THE KEISLER-SHELAH ISOMORPHISM THEOREM AND THE CONTINUUM HYPOTHESIS

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Abstract. We show that Keisler's isomorphism theorem implies CH. We also provide some consistency results about Keisler and Shelah isomorphism theorems in the absence of CH.

§ 1. introduction

The Keisler-Shelah isomorphism theorem provides a characterization of elementary equivalence. It says that two models of a theory are elementarily equivalent if and only if they have isomorphic ultrapowers. In [3], Keisler has shown, assuming CH, that in a countable language $\mathcal{L}$, two $\mathcal{L}$-models $\mathbf{M}, \mathbf{N}$ of size $\leq 2^{\aleph_0}$, are elementary equivalent if and only if there exist an ultrafilter $\mathcal{U}$ on ω such that $\mathbf{M}^\omega/\mathcal{U} \simeq \mathbf{N}^\omega/\mathcal{U}$. Later Shelah [4] removed the CH assumption in Keisler's theorem, by showing that if $\mathcal{L}$ is a countable language and $\mathbf{M}, \mathbf{N}$ are countable $\mathcal{L}$-models, then $\mathbf{M} \equiv \mathbf{N}$ if and only if there exists an ultrafilter $\mathcal{U}$ on 2^ω such that $\mathbf{M}^\omega/\mathcal{U} \simeq \mathbf{N}^\omega/\mathcal{U}$.

In [6], Shelah has constructed a model of ZFC in which $2^{\aleph_0} = \aleph_2$ and in which there are countable graphs $\mathbf{\Delta} \equiv \mathbf{\Gamma}$ such that for no ultrafilter $\mathcal{U}$ on ω, $\mathbf{\Delta}^\omega/\mathcal{U} \simeq \mathbf{\Gamma}^\omega/\mathcal{U}$. This shows that CH is an essential assumption for Keisler's theorem, even for countable models.

In this paper we discuss Keisler's theorem in the absence of CH and prove some related results. First we show that Keisler's theorem is indeed equivalent to CH by proving the following theorem.

Theorem 1.1. *Suppose $2^{\aleph_0} \geq \aleph_2$. Then Keisler's isomorphism theorem fails.*

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The counterexample we consider for the above theorem comes from the theory of dense linear orders, see Theorem 2.1. It is known from the work of Ellentuck and Rucker [2] that if Martin's axiom+$\neg$CH holds, then there exists an ultrafilter $\mathcal{U}$ on ω such that for any countable structure $\mathbf{M}$, the ultrapower $\mathbf{M}^\omega/\mathcal{U}$ is saturated. In particular if $\mathbf{M} \equiv \mathbf{N}$ are countable models of the same vocabulary, then $\mathbf{M}^\omega/\mathcal{U} \simeq \mathbf{N}^\omega/\mathcal{U}$. We consider models of larger size and ask for the same conclusion. In particular, we prove the following:

Theorem 1.2. *Suppose* $2^{\aleph_0} > \aleph_1 = \mathrm{cf}(2^{\aleph_0})$ *and* $\mathrm{Cov}(\text{meagre}) = 2^{\aleph_0}$. *Then Keisler's isomorphism theorem holds for models of size* $\aleph_1$.

We also prove a related consistency result in the generic extension obtained by adding many Cohen reals, which allows us to remove the cofinality restriction of the above theorem.

§ 2. Keisler's theorem and the CH

In this section we prove the following theorem which immediately implies Theorem 1.1.

Theorem 2.1. *Suppose* $2^{\aleph_0} \geq \aleph_2$. *Then there are models* $\mathbf{M}, \mathbf{N}$ *of the theory* $\mathrm{Th}(\mathbb{Q}, <)$ *of size* $\aleph_0, \aleph_2$ *respectively such that for no ultrafilter* $\mathcal{U}$ *on* ω, $\mathbf{M}^\omega/\mathcal{U} \simeq \mathbf{N}^\omega/\mathcal{U}$.

