GRAPHS REPRESENTED BY EXT

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ABSTRACT. Daniel Herden asked for which graph (μ, R) we can find a family $\{G_\alpha\}$ of Abelian groups such that $\mathrm{Ext}(G_\alpha, G_\beta) = 0$ iff $(\alpha, \beta) \in R$. We show that this is always possible for bipartite graphs in ZFC. We also give a consistent positive answer for the general case.

§ 1. INTRODUCTION

The vanishing and non-vanishing properties of $\mathrm{Ext}(-, \sim)$ are useful tools in some branches of mathematics. For instance, see the book [13]. Here, we assume the objects $-, \sim$ are not necessarily noetherian, so the corresponding Ext-family becomes more mysterious. Despite its ubiquity, there was no correspondence from graph theory to the Ext-family. Our aim in this paper is to present such connection by coding graphs using the vanishing property Ext of a family of (almost free) abelian groups.

Recall the following achievements from literature. Any basic hereditary finite-dimensional algebra over an algebraically closed field, realizes as the path algebra with respect to the Ext-quiver of simple objects. For more details, see the book [1]. In his seminal paper [10], Shelah proved that freeness of Whitehead groups, that is groups $\mathbb{G}$ satisfying the vanishing property $\mathrm{Ext}(\mathbb{G}, \mathbb{Z}) = 0$, is undecidable in ZFC. Göbel and Shelah [5] introduced a method to construct splitters, that is groups $\mathbb{G}$ satisfying $\mathrm{Ext}(\mathbb{G}, \mathbb{G}) = 0$. They applied some splitter arguments to prove

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the existence of enough projectives and injectives objects in the rational cotorsion theories, see [5].

Cotorsion theories were introduced by Salce [9] in 1979. Combining with the splitters, this theory equipped with a lot of applications. For instance, see the book [6]. Göbel, Shelah and Wallutis, proved in [4] that any poset embeds into the lattice of cotorsion pairs of Abelian groups. To be more explicit, let I be any set, and look at the power set $\mathcal{P}(I)$. For any $X \in \mathcal{P}(I)$ they construct Abelian groups $\mathbb{G}_X, \mathbb{H}^X$ such that for all $X, Y \subseteq I$,

$$\mathrm{Ext}(\mathbb{G}_Y, \mathbb{H}^X) = 0 \Longleftrightarrow Y \subseteq X,$$

i.e., there is an order-preserving and an embedding from $(\mathcal{P}(I), \subseteq)$ into the lattice of all cotorsion pairs by mapping the set Y onto the cotorsion pair generated by $\mathbb{G}_Y$. This result is documented very well in the books [3] and [6].

One can consider the partially ordered set $(\mathcal{P}(I), \subseteq)$ as a special case of graphs. In this regard, Daniel Herden asked the third author the following question:

Question 1.1. For which graphs (μ, R), we can find a family of Abelian groups $\{\mathbb{G}_\alpha\}_{\alpha<\mu}$ such that

$$\mathrm{Ext}(\mathbb{G}_\alpha, \mathbb{G}_\beta) = 0 \Longleftrightarrow (\alpha, \beta) \in R.$$

This transports the information contained in a graph within the vanishing and non-Vanishing of a suitable family of groups. The problem becomes more subtle, if we put some restrictions on the family $\{\mathbb{G}_\alpha\}$, say for example if we also impose the groups are almost free.

Section 2 contains the preliminaries and basic notations that we need. In Section 3 we apply Jensen's diamond principle, to give a positive answer for the case of bipartite graphs. The key to prove this, is a criteria on the vanishing of Ext-groups given in Lemma 3.4. The diamond principle helps us to construct the groups $\mathbb{G}_\alpha$, for $\alpha < \mu$, with the strong property that each $\mathbb{G}_\alpha$ is $|\mathbb{G}_\alpha|$-free.

Recall that the diamond principle is a kind of prediction principle whose truth is independent of ZFC. Then in Section 4, we descend from §3 to the ordinary ZFC set theory and prove the following theorem as one of our main results:

Theorem A. Let $\lambda = \lambda^{\aleph_0}, \mu = 2^\lambda$ and let $R \subseteq \mu \times \mu$ be a relation. Then there exist families $\langle \mathbb{G}_\alpha : \alpha < \mu \rangle$, and $\langle \mathbb{K}_\alpha : \alpha < \rangle$ of $\aleph_1$-free Abelian groupsequipped with the following properties:

(1) for all $\alpha < \mu, \mathbb{G}_\alpha$ has size λ and $\mathbb{K}_\alpha$ has size 2^λ,

(2) for all $\alpha, \beta < \mu$,

$$\mathrm{Ext}(\mathbb{G}_\alpha, \mathbb{K}_\beta) = 0 \iff \alpha R \beta.$$

Recall that a graph (μ, R) is called bipartite if the vertex set can be decomposed as $V_1 \cup V_2$ such that all edges go between V_1 and V_2. The main technical tool to prove the above theorem is the "*Shelah's black box*". We can consider black boxes as a general method to generate a class of diamond-like principles provable in ZFC. So, Question 1.1 has a positive answer for the case of bipartite graphs.

In the final section we relay on the theory of forcing, and prove the following theorem:

Theorem B. Suppose GCH holds and the pair (S, R) is a graph where $R \subseteq S \times S$, and let $\lambda > |S|$ be an uncountable regular cardinal. Then there exists a cardinal preserving generic extension of the universe, and there is a family $\{\mathbb{G}_s\}_{s \in S}$ of λ-free Abelian groups such that

$$\mathrm{Ext}(\mathbb{G}_s, \mathbb{G}_t) = 0 \iff sRt.$$

This theorem gives a consistent positive answer to Question 1.1.

For all unexplained definitions and more information, see the books by Eklof-Mekler [3] and Göbel-Trlifaj [6].

§ 2. Preliminary notation

In this section, we set out our notation and discuss some facts that will be used throughout the paper and refer to the book of Eklof and Mekler [3] for more information. We restrict our discussion to the category Mod-$\mathbb{Z}$ of Abealin groups, though most of the notions and results can be extended to module categories over more general rings. For Abelain groups $\mathbb{G}, \mathbb{H}$ we set $\mathrm{Ext}(\mathbb{G}, \mathbb{H}) := \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{G}, \mathbb{H})$ and similarly, $\mathrm{Hom}(\mathbb{G},\mathbb{H}) := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{G},\mathbb{H})$. We need the following well-known fact (see e.g. the book [13, Page 77]):

Fact 2.1. (Yoneda's lemma) Let $\zeta_i := 0 \longrightarrow \mathbb{B} \xrightarrow{g_i} \mathbb{C}_i \xrightarrow{f_i} \mathbb{A} \longrightarrow 0$ be two short exact sequences of Abelian groups. We say ζ_1 is equivalent to ζ_2 if there is a commutative diagram:

$$\begin{array}{ccccccccc} \zeta_2 = 0 & \longrightarrow & \mathbb{B} & \xrightarrow{g_2} & \mathbb{C}_2 & \xrightarrow{f_2} & \mathbb{A} & \longrightarrow & 0 \\ & & =\uparrow & & \cong\uparrow & & =\uparrow & & \\ \zeta_1 = 0 & \longrightarrow & \mathbb{B} & \xrightarrow{g_1} & \mathbb{C}_1 & \xrightarrow{f_1} & \mathbb{A} & \longrightarrow & 0 \end{array}$$

Indeed, this is an equivalent relation, and there is a 1-1 correspondence between the equivalent class of these short exact sequences and $\mathrm{Ext}(\mathbb{A},\mathbb{B})$. In addition, $[\zeta_1] = 0 \in \mathrm{Ext}(\mathbb{A},\mathbb{B})$ iff ζ_1 splits.

Definition 2.2. An abelian group $\mathbb{G}$ is called $\aleph_1$-free if every subgroup of $\mathbb{G}$ of cardinality $< \aleph_1$, i.e., every countable subgroup, is free. More generally, an Abelian group $\mathbb{G}$ is called λ-free if every subgroup of $\mathbb{G}$ of cardinality $< \lambda$ is free.

Definition 2.3. Let κ be a regular cardinal. An Abelian group $\mathbb{G}$ is said to be strongly κ-free if there is a set $\mathcal{S}$ of $< \kappa$-generated free subgroups of $\mathbb{G}$ containing 0 such that for any subset S of $\mathbb{G}$ of cardinality $< \kappa$ and any $\mathbb{N} \in \mathcal{S}$, there is $\mathbb{L} \in \mathcal{S}$ such that $S \cup \mathbb{N} \subset \mathbb{L}$ and $\mathbb{L}/\mathbb{N}$ is free.

By club we mean closed and unbounded.

Definition 2.4. Suppose κ is an uncountable regular cardinal. Let $\mathscr{D}_\kappa$ denote the club filter on κ, i.e.,

$$\mathscr{D}_\kappa = \{A \subseteq \kappa\colon\ A \text{ contains a club subset of } \kappa\}.$$

Let also $\mathcal{P}(\kappa)/\mathscr{D}_\kappa$ denote the resulting quotient Boolean algebra.

It is easily seen that $\mathscr{D}_\kappa$ is a normal κ-complete filter on κ and that it is closed under diagonal intersections, i.e., if $A_i \in \mathscr{D}_\kappa$, for $i < \kappa$, then their diagonal intersection

$$\triangle_{i<\kappa} A_i = \{\xi < \kappa : \forall i < \xi, \xi \in A_i\}$$

is also in $\mathscr{D}_\kappa$. This can be used to prove the following easy lemma.

Lemma 2.5. *Suppose κ is a regular uncountable cardinal, $\delta \leq \kappa$ and let $\Gamma_i \in \mathscr{P}(\kappa)/\mathscr{D}_\kappa$ for $i < \delta$. Then in the Boolean Algebra $\mathscr{P}(\kappa)/\mathscr{D}_\kappa$, the sequence $\{\Gamma_i : i < \delta\}$ has a lub (least upper bound) Γ.*

Proof. For each $i < \delta$ let $A_i \subseteq \kappa$ be such that $\Gamma_i = A_i/\mathscr{D}_\kappa$. If $\delta < \kappa$, then $\Gamma = A/\mathscr{D}_\kappa$ is as required where $A = \bigcup\limits_{i<\delta} A_i$, and if $\delta = \kappa$, then $\Gamma = A/\mathscr{D}_\kappa$ is as required where $A = \triangle_{i<\kappa} A_i$ is the diagonal intersection of the sets $A_i, i < \kappa$. □

The following definition plays an important role in the sequel.

Definition 2.6. Let κ be a regular cardinal. If $\mathbb{G}$ is a $\leq \kappa$-generated Abelian group, a κ-filtration of $\mathbb{G}$ is a sequence $\{\mathbb{G}_\nu : \nu < \kappa\}$ of subgroups of $\mathbb{G}$ whose union is $\mathbb{G}$ such that for all $\nu < \kappa$:

- (a) $\mathbb{G}_\nu$ is a $< \kappa$-generated subgroup of $\mathbb{G}$;
- (b) if $\mu < \nu$, then $\mathbb{G}_\mu \subset \mathbb{G}_\nu$;
- (c) if ν is a limit ordinal, then $\mathbb{G}_\nu = \bigcup_{\mu<\nu} \mathbb{G}_\mu$ i.e., the sequence is continuous.

It is easily seen that if $\{\mathbb{G}_\nu : \nu < \kappa\}$ and $\{\mathbb{H}_\nu : \nu < \kappa\}$ are two κ-filtrations of a group $\mathbb{G}$, then the set

$$\{\nu < \kappa : \mathbb{G}_\nu = \mathbb{H}_\nu\}$$

contains a club subset of κ, in particular, modulo the club filter $\mathscr{D}_\kappa$, the choice of the κ-filtration does not matter. This observation makes the following definition well-defined.

Definition 2.7. Let λ be an uncountable regular cardinals.

(1) If $\mathbb{G}$ is an Abelian group of cardinality λ and $\langle \mathbb{G}_\alpha : \alpha < \lambda \rangle$ is a filtration of $\mathbb{G}$, then

$$\Gamma(\mathbb{G}, \bar{\mathbb{G}}) = \{\delta < \lambda : \mathbb{G}/\mathbb{G}_\delta \text{ is not } \lambda\text{-free}\}.$$

(2) Let

$$\Gamma(\mathbb{G}) = \Gamma(\mathbb{G}, \bar{\mathbb{G}})/\mathscr{D}_\lambda$$

for some (and hence every) filtration $\bar{\mathbb{G}}$ of $\mathbb{G}$.

