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COLOURING OF SUCCESSOR OF REGULAR, AGAIN

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Abstract. We get a version of the colouring property Pr_1 proving $\mathrm{Pr}_1(\lambda,\lambda,\lambda,\partial)$ always when $\lambda=\partial^+,\partial$ are regular cardinals and some stationary subset of λ consisting of ordinals of cofinality $<\partial$ do not reflect in any ordinal $<\lambda$.

0. Introduction

We prove a strong colouring theorem on successor of regular uncountable cardinals, so called Pr_1.

On the history of Pr_1 see [5, Ch. III, §4] and later [6], and then independently Rinot [3] and [7].

Rinot [3, Main result] proved that $\mathrm{Pr}_1(\lambda,\lambda,\lambda,\theta)$ when those are regular cardinals; $\lambda=\theta^{++}$ or just $\theta^+<\lambda$ and λ is a successor of regular or just it has a non-reflecting stationary subset of λ consisting of ordinals of cofinality at least θ. In [7], we have $\mathrm{Pr}_1(\lambda,\lambda,\lambda,(\theta_0,\theta))$ where θ_0 is regular $<\theta=\mathrm{cf}(\theta)$, $\theta^+<\lambda$ and λ is a successor of regular. Earlier [6, 4.2, p. 27] prove that $\mathrm{Pr}_1(\lambda,\lambda,\lambda,\theta)$ when in addition $\lambda=\theta^{++}$.

Much earlier [5, Ch. III, §4] had treated those problems in a general but probably in a not so transparent way, first 4.1 there gives a set of various hypothesis (each with some parameters).

The result here is incomparable with the ones in [3], [7], [6]: the assumption on the stationary set is stronger but the arity – the last parameter, θ is bigger.

The connection between purely combinatorial theorems and topological constructions is known for many years. Several results in general topology were proved using the property $\mathrm{Pr}_1(\lambda,\mu,\sigma,\theta)$, see recently [2], then [7, §1].

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Recall:

Definition 0.1. 1) Assume $\lambda \geq \mu \geq \sigma + \theta_0 + \theta_1, \bar{\theta} = (\theta_0, \theta_1)$, see 0.4(1). Assume further that $\theta_0, \theta_1 \geq \aleph_0$ but σ may be finite.

Let $\mathrm{Pr}_1(\lambda, \mu, \sigma, \bar{\theta})$ mean that there is $\mathbf{c} : [\lambda]^2 \to \sigma$ witnessing it, which means:

$(*)_{\mathbf{c}}$ if (a) then (b), where:

(a) for $\iota = 0, 1, \mathbf{i}_\iota < \theta_\iota$ and $\bar{\zeta}^\iota = \langle \zeta^\iota_{\alpha,i} : \alpha < \mu, i < \mathbf{i}_\iota \rangle$ are sequences of ordinals of λ without repetitions, and $\mathrm{Rang}(\bar{\zeta}^0)$, $\mathrm{Rang}(\bar{\zeta}^1)$ are disjoint and $\gamma < \sigma$

(b) there are $\alpha_0 < \alpha_1 < \mu$ such that $\forall i_0 < \mathbf{i}_0$, $\forall i_1 < \mathbf{i}_1$, $\mathbf{c}\{\zeta^0_{\alpha_0,i_0}, \zeta^1_{\alpha_1,i_1}\} = \gamma$ and $\zeta^0_{\alpha_0,i_0} < \zeta^1_{\alpha_1,i_1}$.

2) Above if $\theta_0 = \theta = \theta_1$ then we may write $\mathrm{Pr}_1(\lambda, \mu, \sigma, \theta)$.

In this paper we prove e.g. that if some stationary $S \subseteq \{\delta < \aleph_2 : \mathrm{cf}(\delta) < \aleph_1\}$ do not reflect then $\mathrm{Pr}_1(\aleph_2, \aleph_2, \aleph_2, \aleph_1)$ holds, which means that countable infinite sequences can be taken in both "sides". Actually, the theorem says that, in particular, $\mathrm{Pr}_1(\lambda, \lambda, \lambda, \partial)$ holds whenever $\partial = \mathrm{cf}(\partial)$ and $\lambda = \partial^+$ and there is a non-reflecting stationay subset of $S^\lambda_{<\kappa}$. We intend to say more on other λ-s in [4].

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Definition 0.2. 1) A filter D on a set I is uniform *when* for every subset A of I of cardinality $< |I|$, the set $I \setminus A \in D$; all our filters will be uniform.

2) A filter D on a set I is weakly θ-saturated *when* $\theta \geq |I|$ and there is no partition of I to θ sets from D^+,

3) We say the filter D on a set I is θ-saturated *when* the Boolean algebra $\mathscr{P}(I)/D$ satisfies the θ-c.c.

Fact 0.3. 1) *If D is a θ-complete filter on λ and is not θ-saturated then it is not weakly θ-saturated; so those properties are equivalent.*

2) *If $\theta = \sigma^+$ and D is a θ-complete filter on θ, then D is not weakly θ-saturated.*

3) *If $n \geq 1$ and $\lambda = \sigma^{+n}$ and D is a (uniform) σ^+-complete filter on λ then D is not weakly σ^{+n}-saturated.*

Proof. 1) Obvious and well known.

2) By [8].

3) Let μ be the minimal cardinal such that D is not μ^+-complete, so clearly $\mu \in [\sigma^+, \lambda]$ hence μ is a successor cardinal. So there is a function f from λ into μ such that for every subset A of μ of cardinality $< \mu$, $f^{-1}(A) = \emptyset$ mod D. Let E be the family of subsets A of μ such that $f^{-1}(A) \in D$. Clearly E is a (uniform) μ-complete filter on μ hence by part (2) is not weakly μ-saturated, let $\langle A_\varepsilon : \varepsilon < \mu \rangle$ be a partition of μ to sets from E^+. Now $\langle f^{-1}(A_\varepsilon) : \varepsilon < \mu \rangle$ witnesses the desired conclusion. $\square_{0.3}$

NOTATION 0.4. 1) We denote infinite cardinals by $\lambda, \mu, \kappa, \theta, \partial$ while σ denotes a finite or infinite cardinal. We denote ordinals by $\alpha, \beta, \gamma, \varepsilon, \zeta, \xi$. Natural numbers are denoted by k, ℓ, m, n and $\iota \in \{0, 1, 2\}$

1A) Let D denote a filter on an infinite set $\mathrm{dom}(D)$.

