will be also denoted by $s \lhd t$.

Definition 2.1. A tree $\langle T, \prec_T\rangle$ is a partially ordered set (poset) in which for each $x \in T$ the set

$$T_{\prec x} = \{y \in T : y \prec_T x\}$$

is well ordered by $\prec_T$.

Definition 2.2. The height of x in the tree T is the order type of $T_{\prec x}$

$$\mathrm{ht}(x) = \mathrm{otp}(T_{\prec x}).$$

Definition 2.3. For each ordinal α the restriction of T to α is

$$T_{<\alpha} = \{t \in T : \mathrm{ht}(t) < \alpha\}.$$

Definition 2.4. The height of the tree T (in symbols $\mathrm{ht}(T)$) is the least β such that

$$\text{there does not exist } t \in T : \mathrm{ht}(t) = \beta.$$

We will need the following lemma [Kunen 1983, Chapter II, Theorem 1.6.] which we will refer to as the Δ-system lemma.

Lemma 2.5. *Let κ be an infinite cardinal, let $\theta > \kappa$ be regular, and satisfy for all $\alpha < \theta$ ($|\alpha^{<\kappa}| < \theta$). Assume that $|\mathcal{A}| \ge \theta$, and for all $x \in \mathcal{A}$ ($|x| < \kappa$). Then there is a $\mathcal{D} \subseteq \mathcal{A}$, such that $|\mathcal{D}| = \theta$, and $\mathcal{D}$ forms a Δ-system, i.e., there is a kernel set y such that*

$$\textit{for all } x \neq x' \in \mathcal{D} : x \cap x' = y.$$

3. The forcing

Now we can state our main theorem.

Theorem 3.1. *Let $S_\bullet$ be a set of infinite cardinals such that $\omega, \omega_1 \notin S_\bullet$. Assume* ***CH***, *and that either Case* 1:

(i) $\omega_2 \in S_\bullet$,

(ii) $2^{\omega_1} = \omega_2$,

(iii) $\{\sup C : C \in [S_\bullet]^{<\omega_2}\} \subseteq S_\bullet$,

or Case 2:

(i) *there exists an inaccessible* κ *such that* $S_\bullet \cap (\omega_1, \kappa) = \varnothing$,

(ii) $\{\sup C : C \in [S_\bullet]^{<\kappa}\} \subseteq S_\bullet$.

Then there exists a forcing extension $V^{\mathbb{P}}$ *such that*

$$V^{\mathbb{P}} \models S_\bullet = \mathrm{Sp}_{\omega_1}, \text{ where } \mathbb{P} \text{ only collapses cardinals in } (\omega_1, \kappa) \text{ in Case } 2.$$

The key will be Lemma 3.27. After Lemma 3.30 we will put together the pieces in a short argument. Before these we need some preparation.

Definition 3.2. In Case 1 (i.e., $\omega_2 \in S_\bullet$) define the cardinal κ to be ω_2.

Corollary 3.3. *No cardinal* $\mu \notin (\omega_1, \kappa)$ *is collapsed.*

Theorem 3.4. *Suppose that all conditions from Theorem 3.1 hold, and* κ *is defined in Definition 3.2. Assume further that* λ *is a cardinal which is an upper bound of* $S_\bullet$ *such that* $\lambda^{<\kappa} = \lambda$ *(thus* $\mathrm{cf}(\lambda) \geq \kappa$*). Then there exists a forcing extension* $V^{\mathbb{P}}$ *with*

$$V^{\mathbb{P}} \models (S_\bullet = \{\mu : \text{there exists a Kurepa tree } T \text{ such that } |\mathcal{B}(T)| = \mu\}) \wedge (2^{\omega_1} = \lambda).$$

Definition 3.5. Let $S_\bullet^+ = S_\bullet \cup \{\kappa, \lambda\}$.

Definition 3.6. For a cardinal $\theta \in S_\bullet$ let $\mathbb{Q}_\theta$ be the following notion of forcing. The triplet $p = \langle T_p, u_p, \bar{\eta}_p \rangle$ is an element of $\mathbb{Q}_\theta$ if and only if

(a) T_p is a countable tree of height δ for some $\delta < \omega_1$ on the underlying set $\omega \cdot \delta$, where the β-th level is $[\omega \cdot \beta, \omega \cdot (\beta+1))$, i.e., $T_{p,\leq\beta} \setminus T_{p,<\beta} = [\omega\cdot\beta, \omega\cdot(\beta+1))$ for each $\beta < \delta$,

(b) for each $t \in T_p$ and $\beta < \delta$ there exists $t' \in T_p \setminus T_{p,<\beta}$ such that $t \prec_{T_p} t'$,

(c) $u_p \in [\theta]^{\leq\omega}$,

(d) $\bar{\eta}_p = \langle \eta_{p,\alpha} : \alpha \in u_p \rangle$, where $\eta_{p,\alpha} \subseteq T_p$ is a branch in $T_{p,<\gamma}$ for some $\gamma \in \{\beta + 1 : \beta < \delta = \mathrm{ht}(T_p)\}$ (we do it for a technical reason, we also could have stored only the maximal element instead of a chain with a maximal element).

Then $\mathbb{Q}_\theta$ is a poset with the obvious order, i.e., $q \leq p$, if T_q is an end-extension of T_p, formally $T_{q,<\mathrm{ht}(T_p)} = T_p$, and for each $\alpha \in u_p$ the inclusion $\eta_{p,\alpha} \subseteq \eta_{q,\alpha}$ holds.

Let $\underset{\sim}{T}_\theta$, $\underset{\sim}{\bar{\eta}}_\theta$ be the names for the generic tree and sequence, i.e., denoting the generic filter by $\boldsymbol{G}_\theta$

$$1_{\mathbb{Q}_\theta} \Vdash \underset{\sim}{T}_\theta = \bigcup \{T_p : p \in \boldsymbol{G}_\theta\} \quad \text{and} \quad 1_{\mathbb{Q}_\theta} \Vdash \underset{\sim}{\bar{\eta}}_\theta = \left\langle \underset{\sim}{\eta}_{\theta,\alpha} = \bigcup \{\eta_{p,\alpha} : p \in \boldsymbol{G}_\theta\} : \alpha \in \theta \right\rangle.$$

Definition 3.7. For a cardinal $\theta \in S_\bullet$ let $\mathbb{Q}^*_\theta \subseteq \mathbb{Q}_\theta$ be the following subposet:

$$p \in \mathbb{Q}^*_\theta \text{ if and only if } \operatorname{ht}(T_p) \text{ is a successor, and} \qquad (\text{for all } \alpha \in u_p) : \eta_{p,\alpha} \text{ is a branch through } T_p.$$

Definition 3.8. If $\lambda \notin S_\bullet$ then let $\mathbb{Q}_\lambda$ be the countable supported product of $\langle {}^{<\omega_1}2, \lhd\rangle$ of length λ, i.e.,

$$\mathbb{Q}_\lambda = \{p = \langle \eta_\alpha : \alpha \in u_p\rangle : (\text{for all } \alpha \in u_p)\eta_\alpha \in {}^{<\omega_1} 2 \text{ for some } u_p \in [\lambda]^{\leq\omega}\}.$$

Definition 3.9. If $\kappa \notin S_\bullet$ (and then $\kappa > \omega_2^V$ is inaccessible), then let $\mathbb{Q}_\kappa$ be the countable supported product of $\langle {}^{<\omega_1}\gamma, \lhd\rangle$ $(\gamma < \kappa)$, a forcing which collapses each cardinal in (ω_1, κ):

$$\mathbb{Q}_\kappa = \{p = \langle \eta_\alpha : \alpha \in u_p\rangle : (\text{for all } \alpha \in u_p)\eta_\alpha \in {}^{<\omega_1} \alpha \text{ for some } u_p \in [\kappa]^{\leq\omega}\}.$$

Definition 3.10. We define the posets which we will need later.

(1) For $S \subseteq S^+_\bullet$ let $\mathbb{P}_S$ be the countable supported product of $\mathbb{Q}_\theta$ $(\theta \in S)$, i.e.,

$$\mathbb{P}_S = \{p \text{ is a function} : \operatorname{dom}(p) \in [S]^{\leq\omega} \wedge (\text{for all } \theta \in \operatorname{dom}(p)\, p(\theta) \in \mathbb{Q}_\theta)\}.$$

With a slight abuse of notation for $p \in \mathbb{P}_S$ and $\theta \in S \setminus \operatorname{dom}(p)$ we will mean $1_{\mathbb{Q}_\theta}$ under $p(\theta)$.

(2) For $\theta \in S^+_\bullet$, $U \subseteq \theta$ define its restriction from θ to U, i.e.,

$$\mathbb{Q}_{\theta,U} = \{p \in \mathbb{Q} : u_p \subseteq U\}.$$

(3) For $S \subseteq S^+_\bullet$, $\overline{U} = \langle U_\theta : \theta \in S\rangle \in \prod_{\theta\in S} \mathcal{P}(\theta)$ we define $\mathbb{P}_{S,\overline{U}}$ to be $\mathbb{P}$ restriction to coordinates in U_θ, i.e.,

$$\mathbb{P}_{S,\overline{U}} = \{p \in \mathbb{P}_S : (\text{for all } \theta \in S)\, p(\theta) \in \mathbb{Q}_{\theta,U_\theta}\}.$$

(4) For $S, S' \subseteq S^+_\bullet$, $\overline{U} = \langle U_\theta : \theta \in S\rangle \in \prod_{\theta\in S} \mathcal{P}(\theta)$, $\overline{U}' = \langle U'_\theta : \theta \in S\rangle \in \prod_{\theta\in S'} \mathcal{P}(\theta)$ we define
- $\overline{U} + \overline{U}' = \langle U_\theta \cup U'_\theta : \theta \in S \cup S'\rangle$ (where for $\theta \in S' \setminus S$ under U_θ we mean the empty set, similarly for $\theta \in S \setminus S'$, U'_θ),
- $\overline{U} - \overline{U}' = \langle U_\theta \setminus U'_\theta : \theta \in S\rangle$ (here we also mean the empty set under U'_θ if $\theta \in S \setminus S'$),
- $\overline{\mathrm{id}}_S = \langle \theta : \theta \in S\rangle$,
- for the set X if $\overline{W}_\alpha \in \prod_{\theta\in S} \mathcal{P}(\theta)$ $(\alpha \in X)$ then

$$\sum_{\alpha\in X} \overline{W}_\alpha = \left\langle \bigcup_{\alpha\in X} (W_\alpha)_\theta : \theta \in S \right\rangle.$$

(5) Let $\mathbb{P} = \mathbb{P}_{S^+_\bullet}$.

(6) If $p_0, p_0, \dots, p_n \in \mathbb{P}$ let $\bigwedge_{i \le n} p_i$ denote the greatest lower bound if it exists.

(7) For $p \in \mathbb{P}$, and $S \subseteq S_\bullet^+$, $\bar{U} = \langle U_\theta : \theta \in S \rangle \in \prod_{\theta \in S} \mathcal{P}(\theta)$ define $p \restriction \bar{U} \in \mathbb{P}_S$ to be the following restriction of $p \restriction S$ in the obvious fashion

$$\text{for each } \theta \in S : (p \restriction \bar{U})(\theta) = \langle T_{p(\theta)}, u_{p_\theta} \cap U_\theta, \bar{\eta}_p \restriction U_\theta \rangle.$$

Definition 3.11. For $S \subseteq S_\bullet^+$ define the notion of forcing $\mathbb{P}^*$ ($\mathbb{P}_S^*$, $\mathbb{P}_{S,\bar{U}}^*$, resp.) to be the subposet of $\mathbb{P}$ ($\mathbb{P}_S$, $\mathbb{P}_{S,\bar{U}}$, resp.) consisting of elements p for that $p(\theta) \in \mathbb{Q}_\theta^*$ holds for each $\theta \in S_\bullet \cap \operatorname{supp}(p)$.

Remark 3.12. The notion of forcing $\mathbb{P}^*$ ($\mathbb{P}_S^*$, $\mathbb{P}_{S,\bar{U}}^*$, resp.) is a dense subposet of $\mathbb{P}$ ($\mathbb{P}_S$, $\mathbb{P}_{S,\bar{U}}$, resp.), therefore forcing with $\mathbb{P}^*$ ($\mathbb{P}_S^*$, $\mathbb{P}_{S,\bar{U}}^*$, resp.) yields the same extensions as forcing with $\mathbb{P}$ ($\mathbb{P}_S$, $\mathbb{P}_{S,\bar{U}}$, resp.).

Claim 3.13. Let $S \subseteq S_\bullet^+$, $\bar{U} = \langle U_\theta : \theta \in S \rangle$ be fixed. Then the poset $\mathbb{P}_{S,\bar{U}}$ has the κ-cc property.

Proof. Suppose that $\{p_\alpha : \alpha \in \kappa\} \subseteq \mathbb{P}_{S,\bar{U}}$ is an antichain. Working in V', applying the Δ-system lemma (Lemma 2.5) for the system $\{\operatorname{dom}(p_\alpha) : \alpha \in \kappa\}$ of countable sets ((1) from Definition 3.10), we obtain a set $A \in [\kappa]^\kappa$ such that the $\operatorname{dom}(p_\alpha)$ ($\alpha \in A$) form a Δ-system with kernel $K \subseteq S$. Since K is obviously countable, for each α we have that $\langle T_{p_\alpha(\theta)} : \theta \in K \rangle$ is a countable sequence of countable trees (by (a) from Definition 3.6). This means that by $\boldsymbol{CH}$ we can assume that

$$\langle T_{p_\alpha(\theta)} : \theta \in K \rangle = \langle T_{p_\beta(\theta)} : \theta \in K \rangle \quad (\text{for all } \alpha, \beta \in A). \tag{3-1}$$

Now applying the Δ-system lemma again for the system

$$U_\alpha = \bigcup_{\theta \in S} (\{\theta\} \times u_{p_\alpha(\theta)}) \quad (\alpha \in \kappa)$$

yields a set $A' \in [A]^\kappa$ such that the U_α ($\alpha \in A'$) form a Δ-system with kernel $I \subseteq \bigcup_{\theta \in S} \{\theta\} \times \theta$ (of course, in fact, $I \subseteq \bigcup_{\theta \in K} \{\theta\} \times \theta$). Now by (3-1) it suffices to prove that

(3-2) there exist $\alpha \neq \beta \in A'$ such that (for each $\langle \theta, \delta \rangle \in I) : \eta_{p_\alpha(\theta),\gamma} = \eta_{p_\beta(\theta),\gamma}$,

for which it is enough to prove

$$|\{\langle \eta_{p_\alpha(\theta),\gamma} : \langle \theta, \gamma \rangle \in I \rangle : \alpha \in A'\}| < \kappa. \tag{3-3}$$

Fix $\alpha \in A'$. Now for each $\langle \theta, \gamma \rangle \in I$, if $\theta \in S_\bullet$ then $\eta_{p_\alpha(\theta),\gamma} \in [\omega_1]^{<\omega_1}$ (a branch through $T_{p_\alpha(\theta)}$).

This means that (using that I is countable)

$$\{\langle \eta_{p_\alpha(\theta),\gamma} : \langle \theta, \gamma \rangle \in I, \theta \in S_\bullet \rangle : \alpha \in A'\} \subseteq \prod_{\langle \theta, \gamma \rangle \in I, \theta \in S_\bullet} [\omega_1]^{<\omega_1}, \tag{3-4}$$

which latter set is of size ω_1 by $\boldsymbol{CH}$. Second, if $\theta = \lambda \in (S^+_\bullet \setminus S_\bullet) \cap S$, then

$$\{\langle \eta_{p_\alpha(\theta),\gamma} : \langle \theta, \gamma \rangle \in I, \theta = \lambda \rangle : \alpha \in A'\} \subseteq \prod_{\langle \theta,\gamma\rangle \in I, \theta = \lambda}^{<\omega_1} 2.$$

Finally we have to consider the coordinate $\theta = \kappa$ if $\kappa \in S \setminus S_\bullet$. Then letting $\delta = \sup\{\gamma : \langle \kappa, \gamma \rangle \in I\}$ we have $\delta < \kappa$, because I is countable and κ is inaccessible. Then

$$\{\langle \eta_{p_\alpha(\kappa),\gamma} : \langle \kappa, \gamma \rangle \in I\} \subseteq \prod_{\langle \kappa,\gamma\rangle \in I}^{<\omega_1} \delta, \tag{3-5}$$

and since κ is inaccessible, this case $|\prod_{\langle \kappa,\gamma\rangle \in I}^{<\omega_1} \delta| < \kappa$. We obtain (using $\omega_1 < \kappa$) that

$$|\{\langle \eta_{p_\alpha(\theta),\gamma} : \langle \theta, \gamma \rangle \in I\}| \leq \omega_1 \cdot \omega_1 \cdot \left| \prod_{\langle \kappa,\gamma\rangle \in I}^{<\omega_1} \delta \right| < \kappa,$$

therefore (3-3) holds. □

Now we make the intuition behind the easy idea of first adding the trees and some branches, and then forcing over the extension precise.

