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ON CON($\mathfrak{d}_\lambda >$ COV$_\lambda$(MEAGRE))
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ABSTRACT. We prove the consistency of: for suitable strongly inaccessible cardinal λ the dominating number, i.e., the cofinality of ${}^\lambda\lambda$, is strictly bigger than cov_λ(meagre), i.e. the minimal number of nowhere dense subsets of ${}^\lambda 2$ needed to cover it. This answers a question of Matet.

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§ 0. Introduction

Cardinal characteristics were defined, historically, over the continuum. See celebrated Van Dowen [vD84], for the general topologist perspective and the excellent survey Blass [Bla], Bartoszyński [Bar10] for the set theoretic perspective. In recent years there are mnay results concerning generalized cardinal characteristics. The idea is to imitate the definition of a given characteristic over the continuum, by translating it to uncountable cardinals.

It is reasonable to distinguish regular cardinals and singular cardinals. Among the regular cardinals, it makes sense to distinguish limit cardinals from successor cardinals. In this paper we focus on strongly inaccessible cardinals. These cardinals and their characteristics behave, in many cases, much like $\aleph_0$, but certainly not always. See Landver [Lan92], Cummings-Shelah [CS95] and Matet-Shelah [MS]. Our main result is+ the consistency of $\mathrm{cov}_\lambda(\text{meagre}) < \mathfrak{d}_\lambda$ at a supercompact cardinal λ, and we begin with the following definitions:

We shall define three cardinal invariants (but the paper deals, actually, just with two of them):

Definition 0.1. The bounding and dominating numbers.
Let λ be an inaccessible cardinal.
For $f, g \in {}^\lambda\lambda$ let:

(a) $f \leq^* g$ if $|\{\alpha < \lambda : f(\alpha) > g(\alpha)\}| < \lambda$,
(b) $A \subseteq {}^\lambda\lambda$ is unbounded if there is no $h \in {}^\lambda\lambda$ so that $f \in A \Rightarrow f \leq^* h$,
(c) $A \subseteq {}^\lambda\lambda$ is dominating when for every $f \in {}^\lambda\lambda$ there exists $g \in A$ so that $f \leq^* g$,
(d) the bounding number for λ, denoted by $\mathfrak{b}_\lambda$, is $\min\{|A| : A$ is unbounded in ${}^\lambda\lambda\}$,
(e) the dominating number for λ, denoted by $\mathfrak{d}_\lambda$, is $\min\{|A| : A$ is dominating in ${}^\lambda\lambda\}$.

Notice that the usual definitions of $\mathfrak{b}$ and $\mathfrak{d}$ are $\mathfrak{b}_{\aleph_0}$ and $\mathfrak{d}_{\aleph_0}$ according to Definition 0.1. The definition of $\mathrm{cov}_\lambda(\text{meagre})$ involves some topology.

Definition 0.2. The meagre covering number.
Let λ be a regular cardinal.

(a) ${}^\lambda 2$ is the space of functions from λ into 2,
(b) $({}^\lambda 2)^{[\nu]} = \{\eta \in {}^\lambda 2 : \nu \triangleleft \eta\}$, for $\nu \in {}^{\lambda>}2 := \bigcup\limits_{\alpha<\lambda} {}^\alpha 2$,
(c) $\mathscr{U} \subseteq {}^\lambda 2$ is open in the topology $({}^\lambda 2)_{<\lambda}$, iff for every $\eta \in \mathscr{U}$ there exists $i < \lambda$ so that $({}^\lambda 2)^{[\eta \restriction i]} \subseteq \mathscr{U}$,
(d) $\mathrm{cov}_\lambda(\text{meagre})$ is the minimal cardinality of a family of meagre subsets of $({}^\lambda 2)_{<\lambda}$, which covers this space.

This paper deals with the relationship between $\mathfrak{d}_\lambda$ and $\mathrm{cov}_\lambda(\text{meagre})$. If λ is a successor cardinal then $\mathrm{cov}_\lambda(\text{meagre}) < \mathfrak{d}_\lambda$ is consistent (see (b) below). Matet asked (a personal communication) whether $\mathfrak{d}_\lambda \leq \mathrm{cov}_\lambda(\text{meagre})$ is provable in ZFC, where λ is strongly inaccessible. We give here a negative answer.

For λ a supercompact cardinal and $\lambda < \kappa = \mathrm{cf}(\kappa) < \mu = \mu^\lambda$, we force large $\mathfrak{d}_\lambda$ i.e., $\mathfrak{d}_\lambda = \mu$ and small covering number (i.e., $\mathrm{cov}_\lambda(\text{meagre}) = \kappa$). A similar result should hold also for a wider class of cardinals and we intend to return elsewhere to this subject.

Let us sketch some known results. These results are related to the unequality number and the covering number for category. Recall:

Definition 0.3. The unequality number.

Let κ be an infinite cardinal. The unequality number of $\kappa, \mathfrak{e}_\kappa$, is the minimal cardinal λ satisfying that there is a set $\mathscr{F} \subseteq {}^\kappa\kappa$ of cardinality λ such that there is no $g \in {}^\kappa\kappa$ satisfying $(\forall f \in \mathscr{F})(\exists^\kappa \alpha < \kappa)(f(\alpha) = g(\alpha))$.

For $\kappa = \aleph_0, \mathfrak{e}_\kappa =$ cov$_{\aleph_0}$(meagre); see Bartoszyński (in [Bar87]) and Miller (in [Mil82]).

Now

(a) the statement $\mathfrak{e}_\kappa =$ cov$_\kappa$(meagre) is valid for $\kappa > \aleph_0$, in the case that κ is strongly inaccessible, by [Lan92]. But if κ is a successor cardinal, it may fail,

(b) if $\kappa < \kappa^{<\kappa}$, then cov$_\kappa$(meagre) $= \kappa^+$. This is due to Landver (in [Lan92]).

We intend also to address:

Problem 0.4. Can we replace "supercompact" by "strongly inaccessible"?

Problem 0.5. 1) Can we prove the consistency of cov$_\lambda$(meagre) $< \mathfrak{b}_\lambda$?
2) For λ strongly inaccessible (or just Laver indestructible supercompact) is there a non-trivial λ^+-c.c. $(< \lambda)$-strategically complete forcing notion $\mathbb{Q}$ which is ${}^\lambda\lambda$-bounding?

We thank the referee, Shimoni Garti and Haim Horowitz for helpful comments and pressuring me to expand some proofs and Johannes Schürz and Martin Goldstern for pointing several times problem with the connection to [She20], in particular out in 2019 that an earlier version of the proof of [She20, 2.7=La32] the statement $\circledast'_4$ was insufficient; and later pointing out a problem in earlier version of the end of the proof of [Sheb, 3.24=Le67(1)] which require the addition of "specially solve" additions to [Sheb] (sub-sections $3F, §3G) and allowing $\mathbf{m} \in \mathbf{M}$ to be non-simple) and corresponding changes here. We say more in subsequent works [She17], [Shea] and in preparation [Shec].

A point which in a previous version was just a step along the way, the referee asked to justify fully, was analyzed to be serious. This was done but eventually is separated to [Sheb]. A posteriori the point is that in the parallel case for $\lambda = \aleph_0$, for full memory FS iteration such a claim is true. In fact, by Judah-Shelah [JS88], if $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \alpha(*), \beta < \alpha(*)\rangle$ is FS iteration of Suslin-c.c.c. forcing notion, $\underset{\sim}{\mathbb{Q}}_\beta$ with the generic $\underset{\sim}{\eta}_\beta \in {}^\omega\omega$ and for notational transparency, its definition is with no parameter and $\zeta : \beta(*) \to \alpha(*)$ is increasing and $\mathbb{P} = \langle \mathbb{P}'_\alpha, \underset{\sim}{\mathbb{Q}}'_\beta : \alpha \le \beta(*), \beta < \beta(*)\rangle$ is FS iteration, but $\underset{\sim}{\mathbb{Q}}'_\beta$ is defined exactly as $\underset{\sim}{\mathbb{Q}}_{\zeta(\beta)}$ is but now in $\mathbf{V}^{\mathbb{P}'_\beta}$ rather then in $\mathbf{V}^{\mathbb{P}_{\zeta(\beta)}}$ then $\Vdash_{\mathbb{P}_{\alpha(*)}}$ "$\langle \underset{\sim}{\eta}_{\zeta(\beta)} : \beta < \beta(*)\rangle$ is generic for $\mathbb{P}'_{\beta(*)}$ over $\mathbf{V}$".

Now this is not clear to us for $(< \lambda)$-support iteration of $(< \lambda)$-strategically complete forcing notions. The solution is essentially to change the iteration: to use a "quite generic" $(< \lambda)$-support iteration which "includes" the one we like and use the complete sub-forcing it generates; see [Sheb].

We try to use standard notation. We use $\theta, \kappa, \lambda, \mu, \chi$ for cardinals and $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ for ordinals. We use also i and j as ordinals. We adopt the Cohen convention that

$p \leq q$ means that q gives more information, in forcing notions. The symbol $\triangleleft$ is preserved for "being an initial segment". Also recall ${}^B A = \{f : f$ a function from B to $A\}$ and let ${}^{\alpha>}A = \cup\{{}^\beta A : \beta < \alpha\}$, some prefer ${}^{<\alpha}A$, but ${}^{\alpha>}A$ is used systematically in the author's papers. Lastly, $J^{\rm bd}_\lambda$ denotes the ideal of the bounded subsets of λ.

For exact references to [Sheb] see the introduction there, just before Def 01.1.

The picture of cardinal invariants related to uncountable λ is related but usually quite different than the one for $\aleph_0$, they are more similar if κ is "large" enough, mainly strongly inaccessible.

§ 1. Preliminaries

Definition 1.1. Let λ be supercompact. We say that $h : \lambda \to \mathscr{H}(\lambda)$ is a Laver diamond (for λ) when for every $x \in \mathbf{V}$ there are a normal fine ultrafilter D over $I = [\mathscr{H}(\chi)]^{<\lambda}]$ for some χ such that $x \in \mathscr{H}(\chi)$ and the Mostowski collapse $\mathbf{j}$ on $\mathbf{V}^I/D$ maps $\langle h(\sup(u \cap \lambda)) : u \in I \rangle/D$ to x; (we can use elementary embeddings instead of an ultrafilter).

Notation 1.2. If $\mathbb{P}$ is a forcing notion in $\mathbf{V}$ then $\mathbf{V}^{\mathbb{P}}$ denotes $\mathbf{V}[\mathbf{G}]$ for $\mathbf{G} \subseteq \mathbb{P}$ generic over $\mathbf{V}$; we may write $\mathbf{V}[\mathbb{P}]$ instead.

The most straightforward way to increase $\mathfrak{b}_\lambda$ in the classical case of $\aleph_0$ is Hechler forcing = dominating real forcing. A condition is a function $f_p : \omega \to \omega$ which is separated into a finite stem η_p and the rest of the function. Formally, $p = (\eta_p, f_p)$ where $\eta_p \trianglelefteq f_p$.

If p, q are conditions then $p \le q$ iff $\eta_p \trianglelefteq \eta_q$ and $f_q(n) \ge f_p(n)$ for every $n \notin \mathrm{dom}(\eta_p)$ hence for every n. A generic object adds a function $g : \omega \to \omega$ which dominates the functions from the ground model. By iterating Hechler reals one increases the bounding number $\mathfrak{b}$.

If $\lambda = \lambda^{<\lambda}$ then one can define the generalized Hechler forcing $\mathbb{D}_\lambda$ by replacing ω by λ. The basic step is $(< \lambda)$-complete and λ^+-c.c. and actually λ-centered. Hence one can iterate and increase $\mathfrak{b}_\lambda$.

In [She92, §1,§2] and then Goldstern-Shelah [GS93], Kellner-Shelah [KS12] consider other invariants. Consider two functions $f, g : \omega \to (\omega \backslash \{0\})$ going to infinity such that $f \ge g$ and ask about:

- $\mathfrak{c}^+_{f,g} = \min\{|\mathscr{F}| : \mathscr{F} \subseteq \prod\limits_i [f(i)]^{g(i)}$ and $(\forall \eta \in \prod\limits_i f(i))(\exists g \in \mathscr{F})[\bigwedge\limits_i \eta(i) \in g(i)]$,
- $\mathfrak{c}^-_{f,g} = \min\{\mathscr{F} : \mathscr{F} \subseteq \prod\limits_i f(i)$ and for no $g \in \prod\limits_i [f(i)]^{g(i)}$ do we have $(\forall \eta \in \mathscr{F})(\forall^\infty i)(\eta(i) \in g(i))$.

There are relevant forcing notions; we shall use a λ^+-c.c. one as in c.c.c. creature forcing (see [RS97],[HS]).

For transparency

Convention 1.3. Below $\lambda, \bar{\theta}$ are as in 1.4 below.

Definition 1.4. Let λ be inaccessible, $\bar{\theta} = \langle \theta_\varepsilon : \varepsilon < \lambda \rangle$ be a sequence of regular cardinals $< \lambda$ satisfying $\theta_\varepsilon > \varepsilon$.
1) We define the forcing notion $\mathbb{Q} = \mathbb{Q}_{\bar{\theta}}$ by:

(α) $p \in \mathbb{Q}$ iff:
 - (a) $p = (\eta, f) = (\eta^p, f^p)$,
 - (b) $\eta \in \prod\limits_{\zeta < \varepsilon} \theta_\zeta$ for some $\varepsilon < \lambda$, (η is called the trunk of p),
 - (c) $f \in \prod\limits_{\zeta < \lambda} \theta_\zeta$,
 - (d) $\eta \triangleleft f$.

(β) $p \leq_{\mathbb{Q}} q$ iff:
   (a) $\eta^p \trianglelefteq \eta^q$,
   (b) $f^p \leq f^q$, i.e. $(\forall \varepsilon < \lambda) f^p(\varepsilon) \leq f^q(\varepsilon)$,
   (c) if $\ell g(\eta^p) \leq \varepsilon < \ell g(\eta^q)$ then $\eta^q(\varepsilon) \in [f^p(\varepsilon), \lambda)$, actually follows.

2) The generic is $\underset{\sim}{\eta} = \cup\{\eta^p : p \in \underset{\sim}{\mathbf{G}}_{\mathbb{Q}_{\bar{\theta}}}\}$.

The new forcing defined above is not $(< \lambda)$-complete anymore. By fixing a stem η one can define a short increasing sequence of conditions which goes up to some θ_ζ at the ζ-th coordinate and hence has no upper bound in $\prod\limits_{\zeta<\varepsilon} \theta_\zeta$. However, this forcing is $(< \lambda)$-strategically complete since the COM (= completeness) player can increase the stem at each move.

Remark 1.5. The forcing is parallel to the creature forcing from [She92, §1,§2], [KS12] but they are ${}^\omega\omega$-bounding.

