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MONOCOLORED TOPOLOGICAL COMPLETE GRAPHS IN COLORINGS OF UNCOUNTABLE COMPLETE GRAPHS

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Abstract. If $\kappa > \aleph_0$ then $\kappa \to (\kappa, \mathrm{Top}\, K_\kappa)^2$, i.e., every graph on κ vertices contains either an independent set of κ vertices, or a topological K_κ, iff κ is regular and there is no κ-Suslin tree. Concerning the statement $\omega_2 \to (\mathrm{Top}\, K_{\omega_2})^2_\omega$, i.e., in every coloring of the edges of K_{ω_2} with countably many colors, there is a monochromatic topological K_{ω_2}, both the statement and its negation are consistent with the Generalized Continuum Hypothesis.

The natural generalization of Ramsey's theorem $\aleph_0 \to (\aleph_0, \aleph_0)^2$ fails strongly for most cardinals, as it is well known. If κ is uncountable, then $\kappa \to (\kappa, \kappa)^2$ holds if and only if κ is weakly compact. Erdős and Hajnal in [2] gave a short, elegant proof (applying ultrafilters) of $\kappa \to (\mathrm{Top}\, K_\kappa, \mathrm{Top}\, K_\kappa)^2$ for $\kappa > \aleph_0$, that is, if the pairs of κ are colored with 2 colors, then one of them contains a topological K_κ. The same argument gives $\kappa \to (\mathrm{Top}\, K_\kappa)^2_n$ for any finite n.

Erdős and Hajnal asked if the asymmetric variant $\kappa \to (\kappa, \mathrm{Top}\, K_\kappa)^2$ holds for $\kappa > \aleph_0$. A moment's reflection shows that this implies the above $\kappa \to (\mathrm{Top}\, K_\kappa)^2_n$. Here we answer this question by showing that $\kappa \not\to (\kappa, \mathrm{Top}\, K_\kappa)^2$ for κ is singular and if $\kappa > \aleph_0$ is regular, $\kappa \to (\kappa, \mathrm{Top}\, K_\kappa)^2$ holds if and only if there is no κ-Suslin tree.

Next we address the following natural extension of the Erdős–Hajnal result: does $\kappa \to (\mathrm{Top}\, K_\kappa)^2_\omega$ hold? As the edges of the complete graph K_{ω_1}

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can be covered by countably many circuit-free graphs (Erdős–Kakutani [3]), κ must be at least $\aleph_2$. Our interest is therefore in $\omega_2 \to (\operatorname{Top} K_{\omega_2})^2_\omega$.

In order to show the positive relation, we introduce the following principle.

$$(*) \quad \begin{cases} \text{For every } F\colon [\omega_2]^2 \to \omega \text{ there exist } i < \omega \text{ and } A \in [\omega_2]^{\omega_2} \text{ such that} \\ \text{if } \alpha, \beta \in A,\ \alpha < \beta, \text{ then there is } \gamma > \beta \text{ with } F(\alpha,\gamma) = F(\beta,\gamma) = i. \end{cases}$$

We show that $(*)$ implies $\omega_2 \to (\operatorname{Top} K_\kappa)^2_\omega$. $(*)$ follows from the existence of an $\aleph_1$-dense ideal on ω_2, whose consistency was deduced from the consistency of a huge cardinal by Foreman [4]. Next we give two forcing models for the negation of $(*)$, one without, one with CH. Finally we give a forcing model of GCH in which $[\omega_2]^2$ is the union of countably many ω_2-Suslin trees, from which we deduce $\omega_2 \not\to (\operatorname{Top} K_{\omega_2})^2_\omega$.

We notice that both results lift to larger cardinals: if $\mu > \omega$ is regular then both $\mu^{++} \not\to (\operatorname{Top} K_{\mu^{++}})^2_\mu$ and $\mu^{++} \to (\operatorname{Top} K_{\mu^{++}})^2_\mu$ are consistent (the latter relative to the consistency of a huge cardinal).

Notation. Definitions. We use the notation and definitions of axiomatic set theory. In particular, ordinals are von Neumann ordinals, and each cardinal is identified with the least ordinal of that cardinality.

If S is a set, κ a cardinal, we define $[S]^\kappa = \{x \subseteq S : |x| = \kappa\}$, $[S]^{<\kappa} = \{x \subseteq S : |x| < \kappa\}$. If A is some set of ordinals, then $\operatorname{tp}(A)$ is the order type of A. If A, B are sets of ordinals, then $A < B$ denotes that $x < y$ holds for $x \in A$, $y \in B$. $\mathcal{H}(\theta)$ denotes the set of all sets with transitive closure of size $< \theta$. $<_w$ always denotes some well order on it.

In a partially ordered set $(P, \leq)$ we define $p{\uparrow} = \{q \in P : p < q\}$ and $p{\downarrow} = \{q \in P : q < p\}$. A *tree* $(T, \leq)$ is a partially ordered set in which each $t{\downarrow}$ is well ordered. Define $T_\alpha = \{t \in T : \operatorname{tp}(t{\downarrow}) = \alpha\}$, the *height of* $(T, \leq)$ is $h(T) = \min\{\alpha : T_\alpha = \emptyset\}$. If $\alpha \leq h(T)$ is limit, an *α-branch* is a set $b \subseteq \bigcup\{T_\beta : \beta < \alpha\}$ such that $(b, \leq)$ is totally ordered and $|b \cap T_\beta| = 1$ $(\beta < \alpha)$. A set $A \subseteq T$ is an *antichain* if its elements are pairwise incomparable in $(T, \leq)$. A tree $(T, \leq)$ with $h(T) = \kappa$ is a *κ-Suslin-tree* if it contains neither κ-branches, nor antichains of size κ. It is easy to see that if κ is weakly compact, then there is no κ-Suslin tree. If the axiom of constructibility holds, then the reverse implication holds—for every non-weakly compact strongly inaccessible κ there exists a κ-Suslin tree, as shown by Jensen (cf. [5] and also in [1]).

A *graph* is a pair (V, X), where V is an arbitrary set (the set of *vertices*) and $X \subseteq [V]^2$ (the set of *edges*). We sometimes write simply X rather than (V, X), i.e., identify the graph with its edge set. If (V, X) is a graph, $x \in V$ a vertex, then $N(x) = \{y \in V : \{x, y\} \in X\}$ is the *neighborhood of* x. If V is ordered by $<$, then $N^-(x) = \{y < x : \{x, y\} \in X\}$ and

$N^+(x) = \{y > x : \{x, y\} \in X\}$. A *path* is a sequence $(v_0, v_1, \dots, v_n)$ of distinct vertices such that $\{v_i, v_{i+1}\} \in X$ $(i < n)$. A graph is *connected* if any two vertices are connected by a path.