2) For a set A of ordinals let $\mathrm{nacc}(A) = \{\alpha \in A : \alpha > \sup(A \cap \alpha)\}$ and $\mathrm{acc}(A) = A \setminus \mathrm{nacc}(A)$. For regular cardinals $\lambda > \kappa$ let $S^\lambda_\kappa = \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$, $S^\lambda_{<\kappa} = \{\delta < \lambda : \mathrm{cf}(\delta) < \kappa\}$.

1. A colouring theorem

Our aim is to prove

THEOREM 1.1. $\mathrm{Pr}_1(\lambda, \lambda, \partial, \partial)$ *and moreover* $\mathrm{Pr}_1(\lambda, \lambda, \lambda, \partial)$ *holds provided that*:

(a) $\lambda = \partial^+$

(b) $\partial = \mathrm{cf}(\partial) > \aleph_0$

(c) $\mathscr{W}$ *is a stationary subset of* λ *consisting of ordinals of cofinality* $< \partial$ *reflecting in no ordinal* $< \lambda$.

REMARK 1.2. 1) The case of ∂ colours, i.e. proving only $\mathrm{Pr}_1(\lambda, \lambda, \partial, \partial)$ is easier so we prove it first.

2) Can we weaken clause (c) of 1.1 replacing "reflecting in no ordinal $< \lambda$" by "reflecting in no ordinal of cofinality ∂?"

The answer seem yes provided that we add:

(α) there is a sequence $\langle e_\alpha : \alpha \notin \mathscr{W} \rangle$ such that ($\mathscr{W}$ is as above and) e_α is a club of α of order type $< \partial$ and for $\alpha \in e_\beta \cap \mathscr{W}$ we have $e_\alpha = \alpha \cap e_\beta$

(β) there is no ∂-complete not ∂^+-complete uniform weakly ∂-saturated filter on λ.

PROOF. *Stage A*: We begin as in earlier proofs (e.g. [7]). We let $(\kappa_1, \kappa_2) = (\partial, \lambda)$. Let $S \subseteq S^\lambda_\partial$ be stationary and $h : \lambda \to \lambda$ be such that $\alpha < \lambda \Rightarrow h(\alpha) < 1 + \alpha$, $h{\restriction}(\lambda \backslash S)$ is constantly zero and $S^*_\gamma := \{\delta \in S : h(\delta) = \gamma\}$ is a stationary subset of λ for every $\gamma < \lambda$. Let $F_\iota : \lambda \to \kappa_\iota$ for $\iota = 1, 2$ be such that for every $(\varepsilon_1, \varepsilon_2) \in (\kappa_1 \times \kappa_2)$ the set $W_{\varepsilon_1,\varepsilon_2}(\beta) = \{\gamma \in S^*_\beta : F_\iota(\gamma) = \varepsilon_\iota$ for $\iota = 1, 2\}$ is a stationary subset of λ for every $\beta < \lambda$.

For $\iota = 1, 2$ and $\rho \in {}^{\omega>}\lambda$ let $F_\iota(\rho) = \langle F_\iota(\rho(\ell)) : \ell < \ell g(\rho) \rangle$.

$\odot_0$ without loss of generality if $\delta \in \mathscr{W}$ then δ is divisible by ∂.

Let $\bar{e} = \langle e_\alpha : \alpha < \lambda \rangle$ be such that:

$\odot_1$ (a) if $\alpha = 0$ then $e_\alpha = \emptyset$

(b) if $\alpha = \beta + 1$ then $e_\alpha = \{\beta\}$

(c) if α is a limit ordinal then e_α is a club of α of order type $\mathrm{cf}(\alpha)$ disjoint to S^λ_∂ hence to S.

(d) if α is a limit ordinal then e_α is disjoint to $\mathscr{W}$.

In other cases (not here) instead h we use a sequence $\langle h_\alpha : \alpha < \lambda \rangle$ of functions, $h_\alpha : e_\alpha \to \partial$ and use e.g. $\langle h_{\gamma_\ell(\beta,\alpha)}(\gamma_{\ell+1}(\beta,\alpha)) : \ell < k(\beta,\alpha) \rangle$ and ρ_h, but this is not necessary here.

Now (using $\bar{e}$) for $\alpha < \beta < \lambda$, let

$$\gamma(\beta,\alpha) := \min\{\gamma \in e_\beta : \gamma \geq \alpha\}.$$

Let us define $\gamma_\ell(\beta,\alpha)$:

$$\gamma_0(\beta,\alpha) = \beta, \quad \text{and} \quad \gamma_{\ell+1}(\beta,\alpha) = \gamma(\gamma_\ell(\beta,\alpha),\alpha) \quad \text{(if well defined)}.$$

If $\alpha < \beta < \lambda$, let $k(\beta,\alpha)$ be the maximal $k < \omega$ such that $\gamma_k(\beta,\alpha)$ is defined (equivalently is equal to α) and let $\rho_{\beta,\alpha} = \rho(\beta,\alpha)$ be the sequence

$$\big\langle \gamma_0(\beta,\alpha), \gamma_1(\beta,\alpha), \dots, \gamma_{k(\beta,\alpha)-1}(\beta,\alpha) \big\rangle .$$

Let $\gamma_{\ell t}(\beta,\alpha) = \gamma_{k(\beta,\alpha)-1}(\beta,\alpha)$ where ℓt stands for last.

Let

$$\rho_h = \langle h(\gamma_\ell(\beta,\alpha)) : \ell < k(\beta,\alpha) \rangle$$

and we let $\rho(\alpha,\alpha)$ and $\rho_h(\alpha,\alpha)$ be the empty sequences. Now clearly:

$\odot_2$ if $\alpha < \beta < \lambda$ then $\alpha \leq \gamma(\beta,\alpha) < \beta$ hence

$\odot_3$ if $\alpha < \beta < \lambda, 0 < \ell < \omega$, and $\gamma_\ell(\beta,\alpha)$ is well defined, then

$$\alpha \leq \gamma_\ell(\beta,\alpha) < \beta$$

and

$\odot_4$ if $\alpha < \beta < \lambda$, then $k(\beta,\alpha)$ is well defined and letting $\gamma_\ell := \gamma_\ell(\beta,\alpha)$ for $\ell \leq k(\beta,\alpha)$ we have

$$\alpha = \gamma_{k(\beta,\alpha)} < \gamma_{\ell t}(\beta,\alpha) = \gamma_{k(\beta,\alpha)-1} < \cdots < \gamma_1 < \gamma_0 = \beta$$

and $\alpha \in e_{\gamma_{\ell t}(\beta,\alpha)}$ i.e. $\rho(\beta,\alpha)$ is a (strictly) decreasing finite sequence of ordinals, starting with β, ending with $\gamma_{\ell t}(\beta,\alpha)$ of length $k(\beta,\alpha)$.