Recall

Definition 1.6. 1) We say that a forcing notion $\mathbb{P}$ is α-strategically complete <u>when</u> for each $p \in \mathbb{P}$ in the following game $\Game_\alpha(p, \mathbb{P})$ between the players COM and INC, the player COM has a winning strategy.

A play lasts α moves; in the β-th move, first the player COM chooses $p_\beta \in \mathbb{P}$ such that $p \leq_{\mathbb{P}} p_\beta$ and $\gamma < \beta \Rightarrow q_\gamma \leq_{\mathbb{P}} p_\beta$ and second the player INC chooses $q_\beta \in \mathbb{P}$ such that $p_\beta \leq_{\mathbb{P}} q_\beta$.

The player COM wins a play if he has a legal move for every $\beta < \alpha$.

2) We say that a forcing notion $\mathbb{P}$ is $(< \lambda)$-strategically complete <u>when</u> it is α-strategically complete for every $\alpha < \lambda$.

Basic properties of $\mathbb{Q}_{\bar{\theta}}$ are summarized and proved in [GS12, §2].

The following fact describes some immediate connections between various concepts of completeness:

Fact 1.7.
(a) if $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta \rangle$ is a $(< \lambda)$-support iteration of $(< \lambda)$-strategically complete forcing notions, <u>then</u> $\mathbb{P}_\delta$ is also $(< \lambda)$-strategically complete; (see e.g. [She00]).
(b) If $\mathbb{P}$ is $(< \lambda)$-strategically complete forcing notion <u>then</u> $({}^{\lambda>}\mathrm{Ord})^{\mathbf{V}} = ({}^{\lambda>}\mathrm{Ord})^{\mathbf{V}^{\mathbb{P}}}$, and consequently λ is strongly inaccessible in $\mathbf{V}^{\mathbb{P}}$,
(c) like (a) replacing "$(< \lambda)$-strategically complete" by "$(< \lambda)$-complete"
(d) if $\mathbb{P}$ is $(< \lambda)$-complete then $\mathbb{P}$ is $(< \lambda)$-strategically complete.

Definition 1.8. For an ordinal $\alpha_* = \alpha(*)$ let $\mathbf{Q}_{\lambda,\bar{\theta},\alpha(*)}$ be the class of quintuple $\mathbf{q} = (\bar{u}, \bar{\mathscr{P}}, \bar{\mathbb{P}}, \underset{\sim}{\bar{\mathbb{Q}}}, \underset{\sim}{\bar{\eta}})$ consisting of (omitting α_* means for some α_* and $\ell g(\mathbf{q}) = \alpha_{\mathbf{q}} = \alpha_*$):

(a) $\bar{u} = \langle u_\alpha : \alpha < \alpha_* \rangle$ and $\bar{\mathscr{P}} = \langle \mathscr{P}_\alpha : \alpha < \alpha_* \rangle$ where $\mathscr{P}_\alpha \subseteq [u_\alpha]^{\leq \lambda}, u_\alpha \subseteq \alpha$, without loss of generality $\mathscr{P}_\alpha$ is closed under subsets (but is not necessarily an ideal),
(b) $\langle \mathbb{P}_{0,\alpha}, \underset{\sim}{\mathbb{Q}}_{0,\beta} : \alpha \leq \alpha_*, \beta < \alpha_* \rangle$ is a $(< \lambda)$-support iteration and let $\mathbb{P}_{\mathbf{q},0} = \mathbb{P}_{\mathbf{q},0,\alpha(\mathbf{q})}$,

(c) each $\mathbb{P}_\alpha$ is $(<\lambda)$-strategically complete and λ^+-c.c.,

(d) $\underset{\sim}{\eta}_\beta \in \prod_{\varepsilon<\lambda} \theta_\varepsilon$ is the generic of $\underset{\sim}{\mathbb{Q}}_\beta$ where $\underset{\sim}{\eta}_\beta$, the generic of $\underset{\sim}{\mathbb{Q}}_\beta$ (defined in clause (e) below) is $\cup\{\eta_p : p \in \mathbf{G}_{\mathbb{Q}_\beta}\}$,

(e) if $\mathbf{G} \subseteq \mathbb{P}_\beta$ is generic over $\mathbf{V}$ then $\underset{\sim}{\eta}_\alpha[\mathbf{G}]$ in $(\prod_{\varepsilon<\lambda} \theta_\varepsilon, <_{J^{\rm bd}_\lambda})$ dominates every $\nu \in \prod_{\varepsilon<\lambda} \theta_\varepsilon$ from $\mathbf{V}[\langle \underset{\sim}{\eta}_\gamma : \gamma \in u\rangle]$ when $u \in \mathscr{P}_\alpha$; moreover, in $\mathbf{V}[\mathbf{G}]$:

  $(*)$ $\underset{\sim}{\mathbb{Q}}_\beta[\mathbf{G}]$ is the sub-forcing of $\mathbb{Q}_{\bar\theta}$ consisting of the $p \in \mathbb{Q}_{\bar\theta}$ such that: for some $\bar s, \underset{\sim}{f}, \eta_p$ (so $\eta_p = \eta$, etc.) we have:

   (α) $p = (\eta, f) = (\eta_p, f_p)$ so $\eta \in \prod_{\varepsilon<\zeta} \theta_\varepsilon$ for some $\zeta < \lambda$,

   (β) $\bar s = \langle (u_i, f_i) : i < i_*\rangle$,

   (γ) $i_* < \lambda$,

   (δ) for each $i < i_*$ we have $u_i \in \mathscr{P}_\beta, \eta \triangleleft f_i \in \prod_{\varepsilon<\lambda} \theta_\varepsilon$ and $f_i \in \mathbf{V}[\langle \underset{\sim}{\eta}_\gamma[\mathbf{G}] : \gamma \in u_i\rangle]$,

   (ε) $f = \sup\{f_i : i < i_*\}$, i.e. $\varepsilon < \lambda \Rightarrow f(\varepsilon) = \cup\{f_i(\varepsilon) : i < i_*\}$; we may add $i < i_* \Rightarrow \eta \triangleleft f_i$ and even $i_* < \theta_{\lg(\eta)}$.

(f) <u>notation</u>: so $u_{\mathbf{q},\alpha} = u_\alpha, \mathbb{P}_{\mathbf{q},\alpha} = \mathbb{P}_\alpha$, etc., but when $\mathbf{q}$ is clear from the context we may omit it.

Definition 1.9. For $\mathbf{q} \in \mathbf{Q}_{\lambda,\bar\theta,\alpha(*)}$.
1) We let $\alpha \leq \alpha_*, \mathbb{P}_{1,\alpha} = \mathbb{P}^{\mathbf{q}}_{1,\alpha}$ be essentially the completion of $\mathbb{P}_\alpha$; we express it by:

$(*)_1$ the elements of $\mathbb{P}_{1,\alpha} = \mathbb{P}_{\mathbf{q},1,\alpha}$ are of the form $\mathbf{B}(\ldots, \underset{\sim}{\eta}_{\gamma_i}, \ldots)_{i<i(*)}$ where:

  (α) $i_* = i(*) \leq \lambda$,

  (β) $\gamma_i < \alpha$ for $i < i_*$,

  (γ) $\mathbf{B}$ is a λ-Borel function from ${}^{i(*)}(\prod_{\varepsilon<\lambda} \theta_\varepsilon)$ into $\{0,1\} = \{\text{false}, \text{true}\}$; $\mathbf{B}$ is from $\mathbf{V}$, of course, such that $\nVdash_{\mathbb{P}_{\mathbf{q}}}$ "$\mathbf{B}(\ldots, \underset{\sim}{\eta}_{\gamma_i}, \ldots)_{i<i(*)} = 0$".

$(*)_2$ the order is natural: $\mathbb{P}_{1,\alpha} \models$ "$\mathbf{B}_1(\ldots, \underset{\sim}{\eta}_{\gamma(i,1)}, \ldots)_{i<i(1)} \leq \mathbf{B}_2(\ldots, \underset{\sim}{\eta}_{\gamma(i,2)}, \ldots)_{i<i(2)}$" iff $\Vdash_{\mathbb{P}_\alpha}$ "if $\mathbf{B}_2(\ldots, \underset{\sim}{\eta}_{\gamma(i,2)}[\underset{\sim}{\mathbf{G}}], \ldots)_{i<i(2)}$ is equal to 1 then so is $\mathbf{B}_1(\ldots, \underset{\sim}{\eta}_{\gamma(i,1)}, \ldots)_{i<i(1)}$".

2) For $\mathscr{U} \subseteq \alpha_*$ let $\mathbb{P}_{\mathscr{U}} = \mathbb{P}^{\mathbf{q}}_{\mathscr{U}}$ be the sub-forcing of $\mathbb{P}_{1,\alpha(\mathbf{q})}$ consists of $\{\mathbf{B}(\ldots, \underset{\sim}{\eta}_{\gamma(i)}, \ldots)_{i<i(*)} \in \mathbb{P}_{1,\alpha(\mathbf{q})} : i(*) \leq \lambda$ and $\gamma_i \in \mathscr{U}$ for every $i < i(*)\}$.

Claim 1.10. *1) For any sequence $\langle u_\alpha, \mathscr{P}_\alpha : \alpha < \alpha_*\rangle$ as above, i.e. as in clause (a) of Definition 1.8, there is one and only one $\mathbf{q} \in \mathbb{Q}_{\lambda,\bar\theta,\alpha_*}$ with $u_{\mathbf{q},\alpha} = u_\alpha, \mathscr{P}_{\mathbf{q},\alpha} = \mathscr{P}_\alpha$ for $\alpha < \alpha_*$.*
1A) For $\alpha \leq \alpha_*$, the forcing notions $\mathbb{P}_{\mathbf{q},\alpha}, \mathbb{P}_{\mathbf{q},\mathscr{U}}$'s are well defined and are as demanded in Definition 1.9.
2) For every $\alpha \leq \alpha_*$ the set $\mathbb{P}^\bullet_{\mathbf{q},\alpha}$ of $p \in \mathbb{P}_{\mathbf{q},\alpha}$ satisfying the following is dense:

$(*)$ *if $\beta \in \mathrm{dom}(p)$, then $q = p(\beta)$ is a $\mathbb{P}_\beta$-name of a member of $\underset{\sim}{\mathbb{Q}}_\beta$ such that:*

  (a) *$\eta_q, i_q, \langle u_{q,i} : i < i_q\rangle$ are objects (not just $\mathbb{P}_\beta$-names),*

  (b) *$\underset{\sim}{f}_q = \sup\{\underset{\sim}{f}_i : i < i_q\}$, each $\underset{\sim}{f}_i$ is a $\mathbb{P}_\beta$-name of a member of $\prod_{\varepsilon<\lambda} \theta_\varepsilon$,*

(c) *each* $\underset{\sim}{f}_i$ *has the form* $\mathbf{B}_{q,i}(\ldots,\underset{\sim}{\eta}_{\gamma(i,j)},\ldots)_{j<j(*)\leq\lambda}$ *where* $\{\gamma(i,j):j<j(*)\}\subseteq u_{q,i}$ *and* $\mathbf{B}_q$ *is a Borel function from* ${}^{\partial(*)}(\prod\limits_{\varepsilon<\lambda}\theta_\varepsilon)$ *into* $\prod\limits_{\varepsilon<\lambda}\theta_\varepsilon$,

(d) $p(\beta)=(\underset{\sim}{\eta}_q,\underset{\sim}{f}_q)$.

3) Above for every $v\subseteq\alpha$ *and* $j_*<\lambda$ *the set of* $p\in\mathbb{P}^\bullet_\alpha$ *such that* $v\subseteq\mathrm{dom}(p)\wedge(\forall\beta\in\mathrm{dom}(p))(\ell g(\eta_{p(\beta)})>j_*)$ *is dense.*
4) $\mathbb{P}_{\mathbf{q},0,\alpha}\lessdot\mathbb{P}_{\mathbf{q},1,\alpha}$ *moreover* $\mathbb{P}_{\mathbf{q},0,\alpha}$ *is dense in* $\mathbb{P}_{\mathbf{q},1,\alpha}$ *and* $\mathscr{U}_1\subseteq\mathscr{U}_2\subseteq\alpha_{\mathbf{q}}\Rightarrow\mathbb{P}^{\mathbf{q}}_{\mathscr{U}_1}\lessdot\mathbb{P}^{\mathbf{q}}_{\mathscr{U}_2}\lessdot\mathbb{P}_{\mathbf{q},\alpha}$ *so* $\mathbb{P}_{\mathbf{q},\{\beta:\beta<\alpha\}}=\mathbb{P}_{\mathbf{q},2,\alpha}$ *and* $|\mathbb{P}_{\mathbf{q},\mathscr{U}}|\leq|\mathscr{U}|^\lambda$.
5) If $\alpha<\alpha_*$ *and* $u\in\mathscr{P}_\alpha$ *then* $\underset{\sim}{\eta}_\alpha\in\prod\limits_{\varepsilon<\lambda}\theta_\varepsilon$ *dominates every* $\nu\in(\prod\limits_{\varepsilon<\lambda}\theta_\varepsilon)^{\mathbf{V}[\underset{\sim}{\bar\eta}\restriction u]}$.
6) Assume $\mathbf{G}\subseteq\mathbb{P}_{\mathbf{q}}$ *is generic over* $\mathbf{V},\underset{\sim}{\eta}_\alpha=\underset{\sim}{\eta}_\alpha[\mathbf{G}]$ *and* $\eta'_\alpha\in(\prod\limits_{\varepsilon<\lambda}\theta_\varepsilon)^{\mathbf{V}[\mathbf{G}]}$ *for* $\alpha<\alpha_*$ *and* $\{(\alpha,\varepsilon):\alpha<\alpha_*,\varepsilon<\alpha$ *and* $\eta_\alpha(\varepsilon)\neq\eta'_\alpha(\varepsilon)\}$ *has cardinality* $<\lambda$. *Then for some (really unique)* $\mathbf{G}'$ *we have* $\mathbf{G}'\subseteq\mathbb{P}_{\mathbf{q}}$ *is generic over* $\mathbf{V}$ *and* $\mathbf{V}[\mathbf{G}']=\mathbf{V}[\mathbf{G}]$ *and* $\underset{\sim}{\eta}_\alpha[\mathbf{G}]=\eta'_\alpha$ *for* $\alpha<\alpha_*$.
7) Like (6) for $\mathbb{P}^{\mathbf{q}}_{\mathscr{U}}$

Proof. See [Sheb, 1.11=Lc8, 1.13=Lc11]. $\square_{1.10}$

Theorem 1.11. *For any ordinal* α_* *there is a quadruple* $(\mathbf{q},\delta_*,\mathscr{U}_*,h)$ *such that:*

(A) (a) $\mathbf{q}\in\mathbf{Q}_{\lambda,\bar\theta}$ *and let* $\delta_*=\ell g(\mathbf{q})$
(b) $\mathscr{U}_*\subseteq\delta_*$ *has order type* α_*
(c) h *is the order preserving function from* α_* *onto* $\mathscr{U}_*$
(d) *if* $\alpha\in\mathscr{U}_*$ *then* $\mathscr{U}_*\cap\alpha\in\mathscr{P}_{\mathbf{q},\alpha}$
(e) *if* $\mathrm{cf}(\alpha_*)>\lambda$ *then in* $\mathbf{V}^{\mathbb{P}_{\mathbf{q}}}$ *the set* $\{\underset{\sim}{\eta}_\alpha:\alpha\in\mathscr{U}_*\}$ *is cofinal in* $(\Pi_{\varepsilon<\lambda}\theta_\varepsilon,\leq_{J^{\mathrm{bd}}_\lambda})$

(B) *if* $\mathscr{U}_1\subseteq\mathscr{U}_*,\mathscr{U}_2\subseteq\mathscr{U}_*,\mathrm{otp}(\mathscr{U}_1)=\mathrm{otp}(\mathscr{U}_2)$ *and* g *is the order preserving function from* $\mathscr{U}_1$ *onto* $\mathscr{U}_2$, *then* g *induces an isomorphism* $\hat g$ *from* $\mathbb{P}_{\mathbf{q},\mathscr{U}_1}$ *onto* $\mathbb{P}_{\mathbf{q},\mathscr{U}_2}$ *mapping* $\underset{\sim}{\eta}_\beta$ *to* $\underset{\sim}{\eta}_{g(\beta)}$ *for* $\beta\in\mathscr{U}_1$.