K_κ is the complete graph on κ vertices: $(\kappa, [\kappa]^2)$.

A graph (V, X) contains a *topological* K_κ if there exist distinct vertices $\{v_\alpha : \alpha < \kappa\}$ and paths $\{p_{\alpha,\beta} : \alpha < \beta < \kappa\}$ such that $p_{\alpha,\beta}$ is a path between v_α and v_β, and the paths are vertex disjoint, except, of course, at their extremities.

A similar notion is that the graph (V, X) *contains* K_κ *as a minor*. This happens if there are disjoint vertex sets $\{W_\alpha : \alpha < \kappa\}$ such that each W_α induces a connected subgraph, and there is an edge between W_α and W_β $(\alpha < \beta < \kappa)$. It is easy to see that if (V, X) contains a topological K_κ, then it also contains K_κ as a minor. The reverse implication is proved in [6] by Jung if κ is uncountable regular.

The *comparison graph* of a tree $(T, \leq)$ is (T, X) where X consists of all pairs $\{t, t'\}$ where $t < t't$.

The partition relation symbol $\kappa \to (\alpha, \beta)^2$ holds if the following statement is true: for every $f : [\kappa]^2 \to \{0, 1\}$ either there is a homogeneous set of size α in color 0, or else there is a homogeneous set of size β in color 1. Similarly, $\kappa \to (\alpha)^2_\gamma$ abbreviates the statement that for every coloring $f : [\kappa]^2 \to \gamma$ there is a homogeneous set of size α. That is, the graph $f^{-1}(\tau)$ contains a K_α for some $\tau < \gamma$. We use the ad hoc modification $\kappa \to (\mathrm{Top}\, K_\alpha)^2_\gamma$ to denote that for each $f : [\kappa]^2 \to \gamma$ for some $\tau < \gamma$ the graph $f^{-1}(\tau)$ contains a topological K_α and similarly for $\kappa \to (\alpha, \mathrm{Top}\, K_\beta)^2$. The negation of all these statements is denoted by crossing the arrows.

LEMMA 1. *Let κ be regular, T a tree of height κ, X the comparison graph of T. Then X contains a topological K_κ iff there is a κ-branch in T.*

PROOF. One direction is obvious: if there is a κ branch in T, then this is a K_κ in X.

For the other direction assume that $\{a_\alpha, p_{\alpha,\beta} : \alpha < \beta < \kappa\}$ is a topological K_κ in X.

For each $\alpha < \kappa$, as $|a_\alpha{\downarrow}| < \kappa$, there are only $< \kappa$ many β such that $p_{\alpha,\beta}$ has a vertex in $a_\alpha{\downarrow}$ (apart from a_α). For all other β, all vertices of $p_{\alpha,\beta}$, specifically a_β must be in $a_\alpha{\uparrow}$, that is, $a_\alpha < a_\beta$. We obtained that there is a $U \in [\kappa]^\kappa$ such that $a_\alpha < a_\beta$ for $\alpha, \beta \in U$, $\alpha < \beta$, but this gives a κ-branch. □

LEMMA 2. *If κ is singular, then $\kappa \not\to (\kappa, \mathrm{Top}\, K_\kappa)^2$.*

PROOF. Set $\mu = \mathrm{cf}(\kappa)$, $\sup\{\kappa_\alpha : \alpha < \mu\} = \kappa$. Let V be the disjoint union of the sets V_α $(\alpha < \mu)$, $|V_\alpha| = \kappa_\alpha$ and let X be the graph where two vertices are joined iff they are in the same V_α. Now clearly there is neither an independent set of size μ^+, nor a topological K_κ, or even a connected subgraph of cardinality κ. □

THEOREM 3. *If $\kappa > \omega$ is regular, then the following are equivalent.*
(a) $\kappa \not\to (\kappa, \mathrm{Top}\, K_\kappa)^2$.
(b) *There is a κ-Suslin tree.*

PROOF. First, let $(T, \leq)$ be a κ-Suslin tree. Let X be the comparison graph of T, i.e., $\{s,t\} \in X$ iff s,t are comparable. As $(T, \leq)$ is Suslin, there is no independent set or (by Lemma 1) a topological K_κ in X.

Assume finally that X is a graph on κ with no independent set of cardinal κ, neither a topological K_κ. By Jung's theorem, K_κ is not a minor of X.

Claim 1. *If $W \in [\kappa]^\kappa$, then the number of connected components of $X|W$ is $< \kappa$.*

PROOF. Otherwise, the choice of one vertex from each would give an independent set of size κ. $\square$

Claim 2. *If $W \in [\kappa]^\kappa$, then there is $A \in [W]^{<\kappa}$ such that $X|(W-A)$ contains at least* 2 *connected components of size κ.*

PROOF. Assume that the Claim fails. Then, for some $W \in [\kappa]^\kappa$, the following holds. For every $A \in [W]^{<\kappa}$, all but one of the connected components of $X|(W-A)$ are of size $< \kappa$. By Claim 1, the number of those components is $< \kappa$, as κ is regular, their union still has size $< \kappa$. We have therefore obtained the following. For every $A \in [W]^{<\kappa}$ there is $A \subseteq A' \in [W]^{<\kappa}$ such that $X|(W-A')$ is connected. Using this, we define the increasing sequence $\{A_\alpha : \alpha < \kappa\}$ with $|A_\alpha| < \kappa$ $(\alpha < \kappa)$, such that $\bigcup\{A_\beta : \beta < \alpha\} \subseteq A'_\alpha$ with $|A'_\alpha| < \kappa$ and $X|(W - A'_\alpha)$ connected, $D_\alpha \in [W - A'_\alpha]^{<\kappa}$ is maximal independent and finally $A'_\alpha \cup D_\alpha \subseteq A_\alpha$ is such that $|A_\alpha| < \kappa$ and $B_\alpha = A_\alpha - A'_\alpha$ is connected.

Now, each $X|B_\alpha$ is connected and if $\beta < \alpha < \kappa$, then there is an edge between B_β and B_α, namely if $p \in B_\alpha$ is arbitrary, then there is and edge between p and $D_\beta \subseteq B_\beta$, as D_β is maximal independent. That is, $\{B_\alpha : \alpha < \kappa\}$ establishes K_κ as a minor in $X|W$, a contradiction to our assumption on X. $\square$

Next we are going to build a tree T and define the sets $A(t) \in [\kappa]^{<\kappa}$, $V(t) \in [\kappa]^\kappa$ for $t \in T$. We define T level by level. Let T_0 contain the unique root r with $V(r) = \kappa$.

