$\leq_{\mathrm{SP}}$ CAN HAVE INFINITELY MANY CLASSES
1065

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Abstract. Building off of recent results on Keisler's order, we show that consistently, $\leq_{\mathrm{SP}}$ has infinitely many classes. In particular, we define the property of $\leq k$-type amalgamation for simple theories, for each $2 \leq k < \omega$. If we let $T_{n,k}$ be the theory of the random k-ary, n-clique free random hyper-graph, then $T_{n,k}$ has $\leq k-1$-type amalgamation but not $\leq k$-type amalgamation. We show that consistently, if T has $\leq k$-type amalgamation then $T_{k+1,k} \not\leq_{\mathrm{SP}} T$, thus producing infinitely many $\leq_{\mathrm{SP}}$-classes. The same construction gives a simplified proof of the theorem from [10] that consistently, the maximal $\leq_{\mathrm{SP}}$-class is exactly the class of unsimple theories. Finally, we show that consistently, if T has $< \aleph_0$-type amalgamation, then $T \leq_{\mathrm{SP}} T_{\mathrm{rg}}$, the theory of the random graph.

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§ 0. Introduction

Convention 0.1. T is always a complete theory in a countable language. We will fix a monster model $\mathfrak{C} \models T$ and work within it so $\mathfrak{C} = \mathfrak{C}_T$ but if T is clear from the content we do not mention it (this to make 1.4(A) applicable, if the singular cardinal hypothesis is assumed, not needed.

The first author introduced the following definition in [10], although he had previously investigated the phenomenon in [8] (without giving it a name):

Definition 0.2. Suppose $\lambda \geq \theta$. Define $\mathrm{SP}_T(\lambda, \theta)$ to mean: for every $M \models T$ of size λ, there is a θ-saturated $N \models T$ of size λ extending M.

In this paper, we will restrict to the following special case:

Definition 0.3. 1) Say that (θ, λ) is a nice pair if θ is a regular cardinal $> \beth_\omega$ and $\lambda \geq \theta$ has $\lambda = \lambda^{\aleph_0}$.
2) Given T_0, T_1 complete first order theories, say that $T_0 \leq_{\mathrm{SP}} T_1$ if whenever (θ, λ) is a nice pair, if $\mathrm{SP}_{T_0}(\lambda, \theta)$ then $\mathrm{SP}_{T_1}(\lambda, \theta)$.

Thus, $\leq_{\mathrm{SP}}$ is a pre-ordering of theories which measures how difficult it is to build saturated models. The main case of interest is when $\mathrm{cof}(\lambda) < \theta$.

In [8], the first author proves: the stable theories are the minimal SP-class, and unsimple theories are always maximal. In [10], the first author additionally proves that consistently, unsimple theories are exactly the maximal class.

Recently, there has been substantial progress on Keisler's order $\trianglelefteq$, another pre-ordering of theories which measures how difficult it is to build saturated models; see for instance [6] and [7] by the first author and Malliaris. In particular, in [7] it is shown that Keisler's order has infinitely many classes, these being seperated by certain amalgamation properties. In this paper we use similar ideas to continue investigation of $\leq_{\mathrm{SP}}$.

In §2 we summarize what is already known on $\leq_{\mathrm{SP}}$.

In §3, we introduce several amalgamation-related properties of forcing notions (Definition 2.2), and show that it is preserved under iterations in a suitable sense (Theorem 2.5). In light of this, we define a class of forcing axioms (Definition 2.7); these are closely related to the forcing axiom Ax_{μ_0}, defined by the first author in [9] and used to demonstrate the consistent maximality of unsimple theories under $\leq_{\mathrm{SP}}$ in [10]. However, the forcing axioms we develop are designed specifically for what we want and have been simplified somewhat.

In §4, we define and prove some helpful facts about non-forking diagrams of models.

In §5, we introduce, for each $3 \leq k < \omega$, a property of simple theories called $< k$-type amalgamation (Definition 4.1), and discuss some of its properties. For example, if for $n > k$ we let $T_{n,k}$ be the theory of the k-ary, n-clique free hypergraph, then if $k \geq 3$, $T_{n,k}$ has $< k$-type amalgamation but not $< k+1$-type amalgamation. We also show that if T has $< \aleph_0$-type amalgamation (i.e., $< k$-type amalgamation for all k), then $\mathrm{SP}_T(\lambda, \theta)$ holds whenever we have that there is some $\theta \leq \mu \leq \lambda$ with $\mu^{<\theta} \leq \lambda$ and $2^\mu \geq \lambda$ (Theorem 4.6). This implies that if the singular cardinals hypothesis holds, then whenever T has $< \aleph_0$-type amalgamation, then $T \leq_{\mathrm{SP}} T_{\mathrm{rg}}$, where T_{rg} is the theory of the random graph.

In §6, we put everything together to show that consistently, for all $k \geq 3$, if T has the $< k$-type amalgamation property, then $T_{k,k-1} \not\leq_{\mathrm{SP}} T$ (Theorem 5.2).

In particular, for $k < k'$, $T_{k+1,k} \not\leq_{\rm SP} T_{k'+1,k'}$; this is similar to the situation for Kiesler's order in [7].

By a forcing notion, we mean a pre-ordered set $(P, \leq^P)$ such that P has a least element 0^P (pre-order means that $\leq^P$ is transitive); we are using the convention where $p \leq q$ means q is a stronger condition than p. That is, when we force by P we add a generic ideal, rather than a generic filter. Thus, a finite sequence $(p_i : i < k)$ from P is compatible if it has an upper bound in P.

§ 1. Background

The following theorem is closely related to the classical Hewitt-Marczewski-Pondiczery theorem of topology; it is proved in [1]. It will be central for our investigations.

Theorem 1.1. *Suppose $\theta \leq \mu \leq \lambda$ are infinite cardinals such that θ is regular, $\mu = \mu^{<\theta}$, and $\lambda \leq 2^\mu$. Then there is a sequence $(\mathbf{f}_\gamma : \gamma < \mu)$ from ${}^\lambda\mu$ such that for all partial functions f from λ to μ of cardinality less than θ, there is some $\gamma < \mu$ such that $\mathbf{f}_\gamma$ extends f. Additionally, if $\lambda > 2^\mu$ then this fails.*

We will also want the following technical device, which will allow us to apply Theorem 1.1 to conclude $\mathrm{SP}_T(\lambda, \theta)$ holds. Here is the idea: suppose $M \models T$ with $|M| \leq \lambda$, and we want to find some θ-saturated $N \succeq M$ with $|N| \leq \lambda$. To do this, we will always first find some $N_0 \succeq M$ with $|N_0| \leq \lambda$ which realizes every type over M of cardinality less than θ, and then we iterate θ-many times. The key step is to find N_0, and the following definitions capture when this is possible.

Definition 1.2. 1) Suppose T is a simple theory, θ is a regular uncountable cardinal, and $M_* \preceq M \models T$. then let Γ^θ_{M,M_*} be forcing notion of all partial types $p(x)$ over M of cardinality less than θ which do not fork over M_*, ordered by inclusion. Also, if $p_*(x)$ is a complete type over M_*, then let $\Gamma^\theta_{M,p_*} \subseteq \Gamma^\theta_{M,M_*}$ be the set of all $p(x)$ which extend $p_*(x)$.
2) Given (θ, λ) a nice pair and given μ with $\theta \leq \mu \leq \lambda$, define $\mathrm{SP}^1_T(\lambda, \mu, \theta)$ to mean: for every $M \models T$ of size $\leq \lambda$ and for every countable $M_* \preceq M$, there are complete types $p_\gamma(x) : \gamma < \mu$ over M which do not fork over M_*, such that whenever $p(x) \in \Gamma^\theta_{M,M_*}$, then $p(x) \subseteq p_\gamma(x)$ for some $\gamma < \mu$.
3) Given in addition a fixed countable $M_* \models T$ and type $p_*(x)$ over M_*, define $\mathrm{SP}^1_{T,p_*}(\lambda, \mu, \theta)$ similarly: whenever $M \succeq M_*$ has size at most λ, there are are complete, non-forking extensions $p_\gamma(x) : \gamma < \mu$ of $p_*(x)$ to M, such that whenever $p(x) \in \Gamma^\theta_{M,p_*}$, then $p(x) \subseteq p_\gamma(x)$ for some $\gamma < \mu$.

Note that if $\mu \geq 2^{\aleph_0}$, then $\mathrm{SP}^1_T(\lambda, \mu, \theta)$ if and only if $\mathrm{SP}^1_{T,p_*}(\lambda, \mu, \theta)$ for every complete type $p_*(x)$ over a countable model M_* (the forward direction is unconditional in μ, but for the reverse direction, we need to concatenate witnesses for each $p_*(x)$, of which there are $2^{\aleph_0}$-many). In particular this holds when $\mu = \lambda$, since $\lambda^{\aleph_0} = \lambda$.
The following is an important example.

Example 1.3. Suppose (θ, λ) is a nice pair and suppose μ is a cardinal with $\mu = \mu^{<\theta}$ and $\theta \leq \mu \leq \lambda$. Then $\mathrm{SP}^1_{T_{\mathrm{rg}}}(\lambda, \mu, \theta)$ holds if and only if $\lambda \leq 2^\mu$; and this is equivalent to $\mathrm{SP}^1_{T_{\mathrm{rg}},p_*}(\lambda, \mu, \theta)$ holding for some or any nonalgebraic complete type $p_*(x)$ over a countable model M_*.

Proof. Suppose $M \models T$ has size $\leq \lambda$. See 1.5(C) or recall that:

- $(*)_1$ T_{rg}, the model completion of the theory of graph has elimination of quantifiers
- $(*)_2$ if $A, B \subseteq M$ and $p(x) \in \mathbf{S}^1(A, M)$ then $p(x)$ forks over B (in M) <u>iff</u> some $a \in A \cap B$ realizes $p(x)$

$(*)_3$ any algebraic type $p(x) \in \mathbf{S}^1(A, M)$ is realized in M.

Then the non-algebraic types in $\bigcup\{\mathscr{S}^1(A) : A \subseteq M\}$ correspond naturally to partial functions from M to 2, and so this is just a restatement of Theorem 1.1. □

Theorem 1.4. *Suppose T is a simple theory (in a countable language, as always see 0.1).*

Suppose (θ, λ) is a nice pair:

- (A) $\mathrm{SP}_T(\lambda, \theta)$ *if and only if* $\mathrm{SP}^1_T(\lambda, \lambda, \theta)$.
- (B) *Suppose $p_*(x)$ is a complete type over a countable model M_*, and $\mathrm{SP}^1_{T,p_*}(\lambda, \lambda, \theta)$ holds, and $\mathrm{cof}(\lambda) < \theta$. Then for some μ with $\theta \leq \mu < \lambda, \mathrm{SP}^1_{T,p_*}(\lambda, \mu, \theta)$ holds.*

Proof. (A) forward direction: Suppose $M \models T$ has size $\leq \lambda$, and $M_* \preceq M$ is countable. Choose $N \succeq M$, a θ-saturated model of size λ. Enumerate $N = (a_\alpha : \alpha < \lambda)$.

