–

ARE $\mathfrak{a}$ AND $\mathfrak{d}$ YOUR CUP OF TEA; REVISITED SH:700A

SAHARON SHELAH

Abstract. This was non-essentially revised in late 2020. First point is noting that the proof of [She04b, Th.4.3] which says that the proof giving the consistently $\mathfrak{b} = \mathfrak{d} = \mathfrak{u} < \mathfrak{a}$ gives that also $\mathfrak{s} = \mathfrak{d}$. The proof use a measurable cardinal and a c.c.c. forcing so it give large $\mathfrak{d}$ and assume a large cardinal.

Second point is adding to the results of §2,§3 which say that (in §3 with no large cardinals) we can force $\aleph_1 < \mathfrak{b} = \mathfrak{d} < \mathfrak{a}$. We like to have $\aleph_1 < \mathfrak{s} \leq \mathfrak{b} = \mathfrak{d} < \mathfrak{a}$. For this we allow in §2,3 the sets K_t to be uncountable; this require non-essential changes. In particular, we replace usually $\aleph_0, \aleph_1$ by σ, ∂. Naturally we can deal with $\mathfrak{i}$ and similar invariants.

Third we proof read the work again. To get $\mathfrak{s}$ we could have retain the countability of the member of the I_t-s but the parameters would change with $A \in I_t$, well for a cofinal set of them; but the present seem simpler.

We intend to continue in [S$^+$a].

Original abstract We show that consistently, every MAD family has cardinality strictly bigger than the dominating number, that is $\mathfrak{a} > \mathfrak{d}$, thus solving one of the oldest problems on cardinal invariants of the continuum. The method is a contribution to the theory of iterated forcing for making the continuum large.

Date: 2021-08-16.

2010 *Mathematics Subject Classification.* Primary: 03E17, 03E35; Secondary: 03E75.

Key words and phrases. set theory, general topology, cardinal invariants of the continuum, forcing, iterated forcing, templates, $\mathfrak{a}, \mathfrak{d}$, madness number, dominating number.

This is paper 700a in the author list, created cannibalizing paper 700. The author would like to thank the ISF (IsSrael Science Foundation) for partially supporting this research by grant 1838/19. The author thanks Alice Leonhardt for the beautiful typing [Sh:700] which was . first Typed - 98/Dec/16, last revised 2015-02-20. This paper [Sh:700a] was first typed in 12/2020. References like [She04b, 2.1=Lad.1] means the label of Def 2.1 is ad.1. The reader should note that the version in my website is usually more updated than the one in the mathematical archive .

Annotated Content

§ 0. Introduction

We deal with the theory of iteration of of c.c.c. forcing notions for the continuum and prove CON($\mathfrak{a} > \mathfrak{d}$) and related results. We present it in several perspectives; so §2 + §3 does not depend on §1; and §4 does not depend on §1, §2, §3. In §2 we introduce and investigate iterations which are of finite support but with so called ideal, weakly transitive memory and linear, non well ordered length and prove CON($\mathfrak{a} > \mathfrak{d}$) using a measurable. In §4 we answer also related questions ($\mathfrak{u} < \mathfrak{a}$); in §3, relying on §2 we eliminate the use of a measurable, and in §1 we rely heavily on [She04a].

Very basically, the difference between $\mathfrak{a}$ on the one hand and $\mathfrak{b}, \mathfrak{d}$ on the other hand which we use is that $\mathfrak{a}$ speaks on a set, whereas $\mathfrak{b}$ is witnessed by a sequence and $\mathfrak{d}$ by a quite directed family; it essentially deals with cofinality; so every unbounded subsequence is a witness as well, i.e. the relevant relation is transitive; when $\mathfrak{b} = \mathfrak{d}$ things are smooth, otherwise the situation is still similar. This manifests itself by using ultrapowers for some κ-complete ultrafilter (in model theoretic outlook), and by using "convergent sequence" (see [She87] and later [She09c, §2], [She09b]), or the existence of Av, the average, from [She90]) in §2, §3, respectively. The meaning of "model theoretic outlook", is that by experience set theorists starting to hear an explanation of the forcing tend to think of an elementary embedding $\mathbf{j} : \mathbf{V} \to M$ and then the limit practically does not make sense (though of course we can translate). Note that ultrapowers by e.g. an ultrafilter on κ, preserve any witness for a cofinality of a linear order being $\geq \kappa^+$ (or the cofinality of a κ^+-directed partial order), as the set of old elements is cofinal and a cofinal subset of a cofinal subset is a cofinal subset. On the other hand, the ultrapower always "increase" any set of cardinality at least κ, the completeness of the ultrafilter.

* * *

This (is $\mathfrak{a} \leq \mathfrak{d}$?) is one of the oldest problems and well known on cardinal invariants of the continuum (see [vD] and Roitman [Mil]). It was mostly thought (certainly by me) that consistently $\mathfrak{a} > \mathfrak{d}$ and that the natural way to proceed is by CS iteration $\langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \omega_2 \rangle$ of proper ${}^\omega\omega$-bounding forcing notions, starting with $\mathbf{V} \models ffff$GCH, and $|\mathbb{P}_i| = \aleph_1$ for $i < \omega_2$ and $\underset{\sim}{\mathbb{Q}}_i$ "deal" with one MAD family $\mathscr{A}_i \in \mathbf{V}^{\mathbb{P}_i}, \mathscr{A}_i \subseteq [\omega]^{\aleph_0}$, adding an infinite subset of ω almost disjoint to every $A \in \mathscr{A}_i$. The needed iteration theorem holds by [She98, Ch.V,§4], saying that in $\mathbf{V}^{\mathbb{P}_{\omega_2}}, \mathfrak{d} = \mathfrak{b} = \aleph_1$ and no cardinal is collapsed, <u>but</u> the single step forcing is not known to exist. This has been explained in details in [She00b].

We do not go in this way but in a totally different direction involving making the continuum large, so we still do not know

Problem 0.1. Is ZFC $+ 2^{\aleph_0} = \aleph_2 + \mathfrak{a} > \mathfrak{d}$ consistent?

To clarify our idea, let D be a normal ultrafilter on κ, a measurable cardinal and consider a c.c.c. forcing notion $\mathbb{P}$ and assume we have

(a) a sequence $\bar{\underset{\sim}{f}} = \langle \underset{\sim}{f}_\alpha : \alpha < \kappa^+ \rangle$ of $\mathbb{P}$-names such that $\Vdash_{\mathbb{P}}$ "$\langle \underset{\sim}{f}_\alpha : \alpha < \kappa^+ \rangle$ is $<^*$-increasing cofinal in ${}^\omega\omega$" (so $\bar{\underset{\sim}{f}}$ exemplifies $\Vdash_{\mathbb{P}}$ "$\mathfrak{b} = \mathfrak{d} = \kappa^+$")

(b) a sequence $\langle \underset{\sim}{A}_\alpha : \alpha < \alpha^* \rangle$ of $\mathbb{P}$-names such that $\Vdash_{\mathbb{P}}$ "$\{\underset{\sim}{A}_\alpha : \alpha < \alpha^*\}$ is MAD that is $\alpha \neq \beta \Rightarrow \underset{\sim}{A}_\alpha \cap \underset{\sim}{A}_\beta$ is finite and $\underset{\sim}{A}_\alpha \in [\omega]^{\aleph_0}$".

Now $\mathbb{P}_1 = \mathbb{P}^\kappa/D$ also is a c.c.c. forcing notion by Łoś theorem for $\mathbb{L}_{\kappa,\kappa}$; let $\mathbf{j} : \mathbb{P} \to \mathbb{P}_1$ be the canonical embedding; moreover, under the canonical identification we have $\mathbb{P} \prec_{\mathbb{L}_{\kappa,\kappa}} \mathbb{P}_1$. So also $\Vdash_{\mathbb{P}_1}$ "$\underset{\sim}{f}_\alpha \in {}^\omega\omega$", recalling that $\underset{\sim}{f}_\alpha$ actually consists of ω maximal antichains of $\mathbb{P}$ (or think of $(\mathscr{H}(\chi), \in)^\kappa/D, \chi$ large enough). Similarly $\Vdash_{\mathbb{P}_1}$ "$\underset{\sim}{f}_\alpha <^* \underset{\sim}{f}_\beta$ if $\alpha < \beta < \kappa^+$".

Now, if $\Vdash_{\mathbb{P}_1}$ "$\underset{\sim}{g} \in {}^\omega\omega$", then $\underset{\sim}{g} = \langle \underset{\sim}{g}_\varepsilon : \varepsilon < \kappa\rangle/D, \Vdash_{\mathbb{P}}$ "$\underset{\sim}{g}_\varepsilon \in {}^\omega\omega$" so for some $\alpha < \kappa^+$ we have $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{g}_\varepsilon <^* \underset{\sim}{f}_\alpha$ for $\varepsilon < \kappa$" hence by Łoś theorem $\Vdash_{\mathbb{P}_1}$ "$\underset{\sim}{g} <^* \underset{\sim}{f}_\alpha$" (so before the identification this means $\Vdash_{\mathbb{P}_1}$ "$\underset{\sim}{g} <^* \mathbf{j}(f_\alpha)$"), so $\langle \underset{\sim}{f}_\alpha : \alpha < \kappa^+\rangle$ exemplifies also $\Vdash_{\mathbb{P}_1}$ "$\mathfrak{b} = \mathfrak{d} = \kappa^+$".

On the other hand $\langle \underset{\sim}{A}_\alpha : \alpha < \alpha^*\rangle$ cannot exemplify that $\mathfrak{a} \le \kappa^+$ in $\mathbf{V}^{\mathbb{P}_1}$ because $\alpha^* \ge \kappa^+$ (as ZFC $\models \mathfrak{b} \le \mathfrak{a}$) so $\langle \underset{\sim}{A}_\alpha : \alpha < \kappa\rangle/D$ exemplifies that $\Vdash_{\mathbb{P}_1}$ "$\{\underset{\sim}{A}_\alpha : \alpha < \alpha^*\}$ is not MAD".

Our original idea here is to start with a FS iteration $\bar{\mathbb{Q}}^0 = \langle \mathbb{P}^0_i, \underset{\sim}{\mathbb{Q}}^0_i : i < \kappa^+\rangle$ of nep c.c.c. forcing notions, $\underset{\sim}{\mathbb{Q}}^0_i$ adding a dominating real, (e.g. by dominating real = Hechler forcing), for κ a measurable cardinal and let D be a κ-complete uniform ultrafilter on κ and $\chi \gg \kappa$. Then let $L_0 = \kappa^+, \bar{\mathbb{Q}}^1 = \langle \mathbb{P}^1_i, \mathbb{Q}^1_i : i \in L_1\rangle$ be $\bar{\mathbb{Q}}^0$ as interpreted in $(\mathscr{H}(\chi), \in, <^*_\chi)^\kappa/D$, it looks like $\bar{\mathbb{Q}}^0$ replacing κ^+ by $(\kappa^+)^\kappa/D$. We look at $\mathrm{Lim}(\bar{\mathbb{Q}}^0) = \bigcup\{\ \mathbb{P}^0_i i < \kappa^+\}$ as a subforcing of $\mathrm{Lim}(\bar{\mathbb{Q}}^1)$ identifying $\mathbb{Q}_i$ with $\underset{\sim}{\mathbb{Q}}_{\mathbf{j}_0(i)}, \mathbf{j}_0$ the canonical elementary embedding of κ^+ into $(\kappa^+)^\kappa/D$ (no Mostowski collapse!). We continue to define $\bar{\mathbb{Q}}^n$ and then $\bar{\mathbb{Q}}^\omega$ as the following limit: for the original[1] $i \in \kappa^+$, we use the definition, otherwise we use direct limit ("founding fathers privilege" you may say). So $\mathbb{P}^i = \mathrm{Lim}(\bar{\mathbb{Q}}^i)$ is $\lessdot$-increasing, continuous when $\mathrm{cf}(i) > \aleph_0$; so now we have a kind of iteration with so called ideal, weakly transitive memory and a not well founded base. We continue κ^{++} times. Now in $\mathbf{V}^{\mathrm{Lim}(\bar{\mathbb{Q}}^{\kappa^{++}})}$, the original κ^+ generic reals exemplify $\mathfrak{b} = \mathfrak{d} = \kappa^+$, so we know that $\mathfrak{a} \ge \kappa^+$. To finish assume $p \Vdash$ "$\{\underset{\sim}{A}_\gamma : \gamma < \kappa^+\} \subseteq [\omega]^{\aleph_0}$ is a MAD family". Each name $\underset{\sim}{A}_\gamma$ is a "countable object" and so depends on countably many conditions, so all of them are in $\mathrm{Lim}(\bar{\mathbb{Q}}^i)$ for some $i < \kappa^{++}$. In the next stage, $\bar{\mathbb{Q}}^{i+1}, \langle \underset{\sim}{A}_\gamma : \gamma < \kappa\rangle/D$ is a name of an infinite subset of ω almost disjoint to $\underset{\sim}{A}_\beta$ for each $\beta < \kappa^+$, contradiction.

All this is a reasonable scheme. This is done in §1 but rely on "nep forcing" from [She04a]. But a self contained another approach is in §2,§3, where the meaning of the iteration is more on the surface (and also, in §3, help to eliminate the use of large cardinals). In §4 we deal with the case of an additional cardinal invariant, $\mathfrak{u}$.

Note that just using FS iteration on a non well-ordered linear order L (instead of an ordinal) is impossible by a theorem of Hjorth. On nonlinear orders for iterations (history and background) see [RS]. On iteration with non-transitive memory see [She00a], [She03] and in particular [She03, §3].

Continuing this work J. Brendle has proved the consistency of $\mathrm{cf}(\mathfrak{a}) = \aleph_0$, (note that in 3.6 we have assumed $\lambda = \lambda^{\aleph_0}$ in $\mathbf{V}$ hence $\mathrm{cf}(\lambda) > \aleph_0$ even in $\mathbf{V}^{\mathbb{P}}$).

After this work was arXived, Mejia inform me that concerning consistency of $\aleph_1 < \mathfrak{s} < \mathfrak{b} = \mathfrak{d} < \mathfrak{g}$, but this result was already proved by Diego Mejia [Mej15] and Vera Fischer[FM17]. Now [Mej15] used a measurable, but the template iteration theory in Brendle's version is expanded there and applied in [FM17] where much more is done and the consistency result is proved without large cardinals.

[1] which mean not the ones added by taking ultrapowers

I thank Heike Mildenberger and Juris Steprans for their helpful comments. After publication this was revised simplifying §2.

Notation 0.2. 1) $\mathbb{P}, \mathbb{Q}$ denote forcing notions
2) Let $\mathbb{P} \subseteq \mathbb{Q}$ means that for $p, q \in \mathbb{P}$ we have $p <_{\mathbb{P}} q$ iff $p <_{\mathbb{Q}} q$
3) let $\mathbb{P} \subseteq_{ic} \mathbb{Q}$ iff $\mathbb{P} \subseteq \mathbb{Q}$ and for every $p, q \in \mathbb{P}$ we have p, q are compatible in $\mathbb{P}$ iff they are compatible in $\mathbb{Q}$
4) Let $\mathbb{P} \lessdot \mathbb{Q}$ iff $\mathbb{P} \subseteq_{ic} \mathbb{Q}$ and every maximal anti-chain of $\mathbb{P}$ is a maximal anti-chain of $\mathbb{Q}$

Convention 0.3. *1) When using $\mathfrak{t}, (\mathfrak{t}, \bar{K})$ we mean as in Def 2.1.*
2) When using $(\mathfrak{t}, \bar{K}, \bar{u}, \mathrm{Lim}(\bar{\mathbb{Q}}))$ we mean as in 2.6
3) We may write I_t instead $I^{\mathfrak{t}}_t$ or $I^{\mathbf{q}}_t$ when $\mathfrak{t}, \mathbf{q}$ is clear from the contecxt.
4) Dealing with e.g. $\mathfrak{t}^{\zeta}$ we may write $\mathfrak{t}[\zeta]$ in subscript and superscripts.

§ 1. On Con($\mathfrak{a} > \mathfrak{d}$)

In this section, we look at it in the context of [She04a] and we use a measurable cardinal.

Definition 1.1. 1) Given sets A_ℓ of ordinals for $\ell < n$, we say $\mathscr{T}$ is an $(A_0, \ldots, A_{n-1})$-tree if $\mathscr{T} = \bigcup\limits_{k<\omega} \mathscr{T}_k$ where $\mathscr{T}_k \subseteq \{(\eta_0, \ldots, \eta_\ell, \ldots, \eta_{n-1}) : \eta_\ell \in {}^k(A_\ell)$ for $\ell < n\}$ and $\mathscr{T}$ is ordered by $\bar\eta \le_{\mathscr{T}} \bar\nu \Leftrightarrow \bigwedge\limits_{\ell<n} \eta_\ell \trianglelefteq \nu_\ell$ and we let $\bar\eta \upharpoonright k_1 =: \langle \eta_\ell \restriction k_1 : \ell < n\rangle$ and demand $\bar\eta \in \mathscr{T}_k \wedge k_1 < k \Rightarrow \bar\eta \upharpoonright k_1 \in \mathscr{T}_{k_1}$. We call $\mathscr{T}$ locally countable if $k \in [1,\omega) \wedge \bar\eta \in \mathscr{T}_k \Rightarrow |\{\bar\nu \in \mathscr{T}_{k+1} : \bar\eta \le_{\mathscr{T}} \bar\nu\}| \le \aleph_0$. Let $\lim(\mathscr{T}) = \{\langle \eta_\ell : \ell < n\rangle : \eta_\ell \in {}^\omega(A_\ell)$ for $\ell < n$ and $m < \omega \Rightarrow \langle \eta_\ell \restriction m : \ell < n\rangle \in \mathscr{T}\}$.

Lastly, for $n_1 \le n$ we let $\text{prj}\lim_{n_1}(\mathscr{T}) = \{\langle \eta_\ell : \ell < n_1\rangle :$ for some $\eta_{n_1}, \ldots, \eta_{n-1}$ we have $\langle \eta_\ell : \ell < n\rangle \in \lim(\mathscr{T})\}$; and if n_1 is omitted we mean $n_1 = n-1$.

2)

$\mathfrak{K} = \{\bar{\mathscr{T}} :$ for some sets A, B of ordinals we have
- (i) $\bar{\mathscr{T}} = (\mathscr{T}_1, \mathscr{T}_2)$,
- (ii) $\mathscr{T}_1$ is a locally countable (A,B)-tree,
- (iii) $\mathscr{T}_2$ is a locally countable (A,A,B)-tree, and
- (iv) $\mathbb{Q}_{\bar{\mathscr{T}}} =: (\text{prj}\lim(\mathscr{T}_1),\ \text{prj}\lim(\mathscr{T}_2))$ is a c.c.c. forcing notion absolute under c.c.c. forcing notions (see below)$\}$

2A) We say that $\mathbb{Q}_{\bar{\mathscr{T}}}$ is c.c.c. absolute for c.c.c. forcing if: for c.c.c. forcing notions $\mathbb{P} \lessdot \mathbb{R}$ we have $\mathbb{P} * \underset{\sim}{\mathbb{Q}}_{\bar{\mathscr{T}}} \lessdot \mathbb{R} * \underset{\sim}{\mathbb{Q}}_{\bar{\mathscr{T}}}$ (though not necessarily $\mathbb{Q}^{\mathbf{V}^{\mathbb{P}}}_{\bar{\mathscr{T}}} \lessdot \mathbb{Q}^{\mathbf{V}^{\mathbb{R}}}_{\bar{\mathscr{T}}}$ in $\mathbf{V}^{\mathbb{R}}$) so membership, order, non-order, compatibility, noncompatibility and being predense over p in the universe $\mathbf{V}^{\mathbb{P}}$, are preserved in passing to $\mathbf{V}^{\mathbb{R}}$, note that predense sets belong to $\mathbf{V}^{\mathbb{P}}$ (the $\mathbb{Q}_{\bar{\mathscr{T}}}$'s are snep, from [She04a] with slight restriction). Similarly we define "$\mathbb{Q}_{\bar{\mathscr{T}}_1} \lessdot \mathbb{Q}_{\bar{\mathscr{T}}_2}$ absolute under c.c.c. forcing" (compare with 2.6, clause (A)(a)(iii) in the definition).

3) For a set or class A of ordinals, $\mathfrak{K}^\kappa_A$ is the family of $\bar{\mathscr{T}} \in \mathfrak{K}$ which are a pair of objects, the first an (A,B)-tree and the second an (A,A,B)-tree for some B such that $|\mathscr{T}_1| \le \kappa, |\mathscr{T}_2| \le \kappa$. For a cardinal κ and a pairing function pr with inverses pr_1, pr_2, let $\mathfrak{K}^\kappa_{\text{pr}_1,\gamma} = \mathfrak{K}^\kappa_{\{\alpha:\text{pr}_1(\alpha)=\gamma\}}$ and $\mathfrak{K}^\kappa_{\text{pr}_1,<\gamma} = \mathfrak{K}^\kappa_{\{\alpha:\text{pr}_1(\alpha)<\gamma\}}$. Let $|\bar{\mathscr{T}}| = |\mathscr{T}_1| + |\mathscr{T}_2|$.

4) Let $\bar{\mathscr{T}}, \bar{\mathscr{T}}' \in \mathfrak{K}$, we say $\mathbf{f}$ is an isomorphism from $\bar{\mathscr{T}}$ onto $\bar{\mathscr{T}}'$ <u>when</u> $\mathbf{f} = (f_1, f_2)$ and for $m = 1,2$ we have: f_m is a one-to-one function from $\mathscr{T}_m$ onto $\mathscr{T}'_m$ preserving the level (in the respective trees), preserving the relations $x = y \upharpoonright k, x \ne y \upharpoonright k$ and if $f_2((\eta_1,\eta_2,\eta_3)) = (\eta'_1,\eta'_2,\eta'_3), f_1((\nu_1,\nu_2)) = (\nu'_1,\nu'_2)$ then $[\eta_1 = \nu_1 \Leftrightarrow \eta'_1 = \nu'_1], [\eta_2 = \nu_1 \Leftrightarrow \eta'_2 = \nu'_1]$.

In this case let $\hat{\mathbf{f}}$ be the isomorphism induced by $\mathbf{f}$ from $\mathbb{Q}_{\bar{\mathscr{T}}}$ onto $\mathbb{Q}_{\bar{\mathscr{T}}'}$.

Definition 1.2. For $\bar{\mathscr{T}}', \bar{\mathscr{T}}'' \in \mathfrak{K}$ let $\bar{\mathscr{T}}' \le_{\mathfrak{K}} \bar{\mathscr{T}}''$ mean:

- (a) $\mathscr{T}'_\ell \subseteq \mathscr{T}''_\ell$ (as trees) for $\ell = 1,2$
- (b) if $\ell \in \{1,2\}$ and $\bar\eta \in \mathscr{T}''_\ell \backslash \mathscr{T}'_\ell$ and $\bar\eta \upharpoonright k \in \mathscr{T}'_\ell$ <u>then</u> $k \le 1$
- (c) $\mathbb{Q}_{\bar{\mathscr{T}}'} \lessdot \mathbb{Q}_{\bar{\mathscr{T}}''}$ (absolute under c.c.c. forcing); note that by (a) + (b) we have: $x \in \mathbb{Q}_{\bar{\mathscr{T}}'} \Rightarrow x \in \mathbb{Q}_{\bar{\mathscr{T}}''}$ and $\mathbb{Q}_{\bar{\mathscr{T}}'} \models x \le y \Rightarrow \mathbb{Q}_{\bar{\mathscr{T}}''} \models x \le y$.

Remark 1.3. The definition is tailored such that the union of an increasing chain will give a forcing notion which is the union.

Claim/Definition 1.4. *0) $\le_{\mathfrak{K}}$ is a partial order of $\mathfrak{K}$.*
1) Assume $\langle \bar{\mathscr{T}}[i] : i < \delta\rangle$ is $\le_{\mathfrak{K}}$-increasing and $\bar{\mathscr{T}}$ is defined by $\bar{\mathscr{T}} = \bigcup\limits_i \bar{\mathscr{T}}[i]$ that is $\mathscr{T}_m = \bigcup\limits_{i<\delta} \mathscr{T}_m[i]$ for $m = 1, 2$ then

- (a) $i < \delta \Rightarrow \bar{\mathscr{T}}[i] \le_{\mathfrak{K}} \bar{\mathscr{T}}$
- (b) $\mathbb{Q}_{\bar{\mathscr{T}}} = \bigcup\limits_{i<\delta} \mathbb{Q}_{\bar{\mathscr{T}}[i]}$.

2) Assume $\bar{\mathscr{T}}', \bar{\mathscr{T}} \in \mathfrak{K}$. Then there is $\bar{\mathscr{T}}'' \in \mathfrak{K}$ such that $\bar{\mathscr{T}}' \le_{\mathfrak{K}} \bar{\mathscr{T}}''$ and $\mathbb{Q}_{\bar{\mathscr{T}}''}$ is isomorphic to $\mathbb{Q}_{\bar{\mathscr{T}}'} * \underset{\sim}{\mathbb{Q}}_{\bar{\mathscr{T}}}$ and this is absolute by c.c.c. forcing. Moreover, there is such an isomorphism extending the identity map from $\mathbb{Q}_{\bar{\mathscr{T}}'}$ into $\mathbb{Q}_{\bar{\mathscr{T}}''}$.
3) There is $\bar{\mathscr{T}} \in \mathfrak{K}^{\aleph_0}_\omega$ such that $\mathbb{Q}_{\bar{\mathscr{T}}}$ is the trivial forcing.
4) There is $\bar{\mathscr{T}} \in \mathfrak{K}^{\aleph_0}_\omega$ such that $\mathbb{Q}_{\bar{\mathscr{T}}}$ is the dominating real forcing.

Proof. See [She04a]. $\square_{1.4}$

Claim 1.5. *1) Assume $\bar{\mathscr{T}}[\gamma] \in \mathfrak{K}_{\mathrm{pr}_1,\gamma}$ for $\gamma < \gamma(*)$. Then for each $\alpha \le \gamma(*)$ there is $\bar{\mathscr{T}}\langle\alpha\rangle$ such that $\mathbb{Q}_{\bar{\mathscr{T}}\langle\alpha\rangle}$ is $\mathbb{P}_\alpha$ where $\langle \mathbb{P}_\gamma, \underset{\sim}{\mathbb{Q}}_\beta : \gamma \le \gamma(*), \beta < \gamma(*)\rangle$ is an* FS*-iteration and $\underset{\sim}{\mathbb{Q}}_\beta = (\mathbb{Q}_{\bar{\mathscr{T}}[\beta]})^{\mathbf{V}[\mathbb{P}_\beta]}$ and $\bar{\mathscr{T}}\langle\alpha\rangle \in \mathfrak{K}_{\mathrm{pr}_1,<\alpha}$ and $\bar{\mathscr{T}}\langle\alpha_1\rangle \le_{\mathfrak{K}} \bar{\mathscr{T}}\langle\alpha_2\rangle$ for $\alpha_1 \le \alpha_2 \le \gamma(*)$, $\bar{\mathscr{T}}[\gamma] \le_{\mathfrak{K}} \bar{\mathscr{T}}\langle\alpha\rangle$ for $\gamma < \alpha \le \gamma(*)$. We write $\bar{\mathscr{T}}\langle\alpha\rangle = \sum\limits_{\gamma<\alpha} \bar{\mathscr{T}}[\gamma]$.*
2) In part (1), for each $\gamma < \gamma(*)$ there is $\bar{\mathscr{T}}' \in \mathfrak{K}_{\mathrm{pr}_1,\gamma}$ such that $\bar{\mathscr{T}}', \bar{\mathscr{T}}$ are isomorphic over $\bar{\mathscr{T}}[\gamma]$ hence $\mathbb{Q}_{\bar{\mathscr{T}}'}, \mathbb{Q}_{\bar{\mathscr{T}}}$ are isomorphic over $\mathbb{Q}_{\bar{\mathscr{T}}[\gamma]}$.
3) If in addition $\mathscr{T}[\gamma] \le_{\mathfrak{K}} \mathscr{T}'[\gamma] \in \mathfrak{K}_{\mathrm{pr}_1,\gamma}$ for $\gamma < \gamma(*)$ and $\langle \mathbb{P}_\gamma, \underset{\sim}{\mathbb{Q}}'_\beta : \gamma \le \gamma(*), \beta < \gamma(*)\rangle$ is an FS iteration as above with $\mathbb{P}'_{\gamma(*)} = \mathbb{Q}_{\bar{\mathscr{T}}'}$, then we can find such $\bar{\mathscr{T}}'$ with $\bar{\mathscr{T}} \le_{\mathfrak{K}} \bar{\mathscr{T}}$.

Proof. Straightforward. $\square_{1.5}$

Claim 1.6. *Assume*

- (a) *κ is a measurable cardinal*
- (b) *$\kappa < \mu = \mathrm{cf}(\mu) < \lambda = \mathrm{cf}(\lambda) = \lambda^\kappa$ and $(\forall\alpha < \mu)(|\alpha|^{\aleph_0} < \mu)$ for simplicity.*

Then for some c.c.c. forcing notion $\mathbb{P}$ of cardinality λ, in $\mathbf{V}^{\mathbb{P}}$ we have: $2^{\aleph_0} = \lambda, \mathfrak{d} = \mathfrak{b} = \mu$ and $\mathfrak{a} = \lambda$.

