BETWEEN REDUCED POWERS AND ULTRAPOWERS, II.

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Abstract. We prove that, consistently with ZFC, no ultraproduct of countably infinite (or separable metric, non-compact) structures is isomorphic to a reduced product of countable (or separable metric) structures associated to the Fréchet filter. Since such structures are countably saturated, the Continuum Hypothesis implies that they are isomorphic when elementarily equivalent.

The trivializing effect of the Continuum Hypothesis (CH) to the structure of the continuum has been known at least since the times of Sierpiński and Gödel ([23]). The particular instance of this phenomenon that we are concerned with in the present paper is the existence of highly non-canonical isomorphisms between massive quotient structures of cardinality $\mathfrak{c} = 2^{\aleph_0}$. The operation of taking a reduced product $\prod_{\mathcal{F}} A_n$ of a sequence (A_n) of first-order structures often results in a countably saturated structure.[1] This is the case with the two most commonly used reduced products: ultraproducts associated with nonprincipal ultrafilters on $\mathbb{N}$ and reduced products associated with the Fréchet filter. If each A_n has the cardinality of at most $\mathfrak{c}$, then so does $\prod_{\mathcal{F}} A_n$, and the CH implies that the latter structure is saturated. By a classical theorem of Keisler, elementarily equivalent and saturated first-order structures of the same cardinality are isomorphic (see [5, Theorem 5.1.13]). Therefore CH implies that the isomorphism of such reduced products reduces (no pun intended) to elementary equivalence.

In [13], this observation was combined with computation of the theory of the structure (K denotes the Cantor space)

$$C(K, A) = \{f\colon K \to A \mid \ f \text{ is continuous}\} \tag{0.1}$$

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[1] Saturation is a model-theoretic property that enables transfinite constructions of isomorphisms and automorphisms; see e.g., [5, Chapter 5]. It will not be used in this paper. 'Countable saturation' is often referred to as '$\aleph_1$ saturation'.

for a separable (or countable discrete) structure A to prove that, assuming CH we have[2]

$$\prod_{\mathcal{U}} C(K, A) \cong \prod_{\mathrm{Fin}} A \tag{0.2}$$

for any nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ (~~[13, Corollary 3.7]~~[13, Theorem E]). This result is the basis for ~~[13, Theorem A]~~[13, Theorem B], asserting that under CH there exists an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that the quotient map from $\prod_{\mathrm{Fin}} A$ to $\prod_{\mathcal{U}} A$ has a right inverse for every countable (or separable metric) structure A. In the case when A is a C^*-algebra, this significantly simplifies some intricate arguments in Elliott's classification program for nuclear, separable C^*-algebras (see the upcoming [4], also [46] and [33] for related applications of ultrapowers). Although the assumption of CH can be removed from the applications of (0.2) to the Elliott classification programme (~~[13, Theorem D]~~[13, Theorem A]), the question whether (0.2) can be proved in ZFC remained.

A well-known instance of ~~this~~ the trivializing effect of CH is Parovičenko's theorem from general topology. Stated in the dual, Boolean-algebraic, form, it asserts that CH implies that all atomless, countably saturated, Boolean algebras of cardinality $\mathfrak{c}$ are isomorphic. In [44] it was proved that the conclusion of Parovičenko's theorem is equivalent to CH. An alternative proof of this fact is given by the main result of [19] (or by [28]), asserting that if CH fails then there are $2^{\mathfrak{c}}$ nonisomorphic ultrapowers of the countable atomless Boolean algebra associated with nonprincipal ultrafilters on $\mathbb{N}$~~, and clearly~~. By Łoś's Theorem all of these Boolean algebras are atomless, and they are countably saturated and of cardinality $\mathfrak{c}$, being ultrapowers associated with countably incomplete ultrafilters. Clearly, at most one of ~~them~~ these ultrapowers can be isomorphic to $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$. ~~The following two results show that none of them is isomorphic to $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$~~ Theorem A and Theorem B below show that in two of the most popular models of ZFC ~~:~~ ~~assuming~~ in which CH fails (models of forcing axioms and ~~in the original~~ Cohen's original model of ZFC~~in which CH fails~~), none of these ultrapowers is isomorphic to $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$.

Theorem A. *The Proper Forcing Axiom, PFA, implies that $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ is not isomorphic to an ultraproduct of Boolean algebras associated with a nonprincipal ultrafilter on $\mathbb{N}$.*

The study of quotient Boolean algebras of the form $\mathcal{P}(\mathbb{N})/\mathcal{I}$ for an ideal $\mathcal{I}$ on $\mathbb{N}$, dates back at least to Erdös and Ulam (see [8]). The space $\mathcal{P}(\mathbb{N})$ is identified with the Cantor space, and thus equipped with a canonical compact metrizable topology. In [25] it was proved that $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is countably saturated for every F_σ ideal $\mathcal{I}$ that includes the Fréchet ideal, Fin. The essence of the Just–Krawczyk construction is encapsulated in the concept

[2] In [13], $\prod_{\mathrm{Fin}} A$ was denoted A^∞ and $\prod_{\mathcal{U}} A$ was denoted $A^{\mathcal{U}}$, following the notation favoured by operator-algebraists. In the present paper we adopt the notation favoured by logicians and apologize to any stray operator algebraists (see however Corollary D).

of a *layered ideals* in [11], where it was proved that if $\mathcal{I}$ is a layered ideal that includes Fin then $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ is countably saturated. (This class of ideals properly includes that of the F_σ ideals; for example, for any additively indecomposable countable ordinal α the ideal $\{X \subseteq \alpha |$ the order type of X is less than $\alpha\}$ is layered.) Since the quotient over an analytic ideal that includes Fin necessarily has cardinality $\mathfrak{c}$, these results show that all quotients over layered analytic ideals that include Fin are isomorphic under CH. This conclusion extends to the reduced products of countable Boolean algebras, $\prod_{\mathcal{I}} A_n$, associated with layered ideals ([20, Theorem 2.7]).

Note that $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is canonically isomorphic to the reduced product, $\prod_{\mathcal{I}} A$, where A is taken to be the 2-element Boolean algebra. Also note that all of these reduced products are *projectively definable* in the sense that there is an $n \in \mathbb{N}$ and $\mathbf{\Sigma}^1_n$-formulas φ, $\varphi_\wedge$, $\varphi_\vee$, and $\varphi_\setminus$, such that the set $\{x \in \mathbb{R} | \varphi(x)\}$ equipped with the operations defined by $\varphi_\wedge$, $\varphi_\vee$, and $\varphi_\setminus$ is a Boolean algebra.[3]

Theorem B. *In a model obtained by adding at least $\mathfrak{c}^+$ Cohen reals to a model of ZFC the following holds. If $\mathfrak{B}$ is a ~~Boolean algebra definable from a real~~ projectively definable Boolean algebra then $\mathfrak{B}$ is not isomorphic to an ultraproduct of countable Boolean algebras associated with a nonprincipal ultrafilter on $\mathbb{N}$.*

In particular, for any analytic ideal $\mathcal{I}$ on $\mathbb{N}$, the quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is not isomorphic to an ultraproduct of countable Boolean algebras associated with a nonprincipal ultrafilter on $\mathbb{N}$.

Our other main result applies to a ~~more general~~ wider range of structures. For the order property (OP) and the robust order property see Definition 1.1. Any theory in which the order property is witnessed by an atomic formula has the robust order property.

Theorem C. *There exists a forcing extension in which for every countable theory T that has the robust order property the following holds.*

For every sequence (A_n) of countable ~~models of~~ structures in the language of T, every sequence (B_n) of countable ~~structures in the language of~~models of T, and every nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, the following are true.

1. *The ultraproduct $\prod_{\mathcal{U}} B_n$ is not isomorphic to $\prod_{\mathrm{Fin}} A_n$.*
2. *The ultraproduct $\prod_{\mathcal{U}} B_n$ is not isomorphic to an elementary submodel of $\prod_{\mathrm{Fin}} A_n$.*
3. *If the order property of T is witnessed by a quantifier-free formula ~~and each B_n is a model of the theory of $\prod_{\mathrm{Fin}} A_n$~~ then $\prod_{\mathcal{U}} B_n$ does not embed into $\prod_{\mathrm{Fin}} A_n$.*

Since the original impetus for these results drew from the Elliott classification program of C^*-algebras, we'll explicitly state the relevant corollary.

[3] We avoid using the simpler term projective Boolean algebras in order to avoid confusion with Boolean algebras that are projective objects in the category of Boolean algebras.

If A is a C^*-algebra, then the structure $C(K,A)$ as in (0.1) is isomorphic to the tensor product $A \otimes C(K)$. By ~~[13, Corollary 3.7]~~[13, Theorem E], for a separable C^*-algebra A and a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, CH implies that the ultrapower $(A \otimes C(K))^{\mathcal{U}}$ is isomorphic to $A^\infty := \ell_\infty(A)/c_0(A)$, and the isomorphism extends the identity on A (A is routinely identified with its diagonal copies in $A^{\mathcal{U}}$ and A^∞).

Corollary D. *There exists a forcing extension in which the following holds for every separable C^*-algebra A and every ultrafilter $\mathcal{U}$ on $\mathbb{N}$.*

1. *$(A \otimes C(K))^{\mathcal{U}}$ is not isomorphic to A^∞.*
2. *$(A \otimes C(K))^{\mathcal{U}}$ is not isomorphic to a C^*-subalgebra of B^∞ for any separable C^*-algebra B.*

The related conclusion, that $C(K)^{\mathcal{U}}$ is not isomorphic to a C^*-subalgebra of ℓ_∞/c_0, is known to be relatively consistent with ZFC and its variant (known as *Woodin's condition*) plays an important role in Woodin's proof of automatic continuity for homomorphisms of Banach algebras ([6]).

Closely related to the questions answered in theorems stated above is another question trivialized by CH. This is the question on the existence of a universal structure among the ultrapowers of a fixed countable (or separable metric) structure. Some answers to this question which are easy consequences of earlier work of one of the authors are given in §7.

Some notation. We will write $X \Subset Y$ as a shorthand for '$X \subseteq Y$ and X is finite' (this relation is sometimes denoted $X \in [Y]^{<\aleph_0}$). The ideal of finite subsets of a set X is denoted Fin_X (some authors prefer $[X]^{<\aleph_0}$). For the Fréchet ideal $\mathrm{Fin}_{\aleph_0}$ we write Fin.

Rough outline. Our proofs use model theory (§1, §7) and set theory (§2, §4). In §1 we discuss the order property (OP) of first-order theories, discrete and continuous. ~~Several lemmas about the so-called depletions of partial orderings are proved~~ Standard facts about posets are recalled in §2, and depletions of posets are introduced and studied in 3. In §4 we define a functor $E \mapsto \mathbb{H}_E$ from the category of partial orderings into the category of forcing notions. The material from §~~2~~ 3 is used to prove that the forcing $\mathbb{H}_E$ embeds E into the reduced product $\prod_{n\in\mathbb{N}}(n,<)$ (n is identified with $\{0,\dots,n-1\}$) in a particularly gentle way. Theorem C and Corollary D are proved in §5, while Theorem A and Theorem B are proved in §6. In §7 we make remarks about the existence of a universal model among the ultrapowers of countable models of T associated with ultrafilters on $\mathbb{N}$. Some concluding remarks and questions can be found in §8.

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1. Reduced products, the order property, continuous logic

In this section we recall the pertinent definitions and establish the notation $\triangleleft_\varphi$. It should be emphasized that the first-order theory T is not assumed to be complete.

1.1. **Reduced products.** We will use the following convention. Suppose that A_n are structures of the same countable language, $\bar{a}_n$ is a tuple in A_n for all $n \in \mathbb{N}$, and all of these tuples are of the same sort.[4] Then $\bar{a}$ denotes the ~~element~~ tuple $(\bar{a}_n)$ ~~of~~ in $\prod_n A_n$ of the same sort.

If $\mathcal{F}$ is a filter on $\mathbb{N}$ and A_n, for $n \in \mathbb{N}$, are structures of the same language $\mathcal{L}$, then the reduced product $\prod_{\mathcal{F}} A_n$ is defined as follows. Its domain is the quotient of $\prod_n A_n$ over the relation $\bar{a} \sim_{\mathcal{F}} \bar{b}$ if $\{n | a_n = b_n\} \in \mathcal{F}$. The function symbols in $\mathcal{L}$ are interpreted in the natural way (note that $\sim_{\mathcal{F}}$ is a congruence). If $k \geq 1$ and $R(x(0), \dots, x(k-1))$ is a k-ary relation symbol and $\bar{a}(0), \dots, \bar{a}(k-1)$ is a k-tuple, then we let $\prod_{\mathcal{F}} A_n \models R(\bar{a}(0), \dots, \bar{a}(k-1))$ if and only if the set

$$\{n | A_n \models R(a_n(0), \dots, a_n(k-1))\}$$

belongs to $\mathcal{F}$.