Assume that we have built T up to level α and $t \in T_\alpha$. By Claim 2, there is a set $A \in [V(t)]^{<\kappa}$ such that $V(t) - A$ contains at least two connected components of cardinality κ. By adding those components with size $< \kappa$ to A we obtain a set $A(t) \in [V(t)]^{<\kappa}$ such that all components of $X|(V(t) - A(t))$ have size κ and θ, their number, is at least 2. We add θ immediate successors $\{t_\xi : \xi < \theta\}$ to t and arrange that $\{V(t_\xi) : \xi < \theta\}$ are the above mentioned components of $X|(V(t) - A(t))$.

Assume that $\alpha < \kappa$ is limit and we have constructed $T_{<\alpha}$ and the corresponding sets $A(t), V(t)$ $(t \in T_{<\alpha})$. For each α-branch $b \subseteq T_{<\alpha}$ we place a node s of T_α atop b iff $|\bigcap\{V(t) : t \in b\}| = \kappa$. If this holds, we set $V(s) = \bigcap\{V(t) : t \in b\}$.

It is clear from the construction that if $t < t'$ then $V(t) \supseteq V(t')$.

Claim 3. *If t', t'' are incomparable, then $V(t') \cap V(t'') = \emptyset$ and there is no edge between $V(t')$ and $V(t'')$.*

Proof. Let t be the largest common lower bound of t' and t''. t exists by the way of constructing T_α for α limit. There are immediate successors $t_\xi \neq t_\eta$ of t such that $t_\xi \leq t'$, $t_\eta \leq t''$. Then $V(t') \subseteq V(t_\xi)$, $V(t'') \subseteq V(t_\eta)$ and $V(t_\xi) \cap V(t_\eta) = \emptyset$ and there is no edge between $V(t_\xi)$ and $V(t_\eta)$ and so this hold for $V(t')$, $V(t'')$. □

Claim 4. *There is no antichain of size κ in T.*

Proof. If $\{t_\xi : \xi < \kappa\}$ were an antichain, then picking one vertex from each $V(t_\xi)$ would give an independent set of size κ by Claim 3, a contradiction. □

Claim 5. $|T_\alpha| < \kappa$ $(\alpha < \kappa)$.

Proof. As T_α is an antichain. □

Claim 6. $h(T) = \kappa$.

Proof. Assume indirectly that $T_\alpha = \emptyset$ for some $\alpha < \kappa$. Let α be minimal such. Then α is limit by the way T is constructed. If $\beta \leq \alpha$ is limit, b is a β-branch, set $H(b) = \bigcap\{V(t) : t \in b\}$,

$$U(\beta) = \big\{ b : |H(b)| < \kappa \big\},$$

and

$$R(\beta) = \bigcup \big\{ H(b) : b \in U(\beta) \big\}.$$

By assumption $\kappa = S' \cup S''$ where

$$S' = \bigcup \big\{ A(t) : t \in T_{<\alpha} \big\} \quad \text{and} \quad S'' = \bigcup \big\{ R(\beta) : \beta \leq \alpha \text{ limit} \big\}.$$

We have $|S'| < \kappa$ as $A(t) < \kappa$ $(t \in T_{<\alpha})$ and $\alpha < \kappa$ and $|T_\beta| < \kappa$ $(\beta < \alpha)$ by Claim 5.

Further, by the argument in Claim 3, there are only $< \kappa$ β-branches b for which $H(b) \neq \emptyset$. This implies that $|R(\beta)| < \kappa$ and eventually $|S''| < \kappa$. As we showed both $|S'| < \kappa$ and $|S''| < \kappa$, we have reached a contradiction. □

Claim 7. *There is no κ-branch in T.*

PROOF. Assume indirectly that $b = \{t_\alpha : \alpha < \kappa\}$ is a κ-branch with $b \cap T_\alpha = \{t_\alpha\}$. Pick $s_\alpha \in T_{\alpha+1}$ such that $t_\alpha < s_\alpha$ and $s_\alpha \neq t_{\alpha+1}$. This choice is possible as each t_α has at least two immediate successors. Now $\{s_\alpha : \alpha < \kappa\}$ is an antichain of size κ, contradicting Claim 4. □

By Claims 4, 6, and 7, $(T, \leq)$ is a κ-Suslin tree, and the proof of Theorem 3 is finished. □

A reasonable extension of the above mentioned Erdős–Hajnal theorem $\omega_2 \to (\mathrm{Top}\, K_{\omega_2})^2_n$ $(n < \omega)$ would be $\omega_2 \to (\mathrm{Top}\, K_{\omega_2})^2_\omega$. In order to investigate it, we consider the principle $(*)$ defined in the Introduction.

LEMMA 4. *$(*)$ implies $\omega_2 \to (\mathrm{Top}\, K_{\omega_2})^2_\omega$.*

PROOF. Assume that $F\colon [\omega_2]^2 \to \omega$. Let

$$N_0 \prec N_1 \prec \cdots \prec N_\alpha \prec \cdots \prec \langle \mathcal{H}(\theta); \in, F, <_w \rangle$$

be a continuous sequence of elementary submodels with θ sufficiently large regular and $<_w$ a well ordering of $\mathcal{H}(\theta)$, such that $|N_\alpha| \leq \aleph_1$, $\delta_\alpha = N_\alpha \cap \omega_2 < \omega_2$. Apply $(*)$ to $F|D$ where $D = \{\delta_\alpha : \alpha < \omega_2\}$. This gives $i < \omega$ and $A \in [D]^{\omega_2}$ such that if $\delta_\alpha < \delta_\beta$ are in A then there is $\gamma > \beta$ such that $F(\delta_\alpha, \delta_\gamma) = F(\delta_\beta, \delta_\gamma) = i$. As $\delta_\alpha, \delta_\beta \in N_\gamma < \delta_\gamma$, we have

$$\bigl|\bigl\{\xi : F(\delta_\alpha, \xi) = F(\delta_\beta, \xi) = i\bigr\}\bigr| = \aleph_2.$$

By transfinite recursion one can select $A' \subseteq A$, $|A'| = \aleph_2$ and

$$B = \bigl\{\{\delta_\alpha\} \cup u_{\alpha,\beta} \cup \{\delta_\beta\} : \alpha, \beta \in A', \alpha < \beta\bigr\}$$

such that $A' \cap B = \emptyset$, and $F(\delta_\alpha, u_{\alpha,\beta}) = F(\delta_\beta, u_{\alpha,\beta}) = i$, i.e.,

$$\bigl\{\delta_\alpha : \alpha \in A'\bigr\} \cup \bigl\{u_{\alpha,\beta} : \alpha, \beta \in A', \alpha < \beta\bigr\}$$

form a topological K_{ω_2} in color i. □

LEMMA 5. *If there is an ω_1-complete, $\aleph_1$-dense ideal on ω_2, then $(*)$ holds.*