Claim 3.14. For each $S \subseteq S^+_\bullet$, $\bar{U} = \langle U_\theta : \theta \in S \rangle$ we have

$$\mathbb{P}_{S,\bar{U}} \lessdot \mathbb{P}_S \lessdot \mathbb{P},$$

i.e., $\mathbb{P}_{S,\bar{U}}$ completely embeds into $\mathbb{P}_S$, which completely embeds into $\mathbb{P}$.

Proof. Since $\mathbb{P} \simeq \mathbb{P}_S \times \mathbb{P}_{S^+_\bullet \setminus S}$, it is enough to prove that $\mathbb{P}_{S,\bar{U}} \lessdot \mathbb{P}_S$.

Assume that $A \subseteq \mathbb{P}_{S,\bar{U}}$ is a maximal antichain in $\mathbb{P}_{S,\bar{U}}$, and let $p \in \mathbb{P}_S \setminus \mathbb{P}_{S,\bar{U}}$. Then there exists $a \in A$, $a' \in \mathbb{P}_{S,\bar{U}}$ such that $a' \leq a$, $a' \leq b \restriction \bar{U}$. But then it is straightforward to check that also a' and b have a common lower bound. □

Definition 3.15. Let $S \subseteq S_\bullet$, $\bar{U} = \langle U_\theta : \theta \in S \rangle$, $\theta_0 \in S$, $U'_{\theta_0} \subseteq \theta_0 \setminus U_{\theta_0}$. Then

$$\underset{\sim}{\mathbb{Q}}^\circ_{\theta_0, U'_{\theta_0}} = \underset{\sim}{\mathbb{Q}}^\circ_{(S,\bar{U}),\theta_0,U'_{\theta_0}}$$

denotes the $\mathbb{P}_{S,\bar{U}}$-name for a notion of forcing which adds the branches $\underset{\sim}{\eta}_{\theta_0,\alpha}$ $(\alpha \in U'_{\theta_0})$ to $\underset{\sim}{T}_{\theta_0}$ in the following way

$$1 \Vdash_{\mathbb{P}_{S,\bar{U}}} \underset{\sim}{\mathbb{Q}}^\circ_{\theta_0,U'_{\theta_0}} = \left\{ \begin{array}{l} p = \langle \bar{\eta}_p, u_p \rangle : (u_p \in [U'_{\theta_0}]^{\leq\omega}) \wedge (\bar{\eta}_p = \langle \eta_{p,\alpha} : \alpha \in u_p \rangle), \\ \text{such that each } \eta_{p,\alpha} \text{ is a branch of } \underset{\sim}{T}_{\theta_0,<\delta_\alpha} \\ \text{for some } \delta_\alpha \text{ in } \{\gamma + 1 : \gamma < \omega_1\} \end{array} \right\}.$$

If it is clear from the context we will use $\underset{\sim}{\mathbb{Q}}^\circ_{\theta_0,U'_{\theta_0}}$ not mentioning S and $\bar{U}$.

Definition 3.16. Let $S \subseteq S_\bullet$, $\overline{U} = \langle U_\theta : \theta \in S\rangle$, $\theta_0 \in S$. If $\theta \in S^+_\bullet \setminus S_\bullet$, and $U'_\theta \subseteq \theta \setminus U_\theta$, then define the $\mathbb{P}_{S,\overline{U}}$-name $\underset{\sim}{\mathbb{Q}}_{\theta,U'_\theta} = \underset{\sim}{\mathbb{Q}}^\circ_{\theta,U'_\theta}$ to be the name for $\mathbb{Q}_{\theta,U'_\theta}$.

Definition 3.17. Let $S \subseteq S^+_\bullet$, $\overline{U} = \langle U_\theta : \theta \in S\rangle$, $\overline{U}' = \langle U'_\theta : \theta \in S\rangle \in \prod_{\theta \in S} \mathcal{P}(\theta)$, where $U_\theta \cap U'_\theta = \varnothing$ for each $\theta \in S$. Then $\underset{\sim}{\mathbb{P}}^\circ_{\overline{U}'} = \underset{\sim}{\mathbb{P}}^\circ_{(S,\overline{U}),\overline{U}'}$ denotes the $\mathbb{P}_{S,\overline{U}}$-name for the countably supported product of $\underset{\sim}{\mathbb{Q}}^\circ_{\theta,U'_\theta}$ $(\theta \in S)$, i.e., a notion of forcing which adds the branches $\underset{\sim}{\eta}_{\theta,\alpha}$ $(\alpha \in U'_\theta)$ to $\underset{\sim}{T}_\theta$ for each $\theta \in S \setminus S_\bullet$, and the sequences $\underset{\sim}{\eta}_{\kappa,\alpha}$ $(\alpha \in U'_\kappa)$ if $\kappa \in S \setminus S_\bullet$, $\underset{\sim}{\eta}_{\lambda,\alpha}$ $(\alpha \in U'_\lambda)$ if $\lambda \in S \setminus S_\bullet$:

$$1 \Vdash_{\mathbb{P}_{S,\overline{U}}} \underset{\sim}{\mathbb{P}}^\circ_{\overline{U}'} = \{p \text{ is a function} : \mathrm{dom}(p) \in [S]^{\leq\omega} \wedge (\text{for all } \theta \in \mathrm{dom}(p)\, p(\theta) \in \underset{\sim}{\mathbb{Q}}^\circ_{\theta,U'_\theta})\}.$$

Again, as in Definition 3.15 if it does not cause any confusion we only use the notation $\underset{\sim}{\mathbb{P}}^\circ_{\overline{U}'}$ not mentioning S and $\overline{U}$.

The following claim is an easy observation.

Claim 3.18. If $\boldsymbol{G}$ is a $\mathbb{P}_{S,\overline{U}}$-generic filter over V (where $S \subseteq S^+_\bullet$, $\overline{U} = \langle U_\theta : \theta \in S\rangle$, $\overline{U}' = \langle U'_\theta : \theta \in S\rangle \in \prod_{\theta \in S} \mathcal{P}(\theta)$, and $U_\theta \cap U'_\theta = \varnothing$ for each $\theta \in S$), then with the notation from [Kunen 2011]

$$\mathbb{P}_{S,\overline{U}+\overline{U}'}/\boldsymbol{G} = \{p \in \mathbb{P}_{S,\overline{U}+\overline{U}'} : \text{for all } q \in \boldsymbol{G}\ p \not\perp q\},$$

the quotient poset $\mathbb{P}_{S,\overline{U}+\overline{U}'}/\boldsymbol{G}$ and the evaluation of $\underset{\sim}{\mathbb{P}}^\circ_{\overline{U}'}$ are isomorphic, i.e.,

$$V[\boldsymbol{G}] \models \underset{\sim}{\mathbb{P}}^\circ_{\overline{U}'}[\boldsymbol{G}] \simeq \mathbb{P}_{S,\overline{U}+\overline{U}'}/\boldsymbol{G}.$$

Since $\mathbb{P}_{S,\overline{U}}$ completely embeds into $\mathbb{P}_{S,\overline{U}+\overline{U}'}$ (by Claim 3.14), [Kunen 2011, Lemma V.4.45] (and [Kunen 2011, Lemma V.4.44.]) implies the following.

Claim 3.19. Let $S \subseteq S^+_\bullet$, $\overline{U} = \langle U_\theta : \theta \in S\rangle$, $\overline{U}' = \langle U'_\theta : \theta \in S\rangle \in \prod_{\theta \in S} \mathcal{P}(\theta)$, where $U_\theta \cap U'_\theta = \varnothing$ for each $\theta \in S$. Then the canonical embedding from $\mathbb{P}_{S,\overline{U}+\overline{U}'}$ to the iteration $\mathbb{P}_{S,\overline{U}} * (\mathbb{P}_{S,\overline{U}+\overline{U}'}/\boldsymbol{G})$ is a dense embedding.

Now putting together Claims 3.18 and 3.19 we have the following, meaning that instead of forcing with $\mathbb{P}_{S,\overline{U}+\overline{U}'}$ we can force with $\mathbb{P}_{S,\overline{U}}$ and then with (the evaluation of) $\underset{\sim}{\mathbb{P}}^\circ_{\overline{U}'}$.

Lemma 3.20. *Let $S \subseteq S^+_\bullet$, $\overline{U} = \langle U_\theta : \theta \in S\rangle$, $\overline{U}' = \langle U'_\theta : \theta \in S\rangle \in \prod_{\theta \in S} \mathcal{P}(\theta)$, where $U_\theta \cap U'_\theta = \varnothing$ for each $\theta \in S$. Then forcing with $\mathbb{P}_{S,\overline{U}+\overline{U}'}$ amounts to the same as forcing with $\mathbb{P}_{S,\overline{U}}$ and then with $\mathbb{P}_{S,\overline{U}+\overline{U}'}/\boldsymbol{G} \simeq \underset{\sim}{\mathbb{P}}^\circ_{\overline{U}'}$.*

Definition 3.21. If $S \subseteq S^+_\bullet$, $\overline{U} = \langle U_\theta : \theta \in S\rangle$, $\overline{U}' = \langle U'_\theta : \theta \in S\rangle \in \prod_{\theta \in S} \mathcal{P}(\theta)$. Now if $\boldsymbol{G}$ is generic over $\mathbb{P} = \mathbb{P}_{S^+_\bullet}$ then we define

- $\boldsymbol{G}_S = \boldsymbol{G} \cap \mathbb{P}_S$,

- $\boldsymbol{G}_{S,\overline{U}} = \boldsymbol{G} \cap \mathbb{P}_{S,\overline{U}}$,
- and $\boldsymbol{G}^{\circ}_{\overline{U}'} \subseteq \mathbb{P}^{\circ}_{\overline{U}'}[\boldsymbol{G}_{S,\overline{U}}] \in V[\boldsymbol{G}_{S,\overline{U}}]$ to be the filter given by the canonical mapping from Claims 3.18, 3.19.

The following are basic observations. Roughly speaking, we isolate a dense ω_1-closed subset of a two-step iteration similarly as in [Kunen 1978].

Claim 3.22. $\mathbb{P}^*$ (and in general each $\mathbb{P}^*_{S,\overline{U}}$) is ω_1-closed, i.e., for each decreasing sequence of type ω has a lower bound. In particular if $\boldsymbol{G}^* \subseteq \mathbb{P}^*$, (or in general $\boldsymbol{G}^*_{S,\overline{U}} \subseteq \mathbb{P}^*_{S,\overline{U}}$) is generic over V, then there is no new sequence of ordinals of type ω.

The last claim and Remark 3.12 obviously implies the following.

Claim 3.23. Forcing with $\mathbb{P}$ (or $\mathbb{P}_{S,\overline{U}}$) doesn't add new sequence of ordinals of type ω, and for a given generic filter $\boldsymbol{G} \subseteq \mathbb{P}$

$$\mathcal{H}(\omega_1)^V = \mathcal{H}(\omega_1)^{V[\boldsymbol{G}]} = \mathcal{H}^{V[\boldsymbol{G}_{S,\overline{U}}]}.$$

Lemma 3.24. *Let $\boldsymbol{G} \subseteq \mathbb{P}_{S,\overline{U}}$ generic over V, $B \in V[\boldsymbol{G}]$ where $B \subseteq \mathcal{H}(\omega_1)$. Then* (*in V*) *there exist $S_* \subseteq S$, $|S_*| < \kappa$ and $\overline{W}_* = \langle W^*_\gamma : \gamma \in S_* \rangle \in \prod_{\gamma \in S_*}[U_\gamma]^{<\kappa}$, such that $B \in V[\boldsymbol{G}_{S_*,\overline{W}_*}]$.*

Problem 3.25. Choose $p \in \boldsymbol{G}$ forcing that $B \subseteq \mathcal{H}(\omega_1)$, and a nice $\mathbb{P}_{S,\overline{U}}$-name for B, obtaining for each $x \in \mathcal{H}(\omega_1)$ an antichain $A_x \subseteq \mathbb{P}_{S,\overline{U}}$ deciding about $x \in B$. Then by κ-cc we have that each $|A_x| < \kappa$, the set $S_* = \bigcup_{x \in \mathcal{H}(\omega_1)} \bigcup_{a \in A_x} \mathrm{dom}(a)$ is of size less than κ (as κ is either inaccessible, or ω_2). Also for $\theta \in S_*$ the set $W^*_\theta = \bigcup_{x \in \mathcal{H}(\omega_1)} \bigcup_{a \in A_x} u_{a(\theta)}$ is smaller that κ. Now it is easy to see that $\overline{W}_* = \langle W^*_\gamma : \gamma \in S_* \rangle$ is as claimed.

Then the following immediately follows from the ω_1-closedness, and κ-cc.

Claim 3.26. Forcing with $\mathbb{P}$ doesn't collapse ω_1, and cardinals at least κ. Moreover, if $\boldsymbol{G} \subseteq \mathbb{P}$ is generic, then

$$V[\boldsymbol{G}] \models \text{“}\kappa = \omega_2\text{”}.$$

Lemma 3.27. *Let $T \in V[\boldsymbol{G}_{S,\overline{U}_*}]$ be a Kurepa tree, $S' \subseteq S$ ($S' \in V$). Then, if $b \in V[\boldsymbol{G}_{S,\overline{U}_*+\mathrm{id}_{S'}}]$ is a branch of T, then there exists a finite set $S'' \subseteq S'$, and $\overline{U}_\bullet = \langle U^\bullet_\theta : \theta \in S'' \rangle$ such that each $U^\bullet_\theta$ is finite, and b is in the model obtained by adding these finitely many $\eta_{\theta,\alpha}$ ($\theta \in S''$, $\alpha \in U^\bullet_\theta$) to $V[\boldsymbol{G}_{S,\overline{U}_*}]$, i.e.,*

$$b \in V[\boldsymbol{G}_{S,\overline{U}_*+\overline{U}_\bullet}].$$

Proof. Let $\dot{T} \in V$ be a $\mathbb{P}_{S,\overline{U}_*}$-name for T. Define

$$\mathbb{P}' = \mathbb{P}_{S,\overline{U}_*+\overline{\mathrm{id}}_{S'}}. \tag{3-6}$$

Suppose that $p_* \in \mathbb{P}'$ forces that $\dot{b} \in V$ is a $\mathbb{P}'$-name for a counterexample (i.e., forcing that for no such $\overline{U}_\bullet$ there exists a $\mathbb{P}_{\overline{U}_*+\overline{U}_\bullet}$-name $\dot{b}'$ — which is of course also a $\mathbb{P}'$-name — with $\dot{b}' = \dot{b}$). Let χ be large enough, and let $\langle N_0, \in\rangle \prec \langle \mathcal{H}(\chi), \in\rangle$ be countable such that $p_*, \dot{b}, \dot{T}, S, S', \overline{V}, \mathbb{P}_{S,\overline{U}_*} \in N_0$.