Proof. By [Sheb, 2.13=Lc52], in particular clause (A)(e) is justified by clause (E) there. Alternatively use [Sheb, 3.43=Li37]. $\square_{1.11}$

§ 2. The forcing

In this section we prove the main result of the paper, which reads as follows:

Theorem 2.1. *Assume*

(a) *λ is supercompact*
(b) *$\lambda < \kappa = \mathrm{cf}(\kappa) < \mu = \mathrm{cf}(\mu) = \mu^\lambda$.*

Then for some forcing notion $\mathbb{P}$ not collapsing cardinals $\geq \lambda$, λ is still supercompact in $\mathbf{V}^{\mathbb{P}}$ and $\mathrm{cov}_\lambda(\mathrm{meagre}) = \kappa, \mathfrak{d}_\lambda = \mu$.

Proof. By Lemma 2.3(1) we force $\boxdot_\lambda$ while maintaining the supercompactness of λ. By Lemma 2.7 we force $\mathfrak{d}_\lambda = \mu \wedge \mathrm{cov}_\lambda(\mathrm{meagre}) = \kappa$ using a forcing notion $\mathbb{P}$ which is λ^+-c.c. and $(< \lambda)$-strategically complete. Notice that λ is still supercompact in the generic extension, so we are done. $\square_{2.1}$

Definition 2.2. For λ supercompact we define $\boxdot_\lambda$ by:

$\boxdot_\lambda$ for any regular cardinal $\chi > \lambda$ and forcing notion $\mathbb{P} \in \mathscr{H}(\chi)$ which is $(< \lambda)$-strategically complete (see Definition 1.6(2)) the following set $\mathscr{S} = \mathscr{S}_{\mathbb{P}} = \mathscr{S}_{\chi,\mathbb{P}}$ is a stationary subset of $[\mathscr{H}(\chi)]^{<\lambda}$:
$\mathscr{S} = \mathscr{S}_{\mathbb{P}} = \mathscr{S}_{\chi,\mathbb{P}}$ is the set of N's such that for some $\lambda_N, \chi_N, \mathbf{j} = \mathbf{j}_N, \mathbb{A} = \mathbb{A}_N, M = M_N, \mathbf{G} = \mathbf{G}_N$ we have (and we may say $(\lambda_N, \chi_N, \mathbf{j}_N, \mathbb{A}_N, M_N, \mathbf{G}_N)$ is a witness for $N \in \mathscr{S}_{\chi,\mathbb{P}}$ or for $(N, \mathbb{P}, \chi)$):
(a) $N \prec (\mathscr{H}(\chi)^{\mathbf{V}}, \in)$ and $\mathbb{P} \in N$,
(b) the Mostowski collapse of N is $\mathbb{A}$ and let $\mathbf{j}_N : N \to \mathbb{A}$ be the unique isomorphism,
(c) $N \cap \lambda = \lambda_N$ and ${}^{(\lambda_N)>}N \subseteq N$ and λ_N is strongly inaccessible,
(d) $\mathbb{A} \subseteq M := (\mathscr{H}(\chi_N), \in), M$ is transitive as well as $\mathbb{A}$,
(e) $\mathbf{G} \subseteq \mathbf{j}_N(\mathbb{P})$ is generic over $\mathbb{A}$ for the forcing notion $\mathbf{j}_N(\mathbb{P})$,
(f) $M = \mathbb{A}[\mathbf{G}]$.

Our first lemma is closed to Laver's indestructibility. It consists of two parts. In the first part we prove that one can force $\boxdot_\lambda$ at a supercompact cardinal λ while preserving its supercompactness. In the second part, we prove that this can be done in an indestructible manner. Namely, any further extension of the universe by a $(< \lambda)$-directed-closed forcing notion will preserve the principle $\boxdot_\lambda$.

Lemma 2.3. *1) If λ is supercompact then after some preliminary forcing of cardinality λ , getting a universe $\mathbf{V}$, in $\mathbf{V}$ the cardinal λ is still supercompact and $\boxdot_\lambda$ from 2.2.*
2) Moreover (in part (1)), the statement $\boxdot_\lambda$ holds also in $\mathbf{V}^{\mathbb{P}}$ when $\mathbf{V}$ satisfies $\boxdot_\lambda$ and $\mathbb{P}$ is a $(< \lambda)$-strategically complete forcing notion and $\mathbb{P}$ is $(< \lambda)$-directed closed, but see 2.4(2).

Remark 2.4. 0) The following is a major point in 2.3 and has caused some confusion.
In 2.3 and $\boxdot_\lambda$ we sometimes say a forcing notion is $(< \lambda)$-strategically complete and sometimes demand in addition that it is also $(< \lambda)$-directed closed. To clarify note:

(A) Inside $\boxdot_\lambda$ and the forcing used in 2.3(1) we demand only being $(<\lambda)$-complete; this is used in the proof of 1.8,
(B) For without loss of generality in the the proof of 2.6 we use the relevant forcing being $(<\lambda)$-directed closed.

1) Recall "$\mathbb{P}$ is a $(<\lambda)$-directed closed" means:

$(*)$ if J is a directed partial order of cardinality $<\lambda$ and $p_s \in \mathbb{P}$ for $s \in J$ and $s \leq_J t \Rightarrow p_s \leq_{\mathbb{P}} p_t$ <u>then</u> the set $\{p_s : s \in J\}$ has an upper bound in $\mathbb{P}$.

2) In 2.3(2) we can weaken the assumption "$\boxplus_\lambda$ and $\mathbb{P}$ is $(<\lambda)$-directed closed" to:

$\boxplus$ if $\chi > \lambda$ and $\mathbb{P} \in \mathscr{H}(\chi)$ then we have (A) $\Rightarrow$ (B) <u>where</u>:
- (A) $N \prec (\mathscr{H}(\chi), \in)$ and $\lambda_N, \chi_N, \mathbf{j}_N, \mathbf{G}_N, \underset{\sim}{\mathbb{Q}}$ satisfies:
  - (a) $\mathbb{P}, \underset{\sim}{\mathbb{Q}} \in N, N \cap \lambda = \lambda_N < \chi_N < \lambda, [N]^{<\lambda_N} \subseteq N$,
  - (b) $\underset{\sim}{\mathbb{Q}}$ is a $\mathbb{P}$-name of $(<\lambda)$-strategically, complete forcing notion
  - (c) $\mathbf{j}$ is the Mostowski Collapse of N, its range is $\mathbb{A}$,
  - (d) $\mathbf{G}$ is a subset of $\mathbf{j}(\mathbb{P} * \underset{\sim}{\mathbb{Q}})$, generic over $\mathbb{A}$
  - (e) $\mathbb{A}[\mathbb{G}] = \mathscr{H}(\chi_N)$.
- (B) $\{p \in \mathbb{P} \cap N$: for some $(p', \underset{\sim}{q}') \in \mathbf{G}$ we have $\mathbf{j}(p) = p'\}$ has a common upper bound in $\mathbb{P}$.

3) We can e.g. restrict χ to be strong limit.

Proof. 1) This is similar to the proof in Laver [Lav78] using Laver's diamond, see Definition 1.1, but as requested we elaborate. By Laver [Lav78] without loss of generality there is a Laver diamond $h : \lambda \to \mathscr{H}(\lambda)$. Let $E = \{\theta : \theta < \lambda$ is a strong limit cardinal and $\alpha < \theta \Rightarrow h(\alpha) \in \mathscr{H}(\theta)\}$, clearly a club of λ and let $\langle \kappa_\varepsilon : \varepsilon < \lambda \rangle$ list $\{\theta \in E : \theta$ is strongly inaccessible$\}$ in increasing order.

As requested, we now define $\mathbf{q}_\varepsilon$ and $\bar{\chi}^\varepsilon$ by induction on $\varepsilon \leq \lambda$ such that:

$(*)$
- (a) $\mathbf{q}_\varepsilon = \langle \mathbb{P}_\zeta, \underset{\sim}{\mathbb{Q}}_\xi : \zeta \leq \varepsilon, \xi < \varepsilon \rangle$ is an Easton support iteration (so $\mathbb{P}_\zeta, \underset{\sim}{\mathbb{Q}}_\xi$ do not depend on ε),
- (b) $\mathbb{P}_\zeta \subseteq \mathscr{H}(\kappa_\zeta)$,
- (c) $\bar{\chi}^\varepsilon = \langle \chi_\zeta : \zeta < \varepsilon \rangle$ where each χ_ξ is a regular cardinal $\in [\kappa_\xi, \kappa_{\xi+1})$,
- (d) $\underset{\sim}{\mathbb{Q}}_\xi \in \mathscr{H}(\chi_{\xi+1})$ is a $\mathbb{P}_\xi$-name of a $(<\kappa_\xi)$-strategically complete forcing notion,
- (e) if $h(\xi) = (\underset{\sim}{\mathbb{Q}}, \chi)$ and the pair $(\underset{\sim}{\mathbb{Q}}, \chi)$ satisfies the requirements on $(\underset{\sim}{\mathbb{Q}}_\xi, \chi_\varepsilon)$ in clauses (c),(d) then $(\underset{\sim}{\mathbb{Q}}_\xi, \chi_\xi) = h(\xi)$.

Concerning clause (b) which says "$\mathbb{P}_\zeta \subseteq \mathscr{H}(\kappa_\zeta)$", note that for ζ a limit ordinal letting $\kappa_{<\zeta} = \cup\{\kappa_\xi : \xi < \zeta\}$ we have $\kappa_{<\zeta}$ is strong limit and:

- if $\kappa_{<\zeta}$ is regular, equivalently strongly inaccessible then $\kappa_{<\zeta} = \kappa_\zeta$ and $\mathbb{P}_\zeta = \cup\{\mathbb{P}_\xi : \xi < \zeta\}$ and so $\mathbb{P}_\zeta \subseteq \cup\{\mathscr{H}(\kappa_\xi) : \xi < \zeta\} = \mathscr{H}(\kappa_{<\zeta}) = \mathscr{H}(\kappa_\zeta)$,
- if $\kappa_{<\zeta}$ is singular, then $\mathbb{P}_\zeta \subseteq \mathscr{H}(\kappa^+_{<\zeta}) \subseteq \mathscr{H}(\kappa_\zeta)$ as κ_ζ is inaccessible $> \kappa_{<\zeta}$.

Easily we can carry the induction so $\mathbf{q}_\lambda$ is well defined, $\mathbb{P}_\lambda = \cup\{\mathbb{P}_\varepsilon : \varepsilon < \lambda\} \subseteq \cup\{\mathscr{H}(\kappa_\varepsilon) : \varepsilon < \lambda\} = \mathscr{H}(\lambda)$ and "$\xi < \lambda \Rightarrow \mathbb{P}_\lambda/\mathbb{P}_\xi$ is $(< \kappa_\xi)$-strategically complete" hence $\mathbb{P}_\lambda/\mathbb{P}_\xi$ adds no new sequence of length $< \kappa_\xi$ of ordinals. Clearly it is enough to prove that in $\mathbf{V}^{\mathbb{P}_\lambda}$ we have $\boxdot_\lambda$.

Toward contradiction assume $\chi, \mathbb{P}, \mathscr{S} = \mathscr{S}_{\chi,\mathbb{P}}$ form a counter-example in $\mathbf{V}^{\mathbb{P}_\lambda}$, hence there are $p_* \in \mathbb{P}_\lambda$ and $\mathbb{P}_\lambda$-names $\underset{\sim}{\chi}, \underset{\sim}{\mathbb{P}}, \underset{\sim}{\mathscr{S}}, \underset{\sim}{E}$ such that $p_* \Vdash_{\mathbb{P}_\lambda}$ "$\underset{\sim}{\chi} > \lambda$ is regular, $\underset{\sim}{\mathbb{P}} \in \mathscr{H}(\underset{\sim}{\chi})$ is $(< \lambda)$-strategically complete and $\underset{\sim}{\mathscr{S}}_{\underset{\sim}{\chi},\mathbb{P}}$ is defined as in $\boxdot_\lambda$ and $\underset{\sim}{E} \subseteq [\mathscr{H}(\underset{\sim}{\chi})^{\mathbf{V}[\mathbb{P}_\lambda]}]^{<\lambda}$ is a club disjoint to $\underset{\sim}{\mathscr{S}}$".

As we can increase p_*, without loss of generality $\underset{\sim}{\chi} = \chi$ and let $x = (\chi, \underset{\sim}{\mathbb{P}})$; and as $\mathbf{V} \models$ "λ is supercompact and h is a Laver diamond" for some $(I, D, \mathbf{M}, \mathbf{j}, \mathbf{j}_0, \mathbf{j}_1)$ we have:

$(*)_1$ (a) $\mathbf{M}$ is a transitive class
- (b) $\mathbf{M}$ is a model of ZFC
- (c) ${}^\chi\mathbf{M} \subseteq \mathbf{M}$
- (d) $\mathbf{j}$ is an elementary embedding from $\mathbf{V}$ into $\mathbf{M}$
- (e) $\mathrm{crit}(\mathbf{j}) = \lambda$
- (f) $\mathbf{j}(h)(\lambda) = (\chi, \underset{\sim}{\mathbb{P}})$
- (g) $I = [\mathscr{H}(\chi_1)]^{<\lambda}$ and $\chi_1 > \chi$
- (h) D is a fine normal ultrafilter on I
- (i) $\mathbf{j}_0$ is the canonical elementary embedding of $\mathbf{V}$ into $\mathbf{V}^I/D$
- (j) $\mathbf{M}$ is the Mostowski Collapse of $\mathbf{V}^I/D$
- (k) $\mathbf{j}_1$ is the canonical isomorphism from $\mathbf{V}^I/D$ onto $\mathbf{M}$
- (l) $\mathbf{j} = \mathbf{j}_1 \circ \mathbf{j}_0$.