PROOF. Let I be an ω_1-complete ideal on ω_2 with $\{A_\alpha : \alpha < \omega_1\}$ dense in I^+. Assume that $F\colon [\omega_2]^2 \to \omega$. For each $\alpha < \omega_2$ there is $i(\alpha) < \omega$ such that

$$B_\alpha = \bigl\{\alpha < \beta < \omega_2 : F(\alpha, \beta) = i(\alpha)\bigr\} \in I^+.$$

There is $U \in [\omega_2]^{\omega_2}$ such that if $\alpha \in U$, then $i(\alpha) = i$ and $A_j \subseteq_I B_\alpha$ for some $i < \omega$, $j < \omega_1$. Clearly U is as required in $(*)$. □

The consistency of the existence of an ω_1-complete $\aleph_1$-dense ideal on ω_2 was established by Foreman in [4]. Notice that by a theorem of Woodin, the existence of an ω_1-complete $\aleph_1$-dense ideal on ω_2 implies CH.

Lemma 6. *It is consistent that* $2^{\aleph_0} = \aleph_2$ *and* $(*)$ *fails.*

Proof. We force with the following notion of forcing. $p = (s, h, f) \in P$ iff $s \in [\omega_2]^{<\omega}$, $h, f\colon [s]^2 \to \omega$, and there are no $\alpha < \beta < \gamma$ with $f(\alpha, \gamma) = f(\beta, \gamma) = h(\alpha, \beta)$. $(s', h', f') \le (s, h, f)$ iff $s' \supseteq s$, $f'|s = f$, $h'|s = h$.

Claim 1. *If* $\xi < \omega_2$, *then* $\{(s, g, f) : \xi \in s\}$ *is dense.*

Proof. If $(s, h, f) \in P$ and $\xi \notin s$ define $s' = s \cup \{\xi\}$, $h', f' : [s']^2 \to \omega$ such that $h' \supseteq h$, $f' \supseteq f$ and the values, i.e., the values $h'(\alpha, \xi)$ and $f'(\alpha, \xi)$ for $\alpha \in s$ are different from the range of h and f and from each other. Clearly, (s', h', f') is a condition and $(s', h', f') \le (s, h, f)$. □

Claim 2. $(P, \le)$ *is ccc.*

Proof. Assume that we are given the conditions $\{p_\xi : \xi < \omega_1\}$ with $p_\xi = (s_\xi, h_\xi, f_\xi)$. By the Δ-system lemma we can assume that $\{s_\xi : \xi < \omega_1\}$ form a Δ-system. As there are finitely many isomorphism types of the structures $(s_\xi; <, h_\xi, f_\xi)$ $(\xi < \omega_1)$, without loss of generality we can assume that the order isomorphism between any $(s_\xi; <)$ and $(s_\eta; <)$ gives an isomorphism between $(s_\xi; <, h_\xi, f_\xi)$ and $(s_\eta; <, h_\eta, f_\eta)$ $(\xi, \eta < \omega_1)$.

We therefore need to show that if $p = (\Delta \cup a, h, f)$ and $p' = (\Delta \cup a', h', f')$ are isomorphic conditions, $a \cap a' = \emptyset$, then p and p' are compatible. Set

$$p^* = (\Delta \cup a \cup a', h^*, f^*),$$

where $h^* \supseteq h \cup h'$, $f^* \supseteq f \cup f'$ are such that the crossing values, i.e.,

$$\{ h^*(\alpha, \beta), f^*(\alpha, \beta) : \alpha \in a, \beta \in a' \}$$

are distinct and disjoint from

$$\mathrm{Ran}(h) \cup \mathrm{Ran}(h') \cup \mathrm{Ran}(f) \cup \mathrm{Ran}(f').$$

In order to show that p^* is a condition, assume that $\alpha < \beta < \gamma$ and $f^*(\alpha, \gamma) = f^*(\beta, \gamma) = h^*(\alpha, \beta)$. By the way p^* was constructed, either $\{\alpha, \beta\} \subseteq \Delta \cup a$ or $\{\alpha, \beta\} \subseteq \Delta \cup a'$ and the same holds for $\{\alpha, \gamma\}$ and $\{\beta, \gamma\}$. This is only possible, if either $\{\alpha, \beta, \gamma\} \subseteq \Delta \cup a$ or $\{\alpha, \beta, \gamma\} \subseteq \Delta \cup a'$, but then we cannot have $f^*(\alpha, \gamma) = f^*(\beta, \gamma) = h^*(\alpha, \beta)$ as p and p' are conditions. □

If G is V-P-generic, set

$$H = \bigcup \{ h : (s, h, f) \in G \}, \quad F = \bigcup \{ f : (s, h, f) \in G \}.$$

We show that $F\colon [\omega_2]^2 \to \omega$ is a witness for the failure of $(*)$. Assume, for the sake of contradiction, that $i < \omega$, and p forces that $|A| = \aleph_2$ and if

$\alpha, \beta \in A$, $\alpha < \beta$ then there exists a $\gamma > \beta$ such that $F(\alpha, \gamma) = F(\beta, \gamma) = i$. There is a set $B \subseteq \omega_2$, $|B| = \aleph_2$, and $p_\xi \le p$ for $\xi \in B$, such that $p_\xi \Vdash \xi \in A$. With the usual arguments we can assume that $p_\xi = (\Delta \cup a_\xi, h_\xi, f_\xi)$ where $\Delta < a_\xi < a_\eta$, $\xi \in a_\xi$, $h_\xi|[\Delta]^2 = h_\eta|[\Delta]^2$, $f_\xi|[\Delta]^2 = f_\eta|[\Delta]^2$ ($\xi < \eta \in B$), and $\xi \in a_\xi$ ($\xi \in B$). Pick $\xi < \eta$ from B.