Let $\delta_\bullet = N_0 \cap \omega_1$. Define the countable set N_1 to be such that $N_0 \in N_1$, and $\langle N_1, \in\rangle \prec \langle \mathcal{H}(\chi), \in\rangle$. Let X be set of the indices of the new branches added to $\langle \underset{\sim}{T}_\theta : \theta \in S'\rangle$ by $\boldsymbol{G}_{S,\overline{U}_*+(\mathrm{id}_{S'})}$ that are in $V[\boldsymbol{G}_{S,\overline{U}_*+\overline{\mathrm{id}}_{S'}}] \setminus V[\boldsymbol{G}_{S,\overline{U}_*}]$, and belong to N_0, i.e.,

$$X = N_0 \cap \{\langle \theta, \alpha\rangle : (\theta \in S') \wedge (\alpha \in \theta \setminus U^*_\theta)\}. \tag{3-7}$$

We fix an enumeration of X and define also the sequence of the first n indices from this countable set, and as well for each n the one-length sequence consisting only the n-th, that is; let $\langle\langle \varrho_n, \xi_n\rangle : n \in \omega, n > 0\rangle$ enumerate X (starting from 1),

$$\begin{aligned} \overline{W}_n &= \langle W_{n,\theta} : \theta \in S' \cap N_0\rangle, \\ W_{n,\theta} &= \{\alpha : \langle \theta, \alpha\rangle = \langle \varrho_j, \xi_j\rangle \text{ for some } j \leq n\}, \\ \bar{w}_n &= \langle w_{n,\theta} : \theta \in S' \cap N_0\rangle, \\ w_{n,\theta} &= \begin{cases} \{\xi_n\} & \text{if } \theta = \varrho_n, \\ \varnothing & \text{otherwise.} \end{cases} \end{aligned} \tag{3-8}$$

Observe that if $p \in \mathbb{P} \cap N_0$, then each $\theta \in \mathrm{dom}(p)$ is an element of N_0 since $\mathrm{dom}(p)$ is countable (by Definition 3.10), and similarly $T_{p(\theta)}, u_{p(\theta)} \subseteq N_0$ (by Definitions 3.6–3.9).

Working in V we will construct an N_0-generic condition in $\mathbb{P}'$, which will derive us to a contradiction. It is enough to prove the following claim.

Claim 3.28. There exists a sequence $\langle \bar{p}_n : n \in \omega\rangle \in V$, $p'_0 \in \mathbb{P}_{S,\overline{U}*}$ and a sequence $\bar{q} = \langle q_n : n \in \omega\rangle$ such that the following holds:

($\boxplus_1$) $\bar{p}_0 = \langle p_{0,l} : l \in \omega\rangle$ is such that
- (a) $p_{0,0} = p_* \restriction \overline{U}_*$,
- (b) $p_{0,l} \in N_0 \cap \mathbb{P}_{S,\overline{U}_*}$ for each $l \in \omega$,
- (c) $\langle p_{0,l} : l \in \omega\rangle$ is $\leq_{\mathbb{P}}$-decreasing,
- (d) $\bar{p}_0 \in N_1$,
- (e) letting $\boldsymbol{G}_0 = \{p \in \mathbb{P}_{S,\overline{U}*} \cap N_0 : (\text{there exist } l)\, p \geq p_{0,l}\}$, the filter $\boldsymbol{G}_0$ is $\mathbb{P}_{S,\overline{U}_*}$-generic over N_0.

($\boxplus_2$) $p'_0 \in \mathbb{P}_{S,\overline{U}*}$ satisfies the following:
- (a) p'_0 is a lower bound of $p_{0,l}$ for each $l \in \omega$ (hence forces a value to $\underset{\sim}{T}_{\theta,<\delta_\bullet}$ for each $\theta \in S \cap N_0$),
- (b) p'_0 forces a value to $\underset{\sim}{T}_{\theta,\leq\delta_\bullet}$ for each $\theta \in S \cap N_0$ such that for every $\delta_\bullet$-branch B in $\underset{\sim}{T}_{\theta,<\delta_\bullet}$ the inclusion $B \in N_1$ implies that B has an upper bound in $\underset{\sim}{T}_{\theta,\leq\delta_\bullet}$,
- (c) p'_0 forces a value to $\dot{T}_{\leq\delta_\bullet}$.

($\boxplus_3$) for every $n > 0$ the sequence $\bar{p}_n = \langle p_{n,l} : l \in \omega \rangle$ has the following properties:

(a) for all $l \in \omega$ $p_{n,l} \in N_0 \cap \mathbb{P}_{S,\overline{U}_* + \bar{w}_n}$,

(b) $p_{n,l} \restriction \overline{U}_* \in \boldsymbol{G}_0$,

(c) $\langle p_{n,l} : l \in \omega \rangle$ is $\leq_{\mathbb{P}}$-decreasing,

(d) $\bar{p}_n \in N_1$,

(e) letting

$$\boldsymbol{G}_n = \left\{ p \in \mathbb{P}_{S,\overline{U}* + \overline{W}_n} \cap N_0 : (\text{there exist } l_0, l_1, \ldots, l_n) p \geq \bigwedge_{j=0}^{n} p_{j,l_j} \right\},$$

the filter $\boldsymbol{G}_n$ is $\mathbb{P}_{S,\overline{U}_* + \overline{W}_n}$-generic over N_0.

($\boxplus_4$) For the sequence $\bar{q} = \langle q_n : n \in \omega \rangle$:

(a) $q_n \in N_0 \cap \mathbb{P}_{S,\overline{U}_* + \overline{\mathrm{id}}_{S'}}$ for each $n \in \omega$,

(b) $q_0 = p_*$,

(c) $\langle q_n : n \in \omega \rangle$ is $\leq_{\mathbb{P}}$-decreasing,

(d) for all n: $q_n \restriction (\overline{U}_* + \overline{W}_n) \in \boldsymbol{G}_n$,

(e) let $\langle \dot{B}_n : n \in \omega \rangle$ enumerate the branches of $\dot{T}_{<\delta_\bullet}$ which has an upper bound in $\dot{T}_{\leq\delta_\bullet}$ (forced by p'_0). Then $q_{n+1} \wedge p'_0$ forces that $\dot{b} \neq B_n$, which will be guaranteed by the following requirement: There exist $\delta < \delta_\bullet$, $t \neq t' \in \dot{T}_{\leq\delta} \setminus \dot{T}_{<\delta}$, such that p'_0 forces B_n δ-th level to be t', and q_{n+1} forces $t \in \dot{b}$, i.e.,

$$p'_0 \Vdash \dot{B}_n \cap (\dot{T}_{\leq\delta} \setminus \dot{T}_{<\delta}) = \{t'\} \quad \text{and} \quad q_{n+1} \Vdash \dot{b} \cap (\dot{T}_{\leq\delta} \setminus \dot{T}_{<\delta}) = \{t\}. \tag{3-9}$$

(Observe that the latter is a statement in N_0.)

Before proving Claim 3.28 we argue why this claim implies Lemma 3.27. First, the claim gives the following condition in $\mathbb{P}_{S,\overline{U}_* + \overline{\mathrm{id}}_{S'}}$. For each $n \in \omega$ let η_{ϱ_n,ξ_n} be the branch in $\underset{\sim}{T}_{\varrho_n,<\delta_\bullet}$ represented by the sequence $\bar{p}_n$, i.e.,

$$\eta_{\varrho_n,\xi_n} = \bigcup \{ \eta_{p_{n,l}(\varrho_n),\xi_n} : l \in \omega \}, \tag{3-10}$$

and note that $\eta_{\varrho_n,\xi_n} \in N_1$ ($n \in \omega$) by ($\boxplus_3$)(d). Therefore by ($\boxplus_2$)(b) we can extend each η_{ϱ_n,ξ_n} to a branch η'_{ϱ_n,ξ_n} in $(T_{p'_0(\varrho_n)})_{<\delta_\bullet+1}$. Define the function $p_\bullet$ to be the extension of p'_0 by the η_{ϱ_n,ξ_n} in the obvious way: (Note that by ($\boxplus_2$) we have $S \cap N_0 \subseteq \mathrm{dom}(p'_0) \subseteq S$, and for each $\theta \in S \cap N_0$ the inclusion $U^*_\theta \cap N_0 \subseteq u_{p'_0(\theta)} \subseteq U^*_\theta$.) Define $p_\bullet$ to be function on $\mathrm{dom}(p'_0)$ such that if $\theta \notin N_0 \cap S'$, then $p_\bullet(\theta) = p'_0(\theta)$, and for $\theta \in N_0 \cap S'$ define $p_\bullet(\theta)$ to be the following proper extension of $p'_0(\theta)$. Let $u_{p_\bullet(\theta)} = u_{p_0(\theta)} \cup (\theta \cap N_0)$, and if $\alpha \notin u_{p'_0(\theta)}$ (when necessarily $\alpha \notin U^*_\theta$) and (by (3-8)) choose $n > 0$ so that

$$\langle \theta, \alpha \rangle = \langle \varrho_n, \xi_n \rangle, \quad \text{and let} \quad \eta_{p_\bullet(\theta),\alpha} = \eta'_{\varrho_n,\xi_n}, \tag{3-11}$$

otherwise

$$\eta_{p_\bullet(\theta),\alpha} = \eta_{p_0(\theta),\alpha} \quad (\text{if } \alpha \in U^*_\theta). \tag{3-12}$$

Observe that as η'_{ϱ_n,ξ_n} was a cofinal branch in $(T_{p_\bullet(\varrho_n)})_{<\delta_\bullet+1} = (T_{p'_0(\varrho_n)})_{<\delta_\bullet+1}$ our function $p_\bullet$ is indeed a condition in $\mathbb{P}_{S,\bar{U}_*+\bar{\mathrm{id}}_{S'}}$. Moreover, the following shows that for all $n \in \omega$, $p_\bullet \leq q_n$. Fix $n \in \omega$, then using $(\boxplus_4)$(d) we have $q_n \restriction (\bar{U}_* + \overline{W}_n) \in \boldsymbol{G}_n$, i.e., there exist $l_0, l_1, \ldots l_n \in \omega$, such that $\bigwedge_{j=0}^n p_{j,l_j} \leq_{\mathbb{P}} q_n \restriction (\bar{U}_* + \overline{W}_n)$. This means that

$$\bigwedge_{j=0}^{n} p_{j,l_j} \leq q_n \restriction (\bar{U}_* + \overline{W}_0) = q_n \restriction (\bar{U}_*),$$

and, for each $0 < j \leq n$,

$$\eta_{q_n(\varrho_j),\xi_j} \subseteq \eta_{p_{j,l_j}(\varrho_j),\xi_j} \subseteq \eta'_{\varrho_j,\xi_j} = \eta_{p_\bullet(\varrho_j),\xi_j}.$$

On the other hand, for $j > n$ we have (recalling $\bar{q} = \langle q_n : n \in \omega\rangle$ is $\leq_{\mathbb{P}}$-decreasing by $(\boxplus_4)$) that

$$\eta_{q_n(\varrho_j),\xi_j} \subseteq \eta_{q_j(\varrho_j),\xi_j} \subseteq \eta'_{\varrho_j,\xi_j} = \eta_{p_\bullet(\varrho_j),\xi_j},$$

therefore $p_\bullet \leq q_n$, indeed.

Now assuming $p_\bullet \in \boldsymbol{G}_{S,\bar{U}_*+\bar{\mathrm{id}}_{S'}}$ will easily yield a contradiction: First recall that p_* (and therefore as well q_0 and $p_\bullet$) forced that $\dot{b}$ is a branch through $\dot{T}$. Then $(\boxplus_2)$(c) implies that p'_0, thus $p_\bullet$ as well determines $\dot{T}_{\leq\delta_\bullet}$, and $p_\bullet$ forces (by $(\boxplus_4)$(e)) that each element of the $\delta_\bullet$-th level of $\dot{T}$ is the upper bound of B_i for some $i \in \omega$. This means that

$$p_\bullet \Vdash (\text{there exist } i \in \omega)\dot{b} \cap \dot{T}_{<\delta_\bullet} = B_i,$$

while at the same time

$$(q_i \wedge p'_0) \Vdash \dot{b} \neq B_i,$$

since (3-9) holds.

This together with $p_\bullet \leq q_i, p'_0$ gives the contradiction. Now we can turn to the proof of the claim.

Proof of Claim 3.28. For the construction of each sequence $\bar{p}_n$ and each q_n we will work in N_1. This will need a lot of preparation.

Recall that $X \subseteq N_0$ denoted the indices of branches added by forcing with $\mathbb{P}_{S,\bar{U}_*+\bar{\mathrm{id}}_{S'}} \cap N_0$ but missing from $V[\boldsymbol{G}_{S,\bar{U}_*}]$ (3-7), and that for each condition p, $\theta \in S_\bullet$, and $\delta < \omega_1$ the δ-th level of $T_{p(\theta)}$ is (a subset of) $[\omega\cdot\delta, \omega\cdot(\delta+1))$. Define $E \subseteq N_0$ as follows:

$$\begin{aligned} e \in E \text{ if and only if } e \in N_0, \text{ and } e = (u_e, \bar{\eta}_e), \text{ where } u_e \in [X]^{\leq\omega}, \\ \bar{\eta}_e = \langle \eta_{e,\theta,\alpha} : \langle\theta,\alpha\rangle \in u_e\rangle, \quad \text{such that} \\ \eta_{e,\theta,\alpha} \subseteq \omega\cdot(\delta_{\theta,\alpha}+1) \text{ for some } \delta_{\theta,\alpha} < \omega_1. \end{aligned} \tag{3-13}$$

Definition 3.29. For each n, $p \in \mathbb{P}_{S,\bar{U}_*+\overline{W}_n}$, and $e \in E$, if for each $\langle \theta, \alpha \rangle \in u_e$ we have $\theta \in \mathrm{dom}(p)$, and for each $i < n$ $\langle \varrho_i, \xi_i \rangle \notin u_e$ holds then define $p \frown e$ as

$$\begin{aligned} &\mathrm{dom}(p \frown e) = \mathrm{dom}(p), \\ &u_{(p\frown e)(\theta)} = u_{p(\theta)} \cup \{\alpha : \langle \theta, \alpha \rangle \in u_e\}) \quad (\text{for all } \theta \in \mathrm{dom}(p \frown e)), \\ &\eta_{(p\frown e)(\theta),\alpha} = \begin{cases} \eta_{p(\theta),\alpha} & \text{if } \alpha \in u_{p(\theta)}, \\ \eta_{e,\theta,\alpha} & \text{if } \langle \theta, \alpha \rangle \in u_e, \end{cases} \end{aligned} \tag{3-14}$$

if this is a condition in $\mathbb{P}$ (i.e., for each $\langle \theta, \alpha \rangle \in u_e$, $\eta_{e,\theta,\alpha}$ is a cofinal branch of $(T_{p(\theta)})_{<\delta+1}$ for some $\delta \le \mathrm{ht}(T_{p(\theta)})$), otherwise $p \frown e = \varnothing$.

Let $\mathcal{D}$ denote the set of dense subsets of $\mathbb{P}_{S,\bar{U}_*+\overline{\mathrm{id}}_{S'}}$. Fix an enumeration

$$\langle \langle J_i, \varepsilon_i \rangle : i \in \omega \rangle \in N_1 \quad \text{of} \quad (\mathcal{D} \cap N_0) \times E,$$

and let $k(D, e)$ denote the index of the pair $\langle D, e \rangle$, i.e.,

$$J_{k(D,e)} = D,\ \varepsilon_{k(D,e)} = e, \tag{3-15}$$

then we also have $k \in N_1$, of course. Fix a function $g \in N_0$:

$$g : \mathbb{P}_{S,\bar{U}_*+\overline{\mathrm{id}}_{S'}} \times \mathcal{D} \to \mathbb{P}_{S,\bar{U}_*+\overline{\mathrm{id}}_{S'}} \tag{3-16}$$

where, for all p, D,

($\bullet_1$) $g(p, D) \in D$,

($\bullet_2$) $g(p, D) \le p$.

(Then $g \in N_0$ obviously implies $(p, D \in N_0 \Rightarrow g(p, D) \in N_0)$.)

We will have to define also the auxiliary sequence $\bar{r} = \langle r_l : l \in \omega \rangle$ with the following properties:

($\circledast_1$) $\bar{r} \in N_1$,

($\circledast_2$) for each l, $r_l \in \mathbb{P}_{S,\bar{U}_*} \cap N_0$,

($\circledast_3$) for each l, $p_{0,l+1} \le r_l \le p_{0,l}$,

($\circledast_4$) if there exists $p \in \mathbb{P}_{S,\bar{U}^*}$ such that $p \le p_{0,l}$, and $p \frown \varepsilon_l$ is a condition extending $p_{0,l}$ in $\mathbb{P}_{S,\bar{U}^*+\overline{\mathrm{id}}_{S'}}$, then r_l is such a condition.