Proof. We choose by induction on $\zeta \le \lambda$ the following objects satisfying the following conditions:

- (a) a sequence $\langle \bar{\mathscr{T}}[\gamma, \zeta] : \gamma < \mu\rangle$
- (b) $\bar{\mathscr{T}}[\gamma, \zeta] \in \mathfrak{K}^\lambda_{\mathrm{pr}_1,\gamma}$
- (c) $\xi < \zeta \Rightarrow \bar{\mathscr{T}}[\gamma, \xi] \le_{\mathfrak{K}} \bar{\mathscr{T}}[\gamma, \zeta]$
- (d) if ζ limit then $\bar{\mathscr{T}}[\gamma, \zeta] = \bigcup\limits_{\xi<\zeta} \bar{\mathscr{T}}[\gamma, \xi]$
- (e) if $\gamma < \mu, \zeta = 1$ then $\mathbb{Q}_{\bar{\mathscr{T}}}[\gamma, \zeta]$ is the $\mathbb{Q}_{\mathrm{dom}}$, dominating real forcing = Hechler forcing

(*f*) if $\gamma < \mu, \zeta = \xi + 1 > 1$ and ξ is even, <u>then</u> $\bar{\mathscr{T}}[\gamma,\zeta]$ is isomorphic to $\bar{\mathscr{T}}\langle\gamma+1,\xi\rangle$ over $\bar{\mathscr{T}}[\gamma,\xi]$ say by $\mathbf{j}_{\gamma,\xi}$ where $\bar{\mathscr{T}}\langle\gamma+1,\xi\rangle =: \sum\limits_{\beta\leq\gamma} \bar{\mathscr{T}}[\beta,\xi]$ and let $\hat{\mathbf{j}}_{\gamma,\xi}$ be the isomorphism induced from $\mathbb{Q}_{\bar{\mathscr{T}}\langle\gamma+1,\xi\rangle}$ onto $\mathbb{Q}_{\bar{\mathscr{T}}}[\gamma,\zeta]$ over $\mathbb{Q}_{\bar{\mathscr{T}}[\gamma,\xi]}$

(*g*) if $\gamma < \mu, \zeta = \xi+1, \xi$ odd, <u>then</u> $\bar{\mathscr{T}}[\gamma,\zeta]$ is almost isomorphic to $(\bar{\mathscr{T}}[\gamma,\xi])^\kappa/D$ over $\bar{\mathscr{T}}_{[\gamma,\xi]}$ which means that we say $\mathbf{j}_{\gamma,\xi}$ is an isomorphism from $(\bar{\mathscr{T}}[\gamma,\xi])^\kappa/D$ onto $\bar{\mathscr{T}}[\gamma,\zeta]$ such that by $\mathbf{j}_{\gamma,\xi}$, $\langle x : \varepsilon < \kappa\rangle/D$ is mapped onto x.

There is no problem to carry the definition. Let $\mathbb{P}_\zeta = \mathbb{Q}_{\bar{\mathscr{T}}\langle\mu,\zeta\rangle}$ where $\bar{\mathscr{T}}\langle\mu,\zeta\rangle =: \sum\limits_{\gamma<\mu} \bar{\mathscr{T}}[\gamma,\zeta]$ for $\zeta \leq \lambda, \mathbb{P} = \mathbb{P}_\lambda$ and $\mathbb{P}_{\gamma,\zeta} = \mathbb{Q}_{\bar{\mathscr{T}}\langle\gamma,\zeta\rangle}$.

Now

$\boxtimes_1$ $|\mathbb{P}| \leq \lambda$.

[Why? As we prove by induction on $\zeta \leq \lambda$ that: each $\bar{\mathscr{T}}[\gamma,\zeta]$ and $\sum\limits_{\gamma\leq\mu} \bar{\mathscr{T}}[\gamma,\lambda]$ has cardinality $\leq \lambda$. Hence for $\gamma < \mu$ we have: the forcing notion $\mathbb{Q}_{\bar{\mathscr{T}}[\gamma,\lambda]}$ in the universe $\mathbf{V}^{\mathbb{Q}_{\bar{\mathscr{T}}\langle\gamma,\lambda\rangle}}$ has cardinality $\leq \lambda^{\aleph_0} = \lambda$.]

$\boxtimes_2$ in $\mathbf{V}^{\mathbb{P}}$ we have $\mathfrak{b} = \mathfrak{d} = \mu$

[Why? Let $\underset{\sim}{\eta}_\gamma$ be the $\mathbb{Q}_{\bar{\mathscr{T}}[\gamma,1]}$-name of the dominating real (see clause (e)). As $\bar{\mathscr{T}}[\gamma,1] \leq_{\mathfrak{K}} \bar{\mathscr{T}}[\gamma,\lambda]$, clearly $\underset{\sim}{\eta}_\gamma$ is also a $\mathbb{Q}_{\bar{\mathscr{T}}[\gamma,\lambda]}$-name of a dominating real, but this is preserved by (forcing by) $\mathbb{P}_\gamma$ hence $\Vdash_{\mathbb{P}_{\gamma+1}}$ "$\underset{\sim}{\eta}_\gamma$ dominates $({}^\omega\omega)^{\mathbf{V}[\mathbb{P}_{\gamma,\lambda}]}$". But $\langle\mathbb{P}_{\gamma,\lambda} : \gamma < \mu\rangle$ is $\lessdot$-increasing with union $\mathbb{P}$ and $\mathrm{cf}(\mu) = \mu > \aleph_0$ so $\Vdash_{\mathbb{P}}$ "$\langle\underset{\sim}{\eta}_\gamma : \gamma < \mu\rangle$ is $<^*$-increasing and dominating", so the conclusion follows.]

We shall prove below that $\mathfrak{a} \geq \lambda$, together this finishes the proof (note that it implies $2^{\aleph_0} \geq \lambda$ hence as $\lambda = \lambda^{\aleph_0}$ by $\boxtimes_1$ we get $2^{\aleph_0} = \lambda$)

$\boxtimes_3$ $\Vdash_{\mathbb{P}}$ "$\mathfrak{a} \geq \lambda$".

So assume $p \Vdash$ "$\underset{\sim}{\mathscr{A}} = \{\underset{\sim}{A}_i : i < \theta\}$ is a MAD family, i.e. ($\theta \geq \aleph_0$ and)

(*i*) $\underset{\sim}{A}_i \in [\omega]^{\aleph_0}$,

(*ii*) $i \neq j \Rightarrow |\underset{\sim}{A}_i \cap \underset{\sim}{A}_j| < \aleph_0$ and

(*iii*) $\underset{\sim}{\mathscr{A}}$ is maximal under $(i) + (ii)$".

Without loss of generality $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{A}_i \in [\omega]^{\aleph_0}$".

As always $\mathfrak{a} \geq \mathfrak{b}$, by $\boxtimes_2$ we know that $\theta \geq \mu$, and toward contradiction assume $\theta < \lambda$. For each $i < \theta$ and $m < \omega$ there is a maximal anti-chain $\langle p_{i,m,n} : n < \omega\rangle$ of $\mathbb{P}$ and a sequence $\langle \mathbf{t}_{i,m,n} : n < \omega\rangle$ of truth values such that $p_{i,m,n} \Vdash_{\mathbb{P}}$ "$n \in \underset{\sim}{A}_i$ iff $\mathbf{t}_{i,m,n}$ is truth". We can find a countable $w_i \subseteq \mu$ such that: $[\bigcup\limits_{m,n<\omega} \mathrm{Dom}(p_{i,m,n}) \subseteq w_i], p_{i,m,n} \in \mathbb{Q}_{\Sigma\{\bar{\mathscr{T}}[\gamma,\lambda]:\gamma\in w_i\}}$, moreover, $\gamma \in \mathrm{Dom}(p_{i,m,n}) \Rightarrow p_{i,m,n}(\gamma)$ is a $\mathbb{Q}_{\sum\{\bar{\mathscr{T}}[\beta,\lambda]:\beta\in\gamma\cap w_i\}}$-name.

Note that $\mathbb{Q}_{\sum\{\bar{\mathscr{T}}[\beta,\lambda]:\beta\in\gamma\cap w_i, i<\theta\}} \lessdot \mathbb{Q}_{\sum\{\bar{\mathscr{T}}_\beta:\beta<\gamma\}}$, see [She04a].

Clearly for some even $\zeta < \lambda$, we have $\{p_{i,m,n} : i < \theta, m < \omega$ and $n < \omega\} \subseteq \mathbb{Q}_{\sum\{\bar{\mathscr{T}}[\beta,\zeta]:\beta<\mu\}}$. Now for some stationary $S \subseteq \{\delta < \mu : \mathrm{cf}(\delta) = \kappa\}$ and w^* we have: $\delta \in S \Rightarrow w_\delta \cap \delta = w^*$ and $\alpha < \delta \in S \Rightarrow w_\alpha \subseteq \delta$. Let $\langle\delta_\varepsilon : \varepsilon < \kappa\rangle$ be an increasing sequence of members of S, and $\delta^* = \bigcup\limits_{\varepsilon<\kappa} \delta_\varepsilon$. The definition of

$\langle \bar{\mathscr{T}}[\gamma,\zeta+1] : \gamma < \mu\rangle, \langle \bar{\mathscr{T}}[\gamma,\zeta+2] : \gamma < \mu\rangle$ was made to get a name of an infinite $\underset{\sim}{A} \subseteq \omega$ almost disjoint to every $\underset{\sim}{A}_\beta$ for $\beta < \theta$ (in fact $(\sum\limits_{\gamma<\mu} \mathbb{Q}_{\bar{\mathscr{T}}[\gamma,\zeta]})^\kappa/D$ can be $\lessdot$-embedded into $\sum\limits_{\gamma<\mu} \mathbb{Q}_{\bar{\mathscr{T}}[\gamma,\zeta+2]}$). $\square_{1.6}$

Remark 1.7. In later proofs in §2 we give more details.

§ 2. On Con($\mathfrak{a} > \mathfrak{d}$) revisited with FS, with ideal memory, non-well ordered length

(Pre 2020 introduction to this section) We first define the FSI-templates, telling us how do we iterate along a linear order L; we think of having for each $t \in L$, a forcing notion $\mathbb{Q}_t$, say adding a generic $\underset{\sim}{\nu}_t$, and $\mathbb{Q}_t$ will really be $\cup\{\mathbb{Q}^{\mathbf{V}[\langle \underset{\sim}{\nu}_s : s \in A\rangle]} : A \in I_t\}$ where I_t is an ideal of subsets of $\{s : s <_L t\}$; so $\mathbb{Q}_t$ in the nice case is a definition, e.g. as in 1.1(2A). In our application this definition is constant, but we treat a more general case, so $\underset{\sim}{\mathbb{Q}}_t$ may be defined using parameters from $\mathbf{V}[\langle \underset{\sim}{\nu}_s : s \in K_t\rangle], K_t$ a subset of $\{s : s <_L t\}$ so the reader may consider only the case $t \in L \Rightarrow K_t = \emptyset$. In part (3) of Definition 2.1 instead distinguishing "ζ successor, ζ limit" we can consider the two cases for each ζ. The depth of L is the ordinal on which our induction rests (as otp(L) is inadequate).

Now (2020) we allow uncountable K_t-s (and similarly $\underset{\sim}{\eta}, \nu$), a non-essential change.

Definition 2.1. 1) An FSI-template (= finite support iteration template) $\mathfrak{t}$ is a sequence $\langle I_t : t \in L\rangle = \langle I^{\mathfrak{t}}_t : t \in L^{\mathfrak{t}}\rangle = \langle I_t[\mathfrak{t}] : t \in L[\mathfrak{t}]\rangle$ such that:

- (a) L is a linear order (or partial, it does not really matter); but we may write $x \in \mathfrak{t}$ instead of $x \in L$ and $x <_{\mathfrak{t}} y$ instead of $x <_L y$
- (b) I_t is an ideal of subsets of $L_t = \{s : L \models s < t\}$, (but see 2.3(4)(b)).

2) Let $\mathfrak{t}$ be an FSI-template.

- (c) We say $\bar{K} = \langle K_t : t \in L^{\mathfrak{t}}\rangle$ is a $\mathfrak{t}$-memory choice (or $(\mathfrak{t}, \bar{K})$ is an FSI-template) <u>if</u>
  - (i) $K_t \in I^{\mathfrak{t}}_t$
  - (ii) $s \in K_t \Rightarrow K_s \subseteq K_t$.
- (d) We say $L \subseteq L^{\mathfrak{t}}$ is $\bar{K}$-closed if $t \in L \Rightarrow K_t \subseteq L$
- (e) for $\bar{K}$ a $\mathfrak{t}$-memory choice and $L \subseteq L^{\mathfrak{t}}$ which is $\bar{K}$-closed we say $\bar{K}' = \bar{K} \restriction L$ if Dom($\bar{K}'$) $= L$ and K'_t is K_t for $t \in L$, (it is a $(\mathfrak{t} \restriction L)$-memory choice, see part (5)).
- (f) We say that A is $\bar{K}$-countable (or, pedantically $(\mathfrak{t}, \bar{K})$-countable) <u>when</u> ($A = \emptyset$ or) there are $t_n \in L^{\mathfrak{t}}$ and $\bar{K}$-closed, $A_n \in I^{\mathfrak{t}}_{t_n}$ for $n < \omega$ such that $A = \cup\{A_n \cup \{t_n\} : n < \omega\}$. We define similarly $\bar{K}$-finite or $(\bar{K}, < \partial)$-finite) for any (infinite) ∂
- (g) Let $K^{\dagger}_t$ be $K_t \cup \{t\}$
- (h) We let $\partial(\mathfrak{t}) = \sup\{|A|^+ + \aleph_0 : A \in I_t$ for some $t \in L\}$ and $\partial(\mathfrak{t}, \bar{K}) = \partial(\mathfrak{t})$. Let $\partial(\bar{K}) = \sup\{|\bar{K}_t| : t \in L^{\mathfrak{t}}\}$.

3) For an FSI-template $\mathfrak{t}$ and $\mathfrak{t}$-memory choice $\bar{K}$ and $\bar{K}$-closed $L \subseteq L^{\mathfrak{t}}$ we define $\mathrm{Dp}_{\mathfrak{t}}(L, \bar{K})$, the $\mathfrak{t}$-depth (or $(\mathfrak{t}, \bar{K})$-depth) of L by defining by induction on the ordinal ζ when $\mathrm{Dp}_{\mathfrak{t}}(L, \bar{K}) \leq \zeta$.

<u>For $\zeta = 0$</u>: $\mathrm{Dp}_{\mathfrak{t}}(L, \bar{K}) \leq \zeta$ when $L = \emptyset$.

<u>For ζ a successor ordinal</u>: $\mathrm{Dp}_{\mathfrak{t}}(L, \bar{K}) \leq \zeta$ iff:

(a) there is L^* such that: $L^* \subseteq L, |L^*| \le 1, (\forall t \in L)(\forall A \in I_t^{\mathfrak{t}})(A \cap L^* = \emptyset)$ hence $L \backslash L^*$ is $\bar{K}$-closed and $\mathrm{Dp}_{\mathfrak{t}}(L\backslash L^*, \bar{K}) < \zeta$ and for every $t \in L^*$ we have:

$\boxtimes_{t,L}$ $L\backslash L^* \in I_t^{\mathfrak{t}}$ and [2] it is $\bar{K}$-closed.

<u>For $\zeta > 0$ a limit ordinal</u>: $\mathrm{Dp}_{\mathfrak{t}}(L, \bar{K}) \le \zeta$ iff:

(b) there is a directed partial order M and a sequence $\langle L_a : a \in M\rangle$ with union L such that the sequence is increasing, i.e., $M \models$ "$a \le b \Rightarrow L_a \subseteq L_b$", each L_b is $\bar{K}$-closed, $(\forall b \in M)(\zeta > \mathrm{Dp}_{\mathfrak{t}}(L_b, \bar{K}))$ and $t \in L \wedge A \in I_t \wedge A \subseteq L \Rightarrow (\exists a \in M)\, A \subseteq L_a$.

So $\mathrm{Dp}_{\mathfrak{t}}(L, \bar{K}) = \zeta$ iff $\mathrm{Dp}_{\mathfrak{t}}(L, \bar{K}) \le \zeta \wedge (\forall \xi < \zeta)\mathrm{Dp}_{\mathfrak{t}}(L, \bar{K}) \nleq \xi$

3A) $\mathrm{Dp}_{\mathfrak{t}}(L, \bar{K}) = \infty$ iff ($\forall$ ordinal ζ) $[\mathrm{Dp}_{\mathfrak{t}}(L, \bar{K}) \nleq \zeta]$.
4) We say $\bar{K}$ is a smooth $\mathfrak{t}$-memory choice or $(\mathfrak{t}, \bar{K})$ is smooth <u>if</u> $\mathrm{Dp}_{\mathfrak{t}}(L^{\mathfrak{t}}, \bar{K}) < \infty$ and $\bar{K}$ a $\mathfrak{t}$-memory choice (and $\mathfrak{t}$ is an FSI-template).
5) If $\bar{K}$ is omitted we mean it is the trivial $\bar{K}$, that is $K_t = \emptyset$ for $t \in L^{\mathfrak{t}}$. We say $\mathfrak{t}$ is smooth if the trivial $\bar{K}$ is a smooth $\mathfrak{t}$-memory choice. For $L \subseteq L^{\mathfrak{t}}$ let $\mathfrak{t} \restriction L = \langle I_t \cap \mathscr{P}(L) : t \in L\rangle$.
6) Let $L_1 \le_{\mathfrak{t}} L_2$ mean $L_1 \subseteq L_2 \subseteq L^{\mathfrak{t}}$ and $t \in L_1 \wedge A \in I_t^{\mathfrak{t}} \Rightarrow A \cap L_2 \subseteq L_1$.

Definition 2.2. Let $\mathfrak{t} = \langle I_t : t \in L^{\mathfrak{t}}\rangle$ be a FSI-template and $\bar{K}$ a $\mathfrak{t}$-memory choice.
1) We say $\bar{L}$ is a $(\mathfrak{t}, \bar{K})$-representation of L (or $(\mathfrak{t}, \bar{K}) - 0$-representation of L) <u>if</u>:

(a) $L \subseteq L^{\mathfrak{t}}$ is $\bar{K}$-closed
(b) $\bar{L} = \langle L_a : a \in M\rangle$
(c) M is a directed partial order
(d) $\bar{L}$ is increasing, that is $a <_M b \Rightarrow L_a \subseteq L_b$
(e) $L = \bigcup\limits_{a \in M} L_a$
(f) each L_a is $\bar{K}$-closed
(g) if $t \in L, A \in I_t^{\mathfrak{t}}, A \subseteq L$ then $(\exists a \in M)(A \subseteq L_a)$

2) We say L^* is a $(\mathfrak{t}, \bar{K})-^*$representation or a $(\mathfrak{t}, \bar{K}) - 1$-representation $+$ of L <u>if</u>:

(a) $L \subseteq L^{\mathfrak{t}}$ is $\bar{K}$-closed
(b) $L^* \subseteq L, L^*$ a singleton
(c) if $t \in L$ and $A \in I_t^{\mathfrak{t}}$ then $A \cap L^* = \emptyset$ (so $(L\backslash L^*) \le_{\mathfrak{t}} L$, see Definition 2.1(6))
(d) if $t \in L^*$ then $L\backslash L^* \in I_t^{\mathfrak{t}}$.

Claim 2.3. *Let $\mathfrak{t}$ be an* FSI-*template and $\bar{K}$ a $\mathfrak{t}$-memory choice.*
0) The family of $\bar{K}$-closed sets is closed under (arbitrary) unions and intersections. Also if $L \subseteq L^{\mathfrak{t}}$ then $L \cup \bigcup\{K_t : t \in L\}$ is $\bar{K}$-closed.
1) If $L_2 \subseteq L^{\mathfrak{t}}$ is $\bar{K}$-closed and L_1 is an initial segment of L_2, <u>then</u> L_1 is $\bar{K}$-closed.
2) If $L_1 \subseteq L_2 \subseteq L^{\mathfrak{t}}$ are $\bar{K}$-closed <u>then</u>

[2] we can use less, it seems not needed at the moment. We can go deeper to names of depth $\le \varepsilon$ inductively on $\varepsilon < \omega_1$, as in [She03, §3], or in a more particular way to make the point that is used here true, and/or make $I_t^{\mathfrak{t}}$ only closed under unions (but not subsets), etc. Note that e.g. $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$ is well defined when $L^{\mathfrak{t}}$ is well ordered.

(α) $\mathrm{Dp}_{\mathfrak{t}}(L_1, \bar{K}) \leq \mathrm{Dp}_{\mathfrak{t}}(L_2, \bar{K})$, *moreover*

(β) $(\exists t \in L_2)[L_1 \in I_t^{\mathfrak{t}}]$ *implies that* $\mathrm{Dp}_{\mathfrak{t}}(L_1, \bar{K}) < \mathrm{Dp}_{\mathfrak{t}}(L_2, \bar{K})$ *or both are* ∞.

3) If $L_1 \subseteq L_2 \subseteq L^{\mathfrak{t}}$ *are* $\bar{K}$*-closed* <u>*then*</u> $\mathfrak{t} \restriction L_2$ *is an* FSI*-template,* L_1 *is* $(\mathfrak{t} \restriction L_2)$*-closed and* $\mathrm{Dp}_{\mathfrak{t}\restriction L_2}(L_1, \bar{K} \restriction L_2) = \mathrm{Dp}_{\mathfrak{t}}(L_1, \bar{K})$.
4) If $(\mathfrak{t}, \bar{K})$ *is smooth and* $A \in I_t, t \in L^{\mathfrak{t}}$ <u>*then*</u>*:*

(a) *there is a* $\bar{K}$*-closed* $B \in I_t$ *such that* $A \subseteq B$

(b) *if* $s \in L_1 \in I_t$ *and* $L_2 \in I_s$ *then* $L_1 \cup L_2 \in I_t$.

Proof. 0), 1) Trivial - read the definitions.
2) We prove by induction on the ordinal ζ that

$(*)_\zeta$(α) if $\mathrm{Dp}_{\mathfrak{t}}(L_2, \bar{K}) = \zeta$ (and $L_1 \subseteq L_2$ are $\bar{K}$-closed subsets of $L^{\mathfrak{t}}$) <u>then</u> $\mathrm{Dp}_{\mathfrak{t}}(L_1, \bar{K}) \leq \zeta$

(β) if in addition $(\exists t \in L_2)(L_1 \in I_t^{\mathfrak{t}})$ <u>then</u> $\mathrm{Dp}_{\mathfrak{t}}(L_1, \bar{K}) < \zeta$.

So assume $\mathrm{Dp}_{\mathfrak{t}}(L_2, \bar{K}) = \zeta$, so $\mathrm{Dp}_{\mathfrak{t}}(L_2, \bar{K}) \not\geq \zeta + 1$ hence one of the following cases occurs.

<u>First Case</u>: $\zeta = 0$.
Trivial; note that clause (β) is empty.

<u>Second Case</u>: ζ is a successor, hence L_2 has a $(\mathfrak{t}, \bar{K})$-*representation L^* such that $\mathrm{Dp}_{\mathfrak{t}}(L_2 \backslash L^*, \bar{K}) < \zeta$; see Definition 2.2(2).

Let $L_2^- =: L_2 \backslash L^*$; if $L_1 \subseteq L_2^-$ then by the induction hypothesis $\mathrm{Dp}_{\mathfrak{t}}(L_1, \bar{K}) \leq \mathrm{Dp}_{\mathfrak{t}}(L_2^-, \bar{K}) < \zeta$, so assume $L_1 \nsubseteq L_2^-$ and so only clause (α) is relevant. Now letting $L_1^- = L_1 \backslash L^*$ we have $[L_1^-, L_2^-$ are $\bar{K}$-closed$] \wedge L_1^- \subseteq L_2^-$ and $\mathrm{Dp}_{\mathfrak{t}}(L_2^-, \bar{K}) < \zeta$ hence $\mathrm{Dp}_{\mathfrak{t}}(L_1^-, \bar{K}) < \zeta$ by the induction hypothesis. Let $L_1^* = L_1 \cap L^*$, so $L_1^* \subseteq L_1, L_1$ is $\bar{K}$-closed, $L_1 \backslash L_1^* = (L_2 \backslash L_2^*) \cap L_1$ is $\bar{K}$-closed, $\mathrm{Dp}_{\mathfrak{t}}(L_1 \backslash L_1^*, \bar{K}) = \mathrm{Dp}_{\mathfrak{t}}(L_1^-, \bar{K}) < \zeta$ and necessarily L_1^* has exactly one element. Also easily: $t \in L_1^*$ implies $L_1^- \in I_t^{\mathfrak{t}}$ so L_1^* is a $(\mathfrak{t}, \bar{K})-^*$representation of L_1. So clearly $\mathrm{Dp}_{\mathfrak{t}}(L_1, \bar{K}) \leq \mathrm{Dp}_{\mathfrak{t}}(L_1^-, \bar{K}) + 1 \leq \zeta$.

<u>Third Case</u>: ζ is limit and $\langle L_a : a \in M \rangle$ is a $(\mathfrak{t}, \bar{K})$-representation of L_2 such that $a \in M \Rightarrow \mathrm{Dp}_{\mathfrak{t}}(L_a, \bar{K}) < \zeta$.

Let $L_a^2 =: L_a$ and $L_a^1 =: L_a \cap L_1$, so $\langle L_a^1 : a \in M \rangle$ is increasing, $\bigcup_{a \in M} L_a^1 = L_1$ and each L_a^1 is $\bar{K}$-closed (as L_a^2, L_1 are $\bar{K}$-closed, see part (0)) and easily $t \in L_1 \wedge A \in I_t^{\mathfrak{t}} \wedge A \subseteq L_1 \Rightarrow (\exists a \in M)(A \subseteq L_a^2 \cap L_1 = L_a^1)$. Also by the definition of Dp at limit ordinals $b \in M \Rightarrow \mathrm{Dp}_{\mathfrak{t}}(L_b^2, \bar{K}) < \zeta$. Hence by the induction hypothesis $\mathrm{Dp}_{\mathfrak{t}}(L_b^1, \bar{K}) < \zeta$. By the last two sentences (and Definition 2.1) we get $\mathrm{Dp}_{\mathfrak{t}}(L_1, \bar{K}) \leq \zeta$, as required in clause (α). For clause (β) we know that there is $t \in L_2$ such that $L_1 \in I_t^{\mathfrak{t}}$, hence by clause (g) of Definition 2.2(1)) for some $b \in M$ we have $L_1 \subseteq L_b$ and we can use the induction hypothesis on ζ for L_1, L_b.
3) Easy.
4) By induction on the depth ζ. The case $\zeta = 0$ is trivial; and the case ζ is a limit ordinal is easy. Lastly for the successor case of 2.1(3) recall $\boxplus_{t,L}$ there. $\square_{2.3}$

Claim 2.4. *1) If for $\ell = 1,2$ we have $\bar{L}^\ell$ is a $(\mathfrak{t},\bar{K})$-representation of L and $\bar{L}^\ell = \langle L^\ell_a : a \in M_\ell\rangle$ and $M = M_1 \times M_2$ <u>then</u> $\bar{L} = \langle L_a \cap L_b : (a,b) \in M\rangle$ is a $(\mathfrak{t},\bar{K})$-representation of L.*
2) If L^*_ℓ is a $(\mathfrak{t},\bar{K})-^*$representation of L for $\ell = 1,2$ <u>then</u> $L^*_1 = L^*_2$.
3) If A is $(\bar{K},<\partial)$-countable <u>then</u> it is $\bar{K}$-closed.
4) If $L \subseteq L^{\mathfrak{t}}$ is $\bar{K}$-closed and $L_1 \subseteq L$ has cardinality $<\partial$ <u>then</u> some $(\bar{K},\partial)$-finite set $L_2 \subseteq L$ includes L_1.
5) If $(\mathfrak{t},\bar{K})$ is an FSI-template then so are $\mathfrak{t}',\mathfrak{t}''$ where $\mathfrak{t}',\mathfrak{t}''$ are FSI-templates satisfying

- (a) $L^{\mathfrak{t}'} = L^{\mathfrak{t}''} = L^{\mathfrak{t}}$
- (b) *for $t \in L^{\mathfrak{t}}$ let $I^{\mathfrak{t}'}_t = \{A \in I^{\mathfrak{t}}_t : A$ is $\bar{K}$-countable$\}$*
- (c) *for $t \in L^{\mathfrak{t}}$ let $I^{\mathfrak{t}''}_t = \{A \subseteq L^{\mathfrak{t}} :$ the set $\{B \subseteq A : B \in I^{\mathfrak{t}}_t$ is $\bar{K}$-closed$\}$ is cofinal in $[A]^{<\partial(\mathfrak{t})}\}$*

Proof. 1) Straightforward, e.g. if $t \in L^{\mathfrak{t}}, A \in I_t$ and $A \subseteq L$ then for $\ell = 1,2$ we can choose $a_\ell \in M_\ell$ such that $A \subseteq L^\ell_{a_\ell}$ and $t \in L_{a_\ell}$. Clearly $A \cup \{t\} \subseteq L^1_{a_1} \cap L^2_{a_2}$.
2)-5) Easy, too. $\square_{2.4}$

Discussion 2.5. This discussion is from the old version, so some "we may" are actually done in the new version.
1) Our next aim is to define iteration for any $\bar{K}$-smooth FSI-template $\mathfrak{t}$; for this we define and prove the relevant things; of course, by induction on the depth. In the following Definition 2.6, in clause (A)(a), we avoid relying on [She04a]; moreover the reader may consider only the case $K_t = \emptyset$, omit $\underset{\sim}{\eta}_t$ and have $\underset{\sim}{\mathbb{Q}}_{t,\bar{\varphi}'_t}$ be the dominating real forcing = Hechler forcing.
2) We may more generally than here allow $\underset{\sim}{\eta}_t$ to be e.g. a sequence of ordinals, and members of $\underset{\sim}{\mathbb{Q}}_{t,\varphi,\underset{\sim}{\eta}_t}$ be $\subseteq \mathscr{H}_{<\aleph_1}(\mathrm{Ord})$, and even K_t large but increasing L, we need more "information" from $\underset{\sim}{\eta}_t \restriction \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L)$. We may require more by changing to: $\underset{\sim}{\mathbb{Q}}_t$ is a definition of nep c.c.c. forcing ([She04a]) or just "Souslin c.c.c. forcing (= snep)" or just absolute enough c.c.c. forcing notion. All those cases do not make real problems (but when the parameter $\underset{\sim}{\eta}_t$ have length $\geq \kappa$ (or just has no bound $<\kappa$) it is changed in the ultra-power! i.e. $\mathbf{j}(\underset{\sim}{\eta}_t)$ has length $>$ length of $\underset{\sim}{\eta}_t$).
3) If we restrict ourselves to σ-centered forcing notions (which is quite reasonable) probably we can in Definition 2.1(3)(a) omit $\boxtimes_{t,L}$ if in Definition 2.6 below in (A)(b) second case we add that $t \in L^* \Rightarrow p \restriction (L \backslash L^*)$ forces a value to $\underset{\sim}{f}_t(p(t))$ where $\underset{\sim}{f}_t : \mathbb{Q}_t \to \omega$ witnesses σ-centerness and is absolute enough (or just assume $\underset{\sim}{\mathbb{Q}}_t \subseteq \omega \times \underset{\sim}{\mathbb{Q}}'_t, f_t(p(t))$ is the first coordinate). More carefully probably we can do this with σ-linked instead σ-centered.