The image of $\bar{a}$ in the reduced product $\prod_{\mathcal{F}} A_n$ under the quotient map is also denoted $\bar{a}$, by a standard and innocuous abuse of notation.

If $\mathcal{F}$ is the Fréchet filter (i.e., the filter of cofinite subsets of $\mathbb{N}$), then $\prod_{\mathcal{F}} A_n$ is denoted $\prod_{\mathrm{Fin}} A_n$. (This is yet another standard and innocuous abuse of notation; Fin denotes the ideal dual to the Fréchet filter, and the reduced products are sometimes defined with respect to the dual ideals.) If $\mathcal{U}$ is an ultrafilter (i.e., a proper filter maximal with respect to the inclusion), then $\prod_{\mathcal{U}} A_n$ is called the ultraproduct.

When all structures A_n are equal to some A, the corresponding reduced products (ultraproducts) are called reduced powers (ultrapowers).

1.2. **The order properties.** This combinatorial property of a first-order theory marks the watershed between well-behaved and wild (see [35]).

Definition 1.1. Suppose that T is a first-order theory.

(1) If $\varphi(\bar{x}, \bar{y})$ is an asymmetric formula (with $\bar{x}$ and $\bar{y}$ of the same sort) in the language of T consider the asymmetric binary relation $\triangleleft_\varphi$ on a model A of T, defined by ~~$\bar{a} \triangleleft_\varphi \bar{b}$ if $A \models \varphi(\bar{a}, \bar{b})$.~~

$$\bar{a} \triangleleft_\varphi \bar{b} \text{ if } A \models \varphi(\bar{a}, \bar{b}). \tag{1.1}$$

Some $\bar{a}_j$, for $j < n$, in A form a $\triangleleft_\varphi$-*chain* if for all $i \neq j$ we have $\bar{a}_i \triangleleft_\varphi \bar{a}_j$ if and only if $i < j$.

[4] If the language is multisorted, then the sort of a tuple is simply its arity. Note that the natural languages associated with the unbounded metric structures, such as C^*-algebras, are multisorted.

(2) If every model of T has an arbitrarily long finite $\triangleleft_\varphi$-chain, we say that the pair (T,φ) *has the order property, OP* ([35]).[5]
(3) The pair (T,φ) has the *robust order property* if it has the order property and in addition for models A_n, for $n\in\mathbb{N}$, of T and $\bar a$ and $\bar b$ in ~~$\prod_{\mathcal{F}} A_n$~~ $\prod_{\mathrm{Fin}} A_n$ we have $\prod_{\mathrm{Fin}} A_n \models \varphi(\bar a,\bar b)$ if and only if the set

$$\{n \mid A_n \not\models \varphi(\bar a_n,\bar b_n)\}$$

is finite. (Note that it is not required that $\prod_{\mathrm{Fin}} A_n$ models T.)
(4) The pair (T,φ) is said to have the *strict order property* (SOP) if the relation $\triangleleft_\varphi$ is a partial ordering on every model of T.

The relation between the order property and the robust order depends on the analysis of the relation between the theories of A_n and the theory of $\prod_{\mathrm{Fin}} A_n$, as given by the Feferman–Vaught theorem ([21] and [22] for continuous logic, also see [12, §16.3]). We will need only the following easy case.

Lemma 1.2. *If a pair (T,φ) has the order property and $\varphi(\bar x,\bar y)$ is atomic, or a conjunction of atomic formulas, then the pair (T,φ) has the robust order property.*

Proof. Fix models $A_n\models T$ for $n\in\mathbb{N}$ and suppose φ is a conjunction of atomic formulas. If $\bar a_n$ and $\bar b_n$ are tuples of the appropriate sort in A_n such that $A_n\models\varphi(\bar a_n,\bar b_n)$, then (writing $\bar a$ for the element of the product that has the representing sequence $(\bar a_n)$), we have $\prod_n A_n\models\varphi(\bar a,\bar b)$ and moreover for any filter $\mathcal{F}$ on $\mathbb{N}$ we have $\prod_{\mathcal{F}} A_n\models\varphi(\bar a,\bar b)$. The assertion follows immediately. □

1.3. **Continuous logic.** For more details on continuous logic see [2] ~~and [14]~~ (see [14] or [12, §16] for operator algebras~~, also [12, §16]~~). That said, this subsection is targeted at the readers already familiar with ~~the continuous logic~~continuous logic, and its aim is to convince these readers that the proofs of the continuous versions of our main results are analogous to the proofs in the discrete case.

The reduced product $\prod_{\mathcal{F}} A_n$ of metric structures of the same language is defined analogously to the discrete case. See e.g., [2, §5] (for ultraproducts) and [12, §16.2 and §D.2.5] for the general case.

The value of a formula $\varphi(\bar x)$ evaluated in a model M, at a tuple $\bar a$ of the appropriate sort, is denoted $\varphi(\bar a)^M$ and defined by recursion on the complexity of φ. In particular, if $\varphi(\bar x,\bar y)$ is a formula (with $\bar x$ and $\bar y$ of the same sort) then the binary relation $\triangleleft_\varphi$ on a model A of T is defined by $\bar a\triangleleft_\varphi\bar b$ if $\varphi(\bar a,\bar b)^A=0$ and $\bar b\triangleleft_\varphi\bar a$ if $\varphi(\bar b,\bar a)^A=1$. The pair (T,φ) has the order property if every model A of T contains arbitrarily long finite $\triangleleft_\varphi$-chains ([16, Definition 5.2]).

~~In continuous logic~~

[5] One says that φ has the order property when T is clear from the context.

Definition 1.3. If T is a theory in a continuous language and φ is a formula of the same language, we say that the order property of the pair (T,φ) is *robust* if for models A_n, for $n\in\mathbb{N}$, of T, and all $\bar a$ and $\bar b$ in ~~$\prod_{\mathcal{F}} A_n$~~ $\prod_{\mathrm{Fin}} A_n$ we have $\prod_{\mathrm{Fin}} A_n \models \varphi(\bar a,\bar b)$ if and only if for all sufficiently small $\varepsilon>0$ the set

$$\{n \mid \varphi^{A_n}(\bar a_n,\bar b_n)<\varepsilon \text{ and } \varphi^{A_n}(\bar b_n,\bar a_n)>1-\varepsilon\}$$

is finite.

Therefore by replacing φ with $f(\varphi)$ for a suitable piecewise continuous function f, the order property of a continuous theory as well as its robustness are witnessed by a discrete (i.e., 0-1 valued) formula. Because of this, we will provide proofs of our results only in the case of discrete theories, with understanding that they carry on virtually unchanged to the continuous context. A proof of the following is analogous to the proof of Lemma 1.2 and therefore omitted.

Lemma 1.4. *If T is a continuous theory, a pair (T,φ) has the order property, and $\varphi(\bar x,\bar y)$ is atomic or a minimum of atomic formulas then the pair (T,φ) has the robust order property.* □

2. Background on ~~partial orderings~~posets

In this section we warm up by stating and proving some well-known results. Consider the following two partial quasi-orderings on $\mathbb{N}^{\mathbb{N}}$ (by $\forall^j$ we denote the quantifier 'for all but finitely many $j\in\mathbb{N}$'):

$$f\le^* g \Leftrightarrow (\forall^\infty j) f(j)\le g(j)$$
$$f<^* g \Leftrightarrow (\forall^\infty j) f(j)< g(j).$$

Any proper initial segment of $(\mathbb{N}^{\mathbb{N}},\le^*)$ has the form $(\{f\in\mathbb{N}^{\mathbb{N}} \mid f\le\eta\},\le^*)$ for some $\eta\in\mathbb{N}^{\mathbb{N}}$. Such an initial segment is isomorphic to $(\prod_n \eta(n),\le^*)$ (if $f\le^*\eta$, then the pointwise minimum of f and η is an element of $\prod_n\eta(n)$ equal to f modulo finite) and these structures will be our main focus. The following is essentially a bounded variant of [9, Proposition 0.1].

Lemma 2.1. *There are $\eta\in\mathbb{N}^{\mathbb{N}}$ and $\Phi\colon(\prod_n n,\le^*)\to(\prod_n\eta(n),<^*)$ such that for all f and g, if $f\le^* g$ and $g\not\le^* f$ then $\Phi(f)<^*\Phi(g)$.*

A morphism Φ as guaranteed by Lemma 2.1 is called *strictly increasing*.

Proof. ~~Let~~ Recursively define η by $\eta(0):=0$ and $\eta(n+1):=\sum_{j\le n} j\eta(j)+1$ for $n\neq 0$. The following is the salient property of the function η:

$$(m+1)\eta(n)>\sum_{j<n} j\eta(j)+m\eta(n)>\sum_{j<n} f(j)\eta(j)+m\eta(n),$$

for any $f\in\prod_n n$. For $f\in\prod_n n$ let ~~$\Phi(f)(n):=\sum_{j\le n} f(j)\eta(j)$. Fix~~

$$\Phi(f)(n):=\sum_{j\le n} f(j)\eta(j).$$

Suppose that f and g ~~in the domain of Φ such that~~ are in $\prod_n n$ and k is such that $f(k) < g(k)$. Using the salient property of η with $m = f(n)$, we obtain the following.

$$\begin{aligned}\Phi(g)(n) &= \sum_{j<n} g(j)\eta(j) + g(n)\eta(n) \\ &> \sum_{j<n} (g(j)+j)\eta(j) + f(n)\eta(n) > \Phi(f)(n).\end{aligned}$$

Therefore, if f and g are in $\prod_n n$, then $f \leq^* g$ ~~but $f \neq^* g$. Let~~ if and only if $\Phi(f) \leq^* \Phi(g)$.

It remains to prove that if $(\forall^\infty)n f(n) \geq g(n)$ and $(\exists^\infty n) f(n) > g(n)$ then $(\forall^\infty n)\Phi(f)(n) > \Phi(g)(n)$. For such f and g and fix m ~~be such that $f(n) \leq g(n)$~~ such that $f(n) \geq g(n)$ for all $n \geq m$ ~~, and let $k > m$ be such that $f(k) < g(k)$. Then $\Phi(g)(k) - \Phi(f)(k) > 0$, and for all $l > k$ we have $\Phi(g)(l) - \Phi(f)(l) > 0$.~~ and $f(m) > g(m)$. By induction we will prove that $\Phi(f)(n) > \Phi(g)(n)$ for all $n \geq m$. For $n = m$ this follows by the first part of the proof. If $\Phi(f)(n) > \Phi(g)(n)$, then

$$\begin{aligned}\Phi(f)(n+1) &= \Phi(f)(n) + f(n+1)\eta(n+1) \\ &> \Phi(g)(n) + f(n+1)\eta(n+1) \geq \Phi(g)(n+1).\end{aligned}$$

By induction, $\Phi(g) <^* \Phi(f)$ as required. □

~~A morphism Φ as guaranteed by Lemma 2.1 is called *strictly increasing*.~~

The universal structure obtained in Lemma 2.2 below is very similar to the Rado graph, also known as the (countably infinite) random graph, and it ought to be well-known. It was however easier to include a proof than to look for it in the literature.

Lemma 2.2. *There exists an injectively universal countable structure* $(C, \lhd)$ *with an asymmetric binary relation* $\lhd$. *This universality property is absolute between transitive models of a sufficiently large fragment of* ZFC.

Proof. Let $C := \mathbb{N}$ and define the relation $\lhd$ as follows. If $m < n$ are in $\mathbb{N}$ and $n = \sum_j d_j(n)3^j$ is the ternary expansion of n (so that $d_j(n) \in \{0,1,2\}$ for all j) then let $m \lhd n$ if $d_m(n) = 1$, $n \lhd m$ if $d_m(n) = 2$, and let m and n unrelated if $d_m(n) = 0$. The structure $(C, \lhd)$ has the following property resembling the random graph:

(*) If F and G, are disjoint finite subsets of C, then there exists $n \in C$ such that $m \in F$ implies $m \lhd n$, $m \in G$ implies $n \lhd m$, and ~~$m \notin F|upG$~~ $m \notin F \cup G$ implies that m and n are unrelated.