Now we can construct the $p_{0,i}$ (and r_i). Let $p_{0,0} = p_* \restriction \bar{U}_*$. For obtaining the $p_{0,l}$ proceed as follows. Assume we have defined $p_{0,0}, p_{0,1}, \ldots, p_{0,l-1}$ (and as well the r_i for $i < l-1$). Now if there exists $p \in \mathbb{P}_{S,\bar{U}^*}$ $p \le p_{0,l-1}$, such that $p \frown \varepsilon_{l-1} \ne \varnothing$ but a condition extending $p_{0,l-1}$, then let $r_{l-1} \in N_0$ be such a p (recall that $\varepsilon_{l-1} \in E \subseteq N_0$ by (3-13)), otherwise define $r_{l-1} = p_{0,l} = p_{0,l-1}$. Lastly, in the former case define $p_{0,l} = g(r_{l-1}, D_{l-1}) \restriction \bar{U}_*$. It is clear from the construction and the definition of g that $p_{0,l-1} \leq r_{l-1} \leq p_{0,l}$, and $r_{l-1}, p_{0,l} \in N_0$, and since every object as well as the series $\langle \varepsilon_i : i \in \omega \rangle$ are elements of N_1, we obtain $\bar{p}_0, \bar{r}_0 \in N_1$ too.

Finally, it is straightforward to check that the filter $\boldsymbol{G}_0$ generated by the $p_{0,l}$ meets every dense subset $D \in N_0$ of $\mathbb{P}_{S,\overline{U}_*}$. We fix a D such that

$$D' = \{p \in \mathbb{P}_{S,\overline{U}_*+\overline{\mathrm{id}}_{S'}} : p \restriction \overline{U}_* \in D\}$$

is clearly a dense subset of $\mathbb{P}_{S,\overline{U}_*+\overline{\mathrm{id}}_{S'}}$ belonging to N_0. This means that if $e \in E$ is the empty sequence, then there exists $i \in \omega$, such that $J_i = D'$, and $\varepsilon_i = e$, therefore $p_{0,i+1} \in D$.

For p'_0, first consider the condition $p''_0 \in N_1$ consisting of only the generic trees given by $\boldsymbol{G}_0$ (for each $\theta \in \mathrm{dom}(p''_0) = N_0 \cap S$ the tree

$$T_{p'_1(\theta)} = \bigcup \{T_{p(\theta)} : p \in \boldsymbol{G}_0\}$$

is of height $\delta_\bullet$, but $u_{p''_0(\theta)=\varnothing}$). Then let $p'''_0 \in \mathbb{P}_{S,\overline{U}_*}$, $p'''_0 \leq p''_0$ be an extension so that for each $\theta \in S' \cap N_0$ the tree $T_{p'_2}(\theta)$ satisfies that for each branch B through $(T_{p'''_0}(\theta))_{<\delta_\bullet} = T_{p''_0(\theta)}$, if $B \in N_1$, then there is an upper bound of B in $T_{p'''_0(\theta)}$. This can be done since N_1 is countable. Moreover, we choose the other part of p'''_0 so that for each $\theta, \alpha \in N_0$, if $\alpha \in U^*_\theta$ the chain $\eta_{p'''_0(\theta),\alpha}$ (with a top element) contains the chain

$$\bigcup \{\eta_{p(\theta),\alpha} : p \in \boldsymbol{G}_0\}$$

which is given by $\boldsymbol{G}_0$ at this coordinate. This can be done as

$$\bigcup \{\eta_{p(\theta),\alpha} : p \in \boldsymbol{G}_0\} \in N_1,$$

since $\boldsymbol{G}_0, \bar{p}_0 \in N_1$. Then clearly $p'''_0 \leq p_{0,l}$ for each $l \in \omega$.

Finally, for the last item of ($\boxplus_2$) first recall that $\mathbb{P}^*_{S,\overline{U}_*}$ is an ω_1-closed dense subposet of $\mathbb{P}_{S,\overline{U}_*}$ by Definition 3.11. Then if a countable increasing sequence in $\mathbb{P}^*_{S,\overline{U}_*}$ (where a first element stronger than p'''_0) decides more and more about the $\delta_\bullet$-th level of $\dot{T}$, then choosing p'_0 to be an upper bound will work (e.g., choose an enumeration $\langle \dot{t}_i : i \in \omega \}$ of the $\delta_\bullet$-th level of $\dot{T}$, let $\langle s_i : i \in \omega \rangle$ enumerate $\dot{T}_{<\delta_\bullet}$ in type ω, and let r_j decide whether the j-th ordered pair in the countable set $\{s_i : i \in \omega\} \times \{\dot{t}_i : i \in \omega\}$ is in $\leq_{\dot{T}}$).

The next step is to construct the $\bar{p}_i$ ($i > 0$) and the q_n. This will be done simultaneously by induction. The induction is carried out in V, but each step can be done in N_1, which will guarantee that each $\bar{p}_n \in N_1$.

It is straightforward to check that choosing $q_0 = p_*$ would satisfy our requirements, as, e.g., $p_{0,0} = p_* \restriction \overline{U}_*$. Then fixing $n > 0$, and assuming that $\bar{p}_i, q_i$ are constructed for each $i < n$, first we construct q_n. Recall that $q_{n-1} \restriction (\overline{U}_* + \overline{W}_{n-1}) \in \boldsymbol{G}_{n-1}$ (by ($\boxplus_4$)(d)).

Recall the definition of the set E (3-13), and let

$$E_{n-1} = \{e \in E : \text{for all } i < n \langle \varrho_i, \xi_i \rangle \notin e\}.$$

Using that $p_* \in \mathbb{P}_{S,\bar{U}_*+\bar{\mathrm{id}}_{S'}}$ forced that $\dot{b}$ is not an element of $V[G_{S,\bar{U}_*+\overline{W}_{n-1}}]$, i.e., there is no $\mathbb{P}_{S,\bar{U}_*+\overline{W}_{n-1}}$-name of it, we argue that

$$D = \big\{p \in \mathbb{P}_{S,\bar{U}_*+\overline{W}_{n-1}} : \text{there exists } e, e' \in E_{n-1} (p ^\frown e \leq q_{n-1}, p ^\frown e' \leq q_{n-1}) \\ \wedge (\text{there exists } \delta < \omega_1, t \neq t' \in \dot{T}_{\leq \delta} \setminus \dot{T}_{<\delta} : (p ^\frown e \Vdash t \in \dot{b}) \wedge (p ^\frown e' \Vdash t' \in \dot{b}))\big\}$$

is dense in $\mathbb{P}_{S,\bar{U}_*+\overline{W}_{n-1}}$ under $q_{n-1} \restriction (\bar{U}_* + \overline{W}_{n-1})$. Indeed, assume on the contrary that $q' \in \mathbb{P}_{S,\bar{U}_*+\overline{W}_{n-1}}$, $q' \leq q_{n-1} \restriction (\bar{U}_* + \overline{W}_{n-1})$ is such that D has no element under q'. Now for every $\delta < \omega_1$, consider the set

$$D_\delta = \big\{p \in \mathbb{P}_{S,\bar{U}_*+\overline{W}_{n-1}} : (p \leq q') \wedge \big(\text{there exists } e \in E_{n-1} : [p ^\frown e \leq q_{n-1}] \\ \wedge [\text{there exists } t_{p,e,\delta} \in \dot{T}_{\leq \delta} \setminus \dot{T}_{<\delta} : p ^\frown e \Vdash t_{p,e,\delta} \in \dot{b}]\big)\big\},$$

which is dense under q' in $\mathbb{P}_{S,\bar{U}_*+\bar{\mathrm{id}}S'}$. Now since for each $\delta < \omega_1$ the sets D and D_δ are disjoint, for $p \in D_\delta$ the witnessing $t_{p,e,\delta}$ doesn't depend on e, therefore $q' \wedge q_{n-1}$ forces that $\dot{b}$ is in $V[\boldsymbol{G}_{S,\bar{U}_*+\overline{W}_{n-1}}]$ (i.e., forces that the $\mathbb{P}_{S,\bar{U}_*+\overline{W}_{n-1}}$-name $\{\langle p, t_{p,\delta}\rangle : p \in D_\delta, \delta < \omega_1\}$ and $\dot{b}$ are equal). Then as our set $D \in N_0$ is indeed dense we have that there exists a condition $q'' \in \boldsymbol{G}_{n-1} \cap D$, witnessed by $t \neq t'$ and e, e'. Finally, if $t \in B_n$ then define $q_n = q'' ^\frown e'$, otherwise we can let $q_n = q'' ^\frown e$, which are both stronger conditions than q_{n-1} by the definition of D. It is straightforward to check ($\boxplus_4$).

As q_n is already defined (and so are $\bar{p}_i, q_i$ for each $i < n$), we turn to the definition of $\bar{p}_n$, which we will do similarly to that of $\bar{p}_0$. Let $p_{n,0} = q_n \restriction (\bar{U}_* + \bar{w}_n)$, assume that $p_{n,0}, p_{n,1}, \ldots, p_{n,l-1}$ are already chosen. If $\varepsilon_{l-1} \notin E_{n-1}$, then $p_{n,l} = p_{n,l-1}$, otherwise proceed as follows. Choose the sequence $\bar{e} = \bar{e}(n, l-1) = \langle e_i : 1 \leq i \leq n+1\rangle \in E^{n+1 \setminus \{0\}}$ and the sequence $\bar{m} = \bar{m}(n, l-1) = \langle m_i : i \leq n\rangle \in \omega^{n+1}$ with the properties

(1) $e_{n+1} = \varepsilon_{l-1}$ and $m_n = l-1$,

(2) for each $i < n+1$,

$$J_{m_i} = D \wedge \text{"}e_i = (e_{i+1} \text{ plus } (\eta_{p_{i,m_i}(\varrho_i),\xi_i} \text{ attained on } \langle \varrho_i, \xi_i\rangle))\text{"}. \tag{3-17}$$

Provided that the e_j are defined for $j > i$, as well as each m_j for $j \geq i$, let $e_i \in E$ be the element with $u_{e_i} = u_{e_{i+1}} \bigcup \{\langle \varrho_i, \xi_i\rangle\}$, $\bar{\eta}_{e_i} \supseteq \bar{\eta}_{e_{i+1}}$, $\eta_{e,\varrho_i,\xi_i} = \eta_{p_{i,m_i}(\varrho_i),\xi_i}$, and let $m_{i-1} = k(D, e_i)$. Observe that by our procedure, and by the definition of the function k (3-15) we have $e_1 = \varepsilon_{m_0}$, and also

$$\eta_{e_1,\varrho_n,\xi_n} = \eta_{p_{n,l-1}(\varrho_n),\xi_n}. \tag{3-18}$$

At some point later we will use the following fact, hence it is worth noting that for each i, $1 \leq i \leq n$,

$$\bar{e}(i, m_i) \subseteq \bar{e}(n, l-1) \quad \text{and} \quad \bar{m}(i, m_i) \subseteq \bar{m}(n, l-1). \tag{3-19}$$

Finally consider the condition r_{m_0} (from (⊛$_1$)–(⊛$_4$)): if $r_{m_0} {}^\frown e_1$ is a not a condition in $\mathbb{P}_{S,\bar{U}_*+\mathrm{id}\restriction S'}$, then let $p_{n,l} = p_{n,l-1}$, otherwise first define the auxiliary condition

$$r_\bullet = g(r_{m_0} {}^\frown e_1, D), \tag{3-20}$$

and note that in this case $\eta_{(r_{m_0} {}^\frown e_1)(\varrho_n),\xi_n} = \eta_{p_{n,l-1}(\varrho_n),\xi_n}$ by (3-18), and therefore by the properties of g we obtain

$$\eta_{r_\bullet(\varrho_n),\xi_n} \supseteq \eta_{p_{n,l-1}(\varrho_n),\xi_n}. \tag{3-21}$$

Recall that $p_{n,l-1} \restriction \bar{U}_* \in \boldsymbol{G}_0$ by our induction hypotheses (⊞$_3$), and it can be seen from the construction of $p_{0,j}$ that in this case $p_{0,m_0+1} = r_\bullet \restriction \bar{U}_* \in \boldsymbol{G}_0$. Therefore by (3-21) we have that $(r_\bullet \restriction \bar{U}_* + \bar{w}_n) \wedge p_{n,l-1}$ is a condition in $\mathbb{P}_{\bar{U}_*+\bar{w}_n}$, and let

$$p_{n,l} = (r_\bullet \restriction \bar{U}_* + \bar{w}_n) \wedge p_{n,l-1}.$$

Then clearly $p_{n,l} \leq p_{n,l-1}$, and $p_{n,l} \restriction \bar{U}_* \in \boldsymbol{G}_0$. From (⊞$_3$) it only remains to check that (d) and (e) also hold. Since the whole construction of $\bar{p}_n$ took place in N_1 ($k \in N_1$ and so is the enumeration $\langle \langle J_i, \varepsilon_i \rangle : i \in \omega \rangle$, $g \in N_0$), $\bar{p}_n \in N_1$ obviously follows. Verifying the genericity of $\boldsymbol{G}_n$ goes similarly as of $\boldsymbol{G}_0$. Let $D \subseteq \mathbb{P}_{S,\bar{U}_*+\overline{W}_n}$, $D \in N_0$ be a fixed dense set, and $e' \in E$ be the empty sequence. Now, if we choose l so that $J_{l-1} = D' = \{p \in \mathbb{P}_{S,\bar{U}_*+\bar{\mathrm{id}}_{S'}} : p \restriction \bar{U}_* + \overline{W}_n \in D\}$, $\varepsilon_{l-1} = e'$, then it follows from the construction of $p_{k,j}$, that of $\bar{m} = \bar{m}(n, l-1)$ and $\bar{e} = \bar{e}(n, l-1)$, and from (3-19) that

$$p_{i,m_i+1} = (r_\bullet \restriction \bar{U}_* + \bar{w}_i) \wedge p_{i,m_i} \quad \text{if} \quad 1 \leq i \leq n,$$

and

$$p_{0,m_0+1} = g(r_{m_0} {}^\frown e_1) \restriction \bar{U}_*,$$

therefore

$$\bigwedge_{i \leq n} p_{i,m_i} \leq g(r_{m_0} {}^\frown e_1) \restriction (\bar{U}_* + \overline{W}_n) \in D'.$$

□

Lemma 3.30. *Let $T \in V[\boldsymbol{G}_{S,\bar{U}_*}]$ be a Kurepa tree, $S' \subseteq S \cap S_\bullet$ ($S' \in V$), $\boldsymbol{G}^\circ_{\bar{\mathrm{id}}_{S'}-\bar{U}_*} \subseteq \mathbb{P}^\circ_{\bar{\mathrm{id}}_{S'}-\bar{U}_*}$ be generic over $V[\boldsymbol{G}_{S,\bar{U}_*}]$. Suppose that*

$$b \in V[\boldsymbol{G}_{S,\bar{U}_*}][\boldsymbol{G}^\circ_{S',(\bar{\mathrm{id}}_{S'}-\bar{U}_*)}] \setminus V[\boldsymbol{G}_{S,\bar{U}_*}]$$

is a new branch of T, and suppose that $\gamma \geq \kappa$ is a cardinal, and for each $\theta \in S'$ the inequality $|\theta \setminus U^*_\theta| \geq \gamma$ holds. Then the filter $\boldsymbol{G}^\circ_{\bar{\mathrm{id}}_{S'}-\bar{U}_*}$ adds at least $|\gamma|$-many new branches to T.