Note that if $\alpha \in S$, $\alpha < \beta$ then $\gamma_{\ell t}(\beta,\alpha) = \alpha + 1$.

Also

$\odot_5$ if δ is a limit ordinal and $\delta < \beta < \lambda$, *then* for some $\alpha_0 < \delta$ we have: $\alpha_0 \leq \alpha < \delta$ *implies*:

(i) for $\ell < k(\beta,\delta)$ we have $\gamma_\ell(\beta,\delta) = \gamma_\ell(\beta,\alpha)$

(ii)

$$\begin{aligned} \delta \in \mathrm{nacc}(e_{\gamma_{\ell t}(\beta,\delta)}) &\Leftrightarrow \delta = \gamma_{k(\beta,\delta)}(\beta,\delta) = \gamma_{k(\beta,\delta)}(\beta,\alpha) \\ &\Leftrightarrow \neg[\gamma_{k(\beta,\delta)}(\beta,\delta) = \delta > \gamma_{k(\beta,\delta)}(\beta,\alpha)] \end{aligned}$$

(iii) $\rho(\beta,\delta) \trianglelefteq \rho(\beta,\alpha)$; i.e. is an initial segment

(iv) $\delta \in \mathrm{nacc}(e_{\gamma_{\ell t}(\beta,\delta)})$ (here always holds if $\delta \in S$) implies:
- $\rho(\beta,\delta)\hat{}\langle\delta\rangle \trianglelefteq \rho(\beta,\alpha)$ hence
- $\rho_h(\beta,\delta)\hat{}\langle h(\beta,\delta)(\delta)\rangle \trianglelefteq \rho_h(\beta,\alpha)$.

(v) if $\mathrm{cf}(\delta) = \partial$ or $\delta \in \mathscr{W}$ *then* we have $\gamma_{\ell t}(\beta,\delta) = \delta + 1$ so $\delta + 1 \in \mathrm{nacc}(e_{\gamma_{1t}(\beta,\delta)})$

(vi) if $\mathrm{cf}(\delta) = \partial$ or $\delta \in \mathscr{W}$ and $\delta \in e_\gamma$, *then* necessarily $\gamma = \delta + 1$.

Why? Just let

$$\alpha_0 = \mathrm{Max}\big\{ \sup(e_{\gamma_\ell(\beta,\delta)} \cap \delta) + 1 : \ell < k(\beta,\delta) \text{ and } \delta \notin \mathrm{acc}(e_{\gamma_\ell(\beta,\delta)})\big\}.$$

Notice that if $\ell < k(\beta,\delta) - 1$ then $\delta \notin \mathrm{acc}(e_{\gamma_\ell(\beta,\delta)})$ follows.

Note that the outer maximum (in the choice of α_0) is well defined as it is over a finite non-empty set of ordinals. The inner sup is on the empty set (in which case we get zero) or is the maximum (which is well defined) as $e_{\gamma_\ell(\beta,\delta)}$ is a closed subset of $\gamma_\ell(\beta,\delta), \delta < \gamma_\ell(\beta,\delta)$ and $\delta \notin \mathrm{acc}(e_{\gamma_\ell(\beta,\delta)})$ – as this is required. For clauses (v), (vi) recall $\delta \in S^\lambda_\partial \cup \mathscr{W}$ and $e_\gamma \cap S^\lambda_\partial = \emptyset$ and $e_\gamma \cap \mathscr{W} = \emptyset$ when γ is a limit ordinal and $e_\gamma = \{\gamma - 1\}$ when γ is a successor ordinal.

$\odot_6$ (a) if $\alpha<\beta<\lambda$, $\ell<k(\beta,\alpha)$, $\gamma=\gamma_\ell(\beta,\alpha)$ *then* $\rho(\beta,\alpha)=\rho(\beta,\gamma)\hat{}\rho(\gamma,\alpha)$ and $\rho_h(\beta,\alpha)=\rho_h(\beta,\gamma)\hat{}\rho_h(\gamma,\alpha)$

(b) if $\alpha_0 < \dots < \alpha_k$ and $\rho(\alpha_k,\alpha_0) = \rho(\alpha_k,\alpha_{k-1})\hat{}\cdots\hat{}\rho(\alpha_1,\alpha_0)$ *then* this holds for any sub-sequence of $\langle\alpha_0,\dots,\alpha_k\rangle$.

$\odot_7$ let F'_ι be $F_\iota \circ h$ for $\iota = 1,2$; so F'_1 is a function from λ into ∂ and F'_2 is a function from λ into λ.

Stage B: Let

$\boxplus_2$ $\mathbf{T} = \{\bar t : \bar t = \langle t_\alpha : \alpha < \lambda\rangle$ satisfies $t_\alpha \in [\lambda]^{<\partial}$ and[1] $t_\alpha \subseteq \lambda\backslash\alpha\}$.

$\boxplus_3$ for $\varepsilon < \partial$ and $\bar t \in \mathbf{T}$ let $A_{\bar t,\varepsilon}$ be the set of $\gamma < \lambda$ such that for some (α_0,α_1) we have:

(a) $\alpha_0 < \alpha_1 < \lambda$ and[2] $(\zeta,\xi) \in t_{\alpha_0} \times t_{\alpha_1} \Rightarrow \zeta < \xi$

(b) for every $(\zeta,\xi) \in t_{\alpha_0} \times t_{\alpha_1}$ for some ℓ we have:

(α) $\ell < k(\xi,\zeta)$

(β) $\gamma_\ell(\xi,\zeta) = \gamma$

(γ) if $k < k(\xi,\zeta)$ then $F'_1(\gamma) \geq F'_1(\gamma_k(\xi,\zeta))$ and $F'_1(\gamma) \geq \varepsilon$

(δ) if $k < \ell$ then $F'_1(\gamma_k(\xi,\zeta)) < F'_1(\gamma)$.

We define:

$\boxplus_4$ $D = \{A \subseteq \lambda : A$ includes $A_{\bar t,\varepsilon}$ for some $\bar t \in \mathbf{T}, \varepsilon < \partial\}$.