As $\lambda > \theta \geq \beth_\omega$ by [11], there is $\kappa = \mathrm{cf}(\kappa) < \beth_\omega$ such that $\lambda = \lambda^{[\kappa]}$ which means that there is $\mathscr{S} \subseteq [\lambda]^\kappa$ of cardinality λ such that for every $v \in [\lambda]^\kappa$ there is $u \in [u]^\kappa$ such that $u \in \mathscr{S}$. For $u \in \mathscr{P}$ let $p_u(x)$ be $\{\varphi(x, \bar{b}) : \varphi = \varphi(x, \bar{y}) \in \mathbb{L}(T), \bar{b} \in {}^{\ell g(\bar{b})}N$ and for some $w \in [u]^{<\kappa}$ we have $\alpha \in u \backslash h \Rightarrow N \models \varphi[a_\alpha, \bar{b}]$.

Clearly $p_u(x)$ is a type in N. Let $\mathscr{S}_1 = \{u \in \mathscr{S} : u \in S$ and $p_u(x)$ extends $p_*(x)$ and does not fork over $M_*\}$ and for each $u \in \mathscr{S}_1$ let $q_u(x) \in \mathbf{S}(M)$ extends $p_u(x)$ and does not fork over $M_*\}$. Easily $\mathscr{P} = \{q_u(x) : u \in \mathscr{S}_1\}$ is as required.

(A) reverse direction: Suppose $M \models T$ has size $\leq \lambda$. Using $\mathrm{SP}^1_T(\lambda, \lambda, \theta)$, we can find $N \succeq M$ of size λ, such that every partial type $p(x)$ over M of cardinality less than θ is realized in N using every type in M does not fork over some countable submodel of A (we are also using $\lambda = \lambda^{\aleph_0}$, so there are only λ-many countable elementary submodels M_* of M). If we iterate this θ-many times then we will get a θ-saturated model of T.

(B): Suppose towards a contradiction that $\mathrm{SP}^1_{T,p_*}(\lambda, \mu, \theta)$ failed for all $\theta \leq \mu < \lambda$. Write $\kappa = \mathrm{cof}(\lambda)$, and let $(\mu_\beta : \beta < \kappa)$ be a cofinal sequence of cardinals in λ. For each $\beta < \kappa$, choose $M_\beta \succeq M_*$ with $|M_\beta| \leq \lambda$, witnessing that $\mathrm{SP}^1_{T,p_*}(\lambda, \mu_\beta, \theta)$ fails. We can suppose that $(M_\beta : \beta < \kappa)$ is independent over M_*.

Let $N \models T$ have size $\leq \lambda$ such that each $M_\beta \preceq N$. Then by $\mathrm{SP}^1_{T,p_*}(\lambda, \lambda, \theta)$, we can find $(q_\alpha(x) : \alpha < \lambda)$ such that each $q_\alpha(\lambda)$ extends $p_*(*)$, does not fork over M_* and whenever $q(x) \in \Gamma^\theta_{N,p_*}$, then $q(x) \subseteq q_\alpha(x)$ for some $\alpha < \lambda$.

For each $\beta < \kappa$, we can by hypothesis choose $p_\beta(x) \in \Gamma^\theta_{M^\beta,p_*}$ such that $p_\beta(x) \nsubseteq q_\alpha(x)$ for any $\alpha < \mu_\beta$ note that still $p(x) \supseteq p_*(x)$. By the independence theorem for simple theories, $p(x) := \bigcup\limits_{\beta<\kappa} p_\beta(x)$ does not fork over M_*. Hence $p(x) \subseteq q_\alpha(x)$ for some $\alpha < \lambda$. Choose $\beta < \kappa$ with $\alpha < \mu_\beta$; then this implies that $p_\beta(x) \subseteq q_\alpha(x)$, a contradiction. □

Finally, the following theorem is a collection of most of what has been previously known on $\leq_{\rm SP}$.

Theorem 1.5. *Suppose T is a complete first order theory in a countable language.*

Suppose (θ, λ) is a nice pair:

(A) *If* $\lambda = \lambda^{<\theta}$, *then* $\mathrm{SP}_T(\lambda, \theta)$ *holds; if* T *is unsimple then the converse is true as well. Thus unsimple theories are all* $\leq_{\mathrm{SP}}$*-maximal. (This is proved in* [8]*.)*

(B) T_{rg} *is the* $\leq_{\mathrm{SP}}$*-minimal unstable theory. (This is implicit in* [10]*.)*

(C) *If* T *is stable, then* $\mathrm{SP}_T(\lambda, \theta)$ *holds (this is proved in* [8]*).*

(D) *If* λ *is a strong limit with* $\mathrm{cof}(\lambda) < \theta$ *(hence* $\aleph_0 < \mathrm{cof}(\lambda)$*), and if* $\mathrm{SP}_T(\lambda, \theta)$ *holds, then* T *is stable. (This is implicit in* [10]*.) Thus the stable theories are exactly the minimal* $\leq_{\mathrm{SP}}$*-class. Also, under* GCH*, all unstable theories are maximal.*

(E) *If* $\theta \leq \mu \leq \lambda$ *and* $\mu^{<\theta} = \mu$ *and* $\lambda \leq 2^\mu$, *then* $\mathrm{SP}_{T_{\mathrm{rg}}}(\lambda, \theta)$ *holds. (This is an exercise in* [8]*.)*

(F) *It is consistent that there exists a nice pair* (θ, λ) *such that for all simple* $T, \mathrm{SP}_T(\theta, \lambda)$ *holds. Hence, it is consistent that the unsimple theories are exactly the* $\leq_{\mathrm{SP}}$*-maximal class. (This is proved in* [10]*.)*

For the reader's convenience, we prove (A) through (E), making use of the language of SP^1. Theorem (F) will be a special case of our main theorem, namely Theorem 5.2(B).

Proof. (A): By standard arguments, if $\lambda^{<\theta} = \lambda$ then $\mathrm{SP}_T(\lambda, \theta)$ holds. Suppose T is unsimple, and $\mathrm{SP}_T(\lambda, \theta)$ holds, and suppose towards a contradiction that $\lambda^{<\theta} > \lambda$. Choose a formula $\phi(x, y)$ with the tree property (possibly y is a tuple).

Let $\kappa < \theta$ be least such that $\lambda^\kappa > \lambda$. Choose $M \models T$ and $(a_\eta : \eta \in {}^{<\kappa}\lambda)$ such that for all $\eta \in {}^\kappa\lambda, p_\eta(x) := \{\phi(x, a_{\eta \restriction_\beta}) : \beta < \kappa\}$ is consistent, and for all $\eta \in {}^{<\kappa}\lambda$ and for all $\alpha < \beta < \lambda, \phi(x, a_{\eta^\frown(\alpha)})$ and $\phi(x, a_{\eta^\frown(\beta)})$ are inconsistent. Note that each $|p_\eta(x)| < \theta$; but clearly if $N \succeq M$ realizes each $p_\eta(x)$ then $|N| \geq \lambda^\kappa > \lambda$.
(B): Suppose T is unstable; we show $T_{\mathrm{rg}} \leq_{\mathrm{SP}} T$. By (A), this is true if T is unsimple, so we can suppose that T is simple, hence has the independence property via some formula $\phi(x, y)$. Now suppose (θ, λ) is a nice pair. By Theorem 1.4(A), it suffices to show that if $\mathrm{SP}_T(\lambda, \theta)$ holds, then $\mathrm{SP}^1_{T_{\mathrm{rg}}}(\lambda, \lambda, \theta)$ holds. (Note we cannot apply Example 1.3 because possibly $\lambda^{<\theta} > \lambda$.) Choose some $(a_\alpha : \alpha < \lambda)$ from $\mathfrak{C}$ such that for all $\mathbf{f} : \lambda \to 2, \{\phi(x, a_\alpha)^{\mathbf{f}(\alpha)} : \alpha < \lambda\}$ is consistent. By $\mathrm{SP}_T(\lambda, \theta)$ we can find some θ-saturated $M \prec \mathfrak{C}$ with $|M| \leq \lambda$ and each $a_\alpha \in M$.

Suppose $N \models T_{\mathrm{rg}}$ has cardinality λ say $N = \{a_\alpha : \alpha < \lambda\}$ without repetitions. For each $b \in M, p_b(x)$ to be the complete nonalgebraic type over N, defined by putting $R(x, a_\alpha) \in p_b(x)$ if and only if $M \models \phi(b, a_\alpha)$. Then recalling the proof of

1.3 this witnesses $\mathrm{SP}^1_{T_{\mathrm{rg}}}(\lambda, \lambda, \theta)$ holds (since $|M| \leq \lambda$). [redacted]
(C): Suppose T is stable. It suffices to show that $\mathrm{SP}^1_T(\lambda, \lambda, \theta)$ holds. But this is clear: given $M \models T$ of size $\leq \lambda$ and $M_* \preceq M$ countable, there are only $\leq 2^{\aleph_0} \leq \lambda^{\aleph_0} = \lambda$ many types over M that do not fork over M_*, seeing as types over M_* are stationary.

(D): Suppose towards a contradiction that $\mathrm{SP}_T(\lambda, \theta)$ holds for some unstable T. Then in particular $\mathrm{SP}_{T_{\mathrm{rg}}}(\lambda, \theta)$ holds. Let $p_*(x)$ be a complete non-algebraic type over some countable $M_* \models T_{\mathrm{rg}}$. By Theorem 1.4 we can find $\theta \leq \mu < \lambda$ such that $\mathrm{SP}^1_{T_{\mathrm{rg}}, p_*}(\lambda, \mu, \theta)$ holds. By possibly replacing μ with $\mu^{<\theta}$ we can suppose $\mu = \mu^{<\theta}$. Then this contradicts Example 1.3, since $2^\mu < \lambda$.
(E): By Example 1.3 and Theorem 1.4(A).
(F): See [10] or [3]. □

If the singular cardinals hypothesis holds, then we can say more. Recall that

Definition 1.6. The singular cardinals hypothesis states that if λ is singular and $2^{\mathrm{cof}(\lambda)} < \lambda$, then $\lambda^{\mathrm{cof}(\lambda)} = \lambda^+$. (Note that $2^{\mathrm{cof}(\lambda)} \neq \lambda$ since $\mathrm{cof}(2^\kappa) > \kappa$ for all cardinals κ, by König's theorem.)