Define $p^* = (\Delta \cup a_\xi \cup a_\eta, h^*, f^*)$ such that $h^*|(\Delta \cup a_\xi) = h_\xi$, $h^*|(\Delta \cup a_\eta) = h_\eta$, $f^*|(\Delta \cup a_\xi) = f_\xi$, $f^*|(\Delta \cup a_\eta) = f_\eta$. Further, $h^*(\xi, \eta) = i$, and the other crossing values, i.e.,

$$\{ h^*(\alpha, \beta) : \alpha \in a_\xi, \beta \in a_\eta, \{\alpha, \beta\} \ne \{\xi, \eta\} \} \cup \{ f^*(\alpha, \beta) : \alpha \in a_\xi, \beta \in a_\eta \}$$

are different from i, each other, and from the elements of

$$\mathrm{Ran}(h_\xi) \cup \mathrm{Ran}(h_\eta) \cup \mathrm{Ran}(f_\xi) \cup \mathrm{Ran}(f_\eta).$$

Claim 3. *p^* is a condition.*

Proof. The argument in Claim 2 works, except for the case when $h^*(\xi, \eta)$ plays a role, i.e., when $\alpha = \xi$, $\beta = \eta$ (in the condition on $\alpha < \beta < \gamma$). We have to show that we cannot have $f^*(\xi, \gamma) = f^*(\eta, \gamma) = h^*(\xi, \eta)$. But $h^*(\xi, \eta) = i$, and as $a_\xi < a_\eta$, we have $\gamma \in a_\eta$, and $f^*(\xi, \gamma) \ne i$ by construction. □

p^* forces that there is no $\gamma > \eta$ with $F(\xi, \gamma) = F(\eta, \gamma) = i$, a contradiction to $\xi, \eta \in A$. This concludes the proof of Lemma 6. □

Theorem 7. *CH is consistent with the negation of* $(*)$.

Proof. Let V be a model of CH.

We define the following notion of forcing. $(S, f, \mathcal{H}, h) \in P$ if

(a) $S \in [\omega_2]^{\le \aleph_0}$,

(b) $f \colon S \times S \to [\omega]^\omega$ is symmetric, $f(\alpha, \alpha) = \omega$ ($\alpha \in S$),

(c) $\mathcal{H} \subseteq [S]^\omega$, $|\mathcal{H}| \le \omega$, if $H \in \mathcal{H}$ then $\mathrm{tp}(H) = \omega$, if $H \ne H' \in \mathcal{H}$, then $|H \cap H'| < \omega$ ($\mathcal{H}$ is almost disjoint), $h : \mathcal{H} \to \omega$,

(d) if $\alpha \in S$, $H \in \mathcal{H}$, $\alpha \ge \min(H)$, then

$$\left| \{ \beta \in H : h(H) \in f(\alpha, \beta) \} \right| \le 1.$$

We define $(S', f', \mathcal{H}', h') \le (S, f, \mathcal{H}, h)$ iff $S' \supseteq S$, $f = f'|(S \times S)$, $\mathcal{H}' \supseteq \mathcal{H}$ with $H \not\subseteq S$ for $H \in \mathcal{H}' - \mathcal{H}$, and $h = h'|\mathcal{H}$.

Claim 1. *If $\alpha < \omega_2$, then $D = \{(S, f, \mathcal{H}, h) : \alpha \in S\}$ is dense in $(P, \le)$.*

Proof. Assume that $(S, f, \mathcal{H}, h) \in P$ with $\alpha \notin S$. Define $S' = S \cup \{\alpha\}$. Let $\omega = \bigcup\{u_\xi : \xi \in S'\}$ be a partition of ω into infinite parts. We define the symmetric $f' \colon S' \times S' \to [\omega]^\omega$ such that $f' \supseteq f$, $f'(\alpha, \alpha) = \omega$ and $f'(\alpha, \beta) = u_\beta$ ($\beta \in S$). Finally, set $\mathcal{H}' = \mathcal{H}$ and $h' = h$. Now $(S', f', \mathcal{H}', h')$ is a condition: (d) follows from $u_\beta \cap u_{\beta'} = \emptyset$ ($\beta \ne \beta'$) and $\alpha \notin H$ ($H \in \mathcal{H}$).

Clearly, $(S', f', \mathcal{H}', h') \leq (S, f, \mathcal{H}, h)$ and $(S', f', \mathcal{H}', h') \in D$. $\square$

Claim 2. *$(P, \leq)$ is ω_1-closed.*

PROOF. Assume that $p_n = (S_n, f_n, \mathcal{H}_n, h_n)$ and $\langle p_n : n < \omega \rangle$ is a decreasing sequence of conditions. Define $S = \bigcup\{S_n : n < \omega\}$, $f = \bigcup\{f_n : n < \omega\}$, $\mathcal{H} = \bigcup\{\mathcal{H}_n : n < \omega\}$, and $h = \bigcup\{h_n : n < \omega\}$. Now $p = (S, f, \mathcal{H}, h)$ is a condition as if $H \in \mathcal{H}$, $\alpha \in S$, $\alpha \geq \min(H)$, then there is $n < \omega$ such that $H \in \mathcal{H}_n$, $\alpha \in S_n$, and so (d) holds for α and H. Also, $p \leq p_n$ $(n < \omega)$ as if $H \in \mathcal{H} - \mathcal{H}_n$, then $H \not\subseteq S_n$. $\square$

Claim 3. *$(P, \leq)$ is $\aleph_2$-c.c.*

PROOF. With the usual methods it suffices to show that $p_0 = (S \cup S_0, f_0, \mathcal{H}_0, h_0)$ and $p_1 = (S \cup S_1, f_1, \mathcal{H}_1, h_1)$ are compatible if $S < S_0 < S_1$, $\mathrm{tp}(S_0) = \mathrm{tp}(S_1)$, and the order isomorphism $\pi \colon S \cup S_0 \to S \cup S_1$ is an isomorphism of p_0 and p_1. We define $S' = S \cup S_0 \cup S_1$, $f' \supseteq f_0 \cup f_1$ such that $f'(\alpha, \pi(\beta)) \subseteq f_0(\alpha, \beta)$ $(\alpha, \beta \in S_0)$, $\mathcal{H}' = \mathcal{H}_0 \cup \mathcal{H}_1$, $h' = h_0 \cup h_1$. Further, we require that the sets $\{f'(\alpha, \beta) : \alpha \in S_0, \beta \in S_1\}$ be pairwise disjoint. This is possible, as countably many infinite sets have pairwise disjoint infinite subsets.

We next show that $\mathcal{H}'$ is almost disjoint. Assume that $H_0, H_1 \in \mathcal{H}'$ and $|H_0 \cap H_1| = \omega$. Then $H_0 \in \mathcal{H}_0 - \mathcal{H}_1$, $H_1 \in \mathcal{H}_1 - \mathcal{H}_0$ (or vice versa). These mean that $H_0 \subseteq S \cup S_0$ but $H_0 \not\subseteq S$, as $\mathrm{tp}(H_0) = \omega$, we have $|H_0 \cap S| < \omega$. Similarly, $|H_1 \cap S| < \omega$, and these imply that $|H_0 \cap H_1| < \omega$.