To see this, let $n := \sum_{m\in F} 3^m + \sum_{m \in G} 2 \cdot 3^m$.

Given the property (*) of $(C, \lhd)$, every countable $(A, \lhd')$ can be isomorphically embedded into $(C, \lhd)$ by recursion. Since (*) is a first-order property, it is absolute between transitive models of a sufficiently large fragment of ZFC (see e.g., [30, Lemma II.4.3]). □

3. The depletion of a poset

The notion of the *depletion* of a linear ordering ~~and (admittedly rather dull) Lemma 3.3 will be instrumental in a critical place in the proof of Theorem 4.13.~~

appears implicitly in [9].

Definition 3.1. Suppose that $m \geq 2$, A and $F(i)$, for $i < m$, are disjoint sets, and $\leq$ is a partial ordering on the set $E := A \cup \bigcup_{i<m} F(i)$.[6] A binary relation $\ll$ on E defined as follows is called the *depletion of* $\leq$ ~~*given by*~~ *given by* A *and* ~~$F(i)$, *for* $i < m$~~$(F(i) | i < m)$.

If x and y belong to E, we let $x \ll y$ if and only if one of the following applies.

(1) Both x and y belong to $A \cup F(i)$ for some i and $x \leq y$.
(2) There are $i < j$ such that $x \in F(i)$ and $y \in F(j)$ and one of the following holds.
 (a) There exists $a \in A$ such that $x \leq a$ and $a \leq y$
 (b) With $k = j - i$, there are $x_l \in F(i + l)$ for $0 \leq l \leq k$ such that $x_0 = x$, $x_k = y$, and $x_l \leq x_{l+1}$ for all $l < k$.
(3) There are $i > j$ such that $x \in F(i)$ and $y \in F(j)$ and one of the following holds.
 (a) There exists $a \in A$ such that $x \leq a$ and $a \leq y$
 (b) With $k = i - j$, there are $x_l \in F(j + l)$ for $0 \leq l \leq k$ such that ~~$x_0 = x$, $x_k = y$~~$x_0 = y$, $x_k = x$, and $x_l \geq x_{l+1}$ for all $l < k$.

The elements x_i, for $i \leq k$ as in (2b) or (3b) *are called an s-walk between* x *and* y*, or an s-walk with endpoints* x *and* y*.*

In order to help the reader memorize the definition, we state the following trivial lemma without a proof.

Lemma 3.2. *With the notation as in Definition 3.5, if* $s \subseteq t \Subset I$*, and if* $\min(s) = \min(t) = \xi$, $\max(s) = \max(t) = \eta$*, and* $x(\zeta)$*, for* $\zeta \in t$*, is a t-walk, then* $x(\zeta)$*, for* $\zeta \in s$*, is an s-walk with the same endpoints.* □

The (admittedly rather dull) Lemma 3.3 will be instrumental in a critical place in the proof of Theorem 4.13.

Lemma 3.3. *The depletion* $\ll$ *of a partial ordering* $\leq$ *as given in Definition 3.1 is a partial ordering included in* ~~*it*~~$\leq$.[7]

[6] Warning: The definition of the relation $\ll$ depends on the chosen ordering of the index set m. In the context of this definition, m stands for an arbitrary finite linearly ordered set.

[7] 'Included' when identified with its graph—we are set-theorists!

Proof. Fix A, m, $F(i)$, for $i < m$, and an ordering $\leq$ on $E := A \cup \bigcup_{i<m} F(i)$.

It is clear from the definition that $x \ll y$ implies $x \leq y$ and that $\ll$ and $\leq$ agree on $A \cup F(i)$ for every i. Therefore $\ll$ is antisymmetric and reflexive, and it will suffice to prove that it is transitive.

Towards this end, fix x, y, and z such that $x \ll y$ and $y \ll z$. Then $x \leq y$ and $y \leq z$, and therefore $x \leq z$. If x and z belong to $A \cup F(i)$ for some i, then $x \ll z$ by (1). Therefore if at least one of $x \in A$ or $z \in A$ holds then $x \ll z$, and we may assume $x \in F(i)$ and $z \in F(j)$ for distinct i and j. If $y \in A$ then $x \ll z$ by (2a). Similarly, if there exists $a \in A$ such that $x \leq a$ and $a \leq y$, then $x \ll z$. Also, if there exists $a \in A$ such that $y \leq a$ and $a \leq z$, then $x \ll z$.

We can therefore assume that $y \in F(n)$ for some n and both $x \ll y$ and $y \ll z$ are witnessed by instances of (2b). The following claim will help when discussing the possible cases.

Claim 3.4. *Suppose that $i < m$, $0 < k \leq m - i$, $x \in F(i)$ and $y \in F(i+k)$.*

1. *Assume there is no $a \in A$ such that $x \leq a$ and $a \leq y$. Then $x \ll y$ if and only if there are $x_l \in F(i+l)$ for all $0 \leq l \leq k$ such that $x_0 = x$, $x_k = y$, and $x_l \leq x_{l+1}$ for all $0 \leq l < k$.*
2. *Assume there is no $a \in A$ such that $y \leq a$ and $a \leq x$. Then $y \ll x$ if and only if there are $x_l \in F(i+l)$ for all $0 \leq l \leq k$ such that $x_0 = x$, $x_k = y$, and $x_{l+1} \leq x_l$ for all $0 \leq l < k$.*

Proof. (1) For the direct implication, note that the assumptions imply that (2b) of Definition 3.1 applies. Let $x_0 := x$, $x_k := y$, and for $0 < l < k$ let x_l be a witness for (2b) of Definition 3.1. These objects are clearly as required.

For the converse implication, assume that x_l for $0 \leq l \leq k$ are as in the statement of the claim. Then clearly (2b) of Definition 3.1 applies.

The proof of (2) is analogous and therefore omitted. □

Back to our proof. If $i \leq n \leq j$, then part (1) of Claim implies that $x \ll z$. If $i < j < n$, then the witnessing sequence for $x \ll y$ contains $t \in F(j)$, such that $x \ll t$ and $t \ll y$. But then (since $\ll$ implies $\leq$) $t \leq z$, and $t \ll z$ since both t and z belong to $F(j)$. A proof in the case when $n < i < j$ is similar and uses part (2) of the Claim. This proves our claim in the case when $i < j$.

The proof in the case when $i > j$ is analogous. □

Definition 3.5. Given a linear ordering I, poset, disjoint subsets A and $F(\xi)$, for $\xi \in I$, linear ordering $\leq$ on $E := A \cup \bigcup_{\xi \in I} F(\xi)$, and $s \Subset I$, let $\ll_s$ denote the depletion of $\leq$ determined by A and $F(\xi)$, for $\xi \in s$.

By Lemma 3.2, $\ll_s$ is a partial ordering on $D_s := A \cup \bigcup_{\xi \in I} F(\xi)$.

Lemma 3.6. *With the notation as in Definition 3.5, if $s \subseteq t \Subset I$, then for any two x and y in D_s we have that $x \ll_t y$ implies $x \ll_s y$.*

Proof. Fix $x \ll_t y$. A glance at Definition 3.1 shows that we may assume that $x \in F(\xi)$ and $y \in F(\eta)$ for some distinct ξ and η in s, since in any other situation $x \ll_s y$ follows immediately. If there is $z \in A$ such that $x \le z \le y$, then clearly $x \ll_w y$ whenever $\{\xi, \eta\} \subseteq w \Subset I$. We may therefore assume that there is a t-walk with endpoints x and y. We denote the 'steps' of s by $x(\zeta)$, for $\zeta \in t$ and $\xi \le \zeta \le \eta$. Lemma 3.2 implies that $x(\zeta)$, for $\zeta \in s$ and $\xi \le \zeta \le \eta$ is an s-walk with endpoints x and y, hence $x \ll_s y$. □

Proposition 3.7 is a relative to a result of Kurepa ([31]) and to [9, Theorem 7.1] . The role that this proposition plays in the proof of our Theorem 4.13 is analogous to the role that [9, Theorem 7.1] played in the proofs of the main results of [9]. For reader's convenience, we include a proof. If I is a linear ordering, then I^* denotes the converse ordering.

Proposition 3.7. *Suppose that κ is an uncountable cardinal, A and $F(\xi)$, for $\xi < \kappa$, are disjoint, and $\le$ is a partial ordering of $E := A \cup \bigcup_{\xi<\kappa} F(\xi)$. In addition suppose that A is countable, all $F(\xi)$ are finite, and E has neither κ nor κ^*-chains.*

Then there exists a cofinal $X \subseteq \kappa$ such that for any two distinct elements ξ and η of X there is $s \Subset \kappa$ such that $\{\xi, \eta\} \subseteq s$ and there is no s-walk with endpoints in $F(\xi)$ and $F(\eta)$.[8]

Proof. We will choose X recursively. Let $0 \in X$, and assume that a proper initial segment of X has been chosen and that it satisfies the requirements. For simplicity of notation, we will denote this proper initial segment by X and explain how an ordinal greater than all of its elements can be added to X while preserving the requirements on X.

If X has no maximal element, then add $\alpha := \sup(X)$ to X. In order to verify that the new X still satisfies the requirements, fix $\xi \in X$. We need to prove that there exists t such that there is no t-walk with endpoints in $F(\xi)$ and $F(\alpha)$. Let $\eta = \min(X \setminus (\xi + 1))$ and let $s \Subset \kappa$ be such that $\{\xi, \eta\} \subseteq s$ and there is no s-walk with endpoints $F(\xi)$ and $F(\eta)$. By Lemma 3.2, with $t = s \cup \{\alpha\}$ there is no t-walk with endpoints in $F(\xi)$ and $F(\alpha)$.

Now suppose that X has a maximal element, ξ. We need to find $\eta > \xi$ and $\{\xi, \eta\} \subseteq s \Subset \kappa$ such that there is no s-walk with endpoints in $F(\xi)$ and $F(\eta)$. Suppose, towards a contradiction, that for every $\eta > \xi$ and every $\{\xi, \eta\} \subseteq s \Subset \kappa$ there is an s-walk with endpoints in $F(\xi)$ and $F(\eta)$. Fix such a walk, $x(s, \zeta)$, for $\zeta \in s$.

Let $\mathcal{U}$ be an ultrafilter on Fin_κ which for every $s \Subset \kappa$ includes the set $\{t \Subset \kappa | s \subset t\}$ (such $\mathcal{U}$ exists, since the family of sets of this form has the finite intersection property). Fix $\xi \le \zeta < \kappa$. Since $F(\zeta)$ is finite, there exists a unique $y(\zeta) \in F(\zeta)$ such that $\{s | x(s, \zeta) = y(\zeta)\} \in \mathcal{U}$. Clearly $y(\zeta)$, for $\xi \le \zeta < \kappa$, form a κ-chain or a κ^*-chain (as decided by $\mathcal{U}$); contradiction.

Therefore there exist $\eta > \xi$ and $\{\xi, \eta\} \subseteq s \Subset \kappa$ such that there is no s-walk with endpoints in $F(\xi)$ and $F(\eta)$. Add the minimal such η to X. This completes the recursive construction of X and the proof. □

[8] We emphasize that s is not necessarily a subset of X.

4. Embedding posets, gently

In the present section we assume that the reader is familiar with the basics of forcing as presented in e.g., [30] or [39]. The present section is largely based on [9], and Theorem 4.1 is a close relative to [9, Theorem 9.1].

The category of partially ordered sets is considered with respect to the order-embeddings, i.e., injections $f\colon E \to E'$ such that $a \leq_E b$ if and only if $f(a) \leq_{E'} f(b)$. The category of forcing notions is considered with respect to regular embeddings (also known as complete embeddings, [30, Definition III.3.65]). If a forcing notion $\mathbb{H}_0$ is a regular subordering of a forcing notion $\mathbb{H}_1$, we then write $\mathbb{H}_0 \lessdot \mathbb{H}_1$. Notably, $\mathbb{H}_0 \lessdot \mathbb{H}_1$ is equivalent to the assertion that for every generic filter $G \subseteq \mathbb{H}_1$, $G \cap \mathbb{H}_0$ is also generic. In other words, $\mathbb{H}_1$ can be considered as a two-step iteration of $\mathbb{H}_0$ followed by some other forcing notion.

If κ is an uncountable cardinal, a forcing notion $\mathbb{P}$ is said to have *precaliber* κ if every set of κ conditions in $\mathbb{P}$ has a subset of cardinality κ such that each of its finite subsets has a common lower bound. Precaliber $\aleph_1$ is a strong form of the countable chain condition. For example, if $\mathbb{P}$ has precaliber $\aleph_1$ then it is *productively ccc*, in the sense that the product of $\mathbb{P}$ with any ccc poset is ccc. (We will not need this fact.)