The failure of the singular cardinals hypothesis is a large cardinal axiom; see Chapter 5 of [5].

We want the following simple lemma.

Lemma 1.7. *Suppose the singular cardinals hypothesis holds. Suppose θ is regular, $\lambda \geq \theta$, $\lambda^{<\theta} > \lambda$, and $2^{<\theta} \leq \lambda$. Then for every $\mu < \lambda, \mu^{<\theta} < \lambda$. Further, λ is singular of cofinality $< \theta$.*

Proof. First of all, note that $2^{<\theta} < \lambda$, as otherwise $\lambda^{<\theta} = \lambda$.

Now suppose towards a contradiction there were some $\mu < \lambda$ with $\mu^{<\theta} \geq \lambda$; then necessarily $\mu^{<\theta} > \lambda$, as otherwise again $\lambda^{<\theta} = \lambda$. We can choose μ least with $\mu^{<\theta} > \lambda$. Let $\kappa < \theta$ be least such that $\mu^\kappa > \lambda$.

Note that $2^\kappa < \mu$, as otherwise $2^\kappa = (2^\kappa)^\kappa \geq \mu^\kappa > \lambda$, contradicting $2^{<\theta} < \lambda$. Thus, by a consequence of the singular cardinals hypothesis (Theorem 5.22(ii)(b),(c) of [5]), $\mu^\kappa \leq \mu^+$. But since $\mu < \lambda$, $\mu^+ \leq \lambda$, so this is a contradiction.

To finish, suppose towards a contradiction that $\mathrm{cof}(\lambda) \geq \theta$. Then $\lambda^{<\theta} = \lambda + \sup\{\mu^{<\theta} : \mu < \lambda\} = \lambda$, a contradiction. □

This allows us to more intimately connect SP and SP^1:

Theorem 1.8. *Suppose the singular cardinals hypothesis holds, and suppose (θ, λ) is a nice pair. Then $\mathrm{SP}_T(\lambda, \theta)$ holds if and only if either T is stable, or $\lambda = \lambda^{<\theta}$, or else T is simple and for every complete type $p_*(x)$ over a countable model $M_* \models T$, there is some $\theta \leq \mu < \lambda$ with $\mu^{<\theta} = \mu$ and $2^\mu \geq \lambda$, such that $\mathrm{SP}^1_{T,p_*}(\lambda, \mu, \theta)$ holds.*

Proof. If T is stable or $\lambda = \lambda^{<\theta}$, then $\mathrm{SP}_T(\lambda, \theta)$ holds, by Theorem 1.5(A),(C). If T is unsimple, then $\mathrm{SP}_T(\lambda, \theta)$ fails by Theorem 1.5(A). Thus we can assume T is unstable, simple (hence has the independence property) and $\lambda > \lambda^{<\theta}$.

Note that $\mathrm{SP}_T(\lambda, \theta)$ iff $\mathrm{SP}^1_T(\lambda, \lambda, \theta)$ by Theorem 1.4(A), so it suffices to show that $\mathrm{SP}^1_T(\lambda, \lambda, \theta)$ holds if and only if for every complete type $p_*(x)$ over a countable model M_*, there is some $\theta \leq \mu < \lambda$ with $\mu^{<\theta} = \mu$ and $2^\mu \geq \lambda$, such that $\mathrm{SP}^1_{T,p_*}(\lambda, \mu, \theta)$ holds.

Suppose first $\mathrm{SP}^1_T(\lambda, \lambda, \theta)$ holds, and $p_*(x)$ is given. Since T is unstable with the independence property $\mathrm{SP}^1_T(\lambda, \lambda, \theta)$, clearly implies that $2^{<\theta} \leq \lambda$. Hence, by Lemma 1.7, λ is singular with $\mathrm{cof}(\lambda) < \theta$, and there are cofinally many $\mu < \lambda$ with $\mu^{<\theta} = \mu$. By Theorem 1.5(D), λ is not a strong limit. Thus by Theorem 1.4(B), we can find $\theta \leq \mu < \lambda$ such that $\mu = \mu^{<\theta}$ and $2^\mu \geq \lambda$ and $\mathrm{SP}^1_{T,p_*}(\lambda, \mu, \theta)$ holds.

Conversely, we have in particular that each $\mathrm{SP}^1_{T,p_*}(\lambda, \lambda, \theta)$ holds; since $\lambda = \lambda^{\aleph_0} \geq 2^{\aleph_0}$ we get that $\mathrm{SP}^1_T(\lambda, \lambda, \theta)$ holds. □

§ 2. Forcing Axioms

In this section, we introduce the forcing axioms which will produce the desired behavior in SP. It is well-known that the countable chain condition is preserved under finite support iterations; we aim to find generalizations to the κ-closed, κ^+-c.c. context.

Definition 2.1. For a cardinal θ and sets X, Y, define $P_{XY\theta}$ to the forcing notion of all partial functions from X to Y of cardinality less than θ, ordered by inclusion. Note that $P_{XY\theta}$ has the $|Y^{<\theta}|^+$-c.c. by the Δ-system lemma and is θ-closed.

Definition 2.2. Suppose P, Q are forcing notions, and suppose $k \geq 3$ is a cardinal (typically finite). Then say that $P \to_k Q$ if there is a dense subset P_0 of P and a map $F : P_0 \to Q$ such that for all sequences $(p_i : i < i_*)$ from P_0 with $i_* < k$, if $(F(p_i) : i < i_*)$ is compatible in Q (that is, has a common upper bound), then $(p_i : i < i_*)$ has a least upper bound in P; we write $F : (P, P_0) \to_k Q$. Say that $P \to_k^w Q$ (where w stands for weak) if there is a map $F : P \to Q$ such that whenever $(p_i : i < i_*)$ is a sequence from P with $i_* < k$, if $(F(p_i) : i < i_*)$ is compatible in Q, then $(p_i : i < i_*)$ is compatible in P.

Suppose P is a forcing notion, $\aleph_0 < \theta \leq \mu$ are cardinals with θ regular, and $3 \leq k \leq \theta$ is a cardinal (often finite). Then say that P has the $(< k, \mu, \theta)$-amalgamation property if every ascending chain from P of length less than θ has a least upper bound in P, and for some set X, $P \to_k P_{X\mu\theta}$.

For example, $P_{X\mu\theta}$ has the $(< k, \mu, \theta)$-amalgamation property.
The following lemma sums up several obvious facts.

Lemma 2.3. *Suppose $\aleph_0 < \theta \leq \mu$ are cardinals with $\theta = \mathrm{cf}(\lambda) > \aleph_0$, and $3 \leq k \leq \theta$ is a cardinal.*

(1) *If $P \to_k Q$ and $Q \to_k^w Q'$ then $P \to_k Q'$.*
(2) *If P, Q have the $(< k, \mu, \theta)$-amalgamation property, then P forces that $\check{Q}$ has the $(< k, |\mu|, \theta)$-amalgamation property. (We write $|\mu|$ because possibly P collapses μ to θ.) (This is where we use $k \leq \theta$.)*
(3) *Suppose P has the $(< k, \mu, \theta)$-amalgamation property for some $k \geq 3$. Then P is $< \theta$-complete (hence $< \theta$-distributive) and $(\mu^{<\theta})^+$-c.c.*
(4) *If P is θ-closed and has the least upper bound property, then P has the $(< k, \mu, \theta)$-amalgamation property if and only if $P \to_k^w P_{\lambda\mu\theta}$ for some λ.*

We note the following:

Lemma 2.4. *Suppose $\aleph_0 < \theta \leq \mu$ are cardinals with θ regular, and $3 \leq k \leq \theta$. Then P has the $(< k, \mu, \theta)$-amalgamation property if and only if P has the $(< k, \mu^{<\theta}, \theta)$-amalgamation property.*

Proof. Define $\mu' = \mu^{<\theta}$, and let λ be a cardinal. It suffices to show there is a cardinal λ' such that $P_{\lambda\mu'\theta} \to_k^w P_{\lambda'\mu\theta}$, by Lemma 2.3(1). Write $Y' = {}^{<\theta}\mu$; it suffices to find a set X' such that $P_{\lambda Y'\theta} \to_k^w P_{X'\mu\theta}$.

Let $X' = \lambda \times (\theta + 1)$. Define $F : P_{\lambda Y'\theta} \to P_{X'\mu\theta}$ as follows. Let $f \in P_{\lambda Y'\theta}$ be given. Let $\mathrm{dom}(F(f)) = \{((\gamma, \delta) : \gamma \in \mathrm{dom}(f)$ and either $\delta < \mathrm{dom}(f(\gamma))$ or $\delta = \theta\}$. Define $F(f)(\gamma, \delta) = f(\gamma)(\delta)$ if $\delta < \theta$, and otherwise $F(f)(\gamma, \theta) = \mathrm{dom}(f(\gamma))$. Clearly this works. $\square$

The following is key; it states that the $(< k, \mu, \theta)$-amalgamation property is preserved under $< \theta$-support iterations. Note that it follows that the $(< k, \mu, \theta)$-amalgamation property is preserved under $< \theta$-support products.

Theorem 2.5. *Suppose θ is a regular uncountable cardinal, $\mu \geq \theta$ and $3 \leq k \leq \theta$. Suppose $(P_\alpha : \alpha \leq \alpha_*), (\dot{Q}_\alpha : \alpha < \alpha_*)$ is a $< \theta$-support forcing iteration, such that each P_α forces that $\dot{Q}_\alpha$ has the $(< k, |\mu|, \theta)$-amalgamation property. Then P_{α_*} has the $(< k, \mu, \theta)$-amalgamation property.*

Proof. Let λ be large enough.

Inductively, choose $(P^0_\alpha : \alpha \leq \alpha_*, \dot{Q}^0_\alpha : \alpha < \alpha_*)$ a $< \theta$-support forcing iteration, and $(\dot{F}_\alpha : \alpha < \alpha_*)$, such that each P^0_α is dense in P_α, and each P_α forces $\dot{F}_\alpha : (\dot{Q}_\alpha, \dot{Q}^0_\alpha) \to_k \check{P}_{\lambda\mu\sigma}$.

Claim 2.6. *For each $\gamma_* < \theta$, if $(p_\gamma : \gamma < \gamma_*)$ is an ascending chain from P_{α_*}; then it has a least upper bound p in P_{α_*}, such that* $\mathrm{supp}(p) \subseteq \bigcup_{\gamma<\gamma_*} \mathrm{supp}(p_\gamma)$.