Proof. Without loss of generality, we can assume that $T \subseteq \omega_1$, and λ is a cardinal (in $V[\boldsymbol{G}_{S,\bar{U}_*}]$). First we will choose a system $\overline{W}_0 = \langle W_{0,\theta} : \theta \in S' \rangle \in \prod_{\theta \in S'} \mathcal{P}(\theta)$ with (for all $\theta \in S'$) $|W_{0,\theta}| < \kappa$, and $b \in V[\boldsymbol{G}_{S,\bar{U}_*}][\boldsymbol{G}^\circ_{\overline{W}_0}]$: since $b \in V[\boldsymbol{G}_{S,\bar{U}_*}][\boldsymbol{G}^\circ_{\overline{\mathrm{id}}_{S'} - \bar{U}_*}]$, $S' \in V$ we can use Lemma 3.20 and obtain that

$$b \in V[\boldsymbol{G}_{S,\bar{U}_*}][\boldsymbol{G}^\circ_{\overline{\mathrm{id}}_{S'} - \bar{U}_*}] = V[\boldsymbol{G}_{S,\bar{U}_* + \overline{\mathrm{id}}_{S'}}].$$

And because $b \subseteq \mathcal{H}(\omega_1)^V$, applying Lemma 3.24 with S, and $\bar{U} = \bar{U}_* + \overline{\mathrm{id}}_{S'}$, there exists $S_* \subseteq S$, $\overline{W}_* \in \prod_{S_* \setminus S'} \mathcal{P}(U_\theta) \times \prod_{\theta \in S_* \cap S'} \mathcal{P}(\theta)$ with

$$b \in V[\boldsymbol{G}_{S_*, \overline{W}_*}] \subseteq V[\boldsymbol{G}_{S,\bar{U}_* + \overline{W}_*}] = V[\boldsymbol{G}_{S,\bar{U}_*}][\boldsymbol{G}^\circ_{\overline{W}_* - \bar{U}_*}],$$

where $|S_*| < \kappa$, and $|W^*_\theta| < \kappa$ for each $\theta \in S_*$. Then fixing $\overline{W}_0 \in \prod_{\theta \in S'} \mathcal{P}(\theta)$ so that $W_{0,\theta} = W^*_\theta \setminus U^*_\theta$ if $\theta \in S_*$, and $W_{0,\theta} = \varnothing$ for $\theta \in S \setminus S_*$ has the required properties.

Now, as $|W_{0,\theta}| < \kappa \leq \gamma$, and $\gamma \leq |\theta \setminus U^*_\theta|$ for each $\theta \in S'$ we can fix for each $\alpha < \gamma$ a system $\overline{W}_\alpha = \langle W_{\alpha,\theta} : \theta \in S' \rangle \in \prod_{\theta \in S'} \mathcal{P}(\theta \setminus U^*_\theta)$ such that for every $\theta \in S'$,

(i) $W_{\alpha,\theta} \cap W_{\beta,\theta} = \varnothing$ for every $\alpha < \beta < \gamma$, and

(ii) $|W_{0,\theta}| = |W_{\alpha,\theta}|$ for each $\alpha < \gamma$.

For each $0 < \alpha < \gamma$ define the bijections

$$\pi_\alpha : \bigcup_{\theta \in S'} \{\theta\} \times W_{0,\theta} \to \bigcup_{\theta \in S'} \{\theta\} \times W_{\alpha,\theta},$$

where $\pi_\alpha \restriction \{\theta\} \times W_{0,\theta}$ is a bijection to $\{\theta\} \times W_{\alpha,\theta}$. Then clearly each π_α induces an automorphism $\hat{\pi}_\alpha \in V[\boldsymbol{G}_{S,\bar{U}_*}]$ of $\mathbb{P}^\circ_{\overline{W}_0}$ and $\mathbb{P}^\circ_{\overline{W}_\alpha}$. Moreover, $\hat{\pi}_\alpha$ induces a natural operation $\hat{\pi}^*_\alpha$ from the class of $\mathbb{P}^\circ_{\overline{W}_0}$-names to the class of $\mathbb{P}^\circ_{\overline{W}_\alpha}$-names. Now fix a $\mathbb{P}^\circ_{\overline{W}_0}$-name $\dot{b}_0 \in V[\boldsymbol{G}_{S,\bar{U}_*}]$ for our new branch $b \in V[\boldsymbol{G}_{S,\bar{U}_*}][\boldsymbol{G}^\circ_{\overline{W}_0}]$, and choose an element $p_\bullet \in \mathbb{P}^\circ_{\overline{W}_0}$ forcing that $\dot{b}_0$ is a new branch, i.e.,

$$V[\boldsymbol{G}_{S,\bar{U}_*}] \models p_\bullet \Vdash \dot{b}_0 \in \mathcal{B}(T) \setminus \mathcal{B}^{V[\boldsymbol{G}_{S,\bar{U}_*}]}(T). \tag{3-22}$$

Let

$$\mathbb{P}^\circ_\bullet = \mathbb{P}^\circ_{\sum_{\alpha<\gamma} \overline{W}_\alpha},$$

i.e., adding the branches $\bigcup_{\alpha \in \gamma} W_{\alpha,\theta}$ to $\underset{\sim}{T}_\theta$ for each $\theta \in S'$, which is of course equal to the countably supported product of $\mathbb{P}^\circ_{\overline{W}_\alpha}$ $(\alpha < \gamma)$, and let $\boldsymbol{G}^\circ_\bullet$ denote the generic filter $\boldsymbol{G}^\circ_{\overline{\mathrm{id}}_{S'} - \bar{U}_*} \cap \mathbb{P}^\circ_\bullet$.

We will show that in $V[\boldsymbol{G}_{S,\bar{U}_*}][\boldsymbol{G}^\circ_\bullet] \subseteq V[\boldsymbol{G}_{S,\bar{U}_*}][\boldsymbol{G}^\circ_{\overline{\mathrm{id}}_{S'} - \bar{U}_*}]$ there are at least γ-many new branches of T, i.e.,

$$|\mathcal{B}(T) \cap (V[\boldsymbol{G}_{S,\bar{U}_*}][\boldsymbol{G}^\circ_\bullet] \setminus V[\boldsymbol{G}_{S,\bar{U}_*}])| \geq \lambda,$$

by arguing that

($\otimes_1$) for any $\alpha < \gamma$ (in $V[\boldsymbol{G}_{S,\bar{U}_*}]$),

$$\hat{\pi}_\alpha(p_\bullet) \Vdash_{\mathbb{P}^\circ_\bullet} \hat{\pi}^*_\alpha(\dot{b}_0) \notin V[\boldsymbol{G}_{S,\bar{U}_*}][\boldsymbol{G}^\circ_{\bullet,<\alpha}]$$

(where $\boldsymbol{G}^\circ_{\bullet,<\alpha}$ stands for $\boldsymbol{G}^\circ_\bullet \cap \mathbb{P}^\circ_{\sum_{\beta<\alpha}\overline{W}_\beta}$), and

($\otimes_2$) $|\{\alpha < \gamma : \hat{\pi}_\alpha(p_\bullet) \in \boldsymbol{G}^\circ_\bullet\}| = \gamma$.

This will complete the proof of Lemma 3.30. □

First we will prove ($\otimes_2$), for which recall that we assumed that γ is a cardinal, and choose a system of uncountable regular cardinals $\{\rho_\beta : \beta < \chi < \gamma\}$, and a partition $\langle I_\beta : \beta < \chi\rangle$ of γ with $\mathrm{otp}(I_\beta) = \rho_\beta$ for each $\beta < \chi$ $\big($i.e., $I_\beta \cap I_\delta = \varnothing$ for $\beta < \delta < \rho$, and $\bigcup_{\beta<\rho} I_\beta = \gamma\big)$. Then it is enough to verify, for all $\beta < \chi$

$$|\{\alpha \in I_\beta : \hat{\pi}_\alpha(p_\bullet) \in \boldsymbol{G}^\circ_\bullet\}| = \rho_\beta, \tag{3-23}$$

which can be seen by a standard density argument: Fix $\beta < \varrho$, $\alpha \in I_\beta$, then it suffices to show that

$$D_{\beta,\alpha} = \{p \in \mathbb{P}^\circ_\bullet : p \le \hat{\pi}_\delta(p_\bullet) \text{ for some } \delta > \alpha,\, \delta \in I_\beta\} \text{ is dense,}$$

which obviously holds by the regularity of the uncountable $\rho_\beta = |I_\beta|$ (since for $\delta \in I_\beta$ we have $\hat{\pi}_\delta(p_\bullet) \in \mathbb{P}^\circ_{\overline{W}_\delta}$, $\mathbb{P}^\circ_\bullet$ is the countably supported product of $\mathbb{P}^\circ_{\overline{W}_\alpha}$ $(\alpha < \gamma)$, and $I_\beta \subseteq \gamma$).

For ($\otimes_1$) first consider $\mathbb{P}^\circ_\bullet$ as the product of $\mathbb{P}^\circ_{\sum_{\beta<\gamma,\beta\neq\alpha}\overline{W}_\beta}$ and $\mathbb{P}^\circ_{\overline{W}_\alpha}$. We will need the following claim.

Claim 3.31. For each $p \in \mathbb{P}^\circ_{\overline{W}_\alpha}$, $p \le \hat{\pi}_\alpha(p_\bullet)$, there exist $q_0, q_1 \in \mathbb{P}^\circ_{\overline{W}_\alpha}$ $q_0, q_1 \le p$, and the incomparable elements t_0, t_1 of the tree T such that

$$V[\boldsymbol{G}_{S,\bar{U}_*}][\boldsymbol{G}^\circ_{\bullet,\gamma\setminus\{\alpha\}}] \models (q_i \Vdash_{\mathbb{P}^\circ_{\overline{W}_\alpha}} t_i \in \hat{\pi}^*_\alpha(\dot{b}_0)) \quad \text{for each} \quad i \in \{0,1\},$$

where $\boldsymbol{G}^\circ_{\bullet,\gamma\setminus\{\alpha\}} = \boldsymbol{G}^\circ_\bullet \cap \mathbb{P}^\circ_{\sum_{\beta<\gamma,\beta\neq\alpha}\overline{W}_\beta}$.

Before proving the claim we verify that ($\otimes_1$) follows from it. In fact,

$$\hat{\pi}_\alpha(p_\bullet) \Vdash_{\mathbb{P}^\circ_\bullet} \hat{\pi}^*_\alpha(\dot{b}_0) \notin V[\boldsymbol{G}_{S,\bar{U}_*}][\boldsymbol{G}^\circ_{\bullet,\gamma\setminus\{\alpha\}}].$$

Since $\boldsymbol{G}^\circ_\bullet \subseteq \mathbb{P}^\circ_\bullet$ is generic over $V[\boldsymbol{G}_{S,\bar{U}_*}]$, and $\mathbb{P}^\circ_\bullet$ can be identified with

$$(\mathbb{P}^\circ_{\sum_{\beta<\gamma,\beta\neq\alpha}\overline{W}_\beta}) \times \mathbb{P}^\circ_{\overline{W}_\alpha},$$

by [Kunen 2011, Lemma V.1.1]

$$\boldsymbol{G}^\circ_{\bullet,\gamma\setminus\{\alpha\}} = \boldsymbol{G}^\circ_\bullet \cap \mathbb{P}^\circ_{\sum_{\beta<\gamma,\beta\neq\alpha}\overline{W}_\beta}$$

is generic over $V[\boldsymbol{G}_{S,\bar{U}_*}]$, and $\boldsymbol{G}^\circ_{\bullet,\alpha} = \boldsymbol{G}^\circ_\bullet \cap \mathbb{P}^\circ_{\overline{W}_\alpha}$ is generic over $V[\boldsymbol{G}_{S,\bar{U}_*}][\boldsymbol{G}^\circ_{\bullet,\gamma\setminus\{\alpha\}}]$. For each branch $c \in V[\boldsymbol{G}_{S,\bar{U}_*}][\boldsymbol{G}^\circ_{\bullet,\gamma\setminus\{\alpha\}}]$ of T define (in $V[\boldsymbol{G}_{S,\bar{U}_*}][\boldsymbol{G}^\circ_{\bullet,\gamma\setminus\{\alpha\}}]$)

$$D_c = \{q \in \mathbb{P}^\circ_{\overline{W}_\alpha} : \text{there exists } t \in T \setminus c \text{ such that } q \Vdash_{\mathbb{P}^\circ_{\overline{W}_\alpha}} t \in \hat{\pi}^*_\alpha(\dot{b}_0)\},$$

which is dense under $\hat{\pi}_\alpha(p_\bullet)$ by Claim 3.31, since for a fixed $p \in \mathbb{P}^\circ_{\overline{W}_\alpha}$ at most one t_i can be in the branch c.

Proof of Claim 3.31. First we argue that the statement holds in $V[\boldsymbol{G}_{S,\bar{U}_*}]$, i.e., for each $p \in \mathbb{P}^\circ_{\overline{W}_\alpha}$, $p \leq \hat{\pi}_\alpha(p_\bullet)$, there exist $q_0, q_1 \in \mathbb{P}^\circ_{\overline{W}_\alpha}$, $q_0, q_1 \leq p$, and the incomparable elements t_0, t_1 of the tree T such that

$$V[\boldsymbol{G}_{S,\bar{U}_*}] \models (q_i \Vdash_{\mathbb{P}^\circ_{\overline{W}_\alpha}} t_i \in \hat{\pi}^*_\alpha(\dot{b}_0)) \quad \text{for each} \quad i \in \{0,1\}. \tag{3-24}$$

Now (3-22) implies that

$$V[\boldsymbol{G}_{S,\bar{U}_*}] \models \hat{\pi}_\alpha(p_\bullet) \Vdash_{\mathbb{P}^\circ_{\overline{W}_\alpha}} \hat{\pi}^*_\alpha(\dot{b}_0) \in (\mathcal{B}(T) \setminus \mathcal{B}^{V[\boldsymbol{G}_{S,\bar{U}_*}]}(T))$$

since $\dot{b}_0 \in V[\boldsymbol{G}_{S,\bar{U}_*}]$ is a $\mathbb{P}^\circ_{\overline{W}_0}$-name and $T \in V[\boldsymbol{G}_{S,\bar{U}_*}]$. Suppose that $p \leq \hat{\pi}_\alpha(p_\bullet)$ is a counterexample, but then for the set

$$b' = \{t \in T : \text{there exist } q \in \mathbb{P}^\circ_{\overline{W}_\alpha}, q \leq p \text{ such that } q \Vdash t \in \hat{\pi}^*_\alpha(\dot{b}_0)\} \in V[\boldsymbol{G}_{S,\bar{U}_*}]$$

we have $p \Vdash \hat{\pi}^*_\alpha(\dot{b}_0) = b'$ (since $\hat{\pi}_\alpha(p_\bullet)$ forced that $\hat{\pi}^*_\alpha(\dot{b}_0)$ is a cofinal branch in T), a contradiction. Finally, fixing $p \leq \hat{\pi}_\alpha(p_\bullet)$, if $q_0, q_1 \in \mathbb{P}^\circ_{\overline{W}_\alpha}$, $q_0, q_1 \leq p$, and the incomparable elements $t_0, t_1 \in T$ are such that (3-24) holds, then

$$V[\boldsymbol{G}_{S,\bar{U}_*}][\boldsymbol{G}^\circ_{\bullet,\gamma\setminus\{\alpha\}}] \models (q_i \Vdash_{\mathbb{P}^\circ_{\overline{W}_\alpha}} t_i \in \hat{\pi}^*_\alpha(\dot{b}_0)) \quad \text{for each} \quad i \in \{0,1\},$$

since if $q_i \in \boldsymbol{H} \subseteq \mathbb{P}^\circ_{\overline{W}_\alpha}$ is generic over $V[\boldsymbol{G}_{S,\bar{U}_*}][\boldsymbol{G}^\circ_{\bullet,\gamma\setminus\{\alpha\}}]$, and $t_i \notin \hat{\pi}^*_\alpha(\dot{b}_0)[\boldsymbol{H}]$ (for some $i \in \{0,1\}$), then $\boldsymbol{H}$ is generic over $V[\boldsymbol{G}_{S,\bar{U}_*}]$ too, and the same holds in $V[\boldsymbol{G}_{S,\bar{U}_*}][\boldsymbol{H}]$. □

It is left to argue why Lemma 3.27 and Lemma 3.30 complete the proof of Theorem 3.1 (and Theorem 3.4). Suppose that $T \in V[\boldsymbol{G}]$ is a Kurepa tree (where $\boldsymbol{G} \subseteq \mathbb{P} = \mathbb{P}_{S^+_\bullet, \overline{\mathrm{id}}_{S^+_\bullet}}$ is generic), and assume on the contrary that $|\mathcal{B}^{V[\boldsymbol{G}]}(T)| \notin S_\bullet$. We can also assume that $T \subseteq \mathcal{H}(\omega_1)^V$, and by Lemma 3.24 there exists $S_* \subseteq S^+_\bullet$, $|S_*| < \kappa$, $\overline{W}_* = \langle W^*_\theta : \theta \in S_* \rangle \in \prod_{\theta \in S_*} [\theta]^{<\kappa}$ such that $T \in V[\boldsymbol{G}_{S_*,\overline{W}_*}]$. For estimating $(2^{\omega_1})^{V[\boldsymbol{G}_{S_*,\overline{W}_*}]}$ first a straightforward calculation yields that $|\mathbb{P}_{S_*,\overline{W}_*}| < \kappa$: Since $|\mathbb{P}_{S_*,\langle\varnothing:\theta\in S_*\rangle}| = (|S_*||\omega_1|)^\omega$ which is either $(\omega_1 \cdot \omega_1)^\omega = \omega_1 < \omega_2$ (if $\kappa = \omega_2$, by $\boldsymbol{CH}$), or $\gamma^\omega < \kappa$ (for some $\gamma < \kappa$, if κ is inaccessible). Thus recalling the definition of $\mathbb{Q}_{\theta,W^*_\theta}$, the fact $\sum_{\theta \in S_*} |W^*_\theta| < \kappa$ as κ is regular, and $\sup W^*_\kappa < \kappa$ (if $\kappa \in S_*$) we have the following (in both cases regardless of whether $\kappa = (\omega_2)^V$, or an inaccessible)

$$|\mathbb{P}_{S_*,\overline{W}_*}| = |\mathbb{P}_{S_*,\langle\varnothing:\theta\in S_*\rangle}| \cdot \left((\omega_1) \cdot \left(\sum_{\theta \in S_* \setminus \{\kappa\}} |W^*_\theta|\right)\right)^\omega \cdot (|W^*_\kappa| \cdot \sup W^*_\kappa)^\omega < \kappa.$$

At this point we have to discuss the two cases (i.e., whether $\kappa \in S_\bullet$) differently, arguing that in both cases there are branches outside $V[\boldsymbol{G}_{S_*,\overline{W}_*}]$.