The following lemma shows gives a combinatorial characterization of λ-free groups to be free.

Lemma 2.8. *(*[3, Ch IV, Proposition 1.7]*) Let λ be an uncountable regular cardinals and let $\mathbb{G}$ be a λ-free Abelian group of cardinality λ. The following are equivalent:*

(1) $\mathbb{G}$ is free,

(2) $\mathbb{G}$ has a filtration $\langle \mathbb{G}_\alpha : \alpha < \lambda \rangle$ such that for all $\alpha < \lambda, \mathbb{G}_{\alpha+1}/\mathbb{G}_\alpha$ is free,

(3) $\Gamma(\mathbb{G}) = \emptyset/\mathcal{D}_\lambda$; i.e., $\Gamma(\mathbb{G}, \bar{\mathbb{G}})$ is non-stationary for some (and hence every) filtration $\langle \mathbb{G}_\alpha : \alpha < \lambda \rangle$ of $\mathbb{G}$.

§ 3. Representing a bipartite graph by Ext

Our main result in this section is Theorem 3.12. We start by recalling the following well-known result:

Lemma 3.1. *(See* [3, Ex. IV.22]*) If $\mathbb{G}$ is the union of a continuous chain $\{\mathbb{G}_\alpha : \alpha \leq \beta\}$ of abelian groups such that $\mathbb{G}_0$ and $\mathbb{G}_{\alpha+1}/\mathbb{G}_\alpha$ are $\aleph_1$-free for all $\alpha + 1 < \beta$, then $\mathbb{G}$ is $\aleph_1$-free.*

Definition 3.2. Let $\mathscr{G}$ be a set or class of Abelian groups.

(1) We say $\bar{\mathbb{L}}$ is a construction by $\mathscr{G}$ over $\mathbb{G}$ when:

  (a) $\bar{\mathbb{L}} = \langle \mathbb{L}_\varepsilon : \varepsilon \leq \varepsilon(*) \rangle$ is a $\subseteq$-increasing and continuous sequence of Abelian groups,

  (b) $\mathbb{L}_0 = \mathbb{G}$,

  (c) for every $\varepsilon < \varepsilon(*), \mathbb{L}_{\varepsilon+1}/\mathbb{L}_\varepsilon$ is free or is isomorphic to some member of $\mathscr{G}$.

(2) Omitting "over $\mathbb{G}$" means for $\mathbb{G} = \{0\}$.

(3) We say $\mathbb{L}$ is constructible by $\mathscr{G}$ (over $\mathbb{G}$) when for some $\bar{\mathbb{L}} = \langle \mathbb{L}_\varepsilon : \varepsilon \leq \varepsilon(*) \rangle$, $\bar{\mathbb{L}}$ is a construction by $\mathscr{G}$ (over $\mathbb{G}$) and $\mathbb{L} = \mathbb{L}_{\varepsilon(*)}$.

Notation 3.3. Suppose we have the following data of abelian groups and homomorphisms:

$$\begin{array}{ccc} \mathbb{A} & & \\ \uparrow & & \\ \mathbb{B} & \longrightarrow & \mathbb{C}, \end{array}$$

We denote the corresponding pushout by $\mathbb{A} \oplus_{\mathbb{B}} \mathbb{C}$.

The next lemma gives sufficient conditions for representing bipartite graphs using the functor Ext.

Lemma 3.4. *Let $\mu_1, \mu_2 \leq 2^\lambda$ be such that $cf(\mu_2) > \lambda$ and let $R \subseteq \mu_1 \times \mu_2$. Suppose the following assumptions are satisfied:*

$\boxtimes^1_{\lambda,\mu_1}$ *(a) $\bar{\mathbb{G}} = \langle \mathbb{G}_\alpha : \alpha < \mu_1 \rangle$ and $\bar{\mathbb{G}}^\iota = \langle \mathbb{G}^\iota_\alpha : \alpha < \mu_1 \rangle$ for $\iota = 1, 2$, are sequences of Abelian groups,*

(b) for each α, $\mathbb{G}_\alpha$ is an $\aleph_1$-free Abelian group of cardinality λ and $\mathbb{G}_\alpha = \mathbb{G}^2_\alpha / \mathbb{G}^1_\alpha$, where $\mathbb{G}^1_\alpha \subseteq \mathbb{G}^2_\alpha$ are free Abelian groups of cardinality λ,

(c) if $\alpha < \mu_1$ and $\mathbb{L}$ is constructible by $\{\mathbb{G}_\gamma : \gamma \in \mu_1 \setminus \alpha\}$ over $\mathbb{G}^1_\alpha$ then, there is no homomorphism $\mathbf{g}$ from $\mathbb{G}^2_\alpha$ into $\mathbb{L}$ extending $\mathrm{id}_{\mathbb{G}^1_\alpha}$.

Then we can find a sequence $\bar{\mathbb{K}}$ equipped with the following properties:

$\boxtimes^2_{\lambda,\mu_1}$ *(α) $\bar{\mathbb{K}} = \langle \mathbb{K}_\beta : \beta < \mu_2 \rangle$ is a sequence of $\aleph_1$-free Abelian groups each of cardinality 2^λ,*

(β) $\mathrm{Ext}(\mathbb{G}_\alpha, \mathbb{K}_\beta) = 0$ iff $\alpha R \beta$,

Proof. Let $\langle \mathscr{U}_\varepsilon : \varepsilon < 2^\lambda \rangle$ be a partition of 2^λ such that $\mathscr{U}_\varepsilon \subseteq [\varepsilon, 2^\lambda)$ has cardinality 2^λ. We claim that there are sequences $\bar{\mathbb{K}}_\varepsilon$, $\bar{\mathbf{H}}_{\varepsilon,\beta}$ and $\bar{\mathbf{h}}^*_{\varepsilon,\beta}$, for $\varepsilon \leq 2^\lambda$ and $\beta < \mu_2$ with the following properties:

$\oplus_\varepsilon$ (a) $\bar{\mathbb{K}}_\varepsilon = \langle \mathbb{K}_{\varepsilon,\beta} : \beta < \mu_2 \rangle$ is a sequence of Abelian groups,

(b) for each $\beta < \mu_2$ the sequence $\langle \mathbb{K}_{\zeta,\beta} : \zeta \leq 2^\lambda \rangle$ is $\subseteq$-increasing and continuous,

(c) $\mathbb{K}_{\varepsilon,\beta}$ has cardinality $\leq 2^\lambda$,

(d) $\bar{\mathbf{H}}_{\varepsilon,\beta} = \langle (\alpha_{\beta,\zeta}, h_{\beta,\zeta}) : \zeta \in \mathscr{U}_\varepsilon \rangle$ lists all the pairs (α, h) where $\alpha < \mu_1$, $\alpha R \beta$ and $h \in \mathrm{Hom}(\mathbb{G}^1_\alpha, \mathbb{K}_{\varepsilon,\beta})$,

(e) $\bar{\mathbf{h}}^*_{\varepsilon,\beta} = \langle h^*_{\beta,\zeta} : \zeta \in \mathscr{U}_\varepsilon \rangle$,

(f) if $\varepsilon \leq \zeta \in \mathscr{U}_\varepsilon$, then $h^*_{\beta,\zeta} \in \mathrm{Hom}(\mathbb{G}^2_{\alpha_{\beta,\zeta}}, \mathbb{K}_{\zeta+1,\beta})$ extends $h_{\beta,\zeta}$.

We proceed by a double induction on ζ and β to construct such a sequence. For $\zeta = 0$ and for all $\beta < \mu_2$ set

$$\mathbb{K}_{0,\beta} := \bigoplus \{\mathbb{G}^1_\alpha : \alpha < \mu_1, \neg(\alpha R \beta)\}.$$

For ζ a limit ordinal and $\beta < \mu_2$ set $\mathbb{K}_{\zeta,\beta} = \bigcup_{\varepsilon<\zeta} \mathbb{K}_{\varepsilon,\beta}$. Now suppose that the groups $K_{\varepsilon,\gamma}$ are defined for all $\varepsilon < \zeta + 1$ and $\gamma < \mu_2$. We define $\mathbb{K}_{\zeta+1,\beta}$ for $\beta < \mu_2$ as follows.

Let $\bar{\mathbf{H}}_{\varepsilon,\beta} = \langle (\alpha_{\beta,\zeta}, h_{\beta,\zeta}) : \zeta \in \mathscr{U}_\varepsilon \rangle$ be as in clause (d). We look at the following diagram

$$\begin{array}{ccc}
\mathbb{K}_{\zeta,\beta} & & \\
h_{\beta,\zeta}\Big\uparrow & & \\
\mathbb{G}^1_{\alpha_{\beta,\zeta}} & \xrightarrow{\subseteq} & \mathbb{G}^2_{\alpha_{\beta,\zeta}},
\end{array}$$

and set $\mathbb{K}_{\zeta+1,\beta} := \mathbb{K}_{\zeta,\beta} \oplus_{\mathbb{G}^1_{\alpha_{\beta,\zeta}}} \mathbb{G}^2_{\alpha_{\beta,\zeta}}$. In particular, there is a homomorphism f induced from $h_{\beta,\zeta}$ which commutes the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{K}_{\zeta,\beta} & \xrightarrow{g} & \mathbb{K}_{\zeta+1,\beta} & \longrightarrow & \frac{\mathbb{G}^2_{\alpha_{\beta,\zeta}}}{\mathbb{G}^1_{\alpha_{\beta,\zeta}}} & \longrightarrow & 0 \\
 & & h_{\beta,\zeta}\Big\uparrow & & f\Big\uparrow & & =\Big\uparrow & & \\
0 & \longrightarrow & \mathbb{G}^1_{\alpha_{\beta,\zeta}} & \longrightarrow & \mathbb{G}^2_{\alpha_{\beta,\zeta}} & \longrightarrow & \frac{\mathbb{G}^2_{\alpha_{\beta,\zeta}}}{\mathbb{G}^1_{\alpha_{\beta,\zeta}}} & \longrightarrow & 0
\end{array}$$

We set $h^*_{\beta,\zeta} := f$. By a diagram chasing, $h^*_{\beta,\zeta}$ extends $h_{\beta,\zeta}$, and recall that $g : \mathbb{K}_{\zeta,\beta} \hookrightarrow \mathbb{K}_{\zeta+1,\beta}$ is an embedding. This completes the inductive construction and hence proves the existence of the sequences claimed to exist above, i.e., the construction of $\oplus_\varepsilon$ is now complete.

For $\beta < \mu_2$ let

$$\mathbb{K}_\beta := \mathbb{K}_{2^\lambda,\beta}.$$

In view of Lemma 3.1, the above short exact sequence and the hypotheses, each $\mathbb{K}_\beta$ is an $\aleph_1$-free Abelian group of size 2^λ.