Definition/Claim 2.6. Let $\mathfrak{t}$ be an FSI-template and $\bar{K} = \langle K_t : t \in L^{\mathfrak{t}}\rangle$ be a smooth $\mathfrak{t}$-memory choice.

By induction on the ordinal ζ we shall define and prove:

- (A) [Def] for $L \subseteq L^{\mathfrak{t}}$ which is $\bar{K}$-closed of $(\mathfrak{t},\bar{K})$-depth $\leq \zeta$ we define
  - (a) when $\bar{\mathbb{Q}} = \langle \underset{\sim}{\mathbb{Q}}_{t,\bar{\varphi}_t,\underset{\sim}{\eta}_t} : t \in L\rangle$ is a $(\mathfrak{t},\bar{K})$-iteration of def-c.c.c. forcing notions, but we can let $\underset{\sim}{\eta}_t$ code $\bar{\varphi}_t$, say as $\bar{\varphi} = \underset{\sim}{\eta}(0) \in \mathscr{H}(\aleph_0)$; so we may omit $\bar{\varphi}_t$; note that "def. - c.c.c." is defined below
  - (b) $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$ for $\bar{\mathbb{Q}}$ as in (A)(a), pedantically we should write $\mathrm{Lim}_{\mathfrak{t},\bar{K}}(\bar{\mathbb{Q}})$
  - (c) $\underset{\sim}{\bar{\nu}}$ is the sequence of generics of $\bar{\mathbb{Q}}$

(d) $\bar{u} = \bar{u}[\bar{\mathbb{Q}}]$ is the parameters domain sequence of $\bar{\mathbb{Q}}$
(e) the class $\mathbf{Q} = \mathbf{Q}_{\rm fsi}$ of fsi-templates as well as some related classes
(f) $\partial(\mathbf{q}), \partial^-(\mathbf{q})$ for $\mathbf{q} \in \mathbf{Q}$.
(g) $\mathbf{q}$ is $\lim\langle \mathbf{q}_a : a \in M\rangle$ where M is a directed partial order and $\mathbf{q}_a \in \mathbf{Q}$ is increasing with a.

(B) [Claim] for $L_1 \subseteq L_2 \subseteq L^{\mathfrak{t}}$ which are $\bar{K}$-closed of $(\mathfrak{t}, \bar{K})$-depth $\le \zeta$ and $(\mathfrak{t}, \bar{K})$-iteration of def-c.c.c. forcing notions $\bar{\mathbb{Q}} = \langle \mathbb{Q}_{t,\bar{\varphi}_t,\eta_t} : t \in L_2\rangle$ we prove:

(a) $\bar{\mathbb{Q}} \restriction L_1$ is a $(\mathfrak{t}, \bar{K} \restriction L_1)$-iteration of def-c.c.c. forcing notions

(b) $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_1) \subseteq \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$ as quasi orders

(c) if $L_1 \le_{\mathfrak{t}} L_2$ (see Definition 2.1(6)) and $p \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$, <u>then</u> $p \restriction L_1 \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_1)$ and $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}) \models$ "$p \restriction L_1 \le p$"

(d) if $L_1 \le_{\mathfrak{t}} L_2$ and $p \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$ and $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_1) \models$ "$(p \restriction L_1) \le q$" <u>then</u> $q \cup (p \restriction (L_2 \backslash L_1))$ is a lub of $\{p, q\}$ in $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$; hence $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_1) \lessdot \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$, (used in the proof of clause (B)(j))

(e) $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_1) \lessdot \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$, that [3] is

(i) $p \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_1) \Rightarrow p \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$

(ii) $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_1) \models p \le q \Rightarrow \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}) \models p \le q$

(iii) if $\mathscr{I} \subseteq \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_1)$ is predense in $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_1)$, <u>then</u> $\mathscr{I}$ is predense in $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$

(iv) if $p, q \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$ are incompatible in $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} {\restriction} L_1)$ then they are incompatible in $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$)

(f) assume $L_0 \subseteq L_2$ is $\bar{K}$-closed, $L = L_0 \cap L_1$; if $p \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_0)$ and $q \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L)$ satisfies $(\forall r \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L))[q \le r \to p, r$ are compatible in $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_0)]$ <u>then</u> $(\forall r \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} {\restriction} L_1))[q \le r \to p, r$ are compatible in $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_2)]$

[explanation: this means that if q forces for $\Vdash_{\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_0)}$ that $p \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_0)/\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L)$ <u>then</u> q forces for $\Vdash_{\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_1)}$ that $p \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})/\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_1)$.]

(g) if $\langle L_a : a \in M_1\rangle$ is a $(\mathfrak{t}, \bar{K})-$representation of L_1 <u>then</u> $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_1) = \bigcup\limits_{a \in M_1} \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L^1_a)$

(h) if L^* is a $(\mathfrak{t}, \bar{K})$-*representation of L_1 and $L^* = L \cup \{t\}$, <u>then</u> $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_1) = \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction (L_1 \setminus L^*) * \mathbb{Q}_{t,\eta_t})$

(i) (α) if $p_1, p_2 \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$ and $t \in \mathrm{Dom}(p_1) \cap \mathrm{Dom}(p_2) \Rightarrow p_1(t) = p_2(t)$, <u>then</u> $q = p_1 \cup p_2$ (i.e. $p_1 \cup (p_2 \backslash (\mathrm{Dom}(p_1)))$ belongs to $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$ and is a l.u.b. of p_1, p_2 in it

(β) $p \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$ <u>iff</u> p is a function with domain a finite subset of L_2 such that for every $t \in \mathrm{Dom}(p)$ for some $A \in I^{\mathfrak{t}}_t, A$ is $\bar{K}$-closed and $K_t \subseteq A$ and $\Vdash_{\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction A)}$ "$p(t) \in \mathbb{Q}_{t,\eta_t}$".
[So if $p \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$ then for some $\bar{K}$-countable (even $\bar{K}$-finite, see 2.1)(2)(f)), $L \subseteq L_2$ we have $p \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L)$]

[3] here we do not assume $L_1 \le_{\mathfrak{t}} L_2$,

(γ) $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}) \models p \le q$ iff $p, q \in \mathrm{Lim}_t(\bar{\mathbb{Q}})$ and for every $t \in \mathrm{Dom}(p)$ we have $t \in \mathrm{Dom}(q)$ and for some $\bar{K}$-closed $A \in I^{\mathfrak{t}}_t$ we have $q \restriction A \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction A)$ and $q \Vdash_{\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction A)}$ "$p(t) \le q(t)$ in $\mathbb{Q}_{t,\underset{\sim}{\eta}_t}$ (as interpreted in $\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction A)}$ of course)"

(j) (α) $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$ is a c.c.c. forcing notion
(β) $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}) = \cup\{\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L) : L \subseteq L_2$ is $\bar{K}$-finite$\}$

(k) (α) $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$ has cardinality $\le |L_2|^{\aleph_0} + \partial^-(\mathbf{q})$
(β) for every $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$-name $\underset{\sim}{\rho}$ of a real there is a $\bar{K}$-countable set L such that $\underset{\sim}{\rho}$ is a $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L)$-name

Let us carry the induction.

<u>Part (A)</u>: [Definition]

So assume $\mathrm{Dp}_{\mathfrak{t}}(L, \bar{K}) \le \zeta$. If $\mathrm{Dp}_{\mathfrak{t}}(L) < \zeta$ we have already defined being $(\mathfrak{t}, \bar{K})$-iteration and $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L)$, so assume $\mathrm{Dp}_{\mathfrak{t}}(L) = \zeta$.

<u>Clause (A)(a)</u> For every $t \in L^{\mathfrak{t}}$ we have:

(i) $\underset{\sim}{\eta}_t$ is a $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction K_t)$-name of a real (i.e. from ${}^{\omega}2$, used as a parameter) and $u_t = \omega$ <u>or</u> (see (A)(d)) a function from a set of ordinals u_t (u_t an object, not a name) into $\{0, 1\}$ or into $\mathscr{H}(\aleph_0)$, (legal as K_t is a $\bar{K}$-closed subset of L and $K_t \in I_t$ and $t \in L$ hence by 2.3(2), clause (β) we have $\mathrm{Dp}_{\mathfrak{t}}(K_t, \bar{K}) < \mathrm{Dp}_{\mathfrak{t}}(K_t \cup \{t\}, \bar{K}) \le \mathrm{Dp}_{\mathfrak{t}}(L, \bar{K}) \le \zeta$ so $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_t)$ is a well defined forcing notion by the induction hypothesis and 2.3(2), clause (β))

(ii) $\bar{\varphi}_t$ is a pair of formulas which from the parameters $\underset{\sim}{\eta}_t$ define in $\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction K_t)}$ a forcing notion denoted by $\mathbb{Q}_{t,\bar{\varphi}_t,\underset{\sim}{\eta}_t}$ whose set of elements is $\subseteq \mathscr{H}(\aleph_1)$ or $\subseteq \mathscr{H}_{\aleph_1}(u_t)$

(iii) in $\mathbf{V}_1 = \mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction K_t)}$, if $\mathbb{P}' \lessdot \mathbb{P}''$ are c.c.c. forcing notions[4] <u>then</u> $\mathbb{Q} = \mathbb{Q}_{t,\bar{\varphi}_t,\underset{\sim}{\eta}_t}$ as interpreted in $\mathbf{V}_2 = (\mathbf{V}_1)^{\mathbb{P}'}$ is a c.c.c. forcing notion there, and $\mathbb{P}' * \underset{\sim}{\mathbb{Q}}_{t,\bar{\varphi}_t,\underset{\sim}{\eta}_t}$ is a $\lessdot$-sub-forcing of $\mathbb{P}'' * \underset{\sim}{\mathbb{Q}}_{t,\bar{\varphi}_t,\underset{\sim}{\eta}_t}$ where $\underset{\sim}{\mathbb{Q}}_{t,\bar{\varphi}_t,\underset{\sim}{\eta}_t}$ mean as interpreted in $(\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction K_t)})^{\mathbb{P}'}$ or in $(\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction K_t)})^{\mathbb{P}''}$ respectively (i.e. "$p \le q$","p, q are incompatible", "$\langle p_n : n < \omega\rangle$ is predense" (so the sequence is from the smaller universe) are preserved)

(iv) assume that $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction K_t) \lessdot \mathbb{P}_0 \lessdot \mathbb{P}_\ell \lessdot \mathbb{P}_3$ are c.c.c. forcing notions for $\ell = 1, 2$ and $\mathbb{P}_1 \cap \mathbb{P}_2 = \mathbb{P}_0$. Let $\underset{\sim}{\mathbb{Q}}_\ell$ be the $\mathbb{P}_\ell$-name of $\underset{\sim}{\mathbb{Q}}_{t,\underset{\sim}{\eta}_t}$ as interpreted in $\mathbf{V}^{\mathbb{P}_\ell}$.

If $(p_\ell, \underset{\sim}{q}_\ell) \in \mathbb{P}_\ell * \underset{\sim}{\mathbb{Q}}_\ell$ for $\ell = 0, 1, 2$ and $(p_0, \underset{\sim}{q}_0) \Vdash$ "$(p_\ell, \underset{\sim}{q}_\ell) \in (\mathbb{P}_\ell * \underset{\sim}{\mathbb{Q}}_\ell)/(\mathbb{P}_0 * \underset{\sim}{\mathbb{Q}}_0)$" for $\ell = 1, 2$ and $p_3 \in \mathbb{P}$ is above p'_1, p'_2 <u>then</u> there are $(p_\ell, \underset{\sim}{q}_\ell) \in \mathbb{P}_\ell * \underset{\sim}{\mathbb{Q}}_\ell$ above (p_ℓ, q_ℓ) for $\ell = 0, 1, 2$ satisfying $(p'_0, q'_0) \Vdash$ "$(p'_\ell, q'_\ell) \in (\mathbb{P}_\ell * \underset{\sim}{\mathbb{Q}}_\ell)/(\mathbb{P}_0 * \underset{\sim}{\mathbb{Q}}_0)$" for $\ell = 1, 2$ such that:

- $p_3 \Vdash_{\mathbb{P}_3}$ "$\underset{\sim}{q}_1, \underset{\sim}{q}_2$ are compatible in $\underset{\sim}{\mathbb{Q}}_3$".

<u>Clause (A)(b)</u>:

<u>First Case</u>: $\zeta = 0$.

[4] So the definition $\bar{\varphi}_t$ still defines a forcing notion; We may restrict ourselves to forcing notions which occur in our proof; but does not seem to matter for now.

Trivial.

<u>Second Case</u>: ζ is a successor.

So let L^* be a $(\mathfrak{t},\bar{K})$-*representation of L.

Define $p \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$ <u>iff</u> p is a finite function, $\mathrm{Dom}(p) \subseteq L, p \restriction (L \backslash L^*) \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction (L \backslash L^*))$ and if $t \in L^* \cap \mathrm{Dom}(p)$, then $p(t)$ is a $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction (L \backslash L^*)$-name of a member of $\mathbb{Q}_{t,\bar{\varphi}_t,\underset{\sim}{\eta}_t}$ and the order is $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}) \models p \leq q$ <u>iff</u>

- (i) $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction (L \backslash L^*)) \models$ "$(p \restriction (L \backslash L^*) \leq (q \restriction (L \backslash L^*))$" and
- (ii) if $t \in L^* \cap \mathrm{Dom}(p)$ then for some $\bar{K}$-closed $A \in I^{\mathfrak{t}}_t$ we have $q \restriction A \Vdash_{\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction (L \backslash L^*))}$ "$p(t) \leq q(t)$".

Clearly $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$ is a quasi order. But we should prove that $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$ is well defined, which means that the definition does not depend on the representation.

So we prove

$\boxtimes_1$ if $\mathrm{Dp}_{\mathfrak{t}}(L,\bar{K}) = \zeta$ and for $\ell = 1,2$ we have L^*_ℓ is a $(\mathfrak{t},\bar{K})$-*representation of L with $\mathrm{Dp}_{\mathfrak{t}}(L \backslash L^*_\ell, \bar{K}) < \zeta$ and $\mathbb{Q}^\ell$ is $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L)$ as defined by L^*_ℓ above, <u>then</u> $\mathbb{Q}^1 = \mathbb{Q}^2$.

This is immediate by Claim 2.4(2) and the induction hypothesis clause (B)(h).

<u>Third Case</u>: ζ limit.

So there are a directed partial order M and $\bar{L} = \langle L_a : a \in M \rangle$ a $(\mathfrak{t},\bar{K})$-representation of L such that $a \in M \Rightarrow \mathrm{Dp}_{\mathfrak{t}}(L_a,\bar{K}) < \zeta$. By the induction hypothesis, $a \leq_M b \Rightarrow L_a \subseteq L_b$ and $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_a) \subseteq \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_b)$.

We let $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L) = \bigcup\limits_{a \in M} \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_a)$, so we have to prove

$\boxtimes_2$ the choice of $\bar{L}$ is immaterial.

So we just assume that for $\ell = 1,2$ we have: M_ℓ is a directed partial order, $\bar{L}^\ell = \langle L^\ell_a : a \in M_\ell \rangle, L^\ell_a \subseteq L, M_\ell \models a \leq b \Rightarrow L^\ell_a \subseteq L^\ell_b$ and $(\forall t \in L)(\forall A \in I_t)[A \subseteq L \rightarrow (\exists a \in M_\ell)(A \subseteq L^\ell_a)$ and $\mathrm{Dp}_{\mathfrak{t}}(L^\ell_a,\bar{K}) < \zeta$.

We should prove that $\bigcup\limits_{a \in M_1} \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L^1_a)$, $\bigcup\limits_{a \in M_2} \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L^2_a)$ are equal, as quasi orders of course.

Now let $M =: M_1 \times M_2$ with $(a_1,a_2) \leq (b_1,b_2) \Leftrightarrow a_1 \leq_{M_1} b_1 \wedge a_2 \leq_{M_2} b_2$, clearly it is a directed partial order. We let $L_{(a_1,a_2)} = L^1_{a_1} \cap L^2_{a_2}$, so clearly $L_{(a_1,a_2)} \subseteq L^{\mathfrak{t}}, \mathrm{Dp}_{\mathfrak{t}}(L_{(a_1,a_2)},\bar{K}) < \zeta$ and $(a_1,a_2) \leq_M (b_1,b_2) \Rightarrow L_{(a_1,a_2)} \subseteq L_{(b_1,b_2)}$ and $\langle L_{(a_1,a_2)} : (a_1,a_2) \in M \rangle$ is a $(\mathfrak{t},\bar{K})$-representation of L by Claim 2.4(1). So by transitivity of equality, it is enough to prove for $\ell = 1,2$ that $\bigcup\limits_{a \in M_\ell} \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L^\ell_a)$, $\bigcup\limits_{(a,b) \in M} \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_{(a,b)})$ are equal as quasi orders. By the symmetry in the situation without loss of generality $\ell = 1$.

Now for every $a \in M_1, \bar{L} = \langle L_{(a,b)} : b \in M_2 \rangle$ satisfies: $L^1_a \subseteq L, \mathrm{Dp}(L^1_a,\bar{K}) < \zeta, L^1_a = \bigcup\limits_{b \in M_2} L_{(a,b)}, b_1 \leq_{M_2} b_2 \Rightarrow L_{(a,b_1)} \subseteq L_{(a,b_2)}$.

Fix $a \in M_1$ and notice that $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L^1_a)$, $\bigcup\limits_{b \in L_2} \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_{(a,b)})$ are equal as quasi orders. Next we have to verify that for every $t \in L^1_a$ and $A \in I^{\mathfrak{t}}_t$ for some $b \in L_2$ we have $t \in L_{a,b}$ and $A \subseteq L_{a,b}$. By the assumption on $\langle L^2_b : b \in M_2 \rangle$ for some $b \in M_2$ we have $t \in L_b \wedge A \subseteq L^2_b$, hence $t \in L^1_a \cap L^2_b$ and $A \subseteq L^1_a \cap L^2_b = L_{a,b}$ so this b is as required. Hence by the induction hypothesis for clause (B)(g) we have $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L^1_a)$, $\bigcup\limits_{b \in L_2} \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_{(a,b)})$ are equal as quasi orders

As this holds for every $a \in M_1$ and M_1 is directed we get $\bigcup\limits_{a \in M_1} \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L^1_a)$, $\bigcup\limits_{a \in M_1} \bigcup\limits_{b \in M_2} \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_{(a,b)})$ are equal as quasi orders. But the second is equal to $\bigcup\limits_{(a,b) \in M} \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_{(a,b)})$ so we are done.

<u>Clause (A)(c)</u>

$\underset{\sim}{\bar{\nu}}$ is the sequence of generics of $\bar{\mathbb{Q}}$ means

- (α) $\underset{\sim}{\bar{\nu}} = \langle \underset{\sim}{\nu}_t : t \in L^{\mathfrak{t}} \rangle$,
- (β) for each $t \in L^{\mathfrak{t}}$, $\underset{\sim}{\nu}_t$ is a $\mathrm{Lim}(\mathbb{Q} \restriction K^\dagger_t)$-name of a function from a set of ordinals to $\{0,1\}$ or to $\mathscr{H}(\aleph_0)$; for simplicity with domain u_t recalling $K^\dagger_t = K_t \cup \{t\}$
- (γ) if $L \subseteq L^{\mathfrak{t}}$ is $\bar{K}$-closed then $\bar{\nu} \restriction L$ is a generic for $\mathrm{Lim}(\bar{\mathbb{Q}} \restriction L)$

<u>Clause (A)(d)</u>

$\bar{u} = \bar{u}[\bar{\mathbb{Q}}] = \langle u_t : t \in L^{\mathfrak{t}} \rangle$ is the parameters domain sequence of $\bar{\mathbb{Q}}$ means that each $u_t = u(t)$ is a set of ordinals, for simplicity, recalling that $\Vdash_{\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction K_t)}$ "$\underset{\sim}{\eta}_t$ is a function from u_t to $\mathscr{H}(\aleph_0)$"

<u>Clause (A)(e)</u>

The class $\mathbf{Q} = \mathbf{Q}_{\mathrm{fsi}}$ of fsi-templates is the class of objects $\mathbf{q}$ of the form $(\mathfrak{t}, \bar{K}, \bar{u}, \bar{\mathbb{Q}})$ which are as above with $\mathrm{dom}(\bar{\mathbb{Q}}) = L^{\mathfrak{t}}$ and $\mathbb{P}_{\mathbf{q}} = \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$. We may write $\mathbf{q}$ instead $\mathfrak{t}, (\mathfrak{q}, \bar{K})$ or $\bar{\mathbb{Q}}$.

For ∂ regular uncountable let $\mathbf{Q}^{\mathrm{fsi}}_\partial$ be the class of $\mathbf{q} \in \mathbf{Q}_{\mathrm{fsi}}$ satisfying $\partial(\mathbf{q}) \le \partial$, similarly in other cases.

Let $\mathbf{Q}_{\mathrm{dom}}$ be the class of $\mathbf{q} \in \mathbf{Q}$ such that $K^{\mathbf{q}}_t = \emptyset, \mathbb{Q}^{\mathbf{q}}_t$ is dominating real= Hechler forcing and $\underset{\sim}{\eta}_t$ the generic.

On $\mathbf{Q}_{\mathrm{cln}}$ see 2.18(2), $\boxplus^1_{\mathbf{q}}$.

<u>Clause (A)(f)</u>

Let $\partial(\mathbf{q})$ for $\mathbf{q} \in \mathbf{Q}$ be the minimal infinite cardinal ∂ which is strictly bigger then $|A|, |u_t|$ for $t \in L^{\mathfrak{t}}, A \in I^{\mathfrak{t}}_t$.

We define $\partial^-(\mathbf{q})$ similarly but requiring only "bigger or equal".

<u>Part (B)</u>:

<u>First Case</u>: $\zeta = 0$.

Trivial.

<u>Second Case</u>: ζ successor.

Similar to usual iterations, so easy using the definition and the induction hypothesis except clause (f) which we prove in details.

<u>Clause (f)</u>:

Let p, q, L, L_0 be as in the assumption of clause (f). Let $r \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_1)$ be above q there and we should prove the p, r are compatible in $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_2)$. Let t_* be the maximal member of $L_2, L^-_\ell := L_\ell \setminus \{t_*\}$ hence $L^-_2 = L_2 \setminus \{t_*\} \in I_{t_*}$ and $\mathrm{dp}(L^-_2) < \zeta, L^- := L \setminus \{t_*\}$. If $(t_* \notin L_0 \vee t_* \notin L_1)$ or just $t_* \notin \mathrm{Dom}(p) \cap \mathrm{Dom}(r)$ then by the induction hypothesis applied to $L^-_1, L^-_2, L^-, L^-_0, p \restriction L^-_0, q \restriction L^-, r \restriction L^-_2$

we can find a common upper bound r^* of $p \restriction L_0^-, r \restriction L_1^-$ in $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_2^-)$ and $r^* \cup p \restriction \{t_*\} \cup r \restriction \{t_*\}$ is a common upper bound of p, r as required.

So assume that $t_* \in \mathrm{Dom}(p) \cap \mathrm{Dom}(r) \subseteq L_0 \cap L_1$ and let $\mathbb{P}_0 := \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L^-)$ and $\mathbb{P}_{\ell+1} := \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_\ell^-)$ for $\ell = 0, 1, 2$.

Now we get p', r' by applying the definition in clause (A)(a)(iv) for t_* with $(p \restriction L_0^-, p(t_*)), (r \restriction L_1^-, r(t_*)), (q{\restriction}L^-, q(t_*))$ here standing for $(p_1, \underset{\sim}{q}_1), (p_2, \underset{\sim}{q}_2), (p_0, \underset{\sim}{q}_0)$ there getting (p'_ℓ, q'_ℓ) for $\ell < 3$ as there.

By the induction hypothesis in $\mathbb{P}_3$ for the conditions p'_0, p'_1, p'_2 we can find a common upper bound p^*, so by $(A)(a)(iv)$ conclusion we are done.

Third Case: ζ limit.

So let $\langle L_a^2 : a \in M \rangle$ be a $(\mathfrak{t}, \bar{K})$-representation of L_2 with $a \in M \Rightarrow \mathrm{Dp}_{\mathfrak{t}}(L_a^2, \bar{K}) < \zeta$ and let $L_a^1 = L_1 \cap L_a^2$.

Clause (B)(a):

Trivial.

Clause (B)(b):

Clearly $\mathrm{Dp}_{\mathfrak{t}}(L_1, \bar{K}) \leq \zeta$ by Claim 2.3(2) (α) hence $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_1)$ is well defined by (A)(b) which we have already proved above, that is $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}) = \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_2) = \bigcup\limits_{a \in M_2} \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_a^2)$ as quasi orders.

Clearly $\langle L_a^1 = L_1 \cap L_a^2 : a \in M \rangle$ is a $(\mathfrak{t}, \bar{K})$-representation of L_1 hence by the induction hypothesis (if $\mathrm{Dp}_{\mathfrak{t}}(L_1, \bar{K}) < \zeta$) or by the uniqueness proved in (A)(b) (if $\mathrm{Dp}_{\mathfrak{t}}(L_1, \bar{K}) = \zeta$) we know that $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_1) = \bigcup\limits_{a \in M} \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}{\restriction}L_a^1)$ as quasi orders and by the induction hypothesis for (B)(b) we know $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_a^1) \subseteq \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_a^2)$ as quasi orders (for $a \in M$), and we can easily finish.

Clause (B)(c),(d):

Use the proof of clause (B)(b) noting that $L_a^1 \leq_{\mathfrak{t}} L_a^2$ and so we can use the induction hypothesis, but we elaborate. For clause (c), let $p \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_2)$ so there is an element $a \in M$ such that $p \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_a^2)$. Now $p \restriction L_1 = p \restriction L_a^1$ and as $L_a^1 \leq_{\mathfrak{t}} L_a^2$ by the induction hypothesis we have $p \restriction L_a^1 \leq_{\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_1)} p$ as promised.

For clause (d) we assume in addition that $p \restriction L_1 \leq q$ in $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}{\restriction}L_1)$. Let $r_1 = q \cup (p \restriction (L_2 \setminus L_1)$. Easily r_1 is an upper bound of p, q in $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_2)$. Assume further that r_2 is another common upper bound of p, q. As M is directed we can choose $a \in M$ such that $p, q, r_1, r_2 \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_a^2)$ but $q, p \restriction L^1 \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_a^1)$. Hence by the induction hypothesis $r_1 \leq r_2$ in $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_a^2)$ so we can finish .

Clause (B)(e):

The statements (i) + (ii) hold by clause (b).

The statement (iii) holds: let $\mathscr{I}$ be a predense subset of $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_1)$, let $p \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$, so for some $a \in M$ we have $p \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_a^2)$.

By the induction hypothesis applying clause (B)(e) to L_a^1, L_a^2 we have $\mathrm{Lim}_{\mathfrak{t}}(\mathbb{Q} \restriction L_a^1) \lessdot \mathrm{Lim}_{\mathfrak{t}}(\mathbb{Q} \restriction L_a^2)$, hence as $p \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_a^2)$ clearly there is $q \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_a^1)$ such that p is compatible with r in $\mathrm{Lim}_{\mathfrak{t}}(\mathbb{Q} \restriction L_a^2)$ whenever $\mathrm{Lim}_{\mathfrak{t}}(\mathbb{Q} \restriction L_a^1) \models$ "$q \leq r$". Now by the assumption on "$\mathscr{I} \subseteq \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_1)$ is predense", as $q \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_1)$ (by clause (B)(b)) we can find $q_0 \in \mathscr{I}$ and q_1 such that $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_1) \models q_0 \leq q_1 \wedge q \leq q_1$, so for some $b \in M$ we have $q, q_0, q_1 \in L_b^1$ and $a \leq_M b$ (as M is directed). Now we consider $p, q, L_a^1, L_a^2, L_b^1, L_b^2$ and apply clause (B)(f).

Clause (B)(f):

Easy to check using clause (f) for the L^2_a's, which holds by the induction hypothesis.

Clause (B)(g):

Let $M_2 =: M$ (and recall M_1 that is from clause $(B)(g)$). For each $a_1 \in M_1$, clearly $\mathrm{Dp}_{\mathfrak{t}}(L_{a_1}, \bar{K}) \le \zeta$ as $L_{a_1} \subseteq L_2$ and $\langle L_{a_1} \cap L^2_{a_2} : a_2 \in M_2\rangle$ is a $(\mathfrak{t}, \bar{K})$-representation of L_{a_1} and $\mathrm{Dp}_{\mathfrak{t}}(L_{a_1} \cap L^2_{a_2}, \bar{K}) \le \mathrm{Dp}_{\mathfrak{t}}(L^2_{a_2}, \bar{K}) < \zeta$ hence by (A)(b) we know $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_{a_1}) = \bigcup\limits_{a_2 \in M_2} \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction (L_{a_1} \cap L^2_{a_2}))$. The rest should be clear.

Clause (B)(h):

Easy. If $t \in L^*$ then $L_1 \setminus L^* \in I^{\mathfrak{t}}_t$ hence for some $a \in M$ we have $L_1 \setminus L^* \subseteq L_a$, and the rest should be clear; and if L^* is empty this is easier.

Clause (B)(i):

Easy.

Clause (B)(j):

Sub-clause (β) is clear by the definition of $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$ so we shall deal with sub-clause (α).