Theorem 4.1. *There is a functor from the category of partially ordered sets into the category of forcing notions $E \mapsto \mathbb{H}_E$ with the following properties.*

(1) *$\mathbb{H}_E$ has precaliber κ for every uncountable regular cardinal κ.*
(2) *$\mathbb{H}_E$ forces that ~~E embeds into $(\prod_n n, \leq^*)$~~ there is an embedding $\Upsilon\colon E \to (\prod_n n, \leq^*)$ (thus $a \leq_E b$ if and only if $\Upsilon(a) \leq^* \Upsilon(b)$).*
(3) *If $\kappa > \mathfrak{c}$ is a regular cardinal and neither κ nor its reverse κ^* embed into E, then $\mathbb{H}_E$ forces that κ does not embed into $\prod_{\mathrm{Fin}}(A_n, \triangleleft)$ for every sequence $(A_n, \triangleleft_n)$ of countable structures equipped with an asymmetric binary relation.*

Proof. The proof of this theorem will occupy most of the present section. For $\mathbb{H}_E$ see Definition 4.2, (1) is Lemma 4.4, (2) is Lemma 4.6, and (3) is Theorem 4.13. $\square$

In the Definition 4.2 and elsewhere, if $\mathrm{dom}(f) \subseteq \mathbb{N}$ then $f \restriction m$ denotes the restriction of f to $m = \{0, \dots, m-1\}$. ~~We will also write~~

$$\underline{X \Subset Y}$$

~~as a short for '$X \subseteq Y$ and X is finite' (this relation is sometimes denoted $X \in [Y]^{<\aleph_0}$).~~

Definition 4.2. For a partially ordered set E, $\mathbb{H}_E$ is the forcing notion defined as follows. The conditions of $\mathbb{H}_E$ are triples $p = (D_p, n_p, f_p)$, where $D_p \Subset E$, $n_p \in \mathbb{N}$, and ~~$f_p\colon D_p \to \prod_{m<n} m$~~ $f_p\colon D_p \to \prod_{m<n_p} m$.

The ordering is defined by ~~$p \leq_E q$~~ letting $p \leq q$ (p extends q) if the following conditions hold.

(1) $D_p \supseteq D_q$, $n_p \geq n_q$, $f_p(a) \restriction n_q = f_q(a)$ for all $a \in D_q$, and
(2) for all a and b in D_q, if $a \leq_E b$ then ~~$f_p(a)(j) \leq_E f_p(b)(j)$~~ $f_p(a)(j) \leq f_p(b)(j)$ for all $j \in [n_q, n_p)$.

In order to relax the notation, if (p_ξ) is an indexed family of conditions in $\mathbb{H}_E$ we write $p_\xi = (D_\xi, n_\xi, f_\xi)$.

Lemma 4.3. *Suppose that E is a poset, $R \Subset E$, $m \geq 2$ and p_i, for $i < m$, are conditions in $\mathbb{H}_E$ such that the following holds whenever $i \neq j$.*

(1) *~~Whenever $i \neq j$ we~~ We have $D_i \cap D_j = R$.*
(2) *All $a \in R$ satisfy $f_i(a) \restriction \min(n_i, n_j) = f_j(a) \restriction \min(n_i, n_j)$.*

Then some $q \in \mathbb{H}_E$ extends all p_i.[9]

Proof. Let $D_q := \bigcup_{i<m} D_i$ and $n_q := \max_{i<m} n_i$. If $i < m$ is such that $n_i = n_q$, then for $a \in D_i$ let $f_q(a) = f_i(a)$. Then $f_q(a)$ is well-defined for $a \in R$ by (2). For $i < m$ such that $n_i < n_q$ and for $a \in D_i \setminus R$, let (with $\max \emptyset = 0$)

$$f_i(a)(j) := \max\{f_q(b)(j) \mid b \in R, b \leq_E a\}.$$

for $n_i \leq_E j < n_q$. This defines $q \in \mathbb{H}_E$. We will prove that $q \leq p_i$ for all $i < m$.

Clearly, q and p_i satisfy (1) of Definition 4.2 for all $i < m$. Fix $i < m$.

If $n_i = n_q$ then (2) of Definition 4.2 is vacuous, hence $q \leq_E p_i$.

Suppose $n_i < n_q$. To check that $q \leq p_i$, we need to verify (2) of Definition 4.2. Fix a and b in D_i such that $a \leq_E b$. If there is no $c \in R$ such that $c \leq_E b$, then for all $j \in [n_i, n_q)$ we have $f_q(a)(j) = f_q(b)(j) = 0$. If there is $c \in R$ such that $c \leq_E b$, then $\{c \mid c \leq_E a\} \subseteq \{c \mid c \leq_E b\}$ and by the definition of f_q we have $f_q(a)(j) \leq f_q(b)(j)$.

Thus (2) of Definition 4.2 holds, and $q \leq p_i$. $\square$

Lemma 4.4. *The poset $\mathbb{H}_E$ has precaliber κ for every uncountable regular cardinal κ.*

Proof. Fix a family p_ξ, for $\xi < \kappa$, in $\mathbb{H}_E$. By the Δ-system lemma and passing to a subfamily of the same cardinality, we may assume that there exists $R \Subset E$ such that $D_\xi \cap D_\eta = R$ for all distinct ξ and η below κ. By the pigeonhole principle (using the assumption that κ has uncountable cofinality), we may also assume that there exists n such that $n_\xi = n$ for all ξ. Also, since there are only finitely many possibilities for $f_\xi(a)$, for $a \in R$, we may assume that the functions f_ξ agree on R and therefore we are in the situation of Lemma 4.3. Therefore, after this refining argument, Lemma 4.3 implies that every finite subset of $\{p_\xi \mid \xi < \kappa\}$ has a common lower bound. $\square$

A proof of the following lemma is omitted as straightforward.

Lemma 4.5. *For any poset E the following holds.*

[9] We write $q \leq p$ if q extends p.

(1) *For every n and every $a \in E$, the set*

$$\mathcal{D}(\mathbb{H}_E, n, a) := \{p \in \mathbb{H}_E \mid n_p \geq n, a \in F_p\}$$

is dense in $\mathbb{H}_E$.

(2) *If $b \not\leq_E a$ in E, then for every $n \in \mathbb{N}$ the set*

$$\mathcal{E}(\mathbb{H}_E, n, a, b) := \{p \in \mathbb{H}_E \mid (\exists k \geq n) f_p(a)(k) < f_p(b)(k)\}$$

is dense in $\mathbb{H}_E$. □

Lemma 4.6. *If E is a poset and $G \subseteq \mathbb{H}_E$ is a generic filter, then*

$$\Upsilon_G(a)(j) := f_p(a)(j)$$

for $p \in G$ defines a strictly increasing function $\Upsilon_G \colon E \to (\prod_n n, \leq^*)$.

Proof. By genericity, G intersects all dense sets defined in Lemma 4.5 and therefore Υ is a strictly increasing map from E into $(\prod_n n, \leq^*)$. □

If E is a subordering of E' then every $p \in \mathbb{H}_E$ is (literally) a condition in $\mathbb{H}_{E'}$. We will therefore identify $\mathbb{H}_E$ with a subordering of $\mathbb{H}_{E'}$.

Lemma 4.7. *If E' is a poset and E is a subposet of E', then $\mathbb{H}_E$ is a regular subordering of $\mathbb{H}_{E'}$.*

Proof. The identity map from $\mathbb{H}_E$ into $\mathbb{H}_{E'}$ is clearly an order-embedding. It suffices to prove that there exists a *reduction* (or *projection*) $\pi \colon \mathbb{H}_{E'} \to \mathbb{H}_E$: A map such that for every $p \in \mathbb{H}_{E'}$ we have ~~$p \leq_{E'} \pi(p)$~~ $p \leq \pi(p)$ and every $q \in \mathbb{H}_E$ such that ~~$q \leq_E \pi(p)$~~ $q \leq \pi(p)$ is compatible with p ([30, Lemma III.3.72]). Let

$$\pi_E(p) := (D_p \cap E, n_p, f_p \restriction (D_p \cap E)).$$

Clearly, $p \leq \pi_E(p)$. If ~~$q \leq_E \pi(p)$~~$q \leq \pi(p)$, then $D_q \cap D_p = D_p \cap E$ and $f_p(a)(j) = f_q(a)(j)$ for all $a \in D_p \cap D_q$ and all $j < n_p$. By Lemma 4.3, p and q are compatible. □

In the situation when E is a subordering of E', as in Lemma 4.7, we will need a description of the quotient forcing $\mathbb{H}_{E'}/\dot{G}$, for a generic $G \subseteq \mathbb{H}_E$. If for some $k \in \mathbb{N}$ we have $s \in \prod_{n<k} n$ and $f \in \prod_n n$, then ~~$s \sqsubset f$ stands for $s = f \restriction k$.~~

$s \sqsubset f$ stands for $s = f \restriction k$.

Definition 4.8. If $E \subseteq E'$ are partial orderings and $\Upsilon \colon E \to (\prod_n n, \leq^*)$ is a strictly increasing function, a forcing notion $\mathbb{H}_{E'}(E, \Upsilon)$ is defined as follows. The conditions in $\mathbb{H}_{E'}(E, \Upsilon)$ are the triples $p = (D_p, n_p, f_p)$, where $D_p \Subset E'$, $n_p \in \mathbb{N}$, $f_p \colon D_p \to \prod_{j<n} n$, and for $a \in E$ we have $f_p(a) \sqsubset \Upsilon(a)$.

The ordering is ~~defined by $p \leq_E q$~~ inherited from $\mathbb{H}_E$. Therefore $p \leq q$ (p *extends* q) if the following conditions hold.

(1) $D_p \supseteq D_q$, $n_p \geq n_q$, $f_p(a) \restriction n_q = f_q(a)$ for all $a \in D_q$, and
(2) for all a and b in D_q, if ~~$a \leq_E b$ then $f_p(a)(j) \leq_E f_p(b)(j)$~~ $a \leq_{E'} b$ then $f_p(a)(j) \leq f_p(b)(j)$ for all $j \in [n_q, n_p)$.

Thus $\mathbb{H}_{E'}(E,\Upsilon)$ is a subordering of $\mathbb{H}_{E'}$ consisting of those conditions that 'agree' with Υ on E and $\mathbb{H}_{E'}(E,\Upsilon)$ generically adds an embedding from $E'\setminus E$ into $\prod_n n$. Note that $\mathbb{H}_{E'}(E,\Upsilon)$ is not necessarily separative; this will not cause any issues.

The proofs of the two parts of Lemma 4.10 below are virtually identical to the proofs of [9, Theorem 4.2] and [9, Lemma 4.3], respectively. ~~For $a \in E$ let~~

Definition 4.9. For a poset E' and $a\in E'$ let

$$L(a) := \{b \in E' \mid b \leq_{E'} a\}$$

and

$$R(a) := \{b \in E' \mid b \geq_{E'} a\}.$$

Lemma 4.10. *Suppose E' is a poset, E is a subposet of E', and $\dot G$ is ~~a~~ the canonical name for the $\mathbb{H}_E$-generic filter.*

1. *With the projection $\pi_E\colon \mathbb{H}_{E'}\to\mathbb{H}_E$ as in the proof of Lemma 4.7, the map $p\mapsto(\pi_E(p),p)$ from $\mathbb{H}_E$ into $\mathbb{H}_{E'}*\mathbb{H}_{E'}/\dot G$ is a dense embedding.*
2. *$\mathbb{H}_E$ forces that $\mathbb{H}_{E'}/\dot G$ is forcing-equivalent to ~~$\mathbb{H}_{E'}(E\cap X,\Upsilon_{\dot G})$~~ $\mathbb{H}_{E'}(E,\Upsilon_{\dot G})$ ($\Upsilon_{\dot G}$ is the generic embedding, see Lemma 4.6).*
3. *If $X\subseteq E$ is such that for every $a\in E'\setminus E$ the set $X\cap L(a)$ is cofinal in $L(a)$ and the set $X\cap R(a)$ is coinitial in $R(a)$, then $\mathbb{H}_E$ forces that $\mathbb{H}_{E'}(E,\Upsilon_{\dot G})$ and ~~$\mathbb{H}_{E'}(E\cap X,\Upsilon_{\dot G}\restriction X)$~~ $\mathbb{H}_{E'\setminus(E\setminus X)}(E\cap X,\Upsilon_{\dot G}\restriction X)$ are forcing-equivalent.* $\square$

The following is [9, Lemma 5.1] (see also [3, Lemma 2.5]).