In order to conclude the proof that $(S', f', \mathcal{H}', h')$ is a condition, we have to show that if $H \in \mathcal{H}'$, $\alpha \in S'$, $\alpha \geq \min(H)$ then

$$\big|\{\beta \in H : h'(H) \in f'(\alpha, \beta)\}\big| \leq 1.$$

holds.

Case 1: $H \in \mathcal{H}_0$ and $\alpha \in S \cup S_0$. The inequality holds as p_0 is a condition.

Case 2: $H \in \mathcal{H}_1$ and $\alpha \in S \cup S_1$. The inequality holds as p_1 is a condition.

Case 3: $H \in \mathcal{H}_0$, $\alpha \in S_1$, and $H \cap S \neq \emptyset$. Then $\pi^{-1}(\alpha) > \min(H)$ and (d) holds for H, $\pi^{-1}(\alpha)$, i.e., at most one of the sets $\{f_0(\pi^{-1}(\alpha), \beta) : \beta \in H\}$ may contain $h'(H) = h_0(H)$, so this holds for the system $\{f'(\alpha, \beta) : \beta \in H\}$, as $f'(\alpha, \beta) \subseteq f_0(\pi^{-1}(\alpha), \beta)$ $(\beta \in H)$.

Case 4: $H \in \mathcal{H}_0$, $\alpha \in S_1$, and $H \subseteq S_0$. The inequality holds as $\{f'(\beta, \alpha) : \beta \in H\}$ are disjoint.

Case 5: $H \in \mathcal{H}_1$ and $\alpha \in S_0$. By the condition $\alpha \geq \min(H)$ we necessarily have $\pi(\alpha) > \min(H)$, and, as (d) holds for H and $\pi(\alpha)$, it holds for $\pi^{-1}(H)$, α. Because for any $\gamma \in S_0$, $\eta \in S_1$, $f'(\gamma, \eta) \subseteq f(\alpha, \pi^{-1}(\eta))$, (d) also holds for H and α. $\square$

From Claims 2, 3 we get that forcing with $(P, \leq)$ does not collapse ω_1 or ω_2. If G is generic for $(P, \leq)$, then let

$$F = \bigcup \{ f : (S, f, \mathcal{H}, h) \in G \}.$$

By the above, $F \colon \omega_2 \times \omega_2 \to [\omega]^\omega$ is a symmetric function. We define $F^*(\alpha, \beta) = \min(F(\alpha, \beta))$ for $\alpha < \beta < \omega_2$.

Claim 4. *For F^*, $(*)$ fails in $V[G]$.*

Proof. Assume that

$$p \Vdash A \in [\omega_2]^{\omega_2},\ i < \omega,\ \forall \alpha, \beta \in A\ \exists \gamma > \max(\alpha, \beta),\ F^*(\alpha, \gamma) = F^*(\beta, \gamma) = i.$$

There exist an increasing sequence $\langle x_\alpha < \omega_2 : \alpha < \omega_2 \rangle$ and $\langle p_\alpha : \alpha < \omega_2 \rangle$ with $p_\alpha \leq p$ and $p_\alpha \Vdash x_\alpha \in A$. Let $p_\alpha = (S'_\alpha, f_\alpha, \mathcal{H}_\alpha, h_\alpha)$. Apply the Δ-system lemma and CH to $\{S'_\alpha : \alpha < \omega_2\}$ to obtain $B \in [\omega_2]^{\aleph_2}$ such that $\{S'_\alpha : \alpha \in B\}$ forms a head-tail-tail Δ-system with root S. Let $S_\alpha = S'_\alpha - S$. With a further shrinking we can assume that the structures $(S \cup S_\alpha; <, S, f_\alpha, \mathcal{H}_\alpha, h_\alpha, \{x_\alpha\})$ are isomorphic for $\alpha \in B$. Let $\pi_\alpha \colon S \cup S_0 \to S \cup S_\alpha$ be the isomorphism between p_0 and p_α.

Let $\{\alpha_n : n < \omega\}$ be the first ω elements of B. We define

$$p^* = (S^*, f^*, \mathcal{H}^*, h^*)$$

where

$$S^* = S \cup \bigcup \{S_{\alpha_n} : n < \omega\}, \quad H^* = \{x_{\alpha_n} : n < \omega\},$$

$$\mathcal{H}^* = \bigcup \{\mathcal{H}_{\alpha_n} : n < \omega\} \cup \{H^*\}, \quad h^* \supseteq \bigcup \{h_{\alpha_n} : n < \omega\}$$

is such that $h^*(H^*) = i$, $f^* \colon S^* \times S^* \to [\omega]^\omega$ is such that $f^* \supseteq \bigcup \{f_{\alpha_n} : n < \omega\}$ and

$$\{ f^*(\pi_{\alpha_m}(\alpha), \pi_{\alpha_n}(\beta)) : m < n,\ \alpha, \beta \in S_{\alpha_0} \}$$

are disjoint and do not contain i.

In order to show that p^* is a condition, we have to check (d). Assume that $H \in \mathcal{H}^*$, $\alpha \in S^*$ and $\alpha \geq \min(H)$. If $H = H^*$, then $\alpha \in S_{\alpha_n}$ for some $n < \omega$ (as $\alpha \geq x_0$), and therefore $\{\beta \in H^* : i \in f^*(\alpha, \beta)\} \subseteq S_{\alpha_n}$ and $|S_{\alpha_n} \cap H^*| = 1$, we are done. If $H \neq H^*$, we proceed as in Claim 3.

Once we obtained that p^* is a condition, it is easy to see that $p^* \leq p_{\alpha_n} \leq p$ $(n < \omega)$. By (d), it is immediate that in $V[G]$, there is no $\gamma > x_{\alpha_1}$ such that

$$F^*(x_{\alpha_0}, \gamma) = F(x_{\alpha_1}, \gamma) = i,$$

a contradiction. □

As Claims 2,3, and 4 give that $(*)$ fails in $V[G]$, we are finished. $\square$

THEOREM 8. *It is consistent that GCH holds and* $[\omega_2]^2 = \bigcup\{T_n : n < \omega\}$ *where each* T_n *is an* ω_2*-Suslin tree.*

PROOF. We force with the following notion of forcing. $p = (S, f) \in P$ if $S \in [\omega_2]^{\leq\aleph_0}$, $f\colon S \times S \to [\omega]^\omega$ is such that $f(\alpha,\beta) = f(\beta,\alpha)$ $(\alpha,\beta \in S)$ and $f(\alpha,\alpha) = \omega$ $(\alpha \in S)$. For $\alpha \leq \beta$ in S we define $\alpha \leq_n \beta$ iff $n \in f(\alpha,\beta)$. (More correctly, we should use the notation $\alpha \leq_n^p \beta$.)