Let us check the item $\boxtimes^2_{\lambda,\mu_1}(\beta)$. First, suppose that $\alpha R \beta$ and let $h \in \mathrm{Hom}(\mathbb{G}^1_\alpha, \mathbb{K}_\beta)$. Since $|\mathbb{G}^1_\alpha| \leq \lambda < \mathrm{cf}(\mu_2)$, for some $\varepsilon < 2^\lambda$ we have $\mathrm{Rang}(h) \subseteq K_{\varepsilon,\beta}$, and hence we can assume that $h \in \mathrm{Hom}(\mathbb{G}^1_\alpha, \mathbb{K}_{\varepsilon,\beta})$. By the definition of $\bar{\mathbf{H}}_{\varepsilon,\beta}$, for some $\zeta \in \mathscr{U}_\varepsilon$ we have $(\alpha, h) = (\alpha_{\beta,\zeta}, h_{\beta,\zeta})$. Then by clause (f) of the construction, $h^*_{\beta,\zeta} \in \mathrm{Hom}(\mathbb{G}^2_\alpha, \mathbb{K}_{\zeta+1,\beta})$ extends $h_{\beta,\zeta}$. There is a natural embedding from $\mathrm{Hom}(\mathbb{G}^2_\alpha, \mathbb{G}_{\zeta+1,\beta})$

into $\mathrm{Hom}(\mathbb{G}^2_\alpha, \mathbb{K}_\beta)$, so without loss of generality, $h^*_{\beta,\zeta} \in \mathrm{Hom}(\mathbb{G}^2_\alpha, \mathbb{K}_\beta)$. Now, we look at the exact sequence

$$(+) := 0 \longrightarrow \mathbb{G}^1_\alpha \longrightarrow \mathbb{G}^2_\alpha \longrightarrow \mathbb{G}_\alpha \longrightarrow 0.$$

Applying $\mathrm{Hom}(-, \mathbb{K}_\beta)$ to $(+)$, it induces the following long exact sequence

$$\mathrm{Hom}(\mathbb{G}^2_\alpha, \mathbb{K}_\beta) \xrightarrow{\mathbf{f}} \mathrm{Hom}(\mathbb{G}^1_\alpha, \mathbb{K}_\beta) \longrightarrow \mathrm{Ext}(\mathbb{G}_\alpha, \mathbb{K}_\beta) \longrightarrow \mathrm{Ext}(\mathbb{G}^2_\alpha, \mathbb{K}_\beta) = 0,$$

where the last vanishing $\mathrm{Ext}(\mathbb{G}^2_\alpha, \mathbb{K}_\beta) = 0$ holds as $\mathbb{G}^2_\alpha$ is free. From this,

$$\mathrm{Ext}(\mathbb{G}_\alpha, \mathbb{K}_\beta) = \frac{\mathrm{Hom}(\mathbb{G}^1_\alpha, \mathbb{K}_\beta)}{\mathrm{Rang}(f)}.$$

Thus, $\mathrm{Ext}(\mathbb{G}_\alpha, \mathbb{K}_\beta) = 0$ if and only if $\mathbf{f}$ is surjective. But as we observed above, $\mathbf{f}$ is onto, and hence we conclude that $\mathrm{Ext}(\mathbb{G}_\alpha, \mathbb{K}_\beta) = 0$.

Let $(\alpha, \beta) \in (\mu_1 \times \mu_2) \setminus R$. We need to show $\mathrm{Ext}(\mathbb{G}_\alpha, \mathbb{K}_\beta) \neq 0$. Suppose on the contrary that $\mathrm{Ext}(\mathbb{G}_\alpha, \mathbb{K}_\beta) = 0$, and search for a contradiction. Since $\mathbb{G}^1_\alpha$ is a direct summand of $\mathbb{K}_{0,\beta} \subseteq \mathbb{K}_\beta$, we may assume that $g^1_\alpha = \mathrm{id}_{G^1_\alpha} \in \mathrm{Hom}(\mathbb{G}^1_\alpha, \mathbb{K}_\beta)$. So by our assumption, there is $g^2_\alpha : \mathbb{G}^2_\alpha \to \mathbb{K}_\beta$ extending g^1_α.

We now show that there exists $u \subseteq 2^\lambda$ equipped with:

(a) $|u| = \lambda$,

(b) $0, \alpha, \beta \in u$,

(c) $\varepsilon \in u \Rightarrow \mathrm{Rang}(h_{\beta,\varepsilon}) \subseteq \Sigma\{\mathrm{Rang}(h^*_{\beta,\zeta}) : \zeta \in u \cap \varepsilon\} + \sum\limits_{\gamma \in u} \mathbb{G}^1_\gamma$,

(d) $\mathrm{Rang}(g^2_\alpha) \subseteq \Sigma\{\mathrm{Rang}(h^*_{\beta,\varepsilon}) : \varepsilon \in u\} + \sum\limits_{\gamma \in u} \mathbb{G}^1_\gamma$.

Indeed, by induction on $i < \lambda$ we define an increasing and continuous sequence $\langle u_i : i < \lambda \rangle$ of subsets of 2^λ such that for each $i < \lambda$, the following properties are valid:

- $|u_i| \leq \lambda$,
- $0, \alpha, \beta \in u_0$,
- If i is limit ordinal, then $u_i = \bigcup_{j<i} u_j$,
- $\mathrm{Rang}(g^2_\alpha) \subseteq \Sigma\{\mathrm{Rang}(h^*_{\beta,\varepsilon}) : \varepsilon \in u_0\} + \sum\limits_{\gamma \in u_0} \mathbb{G}^1_\gamma$,
- If $\varepsilon \in u_i$ then $\mathrm{Rang}(h_{\beta,\varepsilon}) \subseteq \Sigma\{\mathrm{Rang}(h^*_{\beta,\zeta}) : \zeta \in u_{i+1} \cap \varepsilon\} + \sum\limits_{\gamma \in u_{i+1}} \mathbb{G}^1_\gamma$.

Set $u := \bigcup_{i<\lambda} u_i$. It is easily seen that u satisfies the required properties. Let $\varepsilon(*) = \mathrm{otp}(u)$, where $\mathrm{otp}(u)$ denotes the order type of the set u and let $\langle \gamma_\varepsilon : \varepsilon < \varepsilon(*) \rangle$ be the increasing enumeration of u.

Set

$$\mathscr{G} := \langle \mathbb{G}_\gamma : \gamma \in u \setminus \alpha \rangle.$$

We define a construction $\bar{\mathbb{L}} := \langle \mathbb{L}_\varepsilon : \varepsilon \leq \varepsilon(*) \rangle$ by $\mathscr{G}$ over $\mathbb{L}_0 = \mathbb{G}^1_\alpha$. To achieve this, we define

$$\mathbb{L}_{1+\epsilon} = \sum_{\zeta<\epsilon} \mathrm{Rang}(h^*_{\beta,\zeta}) + \sum_{\gamma\in u} \mathbb{G}^1_\gamma.$$

For each $\epsilon < \epsilon(\star)$ we have

$$\frac{\mathbb{L}_{\epsilon+1}}{\mathbb{L}_\epsilon} \cong \mathbb{G}_{\alpha_{\beta,\epsilon}},$$

which is in $\mathscr{G}$. This shows that $\bar{\mathbb{L}}$ is indeed a construction. Recall from (d) that $\mathrm{Rang}(g^2_\alpha) \subset \mathbb{L}_{\varepsilon(*)}$. Hence $g^2_\alpha : \mathbb{G}^2_\alpha \to \mathbb{L}_{\varepsilon(*)}$ extends the identity function over $\mathbb{G}^1_\alpha$. This is in contradiction with $\boxtimes^1_{\lambda,\mu_1}(c)$. □

Remark 3.5. Adopt the notation of Lemma 3.4 and suppose in addition that each $\mathbb{G}_\alpha$ is λ-free. In the same vein one can show that $\mathbb{K}_\beta$ is λ-free.

Notation 3.6. Let $\mathbb{G}$ be a reduced torsion free Abelian group, e.g., a free group. The notation $\widehat{\mathbb{G}}$ stands for the $\mathbb{Z}$-adic completion of $\mathbb{G}$.

Remark 3.7. i) $\mathbb{Z}$-adic topology of a free Abelian group is Hausdorff.

ii) For example, each element of $\widehat{\mathbb{Z}}$ can be represented as $\Sigma_n k_n$ where $n!|k_n$.

Lemma 3.8. *Let $\mathbb{G}$ be a free Abelian group with a base $\{e_n : n \in \mathbb{N}\}$. Let $c_0 := 0$. For each $n > 0$ there is a choose $c_n \in \{0, 1, -1\}$ such that:*

- (a) *not all of the c_n's are zero,*
- (b) $\sum_{n\geq 1} n! c_n e_n = 0 \in \widehat{\mathbb{G}}$.

Proof. For notational simplicity, we denote any element of $\mathbb{G}$ by $x^\oplus$ where $x^\oplus = (x_i)_{i<\omega}$ and all but finitely many of x_i's are zero. By definition,

$$\widehat{\mathbb{G}} = \varprojlim_n \mathbb{G}/n\mathbb{G} \subseteq \prod \mathbb{G}/n\mathbb{G}.$$

Thus, any element of $\widehat{\mathbb{G}}$ is represented by a sequence $(x_n^\oplus + n\mathbb{G})$. Let $a_n^\oplus \in \oplus\{0,1,-1\}$ be so that $\sum_{n<m} n!a_n^\oplus \in m!\mathbb{G} = \bigoplus_{n<\omega} m!\mathbb{Z}$ for all m. In other word, we have the following different representation of $0 \in \widehat{\mathbb{G}}$:

$$0 = (0 + m!\mathbb{G})_{m\in\mathbb{N}} = (\sum_{n<m} n!a_n^\oplus + m!\mathbb{G})_{m\in\mathbb{N}}.$$

In particular, there are c_n such that $\sum\limits_{n\geq 1} n!c_n e_n = 0$. □

Recall that for a stationary set $S \subseteq \lambda$, Jensen's diamond $\diamondsuit_\lambda(S)$ asserts the existence of a sequence $\langle S_\alpha : \alpha \in S\rangle$ such that for every $X \subseteq \lambda$ the set $\{\alpha \in S : X \cap \alpha = S_\alpha\}$ is stationary.

Fact 3.9. (See [6, Theorem 9.1.17]) If $\diamondsuit_\lambda(S)$ holds, then there is a decomposition $S = \bigcup_{\beta<\lambda} S_\beta$ such that $\diamondsuit_\lambda(S_\beta)$ holds for all $\beta < \lambda$.

The following lemma plays a key role in our construction.

Lemma 3.10. *(A) Let $\iota \in \{1,2\}$. Suppose $\mathbb{G}_n^\iota$ are free Abelian groups of the same size such that for all $n < \omega$ the following conditions are satisfied:*

(1) $\mathbb{G}_n^1 \subseteq \mathbb{G}_n^2$,

(2) $\mathbb{G}_n^\iota \subseteq \mathbb{G}_{n+1}^\iota$,

(3) $\mathbb{G}_{n+1}^\iota/\mathbb{G}_n^\iota$ *is free,*

(4) $\mathbb{G}_{n+1}^2/(\mathbb{G}_{n+1}^1 + \mathbb{G}_n^2)$ *is free,*

(5) $\mathbb{G}_{n+1}^2 = \big(\mathbb{G}_{n+1}^1 \oplus_{\mathbb{G}_n^1} \mathbb{G}_n^2\big) \oplus \bigoplus\limits_{n<\omega} y_n\mathbb{Z}$,

(6) $\bar{\mathbb{L}} = \langle \mathbb{L}_\varepsilon : \varepsilon \leq \varepsilon(*)\rangle$ *is a construction,*

(7) $\mathbb{L}_\varepsilon$ *and* $\mathbb{L}_{\varepsilon+1}/\mathbb{L}_\varepsilon$ *are* $\aleph_1$*-free.*

Let $\mathbb{G}^\iota = \bigcup\limits_{n<\omega} \mathbb{G}_n^\iota$, *so that* $\mathbb{G}^1 \subseteq \mathbb{G}^2$ *are free Abelian groups and let* $\mathbf{f} \in \mathrm{Hom}(\mathbb{G}^2, \mathbb{L}_{\varepsilon(*)})$. *Then there are free Abelain groups* $\dot{\mathbb{G}}^1 \subseteq \dot{\mathbb{G}}^2$ *such that:*

(a) $\mathbb{G}^1 \subseteq \dot{\mathbb{G}}^1$,

(b) $\mathbb{G}^2 \subseteq \dot{\mathbb{G}}^2$,

(c) $\mathbb{G}^1 = \mathbb{G}^2 \cap \dot{\mathbb{G}}^1$,

(d) $\dot{\mathbb{G}}^\iota/\mathbb{G}^\iota$ *is free,*

(e) $\dot{\mathbb{G}}^2/(\dot{\mathbb{G}}^1 + \mathbb{G}^2)$ *is torsion free,*

(f) for all $n < \omega$, $\dot{\mathbb{G}}^2/(\dot{\mathbb{G}}^1 + \mathbb{G}_n^2)$ *is free,*

(g) there are no $\dot{\mathbf{f}}$ *and* $\langle \dot{\mathbb{L}}_\varepsilon : \varepsilon \leq \varepsilon(*) \rangle$ *equipped with the following properties:*

(g1) $\langle \dot{\mathbb{L}}_\varepsilon : \varepsilon \leq \varepsilon(*) \rangle$ *is a nice construction over* $\dot{\mathbb{G}}^1$,

(g2) $\dot{\mathbf{f}} \in \mathrm{Hom}(\dot{\mathbb{G}}^2, \dot{\mathbb{L}}_{\varepsilon(*)})$,

(g3) $\dot{\mathbf{f}}$ *extends* $\mathbf{f} \cup \mathrm{id}_{\dot{\mathbb{G}}^1}$,

(g4) $\mathbb{L}_\varepsilon \subseteq \dot{\mathbb{L}}_\varepsilon$ *for* $\varepsilon \leq \varepsilon(*)$,

(g5) $\mathbb{L}_\varepsilon = \mathbb{L}_{\varepsilon+1} \cap \dot{\mathbb{L}}_\varepsilon$,

(g6) $\dot{\mathbb{L}}_{\varepsilon+1}/(\mathbb{L}_{\varepsilon+1} + \dot{\mathbb{L}}_\varepsilon)$ *is free.*

(B) Similar result holds if we replace "free" by "$\aleph_1$-free".