Lemma 4.11. *Suppose $\mathbb{P}_0$ and $\mathbb{P}_1$ are forcing notions and $\dot f_j$ is a $\mathbb{P}_j$-name for an element of $\prod_n n$ for $j<2$. If $\mathbb{P}_0\times\mathbb{P}_1\Vdash \dot f_0\leq^*\dot f_1$ then the set of all $p\in\mathbb{P}_0\times\mathbb{P}_1$ such that there exist $m\in\mathbb{N}$ and $h\in\prod_n n$ which satisfy $p\Vdash \dot f_0\leq^m \check h$ and $p\Vdash\check h\leq^m\dot f_1$ is dense in $\mathbb{P}_0\times\mathbb{P}_1$.* $\square$

~~The~~ In combination with Lemma 4.11, the following lemma will be used in a crucial place in the proof of Theorem 4.13~~in combination with Lemma 4.11~~.

Lemma 4.12. *Suppose $(E,\leq)$ is a poset and A,B, and D are subsets of E such that $E=A\cup B$, $D=A\cap B$, and for every $a\in A$ and every $b\in B$ the following conditions hold.*

1. *$a\leq b$ if and only if $a\leq d$ and $d\leq b$ for some $d\in D$, and*
2. *$a\geq b$ if and only if $a\geq d$ and $d\geq b$ for some ~~$d\in A\cap B$~~ $d\in D$.*

Then $\mathbb{H}_D$ forces that ~~the map~~ ($\dot G$ is the canonical name for the generic filter in $\mathbb{H}_D$) $\mathbb{H}_E(D,\Upsilon_{\dot G})$ and $\mathbb{H}_A(D,\Upsilon_{\dot G})\times\mathbb{H}_B(D,\Upsilon_{\dot G})$ are forcing equivalent.

With the assumptions of Lemma 4.12 it can be proved that the function

$$\Xi\colon \mathbb{H}_E(D,\Upsilon_{\dot G})\to\mathbb{H}_A(D,\Upsilon_{\dot G})\times\mathbb{H}_B(D,\Upsilon_{\dot G})$$

defined by $\Xi(p):=(\pi_A(p),\pi_B(p))$ is a dense embedding~~.~~, but we will not need this fact.

Proof. We use the notation from Lemma 4.10 and write $\dot G(X)$ for the canonical name for the generic filter for $\mathbb{H}_X$ (or $\mathbb{H}_X(Y,\Upsilon)$ for some Y and Υ) where X is A, B, D, or E.

By Lemma 4.10 (1) with E and A in place of E' and E, $\mathbb{H}_E$ is forcing equivalent to $\mathbb{H}_A * \mathbb{H}_E(A,\Upsilon_{\dot G(A)})$. By the same lemma with A and D in place of E' and E, ~~the map $p \mapsto (\pi_D(p), p)$ is a dense embedding of $\mathbb{H}_E$ into $\mathbb{H}_D * \mathbb{H}_E(D,\Upsilon_{\dot G(D)})$, and the latter embeds densely into~~ $\mathbb{H}_A$ is forcing equivalent to $\mathbb{H}_D * \mathbb{H}_A(D,\Upsilon_{\dot G(D)})$. Therefore $\mathbb{H}_E$ is forcing equivalent to the iteration

$$\mathbb{H}_D * \mathbb{H}_{BA}(D,\Upsilon_{\dot G(D)}) * \mathbb{H}_E(BA, \Upsilon_{\dot G(B)\dot G(A)}). \tag{4.1}$$

The assumptions imply that ~~$L(a) \cap D$~~ $L(b) \cap D$ is cofinal in ~~$L(a)$ and $R(a) \cap D$~~ $L(b)$ and $R(b) \cap D$ is coinitial in ~~$R(a)$~~$R(b)$, for every ~~$a \in A$. Therefore $\mathbb{H}_B(D,\Upsilon_{\dot G(D)})$ forces that $\mathbb{H}_A(D,\Upsilon_{\dot G(D)})$ is dense in $\mathbb{H}_E(B,\Upsilon_{\dot G(B)})$. Since the former~~ $b \in B$. Therefore by Lemma 4.10 (3) applied with $\mathbb{H}_E(A,\Upsilon_{\dot G(D)})$, E, A, and D in place of $\mathbb{H}_{E'}(E,\Upsilon_{\dot G})$, E', E, and X, we conclude that $\mathbb{H}_A$ forces that $\mathbb{H}_E(A,\Upsilon_{\dot G(A)})$ is forcing equivalent to $\mathbb{H}_B(D,\Upsilon_{\dot G(D)})$ (also recall that $\mathbb{H}_D$ is a regular subordering of $\mathbb{H}_A$, and that $\Upsilon_{G(A)}$ extends $\Upsilon_{G(D)}$). Since $\mathbb{H}_B(D,\Upsilon_{\dot G(D)})$ does not depend on ~~$\dot G(B)$, the natural embedding of the iteration in~~$\dot G(A)$, but only on its intersection with $\mathbb{H}_D$, the iteration in (4.1) ~~into~~ is forcing equivalent to

$$\mathbb{H}_D * (\mathbb{H}_B(D,\Upsilon_{\dot G(D)}) \times \mathbb{H}_A(A,\Upsilon_{\dot G(D)}))$$

~~is a dense embedding~~and $\mathbb{H}_D$ forces that $\mathbb{H}_E(A,\Upsilon_{\dot G(A)})$ is forcing equivalent to the product of $\mathbb{H}_B(D,\Upsilon_{\dot G(D)})$ and $\mathbb{H}_A(A,\Upsilon_{\dot G(D)})$, as claimed. □

In the proof of Theorem 4.13 below, for f and g in $C^{\mathbb{N}}$ (with $(C,\triangleleft)$ as guaranteed by Lemma 2.2) we will write $f \triangleleft^n g$ if ~~$f(j) \leq_E g(j)$ for all~~$f(j) \triangleleft g(j)$ for all $j \geq n$. A proof of Theorem 4.13 is analogous to, but shorter than, the proof of [9, Theorem 9.1] (a baroque writeup of this proof with an ample supply of limiting examples and all sorts of digressions (many of which were warranted) can be found in [9]).

Theorem 4.13. *Suppose κ is a regular cardinal such that ~~$\kappa > \mathfrak{c}$~~ $\lambda^{\aleph_0} < \kappa$ for every cardinal $\lambda < \kappa$ and E is a partial ordering such that neither κ nor κ^* embeds into E. Then $\mathbb{H}_E$ forces that $\prod_{\mathrm{Fin}}(A_n, \triangleleft_n)$ has no κ-chains for any sequence $(A_n, \triangleleft_n)$, for $n \in \mathbb{N}$, of countable sets with asymmetric binary relations.*

Proof. By Lemma 2.2, $\mathbb{H}_E$ forces that $\prod_{\mathrm{Fin}}(A_n, \triangleleft_n)$ has a κ-chain for some sequence $(A_n, \triangleleft_n)$ (not necessarily in the ground model) if and only if $\mathbb{H}_E$ forces that $(C^{\mathbb{N}}, \triangleleft^*) := \prod_{\mathrm{Fin}}(C, \triangleleft)$ has a κ chain. It will therefore suffice to prove the theorem with the additional assumption that $(A_n, \triangleleft_n) = (C, \triangleleft)$ for all n.

Assume that $\dot f_\xi$, for $\xi < \kappa$, is a name for a κ-chain in $(C^{\mathbb N}, \lhd^*)$. (We emphasize that this means that for all $\xi < \eta$, $\mathbb H_E$ forces both $\dot f_\xi \lhd^* \dot f_\eta$ and $\dot f_\eta \not\lhd^* \dot f_\xi$.) The ccc-ness of $\mathbb H_E$ ~~and Lemma 4.7 together imply~~ implies that for every limit ordinal ξ there exists a countable $E(\xi) \subseteq E$ such that $\dot f_\xi$~~is an~~ , $\dot f_{\xi+1}$, and $\dot f_{\xi+2}$ are $\mathbb H_{E(\xi)}$~~-name. By~~ -names (recall that Lemma 4.7 moreover implies that $\mathbb H_{E(\xi)}$ is a regular subordering of $\mathbb H_E$). Since κ is regular and $\lambda^{\aleph_0} < \kappa$ for all $\lambda < \kappa$, the Δ-system lemma ~~(~~for countable sets ~~, using $\kappa > \mathfrak c$) and~~ implies that any family of κ countable sets includes a Δ-system of cardinality κ. By passing to a subfamily, we may assume that the sets $E(\xi)$ form a Δ-system with countable root A.

For a limit ordinal ξ fix $q_\xi \in \mathbb H_E$ and $n \in \mathbb N$ such that

$$q_\xi \Vdash \dot f_\xi \lhd^n \dot f_{\xi+1} \lhd^n \dot f_{\xi+2}. \tag{4.2}$$

Lemma 4.7 implies that $\mathbb H_{E(\xi)}$ is a regular subordering of $\mathbb H_E$, and we may therefore assume $q_\xi \in \mathbb H_{E(\xi)}$. Writing $q_\xi = (D_\xi, n_\xi, f_\xi)$, let $F(\xi) := D_\xi \setminus A$. Fix a well-ordering $<_w$ of C. ~~For a moment~~

Fix $\xi < \kappa$ for a moment and fix a generic filter $G \subseteq \mathbb H_{A\cup F(\xi)}$ such that $q_\xi \in G$~~, and for~~ . For $j \in \mathbb N$ let $h_\xi(j)$ be the $<_w$-least element of C such that ~~$r \Vdash \dot f_{\xi+1}(j) = \check c$~~

$$r \Vdash \dot f_{\xi+1}(j) = \check c$$

for some r in the quotient $\mathbb H_E(A \cup F(\xi), \Upsilon_G)/G$ (see Lemma 4.10).

This defines $h_\xi \in C^{\mathbb N}$ in $V[G]$. Use the ~~Maximal~~ Maximality Principle ([30, Theorem IV.7.1]) to choose ~~a name~~ an $\mathbb H_{A\cup F(\xi)}$-name $\dot h_\xi$ for this function.

Claim 4.14. *The condition q_ξ forces that $\dot f_\xi \lhd^n \dot h_\xi \lhd^n f_{\xi+2}$.*

Proof. If there are $r \le q_\xi$ in $\mathbb H_E$ and $j \ge n$ such that $r \Vdash \dot f_\xi(j) \not\lhd \dot h_\xi(j)$, fix a generic filter G in $\mathbb H_E$ containing r. Then in $V[G]$ we have $\dot f_\xi \not\lhd^n \dot f_{\xi+1}$, although $q_\xi \in G$; contradiction. An analogous argument gives that there is no $j \ge n$ such that some $r \le_E q_\xi$ forces that $\dot h_\xi(j) \not\lhd \dot f_{\xi+2}(j)$. □

By the pigeonhole principle and passing to a subset if necessary, we may assume that n as in (4.2) is the same for all ξ. The pairs $(q_\xi, \dot h_\xi)$ are indexed by limit ordinals below κ. We re-enumerate them preserving the order and obtain conditions q_ξ and names $\dot h_\xi$ for $\xi < \kappa$. Since $\mathbb H_E$ has the ccc, some condition $q \in \mathbb H_E$ forces that κ of the q_ξ's belong to the generic filter. Therefore q forces that this family is a name for a κ-chain in $(C^{\mathbb N}, \lhd^*)$.

~~Every set $F(\xi)$ is finite, and by the pigeonhole principle we can assume that there exists m~~ By Proposition 3.7, there exists a cofinal $X \subseteq \kappa$ such that for ~~all ξ we have~~

$$F(\xi) = \{a(\xi, 0), \dots, a(\xi, m-1)\}$$

and that for all ξ and η the map

$$a(\xi, i) \mapsto a(\eta, i)$$

is an order-isomorphism between $F(\xi)$ and $F(\eta)$. Since $\kappa > \mathfrak{c}$, by another application of the pigeonhole principle we can assume that there are subsets $L(j)$ and $R(j)$ of the root A for $j < m$ such that

$$L(j) = \{b \in A \mid b \leq_E a(\xi, j)\} \text{ and } R(j) = \{b \in A \mid b \geq_E a(\xi, j)\}$$

for all $\xi < \kappa$. Clearly, $L(j) \cap R(j)$ is empty for all j. Therefore, the extension of the map in by the identity map on A is an order-isomorphism between $A \cup F(\xi)$ and $A \cup F(\eta)$.