We also assume that if $\alpha < \beta < \gamma$ are in S, then

(1) if $\alpha \leq_n \beta \leq_n \gamma$, then $\alpha \leq_n \gamma$, and

(2) if $\alpha \leq_n \gamma$, $\beta \leq_n \gamma$, then $\alpha \leq_n \beta$.

Assumption (1) means that $(S, \leq_n)$ is a partially ordered set, (2) tells that it is a tree.

We set $(S', f') \leq (S, f)$ iff $S' \supseteq S$ and $f = f'|(S \times S)$.

Claim 1. $(P, \leq)$ *is transitive.*

PROOF. Straightforward. $\square$

Claim 2. $(P, \leq)$ *is* ω_1*-closed.*

PROOF. If $p_0 \geq p_1 \geq p_2 \geq \cdots$ where $p_n = (S_n, f_n)$, then we let $p = (S, f)$ where $S = \bigcup\{S_n : n < \omega\}$, $f = \bigcup\{f_n : n < \omega\}$. It is easy to see that p is a condition and $p \leq p_n$ $(n < \omega)$. $\square$

Claim 3. *If* $\alpha < \omega_2$, *then* $\{(S, f) : \alpha \in S\}$ *is dense.*

PROOF. Assume that (S, f) is a condition, $\alpha \notin S$. We show that (S, f) has an extension (S', f') such that $S' = S \cup \{\alpha\}$.

Let $\{(n_k, \beta_k) : k < \omega\}$ be an enumeration in which the n_k's are distinct and each $\beta \in S$ occurs as β_k for infinitely many $k < \omega$.

We extend $(S, \leq_{n_k})$ to $(S', \leq_{n_k})$ as follows.

Case 1: $\beta_k < \alpha$. Let $b = \{x : x \leq_{n_k} \beta_k\}$ be the (closed) branch determined by β_k in $(S, \leq_{n_k})$. Define $x \leq_{n_k} \alpha$ if $x \in b$ and no element will be strictly above α in $(S', \leq_{n_k})$. Notice that $\beta_k \leq_{n_k} \alpha$ and so $n_k \in f'(\beta_k, \alpha)$.

In order to show (1), assume that $x \leq_{n_k} y \leq_{n_k} \alpha$. Then $x \leq_{n_k} y \in b$, therefore $x \in b$, and so $x \leq_{n_k} \alpha$. As there are no elements strictly above α, there are no more cases.

In order to show (2), assume that $x < y$ and $x, y \leq_{n_k} \alpha$. Then $x, y \in b$, so $x \leq_{n_k} y$. Again, there are no more cases.

Case 2: $\alpha < \beta_k$. Let $u \in S - \alpha$ be minimal such that $u \leq_{n_k} \beta_k$. Define $x \leq_{n_k} \alpha$ if $x \leq_{n_k} u$ $(x \in S \cap \alpha)$, $\alpha \leq_{n_k} x$ if $u \leq_{n_k} x$ $(x \in S - \alpha)$. Notice that $\alpha \leq_{n_k} \beta_k$ and so $n_k \in f'(\alpha, \beta_k)$.

We first show (1). If $x \leq_{n_k} y \leq_{n_k} \alpha$, then $x \leq_{n_k} y \leq_{n_k} u$, consequently $x \leq_{n_k} u$ and so $x \leq_{n_k} \alpha$.

If $x \leq_{n_k} \alpha \leq_{n_k} y$, then $x \leq_{n_k} u \leq_{n_k} y$ and so $x \leq_{n_k} y$.

If $\alpha \le_{n_k} x \le_{n_k} y$, then $u \le_{n_k} x \le_{n_k} y$, then $u \le_{n_k} y$, and so $\alpha \le_{n_k} y$.

We finally show (2). If $x, y \le_{n_k} \alpha$, then $x, y \le_{n_k} u$, so x, y are comparable by $\le_{n_k}$.

If $x, \alpha \le_{n_k} y$, $x < \alpha$, then $x \le_{n_k} y$, $u \le_{n_k} y$, so $x \le_{n_k} u$ and then $x \le_{n_k} \alpha$.

If $\alpha, x \le_{n_k} y$, $\alpha < x$, then $u, x \le_{n_k} y$, so $u \le_{n_k} x$, and then $\alpha \le_{n_k} x$. □

In order to investigate how conditions can be extended we make the following setup. Assume that $p_0 = (S \cup S_0, f_0)$ and $p_1 = (S \cup S_1, f_1)$ are conditions, $S < S_0 < S_1$, $\mathrm{tp}(S_0) = \mathrm{tp}(S_1)$, and $\pi \colon S \cup S_0 \to S \cup S_1$ is the order isomorphism. We assume that p_0 and p_1 are isomorphic, i.e., if $\alpha, \beta \in S \cup S_0$, then $f_0(\alpha, \beta) = f_1(\pi(\alpha), \pi(\beta))$. We let $\le_n^0$, $\le_n^1$ be the tree orderings corresponding to p_0, p_1. We notice that $\le_n^0$ and $\le_n^1$ agree on S.

Claim 4. *Set $\alpha \le_n \beta$ iff either $\alpha \le_n^0 \beta$ or $\alpha \le_n^1 \beta$. Then $\le_n$ is a tree ordering on $S \cup S_0 \cup S_1$ which restricts to $\le_n^0$ and $\le_n^1$ on $S \cup S_0$, $S \cup S_1$, respectively.*

PROOF. Straightforward. □

Claim 5. *If $\alpha < \beta \in S_0$ are $\le_n^0$-comparable, then there is a tree ordering $\le_n$ on $S \cup S_0 \cup S_1$ with $\alpha \le_n \pi(\beta)$ which restricts to $\le_n^0$, $\le_n^1$ on $S \cup S_0$ and $S \cup S_1$.*

PROOF. Let τ be the $\le_n^0$-least element of S_0 below β. Notice that $\tau \le_n^0 \alpha$ holds also.

We define the relation $\le_n$ on $S \cup S_0 \cup S_1$ as follows. $x \le_n y$ iff either $x \le_n^0 y$, or $x \le_n^1 y$, or $x \in S_0$, $y \in S_1$, $x \le_n^0 \alpha$ and $\pi(\tau) \le_n^1 y$.

Subclaim 1. *$\le_n$ satisfies* (1).

PROOF. Assume that $x \le_n y \le_n z$. We have to prove that $x \le_n z$. This is immediate if either $x, y, z \in S \cup S_0$ or $x, y, z \in S \cup S_1$.

Case 1: $x \in S \cup S_0$, $y \in S_0$, $z \in S_1$. If $x \in S_0$, then $x \le_n^0 y \le_n^0 \alpha$, consequently $x \le_n^0 \alpha$, and so $x \le_n z$.