Moreover, by Definition 1.1

$(*)_2$ $x = \mathbf{j}_1(\langle(\sup(u \cap \lambda) : u \in I\rangle/D)$.

Let $\mathbf{q} = \mathbf{j}(\mathbf{q}_\lambda)$ so $\mathbf{q} = \langle \mathbb{P}_\zeta, \underset{\sim}{\mathbb{Q}}_\xi : \zeta \leq \mathbf{j}(\lambda), \xi < \mathbf{j}(\lambda)\rangle$ and $\zeta < \lambda \Rightarrow \mathbb{P}^{\mathbf{q}}_\zeta = \mathbb{P}_\zeta$, etc.

So

$(*)_3$ in $\mathbf{M}$ the pair $x = (\chi, \underset{\sim}{\mathbb{P}})$ satisfies:
- (a) $\chi \in (\lambda, \mathbf{j}(\lambda)), \mathbf{j}(\lambda)$ is inaccessible
- (b) $\underset{\sim}{\mathbb{P}} \in \mathscr{H}(\chi)$
- (c) $\underset{\sim}{\mathbb{P}}$ is a $\mathbb{P}_\lambda$-name of a $(< \lambda)$-strategically complete forcing notion.

[Why? Because $[\mathbf{M}]^\chi \subseteq \mathbf{M}$ hence $\mathscr{H}(\chi^+)^{\mathbf{V}} \subseteq \mathbf{M}$.]

Now

$(*)_4$ the following sets belong to D:
- (a) $\mathscr{S}_1 = \{u \in I : x \in u$ and $(\mathscr{H}(\chi_1), \in){\restriction}u \prec (\mathscr{H}(\chi_1), \in)\}$
- (b) $\mathscr{S}_2 = \{u \in \mathscr{S}_1 : u \cap \lambda$ is an inaccessible cardinal we call $\lambda_u\}$
- (c) $\mathscr{S}_3 = \{u \in \mathscr{S}_2$: the Mostowski Collapse N^1_u of $(\mathscr{H}(\chi_1), \in){\restriction}u$ is isomorphic to some $(\mathscr{H}(\chi^1_u), \in)\}$.

[Why? As D is a fine and normal ultrafilter on I.]

$(*)_5$ for every formula $\varphi = \varphi(-) \in \mathbb{L}(\{\in\})$ the following are equivalent:

(a) $(\mathscr{H}(\chi_1), \in) \models \varphi[x]$
(b) $(\mathscr{H}(\chi_1), \in)^I/D \models \varphi[\langle h(u \cap \lambda) : u \in I\rangle/D]$
(c) $\mathscr{X}^1_\varphi \in D$ where $\mathscr{X}^1_\varphi = \{u \in I : x \in u$ and $(\mathscr{H}(\chi_1), \in){\restriction} u \models \varphi[x]\}$
(d) $\mathscr{X}^2_\varphi \in D$ where $\mathscr{X}^2_\varphi = \{u \in I : p_*, x \in u$ and $(\mathscr{H}(\chi_u), \in) \models \varphi[\mathbf{j}_u(x)]\}$
(e) $\mathscr{X}^3_\varphi \in D$ where $\mathscr{X}^3_\varphi = \{u \in I : x \in u, \chi^1_u = \mathrm{otp}(\chi_u {\restriction} u)$ and $(\mathscr{H}(\chi^1_u), \in) \models \varphi[\mathbf{j}_u(x)]\}$.

[Why? We have (a) $\Leftrightarrow$ (c) as D is a fine normal ultrafilter on $I = \mathscr{H}(\chi_1)$; we have (c) $\Leftrightarrow$ (d) as $\mathbf{j}_u$ is an isomorphism from $(\mathscr{H}(\chi_1), \in){\restriction} u$ onto $\mathscr{H}(\chi^1_u)$; we have (d) $\Leftrightarrow$ (e) by the choice of D; lastly, (b) $\Leftrightarrow$ (c) by Łos theorem.]

Hence

$(*)_6$ there is N as required in $\mathbf{V}^{\mathbb{P}}$.

[Why? Choose $u \in I$ which belongs to all the sets from D mentioned in $(*)_4 + (*)_5$. Let $\zeta = u \cap \lambda$, so it is inaccessible, even measurable, and $\mathbf{j}_u(x) = \mathbf{j}_u(\chi, \underset{\sim}{\mathbb{P}}) = h(\zeta)$ so (by the choice of $\mathbf{q}$) $h(\zeta) = (\chi, \underset{\sim}{\mathbb{Q}}_\zeta)$ and $\underset{\sim}{\mathbb{Q}}_\zeta$ is a $\mathbb{P}_{\mathbf{q},\zeta}$-name.

Let $\mathbf{G}$ be a subset of $\mathbb{P}_{\mathbf{q}} = \mathbb{P}_\lambda$ to which p_* belongs, $\mathbf{G}_\zeta = \mathbf{G} \cap \mathbb{P}_{\mathbf{q},\zeta}$, hence is a generic subset of $\mathbb{P}_{\mathbf{q},\zeta}$ over $\mathbf{V}$ hence a generic subset of $\mathbf{j}_u(\mathbb{P}_{\mathbf{q}}) \in \mathscr{H}(\chi_\zeta)$ and let $N = ((\mathscr{H}(\chi_1), \in){\restriction} u)[\mathbf{G}], \mathbb{A} = (\mathscr{H}(\chi_\zeta)^{\mathbf{V}[\mathbf{G}_\zeta]}, \in), M = \mathbb{A}^{\underset{\sim}{\mathbb{Q}}_\zeta[\mathbf{G}_\zeta]}$. Easily N is as promised, contradiction to the choice of p_*.]

So we are done proving part (1).

2) Let $\mathbb{Q}$ be a forcing notion in $\mathbf{V}^{\mathbb{P}}$ which is $(<\lambda)$-strategically complete and (even) $(<\lambda)$-directed closed, $\emptyset \in \mathbb{Q}$ is the weakest condition, χ_1 large enough so that $\lambda, \underset{\sim}{\mathbb{Q}} \in \mathscr{H}(\chi_1)$ and it suffices to prove that in $\mathbf{V}^{\mathbb{P}}$, the set $\mathscr{S}_{\chi_1,\mathbb{Q}}$ is stationary. So let $\underset{\sim}{\mathbb{Q}}, \underset{\sim}{E}$ be $\mathbb{P}$-names such that for some $p \in \mathbb{P}$ we have $p \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathscr{H}(\chi_1)$ is $(<\lambda)$-strategically complete, $(<\lambda)$-directed closed, forcing notion, $\underset{\sim}{E}$ a club of $[\mathscr{H}(\chi_1)]^{<\lambda}$ disjoint to $\underset{\sim}{\mathscr{S}}_{\chi_1,\mathbb{Q}}$", no need to use a name for χ_1 as we can increase p.

Let $\chi \gg \chi_1$; now $\mathbb{P} * \underset{\sim}{\mathbb{Q}} \in \mathscr{H}(\chi)$ is a $(<\lambda)$-strategically complete forcing notion and without loss of generality codes (χ_1, E). As $\boxdot_\lambda$ holds in $\mathbf{V}$ we can apply it to the forcing $\mathbb{P}_{\geq p} * \underset{\sim}{\mathbb{Q}}$ so we can find a tuple $(N, \lambda_N, \chi_N, \mathbf{j}_N, \mathbb{A}_N, M_N, \mathbf{G}_N)$ witnessing it, in particular, $(p, \emptyset) \in \mathbf{G}_N, \mathbb{P} * \underset{\sim}{\mathbb{Q}} \in N$ so $\chi_1, \underset{\sim}{E} \in N$. Let $\mathbf{G}_{\mathbb{P}}$ be a subset of $\mathbb{P}$ generic over $\mathbf{V}$ which extends $\{p' : (p', \underset{\sim}{q}') \in \mathbf{G}_N\}$, possible because $\mathbf{G}_N$ is in $\mathbf{V}$, a subset of $\mathbb{P}$ which has an upper bound, this is the only place we use "$\mathbb{P}$ is $(<\lambda)$-directed closed".

Next, let $\mathbf{V}_1 = \mathbf{V}[\mathbf{G}_{\mathbb{P}}], N_1 = N[\mathbf{G}_{\mathbb{P}}], E_1 = \underset{\sim}{E}[\mathbf{G}_{\mathbb{P}}], \mathbb{A}_1 = \mathbb{A}[\mathbf{j}''_N(\mathbf{G}_{\mathbb{P}} \cap N)] = \mathbb{A}[\{p' : (p', \underset{\sim}{q}') \in \mathbf{G}_N\}], \mathbf{G}_1 = \{\underset{\sim}{q}[\mathbf{j}''(G_{\mathbb{P}} \cap N)] : (p, \underset{\sim}{q}) \in \mathbf{G}_{\mathbb{P}}\}$.

Note that here we may use "the forcing $\mathbb{P}$ is $(<\lambda)$-directed closed" because we are proving part (2).

Let $N_2 = N_1 {\restriction} \mathscr{H}(\chi_1)^{\mathbf{V}[\mathbf{G}_{\mathbb{P}}]}, \mathscr{S} = \mathscr{S}_{\underset{\sim}{\mathbb{Q}}}[\mathbf{G}_{\mathbb{P}}], \mathbf{j}_1 =$ the lifting of $(\mathbf{j}{\restriction}(N \cap \mathscr{H}(\chi)))$, to mapping N_1 onto $\mathbb{A}_1$.

Now recalling p forces $\underset{\sim}{E}$ is disjoint to $\underset{\sim}{\mathscr{S}}$ clearly

$(*)$ $N_2 \in E_1$.

hence

$(*)$ $N_1 \notin \mathscr{S}$.

But easily in $\mathbf{V}_1$ we have: $(\lambda_N, \chi_N, \mathbf{j}_1, \mathbb{A}_1, M_1 = M, \mathbf{G}_1)$ witnesses $N_1 \in \mathscr{S} \cap E_1$, a contradiction to the choice of $\underset{\sim}{E}$. $\square_{2.3}$

Discussion 2.5. Suppose that one wishes to force an inequality between two cardinal characteristics. There are two general approaches, which can be labeled as Top-down and Bottom-up. In the Bottom-up strategy one begins with a universe in which many characteristic are small, e.g. by assuming $2^\lambda = \lambda^+$, and then increases some of them while trying to keep the smallness of the rest. In the Top-down strategy one begins with a universe in which many characteristics are large. The forcing aims to decrease some of them while keeping the large value of the rest.

We shall use the Top-down approach, so we begin by increasing $\mathfrak{b}_\lambda$ (and $\mathfrak{d}_\lambda$) to some $\mu = \mathrm{cf}(\mu) > \lambda$. Notice that $\mathfrak{b}_\lambda$ is a relatively small characteristics and, in particular, always $\mathfrak{b}_\lambda \leq \mathfrak{d}_\lambda$. The next step will be to decrease $\mathrm{cov}_\lambda(\mathrm{meagre})$ in such a way that maintains the fact that $\mathfrak{d}_\lambda = \mu$. We shall increase $\mathfrak{b}_\lambda$ by using the generalization to λ of Hechler forcing. This is a standard way to achieve this goal, but we spell out the proof since it demonstrates the way that we employ Lemma 2.3.

Claim 2.6. *Assume that:*

(a) λ *is supercompact*
(b) $\lambda < \mu = \mathrm{cf}(\mu) = \mu^\lambda$.

Then one can force $\mathfrak{b}_\lambda = \mathfrak{d}_\lambda = \mu$ while keeping the supercompactness of λ and the principle $\boxdot_\lambda$.

Proof. Begin with the preparatory forcing of Lemma 2.3 to make λ indestructible and to force $\boxdot_\lambda$ in such a way that it will be preserved by any further $(< \lambda)$-directed-closed forcing. By 2.3 as in the applications of Laver-indestructibility we can assume that GCH holds above λ after the preparatory forcing. In particular, if $\mu = \mathrm{cf}(\mu) > \lambda$ then $\mu^\lambda = \mu$ follows.

Let $\mathbb{D}_\lambda$ be the generalized Hechler forcing. A condition $p \in \mathbb{D}_\lambda$ is a pair (η_p, f_p) such that $\eta_p \in {}^{<\lambda}\lambda$, $f_p \in {}^\lambda\lambda$ and $\eta_p \trianglelefteq f_p$. If $p, q \in \mathbb{D}_\lambda$ then $p \leq q$ iff $\eta_p \trianglelefteq \eta_q$ and $f_p(\alpha) \leq f_q(\alpha)$ for every $\alpha \in \lambda$.

Let $\mathbf{q} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \mu, \beta < \mu \rangle$ be a $(< \lambda)$-support iteration of the generalized Hechler forcing notions for λ. Explicitly, $\underset{\sim}{\mathbb{Q}}_\alpha$ is the $\mathbb{P}_\alpha$-name of $\mathbb{D}_\lambda$ in $\mathbf{V}^{\mathbb{P}_\alpha}$ for every $\alpha < \mu$. Denote the generic λ-Hechler for $\underset{\sim}{\mathbb{Q}}_\alpha$ by $\underset{\sim}{f}^*_\alpha$. So $\mathbb{P}_\mu$ is the limit and choose a generic $\mathbf{G} \subseteq \mathbb{P}_\mu$. We claim that $\mathbf{V}[\mathbf{G}] \models$ "$\mathfrak{b}_\lambda = \mathfrak{d}_\lambda = \mu$" as witnessed by $\langle f^*_\alpha : \alpha < \mu \rangle$. Notice that $2^\lambda = \mu$ in $\mathbf{V}[\mathbf{G}]$, so it is sufficient to prove that $\mathfrak{b}_\lambda = \mu$ in $\mathbf{V}[\mathbf{G}]$.

Since λ is regular, each $\underset{\sim}{\mathbb{Q}}_\alpha$ is $(< \lambda)$-complete. By Fact 1.7, $\mathbb{P}_\alpha$ is $(< \lambda)$-complete as well, for every $\alpha \leq \mu$. Likewise, each $\underset{\sim}{\mathbb{Q}}_\alpha$ is λ-centered so $\mathbb{P}_\mu$ is λ^+-c.c. (see [She78] or [Shed]). It follows that $\mathbf{V}[\mathbf{G}]$ preserves cardinals and cofinalities. Moreover, no new $(< \lambda)$-sequences are introducted. Notice also that $\mathbb{P}_\mu$ is $(< \lambda)$-directed-closed and hence $\mathbf{V}[\mathbf{G}] \models$ "λ is supercompact and $\boxdot_\lambda$ holds".