So let $p_\alpha \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$ for $\alpha < \omega_1$; let $w_\alpha = \mathrm{Dom}(p_\alpha)$ and without loss of generality $\langle w_\alpha : \alpha < \omega_1\rangle$ is a Δ-system with heart w.

A natural way fails because if $\langle L^*_\alpha : \alpha < \omega_1\rangle$ is increasing continuous, $L^*_\alpha \subseteq L_2$ is $\bar{K}$ -closed, $\subseteq$-increasing continuous then $\langle \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}{\restriction}L^*_\alpha) : \alpha < \omega_1\rangle$ is $\lessdot$-increasing but not necessarily continuous.

Let $\{t_0, \dots, t_{n-1}\}$ list w without repetitions such that[5] $\ell < k < n \Rightarrow \neg(t_k \le t_\ell)$ and let L^*_ℓ be defined by induction on $\ell \le 2n+1 = 2(n-1)+3$ as follows:

- $L^*_0 = \emptyset$
- $L^*_1 = \{s : s < t_k \text{ for every } k < n\}$
- $L^*_{2\ell+2} = L^*_{2\ell+1} \cup \{t_\ell\}$ <u>when</u> $\ell < n$ equivalently $2\ell + 2 < 2n + 1$
- $L^*_{2\ell+3} = L^*_{2\ell+2} \cup \{s : s < t_k \text{ for every } k \in \{\ell+1, \dots, n-1\}\}$ <u>when</u> $\ell < n$ equivalently $2\ell + 3 \le 2n + 1$

So $L^*_{2n+1} = L_2$.

Clearly

$\oplus$ $\langle L^*_\ell : \ell \le 2n+1\rangle$ is a $\subseteq$-increasing sequence of initial segments of L_2 hence is $\le_{\mathfrak{t}}$-increasing

We prove by induction on $\ell \le 2n+1$ that

$(*)_\ell$ for some $q_\ell \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}{\restriction}L^*_\ell)$ we have $q_\ell \Vdash_{\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}\restriction L^*_\ell)}$ "$p_\alpha \restriction L^*_\ell \in \underset{\sim}{\mathbf{G}}$ for $\aleph_1$ ordinals α".

[5] As $L^{\mathfrak{t}}$ is a linear order, this mean $t_0 < t_1 < \dots$.

Case 1: $\ell = 0$ trivial, (e.g. the empty $q \in \mathrm{Lim}(\bar{\mathbb{Q}} \restriction L^*_0)$).

Case 2: $\ell = 1$

As $\langle w_\alpha \cap L^*_1 : \alpha < \omega_1 \rangle$ are pairwise disjoint, every $q \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L^*_1)$ is compatible with $p_\alpha \restriction L^*_1$ for all but finitely many α's, so this follows.

Case 3: $\ell = 2i + 3$

Recall $q_{2i+2} \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L^*_{2i+2})$ has been chosen. Now assume $q_{2i+2} \le q \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L^*_{2i+2})$ and $\alpha < \omega_1$. Then $w^* = \{\gamma < \omega_1 : w_\gamma \cap \mathrm{dom}(q) \nsubseteq w\}$ is finite, hence recalling $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L^*_{2i+2}) \lessdot \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L^*_{2i+3})$ by the induction hypothesis, there is $\beta \in (\alpha, \omega_1) \backslash w^*$ such that $p_\beta \restriction L^*_{2i+2}, q \restriction L^*_{2i+2}$ are compatible hence there is $q_1 \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L^*_{2i+2})$ above both.

It suffices to prove that q_1, q, p_β has a common upper bound (as α was an arbitrary countable ordinal).

We define a function r by:

- $\mathrm{dom}(r) = \mathrm{dom}(q_1) \cup \mathrm{dom}(q) \cup \mathrm{dom}(p_\beta)$
- if $s \in \mathrm{dom}(q_1)$ then $r(s) = q_1(s)$
- if $s \in \mathrm{dom}(q) \backslash L^*_{2i+2}$, equivalently $s \in \mathrm{dom}(q) \backslash \mathrm{dom}(q_1)$ then $r(s) = q(s)$
- if $s \in (\mathrm{dom}(p_\beta) \backslash L^*_{2i+2})$, equivalently $s \in \mathrm{dom}(p_\beta) \backslash \mathrm{dom}(q_1)$ then $r(s) = p_\beta(s)$.

It is easy to verify the "equivalently" and as $\mathrm{dom}(q_1) \cap \mathrm{dom}(p_\beta) \subseteq L^*_{2i+2}$, the function r is a well defined function. Also $r \in \mathrm{Lim}_{\mathfrak{t}}(\mathbb{Q} \restriction L^*_{2i+3})$ as its domain belongs to $[L^*_{2i+3}]^{<\aleph_0}$ and each $r(s)$ is as required.

Why is r above q_1? Because $r \restriction L^*_{2i+2} = q_1$.

Why is r above p_β? By 2.6(B)(d) recalling $\oplus$ above.

Why is r above q? By 2.6(B)(d).

So we are done proving this case.

Case 4: $\ell = 2i + 2$

We can find $\mathbf{G} \subseteq \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L^*_{2i+1})$ generic over $\mathbf{V}$ such that $W = \{\alpha : p_\alpha \restriction L^*_{2i+1} \in \mathbf{G}\}$ is uncountable. Let $t = t_i$, for each $\alpha \in W$ there is a $\bar{K}$-closed $A_\alpha \in I^{\mathfrak{t}}_t$ such that $p_\alpha(t)$ is a $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction A_\alpha)$-name. So as $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction A_\alpha) \lessdot \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L^*_{2i+1})$ clearly in $\mathbf{V}[\mathbf{G}], q'_\alpha = p_\alpha(t_\ell)[\mathbf{G}]$ is well defined and by absoluteness (i.e. (A)(a)) is a member $\mathbb{Q}^{\mathbf{V}[\mathbf{G}]}_{t,\eta_t}$.

Also $\mathbf{V}[\mathbf{G}] \models$ "$\underset{\sim}{\mathbb{Q}}^{V[\mathbf{G}]}_{t,\eta_t}$ satisfies the c.c.c." hence for some $\alpha_1 \neq \alpha_2$ from $W, q_{\alpha_1}, q_{\alpha_2}$ are compatible in $\mathbb{Q}^{\mathbf{V}[\mathbf{G}]}_{t,\underset{\sim}{\eta}_t}$, but $\mathbb{Q}^{\mathbf{V}[\mathbf{G}]}_{t,\underset{\sim}{\eta}_t}$ is "too big".

Let $A = A_{\alpha_1} \cup A_{\alpha_2}$ so A is a $\bar{K}$-closed subset of L^*_{2i+1} and it belongs to I_t. So $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction A_{\alpha_\iota}) \lessdot \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction A)$ for $\iota = 1, 2$ hence by absoluteness $q_{\alpha_1}, q_{\alpha_2}$ belong to $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction A)$ and as $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction A) \lessdot \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L^*_{2i+1})$ they are compatible, so we can finish easily.

So we have carried the induction hence the proof of (B)(j).

Clause (k): Easy.

Claim 2.7. *1) Assume*

(a) $\mathfrak{t}$ *is an* FSI-*template,* $\mathrm{Dp}_{\mathfrak{t}}(L, \bar{K}) < \infty$ *i.e.* $\bar{K}$ *is a smooth* $\mathfrak{t}$-*memory choice*

(b) $\bar{\mathbb{Q}} = \langle \mathbb{Q}_{t,\underset{\sim}{\eta}_t} : t \in L \rangle$ *is a* $(\mathfrak{t}, \bar{K})$*-iteration of def-c.c.c. forcing notions, so* $L \subseteq L_{\mathfrak{t}}$ *is* $\bar{K}$*-closed*

$(c)_1$ $L_1, L_2 \subseteq L$ *and* $L_1 < L_2$ *(that is* $(\forall t_1 \in L_1)(\forall t_2 \in L_2)(L^{\mathfrak{t}} \models t_1 < t_2))$ *and* $t \in L_2 \Rightarrow L_1 \in I^{\mathfrak{t}}_t$ *and* L_1, L_2 *are* $\bar{K}$*-closed and* $L = L_1 \cup L_2$*.*

Then

(α) $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$ *is actually a definition of a forcing (in fact a c.c.c. one) so meaningful in bigger universes, moreover for extensions (by c.c.c. forcings)* $\mathbf{V}_1 \subseteq \mathbf{V}_2$ *of* $\mathbf{V} = \mathbf{V}_0$ *(with the same ordinals of course), we* [6] *get* $[\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})]^{\mathbf{V}_1} \subseteq_{\mathrm{ic}} [\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})]^{\mathbf{V}_2}$ *(see 0.2(3)) and every maximal antichain* $\mathscr{I}$ *of* $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$ *from* $\mathbf{V}_1$ *is a maximal antichain of* $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$ *(in* $\mathbf{V}_2$*).*
Recall that L *is* $\bar{K}$*-closed.*

(β) $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$ *is in fact* $\mathbb{Q}_1 * \underset{\sim}{\mathbb{Q}}_2$ *where* $\mathbb{Q}_1 = \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_1)$ *and* $\mathbb{Q}_2 = [\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_2)]^{\mathbf{V}}[\underset{\sim}{G}_{\mathbb{Q}_1}]$ *(composition).*

2) Assume clauses (a), (b) of part (1) and

$(c)_2$ L *has a last element* t^* *and let* $L^- = L \backslash \{t^*\}$*.*

Then for any $\mathbf{G}^- \subseteq \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L^-)$ *generic over* $\mathbf{V}$*, letting* $\eta_{t^*} = \underset{\sim}{\eta}_{t^*}[\mathbf{G}^-] \in \mathbf{V}[\mathbf{G}^-]$ *we have: the forcing notion* $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})/\mathbf{G}^-$ *is equivalent to* $\cup\{\mathbb{Q}^{\mathbf{V}[\mathbf{G}^-_A]}_{t^*,\eta_{t^*}} : A \in I^{\mathfrak{t}}_{t^*}$ *is* $\bar{K}$*-closed*$\}$ *where* $\mathbf{G}^-_A =: \mathbf{G}^- \cap \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction A)$ *and* $\eta_{t^*_1} = \underset{\sim}{\eta}_{t^*}[\mathbf{G}^-]$*.*
3) Assume clauses (a), (b) of part (1) and

$(c)_3$ $\langle L_i : i < \delta \rangle$ *is an increasing continuous sequence of initial segments of* L *with union* L *and* δ *is a limit ordinal.*

Then $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$ *is* $\bigcup\limits_{i<\delta} \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_i)$*, moreover* $\langle \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_i) : i < \delta \rangle$ *is* $\lessdot$*-increasing continuous.*
4) If $\mathfrak{t}$ *is not smooth then* $\mathfrak{t} \restriction L$ *is not smooth for some countable* $L \subseteq L^{\mathfrak{t}}$*, moreover for every* L' *satisfying* $L \subseteq L' \subseteq L^{\mathfrak{t}}$*.*
5) Assume $\mathfrak{t}$ *is smooth and* $\bar{\mathbb{Q}} = \langle \mathbb{Q}_{t,\underset{\sim}{\eta}_t} : t \in L^{\mathfrak{t}} \rangle$*. If* $\bar{\mathbb{Q}}$ *is not a* $(\mathfrak{t}, \bar{K})$*-iteration of def-c.c. forcing notions, then* $\bar{\mathbb{Q}} \restriction L$ *is not a* $(\mathfrak{t} \restriction L, \bar{K} \restriction L)$*-iteration of c.c.c.-definition forcing notions for some* $\bar{K}$*-closed* $L \subseteq L^{\mathfrak{t}}$ *which is the union of* $\leq \aleph_1$ $\bar{K}$*-countable sets.*

Proof. Straightforward (or read [She04a]). $\square_{2.7}$

We now give sufficient conditions for: "if we force by $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$ from 2.6, then some cardinal invariants are small or equal/bigger or equal to some μ". The necessity of such a claim in our framework is obvious; we deal with two-place relations only as this is the case in the popular cardinal invariants, in particular those we deal with.

Claim 2.8. *Assume* $\mathfrak{t}$ *is a smooth* FSI*-template and* $\bar{K} = \langle K_t : t \in L^{\mathfrak{t}} \rangle$ *and* $\bar{\mathbb{Q}} = \langle \mathbb{Q}_{t,\underset{\sim}{\eta}_t} : t \in L^{\mathfrak{t}} \rangle$ *are as in 2.6 and* $\mathbb{P} = \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$*.*
1) Assume

[6] of course possibly $L_1 = \emptyset$

(a) R *is a Borel* [7] *two-place relation*[8] *on* ${}^\omega\omega$ *(we shall use* $<^*$ *for* $\mathfrak{b}$ *and* $\mathfrak{d}$, $\subseteq^*$ *for* $\mathfrak{u}$ *and for* $\mathfrak{s}$ *we use* $\eta R_{\rm spl}\nu$ *meaning* $\mathrm{Rang}(\nu)\cap\mathrm{Rang}(\eta), \mathrm{Rang}(\nu)\setminus\mathrm{Rang}(\eta)$ *are both infinite; the intention is to use this for* $\mathfrak{s}$*)*

(b) $L^* \subseteq L^{\mathfrak{t}}$

(c) *for every* $\bar K$*-countable* $A \subseteq L^{\mathfrak{t}}$ *for some* $t \in L^*$ *we have* $A \in I^{\mathfrak{t}}_t$

(d) *for* $t \in L^*$ *and* $\bar K$*-closed* $A \in I^{\mathfrak{t}}_t$ *which includes* K_t, *in* $\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}\restriction A)}$ *we have* $\Vdash_{\mathbb{Q}_{t,\underset{\sim}{\eta}_t}}$ "$\underset{\sim}{\nu}_t \in {}^\omega\omega$ *is an* R*-cover of the old reals, that is* $\rho \in ({}^\omega\omega)^{\mathbf{V}[\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}\restriction A)]} \Rightarrow \rho R\underset{\sim}{\nu}_t$" *where* $\underset{\sim}{\nu}_t$ *is the generic real of* $\mathbb{Q}_{t,\underset{\sim}{\eta}_t}$ *or just a* $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}\restriction K^\dagger_t)$*-name. We may use* $\underset{\sim}{\nu}_t$ *a* $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}\restriction A_t)$ *with* $A_t \in I^{\mathfrak{t}}_t$

<u>*Then*</u> $\Vdash_{\mathbb{P}}$ "$(\forall\rho \in {}^\omega\omega)(\exists t \in L^*)(\rho R\underset{\sim}{\nu}_t)$, *i.e.* $\{\underset{\sim}{\nu}_t : t \in L^*\}$ *is an* R*-cover, which, if* $R = <^*$ *means* $\mathfrak{d} \le |L^*|$".

1A) If we weaken assumption (d) to $\Vdash_{\mathbb{P}}$ "*for every* $\rho \in {}^\omega\omega$ *for some* $t \in L^{\mathfrak{t}}$ *and* $\nu \in \mathbf{V}({}^\omega\omega)^{\mathbf{V}[\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}\restriction K^\dagger_t)]}$ *we have* $\rho R\nu$" *then* $\Vdash_{\mathbb{P}}$ "$(\forall\rho \in {}^\omega\omega)(\exists t \in L^*)(\exists\nu \in \mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}\restriction K^\dagger_t)})[\rho R\nu]$". *This implies that in* $\mathbf{V}^{\mathbb{P}}$, *if* $R = <^*$ *then* $\mathfrak{d} \le \sum\limits_{t\in L^{\mathfrak{t}}} |\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}\restriction K^\dagger_t)|$; *we could use* $K^\dagger$*-s index by other sets.*

2) Assume

(a) R *is a Borel two-place relation on* ${}^\omega\omega$ *(we shall use* $<^*$ *or* $\subseteq^*$ *as above)*

(b) μ *is a cardinality*

(c) *if*[9] $L^* \subseteq L^{\mathfrak{t}}, |L^*| < \mu$ <u>*then*</u> *for some* $t \in L^{\mathfrak{t}}$ *and* $\bar K$*-closed* $L^{**} \supseteq L^*$ *we have* $L^{**} \in I^{\mathfrak{t}}_t$ *and in* $\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}\restriction L^{**})}, \Vdash_{\mathbb{Q}_{t,\underset{\sim}{\eta}_t}}$ "*some* $\nu \in {}^\omega\omega$ *is an* R*-cover of the old reals"; (usually* $\underset{\sim}{\nu}$ *is the generic real of* $\mathbb{Q}_{t,\underset{\sim}{\eta}_y}$*), this we assume absolutely).*

<u>*Then*</u> $\Vdash_{\mathbb{P}}$ "$(\forall X \in [{}^\omega\omega]^{<\mu})(\exists\nu \in {}^\omega\omega)(\bigwedge\limits_{\rho\in X}\rho R\nu)$" *(so for* $R = <^*$ *this means* $\mathfrak{b} \ge \mu$*).*

3) Assume

(a) R *is a Borel two-place relation* [10] *on* ${}^\omega\omega$ *(we use* $R = \{(\rho,\nu) : \rho,\nu \in {}^\omega 2$ *and* $\rho^{-1}\{1\}, \nu^{-1}\{1\}$ *are infinite with finite intersection*$\}$, *noting that* ${}^\omega 2 \subseteq {}^\omega\omega$*)*

(b) σ,κ,θ *are cardinals and* $\kappa \le \theta \le \lambda$ *and* $\sigma^+ \ge \partial(\mathfrak{t})$ *with* $\mathrm{cf}(\partial(\mathfrak{t})) > \aleph_0$

(c) *if* $t_{i,\phi} \in L^{\mathfrak{t}}$ *for* $i < i(*), \phi < \sigma$ *and* $\kappa \le i(*) < \theta$ *and each* $\{t_{i,\phi} : \phi < \sigma\}$ *is* $\bar K$*-closed,* <u>*then*</u> *we can find* $t_\phi \in L^{\mathfrak{t}}$ *for* $\phi < \sigma$ *such that* $\{t_\phi : \phi < \sigma\} \subseteq L^{\mathfrak{t}}$ *is* $\bar K$*-closed and:*

$(*)$ *for every* $i < i(*)$ *for some* $j < \kappa, j \ne i$ *and the mapping* $t_{i,\phi} \mapsto t_{i,\phi}, t_{j,\phi} \mapsto t_\phi$ *is a partial isomorphism of* $(\mathfrak{t}, \bar K, \bar{\mathbb{Q}})$ *(see Definition 2.9 below).*

<u>*Then*</u> *in* $\mathbf{V}^{\mathbb{P}}$ *we have*

$\boxtimes^R_{\theta,\kappa}$ *if* $\rho_i, \nu_i \in {}^\omega\omega$ *for* $i < i(*)$ *and* $\kappa \le i(*) < \theta$ *and* $i \ne j \Rightarrow \nu_i R\rho_j$, <u>*then*</u> *we can find* $\rho \in {}^\omega\omega$ *such that* $i < i(*) \Rightarrow \nu_i R\rho$.

---

[7] here and below just enough absoluteness is enough, of course

[8] Why not ${}^\omega 2$? Just as this notation is more natural for $\mathfrak{d},\mathfrak{b}$, our main concern here.

[9] We can weaken Clause (c) by saying: for every set X of $< \mu$ names of reals there is $t \in L^{\mathfrak{t}}$ such that for each such name from X ....

[10] so R is defined in $\mathbf{V}$; if R is from $\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}\restriction K)}$, we need partial isomorphism (see below) of $(\mathfrak{t}, \bar{\mathbb{Q}})$ extending id_K

Proof. Straightforward, but being requested we give details:
1) Let $\underset{\sim}{\rho}$ be a $\mathbb{P}$-name of a member of $({}^\omega\omega)^{\mathbf{V}^{\mathbb{P}}}$, so as $\mathbb{P}$ satisfies the c.c.c. (see 2.6(B)(j)(α)), for each n there is a maximal anti-chain $\{p_{n,i} : i < i_n\}$ such that $p_{n,i}$ forces a value of $\underset{\sim}{\rho}(n)$ and, of course, i_n is countable. Let $M = \{a : a$ is a $\bar{K}$-countable subset of $L^{\mathfrak{t}}\}$ partially ordered by inclusion, so obviously M is closed under countable unions and $\cup\{a : a \in M\} = L^{\mathfrak{t}}$; and let $L_a = a$ for $a \in M$ so by $2.6(B)(i)(\beta)$ we have $p \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}) \Leftrightarrow p \in \cup\{\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}{\restriction}L_a) : a \in M\}$ but $\mathbb{P} = \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$, hence for $n < \omega, i < i_n$ for some $a_{n,i} \in M$ we have $p_{n,i} \in \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_{a_{n,i}})$. But M is $\aleph_1$-directed so for some $a \in M$ we have $\{a_{n,i} : n < \omega, i < i_n\} \subseteq \{c : c \leq_M a\}$. Also by 2.6(B)(e) we know $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}{\restriction}L_a) \lessdot \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}) = \mathbb{P}$, so $\underset{\sim}{\rho}$ is a $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L_a)$-name. Now by assumption (c) of what we are proving, as $L_a \subseteq L$ is $\bar{K}$-countable, we can find $t \in L^* \subseteq L^{\mathfrak{t}}$ such that $L_a \in I^{\mathfrak{t}}_t$. Also we know that $K_t \in I^{\mathfrak{t}}_t$ (see Definition 2.1(2)(c) hence $A =: K_t \cup L_a$ belongs to $I^{\mathfrak{t}}_t$ and is $\bar{K}$-closed; and easily also $B = A \cup \{t\}$ is $\bar{K}$-closed.

Clearly $A \subseteq B \subseteq L^{\mathfrak{t}}$ are $\bar{K}$-closed so as above $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction A) \lessdot \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction B) \lessdot \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}) = \mathbb{P}$ and $\underset{\sim}{\rho}$ is a $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}{\restriction}A)$-name (hence also a $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction B)$-name) of a member of ${}^\omega\omega$.

Now by assumption (d), in $\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}\restriction A)}$ we have $\Vdash_{\mathbb{Q}_{t,\underset{\sim}{\eta}_t}}$ "$\underset{\sim}{\rho} R \underset{\sim}{\nu}_t$", hence by 2.7(2) we know that $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction B) = \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction A) * \underset{\sim}{\mathbb{Q}}_{t,\underset{\sim}{\eta}_t}$, so together $\Vdash_{\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}\restriction B)}$ "$\underset{\sim}{\rho} R \underset{\sim}{\nu}_t$" hence by the previous sentence and obvious absoluteness we have $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\rho} R \underset{\sim}{\nu}_t$". So as $\underset{\sim}{\rho}$ was any $\mathbb{P}$-name of a member of $({}^\omega\omega)^{\mathbf{V}^{\mathbb{P}}}$ we are done.
1A) Same proof.
2) So assume $p \Vdash_{\mathbb{P}}$ "$\underset{\sim}{X} \subseteq {}^\omega\omega$ has cardinality $< \mu$". As we can increase p without loss of generality for some $\theta < \mu$ we have $p \Vdash_{\mathbb{P}}$ "$|\underset{\sim}{X}| = \theta$" so we can find a sequence $\langle \underset{\sim}{\rho}_\alpha : \alpha < \theta\rangle$ of $\mathbb{P}$-names of members of $({}^\omega\omega)^{\mathbf{V}^{\mathbb{P}}}$ such that $p \Vdash_{\mathbb{P}}$ "$\underset{\sim}{X} = \{\underset{\sim}{\rho}_\alpha : \alpha < \theta\}$". Let $\{p_{\alpha,n,i} : i < i_{\alpha,n}\}$ be a maximal antichain of $\mathbb{P}$, with $p_{\alpha,n,i}$ forcing a value to $\underset{\sim}{\rho}_\alpha(n)$ and $i_{\alpha,n}$ countable.

Define $M = \{a \subseteq L^{\mathfrak{t}} : a$ is $\bar{K}$-countable$\}$, so for each $\alpha < \theta, n < \omega, i < i_{\alpha,n}$ for some $a_{\alpha,n,i} \in M$ we have $p_{\alpha,n,i} \in \mathrm{Lim}_t(\bar{\mathbb{Q}} \restriction L_{a_{\alpha,n,i}})$. So[11] for some $\bar{K}$-closed $L^{**} \subseteq L^{\mathfrak{t}}$ and $t \in L^{\mathfrak{t}}$ we have $L^{**} \in I^{\mathfrak{t}}_t$ and $L_{a_{\alpha,n,i}} \subseteq L^{**}$ for $\alpha < \theta, n < \omega, i < i_{\alpha,n}$. We now continue as in part (1).
3) So assume $i(*) \in [\kappa, \theta)$ and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\nu}_i, \underset{\sim}{\rho}_i \in {}^\omega\omega$ and $i \neq j \Rightarrow \underset{\sim}{\nu}_i R \underset{\sim}{\rho}_j$". So as above we can find $\bar{K}$-countable $K^*_i \subseteq L^{\mathfrak{t}}$ such that $\underset{\sim}{\nu}_i, \underset{\sim}{\rho}_i$ are $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction K^*_i)$-names; without loss of generality $K^*_i \neq \emptyset$ and K^*_i has cardinality $< \partial(\mathfrak{t})$ hence $\leq \sigma$. Let $\langle t_{i,\phi} : \phi < \sigma\rangle$ be a list of the members of K^*_i possibly with repetitions. Let $f_{i,j}$ be the mapping from K^*_j to K^*_i defined by $f_{i,j}(t_{j,\phi}) = t_{i,\phi}$ if well defined.

We define two-place relations E_1, E_2 on $i(*)$ and on $i(*) \times i(*)$ respectively by:

- (a) iE_1j <u>iff</u> $f_{i,j}$ is a well defined partial isomorphism of $(\mathfrak{t}, \bar{K}, \bar{\mathbb{Q}})$ such that $\hat{f}_{i,j}$ (see claim (B) of 2.9 below) maps $(\underset{\sim}{\rho}_j, \underset{\sim}{\nu}_j)$ to $(\underset{\sim}{\rho}_i, \underset{\sim}{\nu}_i)$
- (b) $(i_1, i_2)E_2(j_1, j_2)$ <u>iff</u> $i_1E_1j_1, i_2E_1j_2$ and $f_{i_1,j_1} \cup f_{i_2,j_2}$ is a partial isomorphism of $(\mathfrak{t}, \bar{\mathbb{Q}})$.

Easily

$\otimes(i)$ E_1, E_2 are equivalence relations over their domains

[11] In the weaker version for some t for every α for some $A \in I^{\mathfrak{t}}_t$ ....

(ii) $f_{j,i} = f^{-1}_{i,j}$ or both are not well defined.

Now we apply assumption (c), and get $\langle t_\phi : \phi < \sigma\rangle$ and let $K^* = \{t_\phi : \phi < \sigma\}$. By $(*)$ of clause (c) and clause (A)(b) of Definition 2.9 below for any $i, j < i(*)$ clearly $K^*_i \cup K^*_j$ and $K^*_i \cup K^*$ are $\bar{K}$-closed (see the definition below). For any $i < i(*)$ let $j_i < \kappa$ be as in $(*)$ of clause (c) which means: $j_i \neq i$ and the following mapping g_i is a partial isomorphism of $(\mathfrak{t}, \bar{K}, \bar{\mathbb{Q}})$: $\mathrm{Dom}(g_i) = \{t_{i,\phi}, t_{j_i,\phi} : \phi < \sigma\}, g_i(t_{i,\phi}) = t_{i,\phi}, g_i(t_{j,\phi}) = t_\phi$.

Let $\underset{\sim}{\nu}, \underset{\sim}{\rho}$ be $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}{\restriction}K^*)$-names such that for some, equivalently any $i, \hat{g}_i$ maps $\underset{\sim}{\nu}_{j_i}, \underset{\sim}{\rho}_{j_i}$ to $\underset{\sim}{\nu}, \underset{\sim}{\rho}$ respectively (this is O.K. as for any i_1, i_2 we have $j_{i_1} E_1 j_{i_2}$ because $g_{i_2} \circ f_{j_{i_2}, j_{i_1}} = g_{i_1} \restriction K^*_{j_{i_1}}$). Now for any $i < i(*)$, as $j_i \neq i$, we know $\Vdash_{\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}{\restriction}(K^*_i \cup K^*_{j_i}))}$ "$\underset{\sim}{\nu}_i R \underset{\sim}{\rho}_{j_i}$", so applying g_i we have $\Vdash_{\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}(K^*_i \cup K^*))}$ "$\underset{\sim}{\nu}_i R \underset{\sim}{\rho}$". So we have proved $\boxtimes^R_{\theta,\kappa}$. $\square_{2.8}$

In 2.9 below we note that isomorphisms (or embeddings) of $\mathfrak{t}$'s tend to induce isomorphisms (or embeddings) of $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$, and deal (in 2.10, 2.11) with some natural operations. In 2.9 we could use two $\mathfrak{t}$'s, but this can trivially be reduced to one.

Definition/Claim 2.9. Assume that $\mathfrak{t}, \bar{K}$ and $\bar{\mathbb{Q}} = \langle \underset{\sim}{\mathbb{Q}}_{t,\underset{\sim}{\eta}_t} : t \in L^{\mathfrak{t}}\rangle$ are as in 2.6. By induction on ζ we define and prove [12]

(A) [Def] we say f is a partial isomorphism of $(\mathfrak{t}, \bar{\mathbb{Q}})$ of Depth $\le \zeta$ if: (omitting ζ means for some ordinal ζ; writing $\mathfrak{t}$ instead of $(\mathfrak{t}, \bar{K}, \bar{\mathbb{Q}})$ means we assume $\mathbb{Q}_{t,\underset{\sim}{\eta}_t} = \mathbb{Q}$, i.e. constant, $K_t = \emptyset$ for every $t \in L^{\mathfrak{t}}$ and may say "$\mathfrak{t}$-partial isomorphism")

- (a) f is a partial one-to-one function from $L^{\mathfrak{t}}$ to $L^{\mathfrak{t}}$
- (b) $\mathrm{Dom}(f), \mathrm{Rang}(f)$ are $(\mathfrak{t}, \bar{K})$-closed sets of depth $\le \zeta$
- (c) for $t \in \mathrm{Dom}(f)$ and $A \subseteq \mathrm{Dom}(f)$ we have $A \in I^{\mathfrak{t}}_t \Leftrightarrow f''(A) \in I^{\mathfrak{t}}_{f(t)}$
- (d) for $t \in \mathrm{Dom}(f)$, we have: f maps K_t onto $K_{f(t)}$ and $f \restriction K_t$ maps $\underset{\sim}{\eta}_t$ to $\underset{\sim}{\eta}_{f(t)}$, more exactly the isomorphism $\hat{f}$ which f induces from $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction K_t)$ onto $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction K_{f(t)})$ does this.