Proof. (A): Set $\dot{\mathbb{G}}^1 := \mathbb{G}^1 \oplus \bigoplus_{n<\omega} z_n\mathbb{Z}$, where $\bigoplus_{n<\omega} z_n\mathbb{Z}$ is a free Abelian group with a base given by $\{z_n : n < \omega\}$. To define $\dot{\mathbb{G}}^2$, let us first set

$$\mathbb{G}^{2,*} := \dot{\mathbb{G}}^1 \oplus_{\mathbb{G}^1} \mathbb{G}^2.$$

For any infinite sequence $\vec{a} = \langle a_n : n < \omega \rangle \in \prod_\omega 2 = {}^\omega 2$, we look at the following

$$\mathbb{G}^{2,\vec{a}} := \langle \mathbb{G}^{2,*} \cup \{\sum_{n<\omega} n!(y_n + a_n z_n)\}\rangle,$$

i.e., the subgroup of $\widehat{\mathbb{G}^{2,*}}$ generated by $\mathbb{G}^{2,*}$ and the distinguished element

$$\sum_{n<\omega} n!(y_n + a_n z_n) \in \widehat{\mathbb{G}^{2,*}}.$$

It is routine to see that the properties $(a)-(f)$ are satisfied with $\mathbb{G}^{2,\vec{a}}$. We are going to show that $\dot{\mathbb{G}}^2 = \mathbb{G}^{2,\vec{a}}$ satisfies in the remaining property (g) for a suitable choice of $\vec{a}$. Suppose on the way of contradiction that for each $\vec{a} = \langle a_n : n < \omega \rangle \in {}^\omega 2$ there is a counterexample $\dot{\mathbf{f}}_{\vec{a}}$ and $\langle \dot{\mathbb{L}}^{\vec{a}}_\varepsilon : \varepsilon \leq \varepsilon(*) \rangle$ to clause (g). In particular, $\dot{\mathbf{f}}_{\vec{a}} \in \mathrm{Hom}(\mathbb{G}^{2,\vec{a}}, \dot{\mathbb{L}}_{\varepsilon(*)})$, and it extends $\mathbf{f} \cup \mathrm{id}_{\dot{\mathbb{G}}^1}$.

For $\vec{a} \in {}^\omega 2$ and $m < \omega$ set

$$(+) \qquad y^{\vec{a}}_m := \sum_{n \geq m} \frac{n!}{m!}(y_n + a_n z_n).$$

Claim: Let $\vec{a}$, $\vec{b}$ and m be such that

- $m < \omega$,
- $a_\ell = b_\ell$ for all $\ell \leq m$,
- $\dot{\mathbf{f}}_{\vec{a}}(y^{\vec{a}}_\ell) = \dot{\mathbf{f}}_{\vec{b}}(y^{\vec{b}}_\ell)$ for all $\ell \leq n$.

Then $\vec{a} = \vec{b}$.

To prove the claim, we proceed by induction on $\ell < \omega$ and show that:

(i) $a_\ell = b_\ell$,

(ii) $\dot{\mathbf{f}}_{\vec{a}}(y^{\vec{a}}_\ell) = \dot{\mathbf{f}}_{\vec{b}}(y^{\vec{b}}_\ell)$.

The case $\ell \leq m$ follows from the assumption, so we may assume that $\ell > m$. Now suppose that $\ell = k+1$ and the result is true for all natural numbers less or equal to k. Let us evaluate $\dot{\mathbf{f}}_{\vec{a}}$ at $y^{\vec{a}}_m$ from (+), and recall that $\dot{\mathbf{f}}_{\vec{a}}$ extends $\mathbf{f} \cup \mathrm{id}_{\dot{\mathbb{G}}^1}$. It turns out that:

$$\dot{\mathbf{f}}_{\vec{a}}(y^{\vec{a}}_m) = \sum_{n \geq m} \frac{n!}{m!} \dot{\mathbf{f}}_{\vec{a}}(y_n) + a_n \dot{\mathbf{f}}_{\vec{a}}(z_n)) = \sum_{n \geq m} \frac{n!}{m!}(y_n + a_n z_n).$$

It immediately follows that for all $i < \omega$

$$(*) \qquad \dot{\mathbf{f}}_{\vec{a}}(y^{\vec{a}}_i) = (i+1)\dot{\mathbf{f}}_{\vec{a}}(y^{\vec{a}}_{i+1}) - (y_i + a_i z_i).$$

We apply $(*)$ at level $i = k$ and combine it with inductive hypothesis to observe that

$$\begin{aligned} 0 &= \dot{\mathbf{f}}_{\vec{a}}(y^{\vec{a}}_k) - \dot{\mathbf{f}}_{\vec{b}}(y^{\vec{b}}_k) \\ &= \ell\dot{\mathbf{f}}_{\vec{a}}(y^{\vec{a}}_\ell) - (y_{\ell-1} + a_{\ell-1}z_{\ell-1}) - \ell\dot{\mathbf{f}}_{\vec{a}}(y^{\vec{a}}_\ell) + (y_{\ell-1} + a_{\ell-1}z_{\ell-1}) \\ &= \ell(\dot{\mathbf{f}}_{\vec{a}}(y^{\vec{a}}_\ell) - \dot{\mathbf{f}}_{\vec{b}}(y^{\vec{b}}_\ell)). \end{aligned}$$

Since $\dot{\mathbb{L}}_{\varepsilon(*)}$ is torsion-free, it follows that $\dot{\mathbf{f}}_{\vec{a}}(y^{\vec{a}}_\ell) - \dot{\mathbf{f}}_{\vec{b}}(y^{\vec{b}}_\ell) = 0$, so (ii) is valid. In order to show (i) we apply $(*)$ at level $i = \ell$:

$$0 = \dot{\mathbf{f}}_{\vec{a}}(y^{\vec{a}}_\ell) - \dot{\mathbf{f}}_{\vec{b}}(y^{\vec{b}}_\ell) = (\ell+1)(\dot{\mathbf{f}}_{\vec{a}}(y^{\vec{a}}_{\ell+1}) - \dot{\mathbf{f}}_{\vec{b}}(y^{\vec{b}}_{\ell+1})) - (a_\ell - b_\ell)z_\ell,$$

i.e.,

$$(a_\ell - b_\ell)z_\ell = (\ell+1)(\dot{\mathbf{f}}_{\vec{a}}(y^{\vec{a}}_\ell) - \dot{\mathbf{f}}_{\vec{b}}(y^{\vec{b}}_\ell)).$$

It follows that $(\ell+1)|(a_\ell - b_\ell)$. Since $a_i, b_i \in \{0,1\}$, we must have $a_\ell - b_\ell = 0$. This completes the proof of the desired claim.

But by Lemma 3.8, we can find distinct $\vec{a}$ and $\vec{b}$ in ${}^\omega 2$ such that $a_0 = b_0$ and $\dot{\mathbf{f}}_{\vec{a}}(y^{\vec{a}}_0) = \dot{\mathbf{f}}_{\vec{b}}(y^{\vec{b}}_0)$. This contradicts the above Claim, so we are done.

(B): The proof of this is similar to (A) and we leave its straightforward modification to the reader. □

Theorem 3.11. *Let $S \subseteq S^{\lambda}_{\aleph_0}$ be stationary non-reflecting and suppose $\diamondsuit_S$ holds. Suppose one of the followings:*

(1) $\mu = \lambda = \mathrm{cf}(\lambda) > \aleph_0$, *or*

(2) $\mu = 2^\lambda, \lambda = \ cf(\lambda) > \aleph_0$.

Then there is a sequence $\bar{\mathbb{G}} = \langle \mathbb{G}_\alpha : \alpha < \mu \rangle$ of Abelian groups satisfying $\boxtimes^1_{\lambda,\mu}$ and each $\mathbb{G}_\alpha$ is strongly λ-free.

Proof. (1): Since S is non-reflecting, given any limit ordinal $\delta < \lambda$, there exists a club $C_\delta \subseteq \delta$ such that $S \cap C_\delta = \emptyset$. Furthermore as $\diamondsuit_S$ holds, and in the light of Fact 3.9 we can find a sequence $\langle S_\varepsilon : \varepsilon < \lambda \rangle$ of subsets of S such that the following two properties are satisfied:

- the sets S_ε are pairwise almost disjoint, i.e., for all $\varepsilon < \zeta < \lambda, S_\varepsilon \cap S_\zeta$ is a bounded subset of λ,
- $\diamondsuit_{S_\varepsilon}$ holds for $\varepsilon < \lambda$.

For $\varepsilon < \lambda$ let

$$\langle (f^\varepsilon_\alpha, \mathbb{H}^{1,\varepsilon}_\alpha, \mathbb{H}^{2,\varepsilon}_\alpha, \bar{\mathbb{L}}_\alpha) : \alpha \in S_\varepsilon \rangle$$

be such that:

- $f^\varepsilon_\alpha : \alpha \to \alpha$ is a function,
- $\mathbb{H}^{1,\varepsilon}_\alpha \subseteq \mathbb{H}^{2,\varepsilon}_\alpha$ are Abelian groups,
- the universe of $\mathbb{H}^{2,\varepsilon}_\alpha$ is α,
- $\bar{\mathbb{L}}_\alpha = \langle \mathbb{L}^\alpha_\varepsilon : \varepsilon \leq \alpha \rangle$ is a construction such that $\mathbb{L}_\alpha$ has universe α,
- each $\mathbb{L}_\varepsilon$ and $\mathbb{L}_{\varepsilon+1}/\mathbb{L}_\varepsilon$ is $\aleph_1$-free,
- for any tuple $(f, \mathbb{H}^1, \mathbb{H}^2, \bar{\mathbb{L}})$, where $f : \lambda \to \lambda$ is a function, $\mathbb{H}^1 \subseteq \mathbb{H}^2$ are Abelian groups where the universe of $\mathbb{H}^2$ is λ and $\bar{\mathbb{L}} = \langle \mathbb{L}_\varepsilon : \varepsilon \leq \kappa \rangle$ is a construction such that the universe of $\mathbb{L}_\kappa$ is κ and $\mathbb{L}_\varepsilon$ and $\mathbb{L}_{\varepsilon+1}/\mathbb{L}_\varepsilon$ are $\aleph_1$-free, then the following set

$$\{\alpha \in S_\varepsilon : (f \restriction \alpha, \mathbb{H}^1 \restriction \alpha, \mathbb{H}^2 \restriction \alpha, \bar{\mathbb{L}} \restriction \alpha) = (f^\varepsilon_\alpha, \mathbb{H}^{1,\varepsilon}_\alpha, \mathbb{H}^{2,\varepsilon}_\alpha, \bar{\mathbb{L}}_\alpha)\}$$

is stationary.

Given $\varepsilon < \lambda$ by induction on $\alpha < \lambda$, we choose the sequences $\bar{\mathbb{G}}^\iota_\varepsilon = \langle \mathbb{G}^\iota_{\alpha,\varepsilon} : \alpha < \lambda \rangle$, for $\iota = 1, 2$ such that:

⊛ (a) $\bar{G}^\iota_\varepsilon$ is an increasing and continuous sequence of Abelian groups $\mathbb{G}^\iota_{\alpha,\varepsilon} = (G^\iota_{\alpha,\varepsilon}, +, 0)$.

(b) $\mathbb{G}^\iota_{0,\varepsilon} = \{0\}$ and for $\alpha < \lambda$, the universe of $\mathbb{G}^2_{\alpha,\varepsilon}$, namely $G^2_{\alpha,\varepsilon}$ is an ordinal $\gamma_{\alpha,\varepsilon} < \lambda$.