Proof. By induction on $\alpha \leq \alpha_*$, we construct $(q_\alpha : \alpha \leq \alpha_*)$ such that each $q_\alpha \in P_\alpha$ with $\mathrm{supp}(q_\alpha) \subseteq \bigcup_{\gamma<\gamma_*} \mathrm{supp}(p_\gamma) \cap \alpha$, and for $\alpha < \beta \leq \alpha_*, q_\beta \restriction_\alpha = q_\alpha$, and for each $\alpha \leq \alpha_*$, q_α is a least upper bound to $(p_\gamma \restriction_\alpha : \gamma < \gamma_*)$ in P_α. At limit stages there is nothing to do; so suppose we have defined q_α. If $\alpha \notin \bigcup_{\gamma<\gamma_*} \mathrm{supp}(p_\gamma)$ then let $q_{\alpha+1} = q_\alpha \frown (0^{\dot{Q}_\alpha})$. Otherwise, since q_α forces that $(p_\gamma(\alpha) : \gamma < \gamma_*)$ is an ascending chain from $\dot{Q}_\alpha$, we can find $\dot{q}$, a P_α-name for an element of $\dot{Q}_\alpha$, such that q_α forces $\dot{q}$ is the least upper bound. Let $q_{\alpha+1} = q_\alpha \frown (\dot{q})$. □

Now suppose $p \in P^0_{\alpha_*}$. Note that $\mathrm{supp}(p) \in [\alpha_*]^{<\theta}$.

It is easy to find, for each $n < \omega$, elements $\mathbf{q}_n(p) \in P^0_{\alpha_*}$ with $\mathbf{q}_0(p) = p$, so that for all $n < \omega$:

- $\mathbf{q}_{n+1}(p) \geq \mathbf{q}_n(p)$;
- For all $\alpha < \alpha_*$, $\mathbf{q}_{n+1}(p) \restriction_\alpha$ decides $\dot{F}_\alpha(\mathbf{q}_n(p)(\alpha))$. (This is automatic whenever $\alpha \notin \mathrm{supp}(\mathbf{a}_n)$, since then P forces that $\dot{F}_\alpha(\mathbf{q}_n(p)) = \emptyset$.)

mn So we can choose $f_{n,\alpha} \in P_{\lambda\mu\sigma}$ such that each $\mathbf{q}_{n+1}(p) \restriction_\alpha$ forces that $\dot{F}_\alpha(\mathbf{q}_n(p)(\alpha)) = \check{f}_{n,\alpha}(p)$.

Let $\mathbf{q}_\omega(p) \in P$ be the least upper bound of $(\mathbf{q}_n(p) : n < \omega)$, which is possible by the claim. Let $P^0 = \{\mathbf{q}_\omega(p) : p \in P^0_{\alpha_*}\}$. For each $q \in P^0$, choose $\mathbf{p}(q) \in P^0_{\alpha_*}$ such that $q = \mathbf{q}_\omega(\mathbf{p}(q))$. For each $n < \omega$, let $\mathbf{p}_n(q) = \mathbf{q}_n(\mathbf{p}(q))$, and for each $\alpha < \alpha_*$, let $f_{n,\alpha}(q) = f_{n,\alpha}(\mathbf{p}(q))$.

Thus we have arranged that for all $q \in P^0$, q is the least upper bound of $(\mathbf{p}_n(q) : n < \omega)$, and for all $n < \omega$ and $\alpha < \alpha_*$, $\mathbf{p}_{n+1}(q) \restriction_\alpha$ forces that $\dot{F}_\alpha(\mathbf{p}_n(q)(\alpha)) = \check{f}_{n,\alpha}(q)$.

Write $X = \omega \times \alpha_* \times \lambda$. Choose $F : P^0 \to P_{X\mu\theta}$ so that for all $q, q' \in P^0$, if $F(q)$ and $F(q')$ are compatible, then for all $n < \omega$ and for all $\alpha < \alpha_*$, $f_{n,\alpha}(q)$ and $f_{n,\alpha}(q')$ are compatible. For instance, let the domain of $F(q)$ be the set of all (n, α, β) such that β is in the domain of $f_{n,\alpha}$, and let $F(q)(n, \alpha, \beta) = f_{n,\alpha}(\beta)$.

Now suppose $(q_i : i < i_*)$ is a sequence from P^0 with $i_* < k$, such that $(F(q_i) : i < i_*)$ are compatible. Write $\Gamma = \bigcup_{i<i_*, n<\omega} \mathrm{supp}(\mathbf{p}_n(q_i))$.

By induction on $\alpha \leq \alpha_*$, we construct a least upper bound s_α to $(\mathbf{p}_n(q_i) \restriction_\alpha : i < i_*, n < \omega)$ in P_α, such that $\mathrm{supp}(s_\alpha) \subseteq \Gamma \cap \alpha$, and for $\alpha < \alpha'$, $s_{\alpha'} \restriction_\alpha = s_\alpha$.

Limit stages of the induction are clear. So suppose we have constructed s_α. If $\alpha \notin \Gamma$ clearly we can let $s_{\alpha+1} = s_\alpha {}^\frown (0^{\dot{Q}_\alpha})$; so suppose instead $\alpha \in \Gamma$. Let $n < \omega$ be given. Then $(f_{n,\alpha}(q_i) : i < i_*)$ are compatible, and s_α forces that $\dot{F}_\alpha(\mathbf{p}_n(q_i)(\alpha)) = \check{f}_{n,\alpha}(\check{q}_i)$ for each $i < i_*$, since $\mathbf{p}_{n+1}(q_i) \restriction_\alpha$ does. Thus s_α forces that $(\mathbf{p}_n(q_i)(\alpha) : i < i_*)$ has a least upper bound $\dot{r}_n$. Now s_α forces that $(\dot{r}_n : n < \omega)$ is an ascending chain in $\dot{Q}_\alpha$, so let $\dot{q}$ be such that s_α forces $\dot{q}$ is a least upper bound to $(\dot{r}_n : n < \omega)$. Let $s_{\alpha+1} = s_\alpha {}^\frown (\dot{q})$.

Thus the induction goes through, and s_{α_*} is a least upper bound $(q_i : i < i_*)$. □

The following class of forcing axioms, for $k = 2$, is related to Shelah's $\mathrm{Ax}\mu_0$ from [9] although the formulation is different. Although it is not relevant for the current paper, we could have allowed $\theta = \aleph_0$ with some minor changes to the proof of Theorem 2.5; this would then give weakenings of Martin's Axiom.

Definition 2.7. Suppose $\aleph_0 < \theta = \theta^{<\theta} \leq \lambda$, and suppose $2 \leq k < \omega$. Then say that $\mathrm{Ax}(k, \theta, \lambda)$ holds if for every forcing notion P such that $|P| \leq \lambda$ and P has the (k, θ, θ)-amalgamation property, if $(D_\alpha : \alpha < \lambda)$ is a sequence of dense subsets of P, then there is an ideal of P meeting each D_α. (By dense, we mean upwards dense: for every $p \in P$, there is $q \in D_\alpha$ with $q \geq p$.) Say that $\mathrm{Ax}(k, \theta)$ holds iff $\mathrm{Ax}(k, \theta, \lambda)$ holds for all $\lambda < 2^\theta$.

By a typical downward Lowenheim-Skolem argument we could drop the condition that $|P| \leq \lambda$ in $\mathrm{Ax}(k, \theta, \lambda)$, but we won't need this. Note that $P_{\theta\mu\theta}$ collapses μ to θ, so this is why there is not a parameter for μ in $\mathrm{Ax}(k, \theta)$. Finally, note that $\mathrm{Ax}(k, \theta, \lambda)$ implies that $2^\theta > \lambda$, easily.

Theorem 2.8. *Suppose $\aleph_0 < \theta \leq \mu \leq \lambda$ are cardinals such that θ is regular and $\mu = \mu^{<\theta}$, and suppose $3 \leq k \leq \theta$. Suppose $\kappa \geq 2^\lambda$ has $\kappa^{<\kappa} = \kappa$. Then there is a forcing notion P with the $(< k, \mu, \theta)$-amalgamation property (in particular, θ-closed and μ^+-c.c.), such that P forces that $Ax(k, \theta)$ holds and that $2^\theta = \kappa$. We can arrange $|P| = \kappa$.*

Proof. Let $(P_\alpha : \alpha \leq \kappa), (\dot{Q}_\alpha : \alpha < \kappa)$ be a $< \theta$-support iteration, such that (viewing P_α-names as P_β-names in the natural way, for $\alpha \leq \beta < \kappa$):

- Each P_α forces that $\dot{Q}_\alpha$ has the $(< k, \mu, \theta)$-amalgamation property;
- Whenever $\alpha < \kappa$, and $\dot{Q}$ is a P_α-name such that $|\dot{Q}| < \kappa$ and P_α forces $\dot{Q}$ has the $(< k, \mu, \theta)$-amalgamation property, then there is some $\beta \geq \alpha$ such that P_β forces that $\dot{Q}_\beta$ is isomorphic to $\dot{Q}$;
- Each $|P_\alpha| \leq \kappa$.

This is possible by the μ^+-c.c., as in the proof of the consistency of Martin's axiom, and using Lemma 2.3(2). The point is that at each stage α, if P_α forces that $|\dot{Q}| = \lambda' < \kappa$, then we can choose a P_α-name $\dot{Q}'$ such that P_α-forces $\dot{Q} \cong \dot{Q}'$ and that $\dot{Q}'$ has universe λ'; then there are only $|P_\alpha|^{\lambda' \cdot \mu} \leq \kappa$-many possibilities for $\dot{Q}'$, up to P_α-equivalence. Thus we can eventually deal with all of them.

P_{α_*} then works, easily. □

We now relate this to model theory.

Definition 2.9. Suppose (θ, λ) is a nice pair, and $\theta \leq \mu \leq \lambda$, and T is simple. Then say that T has $(< k, \lambda, \mu, \theta)$-type amalgamation if whenever $M \models T$ has size $\leq \lambda$, and whenever $M_* \preceq M$ is countable, then Γ^θ_{M,M_*} has the $(< k, \mu, \theta)$-amalgamation property, or equivalently, $\Gamma^\theta_{M,M_*} \rightarrow^w_k P_{X\mu\theta}$ for some set X.

We prove some simple facts.