For $\xi < \eta < \kappa$, let $\tau_{\xi,\eta}$ be the restriction of the relation $\leq_E$ to $F(\xi) \times F(\eta)$. For $n \geq 2$ and an increasing n-tuple $\bar{\xi} := (\xi(j) : j < n)$ of ordinals let $\lll_{\bar{\xi}}$ denote the depletion of $\leq_E$ determined by any two distinct elements $\xi < \eta$ of X there is $\{\xi, \eta\} \subseteq s \Subset \kappa$ such that there is no s-walk (see Definition 3.1with $A := \emptyset$ and $F(\xi(j))$, for $j < n$) with endpoints in $F(\xi)$ and $F(\eta)$.

Suppose that for every n and every n-tuple $\bar{\xi}$ there are $x \in F(\xi(0))$ Fix $\xi \in X$, let $\xi' := \min(X \setminus (\xi + 1))$, and $y \in F(\xi(n-1))$ fix $\{\xi, \xi'\} \subseteq s \Subset \kappa$ such that $x \lll_{\bar{\xi}} y$. Then there is a κ-chain or a κ^*-chain in E.

This is essentially a result of Kurepa ([31]); for a proof see e.g., [9, Theorem 7.1] .

We can therefore assume that

$$\lll_{\bar{\xi}} \cap (F(\xi(0)) \times F(\xi(n-1))) = \emptyset$$

for some n and some n-tuple $\bar{\xi}$. If this applies for a given $\bar{\xi}$, we'll then slightly abuse the language and say that '$\lll_{\bar{\xi}}$ is empty'. Moreover, we can assume that there is such a tuple for an arbitrarily large $\xi(0)$. This is because every end-segment of κ there is order-isomorphic to κ; thus by applying Claim 3.7 to an end-segment we would otherwise obtain a copy of κ or κ^* inside E. Also, if $\lll_{\bar{\xi}}$ is empty and $\bar{\eta}$ extends $\bar{\xi}$, then $\lll_{\bar{\eta}}$ is also empty. We can therefore recursively choose a cofinal $\mathsf{X} \subseteq \kappa$ such that for every pair $\eta < \zeta$ in X there exist an $n = n(\xi, \eta)$ and an increasing n-tuple $\bar{\xi}$ such that $\xi(0) = \eta$, $\xi(n-1) = \zeta$, and $\lll_{\bar{\xi}}$ is empty.

We now fix an n and an increasing n-tuple $\bar{\xi}$ such that $\lll_{\bar{\xi}}$ is empty and no s-walk with endpoints in $F(\xi)$ and $F(\xi')$. We will analyze the relation between the names $\dot{h}_{\xi(0)}$ and $\dot{h}_{\xi(n-1)}$$\dot{h}_{\xi}$ and $\dot{h}_{\xi'}$.

Consider the depletion $\lll_{\bar{\xi}}$ $\lll$ of $\leq_E$ on the set $A' := A \cup \bigcup_{j<n} F(\xi(j))$. Then $x \lll_{\bar{\xi}} y$ implies $x \lll_{\bar{\xi}} y$, and this implies $x \leq_E y$ for all x and $y$given by A and $F(\eta)$, for $\eta \in s$. By Lemma 3.3, $\lll_{\bar{\xi}}$ $\lll$ is a partial ordering on A'. We'll write $\lll$ for $\lll_{\bar{\xi}}$ whenever $\bar{\xi}$ is clear from the context.

$A' := A \cup \bigcup_{\eta \in s} F(\eta)$. Let

$$s = \{\xi(j) | j < n\}$$

be an increasing enumeration of s. We may assume that $\xi(0) = \xi$ and $\xi(n-1) = \xi'$. For $i < j < n$ let

$$A(i,j) := A \cup F(\xi(i)) \cup F(\xi(j)),$$

ordered by $\ll$. Then $\mathbb{H}_{A(i,j)} \lessdot \mathbb{H}_{A'}$ by Lemma 4.7. For ~~every $i < n-1$~~$0 \leq i < n-1$, the ordering on $A(i,i+1)$ agrees with the ordering induced from E, and therefore ~~in addition we have~~ $\mathbb{H}_{A(i,i+1)} \lessdot \mathbb{H}_E$. Since $\dot{h}_{\xi(i)}$ and $\dot{h}_{\xi(j)}$ are $\mathbb{H}_{A(i,j)}$-names ~~,~~ and $\mathbb{H}_E$ forces $\dot{h}_{\xi(i)} \vartriangleleft^* \dot{h}_{\xi(i+1)}$ for all $i < n-1$, ~~this implies~~ we have that $\mathbb{H}_{A'}$ forces $\dot{h}_{\xi(i)} \vartriangleleft^* \dot{h}_{\xi(i+1)}$ for all $i < n-1$. ~~Therefore~~ By transitivity, $\mathbb{H}_{A'}$ forces ~~$\dot{h}(\xi(0)) \vartriangleleft^* \dot{h}(\xi(n-1))$. Since $\dot{h}(\xi(0))$ and $\dot{h}(\xi(n-1))$ are $\mathbb{H}_{A(0,n-2)}$~~ $\dot{h}_{\xi(0)} \vartriangleleft^* \dot{h}_{\xi(n-1)}$. Since $\dot{h}_{\xi(0)}$ and $\dot{h}_{\xi(n-1)}$ are $\mathbb{H}_{A(0,n-1)}$-names and ~~$\mathbb{H}_{A(0,n-2)} \lessdot \mathbb{H}_{A'}$, $\mathbb{H}_{A(0,n-2)}$ forces $\dot{h}(\xi(0)) \vartriangleleft^* \dot{h}(\xi(n-1))$~~ $\mathbb{H}_{A(0,n-1)}$ is a regular subordering of $\mathbb{H}_{A'}$, $\mathbb{H}_{A(0,n-1)}$ forces $\dot{h}_{\xi(0)} \vartriangleleft^* \dot{h}_{\xi(n-1)}$.

~~However, if $G \subseteq \mathbb{H}_A$ is generic, then since $\lll_{\bar{\xi}}$ is empty, by~~ Consider the statement of Lemma 4.12 with E, A, B, and D replaced with $A \cup F(\xi(0)) \cup F(\xi(n-1))$, $A \cup F(\xi(0))$, $A \cup F(\xi(n-1))$, and A, respectively (sorry!). Since there is no s-walk whose endpoints are some $x \in F(\xi(0))$ and some $y \in F(\xi(n-1))$, we have that $x \ll y$ implies there is $a \in A$ such that $x \leq a \leq y$, and that $y \ll x$ imples there is $a \in A$ such that $y \leq a \leq x$. This means that the assumptions of Lemma 4.12 ~~the quotient forcing $\mathbb{H}_{A(0,n-1)}/G$ is isomorphic~~ are satisfied, and that if $G \subseteq \mathbb{H}_A$ is a generic filter then the quotient $\mathbb{H}_{A \cup A(0,n-1)}/G$ is forcing-equivalent to the product ~~of the quotients $\mathbb{H}_{A\cup\xi(0)}/G$ and $\mathbb{H}_{A\cup\xi(n-1)}/G$~~

$$\mathbb{H}_{A\cup F(\xi(0)}(A, \Upsilon_G) \times \mathbb{H}_{A\cup F(\xi(n-1))}(A, \Upsilon_G).$$

Most importantly, the names $\dot{h}_{\xi(0)}$ and $\dot{h}_{\xi(n-1)}$ are added by the two factors of this product. By Lemma 4.11, there ~~exists a $p \in \mathbb{H}_A$~~ exist a condition $p_\xi \in \mathbb{H}_A$ and an $\mathbb{H}_A$-name ~~$\dot{g}$ such that~~

~~$$p \Vdash \dot{h}_{\xi(0)} \leq^* \dot{g} \leq^* \dot{h}_{\xi(n-1)}.$$~~

~~Therefore for every $\xi \in X$ and $\xi(n-1) := \min X \setminus (\xi(0)+1)$ we can find $\bar{\xi}$ such that $\xi(0) = \xi$, $\lll_{\bar{\xi}}$ is empty, and there are $p_\xi \in \mathbb{H}_A$ and an $\mathbb{H}_A$-name $\dot{g}_\xi$ such that~~ $\dot{g}_\xi$ (recall that $\xi = \xi(0)$) such that

$$p_\xi \Vdash \dot{f}_\xi h_{\xi(0)} \leq^* \dot{g}_\xi \leq^* \dot{h}_{\xi(n-1)}.$$

Since $\mathbb{H}_A$ ~~has the ccc, some~~ is countable, there is $q \in \mathbb{H}_A$ ~~forces that the set of p_ξ that belong to the generic filter has cardinality~~ such that $Y = \{\xi \in X | p_\xi = q\}$ is a cofinal subset of X (and of κ). Therefore q forces that $\mathbb{H}_A$ adds a strictly increasing κ-chain $\dot{g}_\xi$, for $\xi < \kappa$, to $(\mathbb{N}^\mathbb{N}, \leq^*)$. Since A is countable, $\mathbb{H}_A$ cannot add more than $\mathfrak{c}$ reals; contradiction. $\square$

The robustness of the order property (Definition 1.1) is used in the following proposition.

Proposition 4.15. ~~*For every theory T and every formula $\varphi(\bar{x}, \bar{y})$ such that*~~ *Suppose that the pair (T, φ) has the* ~~*order property,*~~ *robust order property and E is any poset. Then $\mathbb{H}_E$ forces* ~~*that*~~ *the following.*

(1) *The poset E embeds into $\prod_{\mathrm{Fin}}(A_n, \lhd_\varphi)$ for every sequence (A_n) of models of T.*
(2) *For any nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ there is strictly increasing map from E into $\prod_{\mathcal{U}}(A_n, \lhd_\varphi)$ whose range is linearly ordered by $\lhd_\varphi$.*

Proof. ~~Working in the forcing extension, note that since $A_n \models T$~~ The first part is almost obvious, but proving it in some detail will also provide a proof of the second part.

By Lemma 2.1, there are $\eta \in \mathbb{N}^{\mathbb{N}}$ and $\Phi\colon \prod_{\mathrm{Fin}}(n, \leq^*) \to \prod_{\mathrm{Fin}}(\eta(n), <^*)$ such that $f \leq^* g$ and $g \not\leq^* f$ implies $(\forall^\infty n)\Phi(f)(n) < \Phi(g)(n)$. Since A_n is a model of T, there exists a $\lhd_\varphi$-chain C_n of length $\eta(n)$ in A_n. ~~Therefore there is~~ By identifying this chain with $(\eta(n), \leq)$, we obtain $\Phi\colon \prod_{\mathrm{Fin}}(n, \leq^*) \to \prod_{\mathrm{Fin}}(A_n, \lhd_\varphi)$ such that $f \leq^* g$ and $g \not\leq^* f$ implies $\Phi(f)(n) \lhd_\varphi \Phi(g)(n)$ and $\Phi(g)(n) \not\lhd_\varphi \Phi(f)(n)$ for all but finitely many n. By composing the embedding of E into $\prod_{\mathrm{Fin}}(n, \leq^*)$ provided by Theorem 4.1 with Φ, we obtain an $\mathbb{H}_E$-name ~~$\dot{\Xi}$ for a strictly increasing map from $(\prod_n n, \leq^*)$ into $\prod_{\mathrm{Fin}}(A_n, \lhd_\varphi)$. By Lemma 4.6, if $G \subseteq \mathbb{H}_E$ is a sufficiently generic filter then in $V[G]$ there exists a strictly increasingfunction $\Upsilon_G\colon E \to (\prod_n n, \leq^*)$. By Lemma 2.1, there is a strictly increasing $\Phi\colon (\prod_n n, \leq^*) \to (\prod_n \eta(n), <^*)$ for some $\eta \in \mathbb{N}^{\mathbb{N}}$. Hence $\dot{\Xi} \circ \Phi \circ \Upsilon_G$ is a name for an embedding as required~~ for an embedding of $\Xi\colon E \to \prod_{\mathrm{Fin}}(A_n, \lhd_\varphi)$ (this proves the first part) that in addition has the property that $a <_E b$ implies

$$(\forall^n)(\Xi(f)(n) \lhd_\varphi \Xi(g)(n) \text{ and } \Xi(g)(n) \not\lhd_\varphi \Xi(f)(n)).$$

Let $\mathcal{U}$ be a nonprincipal ultrafiter on $\mathbb{N}$ and let $\pi_{\mathcal{U}}$ denote the quotient map from $\prod_{\mathrm{Fin}} A_n$ to $\prod_{\mathcal{U}} A_n$. Then the displayed formula implies that the restriction of $\pi_{\mathcal{U}}$ to $\Xi[E]$ is strictly increasing. The range of this map is the ultraproduct of the $\lhd_\varphi$-chains C_n, and therefore linearly ordered by Łoś's Theorem. This proves the second part. □

5. Proofs of Theorem C and Corollary D

The proof of Theorem C will use the following result (see [9, Theorem 3.2] for a proof) .