If $\kappa = \omega_2 \in S_\bullet$, then as

$$V \models |\mathbb{P}_{S_*,\overline{W}_*}|^{\omega_1 \cdot |\mathbb{P}_{S_*,\overline{W}_*}|} = \omega_2,$$

we have

$$V[\boldsymbol{G}_{S_*,\overline{W}_*}] \models 2^{\omega_1} = \omega_2,$$

therefore as $|\mathcal{B}^{V[\boldsymbol{G}]}(T)| \notin S_\bullet$, there are branches of T in $V[\boldsymbol{G}]$ not in $V[\boldsymbol{G}_{S_*,\overline{W}_*}]$. On the other hand, if $\kappa \notin S_\bullet$ is inaccessible, then we obtain that

$$V[\boldsymbol{G}_{S_*,\overline{W}_*}] \models |\mathcal{B}(T)| \leq 2^{\omega_1} < \kappa,$$

and as κ remains a cardinal in $V[\boldsymbol{G}]$ (by Claim 3.26), and

$$V[\boldsymbol{G}] \models |\mathcal{B}(T) \cap V[\boldsymbol{G}_{S_*,\overline{W}_*}]| = \omega_1,$$

we conclude that this case there also must be branches of T not in $V[\boldsymbol{G}_{S_*,\overline{W}_*}]$ as T is a Kurepa tree in $V[\boldsymbol{G}]$. Now let $\bar{r} \in \prod_{\theta \in S_\bullet^+ \setminus S_\bullet} \mathcal{P}(\theta)$, $R_\theta = \theta \setminus W_\theta^*$, then

$$\mathbb{P} = \mathbb{P}_{S_\bullet^+,\overline{\mathrm{id}}_{S_\bullet^+}} \simeq (\mathbb{P}_{S_*,\overline{\mathrm{id}}_{S_*}-\bar{r}}) \times (\mathbb{P}_{S_* \cap (S_\bullet^+ \setminus S_\bullet),\bar{r}}) \times (\mathbb{P}_{S_\bullet^+ \setminus S_*,\overline{\mathrm{id}}_{S_\bullet^+ \setminus S_*}}),$$

and there are no new sequences of type ω in $V[\boldsymbol{G}]$ (by Claim 3.23), and the second component is ω_1-closed, the third component has an ω_1-closed dense subset (which thus remain ω_1-closed in $V[\boldsymbol{G}_{S_*,\overline{\mathrm{id}}_{S_*}-\bar{r}}]$) we obtain that each branch of T is added by $\boldsymbol{G}_{S_*,\overline{\mathrm{id}}_{S_*}-\bar{r}} = \boldsymbol{G} \cap \mathbb{P}_{S_*,\overline{\mathrm{id}}_{S_*}-\bar{r}}$ (since an ω_1-closed forcing do not add new branches to Kurepa trees [Kunen 2011, Lemma V.2.26]). We only have to derive a contradiction from

$$V[\boldsymbol{G}_{S_*,\overline{\mathrm{id}}_{S_*}-\bar{r}}] \models |\mathcal{B}(T)| \notin S_\bullet.$$

Now letting $\partial = |\mathcal{B}^{V[\boldsymbol{G}_{S_*,\overline{\mathrm{id}}_{S_*}-\bar{r}}]}(T)| \notin S_\bullet$, $S_*^- = S_* \cap S_\bullet \cap \partial$, $S_*^+ = (S_* \cap S_\bullet) \setminus S_*^-$ by Lemma 3.20 we have

$$V[\boldsymbol{G}_{S_*,\overline{\mathrm{id}}_{S_*}-\bar{r}}] = V[\boldsymbol{G}_{S_*,\overline{W}_*+\overline{\mathrm{id}}_{S_*^-}}][\boldsymbol{G}^\circ_{\overline{\mathrm{id}}_{S_*^+}-\overline{W}_*}].$$

As $\partial \notin S_*^-, S_*^+$, it is enough to prove that in $V[\boldsymbol{G}_{S_*,\overline{W}_*+\overline{\mathrm{id}}_{S_*^-}}]$ there are less than ∂-many branches of T, because if $\boldsymbol{G}^\circ_{\overline{\mathrm{id}}_{S_*^+}-\overline{W}_*}$ adds new branches, then by Lemma 3.30 it adds $\min(S_*^+)$-many new branches (since each $|W_\theta^*| < \kappa \leq \min(S_\bullet) \leq \min(S_*^+)$).

Now if $\partial = \kappa$, then $S_*^- = \emptyset$, we are done, so we can assume that $\partial > \kappa$, and $\sup S_*^- \geq \kappa$. As $|S_*| < \kappa$ (in V), and our conditions (Case 2 (iii), or Case 2 (ii)) states that then $\sup(S_* \cap S_\bullet \cap \partial) \in S_\bullet$ implying $\sup S_*^- < \partial$. Therefore using that $W_\theta^* \subseteq \theta$ we get $\sum_{\theta \in S_*^-} |W_\theta^*| \leq |\sup S_*^-|^2 < \partial$. Now by Lemma 3.27 for each branch b of T in $V[\boldsymbol{G}_{S_*,\overline{W}_*+\overline{\mathrm{id}}_{S_*^-}}] = V[\boldsymbol{G}_{S_*,\overline{W}_*}][\boldsymbol{G}^\circ_{(\overline{\mathrm{id}}_{S_*^-})-\overline{W}_*}]$ there exist $\theta_0, \theta_1, \ldots, \theta_{n-1}$, $U^\bullet_{\theta_0}, U^\bullet_{\theta_1}, \ldots, U^\bullet_{\theta_{n-1}}$ finite such that $b \in V[\boldsymbol{G}_{S_*,\overline{W}_*}][\boldsymbol{G}^\circ_{\overline{U}_\bullet}]$. Therefore, as $|\mathbb{P}^\circ_{\overline{U}_\bullet}| =$

$\omega_1^n = \omega_1$, counting the nice $\mathbb{P}^\circ_{\overline{U}_\bullet}$-names of subsets T for each possible n, sequence of θ, and $\overline{U}_\bullet$,

$$\mathcal{B}(T) \cap (V[\boldsymbol{G}_{S_*,\overline{W}_*}][\boldsymbol{G}^\circ_{(\overline{\mathrm{id}}_{S_*^-})-\overline{W}_*}] \setminus V[\boldsymbol{G}_{S_*,\overline{W}_*}]) \\ \leq (|\sup S_*^-|^{<\omega} \cdot \omega_1^{\omega_1})^{V[\boldsymbol{G}_{S_*,\overline{W}_*}]} \leq \sup S_*^-,$$

which is smaller than ∂, a contradiction.

For $V[\boldsymbol{G}] \models 2^{\omega_1} = \lambda$ we only need to show that $2^{\omega_1} \leq \lambda$. But a similar straightforward calculation yields that $\mathbb{P} = \mathbb{P}_{S_\bullet^+,\overline{\mathrm{id}}_{S_\bullet^+}}$ is of cardinality λ, and then (using κ-cc and the equality $\lambda^{<\kappa} = \lambda$) by counting the possible nice names for subsets of ω_1 we obtain the desired inequality.

Remark 3.32. If $S_\bullet$ also satisfies

(3-25) $$\text{for all } \mu \in S_\bullet : \mathrm{cf}(\mu) < \kappa \to \mu^+ \in S_\bullet,$$

and $\boldsymbol{GCH}$ holds in V then $S_\bullet \setminus \{\lambda\}$ is the spectrum for the Jech–Kunen trees in $V[\boldsymbol{G}]$. (A tree T of height ω_1 and power ω_1 is a Jech–Kunen tree if $\omega_1 < |\mathcal{B}(T)| < 2^{\omega_1}$.) For more on Jech–Kunen trees see also [Shelah and Jin 1992; 1993; Jin and Shelah 1994]. Note that $\boldsymbol{CH}$ in the final model implies that the product of countably many Jech–Kunen trees is a Jech–Kunen tree, so is the diagonal product of ω_1-many Jech Kunen trees, hence (3-25) cannot be dropped.

One can obtain similar cardinal arithmetic conditions for Sp_μ with μ large.

4. The necessity of the inaccessible cardinal

In this section we prove that if ω_2 is not an element of the spectrum, then ω_2 is inaccessible in L. The idea of using transitive collapses of elementary submodels of constructible sets as nodes of a tree goes back to Solovay's original unpublished argument for the consistency strength of the negation of the Kurepa hypothesis. Although the next proof is deemed to be well-known, for the sake of completeness we include the proof as there is probably no known source to cite.

Theorem 4.1. *Suppose that ω_2^V is a successor in L. Then there exists a Kurepa tree T with $\mathcal{B}^V(T) = \omega_2$.*

Proof. We will use an extension of L, an inner model between L and V, what serves as the motivation for the following definition of relative constructibility, which can be found in [Kanamori 2003].

Definition 4.2. For a set A define $L[A] = \bigcup_{\alpha \in ON} L_\alpha[A]$ by transfinite recursion as follows. $L_0[A] = \varnothing$, $L_{\alpha+1}[A] = \mathrm{def}_A(L_\alpha[A])$, and α limit $L_\alpha[A] = \bigcup_{\beta<\alpha} L_\beta[A]$ (where $\mathrm{def}_Y(X)$ are the subsets of X that can be defined in the structure $(X, \in\restriction (X \times X), Y \cap X)$ by parameters from X; see [Kanamori 2003, Chapter 1, §3]).

The following is an easy exercise, but for the sake of completeness we include the proof.

Claim 4.3. There exists a set $A \subseteq \omega_1$ such that $\omega_1^{L[A]} = \omega_1$, $\omega_2^{L[A]} = \omega_2$.

Proof. If $\omega_2^V = (\lambda^+)^L$, where $|\lambda| = \omega_1$, then in a single subset A of ω_1 we can code a well-ordering of ω_1 in type λ, and also for each $\alpha < \omega_1$ a well-ordering of ω in type α in the obvious fashion, and such that L can read this coding (implying $\omega_1^{L[A]} = \omega_1$, $\omega_2^{L[A]} = \omega_2$): First let $\langle X_\alpha : \alpha \leq \omega_1 \rangle \in L$ be a set of pairwise disjoint sets of ω_1 with $|X_\alpha|^L = \omega$ for each $\alpha < \omega_1$, and $|X_{\omega_1}|^L = \omega_1$, then for each $\alpha < \omega_1$ we can code the well ordering X_α in order type α, and the well ordering of X_{ω_1} in type λ in a subset A' of $\bigcup_{\alpha \leq \omega_1} X_\alpha^2 \subseteq \omega_1^2$. Finally, taking the preimage of this set under a bijection $f \in L$ between ω_1 and ω_1^2, i.e., $A = f^{-1}(A')$ works. □

We have to recall a classical lemma [Kanamori 2003, Theorem 3.3]. Recall that $\mathcal{L}_\in(R_A)$ stands for the (first-order) language of set theory extended by the unary predicate R_A.

Lemma 4.4. *There is a sentence $\sigma \in \mathcal{L}_\in(R_A)$ such that for every transitive set N*

$$(N, \in, X \cap N) \models \sigma \textit{ implies } N = L_\gamma[X] \textit{ for some limit } \gamma.$$

In particular, if $M \prec (L_\beta[X], \in, X \cap L_\beta[X])$, where β is a limit ordinal and π is the collapsing isomorphism from M onto the transitive set $\mathrm{ran}(\pi)$, *then the Mostowski collapse*

$$\mathrm{ran}(\pi) = L_\gamma[\{\pi(x) : x \in M \cap X\}]$$

for some $\gamma \leq \beta$.

The following is immediate.

Claim 4.5. For each infinite ordinal β and $Y \subseteq L_\beta[X]$, if $Y \in L[X]$ and $X \subseteq L_\beta[X]$, then $\mu = (|\beta|^+)^{L[X]}$ implies $Y \in L_\mu[X]$.

(Working in $L[X]$, if $Y \in L_\gamma[X]$, then let $M \prec L_\gamma[X]$ with $\{Y\} \cup L_\beta[X] \subseteq M$, $|M| = |L_\beta[X]|$, and apply the lemma recalling that $\pi \restriction L_\beta[X]$ is the identity.)

Now we can turn to the definition of the tree T, which will be defined by its branches.

Recall that there exists a definable well-order on $L[A]$, which is downward absolute to almost every initial segment of $L[A]$ (to the ones indexed by limit ordinals) [Kanamori 2003, Theorem 3.3]:

Lemma 4.6. *There exists a formula $\varphi \in \mathcal{L}_\in(R_A)$ (i.e., in the language of set theory extended with the unary relation symbol A) which define a well-ordering on $(L[A], \in, A)$, moreover if δ is a limit ordinal, $x, y \in L_\delta[A]$, then*

$$(L[A], \in, A) \models \varphi(x, y) \iff (L_\delta[A], \in, A \cap L_\delta[A]) \models \varphi(x, y).$$

From now on "$x <_{L[A]} y$" abbreviates $\varphi(x, y)$.

We will take Skolem hulls many times, thus we need to introduce the following variant of this standard notion.