The main point is that $\{f^*_\alpha : \alpha < \mu\}$ is a cofinal family in $({}^\lambda\lambda)^{\mathbf{V}[\mathbf{G}]}$. For this, assume that $\Vdash_{\mathbb{P}_\mu}$ "$\underset{\sim}{f} \in {}^\lambda\lambda$". For every $\alpha < \lambda$ fix a maximal antichain $\langle p_{\alpha,i} : i < i_\alpha \leq \lambda \rangle$ of conditions which force a value to $\underset{\sim}{f}(\alpha)$. Let $\delta = \sup(\cup \{\mathrm{dom}(f_{\alpha,i}) : \alpha < \lambda, i < i_\alpha\})$. Since $\lambda < \mu = \mathrm{cf}(\mu)$ we see that $\delta < \mu$, and clearly $\underset{\sim}{f}$ is a $\mathbb{P}_\delta$-name. We conclude, therefore, that $\underset{\sim}{f}$ is dominated by $\underset{\sim}{f}^*_{\delta+1}$ and hence $\{f^*_\alpha : \alpha < \mu\}$ exemplifies $\mathfrak{b}_\lambda = \mu$. This fact completes the proof. $\square_{2.6}$

Our second lemma is the main burden of the proof. The statement of the theorem requires λ to be supercompact, in order to obtain the indestructibility properties given by Lemma 2.3. The combinatorial part given in Lemma 2.7 below requires only strong inaccessibility. However, we assume supercompactness in order to keep $\boxdot_\lambda$.

Lemma 2.7. *Assume that:*

(a) λ *is supercompact,*
(b) $\boxdot_\lambda$ *holds.*

Then there exists a λ^+-c.c. $(<\lambda)$-strategically complete forcing notion $\mathbb{P}$ such that $\Vdash_{\mathbb{P}}$ "$\mathfrak{d}_\lambda = \mu \wedge \mathrm{cov}_\lambda(\text{meagre}) = \kappa$".

Proof. By claim 2.6 without loss of generality $\mathfrak{b}_\lambda = \mathfrak{d}_\lambda = \mu$. In particular, λ is supercompact and $\boxdot_\lambda$ holds in the generic extension. Let $\langle f^*_\alpha : \alpha < \mu\rangle$ witness $\mathfrak{b}_\lambda = \mathfrak{d}_\lambda = \mu$ and without loss of generality $\alpha < \beta < \mu \Rightarrow f^*_\alpha <_{J^{\rm bd}_\lambda} f^*_\beta$.

Recalling Definition 1.8, 1.9, Claim 1.10, Theorem 1.11, in $\mathbf{V}$ there are $\beta(*), \mathbf{q}, \bar{u}, \mathscr{U}_*, \ldots$ such that:

$(*)_1(A)$ $\mathbf{q} \in \mathbf{Q}_{\lambda,\bar{\theta},\beta(*)}$ so in particular we have (in $\mathbf{q}$):

(a) $\langle \mathbb{P}_{0,\alpha}, \underset{\sim}{\mathbb{Q}}_{0,\beta} : \alpha \le \beta(*), \beta < \beta(*)\rangle$ is a $(<\lambda)$-support iteration,
(b) $\bar{u} = \langle u_\beta : \beta < \beta(*)\rangle$, $\bar{\mathscr{P}} = \langle \mathscr{P}_\beta : \beta < \beta(*)\rangle$,
(c) $u_\beta \subseteq \beta$, $\mathscr{P}_\beta \subseteq [u_\beta]^{\le\lambda}$ is closed under subsets,
(d) $\underset{\sim}{\mathbb{Q}}_{0,\beta}$ has generic $\underset{\sim}{\eta}_\beta \in \prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$,
(e) $\underset{\sim}{\mathbb{Q}}_{0,\beta}$ is as in 1.8(e) so is $\subseteq \underset{\sim}{\mathbb{Q}}_{\bar{\theta}}^{\mathbf{V}[\langle \underset{\sim}{\eta}_\alpha : \alpha \in u_\beta\rangle]}$ and $\Vdash_{\mathbb{P}_{\beta+1}}$ "$\underset{\sim}{\eta}_\beta \in \prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$" and $\bar{\underset{\sim}{\eta}} = \langle \underset{\sim}{\eta}_\beta : \beta < \beta(*)\rangle$,
(f) $\mathscr{U}_* \subseteq \beta(*)$ has order type $\gamma(*) = \kappa$ and $\langle \beta^*_i : i \le \kappa\rangle$ lists $\mathscr{U}_* \cup \{\beta(*)\}$ in increasing order,
(g) if $\beta \in \mathscr{U}_*$ then $[\mathscr{U}_* \cap \beta \subseteq u_\beta$ and $[\mathscr{U}_* \cap \beta]^{\le\lambda} \subseteq \mathscr{P}_\beta$ and $\Vdash_{\mathbb{P}_{0,\beta+1}}$ "if $\nu \in \mathbf{V}[\langle \underset{\sim}{\eta}_\alpha : \alpha \in \mathscr{U}_* \cap \beta\rangle] \cap \prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$ then $\nu <_{J^{\rm bd}_\lambda} \underset{\sim}{\eta}_\beta$",
(h) if $\alpha \le \beta(*)$ then $\mathbb{P}_{0,\alpha}$ is $(<\lambda)$-strategically complete and λ^+-c.c.,
(i) $\mathbb{P}_{1,\alpha}, \mathbb{P}_{1,\mathscr{U}}$ are as in 1.9.

(B) letting $\mathbb{P}'_i = \mathbb{P}_{\mathbf{q},1,\{\beta^*_j : j<i\}}$ for $i \le \gamma(*)$ we have:

(a) The sequence $\langle \mathbb{P}'_i : i \le \gamma(*)\rangle$ of forcing notions is $\lessdot$-increasing, and is continuous for ordinals $i \le \gamma(*)$ of cofinality $> \lambda$ see [Sheb, 2.5(8)=Lz48(8)], but the continuity will not be used,
(b) $\mathbb{P}'_i$ is $(<\lambda)$-strategically complete for $i \le \gamma(*)$,
(c) $(\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_{\gamma(*)}]} = \cup\{(\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_i]} : i < \gamma(*)\}$,
(d) The sequence $\langle \mathbb{P}_{1,\beta} : \beta \le \beta(*)\rangle$ is a sequence of forcing notions, $\lessdot$-increasing and if $\beta \le \beta(*)$ then $\mathbb{P}_{0,\beta} \lessdot \mathbb{P}_{1,\beta}$, in fact is dense in it and if $i \le \gamma(*)$ then $\mathbb{P}'_i \lessdot \mathbb{P}_{1,\beta^*_i}$.

We shall mention more properties later.
[Why are there such objects? We apply 1.11 and 1.8 and 1.10, that is [Sheb].]
Also

$(*)_2$ (a) recall $\langle \beta^*_i : i \le \gamma(*)\rangle$ lists $\mathscr{U}_* \cup \{\beta(*)\}$ in increasing order,
(b) for $i < \gamma(*) = \kappa$ let $\underset{\sim}{g}'_i$ be $\underset{\sim}{\eta}_{\beta^*_i}$ (to avoid excessive subscripts),
(c) let $\bar{\underset{\sim}{g}}' = \langle \underset{\sim}{g}'_i : i < \kappa\rangle$,
(d) let $\underset{\sim}{g}_\alpha = \underset{\sim}{\eta}_\alpha$ for $\alpha < \beta(*)$ and $\bar{\underset{\sim}{g}} = \langle \underset{\sim}{g}_\beta : \beta < \beta(*)\rangle$,
(e) $\mathscr{P}_\alpha = \mathscr{P}_{\mathbf{q},\alpha}$ and without loss of generality $u_\alpha = \cup\{u : u \in \mathscr{P}_\alpha\}$ for $\alpha < \beta(*)$.

$(*)_3$ if $u \in \mathscr{P}_\alpha, \alpha < \beta(*)$ then $\Vdash_{\mathbb{P}_{0,\alpha+1}}$ "$\underset{\sim}{g}_\alpha \in \prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$ dominates $(\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\langle \underset{\sim}{g}_\beta : \beta \in u\rangle]}$", the order being modulo $J^{\rm bd}_\lambda$.

[Why? By the choice of the forcing, see 1.4 or $(*)_1(A)(g)$ above.]

$(*)_4$ we have
(a) $\Vdash_{\mathbb{P}'_\kappa}$ "$\bar{\underset{\sim}{g}}' = \langle \underset{\sim}{g}'_i : i < \kappa\rangle$ is $<_{J^{\rm bd}_\lambda}$-increasing and cofinal in $(\prod_{\varepsilon<\lambda} \theta_\varepsilon, <_{J^{\rm bd}_\lambda})$".
(b) moreover this holds even in $\mathbf{V}^{\mathbb{P}_{\beta(*)}}$

[Why? Clause (a) holds by $(*)_3$ noting that $(\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_\kappa]} = \cup\{(\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_i]} : i < \kappa\}$ which holds by 1.11(A)(d).
Clause (b) holds by 1.11(A)(e).]
Now

$(*)_5$ $\Vdash_{\mathbb{P}'_\kappa}$ "cov$_\lambda$(meagre) $\le \kappa$".

[Why? First, notice that we can look at $\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$ instead of ${}^\lambda 2$.

Second, for each $\varepsilon < \lambda, i < \kappa$ the set $B_{\varepsilon,i} = \{\eta \in \prod\limits_{\xi<\lambda} \theta_\xi$: for every $\zeta \in [\varepsilon, \lambda)$ we have $\eta(\zeta) \le \underset{\sim}{g}'_i(\zeta) < \theta_\zeta\}$ is closed nowhere dense, and by $(*)_4$ we have $\mathbf{V}^{\mathbb{P}'_\kappa} \models$ "$\prod_{\zeta<\lambda} \theta_\zeta = \cup\{\underset{\sim}{B}_{\varepsilon,i} : \varepsilon < \lambda, i < \kappa\}$". In fact, $\langle B_{0,i} : i < \kappa\rangle$ suffice.

Alternatively we have $\langle g'_i : i < \kappa\rangle$ is $<_{J^{\rm bd}_\lambda}$-increasing cofinal in $\Pi_{\varepsilon<\lambda}\theta_\varepsilon$ and let $\mathscr{W}_{i,\zeta} := \{\eta : \eta \in {}^\lambda 2$ and for every $\varepsilon \in [\zeta, \lambda)$ we have either $\eta \restriction [\Sigma_{\xi<\varepsilon}\theta_\xi, \Sigma_{\xi\le\varepsilon}\theta_\xi)$ is constantly zero or $\min\{\alpha : \Sigma_{\xi<\varepsilon}\theta_\xi + \alpha \in \eta^{-1}(\{1\})\} < g'_i(\varepsilon)\}$. So $\mathscr{W}_{i,\zeta}$ is a closed nowhere dense subset of ${}^\lambda 2$ and $\cup\{\mathscr{W}_{i,\zeta} : i < \kappa, \zeta < \lambda\} = {}^\lambda 2$ and $\kappa \times \lambda$ has cardinality $\lambda + \kappa = \kappa$ because if $f \in {}^\lambda 2$ then we define $\nu_f \in \Pi_{\varepsilon<\theta}\theta_\varepsilon$ as follows: for $\varepsilon < \lambda$:

(a) if $f\restriction[\Sigma_{\xi<\varepsilon}\theta_\xi, \Sigma_{\xi\le\varepsilon}\theta_\varepsilon)$ is not constantly zero then we let $\nu_f(\varepsilon) = \min\{\alpha : f(\Sigma_{\xi<\varepsilon}\theta_\varepsilon + \alpha) = 1\}$;
(b) if otherwise then let $\nu_f(\varepsilon) = 0$.

So there are $i < \kappa$ and $\varepsilon < \lambda$ such that: $\zeta \in [\varepsilon, \lambda) \Rightarrow \nu_f(\zeta) < g'_i(\zeta)$. Now it is easy to check that $f \in \mathscr{W}_{i,\varepsilon}$].
Lastly,

$(*)_6$ $\Vdash_{\mathbb{P}'_\kappa}$ "cov$_\lambda$(meagre) $\ge \kappa$".

[Why? For $i < \kappa$ let us define the $\mathbb{P}'_{i+1}$-name $\underset{\sim}{\nu}'_i$ of a member of ${}^\lambda 2$ by $\underset{\sim}{\nu}'_i(\varepsilon) = 0$ iff $\underset{\sim}{g}'_i(\varepsilon)$ is even. Now clearly $\Vdash_{\mathbb{P}'_{i+1}}$ "$\underset{\sim}{\nu}'_i$ is a λ-Cohen sequence over $\mathbf{V}^{\mathbb{P}'_i}$". (But let us elaborate; $\underset{\sim}{\nu}'_i$ is also a $\mathbb{P}_{\beta^*_i+1}$-name and $\Vdash_{\mathbb{P}_{\beta^*_i+1}}$ "$\underset{\sim}{\nu}'_i$ is λ-Cohen over $\mathbf{V}^{\mathbb{P}_{\beta^*_i}}$ hence over $\mathbf{V}^{\mathbb{P}'_i}$"; the last hence because $\mathbb{P}'_i \lessdot \mathbb{P}_{1,\beta^*_i}$. As $\mathbb{P}_{\beta^*_i+1} \lessdot \mathbb{P}_{\beta^*_{i+1}}$ and $\mathbb{P}'_{i+1} \lessdot \mathbb{P}_{\beta^*_{i+1}}$ we are done.)

Also every closed nowhere dense subset of ${}^\lambda 2$ from $\mathbf{V}^{\mathbb{P}'_{\gamma(*)}}$ is from $\mathbf{V}^{\mathbb{P}'_i}$ for some $i < \gamma(*)$. So if $p \Vdash$ "$\mathrm{cov}_\lambda(\mathrm{meagre}) < \kappa$" then for some $\zeta < \kappa$ and $\underset{\sim}{A}_\varepsilon(\varepsilon < \zeta)$ we have $p \Vdash$ "$\underset{\sim}{A}_\varepsilon$ is a closed no-where dense subset of ${}^\lambda 2$ for $\varepsilon < \zeta$" and $p \Vdash$ "$\bigcup\limits_{\varepsilon<\zeta} \underset{\sim}{A}_\varepsilon$ is equal to the set of ${}^\lambda 2$". Without loss of generality each $\underset{\sim}{A}_\varepsilon$ is a $\mathbb{P}_{i(\varepsilon)}$-name, $i(\varepsilon) < \kappa$ and recall that κ is regular. Hence $i = \sup\{i(\varepsilon) : \varepsilon < \zeta\} < \kappa$ and $\underset{\sim}{\eta}'_i$ gives a contradiction to the choice of $\langle \underset{\sim}{A}_\varepsilon : \varepsilon < \zeta\rangle$; so $(*)_6$ holds indeed.]