Lemma 2.10. *Suppose T fails the $(< k, \lambda, \mu, \theta)$-amalgamation property, and P has the $(< k, \mu, \theta)$-amalgamation property. Then P forces that $\check{T}$ fails the $(< k, \lambda, \mu, \theta)$-amalgamation property.*

Proof. It suffices to show that if Q is a forcing notion and P forces that $\check{Q} \rightarrow^w_k \check{P}_{\check{X}\mu\theta}$, then $Q \rightarrow^w_k P_{X'\mu\theta}$ for some X', by Lemma 2.3(4). (We then apply this to $Q = \Gamma^\theta_{M,M_*}$ witnessing the failure of $(< k, \mu, \theta)$-amalgamation.)

Choose some $F_* : (P, P_0) \rightarrow_k P_{X_*\mu\theta}$, and let $\dot{G}$ be a P-name so that P forces $\dot{F} : \check{Q} \rightarrow^w_k P_{\check{Y}\mu\theta}$. For every $q \in Q$, choose $\mathbf{p}(q) \in P_0$ such that $\mathbf{p}(q)$ decides $\dot{F}(\check{q})$, say $\mathbf{p}(q)$ forces that $\dot{F}(\check{q}) = f(q)$. Choose $F : Q \rightarrow P_{X\mu\theta}$ so that if $F(q)$ and $F(q')$ are compatible, then $f(q)$ and $f(q')$ are compatible, and $F_*(\mathbf{p}(q))$ and $F_*(\mathbf{p}(q'))$ are compatible.

Suppose $(q_i : i < i_*)$ is a sequence from Q with $(F(q_i) : i < i_*)$ compatible in $P_{X\mu\theta}$. Then $(F_*(\mathbf{p}(q_i)) : i < i_*)$ are all compatible in $P_{X_*\mu\theta}$, so $(\mathbf{p}(q_i) : i < i_*)$ are compatible in P_0 with the least upper bound p. Then p forces each $\dot{F}(\check{q}_i) = f(q_i)$. But also (by choice of F), $(f(q_i) : i < i_*)$ are compatible in $P_{Y,\mu,\theta}$, so p forces that $(\check{q}_i : i < i_*)$ is compatible in $\check{Q}$, i.e. $(q_i : i < i_*)$ is compatible in Q. □

Theorem 2.11. *Suppose T simple, and $\aleph_0 < \theta = \theta^{<\theta} \leq \lambda = \lambda^{\aleph_0}$, and $Ax(k, \theta)$ holds. Suppose $2^\theta > \lambda^{<\theta}$, and suppose $3 \leq k \leq \aleph_0$. Then the following are equivalent:*

(A) *T has $(< k, \lambda, \theta, \theta)$-type amalgamation;*
(B) *$SP^1_T(\lambda, \theta, \theta)$ holds.*

Proof. (B) implies (A) is obvious. For (A) implies (B): let $M \models T$ have size at most λ and let $M_* \preceq M$ be countable. Let P be the $< \theta$-support product of Γ^θ_{M,M_*}; then P has the $(< k, \theta, \theta)$-amalgamation property and $|P| \leq \theta^{<\theta}$. For each $p(x) \in \Gamma^\theta_{M,M_*}$ let D_p be the dense subset of P consisting of all $f \in P$ such that for some $\gamma \in \mathrm{dom}(f)$, $f(\gamma)$ extends $p(x)$. By $\mathrm{Ax}(k, \lambda^{<\theta}, \theta)$ we can choose an ideal I of P meeting each D_p. This induces a sequence $(p_\gamma(x) : \gamma < \theta)$ of partial types over M that do not fork over M_*, such that for all $p(x) \in \Gamma^\theta_{M,M_*}$ there is $\gamma < \theta$ with $p(x) \subseteq p_\gamma(x)$. To finish, extend each $p_\gamma(x)$ to a complete type over M not forking over M_*.

The final claim follows from Theorem 1.4(A). □

§ 3. Non-Forking Diagrams

Suppose T is a simple theory in a countable language. We wish to study various type amalgamation properties of T; in particular we will be looking at systems of types $(p_s(x) : s \in P)$ over a system of models $(M_s : s \in P)$, for some $P \subseteq \mathcal{P}(I)$ closed under subsets. For this to be interesting, we need $(M_s : s \in P)$ to be independent in a suitable sense, which we define in this section.

The following definition is similar to the first author's definition of independence in [8] in the context of stable theories, see Section XII.2. In fact we are modeling our definition after Fact 2.5 there (we cannot take the definition exactly from [8] because we allow P to contain infinite subsets of I).

Definition 3.1. Let T be simple.

Suppose I is an index set and $P \subseteq \mathcal{P}(I)$ is downward closed. Say that $(A_s : s \in P)$ is a diagram (of subsets of $\mathfrak{C}$) if each $A_s \subseteq \mathfrak{C}$ and $s \subseteq t$ implies $A_s \subseteq A_t$. Say that $(A_s : s \in P)$ is a non-forking diagram if for all $s_i : i < n, t \in P$, $\bigcup_{i<n} A_{s_i}$ is free from A_t over $\bigcup_{i<n} A_{s_i \cap t}$.. Say that $(A_s : s \in P)$ is a continuous diagram if for every $X \subseteq P$, $\bigcap_{s \in X} A_s = A_{\bigcap X}$. (If X is finite then this is a consequence of non-forking.)

Note that $(A_s : s \in P)$ is continuous if and only if for every $a \in \bigcup_{s \in P} A_s$, there is some least $s \in P$ with $a \in A_s$. Also note that if $(A_s : s \in P)$ is non-forking (continuous) and $Q \subseteq P$ is downward closed then $(A_s : s \in Q)$ is non-forking (continuous).

Lemma 3.2. *Suppose $(A_s : s \in P)$ is a diagram of subsets of $\mathfrak{C}$. Then the following are equivalent:*

(A) *For all downward-closed subsets $S, T \subseteq P$, $\bigcup_{s \in S} A_s$ is free from $\bigcup_{t \in T} A_t$ over $\bigcup_{s \in S \cap T} A_s$.*
(B) *For all $s_i : i < n, t_j : j < m$ from P, $\bigcup_{i<n} A_{s_i}$ is free from $\bigcup_{j<m} A_{t_j}$ over $\bigcup_{i<n, j<m} A_{s_i \cap t_j}$.*
(C) *$(A_s : s \in P)$ is non-forking.*

Proof. (A) implies (B) implies (C) is trivial. For (B) implies (A), use local character of nonforking and monotonicity.

We show (C) implies (B). So suppose $(A_s : s \in P)$ is non-forking. By induction on m, we show that for all n, if $s_i : i < n, t_j : j < m$ are from P, then $\bigcup_{i<n} A_{s_i}$ is free from $\bigcup_{j<m} A_{s_j}$ over $\bigcup_{i<n, j<m} A_{s_i \cap t_j}$. $m = 1$ is the definition of non-forking diagrams. Suppose true for all $m' \leq m$ and we show it holds at $m+1$; so we have $s_i : i < n, t_j : j < m+1$. Let $A_* = \bigcup_{i<n} A_{s_i}$ and let $B_* = \bigcup_{j<m} A_{t_j}$. By inductive hypothesis applies at $(s_i : i < n, t_m), (t_j : j < m)$, we get that $A_* \cup A_{t_m}$ is free from B_* over $(A_* \cup A_{t_m}) \cap B_*$. By monotonicity, A_* is free from $B_* \cup A_{t_m}$ over $(A_* \cap B_*) \cup A_{t_m}$. By the inductive hypothesis applied at $(s_i : i < n), t_m$, we get that A_* is free from A_{t_m} over $A_* \cap A_{t_m}$, so by monotonicity we get that A_* is free from $(A_* \cap B_*) \cup A_{t_m}$ over $A_* \cap (B_* \cup A_{t_m})$. □

The following lemma is similar to Lemma 2.3 from [8] Section XII.2.

Lemma 3.3. *Suppose $P \subseteq \mathcal{P}(I)$ is downward closed and $(A_s : s \in P)$ is a continuous diagram of subsets of $\mathfrak{C}$. Suppose there is a well-ordering $<_*$ of $\bigcup_s A_s$ such* *that for all $a \in \bigcup_s A_s$, a is free from $\{b \in \bigcup_s A_s : b <_* a\}$ over $\{b \in s_a : b <_* a\}$, where s_a is the least element of P with $a \in A_{s_a}$. Then $(A_s : s \in P)$ is non-forking.*

Proof. Let $(a_\alpha : \alpha < \alpha_*)$ be the $<_*$-increasing enumeration of $\bigcup_s A_s$, and let s_α be the least element of P with $a_\alpha \in A_{s_\alpha}$. For each $\alpha \leq \alpha_*$ and for each $s \in P$ let $A_{s,\alpha} = A_s \cap \{a_\beta : \beta < \alpha\}$. We show by induction on α that $(A_{s,\alpha} : s \in P)$ is non-forking. In fact we show (B) holds of Lemma 3.2 (due to symmetry it is easier).

Limit stages are clear. So suppose we have shown $(A_{s,\alpha} : s \in P)$ is non-forking. Let $(s_i : i < n), (t_j : j < m) \in P$ be given. We wish to show that $\bigcup_{i<n} A_{s_i,\alpha+1}$ is free from $\bigcup_{j<m} A_{t_j \cap \alpha+1}$ over $\bigcup_{i<n, j<n} A_{s_i \cap t_j, \alpha+1}$. If $a_\alpha \notin s_i$ and $a_\alpha \notin t_j$ for each $i < n$ then we conclude by the inductive hypothesis. If $a_\alpha \in s_{i_*} \cap t_{j_*}$ for some $i_* < n$, $j_* < m$, then we conclude by the inductive hypothesis and the fact that a_α is free from $\bigcup_{i<n} A_{s_i,\alpha} \cup \bigcup_{j<m} A_{t_j,\alpha}$ over $A_{s_{i_*} \cap t_{j_*},\alpha}$, since $s_{i_*} \cap t_{j_*}$ contains s_α. If $a_\alpha \in s_i$ for some $i < n$ and $a_\alpha \notin t_j$ for any $j < m$, then reindex so that there is $0 < i_* \leq n$ so that $a_\alpha \in s_i$ iff $i < i_*$. Now a_α is free from $\{a_\beta : \beta < \alpha\}$ over s_α, so by monotonicity, $\bigcup_{i<n} A_{s_i,\alpha+1}$ is free from $\bigcup_{j<m} A_{s_j,\alpha+1}$ over $\bigcup_{i<n} A_{s_i,\alpha}$; use transitivity and the inductive hypothesis to finish. □

For the proof of the following, the reader may find it helpful to bear in mind the special case when T is supersimple, so that every type does not fork over a finite subset of its domain. In that case we can in fact get $(M_s : s \in [\lambda]^{<\aleph_0})$ to cover $\mathbf{A}$.