Proof. Let $\mathbf{M} = (\mathbb{Q}, <)$ and let $\mathbf{N}$ be a dense linear order of cardinality $\aleph_2$ such that for some $a, b \in \mathbf{N}$ we have $\mathrm{cf}(\mathbf{N}_a) = \aleph_1$ and $\mathrm{cf}(\mathbf{N}_b) = \aleph_2$, where for each $c \in \mathbf{N}$,

$$\mathbf{N}_c = \{d \in \mathbf{N} : d <_{\mathbf{N}} c\}.$$

We show that $\mathbf{M}$ and $\mathbf{N}$ are as required. Suppose, towards a contradiction, that for some ultrafilter $\mathcal{U}$ on ω, there exists an isomorphism $f : \mathbf{N}^\omega/\mathcal{U} \simeq \mathbf{M}^\omega/\mathcal{U}$. To simplify the notation, let us set $\mathbf{M}_* = \mathbf{M}^\omega/\mathcal{U}$ and $\mathbf{N}_* = \mathbf{N}^\omega/\mathcal{U}$. Let

$$a_* = [\langle a : n < \omega \rangle]_{\mathcal{U}} \in \mathbf{N}_*$$

and

$$b_* = [\langle b : n < \omega \rangle]_{\mathcal{U}} \in \mathbf{N}_*.$$

By the choice of elements a and b we have:

Claim 2.2. $\mathrm{cf}((\mathbf{N}_*)_{a_*}) = \aleph_1$ *and* $\mathrm{cf}((\mathbf{N}_*)_{b_*}) = \aleph_2$.

Proof. Let us show that $\mathrm{cf}((\mathbf{N}_*)_{a_*}) = \aleph_1$. Pick a $<_{\mathbf{N}}$-increasing sequence $\langle a_i : i < \omega_1\rangle$ which is $<_{\mathbf{N}}$-cofinal in a. Then the sequence $\langle (a_i)_* : i < \omega_1\rangle$, where $(a_i)_* = [\langle a_i : n < \omega\rangle]_{\mathcal{U}}$, witnesses that $\mathrm{cf}((\mathbf{N}_*)_{a_*}) \leq \aleph_1$. Now suppose that $\langle [f_n]_{\mathcal{U}} : n < \omega\rangle$ is a $<_{\mathbf{N}_*}$-increasing sequence below a_*. Without loss of generality, each $f_n : \omega \to \mathbf{N}_a$. For each n, pick $i_n < \omega_1$ such that for all $k \in \omega$, $f_n(k) <_{\mathbf{N}} a_{i_n}$. Set $i_* = \sup_{n<\omega} i_n$. Then $i_* < \omega_1$ and for each $n < \omega$, $[f_n]_{\mathcal{U}} <_{\mathbf{N}_*} (a_i)_*$. Thus the sequence $\langle [f_n]_{\mathcal{U}} : n < \omega\rangle$ is bounded below a_*. It follows that $\mathrm{cf}((\mathbf{N}_*)_{a_*}) = \aleph_1$ as claimed. The proof of $\mathrm{cf}((\mathbf{N}_*)_{b_*}) = \aleph_2$ is the same. □

Set $a_\dagger = f(a_*)$ and $b_\dagger = f(b_*)$.

Claim 2.3. $\mathrm{cf}((\mathbf{M}_*)_{a_\dagger}) = \aleph_1$ *and* $\mathrm{cf}((\mathbf{M}_*)_{b_\dagger}) = \aleph_2$.

Proof. It is trivial by the choice of $a_\dagger$ and $b_\dagger$. □

Claim 2.4. *There is a function $F : \mathbf{M}^3 \to \mathbf{M}$ such that for every $c, d \in \mathbf{M}$, the formula $F(x, c, d)$ defines an automorphism of $\mathbf{M}$ which maps c to d.*

Proof. Define F by $F(x, y, z) = x - y + z$. The function F is easily seen to be as required. □

It follows from Claim 2.4 that for some function F_*, $(\mathbf{M}_*, F_*) = (\mathbf{M}, F)^\omega/\mathcal{U}$. Then by the choice of F, the function F_* has the following property:

($*$): $F_* : \mathbf{M}_*^3 \to \mathbf{M}_*$ is a function such that for all $c, d \in \mathbf{M}_*$, the formula $F_*(x, c, d)$ defines an automorphism of $\mathbf{M}_*$ which maps c to d.