Theorem (Galvin). *For every uncountable cardinal κ there exists a partial ordering E_κ such that E_κ has no infinite chains but for every linear ordering $\mathcal{L}$ such that there are neither κ-chains nor κ^*-chains in $\mathcal{L}$ there is no strictly increasing map $\Phi\colon E \to \mathcal{L}$.* □

Proof of Theorem C. Fix a theory T ~~that has~~ with the robust order property ~~, a sequence (A_n) of countable models of T,~~ and an ultrafilter $\mathcal{U}$ on $\mathbb{N}$. We will prove that the Levy collapse of the continuum to $\aleph_1$ followed by $\mathbb{H}_E$

for an appropriate choice of a poset E provided by Galvin's theorem forces all three statements assumptions of Theorem C. The These proofs have a common initial segment that we now present.

By Lemma 2.1 there are $\eta \in \mathbb{N}^{\mathbb{N}}$ and a strictly increasing

$$\Phi\colon (\prod_n n, \leq^*) \to (\prod_n \eta(n), <^*).$$

Since (T,φ) has the robust order property, each B_n has a $\lhd_{\varphi^\infty}$-chain of length $\eta(n)$, and there exists an embedding $\Xi\colon \prod_{\mathcal{U}}(\eta(n),<) \to \prod_{\mathcal{U}}(B_n, \lhd_{\varphi^\infty})$.

Thus Φ followed by the quotient map from $(\prod_n \eta(n), <^*)$ onto the ultrapower $\prod_{\mathcal{U}}(\eta(n),<)$ and Ξ gives a strictly increasing map

$$\Omega\colon (\prod_n n,\leq^*) \to \prod_{\mathcal{U}}(B_n,\lhd_{\varphi^\infty}).$$

In the extension by the Levy collapse of the continuum to $\aleph_1$ choose the poset E as follows. Let , let $\kappa > \mathfrak{c}$ be a regular cardinal . By a result of Galvinthat appears in [9, Theorem 3.2], there exists a partial ordering E_κ such that E_κ has no infinite chains but for every linear ordering $\mathcal{L}$ and a strictly increasing map $\Phi\colon E \to \mathcal{L}$, there exists a κ-chain or a κ^*-chain in $\mathcal{L}$. ($\kappa = \aleph_2$ will do). Let E be the poset as guaranteed by Galvin's theorem stated at the beginning of this section. We will prove that the Levy collapse followed by $\mathbb{H}_E$ is the forcing notion as promised in the statement of Theorem C.

By Theorem 4.1, Fix a sequence (A_n) of countable structures in the language of T and a sequence (B_n) of countable models of T. Proposition 4.15 implies that $\mathbb{H}_E$ adds a strictly increasing map $\Upsilon_G\colon E \to (\prod_n n, \leq^*)$.

If $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$ and $\Psi\colon (\prod_n \eta(n),<^*) \to \prod_n \eta(n)/\mathcal{U}$ is the quotient map, then $\Psi\circ\Upsilon_G$ is strictly increasing. Since $\prod_n\eta(n)/\mathcal{U}$ is a linear ordering, it has a κ-chain by from E into $\prod_{\mathcal{U}}(B_n,\lhd_\varphi)$ whose range is linearly ordered by $\lhd_\varphi$. By the choice of E. By composing with Ω, we obtain , there exists a κ-$\lhd_{\varphi^\infty}$ $\lhd_\varphi$-chain in $\prod_{\mathcal{U}}(B_n,\lhd_{\varphi^\infty})$or a κ^*-$\lhd_\varphi$-chain in $\prod_{\mathcal{U}}(B_n,\lhd_\varphi)$.

However, On the other hand, by Theorem 4.1 implies that there are no there are neither κ-chains nor κ^*-chains in $\prod_{\mathrm{Fin}}(A_n,\lhd_\varphi)$. Therefore $\prod_{\mathcal{U}} B_n$ cannot be isomorphic to $\prod_{\mathrm{Fin}} A_n$

From this point on the proofs of (1)–(3) differ.

(1) and (2): Since elementary embeddings preserve $\lhd_\varphi$, $\prod_{\mathcal{U}}(B_n,\lhd_\varphi)$ is not isomorphic to $\prod_{\mathrm{Fin}}(A_n,\lhd_\varphi)$ or to an elementary submodel of $\prod_{\mathrm{Fin}} A_n$. This proves parts and of Theorem C. thereof.

To prove (3) , note that if the formula If φ is quantifier-free, then even non-elementary embeddings preserve $\lhd_\varphi$ and $\prod_{\mathcal{U}} B_n$ cannot even be isomorphic to a submodel of $\prod_{\mathrm{Fin}} A_n$. □

Proof of Corollary D. Suppose that A is a separable C^*-algebra and $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$. If $\mathcal{U}$ is principal, then $(A \otimes C(K))^{\mathcal{U}}$ is isomorphic to $A \otimes C(K)$ while A^∞ is nonseparable. We may therefore assume that A is infinite-dimensional and that $\mathcal{U}$ is nonprincipal.

The theory of infinite-dimensional C^*-algebras has the order property witnessed by an atomic formula ([15, Lemma 5.3]). Therefore the theory of $A \otimes C(K)$ has the robust order property, and Theorem C (3) implies that $(A \otimes C(K))^{\mathcal{U}}$ does not embed into B^∞ for any C^*-algebra B. □

6. Proofs of Theorem A and Theorem B: Tie points

The contents of this section is rather accurately described by its title.

Definition 6.1. Suppose X is a compact Hausdorff space. A point $x \in X$ is a *tie point* if there are closed subsets A and B of X such that $A \cup B = X$ and $A \cap B = \{x\}$ (in symbols, $A \bowtie_x B$).

Two subsets $\mathcal{I}$ and $\mathcal{J}$ of a Boolean algebra $\mathfrak{B}$ are *orthogonal* if $a \wedge b = 0_{\mathcal{B}}$ for all $a \in \mathcal{I}$ and all $b \in \mathcal{J}$. The following is proved by parsing the definitions.

Proposition 6.2. *Suppose $\mathfrak{B}$ is a Boolean algebra. The following are equivalent for an ultrafilter $\mathcal{U}$ on $\mathfrak{B}$.*

1. *The complement of $\mathcal{U}$ is equal to the union of two orthogonal ideals.*
2. *$\mathcal{U}$ is a tie-point in the Stone space of $\mathfrak{B}$.* □

Definition 6.3. By analogy with true P-points, an ultrafilter $\mathcal{U}$ in a Boolean algebra is called a *true tie point* if the ideals as in Proposition 6.2 (2) can be chosen so that each one of them is generated by a linearly ordered subset.

The salient point of the proof of the following is the observation that true tie points are Σ_1-definable, but the reader may choose to ignore this remark (at the risk of their own loss).

Lemma 6.4. *Every ultraproduct of countable atomless Boolean algebras has a true tie point.*

Proof. Every ultrafilter in a countable atomless Boolean algebra is a true tie point, since the generating sets of order type ω can be chosen by recursion. Suppose $\prod_{\mathcal{U}} C_n$ is an ultraproduct of countable atomless Boolean algebras. If $\mathcal{U}$ is principal, then $\prod_n C_n$ is isomorphic to one of the C_n's and the assertion follows from the first sentence of this proof.

Now assume $\mathcal{U}$ is nonprincipal. For every C_n fix a true tie point p_n and linearly ordered generating sets $\mathcal{A}_n$ and $\mathcal{B}_n$ for the ideal $C_n \setminus p_n$. Then $(C_n, \mathcal{A}_n, \mathcal{B}_n)$ is an expansion of C_n to the language with two additional unary predicates. Each one of these structures satisfies the following: Both $\mathcal{A}_n$ and $\mathcal{B}_n$ are linearly ordered, $A \wedge B = \emptyset$ for all $A \in \mathcal{A}_n$ and $B \in \mathcal{B}_n$, and for every $X \in C_n$ either X or its complement belongs to $\mathcal{A}_n \cup \mathcal{B}_n$. These are all first-order statements, and they imply that the complement of $\mathcal{A}_n \cup \mathcal{B}_n$ is an ultrafilter.

The ultraproduct $\prod_{\mathcal{U}}(C_n, \mathcal{A}_n, \mathcal{B}_n)$ is an expansion of $\prod_n C_n$ and by Łoś's Theorem the sets $\mathcal{A} := \prod_{\mathcal{U}} \mathcal{A}_n$ and $\mathcal{B} := \prod_{\mathcal{U}} \mathcal{B}_n$ generate ideals of $\prod_{\mathcal{U}} C_n$ whose complement is a true tie point. □

Proof of Theorem A. We need to prove that PFA implies $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ is not isomorphic to an ultraproduct of Boolean algebras associated with a non-principal ultrafilter on $\mathbb{N}$. By [43] (see [7, Corollary 1.9]), PFA implies that there are no tie points in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$, while there are tie points in an ultra-product of countable atomless Boolean algebras by Lemma 6.4. □

The following will be used in the proof of Theorem B.

Lemma 6.5. *The poset for adding at least $\mathfrak{c}^+$ Cohen reals forces that every projectively definable atomless Boolean algebra ~~$\mathcal{B}$ definable from a real~~ $\mathfrak{B}$ has no true tie points.*

Proof. Fix an $n \in \mathbb{N}$ and $\mathbf{\Sigma}^1_n$-formulas φ, $\varphi_\wedge$, $\varphi_\vee$, and $\varphi_\setminus$, which define $\mathfrak{B}$. We will only need the $\mathbf{\Delta}^1_{n+1}$ formula $\varphi_<$, that defines the relation $a < b$ in $\mathfrak{B}$. By passing to an intermediate forcing extension, without a loss of generality we may assume that ~~$\mathcal{B}$ is~~ the reals coding these formulas are in the ground model. Let $\kappa \geq \mathfrak{c}^+$ be the number of the Cohen reals added. By genericity, no nonprincipal ultrafilter on $\mathbb{N}$ is generated by fewer than κ subsets of $\mathbb{N}$. (After adding κ Cohen reals, for every $\mathcal{X} \subseteq \mathcal{U}$ of cardinality less than κ there is a Cohen real Y generic over $V[\mathcal{X}]$. For every infinite $X \subseteq \mathbb{N}$, the set of all $Y \subseteq \mathbb{N}$ such that $X \cap U$ and $X \setminus Y$ are both infinite is comeager. Therefore $\mathcal{X}$ does not 'decide' whether $Y \in \mathcal{U}$ or $\mathbb{N} \setminus Y \in \mathcal{U}$.) Assume p is a true tie point in ~~$\mathcal{B}$~~ $\mathfrak{B}$ and let $\mathcal{A}$ and $\mathcal{B}$ be the linearly ordered (modulo $\mathcal{I}$) sets whose complements generate $\mathfrak{B} \setminus p$. By genericity, at least one of $\mathcal{A}$ and $\mathcal{B}$ has cofinality greater than $\mathfrak{c}$. By interchanging $\mathcal{A}$ and $\mathcal{B}$, we may assume that the cofinality of $\mathcal{A}$ is $\kappa > \mathfrak{c}$.

The proof is completed by Kunen's isomorphism of names argument ([29]) ~~implies that $\mathcal{B}$ does not contain a well-ordered~~ that we now sketch.