(B) [Claim] f induces naturally an isomorphism which we call $\hat{f}$ from $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}{\restriction}\mathrm{Dom}(f))$ onto $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}}{\restriction}\mathrm{Rang}(f))$.

Proof. Straightforward, recalling we are assuming that $\bar{\varphi}_t$ is definable from $\underset{\sim}{\eta}_t$. $\square$

Definition 2.10. 1) We say $\mathfrak{t} = \mathfrak{t}^1 + \mathfrak{t}^2$ if

- (a) $L^{\mathfrak{t}} = L^{\mathfrak{t}^1} + L^{\mathfrak{t}^2}$ (as linear orders)
- (b) for $t \in L^{\mathfrak{t}^1}, I^{\mathfrak{t}^1}_t = I^{\mathfrak{t}}_t$
- (c) for $t \in L^{\mathfrak{t}^2}, I^{\mathfrak{t}^2}_t = \{A \subseteq L^{\mathfrak{t}} : A \cap L^{\mathfrak{t}^2} \in I^{\mathfrak{t}^2}_t\}$.

So $\mathfrak{t}^1 + \mathfrak{t}^2$ is well defined if $\mathfrak{t}^1, \mathfrak{t}^2$ are disjoint, i.e. $L^{\mathfrak{t}^1} \cap L^{\mathfrak{t}^2} = \emptyset$.
2) We say $\mathfrak{t}^1 \le_{\mathrm{wk}} \mathfrak{t}^2$ iff

[12] if $K_t = \emptyset$ and all $\mathbb{Q}_{t,\eta}$ have the same definition of forcing notion, as in our main case, we can separate the definition and claim.

(a) $L^{\mathfrak{t}^1} \subseteq L^{\mathfrak{t}^2}$ (as linear orders) and $t \in L^{\mathfrak{t}^1} \Rightarrow I_t^{\mathfrak{t}^1} \subseteq I_t^{\mathfrak{t}^2}$

(b) if [13] $s \in L^{\mathfrak{t}^1}$ then $I_s^{\mathfrak{t}^1} = \{A \in I_t^{\mathfrak{t}^2} : A \subseteq L^{\mathfrak{t}^1}\}$

3) If $\langle \mathfrak{t}^\zeta : \zeta < \xi \rangle$ is $\leq_{\rm wk}$-increasing, ξ a limit ordinal, we define $\mathfrak{t}^\xi =: \bigcup\limits_{\zeta<\xi} \mathfrak{t}^\zeta$ by

$$L^{\mathfrak{t}^\xi} = \bigcup_{\zeta<\xi} L^{\mathfrak{t}^\zeta} \qquad \text{(as linear orders)}$$

$$I_t^{\mathfrak{t}^\xi} = \cup\{I_t^{\mathfrak{t}^\zeta} : \zeta < \xi \text{ and } t \in L_\zeta^{\mathfrak{t}}\}$$

Clearly $\zeta < \xi \Rightarrow \mathfrak{t}^\zeta \leq_{\rm wk} \mathfrak{t}^\xi$. Such $\mathfrak{t}^\xi$ is called the limit of $\langle \mathfrak{t}^\zeta : \zeta < \xi \rangle$; now a $\leq_{\rm wk}$-increasing sequence $\langle \mathfrak{t}^\zeta : \zeta < \xi \rangle$ is called continuous if for every limit ordinal $\delta < \xi$ we have $\mathfrak{t}^\delta = \bigcup\limits_{\zeta<\delta} \mathfrak{t}^\zeta$.

4) If $\langle \mathfrak{t}^\zeta : \zeta < \xi \rangle$ are pairwise disjoint (that is $\zeta \neq \varepsilon \Rightarrow L^{\mathfrak{t}^\zeta} \cap L^{\mathfrak{t}^\varepsilon} = \emptyset$) we define $\sum\limits_{\zeta<\xi} \mathfrak{t}^\zeta$ by induction on ξ naturally: for $\xi = 1$ it is $\mathfrak{t}^0$, for ξ limit it is $\bigcup\limits_{\varepsilon<\xi} (\sum\limits_{\zeta<\varepsilon} \mathfrak{t}^\zeta)$ and for $\xi = \varepsilon + 1$ it is $(\sum\limits_{\zeta<\varepsilon} \mathfrak{t}^\zeta) + \mathfrak{t}^\varepsilon$, so $\xi_1 \leq \xi_2 \Rightarrow \sum\limits_{\zeta<\xi_1} \mathfrak{t}^\zeta \leq_{\rm wk} \sum\limits_{\zeta<\xi_2} \mathfrak{t}^\zeta$ (even an initial segment).

5) We can replace in 0) - 4) above $\mathfrak{t}^\zeta$ by $(\mathfrak{t}^\zeta, \bar{K}^\zeta)$.

Claim 2.11. *Let $\mathfrak{t}$ be an* FSI-*template.*

1) If $L^{\mathfrak{t}} = \emptyset$ or just is well ordered <u>then</u> $\mathfrak{t}$ is smooth.

2) If $\mathfrak{t}^1, \mathfrak{t}^2$ are disjoint FSI-*templates, <u>then</u> $\mathfrak{t}^1 + \mathfrak{t}^2$ is an* FSI-*template and $\ell \in \{1,2\} \Rightarrow \mathfrak{t}^\ell \leq_{\rm wk} \mathfrak{t}^1 + \mathfrak{t}^2$.*

3) If $\mathfrak{t}^1, \mathfrak{t}^2$ are disjoint smooth FSI-*templates <u>then</u> $\mathfrak{t} = \mathfrak{t}^1 + \mathfrak{t}^2$ is a smooth* FSI-*template; moreover, ${\rm Dp}_{\mathfrak{t}}(L^{\mathfrak{t}}) \leq {\rm Dp}_{\mathfrak{t}^1}(L^{\mathfrak{t}^1}) + {\rm Dp}_{\mathfrak{t}^2}(L^{\mathfrak{t}^2})$ and ${\rm Dp}_{\mathfrak{t}}(L^{\mathfrak{t}^\ell}) = {\rm Dp}_{\mathfrak{t}^\ell}(L^{\mathfrak{t}^\ell})$.*

4) If $\langle \mathfrak{t}^\zeta : \zeta < \xi \rangle$ is an $\leq_{\rm wk}$-increasing (2.10(2)) sequence of FSI-*templates and ξ is a limit ordinal, <u>then</u> $\mathfrak{t}^\xi =: \bigcup\limits_{\zeta<\xi} \mathfrak{t}^\zeta$ is an* FSI-*template and $\zeta < \xi \Rightarrow \mathfrak{t}^\zeta \leq_{\rm wk} \mathfrak{t}^\xi$.*

5) If $\langle \mathfrak{t}^\zeta : \zeta < \xi \rangle$ is an increasing continuous (see Definition 2.10(3)) sequence of smooth FSI-*templates and ξ is a limit ordinal, <u>then</u> $\mathfrak{t}^\xi =: \bigcup\limits_{\zeta<\xi} \mathfrak{t}^\zeta$ is a smooth* FSI-*template and $\zeta < \xi \Rightarrow \mathfrak{t}^\zeta \leq_{\rm wk} \mathfrak{t}^\xi$ and ${\rm Dp}_{\mathfrak{t}^\xi}(L^{\mathfrak{t}^\xi}) \leq \sup\{{\rm Dp}_{\mathfrak{t}[\zeta]}(L^{\mathfrak{t}[\zeta]}) + 1 : \zeta < \xi\}$.*

6) If $\langle \mathfrak{t}^\zeta : \zeta < \xi \rangle$ is a sequence of pairwise disjoint [smooth] FSI-*templates, <u>then</u> $\sum\limits_{\zeta<\xi} \mathfrak{t}^\zeta$ is a [smooth]* FSI-*template and $\langle \sum\limits_{\zeta<\varepsilon} \mathfrak{t}^\zeta : \varepsilon \leq \xi \rangle$ is increasing continuous.*

7) In parts (1)-(6) we can expand $\mathfrak{t}^\zeta$ by $\bar{K}^\zeta$.

8) Assume $\mathbf{J}$ is a linear order, $\mathfrak{t}_x$ is a smooth FSI-template for every $x \in \mathbf{J}$ and $\langle L^{\mathfrak{t}_x} : x \in \mathbf{J} \rangle$ are pairwise disjoint (for notational simplicity) and we define $\mathfrak{t}$ by: $L^{\mathfrak{t}} = \sum\limits_{x\in\mathbf{J}} L^{\mathfrak{t}_x}$ (so $L^{\mathfrak{t}} \models s < t$ iff $(\exists x,y)(s \in L^{\mathfrak{t}_x} \wedge t \in L^{\mathfrak{t}_y} \wedge x <_{\mathbf{J}} y) \vee (\exists x \in \mathbf{J})(L^{\mathfrak{t}_x} \models s < t)$) and $I_t^{\mathfrak{t}} = \{A \subseteq L^{\mathfrak{t}} : (\forall s \in A)(s <_{L^{\mathfrak{t}}} t)$ and letting $x \in \mathbf{J}$ be such that $t \in \mathfrak{t}^x$ we have $A \cap L^{\mathfrak{t}_x} \in I_t^{\mathfrak{t}_x}$ and $\{y : y <_{\mathbf{J}} x, A \cap L^{\mathfrak{t}_y} \neq \emptyset\}$ is finite$\}$. Then $\mathfrak{t}$ is a smooth FSI-template (we can expand by $\bar{K}$'s) (use in §3).

Proof. Easy, e.g. part (3) is proved by induction on ${\rm Dp}_{\mathfrak{t}}(L^{\mathfrak{t}})$ and part (6) by induction on ξ and in part (7) let M be $[\mathbf{J}]^{<\aleph_0}$ ordered by inclusion and $L_{\{x(1),\ldots,x(n)\}} = \cup\{L^{\mathfrak{t}_{x(\ell)}} : \ell = 1,\ldots,n\}$ for any $x(1),\ldots,x(n) \in \mathbf{J}$. $\square_{2.11}$

[13] we may restrict ourselves to FSI-templates $\mathfrak{t}$ of globally countable, i.e., such that $A \in I_t^{\mathfrak{t}}$ and $t \in L^{\mathfrak{t}} \Rightarrow |A| \leq \aleph_0$

* * *

Discussion 2.12. 1) To prove our desired result CON($\mathfrak{a} > \mathfrak{d}$) we need to construct an FSI-template $\mathfrak{t}$ of the right form. Now we do it using a measurable cardinal. The point is that if we are given $\langle \langle t_{i,n} : n < \omega \rangle : i < i(*) \rangle, L^{\mathfrak{t}}, i(*) \geq \kappa$ and D is a normal ultrafilter on κ, then in $\mathfrak{t}^\kappa/D$ the ω-sequence $\langle \langle t_{i,n} : i < \kappa \rangle / D : n < \omega \rangle$ is as required in 2.8(3)(c), considering $\mathfrak{t}^\kappa/D$ an extension of $\mathfrak{t}$.
2) We shall deal with $\mathfrak{s}$ only in 2.18(2).
3) Note that our main old conclusion (i.e. 2.18(1)) has two proofs. The first is shorter and depends on §1 and 2.16, 2.17. The second is longer but does not.

Definition 2.13. 1) For a $\mathbf{q} \in \mathbf{Q}$ and $\partial = \partial(\mathbf{q}) < \kappa$ and an ∂^+-complete ultrafilter D on κ (hence $(2^\partial)^+$-complete), we define $\mathfrak{t}^* =: \mathfrak{t}^\kappa/D, \mathbf{j}_{D,\mathfrak{t}}$ and $\mathbf{j}_{D,\mathfrak{t}}(\mathfrak{t})$ as follows:

(a) we define $\mathfrak{t}^*$ by:

$$L^{\mathfrak{t}^*} = (L^{\mathfrak{t}})^\kappa/D \text{ as a linear order}$$

and if $t^* = \langle t_i : i < \kappa \rangle / D$ where $t_i \in L^{\mathfrak{t}}$ then we let $I^{\mathfrak{t}^*}_{t^*} = \{A :$ we can find $A_i \in I^{\mathfrak{t}}_{t_i}$ for $i < \kappa$ such that $A \subseteq \prod\limits_{i<\kappa} A_i/D\}$

(b) We then let $\mathbf{j}_{D,\mathfrak{t}}$ be the canonical embedding of $\mathfrak{t}$ into $\mathfrak{t}^\kappa/D$ that is $\mathbf{j}_{D,\mathfrak{t}}(t) = \langle t : i < \kappa \rangle / D$ for every $t \in L^{\mathfrak{t}}$ and

(c) let $\mathfrak{t}' = \mathbf{j}_{D,\mathfrak{t}}(\mathfrak{t})$ be defined by $L^{\mathfrak{t}'} = L^{\mathfrak{t}^*} \restriction \{\mathbf{j}_{D,\mathfrak{t}}(s) : s \in L^{\mathfrak{t}}\}, I^{\mathfrak{t}'}_{\mathbf{j}_{D,\mathfrak{t}}(s)} = \{\{\mathbf{j}_{D,\mathfrak{t}}(t) : t \in A\} : A \in I^{\mathfrak{t}}_s\}$.

2) Similarly for $\mathbf{q} \in \mathbf{Q}$ instead $\mathfrak{t}$.

Remark 2.14. We may allow $\partial(\mathbf{q}) \geq \kappa$ but presently not worth the trouble.

Claim 2.15. *In Definition 2.13:*
1) $\mathfrak{t}^\kappa/D$ is also an FSI- *template and $\mathbf{j}_{D,\mathfrak{t}}(\mathfrak{t}) \leq_{\rm wk} \mathfrak{t}^\kappa/D$ and $\mathbf{j}_{D,\mathfrak{t}}$ is an isomorphism from $\mathfrak{t}$ onto $\mathbf{j}_{D,\mathfrak{t}}(\mathfrak{t})$.*
2) If $\mathfrak{t}$ is a smooth FSI*-template <u>then</u> $\mathfrak{t}^\kappa/D$ is a smooth* FSI*-template.*
3) Moreover, $\mathrm{Dp}_{\mathfrak{t}^\kappa/D}(L^{\mathfrak{t}^\kappa/D}) \leq (\mathrm{Dp}_{\mathfrak{t}}(L^{\mathfrak{t}}))^\kappa/D$.
4) Similarly we define $\mathbf{q}^\kappa/D$ for $\mathbf{q} \in \mathbf{Q}$; so u_t is increased if $u^{\mathbf{q}}_t$. is of cardinality $\geq \kappa$ and similarly $K^{\mathbf{q}}_t$

Proof. Straightforward. $\square_{2.15}$

Now 2.16, 2.17 below are used only in the short proof of 2.18 depending on §1, so you may ignore them.

Definition 2.16. Fix $\aleph_0 < \kappa < \mu = \mathrm{cf}(\mu) < \lambda = \mathrm{cf}(\lambda) = \lambda^\kappa$ and D a κ-complete (or just $(2^{\aleph_0})^+$-complete) uniform ultrafilter on κ. We define by induction on $\zeta \leq \lambda$, a smooth FSI-template $\mathfrak{t}_{\gamma,\zeta}$ for $\gamma < \mu$ such that:

(a) $\mathfrak{t}_{\gamma,\zeta}$ is a smooth FSI-template
(b) if $\gamma_1 \neq \gamma_2$ then $\mathfrak{t}_{\gamma_1,\zeta}, \mathfrak{t}_{\gamma_2,\zeta}$ are disjoint, i.e. $L^{\mathfrak{t}_{\gamma_1,\zeta}} \cap L^{\mathfrak{t}_{\gamma_2,\zeta}} = \emptyset$
(c) for $\xi < \zeta$ we have $\mathfrak{t}_{\gamma,\xi} \leq_{\rm wk} \mathfrak{t}_{\gamma,\zeta}$
(d) if ζ is limit then $\mathfrak{t}_{\gamma,\zeta} = \bigcup\limits_{\xi<\zeta} \mathfrak{t}_{\gamma,\xi}$, see 2.10(3), 2.11(6).

(e) if $\zeta = \xi + 1$ and ξ is even, then there is an isomorphism $\mathbf{j}_{\gamma,\zeta}$ from $\sum\limits_{\beta\le\gamma} \mathfrak{t}_{\beta,\xi}$ onto $\mathfrak{t}_{\gamma,\zeta}$ which is the identity over $\mathfrak{t}_{\gamma,\xi}$

(f) if $\zeta = \xi + 1$ and ξ is odd, then there is an isomorphism $\mathbf{j}_{\gamma,\zeta}$ from $(\mathfrak{t}_{\gamma,\xi})^\kappa/D$ onto $\mathfrak{t}_{\gamma,\zeta}$ which extends the inverse of $\mathbf{j}_{D,\mathfrak{t}_{\gamma,\xi}}$.

Observation 2.17. *The definition is 2.16 is legitimate.*

Proof. By the previous claims. $\square_{2.17}$

Conclusion 2.18. *Assume: κ is measurable,*[14] $\kappa < \mu = \mathrm{cf}(\mu) < \lambda = \mathrm{cf}(\lambda) = \lambda^\kappa$.
1) For some c.c.c. forcing notion $\mathbb{P}$ of cardinality λ, in $\mathbf{V}^{\mathbb{P}}$ we have $\mathfrak{a} = \lambda$, $\mathfrak{b} = \mathfrak{d} = \mu$ hence $\mathfrak{s} \le \mu$.
2) If in addition $\partial = \mathrm{cf}(\partial) < \kappa$ then for some $\mathbb{P}$ as above in addition we have $\mathfrak{s} \ge \partial$ (hence $\partial \le \mathfrak{s} \le \mu$)

Proof. 1) Short Proof: (depending on §1).

Let $\mathfrak{t}_{\gamma,\zeta}$ (for $\gamma < \mu, \zeta \le \lambda$) be as in 2.16. Let $\mathfrak{t} = \sum\limits_{\gamma<\mu} \mathfrak{t}_{\gamma,\lambda}$ and let $\bar{K} = \langle K_t : t \in L^{\mathfrak{t}}\rangle, K_t = \emptyset$ and let $\bar{\mathbb{Q}} = \langle \underset{\sim}{\mathbb{Q}}_t : t \in L^{\mathfrak{t}}\rangle$ with $\underset{\sim}{\mathbb{Q}}_t$ being constantly the dominating real forcing (= Hechler forcing).

Lastly, let $\mathbb{P} = \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}})$.

The rest is as in the end of §1. But if we like to use 2.6, etc. we need

$\boxplus$ $\mathbb{Q}_{\mathrm{dom}}$ is as required in 2.6(A)(a)(i)-(iv), i.e. def - c.c.c.

We elaborate concerning why $\mathbb{Q}_{\mathrm{dom}}$ satisfying sub-clause (iv) of the full definition of 2.6(A)(a).

Given p_ℓ assume

(a) $\mathbb{P}_0 \lessdot \mathbb{P}_\ell \lessdot \mathbb{P}_3$ (for $\ell = 1, 2$) be c.c.c. forcing
(b) $\underset{\sim}{\mathbb{Q}}_\ell$ the $\mathbb{P}_\ell$-name of $\mathbb{Q}_{\mathrm{dom}}$ with the generic $\underset{\sim}{\nu}_\ell$, (in a sense they are the same name)
(c) $(p_\ell, \underset{\sim}{q}_\ell) \in \mathbb{P}_\ell * \underset{\sim}{\mathbb{Q}}_\ell$ for $\ell = 0, 1, 2$
(d) $(p_0, \underset{\sim}{q}_0) \Vdash$ "$(p_\ell, \underset{\sim}{q}_\ell) \in (\mathbb{P}_\ell * \underset{\sim}{\mathbb{Q}}_\ell)/(\mathbb{P}_0 * \mathbb{Q}_0)$" for $\ell = 1, 2$
(e) $p_3 \in \mathbb{P}_3$ is a common upper bound of p_1, p_2.

Of course

$(*)_1$ we can replace $(p_\ell, \underset{\sim}{q}_\ell)$ for $\ell < 3$ by $(p'_\ell, \underset{\sim}{q}'_\ell)$ above (p_ℓ, q_ℓ) for $\ell = 0, 1, 2$ and (c),(d) still holds
$(*)_2$ without loss of generality there is $\nu_1 \in {}^{\omega>}\omega$ such that $p_1 \Vdash$ "q_1 has trunk ν_1".

[Why? Let $\mathbf{G}_1 * \mathbf{G}^1 \subseteq \mathbb{P}_1 * \underset{\sim}{\mathbb{Q}}_1$ be generic over $\mathbf{V}$ such that $(p_1, \underset{\sim}{q}_1), (p_0, \underset{\sim}{q}_0) \in \mathbf{G}_1$, and $p_3 \in \mathbb{P}_3$ is a common upper bound of p_1, p_2.

We can find ν_1 and $p'_1 \in \mathbf{G}_1$ above p_1 such that $p'_1 \Vdash_{\mathbb{P}_1}$ "$\mathrm{tr}(\underset{\sim}{q}_1) = \nu_1$". Let $q'_1 = q_1$ and choose $(p'_0, \underset{\sim}{q}'_0) \in \mathbf{G}_1 * \mathbf{G}^1$ above $(p_0, \underset{\sim}{q}_0)$ such that $(p'_0, \underset{\sim}{q}'_0) \Vdash$ "$(p'_1, \underset{\sim}{q}'_1) \in (\mathbb{P}_1 * \underset{\sim}{\mathbb{Q}}_1)/(\mathbb{P}_0 * \underset{\sim}{\mathbb{Q}}_0)$".

Let $(p'_2, \underset{\sim}{q}'_2) = (p_2, \underset{\sim}{q}_2)$, so clearly we are done.]

---

[14] Instead $\lambda = \lambda^\kappa$ it suffice to demand $\lambda = \lambda^{\aleph_0} = \lambda^\kappa/D$. This holds for any strong limit cardinal $> \kappa$ of cofinality $\neq \aleph_0, \neq \kappa$.

$(*)_3$ without loss of generality for some $\nu_2, p_2 \Vdash$ "$\mathrm{tr}(\underset{\sim}{q}'_2) = \nu_2$" (and $(*)_1$ still holds); we shall not repeat such statements.

[Similarly as in the proof of $(*)_2$ because in the proof there $(p_2, \underset{\sim}{q}_2)$ was not changed and we can interchange $\mathbb{P}_1, \mathbb{P}_2$.]

$(*)_4$ without loss of generality for some ν_0 of length $\geq \ell g(\nu_1), \ell g(\nu_2)$ we have $p_0 \Vdash$ "$\mathrm{tr}(\underset{\sim}{q}_0) = \nu_0$".

[Why? As we can just increase $(p_0, \underset{\sim}{q}_0)$, not change $p_1, \underset{\sim}{q}_1, p_2, \underset{\sim}{q}_2$.]

$(*)_5$ without loss of generality $\mathrm{tr}(\underset{\sim}{q}_\ell) = \nu_0$

[Why? By the properties of $\mathbb{Q}_\ell$.]

Now $\underset{\sim}{q}_1, \underset{\sim}{q}_2$ are two $\mathbb{P}_3$-names of members of $\underset{\sim}{\mathbb{Q}}_3$ with the same trunk hence $\Vdash_{\mathbb{P}_3}$ "$\underset{\sim}{q}_1, \underset{\sim}{q}_2$ are compatible" so we are done.

<u>Alternative presentation of the proof of 2.18(1), self contained not depending on 2.16, 2.17:</u>

We define an FSI-template $\mathfrak{t}^\zeta = \mathfrak{t}[\zeta]$ from $\mathbf{Q}_{\mathrm{dom}}$ for $\zeta \leq \lambda$ by induction on ζ.

<u>Case 1</u>: For $\zeta = 0$.

Let $\mathfrak{t}^\zeta$ be defined as follows:

$$L^{\mathfrak{t}[\zeta]} = \mu$$

$$I_\alpha^{\mathfrak{t}[\zeta]} = \{A : A \subseteq \alpha\} \text{ for } \alpha < \mu$$

<u>Case 2</u>: For $\zeta = \xi + 1$.

We choose $\mathfrak{t}^\zeta$ such that there is an isomorphism $\mathbf{j}_\zeta$ from $L^{\mathfrak{t}[\zeta]}$ onto $(L^{\mathfrak{t}[\xi]})^\kappa/D$, satisfying $\mathbf{j}_\zeta \restriction L^{\mathfrak{t}[\xi]}$ is the canonical embedding $\mathbf{j}_{D,\mathfrak{t}[\xi]}$, that if $x \in L^{\mathfrak{t}[\zeta]}$ then $\mathbf{j}_\zeta(x) = \langle x_\varepsilon : \varepsilon < \kappa \rangle / D \in (L^{\mathfrak{t}[\xi]})^\kappa/D$ and: $A \in I_x^{\mathfrak{t}[\zeta]}$ <u>iff</u> for some $\bar{A} = \langle A_\varepsilon : \varepsilon < \kappa \rangle$ we have $A_\varepsilon \in I_{x_\varepsilon}^{\mathfrak{t}[\xi]}$ and $\{y : y \in A\} \subseteq \{\langle y_\varepsilon : \varepsilon < \kappa \rangle / D : \{\varepsilon < \kappa : y_\varepsilon \in A_\varepsilon\} \in D\}$.

<u>Case 3</u>: ζ limit[15].

We choose $\mathfrak{t}^\zeta$ as follows:

$$L^{\mathfrak{t}[\zeta]} = \bigcup_{\xi < \zeta} L^{\mathfrak{t}[\xi]} \text{ as linear orders.}$$

$I_x^{\mathfrak{t}[\zeta]}$ is

<u>Subcase 3A</u>: If $x \in L^{\mathfrak{t}[0]}$ then $\{A : A \subseteq \{s : L^{\mathfrak{t}[\zeta]} \models \text{"} s < x \text{"}\}\}$.

<u>Subcase 3B</u>: If $x \notin L^{\mathfrak{t}[0]}$ but $x \in L^{\mathfrak{t}[\zeta]}$ then $I_x^{\mathfrak{t}[\zeta]}$ is [16] (we rely on $L^{\mathfrak{t}^0}$ is well ordered):

---

[15] we may do one of the following changes (but not both): (a) in subcase 3B use $I_x^{\mathfrak{t}[\zeta]} = \{A$: for some $\xi < \zeta, x \in L^{\mathfrak{t}^\xi}$ and $A \in I_x^{\mathfrak{t}^\xi}\}$ and/or (b) in sub-case 3A behave as in sub-case 3B.

[16] so members of $L^{\mathfrak{t}[0]}$ have the "veteranity privilege", i.e. "founding father right"; members t of $L^{\mathfrak{t}^0}$ have the maximal $I_t^{\mathfrak{t}[\zeta]}$.

$\{A:$ for some $\xi < \zeta$ we have $x \in L^{\mathfrak{t}^\xi}$ and if $y = \mathrm{Min}\{y \in L^{\mathfrak{t}^0} : L^{\mathfrak{t}[\zeta]} \models$ "$x < y$"$\}$ which is $\in L^{\mathfrak{t}[0]}$ (and is always well defined see clause (b) of $\oplus$ below) then $A \backslash \{t \in L^{\mathfrak{t}[\zeta]} : L^{\mathfrak{t}[\zeta]} \models$ "$t < z$" for some z such that $L^{\mathfrak{t}^0} \models$ "$z < y$"$\}$ belongs to $I_x^{\mathfrak{t}^\xi}$ (hence is $\subseteq L^{\mathfrak{t}^\xi}$)$\}$.

We now prove by induction on $\zeta \leq \lambda$ that:

$\oplus$ (a) $\mathfrak{t}^\zeta$ is an FSI-template

(b) $L^{\mathfrak{t}[0]}$ is a cofinal subset of $L^{\mathfrak{t}[\zeta]}$

(c) $\mathfrak{t}^\zeta$ is smooth

(d) $\mathfrak{t}^\xi \leq_{\mathrm{wk}} \mathfrak{t}^\zeta$ for $\xi < \zeta$

(e) if $x \in L^{\mathfrak{t}[\zeta]}$ then $\{z :$ for some $y \in L^{\mathfrak{t}^0}$ we have $L^{\mathfrak{t}[\zeta]} \models$ "$z \leq y < x$"$\} \in I_x^{\mathfrak{t}[\zeta]}$

(f) $L^{\mathfrak{t}[\zeta]}$ has cardinality $\leq (\mu + |\zeta|)^\kappa$

(g) we have $\mathfrak{t}^\zeta = \sum\limits_{\gamma < \mu} \mathfrak{s}^{\gamma,\zeta}$ where $\mathfrak{s}^{\gamma,\zeta} = \mathfrak{t}^\zeta {\restriction} \{x \in L^{\mathfrak{t}[\zeta]} : L^{\mathfrak{t}_\zeta} \models x < \gamma$ and $\beta \leq x$ if $\beta < \gamma\}$.

(h) the sequence $\langle \mathfrak{s}^{\gamma,\zeta} : \zeta \leq \lambda \rangle$ is $\leq_{\mathrm{wk}}$-increasing continuous.

[Why? Easy, e.g. why clauses (a)+(c) hold? For $\zeta = 0$ by 2.11(1). For $\zeta = \xi + 1$ by 2.15(2) noting that for $t \in L^{\mathfrak{t}[0]}$ the desired value of $I_t^{\mathfrak{t}}$ holds. For ζ limit, for any $t \in L^{\mathfrak{t}^0}$ clearly $\mathfrak{s}^{\gamma,\zeta}$ is the union of the increasing continuous sequence $\langle \mathfrak{s}^{\gamma,\varepsilon} : \varepsilon < \zeta \rangle$ hence is a smooth FSI-template by clause (h) and 2.11(5). Now also $\mathfrak{t}^\zeta$ is a smooth FSI-template by 2.11(6). So $\oplus$ holds indeed.]