In particular $F_*(x, a_\dagger, b_\dagger)$ defines an automorphism of $\mathbf{M}_*$ which maps $a_\dagger$ to $b_\dagger$. Thus we must have

$$\mathrm{cf}((\mathbf{M}_*)_{a_\dagger}) = \mathrm{cf}((\mathbf{M}_*)_{b_\dagger}),$$

which contradicts Claim 2.3. □

By the above result and Keisler's theorem, we have the following corollary.

Corollary 2.5. *The following are equivalent:*

(a) CH,

(b) *Keisler's isomorphism theorem: if $\mathcal{L}$ is a countable language and $\mathbf{M}, \mathbf{N}$ are $\mathcal{L}$-models of size $\leq 2^{\aleph_0}$, then $\mathbf{M} \equiv \mathbf{N}$ if and only if there exists an ultrafilter $\mathcal{U}$ on ω such that $\mathbf{M}^\omega/\mathcal{U} \simeq \mathbf{N}^\omega/\mathcal{U}$.*

§ 3. Keisler-Shelah theorem for models of cardinality $\aleph_1$

In this section, we ask to what extent the Keisler and Shelah isomorphism theorems can hold for models of uncountable cardinality. We prove some theorems which by the result of the previous section are, in some sense, optimal. Our proofs rely on the following lemma.

Lemma 3.1. *(see* [1, Theorem 7.13]*) Suppose* $\mathrm{Cov}(\text{meagre}) = 2^{\aleph_0}$*. Then* $\mathrm{MA}_\kappa(\text{countable})$ *holds for all* $\kappa < 2^{\aleph_0}$*.*

Let us start by proving Theorem 1.2.

Theorem 3.2. *Suppose* $2^{\aleph_0} > \aleph_1 = \mathrm{cf}(2^{\aleph_0})$ *and* $\mathrm{Cov}(\text{meagre}) = 2^{\aleph_0}$*. Suppose* $\mathbf{M}_0 \equiv \mathbf{M}_1$ *are models of size* $\leq \aleph_1$ *of the same countable vocabulary* $\mathcal{L}$*. Then for some ultrafilter* $\mathcal{U}$ *on* ω, $\mathbf{M}_0^\omega/\mathcal{U} \simeq \mathbf{M}_1^\omega/\mathcal{U}$.

Proof. Before giving the details of the proof, let us sketch the main idea. We would like to find an ultrafilter $\mathcal{U}$ on ω and enumerations $\langle g^0_\alpha : \alpha < 2^{\aleph_0}\rangle$ and $\langle g^1_\alpha : \alpha < 2^{\aleph_0}\rangle$ of $\mathbf{M}_0^\omega$ and $\mathbf{M}_1^\omega$ respectively, such that

$$((\mathbf{M}_0)^\omega/\mathcal{U}, [g^0_0]_\mathcal{U}, \cdots, [g^0_\alpha]_\mathcal{U}, \cdots) \equiv ((\mathbf{M}_1)^\omega/\mathcal{U}, [g^1_0]_\mathcal{U}, \cdots, [g^1_\alpha]_\mathcal{U}, \cdots). \tag{3.1}$$

This will show that the function $\langle([g^0_\alpha]_\mathcal{U}, [g^1_\alpha]_\mathcal{U}) : \alpha < 2^{\aleph_0}\rangle$ is an isomorphism between $\mathbf{M}_0^\omega/\mathcal{U}$ and $\mathbf{M}_1^\omega/\mathcal{U}$. On the other hand, 3.1 means that for all $\mathcal{L}$-formula $\phi(x_0, \cdots, x_{n-1})$ and all $\beta_0, \cdots, \beta_{n-1} < 2^{\aleph_0}$,

$$\left\{k < \omega : \mathbf{M}_0 \models \phi(g^0_{\beta_0}(k), \cdots, g^0_{\beta_{n-1}}(k)) \Leftrightarrow \mathbf{M}_1 \models \phi(g^1_{\beta_0}(k), \cdots, g^1_{\beta_{n-1}}(k))\right\} \in \mathcal{U}.$$