(c) The following holds:

$(c1)$: $\mathbb{G}^1_{\alpha,\varepsilon} \subseteq \mathbb{G}^2_{\alpha,\varepsilon}$ are free.

$(c2)$: If $\alpha < \beta$ then $\mathbb{G}^2_{\alpha,\varepsilon} \cap \mathbb{G}^1_{\beta,\varepsilon} = \mathbb{G}^1_{\alpha,\varepsilon}$.

$(c3)$: If $\alpha < \beta$ and $\alpha \notin S_\varepsilon$, then $\mathbb{G}^2_\beta / (\mathbb{G}^1_\beta + \mathbb{G}^2_\alpha)$ is free.

(d) If $\alpha < \beta$ and $\alpha \notin S_\varepsilon$, then $\mathbb{G}^1_{\beta,\varepsilon}/\mathbb{G}^1_{\alpha,\varepsilon}$ and $\mathbb{G}^2_{\beta,\varepsilon}/\mathbb{G}^2_{\alpha,\varepsilon}$ are free.

(e) if $\delta \in S_\varepsilon$, then there are no $\dot{\mathbf{f}}$ and $\langle \dot{\mathbb{L}}_\zeta : \zeta \le \delta \rangle$ equipped with the following properties:

$(e1)$ $\langle \dot{\mathbb{L}}_\zeta : \zeta \le \delta \rangle$ is a nice construction over $\mathbb{G}^1_{\delta+1,\varepsilon}$,

$(e2)$ $\dot{\mathbf{f}} \in \mathrm{Hom}(\mathbb{G}^2_{\delta+1,\varepsilon}, \dot{\mathbb{L}}_\delta)$,

$(e3)$ $\dot{\mathbf{f}}$ extends $\mathbf{f}^\varepsilon_\delta \cup \mathrm{id}_{\mathbb{G}^1_{\delta+1,\varepsilon}}$,

$(e4)$ $\mathbb{L}^\delta_\zeta \subseteq \dot{\mathbb{L}}_\zeta$ for $\zeta \le \delta$,

$(e5)$ $\mathbb{L}^\delta_\zeta = \mathbb{L}^\delta_{\zeta+1} \cap \dot{\mathbb{L}}_\zeta$,

$(e6)$ $\dot{\mathbb{L}}_{\zeta+1}/(\mathbb{L}^\delta_{\zeta+1} + \dot{\mathbb{L}}_\zeta)$ is free.

For $\alpha = 0$, set $\mathbb{G}^1_{0,\varepsilon} = \mathbb{G}^2_{0,\varepsilon} = \{0\}$. For limit ordinal δ, set $\mathbb{G}^\iota_{\delta,\varepsilon} = \bigcup\limits_{\alpha<\delta} \mathbb{G}^\iota_{\alpha,\varepsilon}$. Let us show that items (c) and (d) continue to hold. By the induction hypothesis and for all $\alpha < \beta < \delta$ with $\alpha \notin S$, $\mathbb{G}^\iota_{\beta,\varepsilon}/\mathbb{G}^\iota_{\alpha,\varepsilon}$ and $\mathbb{G}^2_\beta/(\mathbb{G}^1_\beta + \mathbb{G}^2_\alpha)$ are free, and since we have a club C_δ of δ which is disjoint to S, it immediately follows that the groups $\mathbb{G}^\iota_{\delta,\varepsilon}$, $\mathbb{G}^\iota_{\delta,\varepsilon}/\mathbb{G}^\iota_{\alpha,\varepsilon}$ and $\mathbb{G}^2_{\delta,\varepsilon}/(\mathbb{G}^1_{\delta,\varepsilon} + \mathbb{G}^2_{\alpha,\varepsilon})$ are free for all $\alpha < \delta$ with $\alpha \notin S$.

Let $\iota = 1, 2$. Now suppose that $\delta < \lambda$ and we have defined the groups $\mathbb{G}^\iota_{\alpha,\varepsilon}$ and ordinals $\alpha \le \delta$. We would like to define the groups $\mathbb{G}^1_{\delta+1,\varepsilon}$ and $\mathbb{G}^2_{\delta+1,\varepsilon}$ so that the (a)-(e) continue to hold.

In the case $\delta \notin S_\varepsilon$, we set

$$\mathbb{G}^1_{\delta+1,\varepsilon} := \mathbb{G}^1_{\delta,\varepsilon}$$

and

$$\mathbb{G}^2_{\delta+1,\varepsilon} := (\mathbb{G}^1_{\delta+1,\varepsilon} \oplus_{\mathbb{G}^1_{\delta+1,\varepsilon}} \mathbb{G}^2_{\delta,\varepsilon}) \oplus \bigoplus_{n<\omega} y_{\delta,n}\mathbb{Z}.$$

It is not difficult to show that items (a)-(d) continue to hold and there is nothing to prove for case (e).

Now suppose that $\delta \in S_\varepsilon$. We define the groups $\mathbb{G}^1_{\delta+1,\varepsilon}$ and $\mathbb{G}^2_{\delta+1,\varepsilon}$ such that items (a)-(e) above continue to hold, and further we have:

- (f) $\mathbb{G}^2_{\delta+1,\varepsilon}/(\mathbb{G}^1_{\delta,\varepsilon} + \mathbb{G}^2_{\delta,\varepsilon})$ is not free,
- (g) if $\gamma \in \delta \setminus S_\varepsilon$, then $\mathbb{G}^2_{\delta+1,\varepsilon}/(\mathbb{G}^1_{\delta,\varepsilon} + \mathbb{G}^2_{\gamma,\varepsilon})$ is free.

As $\delta \in S_\varepsilon$, we have $\mathrm{cf}(\delta) = \aleph_0$, so let $\langle \gamma_{\delta,n} : n < \omega \rangle$ be an increasing sequence of successor ordinals $< \delta$ with limit δ.

For any $n < \omega, \gamma_{\delta,n} \notin S_\varepsilon$, it is easily seen that the Abelian groups $\mathbb{G}^1_{\gamma_{\delta,n},\varepsilon} \subseteq \mathbb{G}^2_{\gamma_{\delta,n},\varepsilon}$ satisfy the hypotheses of Lemma 3.10, hence by the lemma, we can find the groups $\mathbb{G}^1_{\delta+1,\varepsilon} \subseteq \mathbb{G}^2_{\delta+1,\varepsilon}$ such that there for all constructions $\mathbb{L} \supseteq \mathbb{G}^1_{\delta+1,\varepsilon}$ as described in (e), there is no $\mathbf{f} \in \mathrm{Hom}(\mathbb{G}^2_{\delta+1,\varepsilon}, \mathbb{L})$ such that $\mathbf{f} \supseteq f^\varepsilon_\delta \cup \mathrm{id}_{\mathbb{G}^1_{\delta+1,\varepsilon}}$.

This finishes our inductive construction. For $\iota = 1, 2$ and $\varepsilon < \lambda$, we define:

$$\mathbb{G}^\iota_\varepsilon = \bigcup_{\alpha<\lambda} \mathbb{G}^\iota_{\alpha,\varepsilon}.$$

Set also

(h1): $\mathbb{G}_{\alpha,\varepsilon} = \mathbb{G}^2_{\alpha,\varepsilon}/\mathbb{G}^1_{\alpha,\varepsilon}$,

(h2): $\mathbb{G}_\varepsilon = \bigcup_{\alpha<\lambda} \mathbb{G}_{\alpha,\varepsilon}$,

(h3): $\bar{\mathbb{G}} = \langle \mathbb{G}_\varepsilon : \varepsilon < \lambda \rangle$.

Let us show that $\boxtimes^1_{\lambda,\mu_1}$ is satisfied. By items (c) and (d) of the construction, $\mathbb{G}^1_\varepsilon \subseteq \mathbb{G}^2_\varepsilon$ are free and $\mathbb{G}_\varepsilon$ is strongly λ-free as witnessed by the sequence

$$\mathcal{S}_\varepsilon = \{\mathbb{G}_{\alpha,\varepsilon} : \alpha \in \lambda \setminus S_\varepsilon\}.$$

Let us now show that $\boxtimes^1_{\lambda,\mu_1}(c)$ is satisfied as well. Suppose by contradiction that there are $\varepsilon < \lambda$, a construction $\bar{\mathbb{L}} = \langle \mathbb{L}_\varepsilon : \varepsilon \leq \lambda \rangle$ by $\{\mathbb{G}_\alpha : \alpha \in \lambda \setminus \varepsilon\}$ over $\mathbb{G}^1_\varepsilon$ and there is a homomorphism $\mathbf{g}$ from $\mathbb{G}^2_\varepsilon$ into $\mathbb{L}_\lambda$ which extends $\mathrm{id}_{\mathbb{G}^1_\varepsilon}$. Without loss of generality we can assume that $\mathbb{L}_\lambda$ has size λ and that its universe is λ. The set

$$E = \{\delta < \lambda : \mathbf{g} \restriction \delta : \delta \to \delta \text{ and } \bar{\mathbb{L}} \restriction \delta = \bar{\mathbb{L}}_\delta \restriction \delta \text{ and } \mathbb{L}_\lambda \restriction \delta \text{ is a subgroup of } \mathbb{L}\}$$

is a club, thus we can find some $\delta \in E \cap S_\varepsilon$ such that:

(i1): $\mathbf{g} \restriction \delta = f^\varepsilon_\delta$,

(i2): $\mathbb{L}_\kappa \restriction \delta = \mathbb{H}^{2,\varepsilon}_\delta$,

(i3): $\mathbb{H}^{1,\varepsilon}_\delta = \mathbb{G}^1_{\delta,\varepsilon}$,

(i4): $\mathbb{G}^2_\varepsilon \cap \delta = \mathbb{G}^2_{\delta,\varepsilon}$.

Now note that $f = \mathbf{g} \restriction \mathbb{G}^2_{\delta+1,\varepsilon} : \mathbb{G}^2_{\delta+1,\varepsilon} \to \mathbb{L}_\lambda$ is such that $f \supseteq f^\varepsilon_\delta \cup \mathrm{id}_{\mathbb{G}^1_{\delta+1,\varepsilon}}$, which is in contradiction with clause (e) of the construction. Thus $\boxtimes^1_{\lambda,\mu_1}(c)$ is satisfied as well.

(2): The proof is similar to the proof of (1), this time, we find the following family

$$\langle S_\varepsilon : \varepsilon < 2^\lambda \rangle$$

of almost disjoint subsets of λ such that $\diamondsuit_{S_\varepsilon}$ holds for all ε. By [2], such a sequence exists. □

Theorem 3.12. *Adopt the assumptions of Theorem 3.11. Then there are sequences $\langle \mathbb{G}_\alpha : \alpha < \mu \rangle$ and $\langle \mathbb{K}_\alpha : \alpha < \mu \rangle$ of λ-free Abelian groups such that for all $\alpha < \mu, |\mathbb{G}_\alpha| = \lambda$, $|\mathbb{K}_\alpha| = 2^\lambda$ and for all $\alpha, \beta < \mu$,*

$$\mathrm{Ext}(\mathbb{G}_\alpha, \mathbb{K}_\beta) = 0 \Leftrightarrow \alpha R \beta.$$

Proof. This follows from Lemma 3.4 and Theorem 3.11. □

§ 4. Representing a bipartite graph by Ext in ZFC

In this section we show that it is possible to remove the diamond principle from the construction of Section 3 and get ZFC result. The main result is Theorem 4.4. This answers Herden's question for the case of bipartite graphs. We will do this by using a simple version of Shelah's black box. In this case, the groups $\mathbb{G}_\alpha$ that we construct are not λ-free, but just $\aleph_1$-free.

Let us start by stating the version of the black box we are using in this paper.

Theorem 4.1. *Let χ, λ and μ be infinite cardinals such that $\lambda = \mu^+, \mu^{\aleph_0} = \mu$, and $E_0, \ldots, E_{m-1}$ are pairwise disjoint stationary subsets of λ consisting of ordinals of cofinality ω, and $\chi > \lambda$. Let N be an expansion in a countable language of*

$(H(\chi), \in, \lhd, \lambda)$ where $H(\chi)$ is the collection of sets of hereditary cardinality less than χ and $\lhd$ is a well ordering of $H(\chi)$. Then there is a family of countable sets $\{(M_i, X_i) : i \in I\}$ such that the following properties hold:

(a) $M_i \prec N$ and $X_i \subset \lambda$.