Now note:

$\boxplus_5$ (a) if $\bar s, \bar t \in \mathbf{T}, \varepsilon \leq \zeta < \partial$ and $(\forall\alpha < \lambda)(s_\alpha \subseteq t_\alpha)$, *then* $A_{\bar t,\zeta} \subseteq A_{\bar s,\varepsilon}$

(b) if $\bar s \in \mathbf{T}$, $\varepsilon < \partial$, g is an increasing function from λ to λ and $\bar t = \langle t_\alpha : \alpha < \lambda\rangle$ is defined by $t_\alpha = s_{g(\alpha)}$ *then* $A_{\bar t,\varepsilon} \subseteq A_{\bar s,\varepsilon}$.

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[1] if instead we demand $\alpha \neq \beta < \lambda \Rightarrow t_\alpha \cap t_\beta = \emptyset$ then we shall get the same filter D.

[2] If we choose to add here "$t_{\alpha_0} \subseteq \alpha_1$", *then* we would have a problem in proving clause $\boxplus_5$(b).

[Why? Read the definitions.]

$\boxplus_6$ (a) the intersection of any $< \partial$ members of D is a member of D, equivalently includes the set $A_{\bar{t},\zeta}$ for some $\bar{t} \in \mathbf{T}$, $\zeta < \partial$

(b) for every $\beta < \lambda$ for some $\bar{t} \in \mathbf{T}$, $A_{\bar{t},0} \subseteq [\beta, \lambda)$

(c) if $\bar{t} \in \mathbf{T}$ and $\alpha < \lambda \Rightarrow t_\alpha \neq \emptyset$ then $\cap\{A_{\bar{t},\varepsilon} : \varepsilon < \partial\} = \emptyset$

(d) D is upward closed.

(e) λ belongs to D

[Why? For clause (a) assume $A_\varepsilon \in D$ for $\varepsilon < \varepsilon(*) < \partial$ then for some $\zeta_\varepsilon < \partial$ and $\bar{t}_\varepsilon \in \mathbf{T}$ we have $A_\varepsilon \supseteq A_{\bar{t}_\varepsilon,\zeta_\varepsilon}$. Define $t_\alpha = \bigcup\{t^\varepsilon_\alpha : \varepsilon < \varepsilon(*)\}$ for $\alpha < \lambda$ and $\zeta = \sup\{\zeta_\varepsilon : \varepsilon < \varepsilon(*)\}$; as the cardinal ∂ is regular, clearly $|t_\alpha| \leq \sum_{\varepsilon<\varepsilon(*)} |t^\varepsilon_\alpha| < \partial$ and obviously $t_\alpha \subseteq [\alpha, \lambda)$ hence $\bar{t} = \langle t_\alpha : \alpha < \lambda\rangle \in \mathbf{T}$ and similarly $\zeta < \partial$. Easily $A_{\bar{t},\zeta} \subseteq A_{\bar{t}_\varepsilon,\zeta_\varepsilon}$ for every $\varepsilon < \varepsilon(*)$, see $\boxplus_5$(a) so we are done proving clause (a). For clause (b) define $t_\alpha = \{\beta + \alpha + 1\}$ and recalling $\boxplus_3$(b)(β) and $\odot_4$ check that $A_{\bar{t},0} \subseteq [\beta, \lambda)$. Also clause (c) obviously holds because $\gamma \in A_{\bar{t},\varepsilon} \Rightarrow F'_1(\gamma) \geq \varepsilon$ by $\boxplus_3$(b)(γ) and F'_1 is a function from λ to ∂ and clauses (d), (e) hold trivially by the definition.]

$\boxplus_7$ (a) $\emptyset \notin D$

(b) D is a filter on λ, equivalently $A_{\bar{t},\varepsilon} \neq \emptyset$ for every $\bar{t}, \varepsilon$; also D is uniform ∂-complete, not ∂^+-complete.

[Why? Clause (a) is a major point, proved in Stage C below. That is, by $\boxplus_6$(a),(d) the only missing point is to show $A_{\bar{t},\zeta} \neq \emptyset$, (in fact, $|A_{\bar{t},\zeta}| = \lambda$). For clause (b) by (a) and $\boxplus_6$(a),(d),(e), D is a ∂-complete filter and the statement that D is uniform holds by $\boxplus_6$(b) and not ∂^+-complete holds by $\boxplus_6$(c).]

Note also

$\boxplus_8$ D is not weakly ∂-saturated.

[Why? By $\boxplus_7 + \boxplus_6$(c) and clause (c) in the assumptions of the theorem. That is it is known that if D fail this statement (and has the properties listed before) then there is no $\mathscr{W}$ as in clause (c) of the theorem. That is, considering the forcing notion $\mathbb{P} = D^+$ with inverse inclusion. Toward contradiction assume that the conclusion fails: by 0.4 the forcing notion $\mathbb{P}$ satisfies the δ-cc. Now, in $\mathbf{V}^{\mathbb{P}}$, the generic set $\mathbf{G}$ is an ultrafilter on the Boolean algebra $\mathscr{P}(\lambda)^{\mathbf{V}}$ and let $\mathbf{j}$ be the canonical embedding from $\mathbf{V}$ into the Mostowski collapse of $\mathbf{V}^\lambda/\mathbf{G}$ (we are using only functions from $\mathbf{V}$), now the contradiction will be clear. If ∂ is a successor cardinal we can use 0.3(2).]

Stage C: In this stage we accomplish the proof of the missing point in $\boxplus_7$(a) from above, so we shall prove "$A_{\bar{t},\varepsilon}$ is non-empty (in fact, has cardinality λ)" *when*:

$\boxplus$ (a) $t_\alpha \subseteq \lambda\backslash\alpha$ for $\alpha < \lambda$

(b) $|t_\alpha| < \partial$

(c) $\varepsilon < \partial$.

To start we note that:

$(*)_1$ without loss of generality $t_\alpha \neq \emptyset$ and $\alpha < \min(t_\alpha)$.

[Why? First, recalling $\boxplus_5$(a) we can replace $\bar t$ by $\bar t = \langle t_\alpha \cup \{\alpha\} : \alpha < \lambda\}$, so we may assume that each t_α is not empty. Second, let $\bar t' = \langle t'_\alpha : \alpha < \lambda\rangle$, $t'_\alpha = t_{\alpha+1}$, so easily $\bar t'$ satisfies $(*)_1$ and $A_{\bar t',\varepsilon} \subseteq A_{\bar t,\varepsilon}$ by clause $\boxplus_5$(b).]