We define by induction on $\alpha < 2^{\aleph_0}$, a sequence $\langle(\mathcal{U}_\alpha, g^0_\alpha, g^1_\alpha) : \alpha < 2^{\aleph_0}\rangle$, where $\langle\mathcal{U}_\alpha : \alpha < 2^{\aleph_0}\rangle$ is an increasing and continuous chain of filters on ω such that 3.1

holds whenever $\mathcal{U}$ replaced by $\mathcal{U}_{\alpha+1}$. To make sure that g^0_α's and g^1_α's enumerate all elements of $\mathbf{M}_0^\omega$ and $\mathbf{M}_1^\omega$ respectively, we use a back and forth construction. To make sure that the construction continues to work at all levels below $2^{\aleph_0}$, we use the assumption Cov(meagre) $= 2^{\aleph_0}$ and proceed in such a way that $\mathcal{U}_\alpha$ is generated by $\leq \aleph_0 + |\alpha|$ many elements.

Let us now go into the details of the proof. Let $\langle \lambda_i : i < \omega_1 \rangle$ be an increasing and continuous sequence of cardinals $\geq \aleph_1$ cofinal in $2^{\aleph_0}$ and for $\ell < 2$ let $\langle \mathbf{M}^\ell_i : i < \omega_1 \rangle$ be an increasing and continuous chain of elementary submodels of $\mathbf{M}_\ell$ such that for all $i < \omega_1, ||\mathbf{M}^\ell_i|| = \aleph_0$ and $\mathbf{M}_\ell = \bigcup\limits_{i<\omega_1} \mathbf{M}^\ell_i$. Let $\langle f^\ell_\alpha : \alpha < 2^{\aleph_0} \rangle$ list elements of $\mathbf{M}^\omega_\ell$ such that

$$\alpha < \lambda_i \implies f^\ell_\alpha \in (\mathbf{M}^\ell_i)^\omega.$$

Let also $\langle X_\alpha : \alpha < 2^{\aleph_0} \rangle$ enumerate $\mathcal{P}(\omega)$. By induction on $\alpha < 2^{\aleph_0}$ and using a back and forth construction, we build the triple $(\mathcal{U}_\alpha, g^0_\alpha, g^1_\alpha)$ such that:

(a) $g^0_\alpha \in \mathbf{M}^\omega_0$, furthermore if $\alpha < \lambda_i$, then $g^0_\alpha \in (\mathbf{M}^0_i)^\omega$,

(b) $g^1_\alpha \in \mathbf{M}^\omega_1$, furthermore if $\alpha < \lambda_i$, then $g^1_\alpha \in (\mathbf{M}^1_i)^\omega$,

(c) for $i < \omega_1$ and $\ell < 2$, $\{g^\ell_\alpha : \alpha < \lambda_i\} = \{f^\ell_\alpha : \alpha < \lambda_i\}$,

(d) $\mathcal{U}_\alpha$ is a filter on ω generated by $\leq \aleph_0 + |\alpha|$ sets containing all co-finite subsets of ω,

(e) if $\phi(x_0, \cdots, x_{n-1})$ is a formula of $\mathcal{L}$ and $\beta_0, \cdots, \beta_{n-1} \leq \alpha$, then the set $Y_{\phi, \langle \beta_0, \cdots, \beta_{n-1} \rangle}$ defined as

$$\left\{ k < \omega : \mathbf{M}_0 \models \phi(g^0_{\beta_0}(k), \cdots, g^0_{\beta_{n-1}}(k)) \Leftrightarrow \mathbf{M}_1 \models \phi(g^1_{\beta_0}(k), \cdots, g^1_{\beta_{n-1}}(k)) \right\},$$

belongs to $\mathcal{U}_{\alpha+1}$,

(f) if $\alpha < \beta < 2^{\aleph_0}$, then $\mathcal{U}_\alpha \subseteq \mathcal{U}_\beta$.

(g) if α is a limit ordinal, then $\mathcal{U}_\alpha = \bigcup\limits_{\beta<\alpha} \mathcal{U}_\beta$,

(h) for all $\alpha < 2^{\aleph_0}$, either $X_\alpha \in \mathcal{U}_{\alpha+1}$ or $\omega \setminus X_\alpha \in \mathcal{U}_{\alpha+1}$.

As in Shelah [5, Ch VI, §3], there is no problem in carrying the induction, however let us elaborate the main point of the proof. The only difficulty in carrying the induction is clause (e). Thus suppose that $\alpha < 2^{\aleph_0}$ and the construction is done up to α. Let also $i < \omega_1$ be such that $\alpha < \lambda_i$. First suppose that α is an even ordinal.