Theorem 3.4. *Suppose T is a simple theory in a countable language, and suppose $\mathbf{A}$ is a set of cardinality λ, where $\lambda = \lambda^{\aleph_0}$. Then we can find a continuous, non-forking diagram of models $(M_s : s \in [\lambda]^{\leq\aleph_0})$ such that $\mathbf{A} \subseteq \bigcup_s M_s$, and such that for all $S \subseteq \lambda$, $\bigcup_{s \in [S]^{\leq\aleph_0}} M_s$ has size at most $|S| \cdot \aleph_0$.*

Proof. Enumerate $\mathbf{A} = (a_\alpha : \alpha < \lambda)$.

We define $(\mathrm{cl}(\{\alpha\}) : \alpha < \lambda)$ inductively as follows, where each $\mathrm{cl}(\{\alpha\})$ is a countable subset $\alpha+1$ with $\alpha \in \mathrm{cl}(\{\alpha\})$. Suppose we have defined $(\mathrm{cl}(\{\beta\}) : \beta < \alpha)$. Choose a countable set $\Gamma \subseteq \alpha$ such that a_α is free from $\{a_\beta : \beta < \alpha\}$ over $\bigcup_{\beta \in \Gamma} a_\beta$; put $\mathrm{cl}(\{\alpha\}) = \{\alpha\} \cup \bigcup_{\beta \in \Gamma} \mathrm{cl}(\{\beta\})$. (So, if T is supersimple, each Γ can be chosen to be finite.)

Now, for each $s \subseteq \lambda$, let $\mathrm{cl}(s) := \bigcup_{\alpha \in s} \mathrm{cl}(\{\alpha\})$. Say that $A \subseteq \lambda$ is closed if $\mathrm{cl}(A) = A$; this satisfies the usual properties of a set-theoretic closure operation, that is $\mathrm{cl}(A) \supseteq A$, and $A \subseteq B$ implies $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$, and $\mathrm{cl}^2(A) = \mathrm{cl}(A)$, and cl is finitary: in fact $\mathrm{cl}(A) = \bigcup_{\alpha \in A} \mathrm{cl}(\{\alpha\})$, which is even stronger. Finally, $|\mathrm{cl}(A)| \leq |A| + \aleph_0$.

For each $s \in [\lambda]^{\leq\omega}$, let $A_s = \{a_\alpha : \alpha < \lambda \text{ and } \mathrm{cl}(\{\alpha\}) \subseteq s\}$. Since each $a_\alpha \in A_{\mathrm{cl}(\{\alpha\})}$, clearly $\bigcup_s A_s = \mathbf{A}$. I claim that $(A_s : s \in [\lambda]^{\leq\omega})$ is a non-forking diagram of sets. But this follows from Lemma 3.3, since each a_α is free from $\{a_\beta : \beta < \alpha\}$ over $A_{\mathrm{cl}(\{\alpha\})} \cap \{a_\beta : \beta < \alpha\}$.

For each $\alpha \leq \lambda$, let $\mathcal{A}_\alpha = \{\mathrm{cl}(s) : s \in [\alpha]^{<\omega}\}$. I show by induction on $\alpha \leq \lambda$ that $(\mathcal{A}_\alpha, \subset)$ is well-founded. Note that since $\mathbf{A} = \bigcup \mathcal{A}_\lambda$, it will follow that $(A_s : s \in [\lambda]^{\leq\aleph_0})$ is continuous. Since $\mathcal{A}_\alpha$ is an end extension of $\mathcal{A}_\beta$ for $\alpha > \beta$, the limit stage is clear. So suppose we have shown $(\mathcal{A}_\alpha, \subset)$ is well-founded.

Write $X = \mathrm{cl}(\{\alpha\}) \cap \alpha$; note that $\mathrm{cl}(X) = X$. Now suppose $s, t \in [\alpha]^{<\omega}$. I claim that $\mathrm{cl}(s \cup \{\alpha\}) \subseteq \mathrm{cl}(t \cup \{\alpha\})$ iff $\mathrm{cl}(s \cup X) \subseteq \mathrm{cl}(t \cup X)$. But this is clear, since

$\mathrm{cl}(s \cup \{\alpha\}) = \mathrm{cl}(s) \cup X \cup \{\alpha\}$, and $\mathrm{cl}(t \cup \{\alpha\}) = \mathrm{cl}(t) \cup X \cup \{\alpha\}$, and $\mathrm{cl}(s \cup X) = \mathrm{cl}(s) \cup X$, and $\mathrm{cl}(t \cup X) = \mathrm{cl}(t) \cup X$.

Thus it follows from the inductive hypothesis that $(\{\mathrm{cl}(s \cup \{\alpha\}) : s \in [\alpha]^{<\omega}\}, \subset)$ is well-founded, and hence that $\mathcal{A}_{\alpha+1}$ is well-founded; hence $\mathcal{A}_\lambda$ is well-founded.

Let $<_*$ be a well-order of $\mathcal{A}_\lambda$ refining $\subset$. Now by induction on $<_*$, choose countable models $(M(A) : A \in \mathcal{A}_\lambda)$ so that $M(A) \supseteq A$ and such that $M(A)$ is free from $\mathbf{A} \cup \bigcup\{M(B) : B \in \mathcal{A}_\lambda, B <_* A\}$ over $A \cup \bigcup\{M(B) : B \in \mathcal{A}, B \subset A\}$. Finally, given $s \in [\lambda]^{\leq\omega}$, let $M_s := M(A_s)$. This is a non-forking diagram of models, using Lemma 3.3, and it is clearly continuous.

The final claim follows, since for all $S \subseteq \lambda$, $\{t \in \mathcal{A} : t \subseteq S\}$ has size at most $|S| \cdot \aleph_0$. $\square$

§ 4. Amalgamation properties

Suppose T is a simple theory in a countable language. We now explain what we mean by T having type amalgamation.

Definition 4.1. Given $\Lambda \subseteq {}^n m$, let P_Λ be the set of all partial functions from n to m which can be extended to an element of Λ; so P_Λ is a downward-closed subset of $n \times m$, and Λ is the set of maximal elements of P_λ.

Suppose $(M_u : u \subseteq n)$ is a non-forking diagram of models. Then by a $(\Lambda, \overline{M})$-array, we mean a non-forking diagram of models $(N_s : s \in P_\Lambda)$, together with maps $(\pi_s : s \in P_\Lambda)$ such that each each $\pi_s : M_{\text{dom}(s)} \cong N_s$, and such that $s \subseteq t$ implies $\pi_s \subseteq \pi_t$.

Definition 4.2. Suppose $\Lambda \subseteq {}^n m$. Then T has Λ-type amalgamation if, whenever $(M_u : u \subseteq n)$ is a non-forking diagram of models, and whenever $p(x)$ is a complete type over M_n in finitely many variables which does not fork over M_0, and whenever $(N_s, \pi_s : s \in P_\Lambda)$ is a $(\Lambda, \overline{M})$-array, then $\bigcup_{\eta \in \Lambda} \pi_\eta(p(x))$ does not fork over N_0.

Suppose $3 \leq k \leq \aleph_0$; then say that T has $< k$-type amalgamation if whenever $|\Lambda| < k$, then T has Λ-type amalgamation.

The following lemma is straightforward.

Lemma 4.3. *Suppose $\Lambda \subseteq {}^n m$. Then in the definition of Λ-type amalgamation, the following changes would not matter:*

(A) *We could restrict to just countable models M_u.*
(B) *We could allow $p(x)$ to be any partial type, or insist it is a single formula. Also, we could replace x by a tuple $\overline{x}$ of arbitrary cardinality.*

Example 4.4. Every simple theory has < 3-type amalgamation. T_{rg} has $< \aleph_0$-type amalgamation.

Example 4.5. Suppose $\ell > k \geq 2$. Let $T_{\ell,k}$ be the theory of the random k-ary, ℓ-clique free hypergraph; these examples were introduced by Hrushovski [4], where he proved $T_{\ell,k}$ is simple if and only if $k \geq 3$.

For $k \geq 3$, $T_{\ell,k}$ has $< k$-type amalgamation but not $< k+1$-type amalgamation.

Proof. First suppose $\Lambda \subseteq {}^n m$ with $|\Lambda| < k$, and $(M_u : u \subseteq n)$ are given, and suppose $p(\overline{x})$ is a complete type over M_n. Suppose towards a contradiction there were a $(\Lambda, \overline{M})$-array $(N_s, \pi_s : s \in P_\Lambda)$ with $\bigcup_{\eta \in \Lambda} \pi_\eta[p(\overline{x})]$ inconsistent. Write $q(\overline{x}) = \bigcup_{\eta \in \Lambda} \pi_\eta[p(\overline{x})]$; then $q(\overline{x})$ must create some ℓ-clique $(a_i :< \ell_0), (x_j : j < \ell_1)$, where $\ell_0 + \ell_1 = \ell$, and each $a_i \in N_\eta$ for some $\eta \in \Lambda$, and each $x_j \in \overline{x}$. Clearly we have each $\ell_0, \ell_1 > 0$.

For each $i < \ell_0$, let $h(i)$ be the least $s \in P_\Lambda$ with $a_i \in N_s$. The following must hold:

(I) For every $u \in [\ell_0]^{<k}$, $h[u] \in P_\Lambda$;
(II) $h[\ell_0] \not\in P_\Lambda$.

By (II), for each each $\eta \in \Lambda$ we must have $h[\ell_0] \not\subseteq \eta$; thus we can choose $i_\eta < \ell_0$ such that $h(i_\eta) \not\subseteq \eta$. Let $u = \{i_\eta : \eta \in \Lambda\} \in [\ell_0]^{<k}$. Clearly then $h[u] \not\in P_\Lambda$, but this contradicts (I).

Now we show that $T_{\ell,k}$ fails $< k+1$-type amalgamation. Indeed, let $\Lambda \subseteq {}^k 2$ be the set of all $f : k \to 2$ for which there is exactly one $i < k$ with $f(i) = 1$; so $|\Lambda| = k$. Also, let $(M_u : u \subseteq k)$ be a non-forking diagram of models so that there are $a_i \in M_{\{i\}}$ for $i < k$ and there are $b_j \in M_0$ for $n < \ell - k - 1$, such that every k-tuple of distinct elements from $(a_i, b_j : i < k, j < \ell - k - 1)$ is in R except for $(a_i : i < k)$. Let $p(x)$ be the partial type over M_k which asserts that $R(x, \overline{a})$ holds for every $k-1$-tuple of distinct elements from $(a_i, b_j : i < k, j < \ell - k - 1)$.