(b) Let $\delta(i) := \sup(M_i \cap \lambda)$. If $\delta(i) = \delta(j)$, then $(M_i, X_i) \cong (M_j, X_j)$ and $M_i \cap M_j \cap \lambda$ is an initial segment of $M_i \cap \lambda$.

(c) For all $X \subset \lambda$, all $\ell < m$, the following set

$$\{\delta \in E_\ell : \exists i \text{ such that } \delta(i) = \delta \text{ and } (M_i, X_i) \equiv_{M_i \cap \lambda} (N, X)\}$$

is stationary in λ.

Proof. See [3, Page 444]. □

Notation 4.2. Given a torsion-free group $\mathbb{G}$ and a subgroup $\mathbb{H} \subseteq \mathbb{G}$. By $\mathbb{H}_*$ we mean the pure-closure, i.e., the smallest pure subgroup of $\mathbb{G}$ containing $\mathbb{H}$. In fact,

$$\mathbb{H}_* = \{g \in \mathbb{G} : ng \in \mathbb{H} \textit{ for some nonzero } n \in \mathbb{Z}\}.$$

Theorem 4.3. *Adopt one of the following assumptions:*

(1) If $\lambda = \lambda^{\aleph_0}, \mu = \lambda$,

(2) If $\lambda = \lambda^{\aleph_0}, \mu = 2^\lambda$.

Then $\boxtimes^1_{\lambda,\mu}$ from Lemma 3.4 holds.

Proof. (1): We are going to use a result of Solovay (see [3, Corollary II.4.9]). This enable us to find a partition $\langle S_\varepsilon : \varepsilon < \lambda \rangle$ of λ into λ many disjoint stationary sets. Let $\mathbb{C}_\varepsilon := \bigoplus_{\nu<\lambda, n<\omega} (y_{\varepsilon,\nu,n}\mathbb{Z} \oplus z_{\varepsilon,\nu,n}\mathbb{Z})$ and and recall that its $\mathbb{Z}$-adic completion denoted by $\widehat{\mathbb{C}_\varepsilon}$. Clearly, $\widehat{\mathbb{C}_\varepsilon}$ has cardinality λ, so we identify it with λ. Let also

$$Y_\varepsilon := \{y_{\varepsilon,\nu,n} : \nu < \lambda, n < \omega\} \cup \{z_{\varepsilon,\nu,n} : \nu < \lambda, n < \omega\}.$$

Set $\chi = \lambda^{+3}$. The initial structure for the Black box, corresponding to the stationary set S_ε is as follows:

$$N_\varepsilon := (\mathscr{H}(\chi), \in, \lhd, \lambda, \widehat{C_\varepsilon}, Y_\varepsilon),$$

where $\widehat{C_\varepsilon}$ denotes the 3-ary relation on λ which is the graph of the addition operation on the group $\widehat{\mathbb{C}_\varepsilon}$. We take the first bijection $\mathbf{g}_\varepsilon : \lambda \times \lambda \to \lambda$ with respect to $\lhd$, and use it to identify each $X \subseteq \lambda$ with a subset of $\widehat{C_\varepsilon} \times \widehat{C_\varepsilon}$. Let $\{(M_i^\varepsilon, X_i^\varepsilon) : i \in I_\varepsilon\}$ be as in the statement of Theorem 4.1 when $m = 1$ and $E_0 = S_\varepsilon$, and note that for each $i \in I_\varepsilon$ and $\varepsilon < \lambda$, $\mathbf{g}_\varepsilon \in M_i^\varepsilon$.

Let $\iota = 1, 2$. We proceed as in the previous section and for a given $\varepsilon < \lambda$, by induction on $\alpha < \lambda$, we choose the sequences $\bar{\mathbb{G}}_\varepsilon^\iota = \langle \mathbb{G}_{\alpha,\varepsilon}^\iota : \alpha < \lambda \rangle$ equipped with the following five items:

(a) $\bar{\mathbb{G}}_\varepsilon^\iota$ is an increasing and continuous sequence of Abelian groups $\mathbb{G}_{\alpha,\varepsilon}^\iota = (G_{\alpha,\varepsilon}^\iota, +, 0)$.

(b) $\mathbb{G}_{0,\varepsilon}^\iota = \{0\}$ and for $\alpha < \lambda$, the universe of $\mathbb{G}_{\alpha,\varepsilon}^2$, namely $G_{\alpha,\varepsilon}^2$ is an ordinal $\gamma_{\alpha,\varepsilon} < \lambda$.

(c) The following three properties hold:

(c1): $\mathbb{G}_{\alpha,\varepsilon}^1 \subseteq \mathbb{G}_{\alpha,\varepsilon}^2$ are $\aleph_1$-free,

(c2): if $\alpha < \beta$ then $\mathbb{G}_{\alpha,\varepsilon}^2 \cap \mathbb{G}_{\beta,\varepsilon}^1 = \mathbb{G}_{\alpha,\varepsilon}^1$,

(c3): if $\alpha < \beta$ and $\alpha \notin S_\varepsilon$, then $\mathbb{G}_\beta^2/(\mathbb{G}_\beta^1 + \mathbb{G}_\alpha^2)$ is $\aleph_1$-free.

(d) If $\alpha < \beta$ and $\alpha \notin S_\varepsilon$, then $\mathbb{G}_{\beta,\varepsilon}^1/\mathbb{G}_{\alpha,\varepsilon}^1$ and $\mathbb{G}_{\beta,\varepsilon}^2/\mathbb{G}_{\alpha,\varepsilon}^2$ are $\aleph_1$-free.

(e) If $\delta \in S_\varepsilon$, then there are no f and $\langle \dot{\mathbb{L}}_\zeta : \zeta \leq \delta \rangle$ equipped with the following properties:

(e1) $\langle \dot{\mathbb{L}}_\zeta : \zeta \leq \delta \rangle$ is a nice construction over $\mathbb{G}_{\delta+1,\varepsilon}^1$,

(e2) $f \in \mathrm{Hom}(\mathbb{G}_{\delta+1,\varepsilon}^2, \dot{\mathbb{L}}_\delta)$,

(e3) f extends $f_{\delta,i}^\varepsilon \cup \mathrm{id}_{\mathbb{G}_{\delta+1,\varepsilon}^1}$, where $i \in I_\varepsilon$ and

$$\langle f_{\delta,i}^\varepsilon, \bar{\mathbb{L}}_{\delta,i}^\varepsilon = \langle \mathbb{L}_{\delta,i,\zeta}^\varepsilon : \zeta \leq \delta \rangle \rangle$$

is coded by X_i^ε, under the identification given by $\mathbf{g}_\varepsilon$. Also, $f_{\delta,i}^\varepsilon$ is in $\mathrm{Hom}(\mathbb{G}_{\delta,\varepsilon}^2, \mathbb{L}_{\delta,i,\delta}^\varepsilon)$ and $\bar{\mathbb{L}}_{\delta,i}^\varepsilon$ is a construction,

(e4) $\mathbb{L}_\zeta^\delta \subseteq \dot{\mathbb{L}}_\zeta$ for $\zeta \leq \delta$,

(e5) $\mathbb{L}_\zeta^\delta = \mathbb{L}_{\zeta+1}^\delta \cap \dot{\mathbb{L}}_\zeta$,

(e6) $\dot{\mathbb{L}}_{\zeta+1}/(\mathbb{L}_{\zeta+1}^\delta + \dot{\mathbb{L}}_\zeta)$ is free.

For $\alpha = 0$, set $\mathbb{G}^1_{0,\varepsilon} = \mathbb{G}^2_{0,\varepsilon} = \{0\}$. For the limit ordinal δ, we set $\mathbb{G}^\iota_{\delta,\varepsilon} := \bigcup_{\alpha<\delta} \mathbb{G}^\iota_{\alpha,\varepsilon}$. Let us show that items (c) and (d) continue to hold. By the induction hypothesis and for all $\alpha < \beta < \delta$ with $\alpha \notin S$, $\mathbb{G}^\iota_{\beta,\varepsilon}/\mathbb{G}^\iota_{\alpha,\varepsilon}$ and $\mathbb{G}^2_\beta/(\mathbb{G}^1_\beta + \mathbb{G}^2_\alpha)$ are $\aleph_1$-free, and since we have a club C_δ of δ which is disjoint to S, it immediately follows from Lemma 3.1 that the groups $\mathbb{G}^\iota_{\delta,\varepsilon}$, $\mathbb{G}^\iota_{\delta,\varepsilon}/\mathbb{G}^\iota_{\alpha,\varepsilon}$ and $\mathbb{G}^2_{\delta,\varepsilon}/(\mathbb{G}^1_{\delta,\varepsilon} + \mathbb{G}^2_{\alpha,\varepsilon})$ are $\aleph_1$-free for all $\alpha < \delta$ with $\alpha \notin S_\varepsilon$.

Now suppose that $\delta < \lambda$ and we have defined the groups $\mathbb{G}^\iota_{\alpha,\varepsilon}$ for $\iota = 1, 2$ and ordinals $\alpha \leq \delta$. We would like to define the groups $\mathbb{G}^1_{\delta+1,\varepsilon}$ and $\mathbb{G}^2_{\delta+1,\varepsilon}$ so that the (a)-(e) continue to hold.

If $\delta \notin S_\varepsilon$, then we set

$$\mathbb{G}^1_{\delta+1,\varepsilon} := \mathbb{G}^1_{\delta,\varepsilon}$$

and

$$\mathbb{G}^2_{\delta+1,\varepsilon} := (\mathbb{G}^1_{\delta+1,\varepsilon} \oplus_{\mathbb{G}^1_{\delta+1,\varepsilon}} \mathbb{G}^2_{\delta,\varepsilon}) \oplus \bigoplus_{n<\omega} y_{\varepsilon,\delta,n}\mathbb{Z}.$$

We leave to the reader to check that the items presented from (a) to (d) all are valid, and recall that there is nothing to prove for case (e).

Now suppose that $\delta \in S_\varepsilon$. We define the groups $\mathbb{G}^1_{\delta+1,\varepsilon}$ and $\mathbb{G}^2_{\delta+1,\varepsilon}$ such that items (a)-(e) above continue to hold, and further we have:

- (f) $\mathbb{G}^2_{\delta+1,\varepsilon}/(\mathbb{G}^1_{\delta,\varepsilon} + \mathbb{G}^2_{\delta,\varepsilon})$ is not free,
- (g) if $\gamma \in \delta \setminus S_\varepsilon$, then $\mathbb{G}^2_{\delta+1,\varepsilon}/(\mathbb{G}^1_{\delta,\varepsilon} + \mathbb{G}^2_{\gamma,\varepsilon})$ is $\aleph_1$-free.

As $\delta \in S_\varepsilon$, we have $\mathrm{cf}(\delta) = \aleph_0$, so let $\langle \gamma_{\delta,n} : n < \omega \rangle$ be an increasing sequence of successor ordinals $< \delta$ with limit δ.

Let $n < \omega$ be such that $\gamma_{\delta,n} \notin S_\varepsilon$. It turns out that the Abelian groups $\mathbb{G}^1_{\gamma_{\delta,n},\varepsilon} \subseteq \mathbb{G}^2_{\gamma_{\delta,n},\varepsilon}$ are suited well in the hypotheses of Lemma 3.10.