Let $g^0_\alpha = f^0_{\gamma_\alpha}$, where γ_α is the least ordinal such that $f^0_{\gamma_\alpha} \notin \{g^0_\beta : \beta < \alpha\}$. Note that $\gamma_\alpha < \lambda_i$. Let $\mathbb{P}$ be the forcing notion consisting of all maps $p : \mathrm{dom}(p) \to M^1_i$, where $\mathrm{dom}(p)$ is a finite subset of ω, ordered by inclusion. $\mathbb{P}$ is countable. Define the following dense sets:

- $D_n = \{p \in \mathbb{P} : n \in \mathrm{dom}(p)\}$, where $n < \omega$.
- For any $A \in \mathcal{U}_\alpha$, any finite sequence $\vec{\phi} = \langle \phi_\iota(x_0, \cdots, x_{n_\iota - 1}, y) : \iota \in I \rangle$, any finite sequence $\langle \vec{\beta}^\ell_\iota : \iota \in I, \ell \in J_\iota \rangle$, where $\vec{\beta}^\ell_\iota = \langle \beta^\ell_{\iota,0}, \cdots, \beta^\ell_{\iota, n_\iota - 1} \rangle$ consists of ordinals less than α and $m < \omega$ let $\Sigma_{A, \vec{\phi}, \langle \vec{\beta}^\ell_\iota : \iota \in I, \ell \in J_\iota \rangle, m}$ be the set of all condition $p \in \mathbb{P}$ such that for some $k \in \mathrm{dom}(p) \cap A$ with $k > m$ and all $\iota \in I$ and $\ell \in J_\iota$:

$$\mathbf{M}^0_i \models \phi_\iota(g^0_{\beta^\ell_{\iota,0}}(k), \cdots, g^0_{\beta^\ell_{\iota,n_\iota-1}}(k), g^0_\alpha(k)) \Leftrightarrow \mathbf{M}^1_i \models \phi_\iota(g^1_{\beta^\ell_{\iota,0}}(k), \cdots, g^1_{\beta^\ell_{\iota,n_\iota-1}}(k), p(k)).$$

As MA_{λ_i}(countable) holds, there exists a filter $\mathbf{G} \subseteq \mathbb{P}$ meeting all the above dense sets. Set $g^1_\alpha = \bigcup_{p \in \mathbf{G}} p$. Let $\mathcal{U}'_{\alpha+1}$ be the filter generated by

$$\mathcal{U}_\alpha \cup \{Y_{\phi, \langle \beta_0, \cdots, \beta_{n-1} \rangle} : \phi, \beta_0, \cdots, \beta_{n-1} \text{ as in clause (e)}\}.$$

By the choice of the sets $\Sigma_{A, \vec{\phi}, \langle \vec{\beta}^\ell_\iota : \iota \in I, \ell \in J_\iota \rangle, m}$, the above set has the finite intersection property and hence $\mathcal{U}'_{\alpha+1}$ is a proper filter. Now let $\mathcal{U}_{\alpha+1}$ be the filter generated by $\mathcal{U}'_{\alpha+1} \cup \{X_\alpha\}$ if this is a proper filter and let $\mathcal{U}_{\alpha+1}$ be the filter generated by $\mathcal{U}'_{\alpha+1} \cup \{\omega \setminus X_\alpha\}$ otherwise. If α is an odd ordinal, proceed in the same way, changing the role of the indices 0 and 1.

This completes the induction construction. Set

$$\mathcal{U} = \bigcup \{\mathcal{U}_\alpha : \alpha < 2^{\aleph_0}\}.$$

Then $\mathcal{U}$ is a non-principal ultrafilter on ω and $\mathbf{M}^\omega_0 / \mathcal{U} \simeq \mathbf{M}^\omega_1 / \mathcal{U}$ as witnessed by the function

$$\langle ([g^0_\alpha]_\mathcal{U}, [g^1_\alpha]_\mathcal{U}) : \alpha < 2^{\aleph_0} \rangle.$$

This completes the proof of the theorem. □

We close the paper by proving the following consistency result, which is an analogue of Theorem 1.2, but the cofinality restriction on $2^{\aleph_0}$ is removed.