Now

$(*)_2$ we can find $\mathscr{U}_1^{\rm dn}, \varepsilon^{\rm dn}$ such that:

(a) $\mathscr{U}_1^{\rm dn} \subseteq \mathscr{W}$ is stationary in λ

(b) $\alpha < \delta \in \mathscr{U}_1^{\rm dn} \Rightarrow t_\alpha \subseteq \delta$

(c) $\varepsilon^{\rm dn} < \partial$

(d) if $\delta \in \mathscr{U}_1^{\rm dn}$ *then* for arbitrarily large $\alpha < \delta$ we have $\zeta \in t_\alpha \Rightarrow {\rm Rang}(F_1(\rho_h(\delta,\zeta))) \subseteq \varepsilon^{\rm dn} < \kappa_1 = \partial$.

[Why? Clearly $E_0 = \{\delta < \lambda : \delta$ is a limit ordinal such that $\alpha < \delta \Rightarrow t_\alpha \subseteq \delta\}$ is a club of λ. For every $\delta \in \mathscr{W} \cap E_0$ and $\alpha < \delta$ we can find $\varepsilon^{\rm dn}_{\delta,\alpha}$ as in clauses (c), (d) of $(*)_2$ (because $|t_\alpha| < \partial$) and so recalling that ${\rm cf}(\delta) < \partial$ it follows that there is $\varepsilon^{\rm dn}_\delta$ such that $\delta = \sup\{\alpha < \delta : \varepsilon^{\rm dn}_{\delta,\alpha} = \varepsilon^{\rm dn}_\delta\}$. Then recalling $\lambda = {\rm cf}(\lambda) > \partial$ we can choose $\varepsilon^{\rm dn}$ such that the set $\mathscr{U}_1^{\rm dn} = \{\delta \in \mathscr{W} \cap E_0 : \varepsilon^{\rm dn}_\delta = \varepsilon^{\rm dn}\}$ is stationary. So $(*)_2$ holds indeed.]

$(*)_3$ We can find $\mathscr{U}_1^{\rm up}, \alpha_1^*, \varepsilon^{\rm up}$ such that:

(a) $\mathscr{U}_1^{\rm up} \subseteq S_0^*$ is stationary

(b) $h{\restriction}\mathscr{U}_1^{\rm up}$ is constantly 0, actually follows by (a), see Stage A

(c) $\alpha_1^* < \lambda$ satisfies $\alpha_1^* < \min(\mathscr{U}_1^{\rm up})$ and $\varepsilon^{\rm up} < \partial$

(d) if $\delta \in \mathscr{U}_1^{\rm up}$ and $\alpha \in [\alpha_1^*, \delta)$ and $\beta \in t_\delta$ *then*:

- $\rho_{\beta,\delta}{}^\frown\langle\delta\rangle \trianglelefteq \rho_{\beta,\alpha}$
- ${\rm Rang}(F_1(\rho_h(\beta,\delta))) \subseteq \varepsilon^{\rm up}$.

[Why? For every $\delta \in S_0^* \subseteq S$ and $\zeta \in t_\delta$ let $\alpha_{1,\delta,\zeta} < \delta$ be such that $(\forall\alpha)(\alpha \in [\alpha_{1,\delta,\zeta}, \delta) \Rightarrow \rho_{\zeta,\delta}{}^\frown\langle\delta\rangle \trianglelefteq \rho_{\zeta,\alpha})$, it exists by $\odot_5$ of Stage A.

Let

- $\alpha_{1,\delta} = \sup\{\alpha_{1,\delta,\zeta} : \zeta \in t_\delta\}$
-
$$\varepsilon^{\rm up}_\delta = \sup\big\{F'_1(\gamma_\ell(\zeta,\delta))(\ell) + 1 : \zeta \in t_\delta \text{ and } \ell < k(\zeta,\delta)\big\}$$
$$= \bigcup\big\{\sup {\rm Rang}(F_1(\rho_h(\zeta,\delta))) + 1 : \zeta \in t_\delta\big\};$$

as ${\rm cf}(\delta) = \partial$ and $\partial = {\rm cf}(\partial) > |t_\delta|$, necessarily $\alpha_{1,\delta} < \delta$ and $\varepsilon^{\rm up}_\delta < \partial$.

Lastly, there are $\alpha_1^* < \lambda$ and $\varepsilon^{\rm up} < \kappa_1 = \partial$ and $\mathscr{U}_1^{\rm up} \subseteq S_0^*$ as required by using Fodor lemma. So $(*)_3$ holds indeed.]

Now let $E = \big\{\delta < \lambda : \delta$ is a limit ordinal $> \alpha_1^*$ such that $\delta = \sup(\mathscr{U}_1^{\rm dn} \cap \delta)$ and $\alpha < \delta \Rightarrow t_\alpha \subseteq \delta\big\}$, it is a club of λ because $\alpha_1^* < \lambda$ by $(*)_3$(c) and $\mathscr{U}_1^{\rm dn}$ is an unbounded subset of λ by $(*)_2$(a), and t_α is a subset of λ of cardinality $< \partial$ hence is bounded.

Choose $\varepsilon(*) = \max\{\varepsilon^{\rm up} + 1, \varepsilon^{\rm dn} + 1, \varepsilon + 1\}$ where ε is from $\boxplus$(c), so $\varepsilon(*) < \partial$ and choose $\delta_2 \in E \cap S$ such that $F'_1(\delta_2) = \varepsilon(*)$. Next choose $\alpha_2 \in \mathscr{U}_1^{\rm up} \backslash (\delta_2 + 1)$ and let $\alpha^* \in (\alpha_1^*, \delta_2)$ be large enough such that $\zeta \in$

$(\alpha^*,\delta_2) \wedge \xi \in t_{\alpha_2} \Rightarrow \rho(\xi,\delta_2)\hat{}\langle\delta_2\rangle \triangleleft \rho(\xi,\zeta)$. Now choose $\delta_1 \in \mathscr{U}_1^{\rm dn} \cap (\alpha^*,\delta_2)$ and $\alpha^{**} \in (\alpha^*,\delta_1)$ be such that $\alpha \in (\alpha^{**},\delta_1) \wedge \xi \in t_{\alpha_2} \Rightarrow \rho(\xi,\delta_1)\hat{}\langle\delta_1\rangle \triangleleft \rho(\xi,\alpha)$.