The notation $\Sigma_{\delta,\varepsilon}$ stands for the following:

$$\{i \in I_\varepsilon : \delta(i) = \delta \textit{ and } X^\varepsilon_i \textit{ codes } \langle f^\varepsilon_{\delta,i}, \bar{\mathbb{L}}^\varepsilon_{\delta,i} = \langle \mathbb{L}^\varepsilon_{\delta,i,\zeta} : \zeta \leq \delta \rangle \rangle \textit{ as in } (e3)\}.$$

Given any $i \in \Sigma_{\delta,\varepsilon}$, and according to Lemma 3.10, we can find the $\aleph_1$-free groups $\mathbb{G}^{1,i}_{\delta+1,\varepsilon} \subseteq \mathbb{G}^{2,i}_{\delta+1,\varepsilon}$ such that there for all constructions $\langle \dot{\mathbb{L}}_\zeta : \zeta \leq \delta \rangle$ over $\mathbb{G}^{1,i}_{\delta+1,\varepsilon}$ as in item (e), if we set $\mathbb{L} = \dot{\mathbb{L}}_\delta$, then there is no $\mathbf{f} \in \mathrm{Hom}(\mathbb{G}^{2,i}_{\delta+1,\varepsilon}, \mathbb{L})$ such that

$\mathbf{f} \supseteq f^{\varepsilon}_{\delta,i} \cup \mathrm{id}_{\mathbb{G}^{1,i}_{\delta+1,\varepsilon}}$. For $\iota = 1, 2$ we look at

$$\mathbb{G}^{\iota}_{\delta+1,\varepsilon} := \langle \mathbb{G}^{\iota}_{\delta,\varepsilon} \cup \bigcup \{\mathbb{G}^{\iota,i}_{\delta+1,\varepsilon} : i \in \Sigma_{\delta,\varepsilon}\}\rangle_{*},$$

i.e., the pure closure of $\langle \mathbb{G}^{\iota}_{\delta,\varepsilon} \cup \bigcup \{\mathbb{G}^{\iota,i}_{\delta+1,\varepsilon} : i \in \Sigma_{\delta,\varepsilon}\}\rangle$ in $\widehat{\mathbb{C}_{\varepsilon}}$. Let us show that the hypothesis (a)-(e) hold. We first show that the group $\frac{\mathbb{G}^2_{\delta+1,\varepsilon}}{\mathbb{G}^2_{\alpha,\varepsilon}}$ is $\aleph_1$-free, provided that α is a successor ordinal. Let $\mathbb{K}$ be any countable subgroup of $\mathbb{G}^2_{\delta+1,\varepsilon}$. We are going to show that $\frac{\mathbb{K}+\mathbb{G}^2_{\delta+1,\varepsilon}}{\mathbb{G}^2_{\alpha,\varepsilon}}$ is free. There is an ω-sequence $\{i_m : m \in \omega\}$ together with a countable subgroup $\mathbb{I} \subset \mathbb{G}^2_{\delta,\varepsilon}$ such that $\mathbb{K}$ is the subgroup generated by $\mathbb{I}$ together with some countable subset $\{w_{n,i_m} : n, m \in \omega\}$ of $\bigcup\limits_{i\in\Sigma_{\delta,\varepsilon}} \mathbb{G}^{\iota,i}_{\delta+1,\varepsilon}$. We can assume that for all n, m, $y_{\varepsilon,\alpha_{n,i_m}} \in \mathbb{I}$. Choose an increasing sequence of ordinals $\{\alpha_k : k < \omega\}$ with limit δ such that $\alpha_0 = \alpha$ and for all $m \in \omega$ and all but finitely many of n, $\alpha_{n,i_m} \in \{\alpha_k : k < \omega\}$. Notice that for any successor ordinal $\gamma < \delta$

$$\mathbb{I}/(\mathbb{I} \cap \mathbb{G}^2_{\gamma,\varepsilon}) \cong (\mathbb{I} + \mathbb{G}^2_{\gamma,\varepsilon})/\mathbb{G}^2_{\gamma,\varepsilon}$$

which is free by the induction hypothesis. So for such γ, $\mathbb{I}\cap\mathbb{G}^2_{\gamma,\varepsilon}$ is a direct summand of $\mathbb{I}$. Inductively choose subgroups $\mathbb{I}_k$ so that:

$$\mathbb{I} \cap \mathbb{G}^2_{\alpha_k+1,\varepsilon} \oplus \mathbb{I}_k = \mathbb{I} \cap \mathbb{G}^2_{\alpha_{k+1},\varepsilon}$$

for all k. Hence

$$\mathbb{I} = \bigoplus_k \mathbb{I}_k \oplus \bigoplus \mathbb{Z} y_{\varepsilon,\alpha_k,n}.$$

In view of Theorem 4.1(b), we are able to choose $\{n(m) : m < \omega\}$ so that:

- the collections $\{\alpha_{n,i_m} : n(m) < n\}$ are pairwise disjoint.
- $\{\alpha_{n,i_m} : n(m) < n\} \subset \{\alpha_k : k < \omega\}$.

We observe that $\frac{\mathbb{K}+\mathbb{G}^2_{\delta,\varepsilon}}{\mathbb{G}^2_{\delta,\varepsilon}}$ is isomorphic to the direct sum of $\bigoplus_k \mathbb{I}_k$ together with the group freely generated by

$$\{w_{n,i_m} : n(m) \leq n \textit{ and } m \in \omega\}$$

and

$$\{y_{\varepsilon,\alpha_k,n} : \forall m < \omega \text{ and } n(m) \leq n, \alpha_k \neq \alpha_{n,i_m}\}.$$

From this, the claim follows. By a similar argument, the group $\frac{\mathbb{G}^1_{\delta+1,\varepsilon}}{\mathbb{G}^1_{\alpha,\varepsilon}}$ is $\aleph_1$-free, provided that α is a successor ordinal.

In the same vein, we also observe that the group $\mathbb{G}^1_{\delta+1,\varepsilon}$ is $\aleph_1$-free and the hypotheses (a)-(e) continue to hold.

The rest of the argument is similar to Theorem 3.11. Let us elaborate the main idea of the proof. For $\iota = 1, 2$ and $\varepsilon < \lambda$, we define:

$$\mathbb{G}^\iota_\varepsilon = \bigcup_{\alpha<\lambda} \mathbb{G}^\iota_{\alpha,\varepsilon}.$$

Set also

(h1): $\mathbb{G}_{\alpha,\varepsilon} = \mathbb{G}^2_{\alpha,\varepsilon}/\mathbb{G}^1_{\alpha,\varepsilon}$,

(h2): $\mathbb{G}_\varepsilon = \bigcup\limits_{\alpha<\lambda} \mathbb{G}_{\alpha,\varepsilon}$,

(h3): $\bar{\mathbb{G}} = \langle \mathbb{G}_\varepsilon : \varepsilon < \lambda \rangle$.

Let us show that $\boxtimes^1_{\lambda,\mu_1}$ is satisfied. By items (c) and (d) of the construction, $\mathbb{G}^1_\varepsilon \subseteq \mathbb{G}^2_\varepsilon$ are $\aleph_1$-free and $\mathbb{G}_\varepsilon$ is $\aleph_1$-free as well. Let us now show that $\boxtimes^1_{\lambda,\mu_1}(c)$ is satisfied as well. Suppose by contradiction that there are $\varepsilon < \lambda$, a construction $\mathbb{L}$ by $\{\mathbb{G}_\alpha : \alpha \in \lambda \setminus \varepsilon\}$ over $\mathbb{G}^1_\varepsilon$, witnessed by $\langle \mathbb{L}_\varepsilon : \varepsilon \leq \lambda \rangle$, and there is a homomorphism $\mathbf{g}$ from $\mathbb{G}^2_\varepsilon$ into $\mathbb{L}$ which extends $\mathrm{id}_{\mathbb{G}^1_\varepsilon}$. Without loss of generality we can assume that $\mathbb{L}$ has size λ and that its universe is λ. Let $X \subseteq \lambda$ code $\langle \mathbf{g}, \langle \mathbb{L}_\varepsilon : \varepsilon \leq \lambda \rangle \rangle$. The set

$$E = \{\delta < \lambda : \mathbf{g} \restriction \delta : \delta \to \delta,\ X \cap \delta \text{ codes } \langle \mathbf{g} \restriction \delta, \langle \mathbb{L}_\zeta : \zeta \leq \delta \rangle \rangle \text{ and } \mathbb{L} \restriction \delta \text{ is a subgroup of } \mathbb{L}\}$$

is a club, thus we can find some $\delta \in E \cap S_\varepsilon$ and some i with $\delta(i) = i$ such that $(M^\varepsilon_i, X^\varepsilon_i) \equiv_{M^\varepsilon_i \cap \lambda} (N_\varepsilon, X)$. It then follows that $M^\varepsilon_i \cap \mathbb{G}^2_\varepsilon = M^\varepsilon_i \cap \mathbb{G}^{2,i}_{\delta,\varepsilon}$, and since $(M^\varepsilon_i, X^\varepsilon_i) \equiv_{M^\varepsilon_i \cap \lambda} (N_\varepsilon, X)$, we can easily observe that X^ε_i codes

$$\langle \mathbf{g} \restriction M^\varepsilon_i \cap \mathbb{G}^{2,i}_{\delta,\varepsilon}, \langle \mathbb{L}_\zeta \cap M^\varepsilon_i : \zeta \leq \lambda \rangle \rangle.$$

Now by elementarily and the choice of $\mathbf{g}$,

$$(M^\varepsilon_i, X^\varepsilon_i) \models \text{“}X^\varepsilon_i \text{ codes a homomorphism } f^\varepsilon_{\delta,i} \text{ from } \mathbb{G}^{2,i}_{\delta,\varepsilon}.$$

It follows that $i \in \Sigma_{\delta,\varepsilon}$, and hence by our construction, there is no $\mathbf{f} \in \mathrm{Hom}(\mathbb{G}^{2,i}_{\delta+1,\varepsilon}, \mathbb{L})$ such that $\mathbf{f}$ extends $f^\varepsilon_{\delta,i} \cup \mathrm{id}_{\mathbb{G}^{1,i}_{\delta+1,\varepsilon}}$. This is not possible, as $\mathbf{g}$ is such an extension. This shows that there is no homomorphism $\mathbf{g}$ as above and the result follows.

(2): This is similar to the proof of (1). Take a sequence $\langle S_\varepsilon : \varepsilon < 2^\lambda \rangle$ of almost disjoint subsets of λ such that each S_ε is stationary and proceed as before. □

Now, we are ready to prove Theorem A) from introduction:

Theorem 4.4. *Let $\lambda = \lambda^{\aleph_0}, \mu = 2^\lambda$ and let $R \subseteq \mu \times \mu$ be a relation. Then there are families $\langle \mathbb{G}_\alpha : \alpha < \mu \rangle$, and $\langle \mathbb{K}_\alpha : \alpha < \rangle$ of $\aleph_1$-free Abelian groups such that:*

(1) for all $\alpha < \mu, \mathbb{G}_\alpha$ has size λ and $\mathbb{K}_\alpha$ has size 2^λ,

(2) for all $\alpha, \beta < \mu$,

$$\mathrm{Ext}(\mathbb{G}_\alpha, \mathbb{K}_\beta) = 0 \iff \alpha R \beta.$$

Proof. This follows by Lemma 3.4 and Theorem 4.3. □

§ 5. Representing A General Graph by Ext

In this section we consider general graphs and discuss if they can be represented by Ext as before. We do not know the result in ZFC, but we show the following consistency result which shows that there are no restrictions on such graphs in ZFC, as promised by Theorem B) from introduction:

Theorem 5.1. *Suppose GCH holds and the pair (S, R) is a graph where $R \subseteq S \times S$, and let $\lambda > |S|$ be an uncountable regular cardinal. Then there exists a cardinal preserving generic extension of the universe, and there is a family $\{\mathbb{G}_s\}_{s \in S}$ of λ-free Abelian groups such that*

$$\mathrm{Ext}(\mathbb{G}_s, \mathbb{G}_t) = 0 \iff sRt.$$

The rest of this section is devoted to the proof of the above theorem. Before we go into the details, let us sketch the idea of the proof:

Discussion 5.2. We first define a forcing notion $\mathbb{P}_*$ which adds a sequence $\langle \mathbb{G}_s : s \in S \rangle$ of λ-free Abelian groups of size λ such that for each $s, t \in S$ if $(s,t) \notin R$, then for some Abelian group $\mathbb{H}_{s,t}$ of size λ there exists an exact sequence

$$0 \longrightarrow \mathbb{G}_s \longrightarrow \mathbb{H}_{s,t} \longrightarrow \mathbb{G}_t \longrightarrow 0$$

which does not split. We apply this along with Fact 2.1 to deduce that $\mathrm{Ext}(\mathbb{G}_s, \mathbb{G}_t) \neq 0$. Then working in the generic extension by $\mathbb{P}_*$, we define a cardinal preserving λ-support iteration forcing notion of size λ^+, which makes $\mathrm{Ext}(\mathbb{G}_s, \mathbb{G}_t) = 0$ for all $s, t \in S$ with sRt. This is done by adding a splitter for any exact sequence

$$0 \longrightarrow \mathbb{G}_s \longrightarrow \mathbb{H} \longrightarrow \mathbb{G}_t \longrightarrow 0,$$

where $\mathbb{H}$ is an Abelian group of size λ. By using a suitable book-keeping argument, we make sure that at the end all such exact sequences are considered for all pairs $(s,t) \in R$. We also show that the exact sequence

$$0 \longrightarrow \mathbb{G}_s \longrightarrow \mathbb{H}_{s,t} \longrightarrow \mathbb{G}_t \longrightarrow 0$$

still fails to split after the iteration, which will complete the proof.