Of course, we let $\bar{K}^\zeta = \langle K_t^\zeta : t \in L^{\mathfrak{t}^\zeta} \rangle, K_t^\zeta = \emptyset$ and $\mathbb{Q}_t$ is the dominating real forcing.

Lastly, let for $\zeta \leq \lambda, \mathbb{P}_\zeta = \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbb{Q}} \restriction L^{\mathfrak{t}^\zeta})$.

$\odot$ Now

(α) $\mathbb{P}_\lambda$ is a c.c.c. forcing notion of cardinality $\leq \lambda^{\aleph_0}$ hence $\mathbf{V}^{\mathbb{P}_\lambda} \models 2^{\aleph_0} \leq \lambda$ by 2.4(B)(j) as $\lambda = \lambda^\kappa$

(β) in $\mathbf{V}^{\mathbb{P}_\lambda}$ we have $\mathfrak{d} \leq \mu$, by 2.8(1) applied with $R = <^*$ and $L^* = L^{\mathfrak{t}[0]}$ using $(*)(b) + (e)$

(γ) in $\mathbf{V}^{\mathbb{P}_\lambda}$ we have $\mathfrak{b} \geq \mu$ by 2.8(2) applied with $R = <^*$

(δ) $\mathfrak{b} = \mathfrak{d} = \mu$ and $\mathfrak{a} \geq \mu$ by $(\beta) + (\gamma)$ as it is well known that $\mathfrak{b} \leq \mathfrak{d}$ and $\mathfrak{b} \leq \mathfrak{a}$.

[Why? e.g. why clause (β) holds? Applying 2.8(1), we let $R = <^*, L^* = L^{\mathfrak{t}[0]}$ and we have to verify clauses (a)-(d) there. They are easy, e.g. for clause (c) there, if $A \subseteq L^{\mathfrak{t}}$ is $\bar{K}$-countable then there is $t \in L^*$ as promised because $L^{\mathfrak{t}[0]}$ is cofinal and is of order type μ which is a regular uncountable cardinal.]

But in order to sort out the value of $\mathfrak{a}$ we intend to use 2.8(3) with θ there chosen as λ here.

But why the demand (c) from 2.8(3) holds? Recall that every $A \in L^{\mathfrak{t}[\zeta]}$ is $\bar{K}$-closed. So assume $i(*) \in [\kappa, \lambda)$ and $t_{i,n} \in L^{\mathfrak{t}^\lambda}$ for $i < i(*), n < \omega$ be given. As λ is regular $> i(*)$, necessarily for some $\xi < \lambda$ we have $\{t_{i,n} : i < i(*), n < \omega\} \subseteq L^{\mathfrak{t}^\xi}$.

Now let $t_n \in L^{\mathfrak{t}^{\xi+1}}$ be such that $\mathbf{j}_{\xi+1}(t_n) = \langle t_{i,n} : i < \kappa\rangle/D$; easily $\langle t_n : n < \omega\rangle$ is as required (note that the number of isomorphism types of ω-sequences $\langle t_n : n < \omega\rangle$ in $\mathfrak{t}$ is trivially [17] $\le \beth_2$).

So

(ε) in $\mathbf{V}^{\mathbb{P}_\lambda}$ we have $\mathfrak{a} \ge \kappa \Rightarrow \mathfrak{a} \ge \lambda$ by 2.8(3), see there.

We are assuming $\kappa \le \mu$ and by $\odot(\gamma)$ we have $\mu \le \mathfrak{b}$ and always $\mathfrak{b} \le \mathfrak{a}$ so together $\kappa \le \mathfrak{a}$. Recallin (ε) we are done.

2) We indicate how to adapt the second proof of part (1). For ∂ a regular uncountable cardinal we consider only $\mathbf{q} \in \mathbf{Q}^{\rm cln}_\partial$ which mean:

$\boxplus^1_{\mathbf{q}}$ Let $\mathbf{q} \in \mathbf{Q}_{\rm cln}$ mean
- (a) $\mathbf{q} \in \mathbf{Q}_{\rm fsi}$
- (b) $\partial(\mathbf{q}) \le \partial$
- (c) for every $t \in L^{\mathbf{q}}$ one of the following occurs
  - (α) $K^{\mathbf{q}}_t = \emptyset$ and $\mathbb{Q}^{\mathbf{q}}_t$ is dominating real forcing= Hechler forcing
  - (β) $K^{\mathbf{q}}_t$ has cardinality $< \partial$ and $I^{\mathbf{q}}_t = \mathscr{P}(K^{\mathbf{q}}_t)$ and $\mathbb{Q}^{\mathbf{q}}_{t,\eta}$ is an explicitly linked $(< \partial)$-forcing notion with universe $\gamma^{\tilde{\mathbf{q}}}_t < \partial$; see below

Where

$\boxplus_2$ We say that the forcing notion $\mathbb{Q}$ is an explicitly linked $(< \partial)$-forcing notion with universe γ when:
- (a) the set of elements of $\mathbb{Q}$ is the ordinal γ
- (b) for each $n < \omega$ the set $\{\omega\alpha + n : \omega\alpha + n < \gamma\}$ is a set of pairwise compatible elements of $\mathbb{Q}$

Next

$\boxplus_3$ the relevant claims 2.6-2.11 apply for all $\mathbf{q} \in \mathbf{Q}^{\rm cln}_\partial$ with minor changes; mainly recalling $\boxplus^1_{\mathbf{q}}(d)(\beta)$.

We choose $\mathfrak{t}^\zeta, \langle \mathfrak{s}^{\alpha,\zeta} : \alpha < \mu\rangle$ by induction on $\zeta \le \lambda$ as we have defined $\mathfrak{t}^\zeta$ in the second proof of part (1), but the second case splits to two, that is:

Case 1 $\zeta = 0$

As above

Case 2 $\zeta = \xi + 1$ and ξ is even.

As in the successor case above

Case 3 ζ is a limit ordinal

As above

Case 4 $\zeta = \xi + 1$ and ξ is odd

Now let us define $\mathbf{q}_\zeta$. We let

$\odot$
- (a) $L^{\mathbf{q}[\zeta]} = L^{\mathbf{q}[\xi]} \cup \{(\mathbf{q}^\xi, \alpha, \varepsilon) : \varepsilon < (\mu + |\xi|)^\kappa\}$ and the order is defined by (in addition to the old order)
  - (α) $t = (\mathbf{q}^\xi, \alpha, \varepsilon)$ is below $\alpha + 1 \in \mu = L^{\mathfrak{t}[0]}$ above α
  - (β) moreover t is above any $s \in L^{\mathfrak{t}[\xi]}$ which is below $\alpha + 1$
  - (γ) we let $(\mathbf{q}^\xi, \alpha_1), \varepsilon_1) < (\mathbf{q}^\xi, \alpha_2), \varepsilon_2)$ iff $(\alpha_1 < \alpha_2) \vee (\alpha_1 = \alpha_2 \wedge \varepsilon_1 < \varepsilon_2)$
- (b) $\mathfrak{t}^\zeta, \langle \mathfrak{s}^{\alpha,\zeta} : \alpha < \mu\rangle$ are as in $\oplus$ above

[17] in fact, it is $\le 2^{\aleph_0}$ by the construction, but irrelevant here

(c) We define $\mathbf{q}^\zeta$ by induction on $\zeta \le \lambda$. The new point is when $\zeta = \xi+1, \xi$ odd.

(d) In this case for $\alpha < \mu$ let $\langle(\xi, \alpha, \gamma, L_{\xi,\alpha,\varepsilon}, \underset{\sim}{\mathbb{Q}}_{\xi,\alpha,\varepsilon}) : \varepsilon < (\mu+|\xi|)^\kappa\rangle$ list the quintuples $(\xi, \alpha, \gamma, L, \underset{\sim}{\mathbb{Q}})$ such that L is a $\bar{K}^{\mathfrak{t}[\xi]}$-closed subset of $L^{\mathfrak{s}[\xi,\alpha]}$ of cardinality $< \partial, \gamma < \partial$ and $\underset{\sim}{\mathbb{Q}}$ is a canonical $\mathrm{Lim}_{\mathbf{q}[\zeta]}(\mathbb{Q}^{\mathbf{q}[\partial]} \restriction L)$-name of a forcing notion as in $\boxplus_2$ with universe $\gamma\}$.

(e) lastly, if $t = (\mathbf{q}^\xi, \alpha, \varepsilon)$ we let $K_t = L_{\xi,\alpha,\varepsilon}$ and $\underset{\sim}{\mathbb{Q}}^{\mathbf{q}[\xi]}_t = \underset{\sim}{\mathbb{Q}}_{\xi,\alpha,\varepsilon}$

The rest should be clear. $\square_{2.18}$

§ 3. Eliminating the measurable

Without a measurable cardinal our problem is to verify condition (c) in 2.8(3). Toward this it is helpful to show that for some $\aleph_1$-complete filter D on κ, for any $i(*) \in [\kappa, \lambda)$ and $t_{i,\phi} \in L^{\mathfrak{t}}$, for $i < i(*), \phi < \sigma$, we have: for some $B \in D^+$ for every $j < i(*)$ some $A \in D$ satisfies: "for any $i_0, i_1 \in A \cap B$, the mapping $t_{j,\phi} \mapsto t_{j,\phi}$; $t_{i_0,\phi} \mapsto t_{i_1,\phi}$ is a partial isomorphism of $\mathfrak{t}$". So D behaves as an $\aleph_1$-complete ultrafilter for our purpose.

[If you know enough model theory, this is the problem of finding convergent sequences, see [She90, Ch.I,§2, II], [She09c, §2], [She09b]). The later had generalize what we know on stable first order T with $\kappa = \kappa_r(T)$ (see [She90, Ch.II] κ is regular and $\leq |T|^+$) any indiscernible sequence (equivalently set) $\langle \bar{a}_\alpha : \alpha < \alpha^* \rangle$ of cardinality $\geq \kappa$, is convergent; why? as for any $\bar{\mathbf{b}} \in {}^{\kappa>}\mathfrak{C}$, for all but $< \kappa$ ordinals $\alpha < \alpha^*, \bar{\mathbf{b}}\hat{\ }\bar{a}_\alpha$ has a fixed type so average is definable. The present is closed to [She78], [Sheb]. (The general case is harder to prove existence which we do there under the relevant assumptions).]

Claim 3.1. *Assume $2^{\aleph_0} < \mu = \mathrm{cf}(\mu) < \lambda = \mathrm{cf}(\lambda) = \lambda^{\aleph_0}$. Then for some $\mathbb{P}$ we have*

(a) *$\mathbb{P}$ is a c.c.c. forcing notion of cardinality λ*
(b) *in $\mathbf{V}^{\mathbb{P}}$ we have $\mathfrak{b} = \mathfrak{d} = \mu$ and $\mathfrak{a} = 2^{\aleph_0} = \lambda$.*

Remark 3.2. About combining 3.1 with the end of §2, that is adding $\partial = (2^\sigma)^+ < \mu$ and getting also $\sigma \leq \mathfrak{s} \leq \mu$ (and even $\mathfrak{s} < \mu$) see [S$^+$a], [S$^+$b] and more) and [S$^+$c].

Proof. We rely on 2.6 + 2.8. Let L_0^+ be a linear order isomorphic to λ, let L_0^- be a linear order anti-isomorphic to λ (and $L_0^- \cap L_0^+ = \emptyset$) and let $L_0 = L_0^- + L_0^+$.

Let $\mathbf{J}$ be the following linear order:

(a) its set of elements is ${}^{\omega>}(L_0)$
(b) the order is: $\eta <_{\mathbf{J}} \nu$ iff for some $n < \omega$ we have $\eta \restriction n = \nu \restriction n$ and $\ell g(\eta) = n \wedge \nu(n) \in L_0^+$ or $\ell g(\nu) = n \wedge \eta(n) \in L_0^-$ or we have $\ell g(\eta) > n \wedge \ell g(\nu) > n \wedge L_0 \models \eta(n) < \nu(n)$.

[See more on such orders Laver [Lav71] and [Shea, §2], [She09a, §5] but we are self contained.]

Note that

$\boxdot_1$ every interval of $\mathbf{J}$ as well as $\mathbf{J}$ itself has cardinality λ
$\boxdot_1^+$ if $\aleph_0 < \theta = \mathrm{cf}(\theta) < \lambda$ or $\theta = 1$ or $\theta = 0$ and $\langle t_i : i < \theta \rangle$ is a strictly decreasing sequence in $\mathbf{J}$ then $\mathbf{J} \restriction \{y \in \mathbf{J} : (\forall i < \theta)(y <_{\mathbf{J}} t_i)\}$ has cofinality λ if it is non-empty
$\boxdot_1^-$ the inverse of $\mathbf{J}$ satisfies $\boxdot_1^+$, moreover is isomorphic to $\mathbf{J}$
$\boxdot_2$ if $\theta = \mathrm{cf}(\theta) > \aleph_0$ and $s_\alpha, t_\alpha \in \mathbf{J}$ for $\alpha < \theta$ then we can find a function $f : \theta \to \theta$ which is regressive and a club E of θ such that: if $\alpha_\ell < \beta_\ell$ are from E for $\ell = 1, 2$ and $f(\alpha_1) = f(\beta_1) = f(\alpha_2) = f(\beta_2)$ then: $t_{\alpha_1} <_{\mathbf{J}} s_{\beta_1} \Leftrightarrow t_{\alpha_2} <_{\mathbf{J}} s_{\beta_2}$ and $t_{\alpha_1} = s_{\beta_1} \Leftrightarrow t_{\alpha_2} = s_{\beta_2}$ (we can add $t_{\alpha_1} <_{\mathbf{J}} t_{\beta_1} \Leftrightarrow t_{\alpha_2} <_{\mathbf{J}} t_{\beta_2}$, etc., but this can be deduced using the above several times).

We now define by induction on $\zeta < \mu$ an FSI-templates $\mathfrak{t}_\zeta = \mathfrak{t}[\zeta]$ such that

$\odot^1_\zeta$ the set of members of $L^{\mathfrak{t}_\zeta}$ is a set of finite sequences starting with ζ hence disjoint to $L^{\mathfrak{t}[\varepsilon]}$ for $\varepsilon < \zeta$; for $x \in L^{\mathfrak{t}[\zeta]}$ let $\xi(x) = \zeta$.

Defining $\mathfrak{t}_\zeta$:

Case 1: $\zeta = 0$ or ζ successor or $\mathrm{cf}(\zeta) = \aleph_0$.

$\odot_2$ Let $L^{\mathfrak{t}[\zeta]} = \{\langle \zeta \rangle\}$ and $I^{\mathfrak{t}[\zeta]}_{\langle \zeta \rangle} = \{\emptyset\}$.

Case 2: $\mathrm{cf}(\zeta) > \aleph_0$

First

$\odot_3$ Let $h_\zeta : \mathbf{J} \to \zeta$ be a function such that: $\varepsilon < \zeta \Rightarrow h^{-1}_\zeta\{\varepsilon\}$ is a dense subset of $\mathbf{J}$, specifically $\nu = \eta\char`^\langle s \rangle \in \mathbf{J} \wedge (\mathrm{otp}(L^+_0 \restriction \{t : t <_{L^+_0} s\}) = i < \zeta \vee \mathrm{otp}(\text{the inverse of } (L^-_0)_s) = i < \zeta) \Rightarrow h_\zeta(\eta\char`^\langle s \rangle) = i$ and otherwise $h_\zeta(\nu) = 0$, Let $h(\zeta, \eta) = h_\zeta(\eta)$ for $\eta \in \mathbf{J}$.

Second

$\odot_4$ The set of elements of $\mathfrak{t}_\zeta$, that is of $L^{\mathfrak{t}[\zeta]}$ is

$$\{\langle \zeta \rangle\} \cup \{\langle \zeta \rangle\char`^\langle \eta \rangle\char`^x : \eta \in \mathbf{J} \text{ and } x \in \bigcup_{\varepsilon \le h_\zeta(\eta)} L^{\mathfrak{t}_\varepsilon}\}.$$

Third

$\odot_5$ The order $<_{\mathfrak{t}_\zeta}$ defined by $\langle \zeta \rangle$ is maximal and:
$\langle \zeta \rangle\char`^\langle \eta_1 \rangle\char`^x_1 <_{\mathfrak{t}[\zeta]} \langle \zeta \rangle\char`^\langle \eta_2 \rangle\char`^x_2$ iff at least one of the following holds:
(a) $\eta_1 <_{\mathbf{J}} \eta_2$
(b) $\eta_1 = \eta_2 \wedge \xi(x_1) < \xi(x_2)$
(c) $(\eta_1 = \eta_2$
$\wedge\, \xi(x_1) = \xi(x_2) \wedge x_1 <_{\mathfrak{t}_{\xi(x_1)}} x_2)$.

Lastly,

$(*)_1$ for $y \in L^{\mathfrak{t}[\zeta]}$ we define the ideal $I = I^{\mathfrak{t}[\zeta]}_y$:

(α) if $y = \langle \zeta \rangle$ then $I = \{Y : Y \subseteq L^{\mathfrak{t}[\zeta]} \setminus \{\langle \zeta \rangle\}\}$

(β) if $y = \langle \zeta \rangle\char`^\langle \nu \rangle\char`^x$, then I is the family of countable sets Y satisfying the following conditions:

(i) $Y \subseteq L^{\mathfrak{t}[\zeta]}$

(ii) $(\forall z \in Y)(z <_{\mathfrak{t}[\zeta]} y)$

(iii) the set $\{\eta \in \mathbf{J} : (\exists x)(\langle \zeta \rangle\char`^\langle \eta \rangle\char`^x \in Y)\}$ is finite.

(iv) if $\nu <_{\mathbf{J}} \eta$ and $z \in L^{\mathfrak{t}[h(\nu)]}$ the $\langle \zeta \rangle\char`^\langle \eta \rangle\char`^z \notin Y$

(v) if $\eta \le_{\mathbf{J}} \nu$ then the set $\{z \in L^{\mathfrak{t}} : \langle \zeta \rangle\char`^\langle \eta \rangle\char`^z \in Y\}$ belongs to $I^{\mathfrak{t}[h(\zeta,\eta)]}_x$

Why is $\mathfrak{t}_\zeta$ really an FSI-template? We prove, of course, by induction on ζ that:

$(*)^2_\zeta$ (i) $L^{\mathfrak{t}_\zeta}$ is a linear order

(ii) $I^{\mathfrak{t}_\zeta}_t$ is an ideal of subsets of $\{s \in I^{\mathfrak{t}_\zeta}_t : s < t\}$

(iii) $\mathfrak{t}_\zeta$ is an FSI-template,

(iv) $\mathfrak{t}_\zeta$ is disjoint to $\mathfrak{t}_\varepsilon$ for $\varepsilon < \zeta$

[Why? By 2.11(8) and looking at the definitions.]

Next we prove by induction on ζ, that $\mathfrak{t}_\zeta$ is a smooth FSI-template. Arriving at ζ

$(*)^3_\zeta$ for $\eta \in \mathbf{J}$ and $\varepsilon \leq h_\zeta(\eta) + 1$, we have $\mathfrak{t}_\zeta \restriction \{\langle\zeta\rangle\hat{}\langle\eta\rangle\hat{}\rho : \rho \in \bigcup\limits_{\xi<\varepsilon} \mathfrak{t}_\xi\}$ is a smooth FSI-template.

[Why? We prove this by induction on ε; for $\varepsilon = 0$ by 2.11(1), for ε successor by 2.11(3) for ε limit by 2.11(5) and 2.11(6).]

$(*)^4_\zeta$ for $Z \subseteq \mathbf{J}$ we have $\mathfrak{t}_\zeta \restriction (\bigcup\limits_{\eta \in Z} \{\langle\zeta\rangle\hat{}\langle\eta\rangle\hat{}\rho : \rho \in \bigcup\limits_{\xi<h_\zeta(\eta)} \mathfrak{t}_\xi\})$ is a smooth FSI-template.

[Why? By induction on $|Z|$, for $|Z| = 0, |Z| = n+1$ by 2.11(3), for $|Z| \geq \aleph_0$ by 2.11(5).]

$(*)^5_\zeta$ $\mathfrak{t}_\zeta \restriction (L^{\mathfrak{t}_\zeta} \backslash \{\langle\zeta\rangle\})$ is a smooth FSI-template.

[Why? By $(*)^4_\zeta$ for $Z = \mathbf{J}$.]

$(*)^6_\zeta$ $\mathfrak{t}_\zeta$ is a smooth FSI-template.

[Why? By 2.11(3).]

$(*)^7_\zeta$ if $K \subseteq L^{\mathfrak{t}_\zeta}$ is countable and $t \in L^{\mathfrak{t}_\zeta}$ then the ideal $I^{\mathfrak{t}_\zeta}_t \cap \mathscr{P}(K)$ is generated by a countable family of subsets of K.

[Why? Check by induction on ζ.]

Now for $\zeta \leq \mu$ let

$(*)^8_\zeta$ $\mathfrak{s}_\zeta =: \sum\limits_{\varepsilon<\zeta} \mathfrak{t}_\varepsilon$, i.e.

(i) the set of elements of $\mathfrak{s}_\zeta$ is $\bigcup\limits_{\varepsilon<\zeta} L^{\mathfrak{t}_\varepsilon}$

(ii) for $x, y \in \mathfrak{s}_\zeta$ we have $x <_{\mathfrak{s}_\zeta} y$ iff $\xi(x) < \xi(y) \vee (\xi(x) = \xi(y) \wedge x <_{\mathfrak{t}_{\xi(x)}} y)$

(iii) $I^{\mathfrak{s}_\zeta}_y = \{Y \subseteq^{\mathfrak{s}[\zeta]}: (\forall z \in Y)(z <_{\mathfrak{s}_\zeta} y)$ and $\{z \in Y : \xi(z) = \xi(y)\} \in I^{\mathfrak{t}[\xi(z)]}_y\}$

$(*)^9_\zeta$ $\mathfrak{s}_\zeta$ is a smooth FSI-template.

[Why? Just easier than the proof above.]

$(*)^{10}_\zeta$ if $K \subseteq L^{\mathfrak{s}_\zeta}$ is countable and $t \in L^{\mathfrak{s}_\zeta}$, <u>then</u> the ideal $I^{\mathfrak{s}_\zeta}_t \cap \mathscr{P}(K)$ of subsets of K is generated by a countable family of subsets of K.

[Why? By $(*)^7_\zeta$ and the definition of $\mathfrak{s}_\zeta$ and of the $\mathfrak{t}_\varepsilon$-s.]

Let [18] $\sigma = \aleph_0, \partial = (2^\sigma)^+$, we shall prove below by induction on ζ that $\mathfrak{s}_\zeta, \mathfrak{t}_\zeta$ are $(\lambda, \theta, \sigma)$-good (see definition below and Sub-claim 3.5); then we can finish the proof as in 2.18 (and $(*)^7_\zeta$ and $(*)^{10}_\zeta$) $\square_{3.1}$

Definition 3.3. 1) Assume [19] $\lambda \geq \partial \geq \tau > \sigma$, ∂ is regular uncountable and $(\forall \alpha < \partial)[|\alpha|^\sigma < \partial]$ and $\mathbf{W} \subseteq \mathscr{P}(\mathscr{P}(\sigma))$. We say that a smooth FSI-template $\mathfrak{t}$ is $(\lambda, \partial, \tau, \sigma, \mathbf{W})$-good <u>if</u> :

- $\oplus$ assume that $t_{\alpha,\phi} \in L^{\mathfrak{t}}$ for $\alpha < \partial, \phi < \sigma, \{t_{\alpha,\phi} : \phi < \sigma\}$ is $\bar{K}$-closed, <u>then</u> we can find $\mathscr{W} \in \mathbf{W}$ and a club C of ∂ and a pressing down function h on C such that:
- $\oplus'$ if $S \subseteq C$ is stationary in ∂, $(\forall \delta \in S)[\mathrm{cf}(\delta) > \sigma \wedge (\tau = \partial \rightarrow \mathrm{cf}(\delta) = \tau)]$ and $h{\restriction}S$ is constant <u>then</u>:
  - $\boxtimes^1_S$ for every $\alpha < \beta$ in S and $w \in \mathscr{W}$, the truth value of the following statements does not depend on (α, β): (but may depend on ϕ, ϵ and $w \in \mathscr{W}$)
    - (i) $t_{\alpha,\phi} = t_{\beta,\epsilon}$
    - (ii) $t_{\alpha,\phi} <_{L^{\mathfrak{t}}} t_{\beta,\epsilon}$
    - (iii) $\{t_{\alpha,\varkappa} : \varkappa \in w\} \in I^{\mathfrak{t}}_{t_{\alpha,\epsilon}}$
    - (iv) $\{t_{\beta,\varkappa} : \varkappa \in w\} \in I^{\mathfrak{t}}_{t_{\alpha,\phi}}$
    - (v) $\{t_{\alpha,\varkappa} : \varkappa \in w\} \in I^{\mathfrak{t}}_{t_{\beta,\phi}}$
  - $\boxtimes^2_S$ let $\delta^* \leq \partial$ be such that $\mathrm{cf}(\delta^*) = \tau$ and $\sup(S \cap \delta^*) = \delta^*$; if $\partial \leq \beta^* < \lambda$ and $s_{\beta,\phi} \in L^{\mathfrak{t}}$ for $\beta \in [\partial, \beta^*), \phi < \omega$ <u>then</u> we can find $t_\phi \in L^{\mathfrak{t}}$ for $\phi < \omega$ such that for every $\beta < \beta^*$, for every large enough $\alpha \in S \cap \delta^*$ for some $\mathfrak{t}$-partial $\otimes$ isomorphism f we have $f(t_\phi) = t_{\alpha,\phi}, f(s_{\beta,\phi}) = s_{\beta,\phi}$.

2) We say $\mathfrak{t}$ is strongly $(\lambda, \theta, \tau, \sigma)$-good <u>if</u> above we have $\mathbf{W} = \mathscr{P}(\mathscr{P}(\sigma))$.

3) We may omit $\mathbf{W}$ in part (1) when $\mathbf{W} = \{\mathscr{W} : \mathscr{W}$ is an ideal of the Boolean algebra $\mathscr{P}(\sigma)$ generated by $\leq \sigma$ sets$\}$

4) Above we may omit τ if $\tau = \theta$.

Observation 3.4. *In Def. 3.3, instead "h regressive" it is enough to demand: for some sequence $\langle X_\alpha : \alpha < \theta\rangle$ of sets, increasing continuous, $|X_\alpha| < \theta$ and for every (or club of) $\delta < \theta$, if $\mathrm{cf}(\delta) > \aleph_0$ then $h(\delta) \in \mathscr{H}_{<\aleph_1}(X_\delta)$.*

Claim 3.5. *1) In the proof of 3.1;*

- (i) $\mathfrak{t}_\zeta$ *is strongly* $(\lambda, \partial, \sigma)$*-good*
- (ii) $\mathfrak{s}_\zeta$ *is strongly* $(\lambda, \partial, \sigma)$*-good*
- (iii) *if* $\mathrm{cf}(\zeta) \neq \theta$ *then* $\mathfrak{s}_\zeta$ *is also strongly* (λ, θ)*-good.*

2) Assume $\lambda = \mathrm{cf}(\lambda) > \mu = \mathrm{cf}(\mu), \mathbf{J}, \bar{\mathfrak{t}}_\varepsilon(\varepsilon < \mu), \mathfrak{s}_\zeta(\zeta \leq \mu)$ are as in the proof of 3.1. If $\partial = (2^\sigma)^+ < \mu$ then clauses (i), (ii), (iii) above hold.

[18] but if you like to avoid using $(*)^7_\zeta, (*)^{10}_\zeta$ and $\mathscr{W}$ below just use $\partial = \beth_2^+$. In fact even without $(*)^7_\zeta + (*)^{10}_\zeta$ above, countable $\mathscr{W}$ suffice but then we have to weaken the notion of isomorphisms, and no point.

[19] we ignore here $\bar{K}$ and $\{(t, \bar{\varphi}_t, \underset{\sim}{\eta}_t) : t \in L^{\mathfrak{t}}\}$ using the default values

Proof. 1) Recall that $\partial = (2^\sigma)^+$ (see before Definition 3.3).

First note that

$(*)_1$ for every $\zeta < \mu$ there is a sequence $\bar{\varrho}_\zeta$ such that:

(a) $\bar{\varrho}_\zeta = \langle \varrho_{\zeta,s} : s \in L^{\mathfrak{t}[\zeta]} \rangle$

(b) $\varrho_{\zeta,s} \in {}^{\omega>}\lambda$

(c) the truth value of $L^{\mathfrak{t}[\zeta]} \models$ "$s < t$" depends only on

(α) $\lg(\varrho_{\zeta,s})$

(β) $\lg(\varrho_{\zeta,t})$

(γ) the truth values of $\varrho_{\zeta,s}(k) < \varrho_{\zeta,t}(\ell)$, $\varrho_{\zeta,s}(k) = \varrho_{\zeta,t}(\ell)$, $\varrho_{\zeta,s}(k) > \varrho_{\zeta,t}(\ell)$ for the relevant k, ℓ

[Why? Read the definition of $\mathfrak{t}_\zeta$.]