[Why is this possible? First as $\alpha^{**} > \alpha^*$ it is enough to have $\alpha \in (\alpha^{**},\delta_1) \Rightarrow \rho(\delta_2,\delta_1)\hat{}\langle\delta_1\rangle \triangleleft \rho(\delta_2,\alpha)$. Second here $\mathrm{cf}(\delta_1) < \partial$ however this condition holds because $\delta_1 \in \mathscr{U}_1^{dn} \subseteq \mathscr{W}$ so necessarily $\gamma_{\rm lt}(\delta_2,\delta_1) = \delta_1 + 1$ by $\odot_5$(vi)].

Next let $\ell_* < \ell g(\rho(\alpha_2,\delta_1)$ be such that:

$(*)_4$ (a) $\varepsilon(\bullet) := F_1(\rho_h(\alpha_2,\delta_1))(\ell_*) = \max \mathrm{Rang}\, F_1(\rho_h(\alpha_2,\delta_1))$

(b) under this restriction ℓ_* is minimal.

Lastly, choose $\alpha_1 \in (\alpha^{**},\delta_1)$ which is as in $(*)_2$(d) with respect to δ_1, i.e. such that:

$(*)_5$ if $\zeta \in t_{\alpha_1}$ then $\mathrm{Rang}\, F_1(\rho_h(\delta_1,\zeta)) \subseteq \varepsilon^{\rm dn}$.

Now we shall prove that the pair (α_1,α_2) is as required. So let $(\zeta,\xi) \in t_{\alpha_1} \times t_{\alpha_2}$; now by our choices

$(*)_6$ $\rho(\xi,\zeta) = \rho(\xi,\alpha_2)\hat{}\rho(\alpha_2,\delta_2)\hat{}\rho(\delta_2,\delta_1)\hat{}\rho(\delta_1,\zeta)$ and

$$\rho(\alpha_2,\delta_1) = \rho(\alpha_2,\delta_2)\hat{}\rho(\delta_2,\delta_1)$$

So

$(*)_7$ $\mathrm{Rang}(F_1(\rho_h(\xi,\alpha_2)) \subseteq \varepsilon^{\rm up} \leq \varepsilon(*)$

[Why? by $(*)_3$(a), the choice of $\alpha_2 \in \mathscr{U}_1^{\rm up}$ and ξ being from t_{α_2}]

$(*)_8$ $\mathrm{Rang}(F_1(\rho_h(\delta_1,\zeta)) \subseteq \varepsilon^{\rm dn} \leq \varepsilon(*)$

[Why by $(*)_2$(d) and the choice of α_1 (and ζ being a member of t_{α_1}]

$(*)_9$ $\varepsilon(*) = F_1 \circ h(\delta_2) \in \mathrm{Rang}(F_1(\rho_h(\alpha_2,\delta_1)))$, see $(*)_6$ and (before and after) $\odot_1$.

[Why? Recall that δ_2 was chosen in $E \cap S$ such that $F_1'(\delta_2) = \varepsilon(*)$.]

Hence

$(*)_{10}$ $\varepsilon \leq \varepsilon(*) \leq \varepsilon(\bullet) < \partial$

Putting those together, we can finish this stage by:

$(*)_{11}$ in $\boxplus_3$(b) for our $\bar{t}$ and the pair (α_1,α_2), our $\varepsilon(\bullet)$ (chosen in $(*)_4$(a)) is gotten, witnessing $\gamma_{\ell_\bullet}(\alpha_2,\delta_1) \in A_{\bar{t},\varepsilon(*)} \subseteq A_{\bar{t},\varepsilon}$

[Why? As first $\varepsilon < \varepsilon(*)$, by the choice of $\varepsilon(*)$, and second if $(\zeta,\xi) \in t_{\alpha_1} \times t_{\alpha_2}$ then $\ell = \ell g(\rho(\xi,\alpha_2)) + \ell_*$ is as required in $\boxplus_3$(b) for $\bar{t}$ by $(*)_6$–$(*)_{10}$]

So we are done proving $\boxplus_7$(a).

Stage D: By $\boxplus_8$

$\circledast_1$ there is $F_* : \lambda \to \partial$ such that $\varepsilon < \partial \Rightarrow F_*^{-1}(\{\varepsilon\}) \neq \emptyset \bmod D$.

We first deal with the easier version with ∂ colours, i.e. proving $\mathrm{Pr}_1(\lambda,\lambda,\partial,\partial)$.

We now define the colouring $\mathbf{c}_1 : [\lambda]^2 \to \partial$ by:

$\circledast_2$ if $\alpha < \beta < \lambda$ then $\mathbf{c}_1\{\alpha,\beta\}$ is $F_*(\gamma_{\ell(\beta,\alpha)}(\beta,\alpha))$ where $\ell(\beta,\alpha) = \min\{\ell < k(\beta,\alpha) : F_1'(\gamma_\ell(\beta,\alpha)) = \max \mathrm{Rang}(F_1'(\rho(\beta,\alpha)))\}$.

To prove that the colouring $\mathbf{c}_1$ really witnesses $\mathrm{Pr}_1(\lambda,\lambda,\partial,\partial)$, our task is to prove:

$\circledast_3$ given $\bar{t} \in \mathbf{T}$ and $\iota < \partial$ there are $\alpha < \beta$ such that:

- $\zeta \in t_\alpha \wedge \xi \in t_\beta \Rightarrow \mathbf{c}_1\{\zeta,\xi\} = \iota$.

[Why does $\circledast_3$ hold? Let $B_\iota = \{\gamma < \lambda : F_*(\gamma) = \iota\}$. By the choice of F_* we know that $B_\iota \neq \emptyset \bmod D$. Focus on $A_{\bar{t},\varepsilon}$ for our specific $\bar{t} \in \mathbf{T}$ and any $\varepsilon < \partial$. Since $A_{\bar{t},\varepsilon} \in D$ we conclude that $B_\iota \cap A_{\bar{t},\varepsilon} \neq \emptyset$.