Definition 4.7. Let $(M, \in, X, \partial)$, $M \subseteq L[A]$ be a set model of the language $\mathcal{L}_\in(R_A, c_\partial)$ with $\varnothing \in M$, $M' \subseteq M$ such that the well-ordering formula $\varphi \in \mathcal{L}_\in(R_A)$ from Lemma 4.6 is absolute to M, i.e.,

(4-1) $\quad$ (for all $x, y \in M$) : $(L[A], \in, A) \models \varphi(x, y) \iff (M, \in, X) \models \varphi(x, y)$,

e.g., when $(M, \in, X) = (L_\zeta[A], \in, A \cap L_\zeta[A])$ for some limit ordinal ζ. Then the Skolem-hull of M' in $(M, \in, X, \partial)$ (in symbols, $\mathfrak{H}^{(M,\in,X,\partial)}(M')$) is the closure of M' under the functions $f_\psi^{(M,\in,X,\partial)}$ for each formula $\psi(v_0, v_1, \dots, v_{n_\psi}) \in \mathcal{L}_\in(R_A, c_\partial)$ with $n_\psi + 1$ free variables, where the function $f_\psi^{(M,\in,X,\partial)}$ satisfies the following:

$$f_\psi^{(M,\in,X,\partial)} : M^{n_\psi} \to M$$

is defined so that for every $\langle x_1, x_2, \dots, x_{n_\psi}\rangle \in M^{n_\psi}$: if there exist $y! \in M$ such that

$$(M, \in, X, \partial) \models \psi(y, x_1, x_2, \dots, x_{n_\psi}),$$

then let $f_\psi^{(M,\in,X,\partial)}(x_1, x_2, \dots, x_{n_\psi})$ be the unique such y, otherwise let

$$f_\psi^{(M,\in,X,\partial)}(x_1, x_2, \dots, x_{n_\psi}) = \varnothing.$$

Then the fact that for each formula ψ' we can define the formula saying that y is the least y (with respect to the well-order given by φ) satisfying $\psi'(y, x_1, x_2, \dots, x_{n_{\psi'}})$ together with the Tarski–Vaught criterion implies that the closure is an elementary submodel of M, in symbols, $M' \prec (M, \in, X, \partial)$.

Observe that this closure only depends on the isomorphism class of $(M, \in, X, \partial)$ by the absoluteness of the well-ordering formula φ (4-1).

Choose $\xi < \omega_2$ such that

(4-2) $\quad$ ξ is the minimal ordinal (for all $\alpha < \omega_1$)

there exist $f_\alpha \in L_\xi[A]$ bijection between ω and α

(which can be done due to Claim 4.5, in fact $\xi = \omega_1$, but we won't use this equality, hence we don't argue that).

Now we will define an operation which assigns for each $\delta \in [\xi, \omega_2)$ the ordinal $\delta' < \omega_2$ in the following way. We would like to choose δ' so that in $L_{\delta'}[A]$ it is true that for each set x there exists a surjection from ω_1 to x, and for $\delta'' \neq \delta'$ the structures $(L_{\delta'}[A], \in, A, \delta)$ and $(L_{\delta''}[A], \in, A, \delta)$ cannot be elementarily equivalent.

Definition 4.8. Fix $\delta \in [\xi, \omega_2)$, and define δ' to be the least ordinal such that

(a) $\delta \in L_{\delta'}[A]$,

(b) for each $x \in L_{\delta'}[A]$ there is a bijection $f \in L_{\delta'}[A]$ between ω_1 and x,

(c) taking the sentence σ from Lemma 4.4 $(L_{\delta'}[A], \in, A) \models \sigma$.

(Using Claim 4.5 and $(|L_\alpha[A]| = |\alpha|)^{L[A]}$ for $\alpha \geq \omega$ it is easy to see that we can do this closure operation, and there is such a $\delta' < \omega_2$.) Then we have

$$(\delta' \text{ is a limit }) \bigwedge (L_{\delta'}[A] \models \text{“}\omega_1 \text{ is the largest cardinal”}), \tag{4-3}$$

and also the desired uniqueness by our next claim.

Claim 4.9. There is a statement $\sigma' \in \mathcal{L}_\in(R_A, c_\partial)$ such that for each $\delta \in [\xi, \omega_2)$ $(L_{\delta'}[A], \in, A, \delta) \models \sigma'$, moreover, for each $\delta > \omega_1$ and $\delta'' > \delta$,

$$((L_{\delta''}[A], \in, A, \delta) \models \sigma') \Rightarrow (\delta'' = \delta').$$

Proof. First define $\sigma'' = \sigma \wedge$ (for all y there exist $f : \omega_1 \to y$ bijection) and let σ' be the following sentence:

$$\sigma' = \sigma'' \wedge (\neg(\exists X)(X \text{ is transitive}) \wedge (\sigma'')^X \wedge (\delta \in X))$$

(where under ψ^X we always mean the formula $\psi \in \mathcal{L}_\in(R_A, c_\partial)$ relativized to X, and σ is from Lemma 4.4). □

Now fix $\delta \in [\xi, \omega_2)$, and for each ordinal $0 < \alpha < \omega_1$ define $M_{\delta,\alpha}$ to be the Skolem-hull

$$M_{\delta,\alpha} = \mathfrak{H}^{(L_{\delta'}[A], \in, A, \delta)}(\alpha) \quad (\text{for each } \alpha < \omega_1). \tag{4-4}$$

Also define

$$M_{\delta,0} = \varnothing. \tag{4-5}$$

Then

$$M_{\delta,\alpha} \prec (L_{\delta'}[A], \in, A, \delta) \quad (\text{for each } \alpha > 0). \tag{4-6}$$

Observe that whenever $M^* \prec (L_{\delta'}[A], \in, A, \delta)$ we have for the Skolem functions from Definition 4.7 that $f_\psi^{(L_{\delta'}[A], \in, A, \delta)} \restriction (M^*)^{n_\psi} = f_\psi^{(M^*, \in, A \cap M^*, \delta)}$, hence

$$\text{for all } M' \subseteq M^* \prec (L_{\delta'}[A], \in, A, \delta) : \mathfrak{H}^{(L_{\delta'}[A], \in, A, \delta)}(M') = \mathfrak{H}^{(M^*, \in, A \cap M^*, \delta)}(M'). \tag{4-7}$$

Now as we defined $\langle M_{\delta,\alpha} : \alpha < \omega_1 \rangle$ note that

$$(M \prec (L_{\delta'}[A], \in, A, \delta)) \wedge (|M| = \omega) \to (M \cap \omega_1 \in \omega_1), \tag{4-8}$$

in particular,

$$M_{\delta,\alpha} \cap \omega_1 \in \omega_1, \tag{4-9}$$

since (4-2) together with $\xi \leq \delta < \delta'$ implies that in $L_{\delta'}[A]$ there is an enumeration of each ordinal less than ω_1 (and $M_{\delta,\alpha}$ is countable). This implies that

$$(C_\delta = \{\alpha < \omega_1 : M_{\delta,\alpha} \cap \omega_1 = \alpha\} \text{ is a club in } \omega_1) \wedge (0 \in C_\delta).$$

It is easy to see that

(4-10) $$\text{for all } \alpha < \omega_1 : M_{\delta,\alpha} = M_{\delta,\min(C_\delta \setminus \alpha)}.$$

For later use we verify the following statement.

Claim 4.10. $$\bigcup_{\alpha<\omega_1} M_{\delta,\alpha} = L_{\delta'}[A].$$

Proof. Since the union of an increasing chain of elementary submodels is an elementary submodel, we have $M_{\omega_1} = \bigcup_{\alpha<\omega_1} M_{\delta,\alpha} \prec (L_{\delta'}[A], \in, A, \delta)$. Now recall, that in $L_{\delta'}[A]$ every set x admits a surjection from ω_1 onto x, therefore $\omega_1 \subseteq M_{\omega_1}$ implies that M_{ω_1} is transitive. Then by Lemma 4.4 and $M_{\omega_1} \models \sigma$ we have $M_{\omega_1} = L_{\delta''}[A]$ for some $\delta'' > \delta$. But then either $M_{\omega_1} \in L_{\delta'}[A]$, or $M_{\omega_1} = L_{\delta'}[A]$, and because the former would contradict Claim 4.9, we arrive at our conclusion. □

For each $\alpha \in C_\delta$ and $\beta < \omega_1$, if $\alpha = \max(C_\delta \cap (\beta + 1))$, then let $N_{\delta,\beta,\alpha}$ be the range of the Mostowski-collapse $\pi_{\delta,\alpha}$ of $(M_{\delta,\alpha}, \in)$, and let $A_{\delta,\beta,\alpha} = \pi_{\delta,\alpha}(A)$, $\partial_{\delta,\beta,\alpha} = \pi_{\delta,\alpha}(\delta)$:

(4-11) $$\pi_{\delta,\alpha} : M_{\delta,\alpha} \to N_{\delta,\beta,\alpha},$$

which is of course not only an isomorphism between $(M_{\delta,\alpha}, \in)$ and $(N_{\delta,\beta,\alpha}, \in)$, but witnesses

(4-12) $$(M_{\delta,\alpha}, \in, A \cap M_{\delta,\alpha}, \delta) \simeq (N_{\delta,\beta,\alpha}, \in, A_{\delta,\beta,\alpha}, \partial_{\delta,\beta,\alpha}).$$

Now we are ready to construct the tree T. For a fixed $\delta \in [\xi, \omega_2)$, $\alpha \in C_\delta$, $\beta < \omega_1$, if $0 < \alpha = \max(C_\delta \cap (\beta + 1))$ holds then we define

(4-13) $$t_{\delta,\beta,\alpha} = (N_{\delta,\beta,\alpha}, \in, A_{\delta,\beta,\alpha}, \partial_{\delta,\beta,\alpha}),$$

i.e., the structure $(N_{\delta,\beta,\alpha}, \in)$ extended by the one-place relation for the image of $A \in M_{\delta,\alpha}$ under the collapsing isomorphism, and the constant symbol for $\partial_{\delta,\beta,\alpha}$. For $\max(C_\delta \cap (\beta + 1)) = 0$ let $t_{\delta,\beta,0} = \varnothing$.

Observe that given $t = t_{\delta,\beta,\alpha}$ we can decode α from t, as α is the first uncountable ordinal of t.

Definition 4.11. Define

$$T = \{(\beta, t_{\delta,\beta,\alpha}) : \delta \in [\xi, \omega_2), \beta < \omega_1, \alpha = \max(C_\delta \cap (\beta + 1))\},$$

with the partial order $(\beta_0, t_{\delta_0,\beta_0,\alpha_0}) \leq_T (\beta_1, t_{\delta_1,\beta_1,\alpha_1})$ if and only if either $\alpha_0 = 0$ (thus $t_{\delta_0,\beta_0,\alpha_0}$ is the empty structure), or

(i) $\beta_0 \le \beta_1$, and

(ii) taking the Skolem-hull M of α_0 in

$$t_{\delta_1,\beta_1,\alpha_1} = (N_{\delta_1,\beta_1,\alpha_1}, \in, A_{\delta_1,\beta_1,\alpha_1}, \partial_{\delta_1,\beta_1,\alpha_1}),$$

i.e., $M = \mathfrak{H}^{t_{\delta_1,\beta_1,\alpha_1}}(\alpha_0)$ is isomorphic to $t_{\delta_0,\beta_0,\alpha_0}$:

$$(M, \in, A_{\delta_1,\beta_1,\alpha_1} \cap M, \partial_{\delta_1,\beta_1,\alpha_1}) \simeq (N_{\delta_0,\beta_0,\alpha_0}, \in, A_{\delta_0,\beta_0,\alpha_0}, \partial_{\delta_0,\beta_0,\alpha_0}),$$

and

(iii) if $\alpha_0 < \alpha_1$, then there is no proper elementary submodel

$$M \prec (N_{\delta_1,\beta_1,\alpha_1}, \in, A_{\delta_1,\beta_1,\alpha_1}, \partial_{\delta_1,\beta_1,\alpha_1})$$

with

$$\alpha_0 \cup \{\alpha_0\} \subseteq M \quad \text{and} \quad M \cap \alpha_1 \subseteq \beta_0.$$

Roughly speaking, in level β we have (isomorphism types of) initial segments M of models of the form $(L_{\Delta'}[A], \in, A, \Delta)$ (for some $\Delta \in [\xi, \omega_2)$), such that $M \cap \omega_1 \le \beta$, and M is maximal with respect to this condition. We need to check that T is a tree, its levels are countable, and that it has only ω_2-many branches even in V.

The following claim is a standard calculation, but for the sake of completeness we include the proof.

Claim 4.12. Let $\delta \in [\xi, \omega_2)$ be fixed, $\beta_0 \le \beta_1 < \omega_1$, let $\alpha_1 = \max(C_\delta \cap (\beta_1 + 1))$, $\alpha_0 = \max(C_\delta \cap (\beta_0 + 1))$. Then $(\beta_0, t_{\delta,\beta_0,\alpha_0}) \le_T (\beta_1, t_{\delta,\beta_1,\alpha_1})$.

Moreover, the embedding $\pi_{\beta_0,\beta_1} : N_{\delta,\beta_0,\alpha_0} \to N_{\delta,\beta_1,\alpha_1}$ is unique.

Proof. First observe that by (4-4) and (4-7) for $\delta \in [\xi, \omega_2)$, $\alpha_0 < \alpha_1$,

$$\mathfrak{H}^{(M_{\delta,\alpha_1}, \in, A, \delta)}(\alpha_0) = \mathfrak{H}^{(L_{\delta'}[A], \in, A, \delta)}(\alpha_0) = M_{\delta,\alpha_0},$$

therefore since $\beta_1 < \omega_1$ is such that $\alpha_1 = \max(C_\delta \cap (\beta_1 + 1))$, then applying (the restriction of) the collapsing isomorphism π_{δ,α_1} to the left side, we obtain

$$(\mathfrak{H}^{(N_{\delta,\beta_1,\alpha_1}, \in, A_{\delta,\beta_1,\alpha_1}, \partial_{\delta,\beta_1,\alpha_1})}(\alpha_0), \in) \simeq (M_{\delta,\alpha_0}, \in)$$

and because $\beta_0 < \beta_1$ is such that $\alpha_0 = \max(C_\delta \cap (\beta_0 + 1))$, then applying the isomorphism π_{δ,α_0} to the right side (which fixes α_0) we obtain

$$(\mathfrak{H}^{(N_{\delta,\beta_1,\alpha_1}, \in, A_{\delta,\beta_1,\alpha_1}, \partial_{\delta,\beta_1,\alpha_1})}(\alpha_0), \in) \simeq (N_{\delta,\alpha_0,\beta_0}, \in).$$

Finally, since $\pi_{\delta,\alpha_1}(A) = A_{\delta,\beta_1,\alpha_1}$, $\pi_{\delta,\alpha_0}(A) = A_{\delta,\beta_0,\alpha_0}$, and $\pi_{\delta,\alpha_1}(\delta) = \partial_{\delta,\beta_1,\alpha_1}$, $\pi_{\delta,\alpha_0}(\delta) = \partial_{\delta,\beta_0,\alpha_0}$, we have

$$(\mathfrak{H}^{N_{\delta,\beta_1,\alpha_1}}(\alpha_0), \in A_{\delta,\beta_1,\alpha_1}, \partial_{\delta,\beta_1,\alpha_1}) \text{ is isomorphic to } (N_{\delta,\beta_0,\alpha_0}, \in, A_{\delta,\beta_0,\alpha_0}, \partial_{\delta,\beta_0,\alpha_0}),$$

therefore (ii) holds. The uniqueness easily follows from the facts that the embedding of $(N_{\delta,\beta_0,\alpha_0}, \in, A_{\delta,\beta_0,\alpha_0}, \partial_{\delta,\beta_0,\alpha_0})$ has to fix the ordinals less than α_0, and elementary embeddings uniquely extend to Skolem-hulls.

For (iii) suppose that $\alpha_0 < \alpha_1$, and note that

$$(N_{\delta,\beta_1,\alpha_1}, \in) \models \text{“}\alpha_1 \text{ is the least uncountable ordinal , } \alpha_0 \text{ is countable”},$$

and for $M \prec (N_{\delta,\beta_1,\alpha_1}, \in, A_{\delta,\beta_1,\alpha_1}, \partial_{\delta,\beta_1,\alpha_1})$ if $\alpha_0 \cup \{\alpha_0\} \subseteq M$ then consider the corresponding submodel $M' \prec (M_{\delta,\alpha_1}, \in, A, \delta)$, for which $M' \supseteq M_{\delta,\alpha_0+1}$. But (recalling (4-8)) since $\max(C_\delta \cap (\beta_0+1)) = \alpha_0$ we obtain $\beta_0 \cup \{\beta_0\} \subseteq M' \subseteq M_{\delta,\alpha_1}$, that can happen only if β_0 is smaller than the least uncountable ordinal in $N_{\delta,\beta_1,\alpha_1}$, α_1. But then $\beta_0 \in M \cap \alpha_1$. □

The next claim will verify that T is a tree of height ω_1 (for the transitivity of $\leq_T$ use the claim two times).