The reader may look at some explanation in 2.9.

Now we come to the main and last point recalling $\langle f^*_\alpha : \alpha < \mu\rangle$ from Claim 2.6

$(*)_7$ $\Vdash_{\mathbb{P}'_{\gamma(*)}}$ "no $\underset{\sim}{f} \in ({}^\lambda\lambda)$ dominates $\{f^*_\alpha : \alpha < \mu\}$".

We shall show that it suffices to prove $(*)_7$ for proving Lemma 2.3(2), and that $(*)_7$ holds, thus finishing.

Why it suffices? As $\langle f^*_\alpha : \alpha < \mu\rangle$ is $<_{J^{\mathrm{bd}}_\lambda}$-increasing and $\mathrm{cf}(\mu) = \mu > \lambda$, this implies $\Vdash_{\mathbb{P}'_\kappa}$ "$\mathfrak{d}_\lambda \geq \mu$". Also in $\mathbf{V}, \mu^\lambda = \mu > \kappa > \lambda$ and $|\mathbb{P}'_{\gamma(*)}| = \kappa^\lambda$ by (A)(g) of 1.10(4) which is $\leq \mu$ and $\mathbb{P}'_\kappa$ satisfies the λ^+-c.c. hence $\Vdash_{\mathbb{P}'_\kappa}$ "$2^\lambda = \mu$", hence together $\Vdash_{\mathbb{P}'_\kappa}$ "$\mathfrak{d}_\lambda = \mu$". Also by $(*)_1(B)(b)$, "$\mathbb{P}'_{\gamma(*)}$ is $(< \lambda)$-strategically complete and λ^+-c.c." and by $(*)_5 + (*)_6$ we know that "$\mathrm{cov}_\lambda(\mathrm{meagre}) = \kappa$" so we are done; hence $(*)_7$ is really the last piece missing.

The rest of the proof is dedicated to proving that $(*)_7$ holds.

We shall use further nice properties of $\mathbb{P}'_j, \underset{\sim}{g}'_i(j \leq \gamma(*), i < \gamma(*))$ which hold by $(*)_1 + (*)_2$ (and $(*)_3, (*)_4$) and their proof, i.e. 1.10, 1.11 and see [Sheb, 2.12=Lc51, 2.13=Lc52].

$\boxplus_1$ (a) (α) $\langle \underset{\sim}{g}'_\gamma : \gamma < \gamma(*)\rangle$ is generic for $\mathbb{P}'_{\gamma(*)}$, i.e., if $\mathbf{G}$ is a subset of $\mathbb{P}'_{\gamma(*)}$ generic over $\mathbf{V}$ and $g'_i = \underset{\sim}{g}'_i[\mathbf{G}]$ <u>then</u> $\mathbf{V}[\mathbf{G}] = \mathbf{V}[\langle g'_i : i < \gamma(*)\rangle]$

(β) if in addition $\nu \in ({}^\lambda\lambda)^{\mathbf{V}[\mathbf{G}]}$ <u>then</u> for some $\rho \in ({}^\lambda\gamma(*))^{\mathbf{V}}$ and λ-Borel function $\mathbf{B} \in \mathbf{V}$ we have $\nu = \mathbf{B}(\langle g'_{\rho(\varepsilon)} : \varepsilon < \lambda\rangle)$

(b) if in $\mathbf{V}[\mathbf{G}], g''_\gamma \in \prod\limits_{\zeta<\lambda} \theta_\zeta$ for $\gamma < \gamma(*)$ and the set $\{(\gamma, \zeta) : \gamma < \gamma(*)$ and $\zeta < \lambda$ and $g''_\gamma(\zeta) \neq g'_\gamma(\zeta)\}$ has cardinality $< \lambda$ then $\bar{g}'' = \langle g''_\gamma : \gamma < \gamma(*)\rangle$ is generic for $\mathbb{P}'_{\gamma(*)}$ and $\mathbf{V}[\bar{g}''] = \mathbf{V}[\bar{g}']$; similarly for $\mathbb{P}_{\beta(*)}$

(c) $\Vdash_{\mathbb{P}'_\gamma}$ "$\underset{\sim}{g}'_\gamma$ dominates $(\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_\gamma]}$"

(d) if $\langle \zeta(\gamma) : \gamma < \gamma(*)\rangle$ is an increasing sequence of ordinals $< \gamma(*)$ (from $\mathbf{V}$!), then $\langle g'_{\zeta(\gamma)} : \gamma < \gamma(*)\rangle$ is generic for $\mathbb{P}'_{\gamma(*)}$ (over $\mathbf{V}$);

(e) if $\gamma \leq \gamma(*)$ then $\mathbb{P}'_\gamma$ is $(< \lambda)$-strategically complete and satisfies the λ^+-c.c.

We shall use $\boxplus_1$ freely, recalling $\boxplus_1(d)$ was the reason for [Sheb].

To prove $(*)_7$ assume toward contradiction that this fails, and hence for some condition $p^* \in \mathbb{P}'_{\gamma(*)}$ and $\mathbb{P}'_{\gamma(*)}$-name $\underset{\sim}{f}$ and λ-Borel function $\mathbf{B}$ and $\rho_\bullet \in {}^\lambda\gamma(*)$ we have:

$\circledast_0$ $p^* \Vdash_{\mathbb{P}'_{\gamma(*)}}$ "$\underset{\sim}{f} \in {}^\lambda\lambda$ and dominates $\{f^*_\alpha : \alpha < \mu\}$, equivalently $({}^\lambda\lambda)^{\mathbf{V}}$" and for some λ-Borel function $\mathbf{B}$ from $\mathbf{V}$ we have $\underset{\sim}{f} = \mathbf{B}(\langle \underset{\sim}{g}'_{\rho_\bullet(i)} : i < \lambda\rangle)$.

Now let χ be regular large enough and we choose $\bar{N} = \langle N_\varepsilon : \varepsilon < \lambda\rangle$ such that:

$\circledast_1$ (a) N_ε is as in $\boxdot_\lambda$ for the forcing notion $\mathbb{P}_{0,\beta(*)}$ (equivalently $\mathbb{P}_{1,\beta(*)}$, <u>not</u> $\mathbb{P}'_\kappa$), that is $N_\varepsilon \in \mathscr{S}_{\chi,\mathbb{P}_{1,\beta(*)}}$ see $\boxdot_\lambda$ of 2.2,
(b) $\bar{N} \restriction \varepsilon \in N_\varepsilon$ and $\mathrm{otp}(N_\varepsilon \cap \kappa) < \theta_{(\lambda_\varepsilon)}$ hence $\bigcup_{\zeta<\varepsilon} N_\zeta \subseteq N_\varepsilon$ where $\lambda_\varepsilon := \mathrm{otp}(N_\varepsilon \cap \lambda) > \lambda^-_\varepsilon := \Sigma\{\|N_\zeta\| : \zeta < \varepsilon\} \geq \Sigma\{\lambda_\zeta : \zeta < \varepsilon\}$,
(c) $\bar{\theta}, \mathbf{q}, \mathscr{U}_*, p^*, \underset{\sim}{f}, \mathbf{B}, \rho$ belong to N_ε.

Next choose $f^* \in {}^\lambda\lambda$, i.e. $\in ({}^\lambda\lambda)^{\mathbf{V}}$, such that:

$\circledast_2$ for arbitrarily large $\varepsilon < \lambda$ for some $\zeta \in [\lambda^-_\varepsilon, \lambda_\varepsilon)$ we have $f^*(\zeta) > \lambda_\varepsilon$, (we can demand more: for every $\varepsilon < \lambda$).

For $\varepsilon < \lambda$ let $(\lambda_\varepsilon, \chi_\varepsilon, \mathbf{j}_\varepsilon, M_\varepsilon, \mathbb{A}_\varepsilon, \mathbf{G}^+_\varepsilon)$ be a witness for $(N_\varepsilon, \mathbb{P}_{1,\beta(*)}, \chi)$ recalling $\boxdot_\lambda$ from Definition 2.2 so $\lambda_\varepsilon \in (\varepsilon, \lambda)$ is strongly inaccessible and $\varepsilon < \zeta < \lambda \Rightarrow \lambda_\varepsilon < \lambda^-_\zeta < \lambda_\zeta$, recalling $\circledast_1$ and noting $\langle \lambda^-_\varepsilon : \varepsilon < \lambda\rangle$ is an increasing and a continuous sequence of cardinals below λ. Let $\mathbf{G}^-_\varepsilon = \mathbf{G}^+_\varepsilon \cap \mathbb{P}'_{\varphi(*)}$.

Let (for $\varepsilon < \lambda$):

$\circledast_3$ (a) $v_\varepsilon = N_\varepsilon \cap \gamma(*)$
(b) $\kappa_\varepsilon = \kappa(\varepsilon) = \mathrm{otp}(v_\varepsilon)$ and so $\kappa(\varepsilon) = \mathbf{j}_\varepsilon(\gamma(*))$, etc.
(c) $\bar{\gamma}^\varepsilon = \langle \gamma_i(\varepsilon) : i < \kappa(\varepsilon)\rangle$ list v_ε in increasing order
(d) for $i < \mathrm{otp}(v_\varepsilon)$, equivalently $i < \mathbf{j}_\varepsilon(\gamma(*)) = \kappa*\varepsilon)$ let $\eta^\varepsilon_i = (\mathbf{j}_\varepsilon(\underset{\sim}{g}'_{\gamma_i(\varepsilon)}))^{\mathbb{A}_\varepsilon[\mathbf{G}_\varepsilon]} \in \prod_{\zeta<\lambda_\varepsilon} \theta_\zeta$ and let $\bar{\eta}^\varepsilon = \langle \eta^\varepsilon_i : i < \kappa_\varepsilon\rangle$.

Note that clearly

$\circledast_4$ (a) $\bar{\eta}^\varepsilon$ is generic for $(\mathbb{A}_\varepsilon, \mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)}))$, moreover
(b) for each $\varepsilon < \lambda$, if we change $\eta^\varepsilon_i(\zeta)$ (legally, i.e. to an ordinal $< \theta_\zeta$) for $< \lambda_\varepsilon$ pairs $(i,\zeta) \in \mathrm{otp}(v_\varepsilon) \times \lambda_\varepsilon$ and get $\bar{\eta}'$, <u>then</u> also $\bar{\eta}'$ is generic for $(\mathbb{A}_\varepsilon, \mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)}))$, clearly $N_\varepsilon[\bar{\eta}^\varepsilon] = N_\varepsilon[\bar{\eta}']$,
(c) $\mathbf{G}^+_\varepsilon$ is a subset of $\mathbb{P}_{1,\beta(*)} \cap N_\varepsilon$ generic over N_ε such that $\mathscr{H}(\chi_\varepsilon) = \mathbb{A}_\varepsilon[\mathbf{j}_\varepsilon"(\mathbf{G}^+_\varepsilon)]$; let $\mathbf{G}^-_\varepsilon = \mathbf{G}^+_\varepsilon \cap \mathbb{P}'_{\gamma(*)}, [\mathbf{G}_\varepsilon] = \eta^\varepsilon_i$,
(d) like $\boxplus_1$ with $\mathbf{V}, \mathbb{P}_{1,\beta(*)}, \mathbf{G}, \lambda$ there standing for $\mathbb{A}_\varepsilon, \mathbf{j}_\varepsilon(\mathbb{P}_{1,\beta(*)}), \mathbf{G}^+_\varepsilon \cap \mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)}), \lambda_\varepsilon$ here.

Hence we have

$\circledast'_4$ for $\varepsilon < \lambda$,

(a) let $\Xi^-_\varepsilon := \{\bar\nu : \bar\nu = \langle \nu_i : i < i(\varepsilon)\rangle$ and for some $\mathbf{G}^- \subseteq \mathbb{P}'_{\gamma(*)} \cap N_\varepsilon$ generic over N_ε we have $\nu_i \in \Pi_{\xi<\lambda_\varepsilon}\theta_\xi$ satisfies $\xi < \lambda_\varepsilon \Rightarrow$ some $\psi \in \mathbf{G}^-$ forces $\underset{\sim}{g}'_{\gamma_i(\varepsilon)} \restriction \xi = \nu_i \restriction \xi\ \}$ and let $\Xi_\varepsilon = \{\bar\nu \in \Xi$: there is a subset $\mathbf{G}^+$ of $\mathbb{P}_{1,\beta(*)} \cap N_\varepsilon$ extending $\mathbf{G}^-$ generic over N_ε such that the Mostowski collapse of $N_\varepsilon[\mathbf{G}^+]$ is $\mathscr{H}(\chi_\varepsilon)\}$;

(b) Let Ξ^+_ε be the set of pairs $(\bar\nu, \mathbf{G}^+) = (\bar\nu, \mathbf{G}(+))$ such that: $\bar\nu \in \Xi_\varepsilon$ and $\mathbf{G}^+ \subseteq \mathbb{P}_{1.\beta(*)} \cap N_\varepsilon$ is as in clause 9a).
We may write $(\bar\nu, \mathbf{G}^+_{\bar\nu})$ or just $\bar\nu$ though actually $\bar\nu$ does not determined $\mathbf{G}^+_{\bar\nu}$; but of course it determine $\mathbf{G}^-_{\bar\nu} = \mathbf{G}^+_{\bar\nu} \cap \mathbb{P}'_{\varepsilon(*)}$.
Note that $\bar\eta^\varepsilon$ belongs to Ξ_ε moreover $(\bar\eta^\varepsilon, \mathbf{G}) \in \Xi^+_\varepsilon)$ when $\mathbf{G} = \mathbf{G}^+_\varepsilon$ and of course, $\mathbf{G}6-_{\bar\nu} = \mathbf{G}^-_\varepsilon$ when $\bar\nu = \bar\eta^\varepsilon$.
in fact $\mathbf{j}_\varepsilon(\mathbf{G}^+) = \mathbf{G}_\varepsilon$ and $\mathbf{j}_\varepsilon(\mathbf{G}) = \mathbf{G}_\varepsilon \cap \mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)})$

(c) we can choose $\bar p^*_\varepsilon = \langle p^*_{\varepsilon,\psi,\rho} : \psi \in \mathbb{P}'_{\gamma(*)} \cap N_\varepsilon, \rho \in {}^{(\lambda_\varepsilon>)}(\lambda_\varepsilon)\rangle$ such that:
 (α) $p^*_{\varepsilon,\psi,\rho} \in \mathbb{P}_{0,\beta(*)} \cap N_\varepsilon$ moreover, if $j \le \kappa_\varepsilon$ and $\psi \in \mathbb{P}'_{\gamma_j(\varepsilon)}$ then $p^*_{\varepsilon,\psi,\rho} \in \mathbb{P}_{\gamma_j(\varepsilon)}$,
 (β) $\mathbb{P}_{1,\beta(*)} \models \psi \le p^*_{\varepsilon,\psi,\rho}$ moreover, if $\mathbb{P}'_{\gamma(*)} \Vdash$ "$\psi \le \phi$" then $p^*_{\varepsilon,\psi,\rho}, \phi$ are compatible in $\mathbb{P}_{1,\beta(*)}$,
 (γ) $\langle \cup\{\mathrm{dom}(p^*_{\varepsilon,\psi,\rho}) \setminus \mathscr{U}_* : \rho \in {}^{(\lambda_\varepsilon>)}(\lambda_\varepsilon)\} : \psi \in \mathbb{P}'_{\gamma(*)}\rangle$ is a sequence of pairwise disjoint sets,
 (δ) $p^*_{\varepsilon,\psi,\rho}(\gamma)$ has trunk of length $\ge \lg(\rho)$ when $\gamma \in \mathrm{dom}(p^*_{\varepsilon,\psi,\rho})$,
 (ε) for each $\psi \in \mathbb{P}'_{\gamma(*)}$ and $\zeta < \lambda$ the sequence $\langle p^*_{\varepsilon,\psi,\rho} : \rho \in {}^\zeta(\lambda_\varepsilon)\rangle$ is a maximal anti-chain above ψ in $\mathbb{P}_{1,\beta(*)} \cap N_\varepsilon$,
 (ζ) if $\rho_1 \triangleleft \rho \in {}^{(\lambda_\varepsilon>)}(\lambda_\varepsilon)$ then $p^*_{\varepsilon,\psi,\rho_1} \le p^*_{\varepsilon,\psi,\rho}$.