Theorem 3.3. *Suppose $\lambda > \aleph_1$ and $\lambda^{\aleph_0} = \lambda$. Let $\mathbb{P} = \mathrm{Add}(\omega, \lambda)$ be the Cohen forcing for adding λ many new Cohen reals. Then in $V[\mathbf{G}_{\mathbb{P}}]$, the following holds: if $\mathbf{M}_0 \equiv \mathbf{M}_1$ are models of size $\leq \aleph_1$ of the same countable vocabulary $\mathcal{L}$, then for some ultrafilter $\mathcal{U}$ on ω, $\mathbf{M}_0^\omega/\mathcal{U} \simeq \mathbf{M}_1^\omega/\mathcal{U}$.*

Proof. First note that $V[\mathbf{G}_{\mathbb{P}}] \models$ "Cov(meagre) $= \lambda$". We may assume that $\mathrm{cf}(\lambda) > \aleph_1$, as otherwise the result follows from Theorem 1.2. Now suppose that $\mathbf{M}_0 \equiv \mathbf{M}_1$ are models of size $\leq \aleph_1$ of a countable vocabulary in $V[\mathbf{G}_{\mathbb{P}}]$. Then for some $\bar{\lambda} < \lambda$, $\mathbf{M}_0, \mathbf{M}_1 \in V[\mathbf{G}_{\mathbb{P}\restriction\bar{\lambda}}]$. By replacing V by $V[\mathbf{G}_{\mathbb{P}\restriction\bar{\lambda}}]$, we may assume that $\mathbf{M}_0, \mathbf{M}_1 \in V$.

As $|\lambda \cdot \omega_1| = \lambda$, we may assume that $\mathbb{P}$ is $\mathrm{Add}(\omega, \lambda \cdot \omega_1)$ so that forcing with $\mathbb{P}$ adds a sequence $\langle r_{\alpha,i} : \alpha < \lambda, i < \omega_1\rangle$ of reals of order type $\lambda \cdot \omega_1$.

For $i < \omega_1$, set $\mathbb{P}_i = \mathrm{Add}(\omega, \lambda \cdot i)$. As $\mathbb{P}$ is c.c.c., for every $X \subseteq \omega, X \in V[\mathbf{G}_{\mathbb{P}}]$, there exists some $i < \omega_1$ such that $X \in V[\mathbf{G}_{\mathbb{P}_i}]$. Proceed as in the proof of Theorem 1.2 with:

- $\lambda_i = \lambda \cdot (1 + i)$,
- $\langle \mathbf{M}_i^\ell : i < \omega_1\rangle$ as there,
- $\langle f_\alpha^\ell : \alpha < 2^{\aleph_0}\rangle$ is an enumeration of $\mathbf{M}_\ell^\omega$ in such a way that for $\alpha < \lambda \cdot (1+i)$, $f_\alpha^\ell \in \mathbf{M}_i^\ell$.

The rest of the argument is essentially as before. □

References

[1] Blass, Andreas; Combinatorial cardinal characteristics of the continuum. Handbook of set theory. Vols. 1, 2, 3, 395-489, Springer, Dordrecht, 2010.

[2] Ellentuck, Erik; Rucker, R. v. B.; Martin's axiom and saturated models. Proc. Amer. Math. Soc. 34 (1972), 243-249.

[3] Keisler, H. Jerome; Ultraproducts and elementary classes. Nederl. Akad. Wetensch. Proc. Ser. A 64 = Indag. Math. 23 1961 477-495.

[4] Shelah, Saharon; Every two elementarily equivalent models have isomorphic ultrapowers. Israel J. Math. 10 (1971), 224-233.

[5] Shelah, Saharon; Classification theory and the number of nonisomorphic models. Second edition. Studies in Logic and the Foundations of Mathematics, 92. North-Holland Publishing Co., Amsterdam, 1990. xxxiv+705 pp. ISBN: 0-444-70260-1

[6] Shelah, Saharon; Vive la différence. I. Nonisomorphism of ultrapowers of countable models. Set theory of the continuum (Berkeley, CA, 1989), 357-405, Math. Sci. Res. Inst. Publ., 26, Springer, New York, 1992.

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