If $x \in S$, then, as $x \le_n^0 \alpha$ and $\tau \le_n^0 \alpha$, we have $x \le_n^0 \tau$, and so $x \le_n^1 \pi(\tau) \le_n^1 z$.

Case 2: $x \in S_0$, $y, z \in S_1$. This holds if $x \le_n^0 \alpha$, $\pi(\tau) \le_n^1 y \le_n^1 z$, but then $\pi(\tau) \le_n^1 z$, and so $x \le_n z$. □

Subclaim 2. *$\le_n$ satisfies* (2).

PROOF. Assume that $x \le_n z$, $y \le_n z$, and $x < y$. We have to prove that $x \le_n y$. Again, we have no problem if either $\{x, y, z\} \subseteq S \cup S_0$ or $\{x, y, z\} \subseteq S \cup S_1$ so we ignore these possibilities.

Case 1: $x \in S$, $y \in S_0$, $z \in S_1$. Since $x, \pi(\tau) \le_n z$, so $x \le_n^1 \pi(\tau)$, by the fact that π is an isomorphism, $x \le_n^0 \tau \le_n^0 \alpha$. Also, $y \le_n^0 \alpha$, so x, y are comparable under $\le_n^0$.

Case 2: $x, y \in S_0$, $z \in S_1$. Now $x, y \leq_n^0 \alpha$ so they are comparable under $\leq_n^0$.

Case 3: $x \in S_0$, $y, z \in S_1$. By the definition of $\leq_n$ we have $x \leq_n^0 \alpha$ and $\pi(\tau) \leq_n^1 z$. The latter implies $\pi(\tau) \leq_n^1 y$ and so we get $x \leq_n y$. □

As $\leq_n$ satisfies (1) and (2), we are done. □

Claim 6. *$(P, \leq)$ is $\aleph_2$-c.c.*

Proof. Modulo standard arguments one has to show that p_0 and p_1 are compatible, if $p_0 = (S \cup S_0, f_0)$, $p_1 = (S \cup S_1, f_1)$, $S < S_0 < S_1$ and p_0, p_1 are isomorphic with $\pi\colon S \cup S_0 \to S \cup S_1$ as the isomorphism.

Choose, for each $\alpha \leq \beta$ in S_0 the natural numbers $n_i(\alpha, \beta) \in f_0(\alpha, \beta)$ such that $n_i(\alpha, \beta) \neq n_{i'}(\alpha', \beta')$ for $\langle i, \alpha, \beta\rangle \neq \langle i', \alpha'\beta'\rangle$.

If $n = n_i(\alpha, \beta)$, we apply Claim 5 and obtain a tree ordering $\leq_n$ on $S \cup S_0 \cup S_1$ with $\alpha \leq_n \pi(\beta)$ and if n is not of the form $n_i(\alpha, \beta)$ we apply Claim 4. This gives a structure $p = (S \cup S_0 \cup S_1, f')$ such that $f'(\alpha, \pi(\beta)) \supseteq \{n_i(\alpha, \beta) : i < \omega\}$ and so it is infinite, so p is a condition and $p \leq p_0, p_1$. □

If G is generic, we define the tree T_n as the partial order $\leq_n$ on ω_2 where $\alpha \leq_n \beta$ iff $n \in f(\alpha, \beta)$ for some $(S, f) \in G$. It is clear that $\leq_n$ satisfies (1) and (2) so T_n is indeed a tree.

Claim 7. *T_n does not contain an ω_2-branch $(n < \omega)$.*

Proof. Assume p forces that B is an ω_2-branch of T_n. There are $\aleph_2$ distinct ordinals x_α and conditions $p_\alpha \leq p$ such that $p_\alpha \Vdash x_\alpha \in B$. The usual applications of the pigeon hole principle and the Δ-system lemma give two of these conditions, we simply call them p_0 and p_1 such that $p_0 = (S \cup S_0, f_0)$, $p_1 = (S \cup S_1, f_1)$, $S < S_0 < S_1$, the structures $(S \cup S_0, <, S, x_0, f_0)$, $(S \cup S_1, <, S, x_1, f_1)$ are isomorphic under the order isomorphism $\pi\colon S \cup S_0 \to S \cup S_1$. Notice that $\pi(x_0) = x_1$. We proceed as in the proof of Claim 6 except that we choose all $n_i(\alpha, \beta)$ different from n. This way, we get an extension $p' \leq p_0, p_1$ such that $n \notin f'(x_0, x_1)$ and so p' forces that x_0 and x_1 are incomparable in T_n, a contradiction. □

Claim 8. *T_n does not contain antichains of size $\aleph_2$ $(n < \omega)$.*

Proof. Assume that p forces that A is an antichain of size $\aleph_2$ in T_n. There are $\aleph_2$ distinct ordinals x_α and conditions $p_\alpha \leq p$ such that $p_\alpha \Vdash x_\alpha \in A$. The usual applications of the pigeon hole principle and the Δ-system lemma give two of these conditions, we simply call them p_0 and p_1, such that $p_0 = (S \cup S_0, f_0)$, $p_1 = (S \cup S_1, f_1)$, $S < S_0 < S_1$, the structures $(S \cup S_0, <, S, \{x_0\}, f_0)$, $(S \cup S_1, <, S, \{x_1\}, f_1)$ are isomorphic under the order isomorphism $\pi\colon S \cup S_0 \to S \cup S_1$. Notice that $\pi(x_0) = x_1$ and so $n \in f_0(x_0, x_0)$. We proceed as in the proof of Claim 6 so that we choose $n_0(x_0, x_1) = n$.

This way, we get an extension $p' \leq p_0, p_1$ such that $n \in f'(x_0, x_1)$ and so p' forces that $x_0 < x_1$ in T_n, a contradiction. □

With Claims 2, 6 we get that the forcing is cardinal preserving, and with Claims 7 and 8 that $[\omega_2]^2 = \bigcup\{T_n : n < \omega\}$ as required. $\square$

COROLLARY 9. *GCH is consistent with* $\omega_2 \not\to (\text{Top}\, K_{\omega_2})^2_\omega$.

PROOF. By Theorem 8, $[\omega_2]^2 = \bigcup\{X_n : n < \omega\}$ where X_n is the comparison graph of a κ-Suslin tree on ω_2. By Lemma 1, no X_n contains a topological K_κ. If $F(\alpha, \beta) = \min\{n : \{\alpha, \beta\} \in X_n\}$, then $F\colon [\omega_2]^2 \to \omega$ is a coloring witnessing $\omega_2 \not\to (\text{Top}\, K_\kappa)^2_\omega$. $\square$

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