(d) assume $(\bar\nu, \mathbf{G}(+)) \in \Xi^+_\varepsilon$
 (α) for $\bar p^*_\varepsilon$ as in clause (c) and $(\bar\nu, \mathbf{G}(+)) \in \Xi^+_\varepsilon$ let $\varrho_{\varepsilon,(\bar\nu,\mathbf{G}(+)),\psi}$ be the unique $\varrho \in {}^{(\lambda_\varepsilon)}(\lambda_\varepsilon)$ such that $\zeta < \lambda_\varepsilon \Rightarrow p^*_{\varepsilon,\psi,\varrho\restriction\zeta} \in \mathbf{G}_{\bar\nu}$,
 (β) for $\bar p^*_\varepsilon$ as in clause (c) there is $q \in \mathbb{P}_{\beta(*)}$ which is an upper bound of $\{p^*_{\varepsilon,\psi,\rho} : \psi \in \mathbf{G}^-_{(\bar\nu,\mathbf{G}(+))}, \rho \triangleleft \varrho_{\varepsilon,(\bar\nu,\mathbf{G}(+)),\psi}\}$ in $\mathbb{P}_{1,\beta(*)}$,
 (γ) if $q \in \mathbb{P}_{\beta(*)}$ is as above <u>then</u> q is an upper bound of $\mathbf{G}^-_{\bar\nu}$ and q is $(N_\varepsilon, \mathbb{P}'_{\gamma(*)})$-generic naturally and $q \Vdash_{\mathbb{P}_{1,\beta(*)}}$ "$\mathbf{j}_\varepsilon$ can be extended naturally to an isomorphism from $N_\varepsilon[\underset{\sim}{\mathbf{G}}_{\mathbb{P}'_{\gamma(*)}}] = N_\varepsilon[\langle \underset{\sim}{g}'_\gamma : \gamma \in v_\varepsilon\rangle]$ onto $\mathbb{A}_\varepsilon[\bar\nu]$" note that $\mathbb{A}_\varepsilon[\bar\nu]$ is not necessarily equal to $\mathscr{H}(\chi_\varepsilon)$.

(e) Similarly to clause (d) for $\varepsilon < \lambda, j \le \kappa(\varepsilon)$ and $\bar\nu = \langle \nu_i : i < j\rangle$ and $\mathbf{G}^+$.

[Why? See [Sheb, 3.28-3.32=Le53-Le67]]

By the assumption toward contradiction, $\circledast_0$, and $\mathbb{P}'_{\gamma(*)}$ being $(<\lambda)$-strategically complete, recalling $\boxplus_1$, there are $\zeta(*), p^{**}$ and p^+ such that (recall $p^* \in \mathbb{P}'_{\gamma(*)} \lessdot \mathbb{P}_{1,\beta(*)}$ is from $\circledast_0$):

$\circledast_5$ (a) $p^* \le p^{**} \in \mathbb{P}'_{\gamma(*)}$ and $p^+ \in \mathbb{P}_{0,\beta(*)}$ satisfies $\mathbb{P}_{1,\beta(*)} \models$ "$p^{**} \le p^+$"; (we may add that $\mathbb{P}'_{\gamma(*)} \models$ "$p^{**} \le \phi$" $\Rightarrow \phi, p^{**}$ are compatible in $\mathbb{P}_{1,\beta(*)}$).
 (b) $\zeta(*) < \lambda$
 (c) $p^{**} \Vdash_{\mathbb{P}'_{\gamma(*)}}$ "$f^*(\zeta) < \underset{\sim}{f}(\zeta)$ whenever $\zeta(*) \le \zeta < \lambda$" where f^* is from $\circledast_2$

(d) if $\gamma \in \mathrm{Dom}(p^+)$ then $\eta^{p^+(\gamma)}$ is an object (not just a $\mathbb{P}_{0,\gamma}$-name) and has length $\geq \zeta(*)$ (recall that $\eta^{p^+(\gamma)}$ is the trunk of the condition $p^+(\gamma)$, see clause $(\alpha)(b)$ of Definition 1.4(1)).

Note that possibly $\mathrm{Dom}(p^+) \nsubseteq \cup\{v_\varepsilon : \varepsilon < \lambda\}$. Choose $\varepsilon(*) < \lambda$ such that $\lambda_{\varepsilon(*)} > \zeta(*) + |\mathrm{Dom}(p^+)|$ and $\gamma \in \mathrm{Dom}(p^+) \Rightarrow \varepsilon(*) > \ell g(\eta^{p^+(\gamma)})$; recalling clause (d) of $\circledast_5$ and $|\mathrm{Dom}(p^+)| < \lambda$ as $p^+ \in \mathbb{P}_{0,\beta(*)}$ and $\mathbb{P}_{0,\beta(*)}$ is the limit of a $(< \lambda)$-support iteration.

By $\circledast_2$ we can add $(\exists \zeta)[\lambda^-_{\varepsilon(*)} \leq \zeta < \lambda_{\varepsilon(*)} < f^*(\zeta)]$. Our intention is to find $q \in \mathbb{P}_{0,\beta(*)}$ above p^+ which (in $\mathbb{P}_{1,\beta(*)}$) is above some $q' \in \mathbb{P}'_{\gamma(*)}$ which is $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma(*)})$-generic, that is forces $\underset{\sim}{\mathbf{G}}_{\mathbb{P}'_{\gamma(*)}}$ to include a generic subset of $(\mathbb{P}'_{\gamma(*)})^{N_{\varepsilon(*)}}$ hence is induced by some $\bar{\nu}$ as in $\circledast'_4$, recalling $\circledast_4(b)$. Toward this in $\circledast_6$ below the intention is that $p^+_{i(\varepsilon(*))}$ will serve as q.

Let $\kappa(*) = \kappa(\varepsilon(*))$ and γ_i for $i < \kappa(*)$ be such that[1] $\langle \gamma_i : i < i(*)\rangle$ list $\{\beta^*_i : i \in v_{\varepsilon(*)}\} \subseteq \mathscr{U}_{**} = N_{\varepsilon(*)} \cap \mathscr{U}_*$ in increasing order; recall $\mathscr{U}_* = \{\beta^*_i : i < \gamma(*)\}$ and $i < j < \gamma(*) \Rightarrow \beta^*_i < \beta^*_j$ and $v_{\varepsilon(*)} \subseteq \gamma(*)$ has order type $\kappa(\varepsilon(*))$ so $\gamma_i = \gamma_i(\varepsilon(*))$ from $\circledast_3$. Next let $\gamma_{i(*)} = \kappa(*)$ so $\{\mathbf{j}_{\varepsilon(*)}(\gamma) : \gamma \in v_{\varepsilon(*)}\} = i(*) = \mathbf{j}_{\varepsilon(*)}(\gamma(*))$. Recall $\gamma(*) = \kappa = \mathrm{cf}(\kappa) > \lambda, \mathrm{otp}(v_{\varepsilon(*)}) = \mathrm{otp}(N_{\varepsilon(*)} \cap \gamma(*)) = \mathrm{otp}(N_{\varepsilon(*)} \cap \kappa)$ hence $N_{\varepsilon(*)} \models$ "$i(\varepsilon(*))$ is a regular cardinal $> \lambda_{\varepsilon(*)}$" hence:

(*) $\kappa(*)$ is really a regular cardinal so call it σ.

Now we define a game ∂ as follows[2]:

$\boxplus_2$ (A) each play lasts $\kappa(*) + 1 = \sigma + 1$ moves and in the i-th move:

- (a) if $i = j + 1$ the antagonist player chooses $\xi_j = \xi(j) < \sigma$ such that $j_1 < j \Rightarrow \zeta(j_1) < \xi(j)$,
- (b) then, if $i = j + 1$ the protagonist chooses $\zeta_j = \zeta(j) \in (\xi(j), \sigma)$, <u>but</u> there are more restrictions implicit in $\boxplus_3$ below,
- (c) in any case (that is, also in the case $i = \sigma$) the protagonist also chooses $p^+_i, \bar{\nu}^i$ such that $\boxplus_3$ below holds.

(B) in the end of the play the protagonist wins the play <u>iff</u> he always has a legal move and in the end:

- (a) p^+_σ is $(\mathbb{P}'_{\gamma(*)}, N)$-generic, note the condition is not a member of the same forcing, so we mean that p^+_σ forces that the intersection of the generic with $\mathbb{P}'_{\gamma(*)} \cap N_{\varepsilon(*)}$ is generic over $N_{\varepsilon(*)}$,
- (b) $\{\zeta(i) : i \leq \sigma\} \in \mathbb{A}_{\varepsilon(*)}$; note that trivially it belongs to $M_{\varepsilon(*)} = \mathbb{A}_{\varepsilon(*)}[\mathbf{G}^+_\varepsilon] = \mathscr{H}(\chi_\varepsilon)$, see $\circledast_4(c)$.
- (c) note that we do not demand that $\bar{\nu}' = \langle \nu_{\zeta(i)} : i < \sigma\rangle$ belongs to ξ_ε, we demand only that it belongs to to Ξ^-_ε, however it is still true that it is cofinal in $(\Pi\bar{\theta}, <_{J^{\mathrm{bd}}_{\lambda_\varepsilon}}p$, because $\langle \zeta(i) : i < \sigma\rangle$ belongs to and is cofinal in ${}^\sigma\sigma$

<u>where</u>

[1]This is used in $\boxplus_3$ and the proof of $(*)_6$. Not to be confused with $\bar{\gamma}^\varepsilon$ of $\circledast_3(c)$.

[2]The idea is to scatter the $\eta^{\varepsilon(*)}_{\gamma_i}$'s. Why not use the original places? as then we shall have a problem in $\circledast_6$; the scattering is helpful because we are relying on 1.10 and 1.11.

$\boxplus_3$ (a) $p_i^+ \in \mathbb{P}_{0,\gamma_i}$ and $p_i^+, \bar\nu^i = \langle \nu_{\gamma_j} : j < i\rangle, \mathbf{G}^-_{\varepsilon(*),i} \subseteq \mathbb{P}'_{\gamma_i} \cap N_{\varepsilon(*)}$ are as in $\circledast'_4(e)$ with p_i^+ playing the role of q,

(b) if $j < i$ then $\mathbb{P}_{0,\gamma_i} \models$ "$p_j^+ \le p_i^+$",

(c) if $\gamma \in \cup\{\mathrm{Dom}(p_j^+) : j < i\}$ then $p_i^+ \restriction \gamma \Vdash_{\mathbb{P}_{0,\gamma_i}}$ "$\underset{\sim}{\eta}^{p_i^+(\gamma)}$ has length $\ge i$ and $\ge \lambda_{\varepsilon(*)}$" moreover $\underset{\sim}{\eta}^{p_i^+(\gamma)}$ is an object, $\eta^{p_i^+(\gamma)}$,

(d) $\mathbb{P}_{0,\gamma_i} \models$ "$p^+ \restriction \gamma_i \le p_i^+$", ($p^+$ is from $\circledast_5$(a)),

(e) $\bar\nu^i = \langle \nu_{\gamma_j} : j < i\rangle$ and $\nu_{\gamma_j} \in \prod\limits_{\iota < \lambda_{\varepsilon(*)}} \theta_\iota$ and p_σ^+ is an upper bound of $\Omega_{\varepsilon(*),\bar\nu^i} = \{p^*_{\varepsilon,\psi,\rho} \restriction \gamma_i : \bar\nu \in \Xi_{\varepsilon(*)}$ satisfies $\bar\nu^i \triangleleft \bar\nu$ and $\psi \in \mathbf{G}_{\bar\nu}\}$,

(f) for $j < i$ we have $\nu_{\gamma_j} \trianglelefteq \eta^{p_i^+(\gamma_j)}$ so $p_i^+ \restriction \gamma_j \Vdash$ "$\nu_{\gamma_j} \triangleleft \underset{\sim}{g}'_{\gamma_j}$" recalling $\boxplus_1$,

(g) for $j < i$ we have (recall $\bar\eta^\varepsilon$ from $\circledast_3(d)$) (α) or (β) where:

(α) $\nu_{\gamma_j} = \eta^{\varepsilon(*)}_{\gamma_{\zeta(j)}}$ recalling $\eta^{\varepsilon(*)}_{\gamma_j}$ is from $\circledast_3(d)$,

(β) $\gamma_j \in \mathrm{Dom}(p^+)$ and $\{\iota < \lambda_{\varepsilon(*)} : \eta^{\varepsilon(*)}_{\zeta(j)}(\iota) \ne \nu_{\gamma_j}(\iota)\}$ is a bounded subset of $\lambda_{\varepsilon(*)}$.

We shall prove

$\circledast_6$ in the game $\eth$:

(a) the antagonist has no winning strategy,

(b) at stage i, if $\langle \zeta(j) : j < i\rangle \in \mathbb{A}_\varepsilon$ then the protagonist has a legal move, moreover for any $\zeta(i) \in (\xi(i), \sigma)$ large enough the protagonist can choose it.