Second note that

$(*)_2$ there is a sequence $\bar{\varrho} = \langle \varrho_s : s \in L^{\mathfrak{s}[\mu]} \rangle$ satisfying the parallel of $(*)_1$.

hence

$(*)_3$ if $\bar{s} = \langle s_\phi = s(\phi) : \phi \le \sigma \rangle \in {}^{\sigma+1}(L^{\mathfrak{s}[\mu]})$ then the truth value of $\{s_\phi : \phi < \sigma\} \in I^{\mathfrak{s}[\mu]}$ depends only on

(a) $\lg(\varrho_{s(\phi)})$ for $\phi \le \sigma$

(b) the truth values of $\varrho_{\zeta,s(\varepsilon)}(k) < \varrho_{\zeta,s(\zeta)}(\ell)$, $\varrho_{\zeta,s(\varepsilon)}(k) = \varrho_{\zeta,s(\zeta)}(\ell)$, $\varrho_{\zeta,s(\varepsilon)}(k) > \varrho_{\zeta,s(\zeta)}(\ell)$ for $\varepsilon, \zeta \le \sigma$ and relevant k, ℓ

Why? Again look at the choice of $\mathfrak{s}_\mu$

Now, given $\bar{t}_\alpha = \langle t_{\alpha,\phi} = t[\alpha,\phi] : \phi < \sigma \rangle \in {}^\sigma(L^{\mathfrak{s}[\mu]})$ for $\alpha < \partial$ define

$(*)_4$ $\mathscr{U}_\alpha = \cup\{\mathrm{Rang}(\varrho_{t[\beta,\phi]}) : \beta < \alpha, \phi < \sigma\} \cup \{\infty\}$

Next

$(*)_5$ we define the function $h_\ell, \ell = 0, 1$ with domain $\partial \setminus \{\emptyset\}$, so for $\alpha \in (0, \partial)$ we let:

(a) $h_0(\alpha)$ is equal to the set as $\{(\phi, k, \epsilon, \ell) : \varrho_{t[\alpha,\phi]}(k) < \varrho_{t[\alpha,\epsilon]}(\ell)$ and both are well defined$\}\}$

(b) $h_1(\alpha)$ is the minimal non-zero member β of $\mathscr{U}_\alpha$ such that (if there is no one then it is zero):
for every $\phi < \sigma, k < \lg(t_{\alpha,\phi})(k)$, the following are equal:

(α) the minimal member of $\mathscr{U}_\alpha$ which is $> \varrho_{t[\alpha,\phi]}(k)$

(β) the minimal member of $\mathscr{U}_\beta$ which is $> \varrho_{t[\beta,\phi]}(k)$,

(*b)' similarly for $\ge$ (and so for equal)

Clearly

$(*)_6$ h_0 has range of cardinality $< \partial$ and h_1 is regressive

Lastly

$(*)_7$ if $S \subseteq \partial$ is stationary and $\delta \in S \Rightarrow \mathrm{cf}(\delta) > \sigma$ and h_0, h_1 restricted to S are constant then S is as required.

<u>OLD/ pre 2020 PROOF</u>

We prove this by induction on ζ.

<u>For $\mathfrak{s}_\zeta$:</u>

If $\zeta = 0$ it is empty. Otherwise given $t_{\alpha,\phi} \in \mathfrak{s}_\zeta = \sum\limits_{\varepsilon<\zeta} \mathfrak{t}_\varepsilon$ for $\alpha < \theta, \phi < \omega$ let $h_0^*(\alpha)$ be the sequence consisting of:

(i) $\xi_{\alpha,\phi} =: \mathrm{Min}\{\xi : \xi \in \{\xi(t_{\beta,\epsilon}) : \beta < \delta, \epsilon < \omega\} \cup \{\infty\}$ and $\xi \geq \xi(t_{\alpha,\phi})\}$ for $\phi < \omega$ and

(ii) $u_\alpha = \{(\phi, \epsilon, \varkappa) : \xi(t_{\alpha,\phi}) = \xi_{\alpha,\epsilon} \wedge \varkappa = 1$ or $\xi(t_{\alpha,\phi}) \leq \xi(t_{\alpha,\epsilon}) \wedge \varkappa = 2\}$ and

(iii) $\mathbf{w}_\alpha = \{(n, w) : \phi < \omega, w \subseteq \omega$ and $\{t_{\alpha,\epsilon} : \epsilon \in w\} \in I^{\mathfrak{t}}_{t_{\alpha,\phi}}\}$ that is $h^*_0(\alpha) = \langle u_\alpha, \langle \xi_{\alpha,\phi} : \phi < \omega\rangle, \mathbf{w}_\alpha\rangle$.

If $S_y = \{\delta : \mathrm{cf}(\delta) \geq \aleph_1, h^*_0(\delta) = y\}$ is stationary we define $h^*_1{\restriction}S_y$ such that it codes $h^*_0(\delta)$ and if $\phi(*) < \omega$ and the sequence $\langle \xi(t_{\alpha,\phi(*)}) : \alpha \in S_y\rangle$ is constant call it $\xi_{y,\phi(*)}$ let $u_{y,\phi(*)} = \{\phi : \xi_{\alpha,\phi} = \xi_{y,\phi(*)}\}$, <u>then</u> $h^*_1 \restriction S_y$ codes a function witnessing the (λ, θ)-goodness of $\mathfrak{t}_{\xi_{y,\phi(*)}}$ for $\langle t_{\alpha,\phi} : \phi \in u_{y,\phi(*)}, \alpha \in S_y\rangle$.

Fix S as in $\oplus'$. It is easy to check that this shows $\boxtimes^1_S$ even if $\mathrm{cf}(\zeta) = \theta$. But assume $\mathrm{cf}(\zeta) \neq \theta \wedge \delta^* = \theta$ or $\delta^* < \theta, \mathrm{cf}(\delta^*) = \aleph_1$ (or just $\aleph_0 < \mathrm{cf}(\delta^*) < \theta$), $\delta^* = \sup(S \cap \delta^*)$; we shall prove also the statement from $\boxtimes^2_S$. Let $w_1 = \{\phi :$ the sequence $\langle \xi(t_{\beta,\phi}) : \beta \in S\rangle$ is strictly increasing$\}$, $w_0 = \{\phi : \langle \xi(t_{\beta,\phi}) : \beta \in S\rangle$ is constant$\}$, let $\xi(S, \phi) = \xi_{S,\phi} = \cup\{\xi(t_{\beta,\phi}) : \beta \in S\}$ as $\mathrm{cf}(\zeta) \neq \theta$ it is $< \zeta$ also when $\phi \in w_1$.

Given $\langle \bar{s}_\beta : \beta < \beta^*\rangle, \beta^* < \lambda$ and $\bar{s}_\beta = \bar{s} = \langle s_{\beta,\phi} : \phi < \omega\rangle$ we have to find $\langle t_\phi : \phi < \omega\rangle$ as required in $\boxtimes^2_S$. If $\phi \in w_0$ let $w'_{0,\phi} = \{\epsilon \in w_0 : \xi(t_{\alpha,\phi}) = \xi(t_{\alpha,\epsilon})$ for $\alpha \in S\}$ and to choose $\langle t_\epsilon : \epsilon \in w'_{0,\phi}\rangle$ we use the induction hypothesis on $\mathfrak{t}_{\xi(S,\phi)}$. If $\phi \in w_1$ then we can find $t^*_\phi \in \mathfrak{t}_{\xi_{S,\phi}}$ such that $\{t : t \in \mathfrak{t}_{\xi_{S,\phi}}, t \leq_{\mathfrak{t}_{\xi(S,\phi)}} t^*\}$ is disjoint to $\{t_{\beta,\epsilon} : \beta < \delta^*, \epsilon < \omega\} \cup \{s_{\beta,\epsilon} : \beta < \beta^*$ and $m < \omega\}$ this is possible because the lower cofinality of $L^{\mathfrak{t}_{\xi(S,\phi)}}$ is the same as that of L_0 and is $\lambda = \mathrm{cf}(\lambda) > \theta + |\beta^*|$. Then we choose $\eta^* \in \mathbf{J}$ such that $(\forall x)(\langle\zeta\rangle\hat{}\langle\eta^*\rangle\hat{}x \in \mathfrak{t}_{\xi(S,\phi)} \Rightarrow \langle\zeta\rangle\hat{}\langle\eta^*\rangle\hat{}\langle x\rangle <_{\mathfrak{t}_{\xi(S,\phi)}} t^*)$ and we choose together $\langle t_{\phi'} : \phi' \in w_1, \xi_{S,\phi'} = \xi_{S,\phi}\rangle$ such that $t_\phi \in \{\langle\zeta\rangle\hat{}\langle\eta\rangle\hat{}\langle x\rangle \in \mathfrak{s}_\zeta : \eta <_{\mathbf{J}} \eta^*\}$ taking care of $\mathscr{W}$, (inside $\{\phi \in w_1 : \xi(t_{\alpha,\phi}) = \xi_{S,\epsilon}\}$ and automatically for others, i.e. considering t_{ϕ_1}, t_{ϕ_2} such that $\xi_{S,\phi_1} \neq \xi_{S,\phi_2}$), this is immediate.

<u>For $\mathfrak{t}_\zeta$</u>:

Similar (using $\boxdot_1 + \boxdot_2$). $\square_{3.5}$

* * *

We may like to have "$2^{\aleph_0} = \lambda$ is singular", $\mathfrak{a} = \lambda, \mathfrak{b} = \mathfrak{d} = \mu$. Toward this we would like to have a linear order $\mathbf{J}$ such that if $\bar{x} = \langle x_\alpha : \alpha < \theta\rangle$ is monotonic, say decreasing then for any $\sigma < \lambda$ for some limit $\delta < \theta$ of uncountable cofinality the linear order $\{y \in \mathbf{J} : \alpha < \delta \Rightarrow y <_{\mathbf{J}} x_\alpha\}$ has cofinality $> \sigma$. Moreover, δ can be chosen to suit ω such sequences $\bar{x}$ simultaneously. So every set of ω-tuples from $\mathbf{J}$ of cardinality $\geq \theta$ but $< \lambda$ can be "inflated".

Lemma 3.6. *Assume*

(a) $(2^\sigma)^+ < \mu = \mathrm{cf}(\mu) < \lambda = \lambda^\sigma, \lambda$ *singular*

(b) $(\forall \alpha < \mu)[|\alpha|^{\aleph_0} < \mu]$

(c) $\mu \geq \aleph_{\mathrm{cf}(\lambda)}$ *or at least*

$(c)^-$ *there is $f : \lambda \to \mathrm{cf}(\lambda)$ such that if $\langle \alpha_\varepsilon : \varepsilon < \mu\rangle \in {}^\mu\lambda$ is (strictly) increasing continuous, $\alpha_\varepsilon < \lambda$ and $\gamma < \mathrm{cf}(\lambda)$ <u>then</u> for some $\varepsilon < \mu$ we have $f(\alpha_\varepsilon) \geq \gamma$.*

Then for some c.c.c. forcing notion of cardinality λ we have $\Vdash_{\mathbb{P}}$ "$2^{\aleph_0} = \lambda, \mathfrak{b} = \mathfrak{d} = \mu, \mathfrak{a} = \lambda$".

Proof. Note that $(c) \Rightarrow (c)^-$, just let $\alpha < \lambda \wedge \mathrm{cf}(\alpha) = \aleph_\varepsilon \wedge \varepsilon < \mathrm{cf}(\lambda) \Rightarrow f(\alpha) = \varepsilon$, clearly there is such a function and it satisfies clause $(c)^-$. So we can assume $(c)^-$. Let $\sigma = \mathrm{cf}(\lambda)$ and $\langle \lambda_\varepsilon : \varepsilon < \sigma \rangle$ be a strictly increasing sequence of regular cardinals $> \mu + \sigma$ with limit λ. Let L_0, L_0^+, L_0^- be as in the proof of 3.1, $L_{0,\varepsilon}$ be the unique interval of L_0 of order type (the inverse of λ_ε) $+\lambda_\varepsilon$, so $\langle L_{0,\varepsilon} : \varepsilon < \sigma \rangle$ be ia $\subseteq$-increasing with union $L_0, L_{0,\varepsilon}$ an interval of $L_{0,\xi}$ for $\varepsilon < \xi < \sigma$. We define $g : L_0 \to \mathrm{cf}(\lambda)$ as follows: if $x \in L_0^+$ then $g(x) = f(\mathrm{otp}(\{y \in L_0^+ : y <_L x\}, <))$ and if $x \in L_0^-$ and the order type of $(\{y \in L_0^+ : x <_L y\}, <_L)$ is the inverse of γ then $g(x) = f(\gamma)$ and let

$$\mathbf{J}^* = \{\eta \in {}^{\omega>}(L_0) : \eta(0) \in L_{0,0} \text{ and } \eta(n+1) \in L_{0,g(\eta(n))} \text{ for } n < \omega\}$$

ordered as in the proof of 3.6.

We define $\mathfrak{s}_\zeta, \mathfrak{t}_\zeta$ as there. We then prove that $\mathfrak{s}_\zeta, \mathfrak{t}_\zeta$ are (τ, θ)-good and (λ, τ)-good as there and this suffices repeating the proof of 3.1. $\square_{3.6}$

Discussion 3.7. We may like to separate $\mathfrak{b}$ and $\mathfrak{d}$. So below we adapt the proof of 3.1 to do this (can do it also for 3.6).

A way to do this is to look at the forcing in 3.1 as the limit of the FS iteration $\langle \mathbb{P}_i^*, \underset{\sim}{\mathbb{Q}}_j^* : i \le \mu, j < \mu \rangle$, so the memory of $\mathbb{Q}_j^*$ is $\{i : i < j\}$ where $\underset{\sim}{\mathbb{Q}}_j^*$ is $\mathrm{Lim}_{\mathfrak{t}}[\langle \mathbb{Q}_t : t \in L^{\mathfrak{t}_j} \rangle]$. Below we will use the limit of FS iteration $\langle \mathbb{P}_i^*, \underset{\sim}{\mathbb{Q}}_j^* : j < \mu \times \mu_1 \rangle, \mathbb{Q}_\zeta^*$ has memory $w_\zeta \subseteq \zeta$ where e.g. for $\zeta = \mu\,\alpha + i$ where $i < \mu, w_\zeta = \{\kappa\beta + j : \beta \le \alpha, j \le i, (\beta, j) \neq (\alpha, i)\}$. Let $\mathbb{P}^* = \mathbb{P}^*_{\mu \times \mu_1}$ be $\cup\{\mathbb{P}_i : i < \mu \times \mu_1\}$.

Of course, $\mathbb{Q}_\zeta$ will be defined as $\mathrm{Lim}_{\mathfrak{t}_\zeta}(\bar{\mathbb{Q}})$, the $\mathfrak{t}_\zeta$ defined as above and $\mathfrak{b} = \mu, \mathfrak{d} = \mu_1$. Should be easy. If $\langle \underset{\sim}{A}_\varepsilon : \varepsilon < \varepsilon^{\bar{x}} \rangle$ exemplifies $\mathfrak{a}$ in $\mathbf{V}^{\mathbb{P}^*}$, so $\varepsilon^* \ge \mu$ then for some $(\alpha^*, \beta^*) \in \mu \times \mu_1$ for $\kappa(= \theta)$ of the names they involve $\{\underset{\sim}{\mathbb{Q}}_{\mu\alpha+\beta} : \alpha \le \alpha^*, \beta \le \beta^*\}$ only.

Using indiscernibility on the pairs (α, β) to making them increase we can finish.

Lemma 3.8. *1) In Lemma 3.1, if $\mu = \mathrm{cf}(\mu) \le \mathrm{cf}(\mu_1), \mu_1 < \lambda$, then we can change in the conclusion $\mathfrak{b} = \mathfrak{d} = \mu$ to $\mathfrak{b} = \mu, \mathfrak{d} = \mu_1$.*
2) Similarly for 3.6.

Proof. First assume μ_1 regular.
First Proof: Let $\mu_0 = \mu$. In the proof of 3.1 for $\ell \in \{0, 1\}$ using $\mu = \mu_\ell$ gives $\mathfrak{s}^\ell_{\mu_\ell}$ and without loss of generality $\mathfrak{s}^0_{\mu_0}, \mathfrak{s}^1_{\mu_1}$ are disjoint. Let $\mathfrak{s}$ be $\mathfrak{s}_0 +' \mathfrak{s}_1$ meaning $L^{\mathfrak{s}} = L^{\mathfrak{s}^0_{\mu_0}} + L^{\mathfrak{s}^1_{\mu_1}}$, and for $t \in L^{\mathfrak{s}^\ell_{\mu_\ell}}$ we let $I^{\mathfrak{s}}_t =: I^{\mathfrak{s}^\ell_{\mu_\ell}}_t$ (this is not $\mathfrak{s}_0 + \mathfrak{s}_1$ of 2.11). Now the appropriate goodness can be proved so we can prove $\mathfrak{a} = \lambda$. Easily we get $\mathfrak{d} \ge \mu_1$ and $\mathfrak{b} \le \mu_0$. This is enough to get inequality but to get exact values we turn to the second proof.

Instead of starting with $\langle \mathbb{Q}_i : i < \mu \rangle$ with full memory we start with $\langle \underset{\sim}{\mathbb{Q}}_\zeta : \zeta < \mu \times \mu_1 \rangle, \underset{\sim}{\mathbb{Q}}_\zeta$ with the following "memory" if $\zeta = \mu\alpha + i, i < \kappa, w_\zeta = \{\mu\beta + j : \beta \le \alpha, j \le i, (\beta, j) \neq (\alpha, i)\}$. To deal with the case μ_1 is singular we should use a μ-directed index set (instead $\mu_0 \times \mu_1$) as the product of ordered sets. $\square_{3.8}$

§ 4. On related cardinal invariants

Explanation of §4:

On Th. 4.1 you may wonder: $\mathfrak{u}$ has nothing to do with order or quite directed family, so how can we preserve small $\mathfrak{u}$? True, using the "directed character" of $\mathfrak{b}$ and $\mathfrak{d}$ has been the idea, i.e. in the end we have $\mathbb{P} = \langle \mathbb{P}_i : i < \mu \rangle$ is $\lessdot$-increasing, $\mathbb{P} = \cup\{\mathbb{P}_i : i < \mu\}$ and $\underset{\sim}{\eta}_i$ a $\mathbb{P}_{i+1}$-name of a real dominating $\mathbf{V}^{\mathbb{P}_i}$. But really what we need for a triple $(\mathbb{P}, \underset{\sim}{\bar{\eta}}, \mathbb{P}')$ as $(\mathbb{P}_i, \underset{\sim}{\eta}_i, \mathbb{P}_{i+1})$ above, is that taking ultrapower by the κ-complete ultrafilter D, preserve the property of $\underset{\sim}{\bar{\eta}}$, in our present case $\underset{\sim}{\bar{\eta}}$ has to witness $\mathfrak{u} = \mu$. For being a dominating real this is very natural (Łos theorem). But here we shall use $\langle \underset{\sim}{D}_i : i < \mu \rangle, \underset{\sim}{D}_i$ a $\mathbb{P}_i$-name of an ultrafilter on ω and demand $\text{Rang}(\underset{\sim}{\eta}_i)$ to be mod finite included in every member of $\underset{\sim}{D}_i$ and moreover $\underset{\sim}{\eta}_i$ is generic over $\mathbf{V}^{\mathbb{P}_i}$ for a forcing related to $\underset{\sim}{D}_i$. When we like to preserve something in inductive construction on $\alpha < \lambda$ of $\langle \mathbb{P}^\alpha_i : i < \mu \rangle$, it is reasonable to have strong induction hypothesis more than needed just for the final conclusion. We need here a condition on $(\mathbb{P}^\alpha_{i+1}, \underset{\sim}{\eta}^\alpha_i, \mathbb{P}^\alpha_i, \underset{\sim}{D}^\alpha_i)$ preserved by the ultrapower (as the relevant forcing is c.c.c. nicely enough defined this work).

Secondly, we need in limit α: if $\text{cf}(\alpha) > \aleph_0$ straightforward if not, being generic for the $\mathbb{Q}_i$ has nice enough properties so that we can complete $\bigcup\limits_{\beta<\alpha} \underset{\sim}{D}^\beta_i$ to a suitable ultrafilter.

This explains to some extent the scope of possible applications, of course, in each case the exact inductive assumption on $(\mathbb{P}^\alpha_{i+1}, \underset{\sim}{\eta}^\alpha_i, \mathbb{P}^\alpha_i, \underset{\sim}{Y}^\alpha_i)$ with $\underset{\sim}{Y}^\alpha_i$ a relevant witness, varies.

On continuing §2, §3 so eliminating the measurable here see [S$^+$a], [S$^+$c].

Theorem 4.1. *Assume*

(a) *κ is a measurable cardinal*
(b) $\kappa < \mu = \text{cf}(\mu) < \lambda = \text{cf}(\lambda) = \lambda^\kappa$.

Then for some c.c.c. forcing notion $\mathbb{P}$ of cardinality λ, in $\mathbf{V}^{\mathbb{P}}$ we have: $2^{\aleph_0} = \lambda, \mathfrak{u} = \mathfrak{d} = \mathfrak{b} = \mu$ and $\mathfrak{a} = \lambda$.

Remark 4.2. Recall $\mathfrak{u} = \text{Min}\{|\mathscr{P}| : \mathscr{P} \subseteq [\omega]^{\aleph_0}$ generates a non-principal ultrafilter on $\omega\}$.

Proof. The proof is broken to definitions and claims. $\square_{4.1}$

Definition 4.3. For a filter D on ω (to which all co-finite subsets of ω belong) let $\mathbb{Q}(D)$ be:

$$\begin{aligned}\{T : \quad & T \subseteq {}^{\omega>}\omega \text{ is closed under initial segments, and for some} \\ & \text{tr}(T) \in {}^{\omega>}\omega, \text{ the trunk of } T, \text{ we have :} \\ & (i) \quad \ell \leq \ell g(\text{tr}(T)) \Rightarrow T \cap {}^{\ell}\omega = \{\text{tr}(T) \restriction \ell\} \\ & (ii) \quad \text{tr}(T) \trianglelefteq \eta \in {}^{\omega>}\omega \Rightarrow \{n : \eta\char`^\langle n \rangle \in T\} \in D\}\end{aligned}$$

ordered by inverse inclusion.

Definition 4.4. 1) Assume $S \subseteq \{i < \mu : \text{cf}(i) \neq \kappa\}$ is unbounded in μ (the default value is $\{i < \mu : \text{cf}(i) \neq \kappa\}$).

Let $\mathfrak{K}_{\lambda,S}$ be the family of $\mathfrak{t}$ consisting of $\bar{\mathbb{Q}} = \bar{\mathbb{Q}}^{\mathfrak{t}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \mu\rangle = \langle \mathbb{P}^{\mathfrak{t}}_i, \underset{\sim}{\mathbb{Q}}^{\mathfrak{t}}_i : i < \mu\rangle$ and $\bar{D} = \bar{D}^{\mathfrak{t}} = \langle \underset{\sim}{D}_i : i < \mu \text{ and } \mathrm{cf}(i) \neq \kappa\rangle = \langle \underset{\sim}{D}^{\mathfrak{t}}_i : i \in S\rangle$ and $\bar{\tau}^{\mathfrak{t}} = \langle \underset{\sim}{\tau}^{\mathfrak{t}_i} : i < \mu\rangle$ such that:

- (a) $\bar{\mathbb{Q}}$ is a FS-iteration of c.c.c. forcing notions (and $\mathbb{P}_{\mathfrak{t}} = \mathbb{P}^{\mathfrak{t}}_\mu = \mathrm{Lim}(\bar{\mathbb{Q}}^{\mathfrak{t}}) = \bigcup\limits_{i<\mu} \mathbb{P}^{\mathfrak{t}}_i)$
- (b) if $i \in S$, then $\mathbb{Q}_i = \mathbb{Q}(\underset{\sim}{D}_i)$, see Definition 4.3 above
- (c) $\underset{\sim}{D}_i$ is a $\mathbb{P}_i$-name of a non-principal ultrafilter on ω when $i \in S$
- (d) $|\mathbb{P}_i| \le \lambda$
- (e) for $i \in S$ let $\underset{\sim}{\eta}_i$ be the $\mathbb{P}_{i+1}$-name of the $\underset{\sim}{\mathbb{Q}}_i$-generic real
$$\underset{\sim}{\eta}_i = \cup\{\mathrm{tr}(p(i)) : p \in \underset{\sim}{G}_{\mathbb{P}_{i+1}}\}.$$
and we demand: for $i < j < \mu$ of cofinality $\neq \kappa$ we have
$$\Vdash_{\mathbb{P}_j} \text{“}\mathrm{Rang}(\underset{\sim}{\eta}_i) \in \underset{\sim}{D}_j\text{”}$$
- (f) $\underset{\sim}{\tau}_i$ is a $\mathbb{P}_i$-name of a function from $\underset{\sim}{\mathbb{Q}}_i$ to $\{h : h$ is a function from a finite set of ordinals to $\mathscr{H}(\omega)\}$, such that:
$\Vdash_{\mathbb{P}_i}$ “$p, q \in \underset{\sim}{\mathbb{Q}}_i$ are compatible in $\underset{\sim}{\mathbb{Q}}_i)$ <u>iff</u> the functions $\underset{\sim}{\tau}_i(p), \underset{\sim}{\tau}_i(q)$ are compatible, i.e. $\tau_i(p) \restriction (\mathrm{Dom}(\underset{\sim}{\tau}_i(p)) \cap \mathrm{Dom}(\underset{\sim}{\tau}_i(q)) = \tau_i(q) \restriction (\mathrm{Dom}(\underset{\sim}{\tau}_i(p)) \cap \mathrm{Dom}(\underset{\sim}{\tau}_i(q))$ and <u>then</u> they have a common upper bound r such that $\underset{\sim}{\tau}_i(r) = \underset{\sim}{\tau}_i(p) \cup \underset{\sim}{\tau}_i(q)$”
- (g) if $i \in S \cap \mathrm{Dom}(p), p \in \mathbb{P}_j$ and $i < j \le \mu$ then $\underset{\sim}{\tau}_i(p(i))$ is $\{\langle 0, \mathrm{tr}(p)\rangle\}$; i.e. this is forced to hold
- (h) we stipulate $\mathbb{P}_i = \{p : p$ is a function with domain a finite subset of i such that for each $j \in \mathrm{Dom}(p), \emptyset_{\mathbb{P}_j}$ forces that $p(j) \in \underset{\sim}{\mathbb{Q}}_j$ and it forces a value to $\underset{\sim}{\tau}_j(p(j))\}$
- (i) $\Vdash_{\mathbb{P}_i}$ “$\underset{\sim}{\mathbb{Q}}_i \subseteq \mathscr{H}_{<\aleph_1}(\gamma)$ for some ordinal γ”.

2) Let $\gamma(\mathfrak{t})$ be the minimal ordinal γ such that $i < \mu \Rightarrow \Vdash_{\mathbb{P}_i}$ “if $x \in \underset{\sim}{\mathbb{Q}}_i$ then $\mathrm{dom}(\underset{\sim}{\tau}_i(x)) \subseteq \gamma$”.
3) We let $\tau^{\mathfrak{t}}_i$ be the function with domain $\mathbb{P}_i$ such that $\tau^{\mathfrak{t}}_i(p)$ is a function with domain $\{\gamma(\mathfrak{t})j + \beta : j \in \mathrm{Dom}(p)$ and $p \restriction j \Vdash_{\mathbb{P}_j}$ “$\beta \in \mathrm{Dom}(\underset{\sim}{\tau}_j(p(j))$”$\}$ and let $\tau^{\mathfrak{t}}_i(\gamma(\mathfrak{t})j + \beta)$ be the value which $p \restriction j$ forces on $\underset{\sim}{\tau}^{\mathfrak{t}}_j(\beta)$.

Convention 4.5. *We fix λ, μ, S as in* 4.1, *4.4; so we may write $\mathfrak{K}$ instead $\mathfrak{K}_{\lambda,S}$.*

Obviously

Subclaim 4.6. $\mathfrak{K} \neq \emptyset$.

Proof. Should be clear. $\square_{4.6}$

Recall

Subclaim 4.7. *If in a universe $\mathbf{V}, D$ is a nonprincipal ultrafilter on ω then*

- (a) $\Vdash_{\mathbb{Q}(D)}$ “$\{\mathrm{tr}(p)(\ell) : \ell < \ell g(\mathrm{tr}(p))$ *and* $p \in \underset{\sim}{G}_{\mathbb{Q}(D)}\}$ *is an infinite subset of ω, almost included in every member of D*”
- (b) $\mathbb{Q}(D)$ *is a c.c.c. forcing notion, even σ-centered*

(c) $\underset{\sim}{\eta}_i = \cup\{\mathrm{tr}(p) : p \in \underset{\sim}{G}_{\mathbb{Q}(D)}\} \in {}^{\omega}\omega$ *is forced to dominate* $({}^{\omega}\omega)^{\mathbf{V}}$

(d) $\{p \in \mathbb{Q}(D) : \mathrm{tr}(p) = \eta\}$ *is a directed subsets of* $\mathbb{Q}(D)$ *for every* $\eta \in {}^{\omega>}\omega$.

[Note that this, in particular clause (c), does not depend on additional properties of D; but as we naturally add many Cohen reals (by the nature of the support) we may add more and then can demand e.g. $\underset{\sim}{D}_i$ $(\mathrm{cf}(i) \neq \kappa)$ *is a Ramsey ultrafilter.]*

Definition 4.8. 1) We define $\leq_{\mathfrak{K}}$ by: $\mathfrak{t} \leq_{\mathfrak{K}} \mathfrak{s}$ if ($\mathfrak{t}, \mathfrak{s} \in \mathfrak{K}$ and) $i \leq \mu \Rightarrow \mathbb{P}^{\mathfrak{t}}_i \lessdot \mathbb{P}^{\mathfrak{s}}_i$ and $i < \mu$ and $\mathrm{cf}(i) \neq \kappa \Rightarrow \Vdash_{\mathbb{P}^{\mathfrak{s}}_i}$ "$\underset{\sim}{D}^{\mathfrak{t}}_i \subseteq \underset{\sim}{D}^{\mathfrak{s}}_i$" and $i < \mu \Rightarrow \Vdash_{\mathbb{P}^{\mathfrak{s}}_i}$ "$\underset{\sim}{\mathcal{T}}^{\mathfrak{t}}_i \subseteq \underset{\sim}{\mathcal{T}}^{\mathfrak{s}}_i$".
2) We say $\mathfrak{t}$ is a canonical $\leq_{\mathfrak{K}}$-u.b. of $\langle \mathfrak{t}_\alpha : \alpha < \delta\rangle$ <u>if</u>:

(i) $\mathfrak{t}, \mathfrak{t}_\alpha \in \mathfrak{K}$

(ii) $\alpha \leq \beta < \delta \Rightarrow \mathfrak{t}_\alpha \leq_{\mathfrak{K}} \mathfrak{t}_\beta \leq_{\mathfrak{K}} \mathfrak{t}$

(iii) if $i \in \mu \setminus S$ then $\Vdash_{\mathbb{P}^{\mathfrak{t}}_i}$ "$\underset{\sim}{\mathbb{Q}}^{\mathfrak{t}}_i = \bigcup\limits_{\alpha<\delta} \underset{\sim}{\mathbb{Q}}^{\mathfrak{t}_\alpha}_i$".