Fix an ordinal $\gamma \in B_\iota \cap A_{\bar{t},\varepsilon}$. By the very definition of $A_{\bar{t},\varepsilon}$ in $\boxplus_3$ we choose $\alpha < \beta < \lambda$ such that for every $(\zeta, \xi) \in t_\alpha \times t_\beta$ there exists $\ell < k(\xi, \zeta)$ for which $\gamma_\ell(\xi, \zeta) = \gamma$ and $F_1'(\gamma) \geq F_1'(\gamma_k(\xi, \zeta))$ whenever $k < k(\xi, \zeta)$ and $F_1(\gamma) \geq \varepsilon$ and $F_1'(\gamma) > F_1'(\gamma_k(\xi, \zeta))$ whenever $k < \ell$. Let $\ell(\xi, \zeta)$ be this ℓ, in fact, this ℓ is unique (for the pair (ζ, ξ)).

Now $\mathbf{c}_1\{\zeta, \xi\} = F_*(\gamma_{\ell(\xi,\zeta)}(\xi, \zeta))$ (by $\circledast_2$) which equals $F_*(\gamma)$ (by the choice of $\ell(\xi, \zeta)$) which equals ι (since $\gamma \in B_\iota$). Hence $\circledast_3$ holds and we finish Stage D.]

Stage E: The full theorem: the case of λ colors.

Let h', h'' be functions from ∂ into ∂, ω respectively such that the mapping $\zeta \mapsto (h'(\zeta), h''(\zeta))$ is onto $\partial \times \omega$ and moreover each such pair is gotten ∂ times.

We have to define a colouring $\mathbf{c}_2 : [\lambda]^2 \to \lambda$ exemplifying $\mathrm{Pr}_1(\lambda, \lambda, \lambda, \partial)$.

This is done as follows using h', h'' and F_* from $\circledast_1$:

$\oplus_1$ for $\alpha < \beta < \lambda$ we let

$\bullet_1$ $\zeta = \zeta(\beta, \alpha) := h'(\mathbf{c}_1\{\beta, \alpha\})$, necessarily $< \partial$

$\bullet_2$ $n = n(\beta, \alpha) := h''(\mathbf{c}_1\{\beta, \alpha\})$, necessarily $< \omega$

$\bullet_3$ $m = m(\beta, \alpha)$ is the n-th member of $\{k < k(\beta, \alpha) : F_1'(\gamma_k(\beta, \alpha)) = \zeta\}$ when there is such m and is zero otherwise

$\bullet_4$ we define $\mathbf{c}_2$ as follows: for $\alpha < \beta, \mathbf{c}_2\{\alpha, \beta\}$ is $F_2'(\gamma_{m(\beta,\alpha)}(\beta, \alpha))$ recalling that F_2', a function from λ to λ is from $\odot_2$ from the end of stage A.

To prove that $\mathbf{c}_2$ indeed exemplifies $\mathrm{Pr}_1(\lambda, \lambda, \lambda, \partial)$ it suffice to prove (this is the task of the rest of the proof)

$\oplus_2$ assume $\bar{t} \in \mathbf{T}$ and $j_* < \lambda$ and we shall find $\alpha < \beta$ such that $t_\alpha \subseteq \beta$ and $(\zeta, \xi) \in t_\alpha \times t_\beta \Rightarrow \mathbf{c}_2\{\zeta, \xi\} = j_*$.

Toward this:

$\oplus_3$ (a) we apply $(*)_3$ to our $\bar{t}$, getting $\varepsilon^{\mathrm{up}}, \mathscr{U}_1^{\mathrm{up}}, \alpha_1^*$ as there

(b) we apply $(*)_2$ to our $\bar{t}$ getting $\mathscr{U}_1^{\mathrm{dn}}, \varepsilon^{\mathrm{dn}}$

(c) let $\varepsilon^{\mathrm{md}} = \max\{\varepsilon^{\mathrm{up}} + 1, \varepsilon^{\mathrm{dn}} + 1\}$.

We can find $g_2, \mathscr{U}_2^{\mathrm{up}}, \gamma_*, \alpha_2^*, m_2^*$ such that:

$\oplus_4$ (a) $\gamma_* < \lambda$ satisfies $F_2(\gamma_*) = j_*$ and $F_1(\gamma_*) = \varepsilon^{\mathrm{md}}$

(b) $\mathscr{U}_2^{\mathrm{up}} \subseteq S_{\gamma_*}^*$ is stationary hence $\delta \in \mathscr{U}_2^{\mathrm{up}} \Rightarrow F_2'(\delta) = F_2(h(\delta)) = F_2(\gamma_*) = j_* \wedge F_1'(\delta) = F_1(h(\delta)) = F_1(\gamma_*) = \varepsilon^{\mathrm{md}}$

(c) g_2 is a function with domain $\mathscr{U}_2^{\mathrm{up}}$ such that $\delta \in \mathscr{U}_2^{\mathrm{up}} \Rightarrow \delta < g_2(\delta) \in \mathscr{U}_1^{\mathrm{up}}$

(d) α_2^* satisfies $\alpha_1^* < \alpha_2^* < \min(\mathscr{U}_2^{\mathrm{up}})$

(e) if $\delta \in \mathscr{U}_2^{\mathrm{up}}$ and $\alpha \in [\alpha_2^*, \delta)$ and $\beta \in t_{g_2(\delta)}$ then

- $\rho(g_2(\delta), \delta)\hat{\ }\langle\delta\rangle \trianglelefteq \rho(g_2(\delta), \alpha)$ hence
- $\rho_{\beta,\delta}\hat{\ }\langle\delta\rangle \trianglelefteq \rho_{\beta,\alpha}$

(f) m_2^* satisfies: for every $\delta \in \mathscr{U}_2^{\rm up}$, it is the cardinality of the set $\{\ell < k(g_2(\delta),\delta) : F_1'(\gamma_\ell(g_2(\delta),\delta)) = \varepsilon^{\rm md}\}$ which may be zero.

[Why? First choose γ_* as in clause (a) of $\oplus_4$ (possible by the choice of F_1, F_2 in the beginning of Stage A; hence $\delta \in S_{\gamma_*} \Rightarrow F_2'(\delta) = F_2(h(\delta)) = F_2(\gamma_*) = j_*$ and $F_1'(\delta) = F_1(h(\delta)) = F_1(\gamma_*) = \varepsilon^{\rm md}$ (by the choice of F_1' in $\odot_7$ recalling the definitions of h, F_1'). Second, define $g' : S_{\gamma_*}^* \to \mathscr{U}_1^{\rm up}$ such that $\delta \in S_{\gamma_*}^* \Rightarrow \delta < g'(\delta) \in \mathscr{U}_1^{\rm up}$. Third, for each $\delta \in S_{\gamma_*}^* \backslash (\alpha_1^* + 1)$, find $\alpha_{2,\delta}' < \delta$ above α_1^* and $m_{2,\delta}$ such that the parallel of clauses (e), (f) (with g' here instead of g_2 there) of $\oplus_4$ holds. Fourth, use Fodor lemma to get a stationary $\mathscr{U}_2^{\rm up} \subseteq S_{\gamma_*}^*$ such that $\langle (\alpha_{2,\delta}', m_{2,\delta}) : \delta \in \mathscr{U}_2^{\rm up}\rangle$ is constantly (α_2^*, m_2^*) and lastly let $g_2 = g' \restriction \mathscr{U}_2^{\rm up} \backslash (\alpha_2^* + 1)$. Now it is easy to check that $\oplus_4$ holds indeed.]