Claim 4.13. For a fixed $\delta_1 \in [\xi, \omega_2)$, $\beta_0 \leq \beta_1 < \omega_1$, let $\alpha_1 = \max(C_{\delta_1} \cap (\beta_1+1)$, and fix arbitrary $\alpha_0 \in \omega_1$, $\delta_0 \in [\xi, \omega_2)$. Then $(\beta_0, t_{\delta_0,\beta_0,\alpha_0}) \leq_T (\beta_1, t_{\delta_1,\beta_1,\alpha_1})$ if and only if $t_{\delta_0,\beta_0,\alpha_0} = t_{\delta_1,\beta_0,\max(C_{\delta_1}\cap(\beta_0+1))}$.

Proof. We only have to check the “only if” part, but first observe that Definition 4.11 clearly implies that up to isomorphism there exists only one t for which $(\beta_0, t) \leq (\beta_1, t_{\delta_1,\beta_1,\alpha_1})$. Now the claim is the consequence of the fact that $t_{\delta_*,\beta_0,\alpha_*} \neq t_{\delta_{**},\beta_0,\alpha_{**}}$ implies that they are not isomorphic as structures of the language $\mathcal{L}_\in(R_A, c_\partial)$: For transitive sets N and N' with $X, \partial \in N$, $X', \partial' \in N'$ the structures $(N, \in, X, \partial)$, $(N', \in, X', \partial')$ are isomorphic if and only if $N = N'$, $X = X'$ and $\partial = \partial'$ (since by the uniqueness of the Mostowski collapse we know that $(N, \in) \simeq (N', \in)$ if and only if $N = N'$). □

Lemma 4.14. *For each $\beta < \omega_1$ the β-th level of T is countable.*

Proof. By Claim 4.13 we have that the β-th level of T is

$$T_{\leq\beta} \setminus T_{<\beta} = \{(\beta, t_{\delta,\beta,\alpha}) : \delta \in [\xi, \omega_2), \alpha = \max(C_\delta \cap (\beta+1))\}.$$

For a fixed $\delta \in [\xi, \omega_2)$ fix $\alpha = \max(C_\delta \cap (\beta+1))$ too, and consider the structure

$$t_{\delta,\beta,\alpha} = (N_{\delta,\beta,\alpha}, \in, A_{\delta,\beta,\alpha}, \partial_{\delta,\beta,\alpha}),$$

where $N_{\delta,\beta,\alpha}$ is the Mostowski collapse of $(M_{\delta,\alpha}, \in)$ (by the isomorphism $\pi_{\delta,\alpha}$), and $A_{\delta,\beta,\alpha} = A \cap \alpha$. Now (4-6) states $M_{\delta,\alpha} \prec (L_{\delta'}, \in, A)$ then (recalling $M_{\delta,\alpha} \cap \omega_1 = \alpha$, and $\pi_{\delta,\alpha} \restriction \alpha = \mathrm{id}_\alpha$) by Lemma 4.4

$$N_{\delta,\beta,\alpha} = L_\gamma[A \cap \alpha]$$

for some $\gamma = \gamma(\delta, \alpha) \in (\alpha, \omega_1)$. Now we determine an upper bound γ_α for the set $\{\gamma(\delta, \alpha) : \delta \in [\xi, \omega_2) \wedge \alpha \in C_\delta\}$. If we have such a bound for each possible $\alpha \leq \beta$, then letting γ_∞ denote $\sup\{\gamma_\alpha : \alpha \le \beta\}$, we get

$$\{t_{\delta,\beta,\alpha}) : \delta \in [\xi, \omega_2), \alpha = \max(C_\delta \cap (\beta+1))\}\} \subseteq \{(L_\gamma[A\cap\alpha], \in, A\cap\alpha, \partial) : \gamma \le \gamma_\infty, \alpha \le \beta, \partial < \gamma\},$$

which latter set is obviously countable, this will finish the proof of the lemma.

So fix $\alpha \le \beta$ and $\delta \in [\xi, \omega_2)$ such that $\alpha \in C_\delta$. Now we have two cases depending on whether there is any $(\text{cardinal})^{L[A\cap\alpha]}$ in (α, ω_1). If $\lambda \in (\alpha, \omega_1)$ is a cardinal in the inner model $L[A\cap\alpha]$, then for each δ if $\alpha = \max(C_\delta \cap (\beta+1))$, then the transitive set $N_{\delta,\beta,\alpha}$ cannot contain λ, as $M_{\delta,\alpha}$ sees ω_1 as the largest cardinal, and $\pi_{\delta,\alpha}(\omega_1) = \alpha$. This case choosing $\gamma_\alpha = \lambda$ works.

On the other hand, if $(|\alpha|^+)^{L[A\cap\alpha]} = \omega_1$, then we first prove that $\alpha \in C_\delta$ implies $(|\alpha| = \omega)^{L[A\cap\alpha]}$: otherwise in $M_{\delta,\alpha}$, and in $N_{\delta,\beta,\alpha}$ each ordinal less than α are countable, thus as well in $L[A\cap\alpha]$. Then it is easy to see that the condition

$$(\lambda \text{ is the unique cardinal in } (\omega, \omega_1^V))^{L[A\cap\lambda]}$$

cannot hold for two different λ, therefore α can be defined in $L[A]$. But then using Claim 4.5 with $X = A\cap\alpha$ we have that for each $\zeta \in (\alpha, \omega_1)$ there is a bijection $f_\zeta \in L_{\omega_1}[A\cap\alpha]$ between α and ζ, therefore α can be defined also in $L_{\delta'}[A]$, and $M \prec (L_{\delta'}[A], \in)$ implies $\alpha \in M$, contradicting that $M_{\delta,\alpha} \cap \omega_1 = \alpha$ (which holds by $\alpha \in C_\delta$). Then $(|\alpha| = \omega)^{L[A\cap\alpha]}$ and Claim 4.5 implies that there is an ordinal $\lambda < \omega_1$ such that there exists a bijection between α and ω in $L_\lambda[A\cap\alpha]$, implying

$$N_{\delta,\beta,\alpha} = L_{\gamma(\delta,\alpha)}[A\cap\alpha] \subsetneq L_\lambda[A\cap\alpha],$$

since α is uncountable in $N_{\delta,\beta,\alpha}$. In this case

$$\{\gamma(\delta,\alpha) : \delta \in [\xi, \omega_2) \wedge \alpha \in C_\delta\} \subseteq \gamma_\alpha = \lambda,$$

which completes the proof of Lemma 4.14. □

Now T is obviously a Kurepa tree by the following fact and lemma.

Fact 4.15. The sequence $\langle B_\delta : \delta \in [\xi, \omega_2)\rangle$ lists pairwise distinct cofinal branches in T, where

$$B_\delta = \{(\beta, t_{\delta,\beta,\max(C_\delta\cap(\beta+1))}) : \beta < \omega_1\}.$$

Proof. We only need to prove that $B_\delta \neq B_\gamma$ if $\delta \neq \gamma$. But according to the second statement of Claim 4.12 for each $\beta < \beta' < \omega_1$ there is a unique elementary embedding of $t_{\delta,\beta',\max(C_\delta\cap(\beta'+1))}$ to $t_{\delta,\beta,\max(C_\delta\cap(\beta+1))}$, therefore there is a unique direct-limit of this elementary chain, isomorphic to $\bigcup_{\alpha\in C_\delta} M_{\delta,\alpha}$, which is $(L_{\delta'}[A], \in, A, \delta)$ by Claim 4.10. □

It is only left to prove that each branch of T is of the form B_δ for some $\delta \in [\xi, \omega_2)$ (even in V). The following lemma will complete the proof of Theorem 4.1.

Lemma 4.16. *Let $B \subseteq T$ a cofinal branch in T, $B \in V$. Then $B = B_{\delta_\bullet}$ for a unique $\delta_\bullet \in [\xi, \omega_2)$.*

Proof. Let $t_{\delta_\beta,\beta,\alpha_\beta} = (N_{\delta_\beta,\beta,\alpha_\beta}, \in, A_{\delta_\beta,\beta,\alpha_\beta}, \partial_{\delta_\beta,\beta,\alpha_\beta})$ denote the element in $B \cap (T_{\leq\beta} \setminus T_{<\beta})$. Working in V first we define the following bonding maps: for $\gamma \leq \beta < \omega_1$ let

$$\pi_{\gamma,\beta} : N_{\delta_\gamma,\gamma,\alpha_\gamma} \to N_{\delta_\beta,\beta,\alpha_\beta}$$

be the unique elementary embedding (combining Claim 4.13, and the second statement of Claim 4.12). Since elementary submodels of an elementary submodel are elementary submodels, $\pi_{\beta',\beta} \circ \pi_{\beta'',\beta'}$ is an elementary embedding for each $\beta'' \leq \beta' \leq \beta < \omega_1$, therefore by the uniqueness

$$\text{(for all } \beta'' \leq \beta' \leq \beta < \omega_1) : \ \pi_{\beta',\beta} \circ \pi_{\beta'',\beta'} = \pi_{\beta'',\beta}. \tag{4-14}$$

This elementary chain allows us to define the limit $D = (N_{\omega_1}, \boldsymbol{E}, A_{\omega_1}, \partial_{\omega_1})$ of the directed system $\{t_{\delta_\beta,\beta,\alpha_\beta}, \pi_{\beta',\beta} : \beta' \leq \beta < \omega_1\}$.

Let $\pi_\beta : N_{\delta_\beta,\beta,\alpha_\beta} \to N_{\omega_1}$ be the embedding, $N_\beta = \operatorname{ran}(\pi_\beta)$ (hence $N_{\omega_1} = \bigcup_{\beta<\omega_1} N_\beta$).

First note that $(N_{\omega_1}, \boldsymbol{E})$ is well-founded, otherwise there would be an infinite $\boldsymbol{E}$-decreasing chain in the embedded image of $N_{\delta_\beta,\beta,\alpha_\beta}$ for some (in fact, every large enough) β, contradicting that $(N_{\delta_\beta,\beta,\alpha_\beta}, \in)$ is well-founded. Now (by the $\boldsymbol{E}$-extensionality in N_{ω_1}) we can assume that N_{ω_1} is a Mostowski collapse, i.e., $(N_{\omega_1}, \boldsymbol{E}) = (N_{\omega_1}, \in)$. Then it is easy to see that if $\beta < \omega_1$ for the elementary embedding $\pi_\beta : N_{\delta_\beta,\beta,\alpha_\beta} \to N_{\omega_1}$ we have $\pi_\beta \restriction \alpha_\beta = \mathrm{id}_{\alpha_\beta}$, and $\pi_\beta(\alpha_\beta) = \omega_1$, thus (recalling that $A_{\delta_\beta,\beta,\alpha_\beta} = A \cap \alpha_\beta$) we obtain $(N_{\omega_1}, \boldsymbol{E}, A_{\omega_1}, \partial_{\omega_1}) = (N_{\omega_1}, \in, A, \delta_\bullet)$ for some $\delta_\bullet \in (\omega_1, \omega_2)$. Now we can use Lemma 4.4 (since $(N_{\delta_\beta,\beta,\alpha_\beta}, \in, A_{\delta_\beta,\beta,\alpha_\beta}) \models \sigma$), there exists $\zeta > \delta_\bullet$ such that

$$N_{\omega_1} = L_\zeta[A],$$

and then

$$(N_{\omega_1}, \in, A, \delta_\bullet) = (L_\zeta[A], \in, A, \delta_\bullet).$$

Now because the formula $\sigma' \in \mathcal{L}_\in(R_A, c_\partial)$ from Claim 4.9 holds in $(L_{\delta'}[A], \in, A, \delta)$ (for each $\delta \in [\xi, \omega_2)$) (for our mapping $\delta \mapsto \delta'$ from Definition 4.8) and therefore also in $M_{\delta,\alpha}$, $N_{\delta,\beta,\alpha}$ ($\delta \in [\xi, \omega_2)$), so it must hold in $(N_{\omega_1}, \in, A, \delta_\bullet)$, which means that $\delta_\bullet \geq \xi$, and $\zeta = \delta'_\bullet$, i.e.,

$$(N_{\omega_1}, \in, A, \delta_\bullet) = (L_{\delta'_\bullet}[A], \in, A, \delta_\bullet).$$

Finally, we have to prove that for each $\beta < \omega_1$

$$t_{\delta_\beta,\beta,\alpha_\beta} = (N_{\delta_\beta,\beta,\alpha_\beta}, \in, A_{\delta_\beta,\beta,\alpha_\beta}, \partial_{\delta_\beta,\beta,\alpha_\beta}) = t_{\delta_\bullet,\beta,\max(C_{\delta_\bullet} \cap (\beta+1))}$$

by arguing (having β fixed) that for a large enough γ

$$(\beta, t_{\delta_\bullet,\beta,\max(C_{\delta_\bullet}\cap(\beta+1))}) \leq_T (\gamma, t_{\delta_\gamma,\gamma,\alpha_\gamma}).$$

Let $\alpha = \max(C_{\delta_\bullet} \cap (\beta+1))$, $\alpha' = \min(C_{\delta_\bullet} \setminus (\beta+1))$, $\beta' = \alpha'$, and consider the models $M_{\delta_\bullet,\alpha}, M_{\delta_\bullet,\alpha'} \prec (L_{\delta'_\bullet}[A], \in, A, \delta_\bullet)$. Choose $\gamma \geq \beta'$, $\gamma < \omega_1$ so that $N_\gamma = \pi_\gamma[N_{\delta_\gamma,\gamma,\alpha_\gamma}] \supseteq M_{\delta_\bullet,\alpha'}$. Then

$$\alpha_\gamma \geq \alpha' > \beta + 1, \tag{4-15}$$

and $\alpha' \cup \{\omega_1\} \subseteq N_\gamma \prec (L_{\delta'_\bullet}[A], \in, A, \delta_\bullet)$ with (4-7) imply

$$\mathfrak{H}^{(N_\gamma,\in,A\cap N_\gamma,\delta_\bullet)}(\alpha) = \mathfrak{H}^{(L_{\delta'_\bullet}[A],\in,A,\delta_\bullet)}(\alpha) = M_{\delta_\bullet,\alpha}.$$

Therefore in $(N_\gamma, \in, A \cap N_\gamma, \delta_\bullet) \simeq (N_{\delta_\gamma,\gamma,\alpha_\gamma}, \in, A_{\delta_\gamma,\gamma,\alpha_\gamma}, \partial_{\delta_\gamma,\gamma,\alpha_\gamma})$ there is an elementary submodel isomorphic to $(M_{\delta_\bullet,\alpha}, \in, A \cap M_{\delta_\bullet,\alpha}, \delta_\bullet)$, which latter is isomorphic to $(N_{\delta_\bullet,\beta,\alpha}, \in, A \cap \alpha, \partial_{\delta_\bullet,\beta,\alpha})$, thus (ii) from Definition 4.11 holds.

Similarly, using also (4-10) and the definitions of α, α',

$$\mathfrak{H}^{(N_\gamma,\in,A\cap N_\gamma,\delta_\bullet)}(\alpha+1) = M_{\delta_\bullet,\alpha+1} = M_{\delta_\bullet,\alpha'} \supseteq \alpha' \supseteq \beta \cup \{\beta\},$$

and since the isomorphism between

$$(N_\gamma, \in, A \cap N_\gamma, \delta_\bullet) \quad \text{and} \quad (N_{\delta_\gamma,\gamma,\alpha_\gamma}, \in, A_{\delta_\gamma,\gamma,\alpha_\gamma}, \partial_{\delta_\gamma,\gamma,\alpha_\gamma})$$

fixes the ordinals less than or equal to α' we obtain

$$\mathfrak{H}^{(N_{\delta_\gamma,\gamma,\alpha_\gamma},\in,A_{\delta_\gamma,\gamma,\alpha_\gamma},\partial_{\delta_\gamma,\gamma,\alpha_\gamma})}(\alpha+1) \supseteq \beta \cup \{\beta\}.$$

Therefore recalling (4-15) we obtain that (iii) (of Definition 4.11) holds as well. □

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Received March 23, 2020. Revised December 5, 2020.

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

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Sh:1189

PACIFIC JOURNAL OF MATHEMATICS

Volume 311 No. 2 April 2021