<u>Why $\circledast_6$ suffice?</u>

By clause (a) of $\circledast_6$ we can choose a play $\langle(\xi(i), \zeta(i), p_i^+, \bar\nu^i) : i \le \sigma\rangle$ in which the protagonist wins. Recalling $\mathbb{P}'_{\gamma(*)} \lessdot \mathbb{P}_{1,\beta(*)}$ and $\mathbb{P}_{0,\beta(*)}$ is a dense sub-forcing of $\mathbb{P}_{1,\beta(*)}$, clearly

$\circledast_7$ there is p such that:

(a) $p \in \mathbb{P}'_{\gamma(*)}$,

(b) if $\mathbb{P}'_{\gamma(*)} \models$ "$p \le p'$" hence $p' \in \mathbb{P}'_{\gamma(*)}$ <u>then</u> p', p_σ^+ are compatible in $\mathbb{P}_{1,\beta(*)}$,

(c) p is above p^{**} and it forces that $\underset{\sim}{g}'_{\gamma_i}{\restriction}\lambda_{\varepsilon(*)} = \nu_{\gamma_{\zeta(i)}}$ for $i < \kappa(*)$ and $\mathbf{j}_{\varepsilon(*)}(\underset{\sim}{\mathbf{G}}_{\mathbb{P}'_{\beta(*)}} \cap N_{\varepsilon(*)}) = \mathbf{G}^-_{\langle \nu_{\gamma_i} : i < \sigma\rangle}$.

Then on the one hand

$\circledast'_7$ $p \in \mathbb{P}'_{\gamma(*)}$ being above p^{**} forces $f^* \restriction [\zeta(*), \lambda) < \underset{\sim}{f} \restriction [\zeta(*), \lambda)$ hence $f^* \restriction [\zeta(*), \lambda_{\varepsilon(*)}) < \underset{\sim}{f} \restriction [\zeta(*), \lambda_{\varepsilon(*)})$ recalling that $\zeta(*) < \lambda_{\varepsilon(*)}$, see $\circledast_5$ and the choice of $\varepsilon(*)$ immediately after $\circledast_5$.

On the other hand,

$\circledast''_7$ p is $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma(*)})$-generic.

[Why? As it forces $\underset{\sim}{\eta}_{\gamma_i} \restriction \lambda_{\varepsilon(*)} = \nu_{\gamma_i}$ for $i < i(*)$ and $\langle \nu_{\gamma_i} : i < i(*)\rangle$ is (see $\boxplus_3(g)$ recalling Dom(p^{**}) has cardinality $< \lambda_{\varepsilon(*)}$) "almost equal" to $\langle \eta^{\varepsilon(*)}_{\zeta(i)} : i < i(*)\rangle$ which is a sub-sequence of the sequence from $\circledast_3$. That is $\{(i,\iota) : \iota < \lambda_{\varepsilon(*)}, i < i(*) = \sigma$ and $\nu_{\gamma_i}(\iota) \neq \eta^{\varepsilon(*)}_{\zeta(i)}(\iota)\} \subseteq \cup\{\{(i,\iota) : \iota < \lambda_{\varepsilon(*)}$ and $\nu_{\gamma_i}(\iota) \neq \eta^{\varepsilon(*)}_{\zeta(i)}(\iota)\} : \gamma_i \in v_{\varepsilon(*)} \cap$ Dom$(p^{**})\}$ so is the union of $\leq |\mathrm{Dom}(p^+_\sigma)| < \lambda_{\varepsilon(*)}$ sets each of cardinality $< \lambda_{\varepsilon(*)}$ hence is of cardinality $< \lambda_{\varepsilon(*)}$. Hence by $\circledast_4(d) + \boxplus_1(d)$ the sequence $\bar\nu^{i(*)}$ is generic for $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma(*)})$. By $\boxplus_2$ and the choice of p^+_σ above it is $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma(*)})$-generic. By $\circledast_7(b)$ also p is.

As $\underset{\sim}{f} \in N_{\varepsilon(*)}$ it follows from $\circledast''_7$ that

$\circledast'''_7$ $p \Vdash$ "$\underset{\sim}{f} \restriction \lambda_{\varepsilon(*)}$ is a function from $\lambda_{\varepsilon(*)}$ to $\lambda_{\varepsilon(*)}$".

Together $\circledast'_7 + \circledast'''_7$ gives a contradiction by the choice of f^* in $\circledast_2$ and of $\varepsilon(*)$ above which implies that $\underset{\sim}{f}(\zeta) > f^*(\zeta) > \lambda_{\varepsilon(*)}$ for some $\zeta < \lambda_{\varepsilon(*)}$ hence $\circledast_6$ is enough. In Lemma 2.8 below we show that $\circledast_6$ is true; so we are done. $\square_{2.7}$

Lemma 2.8. *The statement $\circledast_6$ is true.*

Proof. Let us prove $\circledast_6$; first, assuming clause (b) which is proved below, for clause (a) choose any strategy **st** for the antagonist and fix a partial strategy **st**$'$ for the protagonist choosing $(p^+_i, \bar\nu^i)$ depending on the previous choices and $\xi(i) < \kappa_{\varepsilon(*)}$ such that it is a legal move if relevant and possible. So the only freedom left for the protagonist is to choose the $\zeta(i)$. So (recalling $\boxplus_2(A)(a)$) we have in $\mathbf{V}$ a function $F : {}^{\sigma>}\sigma \to \sigma$ (so F depends on **st** *and* **st**') such that:

$(*)_F$ playing the game such that the antagonist uses **st** and the protagonist uses **st**$'$, arriving at the i-th move, $\bar\zeta = \langle \zeta(j) : j < i\rangle$ is well defined and if $\bar\zeta \in N_{\varepsilon(*)}$ <u>then</u> for the protagonist any choice $\zeta_i \in (F(\bar\zeta), \sigma) \cap \mathscr{U}_{**}$ is legal.

Note that F belongs to $\mathscr{H}(\chi_\varepsilon)$ unlike $p^+_\varepsilon, \bar\nu^\varepsilon$.

Now we have to find an increasing sequence $\bar\zeta = \langle \zeta(i) : i < \sigma\rangle$ from $\mathbb{A}_{\varepsilon(*)}$ not just from $M_{\varepsilon(*)} = \mathscr{H}(\chi_{\varepsilon(*)})^{\mathbf{V}}$ such that $F(\bar\zeta \restriction i) < \zeta(i) < \sigma$ and $\bar\zeta \in \mathbb{A}_{\varepsilon(*)}$. Why possible? As $F \in \mathscr{H}(\chi_{\varepsilon(*)})$ and $\mathscr{H}(\chi_{\varepsilon(*)}) = M_{\varepsilon(*)} = \mathbb{A}_{\varepsilon(*)}[\mathbf{G}^+_{\varepsilon(*)}]$ where $\mathbf{G}^+_{\varepsilon(*)}$ is a subset of $\mathbf{j}_{\varepsilon(*)}(\mathbb{P}_{1,\gamma(*)}) \in \mathbb{A}_{\varepsilon(*)}$ generic over $\mathbb{A}_{\varepsilon(*)}$ and $\mathbf{j}_{\varepsilon(*)}(\mathbb{P}_{0,\beta(*)})$ satisfies the $\lambda^+_{\varepsilon(*)}$-c.c. and $\sigma =$ cf$(\sigma) > \lambda_{\varepsilon(*)}$ this[3] is possible. That is, there is a $\mathbf{j}_{\varepsilon(*)}(\mathbb{P}_{0,\beta(*)})$-name $\underset{\sim}{F}_* \in \mathbb{A}_{\varepsilon(*)}$ such that $F = \underset{\sim}{F}_*[\mathbf{G}^+_{\varepsilon(*)}]$ and we define in $\mathbb{A}_{\varepsilon(*)}$ the function $F' : {}^{\sigma>}\sigma \to \sigma$ by $F'(\langle \zeta(j) : j < i\rangle) = \sup\{\xi + 1 : \xi \in \{\zeta(j) + 1 : j < i\}$ <u>or</u> $\xi < \sigma$ and $\nVdash_{\mathbf{j}(\mathbb{P}_{0,\beta(*)})}$ "$\underset{\sim}{F}(\langle \zeta(j) : j < i\rangle) \neq \xi$"$\}$; clearly this is O.K.

We are left with proving $\circledast_6(b)$.

<u>Case 1</u>: $i = 0$.

Let $p^+_0 = p^+ \restriction \gamma_0$.

<u>Case 2</u>: i limit.

By clauses (b) and (c) of $\boxplus_3$, there is $p^+_i \in \mathbb{P}_{0,\gamma_i}$ which is an upper bound (even l.u.b.) of $\{p^+_j : j < i\} \cup \Omega_{\varepsilon(*),\bar\nu^i}$ and it is easily as required. Also $\bar\nu^i$ is well defined and as required.

[3] In fact $\mathbf{V} \models$ "$\mathbb{P}'_\kappa$ satisfies the κ-c.c." suffices.

Case 3: $i = j+1$ and $\gamma_j \notin \mathrm{Dom}(p^+)$.

Clearly γ_i is in $\mathscr{U}_*$ the successor of γ_j and $(\exists \iota)(\gamma_j = \beta^*_\iota \wedge \iota \in v_{\varepsilon(*)})$. As in case 4 below but easier by the properties of the iteration and [Sheb, §3C].

Case 4: $i = j+1$ and $\gamma_j \in \mathrm{Dom}(p^+)$.

Again γ_i is in $\mathscr{U}_*$ the successor of γ_j and $(\exists \iota)(\gamma_j = \beta^*_\iota \wedge \iota \in v_{\varepsilon(*)})$.

First we find p'_j such that:

$\circledast_8$ (a) $p^+_j \le p'_j \in \mathbb{P}_{0,\gamma_j}$,

(b) if $\gamma \in \mathrm{Dom}(p^+_j)$ then $p'_j \restriction \gamma \Vdash$ "$\ell g(\underset{\sim}{\eta}^{p'_j(\gamma)}) > i(*) = \sigma$" (see $\boxplus_3(c)$),

(c) p'_j forces [4] a value to the pair $(\eta^{p^+(\gamma_j)}, \underset{\sim}{f}^{p^+(\gamma_j)} \restriction \lambda_{\varepsilon(*)})$; we call this pair $q_j = (\eta^{q_j}, f^{q_j})$.

[Why? This should be clear.]

Second

$\circledast_9$ p^+_j hence p'_j is $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma_j})$-generic and $\langle \nu_{\gamma_{j(1)}} : j(1) < j\rangle$ induces the generic.

[Why? As in the proof of $\circledast''_7$ of Lemma 2.7 when we assume that we have carried the induction, by $\boxplus_2$, clause (g) and $\circledast_4$.]

Now

$\circledast_{10}$ (a) $f^{q_j} \in (\prod_{\zeta<\lambda_{\varepsilon(*)}} \theta_\zeta)^{\mathbb{A}_{\varepsilon(*)}[\mathbf{G}^+_{\varepsilon(*)}]}$; recalling that F^{q_j} is from clause (c) of $\circledast_9$.

(b) for every large enough $\zeta \in (\xi(i), \sigma)$ we have

- $f^{q_j} \le \eta^{\varepsilon(*)}_\zeta \mod J^{\mathrm{bd}}_{\lambda_\varepsilon}$.

[Why? Clause $\circledast_{10}$(a) holds because $f^{q_j} \in (\prod\limits_{\zeta<\lambda_{\varepsilon(*)}} \theta_\zeta)^{\mathbf{V}}$, hence belongs to $\mathscr{H}(\chi_{\varepsilon(*)})$ which is the universe of $M_{\varepsilon(*)}$ so $f^{q_j} \in M_{\varepsilon(*)}$. But $M_{\varepsilon(*)} = \mathbb{A}_{\varepsilon(*)}[\mathbf{G}_{\varepsilon(*)}]$ and $\bar\eta^{\varepsilon(*)} = \langle \mathbf{j}_{\varepsilon(*)}(\underset{\sim}{\eta}_\gamma) : \gamma \in \gamma(*) \cap N_{\varepsilon(*)}\rangle$; recalling $\bar\eta^{\varepsilon(*)}$ is a generic for $\mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)})$.

For clause $\circledast_{10}$(b) recall $(*)_4(b)$. Hence $N_{\varepsilon(*)}$ satisfies the parallel statement, so $N_{\varepsilon(*)}$ satisfies: if we force by $\mathbb{P}_{\gamma(*)}$ then $\{\underset{\sim}{\eta}_\gamma : \gamma \in \gamma(*) \cap N_{\varepsilon(*)}\}$ is cofinal in $(\Pi_{\varepsilon<\lambda_{\varepsilon(*)}} \theta_\varepsilon, \le_{J^{\mathrm{bd}}_{\lambda_{\varepsilon(*)}}})$.

This is a crucial point: this is justified by clause (A)(e) of 1.11.

Applying $\mathbf{j}_{\varepsilon(*)}$ and recalling $\mathbb{A}_{\varepsilon(*)}[\mathbf{G}^+_{\varepsilon(*)}] = \mathscr{H}(\chi_{\varepsilon(*)})$ we are done proving $(*)_{10}$.

Now we choose $\zeta(j) > \sup\{\zeta(j_1) : j_1 < j\}$ as in clause (b) of $\circledast_{10}$ and $\nu_j = \eta_{\zeta(j)}$; so here we obey the promise "for every large enough $\zeta(i)$.

Next choose $p^+_i \in \mathbb{P}'_{\gamma(*)}$ such that $p^+_i \restriction \gamma_j = p'_j, \eta^{p^+_i(\gamma_i)} = \nu_j$ and $f^{p^+_i(\gamma_j)} \restriction [\lambda_\varepsilon, \lambda) = f^{p^+(\gamma_j)} \restriction [\lambda_\varepsilon, \lambda)$ and $\nu_{\zeta(j)} \triangleleft f^{p^+_i(\gamma_j)}$.

We have carried the induction hence proved $\circledast_6(b)$ so we are done proving 2.8. $\square_{2.8}$

Discussion 2.9. 1) The reader may justly wonder why we use $\mathbf{V}' = \mathbf{V}[\underset{\sim}{\bar g}'] = \mathbf{V}[\bar g \restriction \mathscr{U}_*]$ rather than simply $\mathbf{V}[\underset{\sim}{\bar g}]$. Of course, nothing is lost by it, but why the extra complication?

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[4] recall that $\eta^{p^+(\gamma_j)}$ is an object, not a name and p^+_j is $(N_{\varepsilon(*)}, \mathbb{P}_{\gamma_j})$-generic

2) The answer is that during the proof we used: if $\zeta(i) \in \mathscr{U}_*$ is increasing with $i < \gamma(*)$ then also $\langle \underset{\sim}{g}_{\zeta(i)} : i < \kappa \rangle$ is generic over $\mathbf{V}$ for the sub-forcing of $\mathbb{P}_{1,\beta(*)}$ generated by $\underset{\sim}{\bar{g}} \restriction \mathscr{U}_*$; see $\circledast''_7$ inside the proof of $\circledast_6$ inside 2.8. But using $\mathscr{U}_* = \beta(*)$, we do not know this.
3) Now in the parallel case for $\lambda = \aleph_0$ with FS iteration with full memory, such claim is true, see §0.
4) But we do not know the parallel of (3) for λ, so we use a substitute using $\mathscr{U}_*$, i.e. $\mathbb{P}'_\kappa$.

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