Next

$\oplus_5$ if $\delta \in \mathscr{U}_2^{\rm up}$ *then*:

(a) $F_1'(\delta) = \varepsilon^{\rm md}$

(b) if $\alpha \in [\alpha_2^*, \delta), \xi \in t_{g_2(\delta)}$ *then* $u = \{\ell < k(\xi,\alpha) : F_1'(\gamma_\ell(\xi,\alpha)) = \varepsilon^{\rm md}\}$ has $> m_2^*$ members and if ℓ is the m_2^*-th member of u then $\gamma_\ell(\xi,\alpha) = \delta$.

Why? Clause (a) holds by $\oplus_4$(a),(b). For clause (b) use clause (a) and the demands on m_2^*. That is

(a) $\rho(\xi,\alpha) = \rho(\xi, g_2(\delta))\hat{\ }\rho(g_2(\delta),\delta)\hat{\ }\rho(\delta,\alpha)$ [Why? by $(*)_3, \oplus_4$(e)]

(b) $\mathrm{Rang}(\rho_h(\alpha, g_2(\delta))) \subseteq \varepsilon^{\rm up} \subseteq \varepsilon^{\rm md}$ [Why? by $(*)_2$]

(c) the set $\{\ell < k(g_2(\delta),\delta) : F_1'(\gamma_\ell(g_2(\delta),\delta)) = \varepsilon^{\rm md}\}$ has m_2^* members [why? by $\oplus_4(f)$]

(d) $F_1'(\gamma_0(\delta,\alpha)) = F_1'(\delta) = \varepsilon^{\rm md}$ [Why? by $\oplus_4$(a),(b)]]

(e) if ℓ_* is the m_2^*-th member of $\{\ell : F_1(\gamma_\ell(\xi,\alpha)) = \varepsilon^{\rm md}\}$ then $\gamma_{\ell_*}(\xi,\alpha) = \delta$ [Why? putting the above together]

So $\oplus_5$ holds indeed.

Now choose $\varepsilon(*) < \partial$ such that $h'(\varepsilon(*)) = \varepsilon^{\rm md}$ and $h''(\varepsilon(*)) = m_2^*$.

Next, let $E = \{\delta < \lambda : \delta$ limit ordinal $> \alpha_2^*$ such that $\delta = \sup(\mathscr{U}_1^{\rm dn} \cap \delta)$ and $\alpha < \delta \Rightarrow g_2(\alpha) < \delta\}$.

Lastly,

$\oplus_6$ choose $\delta_1 < \delta_2$ such that

(a) $\delta_1 \in \mathscr{U}_1^{\rm dn} \cap E$

(b) $\delta_2 \in \mathscr{U}_2^{\rm up} \cap E \backslash (\delta_1 + 1)$

(c) $\mathbf{c}_1\{\delta_2, \delta_1\} = \varepsilon(*)$,

[Why does such a pair (δ_1, δ_2) exist? By Stage D applied to $\bar{s} = \langle s_\alpha : \alpha < \lambda\rangle$ where $s_\alpha = \{\min(\mathscr{U}_1^{\rm dn} \cap E \backslash \alpha), \min(\mathscr{U}_2^{\rm up} \cap E \backslash \alpha)\}$.

That is, we can find ordinals $\alpha < \beta < \lambda$ such that: for every $(\zeta,\xi) \in (s_\alpha \times s_\beta)$ we have $\mathbf{c}_1\{\xi,\zeta\} = \varepsilon^{\rm md}$.

Let $\delta_1 = \min(\mathscr{U}_1^{\rm dn} \cap E \setminus \alpha)$ and let $\delta_2 = \min(\mathscr{U}_1^{\rm up} \cap E \setminus \beta)$.

So $(\delta_1, \delta_2) \in (s_\alpha \times s_\beta)$ hence clearly $\delta_1 < \delta_2$, $\mathbf{c}_1\{\delta_1, \delta_2\} = \varepsilon(*)$, $\delta_1 \in \mathscr{U}_1^{\rm dn} \cap E$ and $\delta_1 \in \mathscr{U}_1^{\rm up} \cap E$. So the pair (δ_1, δ_2) is as promised in in $\oplus_6$]

Now let $\beta = g_2(\delta_2)$ and choose $\alpha \in \mathscr{U}_1^{\rm dn} \cap \delta_1 \backslash (\alpha_2^* + 1)$. Easy to check that α, β are as required.

So we have finished proving Theorem 1.1. $\square_{1.1}$

References

[1] Alan Stewart Dow and Saharon Shelah, On the cofinality of the splitting number, *Indag. Math. (N.S.)*, **29** (2018), 382–395.

[2] István Juhász and Saharon Shelah, Strong colorings yield κ-bounded spaces with discretely untouchable points, *Proc. Amer. Math. Soc.*, **143** (2015), 2241–2247.

[3] Assaf Rinot, Complicated colorings, *Math. Res. Lett.*, **21** (2014), 1367–1388.

[4] S. Shelah et al., TBA (in preparation) [Sh:E101] in the author's web site.

[5] Saharon Shelah, *Cardinal Arithmetic*, Oxford Logic Guides, vol. 29, The Clarendon Press, Oxford University Press (New York, 1994).

[6] Saharon Shelah, Colouring and non-productivity of $\aleph_2$-c.c, *Ann. Pure Appl. Logic*, **84** (1997), 153–174.

[7] Saharon Shelah, The colouring existence theorem revisited, *Acta Math. Hungar.*, **159** (2019), 1–26.

[8] Robert M. Solovay, Real valued measurable cardinals. in: *Axiomatic Set Theory*, Proc. of Symposia in Pure Math., vol. XIII, Part 1, Amer. Math. Soc. (Providence, RI, 1971), pp. 397–428.