It is not hard to find a $(\Lambda, \overline{M})$-array $(N_s, \pi_s : s \in P_\Lambda)$ such that, if we write $\pi_{\{(i,0)\}}(a_i) = c_i$, then $R(c_i : i < k)$ holds; but now we are done, since $\bigcup_{f \in \Lambda} \pi_f[p(x)]$ is inconsistent. $\square$

The following is the key consequence of $< k$-type amalgamation.

Theorem 4.6. *Suppose T is a simple theory with $< k$-type amalgamation. Then for all nice pairs (θ, λ), T has $(< k, \lambda, \theta, \theta)$-type amalgamation.*

Proof. By Theorem 3.4, it suffices to show that if $(\mathbf{M}_s : s \in [\lambda]^{<\theta})$ is a continuous non-forking diagram of countable models such that each $|\mathbf{M}_s| < \theta$, then writing $\mathbf{M} = \bigcup_s \mathbf{M}_s$, we have that $\Gamma^\theta_{\mathbf{M},\mathbf{M}_0} \to^w_k P_{X\theta\theta}$ for some X. Let $<_*$ be a well-ordering of $\mathbf{M}$.

Given $A \in [\mathbf{M}]^{<\theta}$ let s_A be the $\subseteq$-minimal $s \in [\lambda]^{<\theta}$ with $A \subseteq M_{s_A}$, possible by continuity.

Let P be the set of all $p(x) \in \Gamma^\theta_{\mathbf{M},\mathbf{M}_0}$ such that for some $s \in [\lambda]^{<\theta}$, $p(x)$ is a complete type over $\mathbf{M}_s$; we write $p(x, \mathbf{M}_s)$ to indicate this. P is dense in $\Gamma^\theta_{\mathbf{M},\mathbf{M}_0}$, so it suffices to show that $P \to^w_k P_{\lambda\theta\theta}$ for some λ.

Choose X large enough, and $F : P \to P_{X\theta\theta}$ so that if $F(p(x), \mathbf{M}_s)$ is compatible with $F(q(x), \mathbf{M}_t)$, then:

- s and t have the same order-type, and if we let $\rho : s \to t$ be the unique order-preserving bijection, then ρ is the identity on $s \cap t$;
- $\mathbf{M}_s$ and $\mathbf{M}_t$ have the same $<_*$-order-type, and the unique $<_*$-preserving bijection from $\mathbf{M}_s$ to $\mathbf{M}_t$ is in fact an isomorphism $\tau : \mathbf{M}_s \cong \mathbf{M}_t$
- For each finite $\overline{a} \in \mathbf{M}_s^{<\omega}$, if we write $s' = s_{\overline{a}}$ and if we write $t' = s_{\tau(\overline{a})}$, then: $\rho[s'] = t'$ and $\tau \restriction_{\mathbf{M}_{s'}} : \mathbf{M}_{s'} \cong \mathbf{M}_{t'}$.
- $\tau[p(x)] = q(x)$.

This is not hard to do. Note that it follows that for every $s' \subseteq s$, $\rho \restriction_{\mathbf{M}_{s'}} : \mathbf{M}_{s'} \cong \mathbf{M}_{\rho[s']}$, since $\mathbf{M}_{s'} = \bigcup\{\mathbf{M}_{s_{\overline{a}}} : \overline{a} \in (\mathbf{M}_{s'})^{<\omega}\}$ and similarly for $\mathbf{M}_{t'}$.

I claim that F works.

So suppose $p_i(x, \mathbf{M}_{s_i}) : i < i_*$ is a sequence from P for $i_* < k$, such that $(F(p_i(x)) : i < i_*)$ is compatible in $P_{\lambda\theta\theta}$.

Let γ_* be the order-type of some or any s_i. Enumerate each $s_i = \{\alpha_{i,\gamma} : \gamma < \gamma_*\}$ in increasing order. Let E be the equivalence relation on γ_* defined by: $\gamma E \gamma'$ iff for all $i, i' < k$, $\alpha_{i,\gamma} = \alpha_{i',\gamma}$ iff $\alpha_{i,\gamma'} = \alpha_{i',\gamma'}$. Let $(E_j : j < n)$ enumerate the equivalence classes of E. For each $i < i_*$, and for each $j < n$, let $X_{i,j} = \{\alpha_{i,\gamma} : \gamma \in E_j\}$. Thus s_i is the disjoint union of $X_{i,j}$ for $j < n$. Moreover, $X_{i,j} \cap X_{i',j'} = \emptyset$ unless $j = j'$; and if $X_{i,j} \cap X_{i',j} \neq \emptyset$ then $X_{i,j} = X_{i',j}$. For each $j < n$, enumerate $\{X_{i,j} : i < i_*\} = (Y_{\ell,j} : \ell < m_i)$ without repetitions. Let $m = \max(m_j : j < n)$; and for each $i < i_*$, define $\eta_i \in {}^n m$ via: $\eta_i(j) =$ the unique $\ell < m_i$ with $X_{i,j} = Y_{\ell,j}$.

Let $\Lambda = \{\eta_i : i < i_*\}$. For each $s \in P_\Lambda$, let $N_s = \mathbf{M}_{t_s}$ where $t_s = \bigcup_{(j,\ell)\in s} Y_{\ell,j}$. Also, define $(M_u : u \subseteq n) := (N_{\eta_0\restriction_u} : u \subseteq n)$. Then the hypotheses on F give commuting isomorphisms $\pi_s : M_{\mathrm{dom}(s)} \cong N_s$ for each $s \in P_\Lambda$, in such a way that $(\overline{N}, \overline{\pi})$ is a $(\lambda, \overline{M})$-array, and each $\pi_{\eta_i}(p_0(x)) = p_i(x)$. It follows by hypothesis on T that $\bigcup_{i<i_*} p_i(x)$ does not fork over N_0, as desired.

□

Corollary 4.7. *Suppose T is simple, with $< \aleph_0$-type amalgamation.*

- (A) *Suppose θ is a regular uncountable cardinal. Then for any $M \models T$ and any $M_0 \preceq M$ countable, Γ^θ_{M,M_0} has the $(< \aleph_0, \theta, \theta)$-amalgamation property.*
- (B) *Suppose (θ, λ) is a nice pair, and suppose that $\theta \leq \mu \leq \lambda$ satisfies $\mu = \mu^{<\theta}$ and $2^\mu \geq \lambda$. Then $SP^1_T(\lambda, \mu, \theta)$ holds.*
- (C) *If the singular cardinals hypothesis holds, then $T \leq_{SP} T_{rg}$.*

Proof. (A) follows immediately from Theorem 4.6, and (C) follows from (B) by Theorem 1.4(A) and Theorem 1.8. So it suffices to verify (B).

Suppose $M \models T$ has $|M| \leq \lambda$, and suppose $M_0 \preceq M$ is countable. Choose some $F : \Gamma^\theta_{M,M_0} \to^w_k P_{\lambda\theta\theta}$. By Corollary 1.1, we can find $(\mathbf{f}_\gamma : \gamma < \mu)$ such that whenever $f \in P_{\lambda\theta\theta}$ then $f \subseteq \mathbf{f}_\gamma$ for some $\gamma < \mu$; for each $\gamma < \mu$, choose $q_\gamma(x)$, a complete type over M not forking over M_0, and extending $\bigcup\{p(x) : F(p(x)) \subseteq \mathbf{f}_\gamma\}$. Then clearly $(q_\gamma(x) : \gamma < \mu)$ witnesses $SP^1_T(\lambda, \mu, \theta)$. □

§ 5. Conclusion

We begin to put everything together. We aim to produce a forcing extension in which, whenever T has $< k$-type amalgamation, then $T_{k,k-1} \not\leq_{SP} T$. We will choose in advance nice pairs (θ_k, λ_k) to witness this. In order to arrange that $SP_T(\lambda_k, \theta_k)$ holds we will use Theorems 2.11 and 4.6. To arrange that $SP_{T_{k,k-1}}(\lambda_k, \theta_k)$ fails, we will use the following.

Theorem 5.1. *Suppose (θ, λ) is a nice pair such that $\theta = \theta^{<\theta}$ and $\lambda > \theta$ is a limit cardinal. Let $3 \leq k < \omega$. Then $P_{\lambda\theta\theta}$ forces that for all $\mu < \lambda$, $\check{T}_{k+1,k}$ fails $(< k+1, \lambda, \mu, \theta)$-type amalgamation.*

Proof. Fix $\theta \leq \mu < \lambda$, and write $P = P_{[\lambda]^k\theta\theta}$. We show that P forces $\check{T}_{k+1,k}$ fails $(< k+1, \lambda, \mu, \theta)$-type amalgamation. Since $P \cong P_{\lambda\theta\theta}$, this suffices.

We pass to a P-generic forcing extension $\mathbb{V}[G]$ of $\mathbb{V}$. Let $R \subseteq [\lambda]^k$ be the set of all v with $\{(v, 0)\} \in G$. Choose $M_0 \preceq M \models T_{k+1,k}$, and $(a_{i,\alpha} : i < k, \alpha < \lambda)$ such that, writing $\overline{a}_s = \{a_{i,\alpha} : (i, \alpha) \in s\}$ for $s \subseteq k \times \lambda$:

- M_0 is countable, and $|M| \leq \lambda$ and each $a_{i,\alpha} \in M \backslash M_0$;
- $a_{i,\alpha} = a_{j,\beta}$ iff $\alpha = \beta$ and $i = j$
- For every $v_* \in [k \times \lambda]^k$, if v_* is not the graph of the increasing enumeration of some $v \in [\lambda]^k$, then $R^M(\overline{a}_{v_*})$ fails. Otherwise, $R^M(\overline{a}_{v_*})$ holds if and only if $v \in R$.

For each $v \in [\lambda]^k$, let $\phi_u(x, \overline{a}_{k\times v})$ be the formula that asserts that $R(x, \overline{a}_u)$ holds for each $u \in [k \times v]^{k-1}$. Note that $\phi_v(x, \overline{a}_{k\times v})$ is consistent exactly when $v \notin R$.

It suffices to show that there is no cardinal λ' and function $F_0 : \Gamma^{\theta}_{M,M_0} \to^{w}_{k+1} P_{\lambda'\mu\theta}$; so suppose towards a contradiction some such F_0 existed. Then we can find $F : [\lambda]^k \backslash R \to P_{\lambda'\mu\theta}$ such that for all sequences $(w_i : i < k+1)$ from $[\lambda]^k \backslash R$, if $(F(w_i) : i < k)$ is compatible in P then $\bigwedge_{i<k} \phi_v(x, \overline{a}_{k\times v})$ is consistent. This is all we will need, and so we can replace λ' by λ (since $|[\lambda]^k| = \lambda$).