Let $\dot{f}_\xi$, for $\xi < \kappa$, be names for the elements of a strictly increasing chain cofinal in $\mathcal{A}$. Since each $\dot{f}_\xi$ is the name of a real, it is coded by a sequence of antichains and the union of the supports of all conditions in these antichains is a countable set, $D_\xi \subseteq \kappa$. Since $\kappa > \mathfrak{c}^+$, by going to a cofinal subset we may assume that the sets D_ξ form a Δ-system with root R. Again using $\kappa > \mathfrak{c}$ (this time with a counting argument) and going to a cofinal subset we may assume that the restrictions of $\dot{f}_\xi$ to R are equal, and that $\dot{f}_\xi$ and $\dot{f}_\eta$ are isomorphic for all ξ and η. This means that for $\xi < \eta$ there is an automorphism $\Phi_{\xi\eta}$ of the poset $\mathbb{C}_\kappa$ for adding κ ~~-chain; contradiction.~~ Cohen reals that sends $\dot{f}_\xi$ to $\dot{f}_\eta$, for any two $\xi < \eta < \kappa$. However, since $\mathbb{C}_\kappa$ forces that $\varphi_<(\dot{f}_\xi, \dot{f}_\xi)$, and the real coding the asymmetric formula $\varphi_<$ is in the ground model, this is a contradiction that completes the proof. □

~~Clearly, Lemma 6.5 can be improved by relaxing its assumption to '$\mathcal{B}$ is definable from a set of at most $\mathfrak{c}^V$ reals'.~~

Proof of Theorem B. In the model obtained by adding at least $\mathfrak{c}^+$ Cohen reals to a model of ZFC, suppose that $\mathfrak{B}$ is a ~~Boolean algebra definable from a real~~projectively definable Boolean algebra. By Lemma 6.5, there are no true tie points in $\mathfrak{B}$. By Lemma 6.4, in every model of ZFC there is a true tie point in any ultraproduct of countable atomless Boolean algebras. □

7. The existence of universal ultrapowers

~~In this sectionwe collect a few easy observations.~~ A set of questions related to the questions resolved in our main results has rather easy answers, collected in this section. Fix a complete first-order theory T in a countable (possibly metric) language. CH implies that all ultrapowers of countable (or separable) models of T associated with nonprincipal ultrafilters on $\mathbb{N}$[10] are isomorphic. This conclusion is by [16, Theorem 5.6] (also [19]) equivalent to CH. In some applications it suffices to know that among the ultrapowers of A there exists one which is injectively universal (i.e., every other ultrapower embeds into it elementarily). Is the existence of such ultrapower equivalent to CH, and is it even consistent with the negation of CH? A partial answer to the first question is given in Corollary E.

For T as in the previous paragraph let

$$\mathbb{M}_T = \{A^{\mathcal{U}} \mid A \models T,\ A \text{ is countable, and } \mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}\}.$$

Since our 'results' are immediate consequences of known results, we do not include the definitions of SOP, SOP_4, and the olive property (references are included below).

~~Fix a first-order theory T. Let~~

$$\mathbb{M}_T = \{A^{\mathcal{U}} \mid A \models T,\ A \text{ is countable, and } \mathcal{U} \in \mathbb{N} \setminus \mathbb{N}\}.$$

~~The Continuum Hypothesis implies that, up to isomorphism, $\mathbb{M}_T$ has exactly one element. If T has the order property, then the converse holds: if $\mathbb{M}_T$ has one element (or even fewer than $2^{\mathfrak{c}}$ elements) up to isomorphism, then the Continuum Hypothesis holds ([15], [19]). We don't know whether, for a T with an order property, the existence of an injectively universal element for $\mathbb{M}_T$ is relatively consistent with the failure of CH.~~

Proposition 7.1. *Suppose that T is a first-order theory with the order property.*

1. *Then T has a universal model of cardinality $\mathfrak{c}$ if and only it has a universal model of cardinality $\mathfrak{c}$ in $\mathbb{M}_T$.*
2. *If T has the SOP, SOP_4, or the olive property, and there exists a cardinal κ such that $\kappa < \mathfrak{c} < 2^\kappa$, then T does not have a universal model in $\mathbb{M}_T$.*
3. *If the assumptions of (2) are strengthened to '$\kappa^+ < \mathfrak{c} < 2^\kappa$ and $\mathfrak{c}$ is regular', then $\mathbb{M}_T$ does not contain a basis consisting of fewer than 2^κ models.*[11]

[10] All utrafilters in this section are nonprincipal and on $\mathbb{N}$.

[11] I.e., every $\mathbb{X} \subseteq \mathbb{M}_T$ such that every element of $\mathbb{M}_T$ embeds into an element of $\mathbb{X}$ has cardinality at least 2^κ.

Proof. (1) It is well-known that every model of T of cardinality $\mathfrak{c}$ is isomorphic to an elementary submodel of an ultrapower of a countable model of T. This follows from the results of [34, Chapter VI.5] or [19].

(2) This was proved in [26] (when T has SOP), [38, §2] (when T has SOP_4), and in [41] (when T has the olive property).

(3) We will prove a stronger statement: For every family M_ξ, $\xi < 2^\kappa$, of elements of $\mathbb{M}_T$ there exists $X \subseteq 2^\kappa$ of cardinality κ and N_ξ such that $M_\xi \prec N_\xi$, $|N_\xi| = \mathfrak{c}$, and N_ξ does not embed into N_η for all $\xi \neq \eta$ in X.

Let $\varphi(\bar{x}, \bar{y})$ be such that (T, φ) has the order property. By the methods of [26], [38], and [41], there exists a function inv_φ from the set

$$\mathrm{Mod}_{\mathfrak{c}}(T) = \{A \mid A \models T \text{ and } |A| = \mathfrak{c}\}$$

into $[\mathcal{P}(\kappa)]^{\mathfrak{c}}$ such that

(1) If $M_0 \in \mathrm{Mod}_{\mathfrak{c}}(T)$ is embeddable into $M_1 \in \mathrm{Mod}_{\mathfrak{c}}(T)$ then $\mathrm{inv}_\varphi(M_0) \subseteq \mathrm{inv}_\varphi(M_1)$.

(2) If $M_0 \in \mathrm{Mod}_{\mathfrak{c}}(T)$ and $S \subseteq \kappa$ then there exists $M_1 \in \mathrm{Mod}_{\mathfrak{c}}(T)$ such that $M_0 \prec M_1$ and $S \in \mathrm{inv}_\varphi(M_1)$.

Fix M_ξ, for $\xi < 2^\kappa$, in $\mathbb{M}_T$. Let S_ξ, for $\xi < 2^\kappa$, be pairwise distinct subsets of κ. By a realizing types argument and (1), there are $N_\xi \in \mathbb{M}_T$ such that $M_\xi \prec N_\xi$ and $S_\xi \in \mathrm{inv}_\varphi(N_\xi)$. By Hajnal's free subset theorem ([24]), there exists $X \subseteq 2^\kappa$ of cardinality 2^κ such that $S_\xi \notin \mathrm{inv}_\varphi(N_\eta)$ for all $\xi \neq \eta$ in X, and therefore N_ξ, for $\xi \in X$, are as required. □

Corollary E. *If T is a first-order theory with the SOP, SOP_4, or the olive property, and there exists a cardinal κ such that $\kappa < \mathfrak{c} < 2^\kappa$, then T does not have a universal model in $\mathbb{M}_T$.*

If in addition there exists a cardinal κ such that $\kappa^+ < \mathfrak{c} < 2^\kappa$ and $\mathfrak{c}$ is regular, then $\mathbb{M}_T$ does not contain a basis consisting of fewer than 2^κ models. □

8. Concluding remarks and questions

The ~~methods of [40], [37], and [36] may be relevant to the question whether $\prod_{\mathcal{U}} A$ can be isomorphic to $\prod_{\mathrm{Fin}} A$ for a countable model~~ question that initiated the research reported here remains open

Question 8.1. *Suppose that A ~~of a theory with~~ is a nontrivial countable (or a separable) structure whose theory has the order property ~~in a model of ZFC + ¬CH .~~ and that $\prod_{\mathcal{U}} A$ is isomorphic to $\prod_{\mathrm{Fin}} A$ for some $\mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}$. Does it follow that the CH holds?*

Our main results show that in some models of ZFC in which CH fails the premise of Question 8.1 fails as well. The methods of [40], [37], and [36] may be relevant to Question 8.1.

Our proof of Theorem B uses a variant of Kunen's well-known proof that after adding $\kappa > \mathfrak{c}$ Cohen reals to a model of ZFC there are no κ-chains in $(\mathbb{N}^{\mathbb{N}}, \rho)$ for any Borel partial ordering ρ on $\mathbb{N}^{\mathbb{N}}$. The proof of Theorem C uses a related (i.e., semicohen; see [27]) forcing notion and a somewhat similar analysis of names. These results are however different, since the forcing $\mathbb{H}_E$ used in the proof of Theorem C can add an ω_2-chain to some Borel poset $(\mathbb{N}^\mathbb{N}, \rho)$ without adding an ω_2-chain to $(\mathbb{N}^\mathbb{N}, \leq^*)$ (see [9, Theorem 2.1]).

The argument of the proof of Theorem B works for many other forcings that add more than $\mathfrak{c}$ reals, as long as one can uniformize the names and in the extension there are no ultrafilters on $\mathbb{N}$ with small generating sets. The latter does not apply to the Sacks forcing. As a matter of fact, after adding $\mathfrak{c}^+$ Sacks reals to a model of CH with countable supports (by either product or iteration), there exists a selective $\aleph_1$-generated ultrafilter on $\mathbb{N}$, and it is a true tie point ([1]). It is therefore not clear whether in the Sacks model(s) $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ is isomorphic to an ultraproduct of countable atomless Boolean algebras.

If so, then this would have to be an $\aleph_1$-generated ultrafilter. The most obvious choice would be an ultrafilter generated by ground-model selective ultrafilters (there are $2^{\aleph_1}$ such ultrafilters by [1]). As all of these ultrafilters 'look the same' (see [47] for an interpretation of this assertion) this suggests the following question.

Question 8.2. *Suppose that in either one of the Sacks models (countable support iteration or countable support product of regular length $\kappa > \mathfrak{c}$), $\mathcal{U}$ and $\mathcal{V}$ are $\aleph_1$-generated selective ultrafilters. Is it true that $(\mathbb{N}, \leq)^\mathcal{U} \cong (\mathbb{N}, \leq)^\mathcal{V}$?*

One could ask an analogous question for countable models of other countable first-order theories with the order property; $(\mathbb{N}, \leq)$ just appears to provide the simplest interesting instance of this question. The ideas from [42, §2 and §4] may be relevant to this problem in the case of Boolean algebras.

The question about the existence universal ultrapowers in the absence of CH tackled in §7 also remains open.

We conclude with a few words on 'definable' reduced products $\prod_\mathcal{F} A_n$. If $\mathcal{F}$ is an analytic filter on $\mathbb{N}$ (i.e., a filter that is analytic as a subset of $\mathcal{P}(\mathbb{N})$, given its Cantor-set topology) that extends the Fréchet filter, then the restriction of $\mathcal{F}$ to any $\mathcal{F}$-positive set is not an ultrafilter. (This is because all analytic sets have the universal property of Baire, unlike the nonprincipal ultrafilters.) Therefore the Feferman–Vaught theorem ([21], and for the metric case [22] or [12, §16.3]) implies that if all A_n are elementarily equivalent, and if $\mathcal{F}$ extends the Fréchet filter then $\prod_\mathcal{F} A_n$ is elementarily equivalent to $\prod_{\operatorname{Fin}} A_n$. Many (but not all) of the reduced products $\prod_\mathcal{F} A_n$ are countably saturated[12] and therefore isomorphic to $\prod_{\operatorname{Fin}} A_n$ if ~~the~~ CH holds. One ~~can~~ could ask for what analytic filters $\mathcal{F}$ is $\prod_\mathcal{F} A_n \cong \prod_{\operatorname{Fin}} A_n$ provable in ZFC.

In the case when each A_n is the two-element Boolean algebra, of quotients of the form $\mathcal{P}(\mathbb{N})/\mathcal{I}$, the isomorphism is provable if and only if there is

[12]A sufficient condition for countable saturation of $\prod_\mathcal{F} A_n$ was isolated in [11, Definition 6.5].

a continuous $f\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ that lifts such an isomorphism ([18]) and in many (conjecturally, all) cases this is equivalent to the Rudin–Keisler isomorphism of the underlying ideals ([10, Corollary 3.4.2] and [18, Corollary 3]).

In the case when all A_n are Boolean algebras, this is a question about abelian C^*-algebras. This is because the category of Boolean algebras is, via the Stone duality, equivalent to the category of compact, zero-dimensional, Hausdorff spacesand the latter category is, by the Gelfand–Naimark duality, equivalent to the category of unital, abelian, C^*-algebras (see [12, §1.3]). By this observation and the main result of [17], PFA implies that two such reduced products are isomorphic if and only if there is an (appropriately defined) 'trivial' isomorphism between them. For example, PFA implies that $\prod_{\mathrm{Fin}} B \not\cong \mathcal{P}(\mathbb{N})/\,\mathrm{Fin}$ if B is the atomless countable Boolean algebra. The ultimate extension of the result of [17] to the coronas of arbitrary separable C^*-algebras was proved in [32] and [45].

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