Note that if $\mathrm{cf}(\delta) > \aleph_0$ then we can add $\Vdash_{\mathbb{P}^{\mathfrak{t}}_i}$ "$\underset{\sim}{\mathbb{Q}}^{\mathfrak{t}}_i = \bigcup\limits_{\alpha<\delta} \underset{\sim}{\mathbb{Q}}^{\mathfrak{t}_\alpha}_i$" for every $i < \mu$, so $\mathfrak{t}$ is totally determined.
3) We say $\langle \mathfrak{t}_\alpha : \alpha < \alpha^*\rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous if: $\alpha < \beta < \alpha^* \Rightarrow \mathfrak{t}_\alpha \leq_{\mathfrak{K}} \mathfrak{t}_\beta$ and for limit $\delta < \alpha^*, \mathfrak{t}_\delta$ is a canonical $\leq_{\mathfrak{K}}$-u.b. of $\langle \mathfrak{t}_\alpha : \alpha < \delta\rangle$. Note that we have not said "the canonical $\leq_{\mathfrak{K}}$-u.b." as for $\delta < \alpha^*, \mathrm{cf}(\delta) = \aleph_0$ we have some freedom in completing $\cup\{\underset{\sim}{D}^{\mathfrak{t}_\alpha}_i : \alpha < \delta\}$ to an ultrafilter (on ω in $\mathbf{V}^{\mathbb{P}^{\mathfrak{t}}_i}$, when $i \in \mu \setminus S$).

Subclaim 4.9. *If* $\mathbb{P}_1 \lessdot \mathbb{P}_2$ *and* $\underset{\sim}{D}_\ell$ *is a* $\mathbb{P}_\ell$*-name of a nonprincipal ultrafilter on* ω *for* $\ell = 1, 2$ *and* $\Vdash_{\mathbb{P}_2}$ "$\underset{\sim}{D}_1 \subseteq \underset{\sim}{D}_2$", <u>*then*</u> $\mathbb{P}_1 * \mathbb{Q}(\underset{\sim}{D}_1) \lessdot \mathbb{P}_2 * \mathbb{Q}(\underset{\sim}{D}_2)$.

Proof. Why? First, we can first force with $\mathbb{P}_1$, so without loss of generality $\mathbb{P}_1$ is trivial and $D_1 \in \mathbf{V}$ is a nonprincipal ultrafilter on ω. Now clearly $p \in \mathbb{Q}(D_1) \Rightarrow p \in \mathbb{Q}(\underset{\sim}{D}_2)$ and $\mathbb{Q}(D_1) \models p \leq q \Rightarrow \mathbb{Q}(\underset{\sim}{D}_2) \models p \leq q$ and if $p, q \in \mathbb{Q}(D_1)$ are incompatible in $\mathbb{Q}(D_1)$ then they are incompatible in $\mathbb{Q}(\underset{\sim}{D}_2)$.

Lastly, in $\mathbf{V}$, let $\mathscr{I} = \{p_\phi : \phi < \omega\} \subseteq \mathbb{Q}(D_1)$ be predense in $\mathbb{Q}(D_1)$, we shall prove that $\mathscr{I}$ is predense in $\mathbb{Q}(\underset{\sim}{D}_2)$ in $\mathbf{V}^{\mathbb{P}_2}$.

For this it suffices to note

$\boxtimes$ if D_1 is a nonprincipal ultrafilter on $\omega, \mathscr{I} \subseteq \mathbb{Q}(D_1)$ and $\eta \in {}^{\omega>}\omega$, <u>then</u> the following conditions are equivalent:

$(a)_\eta$ there is no $p \in \mathbb{Q}(D_1)$ incompatible with every $q \in \mathscr{I}$ which satisfies $\mathrm{tr}(p) = \eta$

$(b)_\eta$ there is a set T such that:

(i) $\nu \in T \Rightarrow \eta \trianglelefteq \nu \in p$

(ii) $\eta \trianglelefteq \nu \trianglelefteq \rho \in T \Rightarrow \nu \in T$

(iii) if $\nu \in T$ then either $\{n : \nu{}^\frown\langle n\rangle \in T\} \in D_1$ or $(\forall n)(\nu{}^\frown\langle n\rangle \notin T) \wedge (\exists q \in \mathscr{I})(\nu = \mathrm{tr}(q))$

(iv) there is a strictly decreasing function $h : T \to \omega_1$

(v) $\eta \in p$.

$\square_{4.9}$

Proof. Proof of $\boxtimes$:

Straightforward.

So as in $\mathbf{V}, \mathscr{I} \subseteq \mathbb{Q}(D_1)$ is predense, for every $\eta \in {}^{\omega>}\omega$ we have $(a)_\eta$ for D_1 hence by $\boxtimes$ we have also $\eta \in {}^{\omega>}\omega \Rightarrow (b)_\eta$, but clearly if T_η witness $(b)_\eta$ in $\mathbf{V}$ for D_1, it witnesses $(b)_\eta$ in $\mathbf{V}^{\mathbb{P}_2}$ for D_2 hence applying $\boxtimes$ again we get: $\eta \in {}^{\omega>}\omega \Rightarrow (a)_\eta$ in $\mathbf{V}^{\mathbb{P}_2}$ for D_2, hence $\mathscr{I}$ is predense in $\mathbb{Q}(D_2)$ in $\mathbf{V}^{\mathbb{P}_2}$. So we have proved Subclaim 4.9. $\square$

Subclaim 4.10. *If $\bar{\mathfrak{t}} = \langle \mathfrak{t}_\alpha : \alpha < \delta \rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous and $\delta < \lambda^+$ is a limit ordinal, then it has a canonical $\leq_{\mathfrak{K}}$-u.b.*

Proof. Why? By induction on $i \leq \mu$, we define $\mathbb{P}^{\mathfrak{t}}_i$ and if $i < \mu$ we then have $\underset{\sim}{\mathbb{Q}}^{\mathfrak{t}_i}, \underset{\sim}{\tau}_i$ and $\underset{\sim}{D}_i$ (if $\mathrm{cf}(i) \neq \kappa$) such that the relevant demands (for $\mathfrak{t} \in \mathfrak{K}$ and for being canonical $\leq_{\mathfrak{K}}$-u.b. of $\bar{\mathfrak{t}}$) hold.

Defining $\mathbb{P}^{\mathfrak{t}}_i$ is obvious: for $i = 0$ trivially, if $i = j+1$ it is $\mathbb{P}^{\mathfrak{t}}_j * \underset{\sim}{\mathbb{Q}}^{\mathfrak{t}}_j$ and if i is limit it is $\cup\{\mathbb{P}^{\mathfrak{t}}_j : j < i\}$.

If $\mathbb{P}^{\mathfrak{t}}_i$ has been defined and $\mathrm{cf}(i) = \kappa$ we let $\underset{\sim}{\mathbb{Q}}^{\mathfrak{t}}_i = \bigcup\limits_{\alpha<\delta} \mathbb{Q}^{\mathfrak{t}_\alpha}_i$ and $\underset{\sim}{\tau}^{\mathfrak{t}_i} = \bigcup\limits_{\alpha<\delta} \underset{\sim}{\tau}^{\mathfrak{t}_\alpha}_i$, easy to check that they are as required. If $\mathbb{P}^{\mathfrak{t}}_i$ has been defined and $\mathrm{cf}(i) \neq \kappa$, then $\bigcup\limits_{\alpha<\delta} D^{\mathfrak{t}_\alpha}_i$ is a filter on ω containing the co-bounded subsets, and we complete it to an ultrafilter, call it $D^{\mathfrak{t}_i}$.

Note that there is such $\underset{\sim}{D}^{\mathfrak{t}_i}$ because:

- (a) for $\alpha < \delta, \mathbb{P}^{\mathfrak{t}_\alpha} \lessdot \mathbb{P}^{\mathfrak{t}}_i$ hence $\Vdash_{\mathbb{P}^{\mathfrak{t}}_i}$ "$\underset{\sim}{D}^{\mathfrak{t}_\alpha}_i$ is a filter on ω to which all co-finite subsets of ω belong and it increases with α".

Note that there will be no need for new values of the $\underset{\sim}{\tau}_i$'s nor any freedom in defining them. As we have proved the relevant demands on $\mathbb{P}^{\mathfrak{t}}_j, \underset{\sim}{\mathbb{Q}}^{\mathfrak{t}}_j$ for $j < i$ clearly $\mathbb{P}^{\mathfrak{t}}_i$ is c.c.c. by using $\langle \underset{\sim}{\tau}_j : j < i \rangle$ and clearly $\langle \mathbb{P}^{\mathfrak{t}}_\zeta, \underset{\sim}{\mathbb{Q}}^{\mathfrak{t}}_\xi : \zeta \leq i, \xi < i \rangle$ is an FS iteration. Now we shall prove that $\alpha < \delta \Rightarrow \mathbb{P}^{\mathfrak{t}_\alpha}_i \lessdot \mathbb{P}^{\mathfrak{t}}_i$.

So let $\mathscr{I}$ be a predense subset of $\mathbb{P}^{\mathfrak{t}_\alpha}_i$ and $p \in \mathbb{P}^{\mathfrak{t}}_i$ and we should prove that p is compatible with some $q \in \mathscr{I}$ in $\mathbb{P}^{\mathfrak{t}}_i$; we divide the proof to cases.

Case 1: i is a limit ordinal.

So $p \in \mathbb{P}^{\mathfrak{t}}_j$ for some $j < i$, let $\mathscr{I}' = \{q \restriction j : q \in \mathscr{I}\}$, so clearly $\mathscr{I}'$ is a predense subset of $\mathbb{P}^{\mathfrak{t}_\alpha}_j$ (as $\mathfrak{t}_\alpha \in \mathfrak{K}$). By the induction hypothesis, in $\mathbb{P}^{\mathfrak{t}}_j$ the condition p is compatible with some $q' \in \mathscr{I}'$; so let $r' \in \mathbb{P}^{\mathfrak{t}}_j$ be a common upper bound of q', p recalling that $q' = q \restriction j$ where $q \in \mathscr{I}$. So $r' \cup (q \restriction [j,i)) \in \mathbb{P}^{\mathfrak{t}}_i$ is a common upper bound of q, p as required.

Case 2: $i = j+1, \mathrm{cf}(j) = \kappa$.

If $j \notin \mathrm{Dom}(p)$ it is trivial. So without loss of generality for some $\beta < \delta, p(j)$ is a $\mathbb{P}^{\mathfrak{t}_\beta}_j$-name of a member of $\underset{\sim}{\mathbb{Q}}^{\mathfrak{t}_\beta}_j$; and without loss of generality $\alpha \leq \beta < \delta$. By the induction hypothesis $\mathbb{P}^{\mathfrak{t}_\beta}_j \lessdot \mathbb{P}^{\mathfrak{t}}_j$ hence there is $p' \in \mathbb{P}^{\mathfrak{t}_\beta}_j$ such that $[p' \leq p'' \in \mathbb{P}^{\mathfrak{t}_\beta}_j \Rightarrow p'', p \restriction j$ are compatible in $\mathbb{P}^{\mathfrak{t}}_j]$.

Let

$$\begin{aligned}\mathscr{J} = \{q' \restriction j : \ & q' \in \mathbb{P}^{\mathfrak{t}_\beta}_i \text{ and } q' \text{ is above some member of } \mathscr{I} \\ & \text{and } q' \restriction j \Vdash_{\mathbb{P}^{\mathfrak{t}_\beta}_j} \text{“}p(j) \leq^{\underset{\sim}{\mathbb{Q}}^{\mathfrak{t}_\beta}_j} q'(j)\text{”}\}.\end{aligned}$$

Now $\mathscr{J}$ is a dense subset of $\mathbb{P}^{\mathfrak{t}_\beta}_j$ (since if $q \in \mathbb{P}^{\mathfrak{t}_\beta}_j$ then $q \cup \{\langle j, p(j)\rangle\}$ belongs to $\mathbb{P}^{\mathfrak{t}_\beta}_i$ hence is compatible with some member of $\mathscr{I}$).

Hence p' is compatible with some $q'' \in \mathscr{J}$ (in $\mathbb{P}^{\mathfrak{t}_\beta}_j$), so there is r such that $p' \le r \in \mathbb{P}^{\mathfrak{t}_\beta}_j, q'' \le r$. As $q'' \in \mathscr{J}$ there is $q' \in \mathbb{P}^{\mathfrak{t}_\beta}_i$ such that $q' \restriction j = q'', q'$ is above some $q^* \in \mathscr{I}$ and $q' \restriction j \Vdash$ "$p(j) \le^{\mathbb{Q}^{\mathfrak{t}_\beta}_j} q'(j)$".

As $\mathbb{P}^{\mathfrak{t}_\beta}_j \models$ "$p' \le r \wedge q' \restriction j = q'' \le r$" and by the choice of p' there is $p^* \in \mathbb{P}^{\mathfrak{t}}_j$ above r (hence above p' and above $q'' = q' \restriction j$), and above $p \restriction j$. Now let $r^* = p^* \cup (q'' \restriction \{j\})$, clearly $r^* \in \mathbb{P}^{\mathfrak{t}}_i$ is above $p \restriction j$ and $r^* \restriction j$ forces that $r^*(j)$ is above $p \restriction \{j\}$. Clearly $r^* \restriction j$ is above r and r^* is also above $q^* \in \mathscr{I}$ so we are done.

Case 3: $i = j+1, j \in S$

Use Subclaim 4.9 above.

So we have dealt with $\alpha < \delta \Rightarrow \mathbb{P}^{\mathfrak{t}_\alpha}_i \lessdot \mathbb{P}^{\mathfrak{t}}_i$.

Clearly we are done. $\square_{4.10}$

Subclaim 4.11. *If $\mathfrak{t} \in \mathfrak{K}$ and E is a κ-complete non-principal ultrafilter on κ, then we can find $\mathfrak{s}$ such that:*

- *(i)* $\mathfrak{t} \le_{\mathfrak{K}} \mathfrak{s} \in \mathfrak{K}$
- *(ii) there is $\langle \mathbf{k}_i, \mathbf{j}_i : i < \mu, \mathrm{cf}(i) \ne \kappa\rangle$ such that:*
  - *(α) $\mathbf{k}_i$ is an isomorphism from $(\mathbb{P}^{\mathfrak{t}}_i)^\kappa/E$ onto $\mathbb{P}^{\mathfrak{s}}_i$*
  - *(β) $\mathbf{j}_i$ is the canonical embedding of $\mathbb{P}^{\mathfrak{t}}_i$ into $(\mathbb{P}^{\mathfrak{t}}_i)^\kappa/E$*
  - *(γ) $\mathbf{k}_i \circ \mathbf{j}_i =$ identity on $\mathbb{P}^{\mathfrak{t}}_i$*
- *(iii) $\underset{\sim}{D}^{\mathfrak{s}_i}$ is the image of $(\underset{\sim}{D}_i)^\kappa/E$ under $\mathbf{k}_i$ and similarly $\underset{\sim}{\tau}^{\mathfrak{s}_i}$ if $i < \mu, \mathrm{cf}(i) \ne \kappa$*
- *(iv) if $i < \mu, \mathrm{cf}(i) = \kappa$, then $\underset{\sim}{\tau}^{\mathfrak{s}_i}$ is defined such that, for $j < \kappa, \mathrm{cf}(j) \ne \kappa$ we have $\mathbf{k}_j$ is an isomorphism from $(\mathbb{P}^{\mathfrak{t}}_i, \gamma', \tau^{\mathfrak{t}}_i)^\kappa/D$ onto $(\mathbb{P}^{\mathfrak{s}}_i, \gamma'', \tau^{\mathfrak{s}_i}_i)$ for some ordinals γ', γ'' (except that we do not require that the map from γ' to γ'' preserves order).*

Proof. Straightforward.

Note that if $\mathrm{cf}(i) = \kappa, i < \mu$ then $\underset{\sim}{\mathbb{Q}}^{\mathfrak{s}}_i$ is isomorphic to $\mathbb{P}^{\mathfrak{s}}_{i+1}/\mathbb{P}^{\mathfrak{s}}_i$ which is c.c.c. as by Łoś theorem for the logic $\mathbb{L}_{\kappa,\kappa}$ we have $\bigcup\limits_{j<i} (\mathbb{P}^{\mathfrak{t}}_j)^\kappa/E \lessdot (\mathbb{P}^{\mathfrak{t}}_{i+1})^\kappa/E$, similarly for $\underset{\sim}{\tau}_i$ which guarantees that the quotient is c.c.c., too (actually $\underset{\sim}{\tau}_i$ is not needed for the c.c.c. here). $\square_{4.11}$

Subclaim 4.12. *If $\mathfrak{t} \in \mathfrak{K}$ then $\Vdash_{\mathbb{P}^{\mathfrak{t}}_\mu}$ "$\mathfrak{u} = \mathfrak{b} = \mathfrak{d} = \mu$".*

Proof. In $\mathbf{V}^{\mathbb{P}^{\mathfrak{t}}_\mu}$, the family $\mathscr{D} = \{\mathrm{Rang}(\underset{\sim}{\eta}_i) : i < \mu \text{ and } \mathrm{cf}(i) \ne \kappa\} \cup \{[n,\omega) : n < \omega\}$ generates a filter on $\mathscr{P}(\omega)^{\mathbf{V}[\mathbb{P}^{\mathfrak{t}}_\mu]}$, as $\mathrm{Rang}(\underset{\sim}{\eta}_i) \in [\omega]^{\aleph_0}, i < j < \mu$ and $\mathrm{cf}(i) \ne \kappa$ and $\mathrm{cf}(j) \ne \kappa \Rightarrow \mathrm{Rang}(\underset{\sim}{\eta}_j) \subseteq^* \mathrm{Rang}(\underset{\sim}{\eta}_i)$.

Also it is an ultrafilter as $\mathscr{P}(\omega)^{\mathbf{V}[\mathbb{P}^{\mathfrak{t}}_\mu]} = \bigcup\limits_{i<\mu} \mathscr{P}(\omega)^{\mathbf{V}[\mathbb{P}^{\mathfrak{t}}_i]}$ and if $i < \mu$, then $\mathrm{Rang}(\underset{\sim}{\eta}_{i+1})$ induces an ultrafilter on $\mathscr{P}(\omega)^{\mathbf{V}[\mathbb{P}^{\mathfrak{t}}_{i+1}]}$. So $\mathfrak{u} \le \mu$. Also $({}^\omega\omega)^{\mathbf{V}[\mathbb{P}^{\mathfrak{t}}_\mu]} = \bigcup\limits_{i<\mu} ({}^\omega\omega)^{\mathbf{V}[\mathbb{P}^{\mathfrak{t}}_i]}, ({}^\omega\omega)^{\mathbf{V}[\mathbb{P}^{\mathfrak{t}}_i]}$ is increasing with i and if $\mathrm{cf}(i) \ne \kappa$ then $\underset{\sim}{\eta}_i \in {}^\omega\omega$ dominates $({}^\omega\omega)^{\mathbf{V}[\mathbb{P}^{\mathfrak{t}}_i]}$ by Subclaim 4.7, so $\mathfrak{b} = \mathfrak{d} = \mu$ as in previous cases.

Lastly, always $\mathfrak{u} \ge \mathfrak{b}$ hence $\mathfrak{u} = \mu$.] $\square_{4.12}$

Now we define $\mathfrak{t}_\alpha \in \mathfrak{K}$ for $\alpha \leq \lambda$ by induction on α satisfying $\langle \mathfrak{t}_\alpha : \alpha \leq \lambda \rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous such that $\mathfrak{t}_{\alpha+1}$ is gotten from $\mathfrak{t}_\alpha$ as in Subclaim 4.11.

Let $\mathbb{P} = \mathbb{P}^{\mathfrak{t}_\lambda}_\mu$, so $|\mathbb{P}| \leq \lambda$ hence $(2^{\aleph_0})^{\mathbf{V}^{\mathbb{P}}} \leq (\lambda^{\aleph_0})^{\mathbf{V}}$ and easily equality holds.

We finish by

Subclaim 4.13. *We have*[20] $\Vdash_{\mathbb{P}^{g\mathfrak{t}_\alpha}_{\lambda}}$ "$\mathfrak{a} \geq \lambda$".

Proof. Why? Assume toward a contradiction that $\theta < \lambda$ and $p \in \mathbb{P}$ and $p \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathscr{A}} = \{\underset{\sim}{A}_i : i < \theta\}$ is a MAD family; i.e.

- (i) $A_i \in [\omega]^{\aleph_0}$
- (ii) $i \neq j \Rightarrow |\underset{\sim}{A}_i \cap \underset{\sim}{A}_j| < \aleph_0$
- (iii) under (i) + (ii), $\underset{\sim}{\mathscr{A}}$ is maximal".

Without loss of generality $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{A}_i \in [\omega]^{\aleph_0}$". As $\mathfrak{a} \geq \mathfrak{b} = \mu$ by Subclaim 4.12, we have $\theta \geq \mu$. For each $i < \theta$ and $m < \omega$ there is a maximal antichain $\langle p_{i,m,n} : n < \omega \rangle$ of $\mathbb{P}$ and there is a sequence $\langle \mathbf{t}_{i,m,n} : n < \omega \rangle$ of truth values such that $p_{i,m,n} \Vdash$ "$(m \in \underset{\sim}{A}_i) \equiv \mathbf{t}_{i,m,n}$". We can find countable $w_i \subseteq \mu$ such that $\bigcup\limits_{m,n<\omega} \mathrm{Dom}(p_{i,m,n}) \subseteq w_i$. Possibly increasing w_i retaining countability, we can find $\langle R_{i,\gamma} : \gamma \in w_i \rangle$ such that:

- (α) w_i has a maximal element and $\gamma \in w_i \backslash \{\max(w_i)\} \Rightarrow \gamma + 1 \in w_i$
- (β) $R_{i,\gamma}$ is a countable subset of $\mathbb{P}^{\mathfrak{t}_\lambda}_\gamma$ and $q \in R_{i,\gamma} \Rightarrow \mathrm{Dom}(q) \subseteq w_i \cap \gamma$
- (γ) for $\gamma_1 < \gamma_2$ in $w_i, q \in R_{i,\gamma_2} \Rightarrow q \restriction \gamma_1 \in R_{i,\gamma_1}$
- (δ) for $\gamma_1 \in w_i, \gamma \in \gamma_1 \cap w_i$ and $q \in R_{i,\gamma_1}$ the $\mathbb{P}^{\mathfrak{t}}_\gamma$-name $q(\gamma)$ involves $\aleph_0$ maximal antichains all included in $R_{i,\gamma}$
- (ε) $\{p_{i,m,n} : m,n < \omega\} \subseteq R_{i,\max(w_i)}$.

As $\mathrm{cf}(\lambda) > \aleph_0$ (as $\mu < \lambda = \mathrm{cf}(\lambda)$ by the assumption of Theorem 4.1) we have $\mathbb{P}^{\mathfrak{t}}_\mu = \bigcup\limits_{\alpha<\lambda} \mathbb{P}^{\mathfrak{t}_\alpha}_\mu$. Clearly for some $\alpha < \lambda$ we have $\cup\{R_{i,\gamma} : i < \theta, \gamma \in w_i\} \subseteq \mathbb{P}^{\mathfrak{t}_\alpha}_\mu$. But $\mathbb{P}^{\mathfrak{t}_\alpha}_\mu \lessdot \mathbb{P}^{\mathfrak{t}_\lambda}_\mu$. So $\Vdash_{\mathbb{P}^{\mathfrak{t}_\alpha}_\mu}$ "$\underset{\sim}{\mathscr{A}} = \{\underset{\sim}{A}_i : i < \theta\}$ is MAD".

Now, letting $\mathbf{j}$ be the canonical elementary embedding of $\mathbf{V}$ into $\mathbf{V}^\kappa/D$, we know:

- $(*)$ in $\mathbf{V}^\kappa/D, \mathbf{j}(\underset{\sim}{\mathscr{A}})$ is a $\mathbf{j}(\mathbb{P}^{\mathfrak{t}_\alpha}_\mu)$-name of a MAD family.

As $\mathbf{V}^\kappa/D$ is κ-closed, for c.c.c. forcing notions things are absolute enough but $\{\mathbf{j}(i) : i < \mu\}$ is not $\{i : \mathbf{V}^\kappa/D \models i < \mathbf{j}(\mu)\}$, so in $\mathbf{V}$, it is forced for $\Vdash_{\mathbf{j}(\mathbb{P}^{\mathfrak{t}_\alpha}_\mu)}$, that $\{\mathbf{j}(\underset{\sim}{A}_i) : i < \mu\}$ is not MAD!

Chasing arrows, clearly $\Vdash_{\mathbb{P}^{\mathfrak{t}_{\alpha+1}}_\mu}$ "$\{\underset{\sim}{A}_i : i < \theta\}$ is not MAD" as required. $\square_{4.13}$

Discussion 4.14. 1) We can now look at other problems, like what can be the order and equalities among $\mathfrak{d}, \mathfrak{b}, \mathfrak{a}, \mathfrak{u}$; have not considered it. I have considered having $\mathfrak{a} = \mu$ but there was a problem.

2) (2020) In 4.1 We can add $\mathfrak{p} = \mathfrak{t} = \mu$ proving as in 4.10. Let me elaborate: in Definition 4.4 (our forcing is $\mathbb{P}_\mu$ for such $\mathfrak{t}$), we have an ultrafilter generated by a sequence of subsets of ω which is decreasing modulo finite; see clauses (c) and (e).

3) So $\mathbb{P}_\mu$ forces $\mathfrak{s}$ is at least μ. But always $\mathfrak{s}$ is at most $\mathfrak{u}$ so in 4.1 we can add $\mathfrak{u} = \mu$.

[20] recall λ is regular; if we allow λ singular we have to use $\mathrm{cf}(\lambda)$.

References

[FM17] Vera Fischer and Diego Alejandro Mejia, *Splitting, bounding, and almost disjointness can be quite different*, Canad. J. Math. **69** (2017), no. 3, 502–531. MR 3679685

[Lav71] Richard Laver, *On fraissé's order type conjecture*, Annals of Mathematics **93** (1971), 89–111.

[Mej15] Diego A. Mejía, *Template iterations with non-definable ccc forcing notions*, Ann. Pure Appl. Logic **166** (2015), no. 11, 1071–1109. MR 3385102

[Mil] Arnold W. Miller, *Arnie miller's problem list*, Set Theory of the Reals (Haim Judah, ed.), Israel Mathematical Conference Proceedings, vol. 6, Proceedings of the Winter Institute held at Bar–Ilan University, Ramat Gan, January 1991, pp. 645–654.

[RS] Andrzej Rosłanowski and Saharon Shelah, *Norms on possibilities III: strange subsets of the real line*.

[S$^+$a] S. Shelah et al., *Tba*, In preparation. Preliminary number: Sh:F2009.

[S$^+$b] ———, *Tba*, In preparation. Preliminary number: Sh:F2029.

[S$^+$c] ———, *Tba*, In preparation. Preliminary number: Sh:F2023.

[Shea] Saharon Shelah, *Combinatorial background for Non-structure*, arXiv: 1512.04767 Appendix of [Sh:e].

[Sheb] ———, *Compactness of the Quantifier on "Complete embedding of BA's"*, arXiv: 1601.03596 Ch. XI of "The Non-Structure Theory" book [Sh:e].

[She78] ———, *Models with second-order properties. I. Boolean algebras with no definable automorphisms*, Ann. Math. Logic **14** (1978), no. 1, 57–72. MR 501097

[She87] ———, *Universal classes*, Classification theory (Chicago, IL, 1985), Lecture Notes in Math., vol. 1292, Springer, Berlin, 1987, pp. 264–418. MR 1033033

[She90] ———, *Classification theory and the number of nonisomorphic models*, 2nd ed., Studies in Logic and the Foundations of Mathematics, vol. 92, North-Holland Publishing Co., Amsterdam, 1990. MR 1083551

[She98] ———, *Proper and improper forcing*, 2nd ed., Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1998. MR 1623206

[She00a] ———, *Covering of the null ideal may have countable cofinality*, Fund. Math. **166** (2000), no. 1-2, 109–136, arXiv: math/9810181. MR 1804707

[She00b] ———, *On what I do not understand (and have something to say). I*, Fund. Math. **166** (2000), no. 1-2, 1–82, arXiv: math/9906113. MR 1804704

[She03] ———, *The null ideal restricted to some non-null set may be $\aleph_1$-saturated*, Fund. Math. **179** (2003), no. 2, 97–129, arXiv: math/9705213. MR 2029228

[She04a] ———, *Properness without elementaricity*, J. Appl. Anal. **10** (2004), no. 2, 169–289, arXiv: math/9712283. MR 2115943

[She04b] ———, *Two cardinal invariants of the continuum ($\mathfrak{d} < \mathfrak{a}$) and FS linearly ordered iterated forcing*, Acta Math. **192** (2004), no. 2, 187–223, Previous title "Are $\mathfrak{a}$ and $\mathfrak{d}$ your cup of tea?" arXiv: math/0012170. MR 2096454

[She09a] ———, *Categoricity and solvability of A.E.C., quite highly*, 2009, arXiv: 0808.3023 Ch. IV of [Sh:h].

[She09b] ———, *Universal Classes: Axiomatic Framework [Sh:h]*, 2009, Ch. V (B) of [Sh:i].

[She09c] ———, *Universal Classes: Stability theory for a model*, 2009, Ch. V of [Sh:i].

Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem, Jerusalem, 91904, Israel, and, Department of Mathematics, Hill Center - Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019 USA

Email address: shelah@math.huji.ac.il

URL: http://shelah.logic.at