Pulling back to $\mathbb{V}$, we can find $p_* \in P$, and P-names $\dot{R}, \dot{M}, \dot{M}_0, \dot{a}_{i,\alpha}, \dot{F}$, such that p_* forces these behave as above.

Write $X = \lambda \backslash \bigcup \mathrm{dom}(p_*)$; so $|X| = \lambda$.

Suppose $v \in [X]^k$. Choose $p_v \in P$ such that $p_v \geq p_* \cup \{(v, 1)\}$ (so p_v forces $v \notin \dot{R}$), and so that p_v decides $\dot{F}(v)$, say p_v forces that $\dot{F}(v) = f_v \in P_{\lambda\mu\theta}$.

Choose $F_* : [\lambda]^k \to P_{\lambda\mu\theta}$ so that for all v, v', if $F_*(v)$ and $F_*(v')$ are compatible, then $p_v, p_{v'}$ are compatible, and $f_v, f_{v'}$ are compatible.

Let $\mathcal{B}$ be the Boolean-algebra completion of $P_{\lambda\mu\theta}$. For each $u \in [\lambda]^{k-1}$, let $\mathbf{b}_u$ be the least upper bound in $\mathcal{B}$ of $(F_*(v) : u \subseteq v \in [\lambda]^k)$. Since $\mathcal{B}$ has the μ^+-c.c., we can find $S(u) \in [\lambda]^{\leq\mu}$ such that $\mathbf{b}_u$ is also the least upper bound in $\mathcal{B}$ of $(F_*(v) : u \subseteq v \in [S(u)]^k)$. By expanding $S(u)$, we can suppose that for all $u \subseteq v \in [\lambda]^k$, $\bigcup \mathrm{dom}(p_v) \subseteq S(u)$.

By Theorem 46.1 of [2], we can find some $v \in [\lambda]^k$ such that for all $u \in [v]^{k-1}$, $S(u) \cap v = u$. Now $(\mathbf{b}_u : u \in [v]^{k-1})$ has an upper bound in $\mathcal{B}$, namely $F_*(v)$; thus we can find $(v_u : u \in [v]^{k-1})$ such that each $u \subseteq v_u \in [S(u)]^k$, and $(F_*(v_u) : u \in [w]^{k-1})$ is compatible in $\mathcal{B}$ (i.e. in $P_{\lambda\mu\theta}$). Thus $(p_{v_u} : u \in [v]^{k-1})$ is compatible in P; write $p = \bigcup_{u\in[v]^{k-1}} p_{v_u}$ (recall $P = P_{[\lambda]^k\theta\theta}$). Note that $v \notin \mathrm{dom}(p)$, since if $v \in \mathrm{dom}(p_{v_u})$ then $v \subseteq \bigcup \mathrm{dom}(p_{v_u}) \subseteq S(u)$, contradicting that $S(u) \cap v = u$. Thus we can choose $p' \geq p$ in P with $p'(v) = 0$.

Now p' forces that each $\dot{F}(v_u) = \check{f}_{v_u}$, and $(f_{v_u} : u \in [v]^{k-1})$ is compatible; thus p' forces that $\phi(x) := \bigwedge_{u \in [v]^{k-1}} \phi_{v_u}(\dot{\bar{a}}_{k \times v_u})$ is consistent. But this is impossible, since if we let v_* be the graph of the increasing enumeration of v, then p' forces that $\dot{R}^{\dot{M}}(\dot{\bar{a}}_{v_*})$ holds, and $\phi(x)$ in particular impliesthat $\dot{R}^{\dot{M}}(x, \dot{\bar{a}}_{u_*})$ holds for all $u_* \in [v_*]^{k-1}$, thus creating a $k+1$-clique. $\square$

Theorem 5.2. *Suppose GCH holds. Then there is a forcing notion P, which forces:*

(A) *For every $k \geq 3$, if T is a simple theory with $< k$-type amalgamation, then $T_{k,k-1} \not\leq_{SP} T$;*
(B) *The maximal $\leq_{SP}$-class is the class of simple theories;*
(C) *If T has $< \aleph_0$-type amalgamation then $T \leq_{SP} T_{rg}$.*

Of course, we can also force to make GCH hold (via a proper-class forcing notion). Thus, (A), (B), (C) can consistently hold.

Proof. Write $\theta_2 = \lambda_2 = \aleph_0$. Choose nice pairs $((\theta_k, \lambda_k) : 3 \leq k \leq \omega)$, such that each $\theta_k > \lambda_{k-1}^{++}$, and each λ_k is singular with $\mathrm{cof}(\lambda_k) < \theta_k$ (so each $\lambda_k^{<\theta_k} = \lambda_k^+$).

We will define a full-support forcing iteration $(P_k : 3 \leq k \leq \omega)$, $(\dot{Q}_k : 3 \leq k < \omega)$; for each $3 \leq k < \omega$, we will have that $|P_k| \leq \lambda_{k-1}^{++}$, and P_k will force that $\dot{Q}_k$ is θ_k-closed and has the θ_k^+-c.c.

Having defined P_k, note that P_k forces that (θ_k, λ_k) remains a nice pair and $\mathrm{cof}(\lambda_k) < \theta_k$ and $\theta_k^{<\theta_k} = \theta_k$, since P_k has the θ_{k-1}^+-c.c. Let $\dot{Q}_k^0 = \check{P}_{\lambda_k \theta_k \theta_k}$. By Theorem 2.8, we can choose a $P_k * \dot{Q}_k^0$-name $\dot{Q}_k^1$ for a forcing notion, such that $P_k * \dot{Q}_k^0$ forces $\dot{Q}_k^1$ has the $(< k, \theta_k, \theta_k)$-amalgamation property, and $\mathrm{Ax}(< k, \theta_k)$ holds, and $2^{\theta_k} = \lambda_k^{++}$, and $|\dot{Q}_k^1| = \lambda_k^{++}$. Let $\dot{Q}_k$ be the P_k-name for $\dot{Q}_k^0 * \dot{Q}_k^1$.

Let P_ω be the iteration of $P_k : 3 \leq k < \aleph_0$ with full supports. Also, for each $3 \leq k < \omega$, write $P_\omega = P_k * \dot{P}_{\geq k}$, where $\dot{P}_{\geq k}$ is the P_k-name for the forcing iteration induced by $(\dot{Q}_{k'} : k' \geq k)$. Note that each $\dot{P}_{\geq k}$ is θ_k-closed, and each P_k is θ_{k-1}^+-c.c.

Given $3 \leq k < \omega$, note that since P_k forces that (θ_k, λ_k) is a nice pair, and $\dot{Q}_k$ is θ_k-closed and θ_k^+-c.c., we have that P_{k+1} forces that (θ_k, λ_k) is a nice pair; since $\dot{P}_{\geq k+1}$ is in particular λ_k^+-closed, we have that P_ω forces that (θ_k, λ_k) is a nice pair.

Now P_{k+1} forces that $SP_T(\lambda_k, \theta_k)$ holds whenever T has $< k$-type amalgamation by Theorem 4.6 and Theorem 2.11, and that $SP_{T_{k,k-1}}(\lambda_k, \theta_k)$ fails by Theorem 5.1 and Theorem 2.10. Since $\dot{P}_{\geq k+1}$ is $(\lambda_k^{<\theta_k})^+$-closed, it does not change this, and so we have that P_ω forces that (θ_k, λ_k) is a nice pair, $SP_T(\lambda_k, \theta_k)$ holds and $SP_{T_{k,k-1}}(\lambda_k, \theta_k)$ fails. Thus we have verified that P_ω forces (A) to hold. (B) follows from (A) in the case $k = 3$, since every simple theory has < 3-type amalgamation, and by Theorem 1.5(A), unsimple theories are maximal in $\leq_{SP}$.

To verify (C), it suffices to show that P_ω forces the singular cardinals hypothesis to hold. This is standard, but we give a full argument.

Claim. Suppose the singular cardinals hypothesis holds and P is κ-closed, κ^+-c.c. Then P forces that the singular cardinals hypothesis holds.

Proof. Let $\mathbb{V}[G]$ be a P-generic forcing extension; we work in $\mathbb{V}[G]$. Suppose λ is singular and $2^{\mathrm{cof}(\lambda)} < \lambda$. Note that $|\lambda^{\mathrm{cof}(\lambda)}| = |[\lambda]^{\mathrm{cof}(\lambda)}| \cdot 2^{\mathrm{cof}(\lambda)} = |[\lambda]^{\mathrm{cof}(\lambda)}|$, so it suffices to show that $|[\lambda]^{\mathrm{cof}(\lambda)}| = \lambda^+$. Note that $|[\lambda]^{\mathrm{cof}(\lambda)} \cap \mathbb{V}| = \lambda^+$ since the singular cardinals hypothesis holds in $\mathbb{V}$ (and $\lambda^+ = (\lambda^+)^{\mathbb{V}}$), and so $|[\lambda]^{\mathrm{cof}(\lambda)}| =$

$\lambda^+ \cdot 2^{\text{cof}(\lambda)}$, since every $X \in [\lambda]^{\text{cof}(\lambda)}$ can be covered some $Y \in ([\lambda]^{\text{cof}(\lambda)} \cap \mathbb{V})$, using that P is κ-closed if $|X| < \kappa$, and that P is κ^+-c.c. if $|X| \geq \kappa$. $\square$

Write $\theta = \sup(\theta_k : 3 \leq k < \omega)$. Note that by a trivial induction together with the claim, for all $3 \leq k < \omega$, P_k forces that the singular cardinals hypothesis holds. Thus, given $3 \leq k < \omega$, since $\dot{P}_{\geq k}$ is θ_k-closed, we have that P_ω forces that the singular cardinals hypothesis holds at all singular cardinals $\lambda < \theta_k$. Since this holds for all k, we get that P forces that the singular cardinal hypothesis holds for all singular $\lambda < \theta$. Also, P_ω is θ^{++}-c.c. (since $|P_\omega| = \theta^+$). Thus to finish it suffices to show that $P_\omega \Vdash 2^\theta = \theta^+$, since then P_ω forces that GCH holds above θ.

Let $\mathbb{V}[G]$ be a P_ω-generic forcing extension of $\mathbb{V}$. Easily, $(2^{<\theta})^{\mathbb{V}[G]} = \theta$; also, since P_ω is ω-closed, $(\theta^+)^{\mathbb{V}[G]} = \theta^+$ (as otherwise it would have countable cofinality) and $(|\theta^\omega|)^{\mathbb{V}[G]} = \theta^+$. But then in $\mathbb{V}[G]$, $2^\theta \leq (2^{<\theta})^\omega = \theta^+$, since we can encode $X \subseteq \theta$ by $(X \cap \theta_k : 3 \leq k < \omega)$. $\square$

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