Let us now go into the details of the proof.

Notation 5.3. Let $\Phi : \lambda^+ \to \mathscr{H}(\lambda^+)$ be such that $\Phi^{-1}[x] \subseteq \lambda^+$ is unbounded for all $x \in \mathscr{H}(\lambda^+)$.

The existence of Φ follows by the GCH assumption. We will use Φ as our book-keeping function. Here, we define the forcing notion $\mathbb{P}_*$.

Definition 5.4. (a) The forcing notion $\mathbb{P}_*$ consists of conditions

$$p = \langle \langle \mathbb{G}^p_{s,\beta} : s \in S, \beta \leq \alpha_p \rangle, E_p, \langle \mathbf{x}^p_{s,t,\beta} : (s,t) \notin R, \beta \leq \alpha_p \rangle \rangle,$$

where

1. $\alpha_p < \lambda$ is an ordinal,
2. for each $s \in S$, $\langle \mathbb{G}^p_{s,\beta} : \beta \leq \alpha_p \rangle$ is an increasing and continuous sequence of free Abelian groups from $\mathscr{H}(\lambda)$,
3. $E_p \subseteq (\alpha_p + 1) \cap S^\lambda_{\aleph_0}$ does not reflect,
4. $\mathbb{G}^p_{s,\beta}/\mathbb{G}^p_{s,\gamma}$ is free when $\gamma < \beta < \alpha_p, \gamma \notin E_p$,
5. if $(s,t) \notin R$, then

$$\mathbf{x}^p_{s,t,\beta} := 0 \longrightarrow \mathbb{G}^p_{t,\beta} \xrightarrow{f^p_{s,t,\beta}} \mathbb{H}^p_{s,t,\beta} \xrightarrow{g^p_{s,t,\beta}} \mathbb{G}^p_{s,\beta} \longrightarrow 0$$

is an exact sequence, all increasing with $\beta \leq \alpha_p$, from $\mathscr{H}(\lambda)$:

$$\begin{array}{ccccccccc}
\mathbf{x}^p_{s,t,\alpha_p} := 0 & \longrightarrow & \mathbb{G}^p_{t,\alpha_p} & \xrightarrow{f^p_{s,t,\alpha_p}} & \mathbb{H}^p_{s,t,\alpha_p} & \xrightarrow{g^p_{s,t,\alpha_p}} & \mathbb{G}^p_{s,\alpha_p} & \longrightarrow & 0 \\
 & & \subseteq\uparrow & & \subseteq\uparrow & & \subseteq\uparrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \subseteq\uparrow & & \subseteq\uparrow & & \subseteq\uparrow & & \\
\mathbf{x}^p_{s,t,1} := 0 & \longrightarrow & \mathbb{G}^p_{t,1} & \xrightarrow{f^p_{s,t,1}} & \mathbb{H}^p_{s,t,1} & \xrightarrow{g^p_{s,t,1}} & \mathbb{G}^p_{s,1} & \longrightarrow & 0 \\
 & & \subseteq\uparrow & & \subseteq\uparrow & & \subseteq\uparrow & & \\
\mathbf{x}^p_{s,t,0} := 0 & \longrightarrow & \mathbb{G}^p_{t,0} & \xrightarrow{f^p_{s,t,0}} & \mathbb{H}^p_{s,t,0} & \xrightarrow{g^p_{s,t,0}} & \mathbb{G}^p_{s,0} & \longrightarrow & 0.
\end{array}$$

(b) Given $p, q \in \mathbb{P}_*$, let $p \leq q$ (q is stronger than p) when

(1) $\alpha_p \leq \alpha_q$,

(2) for all $s \in S$ and $\beta \leq \alpha_p$, $\mathbb{G}^q_{s,\beta} = \mathbb{G}^p_{s,\beta}$,

(3) $E^q \cap (\alpha_p + 1) = E^p$,

(4) If $(s,t) \notin R$ and $\beta \leq \alpha_p$, then $\mathbf{x}^q_{s,t,\beta} = \mathbf{x}^p_{s,t,\beta}$.

The proof of the next lemma is easy.

Lemma 5.5. *Adopt the above notation. Then $\mathbb{P}_*$ is λ-closed and λ^+-c.c. In particular, forcing with $\mathbb{P}_*$ does not add any new sequences of ordinals of size less than λ and it preserves all cardinals.*

Set $E = \bigcup_{p \in \mathbf{G}_*} E_p$. The proof of the next lemma is easy.

Lemma 5.6. *Suppose $\mathbf{G}_* \subseteq \mathbb{P}_*$ is generic over V. Then E is a non-reflecting stationary subset of λ.*

Proof. This is standard, so we just sketch the proof. Let $p \in \mathbb{P}_*$ and suppose that $p \Vdash$"$\dot{C} \subseteq \lambda$ is a club of λ". Let $\chi > \lambda^+$ be large enough regular, $\lhd$ be a well-ordering of $\mathscr{H}(\chi)$ and let $M \prec (\mathscr{H}(\chi), \in, \lhd)$ be an elementary submodel of $\mathscr{H}(\chi)$ such that:

(1) $|M| < \lambda$,

(2) $p, \dot{C}, \mathbb{P}_*, \cdots \in M$,

(3) $M \cap \lambda = \delta$ for some $\delta \in S^\lambda_{\aleph_0}$.

By induction on $n < \omega$ we define an increasing sequence $\langle p_n : n < \omega\rangle$ of conditions, together with a sequence $\langle C_n : n < \omega\rangle$ satisfying the following:

(4) $p \leq p_0$,

(5) p_{n+1} decides $\dot{C} \cap \alpha_{p_n}$ to be C_n,

(6) $p_{n+2} \Vdash$"$\dot{C} \cap (\alpha_{p_n}, \alpha_{p_{n+1}}) \neq \emptyset$",

(7) $\sup\limits_{n<\omega} \alpha_{p_n} = \delta$.

Let $q = \langle\langle \mathbb{G}^q_{s,\beta} : s \in S, \beta \leq \delta\rangle, E_q, \langle \mathbf{x}^q_{s,t,\beta} : (s,t) \notin R, \beta \leq \delta\rangle\rangle$, where

(8) for $s \in S$ and $\beta < \delta, \mathbb{G}^q_{s,\beta} = \mathbb{G}^{p_n}_{s,\beta}$, where n is such that $\alpha_{p_n} > \beta$,

(9) for $s \in S, \mathbb{G}^q_\delta = \bigcup\limits_{n<\omega} \mathbb{G}^{p_n}_{s,\alpha_{p_n}}$,

(10) $E_q = \bigcup\limits_{n<\omega} E_{p_n} \cup \{\delta\}$,

(11) if $(s,t) \notin R$ and $\beta < \delta$, $\mathbf{x}^q_{s,t,\beta} = \mathbf{x}^{p_n}_{s,t,\beta}$, for some n with $\alpha_{p_n} > \beta$,

(12) $\mathbf{x}^q_{s,t,\delta}$ is the exact sequence that is the direct limit of the sequence $\mathbf{x}^{p_n}_{s,t,\alpha_{p_n}}$.

It is easily seen that $q \in \mathbb{P}_*$ is well-defined and it forces $\delta \in \dot{C} \cap \dot{E}$, which completes the proof. □

For each $s \in S$ and $\beta < \lambda$, we set $\mathbb{G}_{s,\beta} = \mathbb{G}^p_{s,\beta}$ for some (and hence any) $p \in \mathbf{G}_*$ with $\alpha_p \geq \beta$. Also, we set

$$\mathbb{G}_s := \bigcup_{\beta<\lambda} \mathbb{G}_{s,\beta}.$$

Let us first show that $\mathbb{G}_s$ is not free.

Lemma 5.7. *Suppose $s \in S$. Then $\mathbb{G}_s$ is a non-free λ-free Abelian group of size λ.*

Proof. It is clear from Definition 5.4(a)(2) that $\mathbb{G}_s$ is a λ-free Abelian group of size λ. Let us show that it is not free. in $\mathbf{V}$, let $\mathbb{A}$ be a λ-free Abelian group of size λ which is not free. We may assume that the universe of $\mathbb{A}$ is λ. Let also $\langle A_\alpha : \alpha < \lambda\rangle$ be a filtration of $\mathbb{A}$. It follows that the set

$$E_{\mathbb{A}} := \{\alpha < \lambda : \mathbb{A}/\mathbb{A}_\alpha \text{ in not } \lambda\text{-free}\}$$

is stationary. An argument similar to the proof of Lemma 5.6 shows that $E_{\mathbb{A}}$ remains stationary in $\mathbf{V}[\mathbf{G}_*]$, and hence $\mathbb{A}$ remains non-free in the generic extension $\mathbf{V}[\mathbf{G}_*]$.

For each $\gamma < \lambda$ set D_γ be the set of all $p \in \mathbb{P}_*$ such that there exists β such that:

- $\alpha_p = \beta + 1 > \gamma$,

- $\mathbb{G}^p_{s,\beta+1} = \mathbb{G}^p_{s,\beta} \oplus \mathbb{A}_\beta$.

It is easily seen that each set D_γ is dense in $\mathbb{P}_*$. Now for each $\gamma < \lambda$ pick some $p_\gamma \in D_\gamma \cap \mathbf{G}_*$ and set $\alpha_{p_\gamma} = \beta_\gamma + 1 > \gamma$. Then clearly

$$\mathbb{G}_s = \mathbb{G}'_s \oplus \mathbb{A},$$

where $\mathbb{G}'_s = \bigcup\limits_{\gamma<\lambda} \mathbb{G}^{p_\gamma}_{s,\beta_\gamma}$. As $\mathbb{A}$ is non-free, it follows that $\mathbb{G}_s$ is also non-free as requested. □

For each $(s,t) \in (S \times S) \setminus R$, we look at the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \subseteq\uparrow & & \subseteq\uparrow & & \subseteq\uparrow & & \\
\mathbf{x}_{s,t,\beta} := 0 & \longrightarrow & \mathbb{G}_{t,\beta} & \xrightarrow{f_{s,t,\beta}} & \mathbb{H}_{s,t,\beta} & \xrightarrow{g_{s,t,\beta}} & \mathbb{G}_{s,\beta} & \longrightarrow & 0 \\
 & & \subseteq\uparrow & & \subseteq\uparrow & & \subseteq\uparrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \subseteq\uparrow & & \subseteq\uparrow & & \subseteq\uparrow & & \\
\mathbf{x}_{s,t,1} := 0 & \longrightarrow & \mathbb{G}_{t,1} & \xrightarrow{f_{s,t,1}} & \mathbb{H}_{s,t,1} & \xrightarrow{g_{s,t,1}} & \mathbb{G}_{s,1} & \longrightarrow & 0 \\
 & & \subseteq\uparrow & & \subseteq\uparrow & & \subseteq\uparrow & & \\
\mathbf{x}_{s,t,0} := 0 & \longrightarrow & \mathbb{G}_{t,0} & \xrightarrow{f_{s,t,0}} & \mathbb{H}_{s,t,0} & \xrightarrow{g_{s,t,0}} & \mathbb{G}_{s,0} & \longrightarrow & 0,
\end{array}$$

where for each $\beta < \lambda, \mathbf{x}_{s,t,\beta} = \mathbf{x}^p_{s,t,\beta}$, for some and hence any $p \in \mathbf{G}_*$ with $\alpha_p \geq \beta$. By taking the corresponding inductive limit, we lead to the following short exact sequence

$$\mathbf{x}_{s,t} := 0 \longrightarrow \mathbb{G}_t \xrightarrow{f_{s,t}} \mathbb{H}_{s,t} \xrightarrow{g_{s,t}} \mathbb{G}_s \longrightarrow 0.$$

In other words, $\mathbf{x}_{s,t} := \lim_{\beta\to\lambda} \mathbf{x}^p_{s,t,\beta}$. The next lemma shows that $\mathbf{x}_{s,t}$ does not split.

Lemma 5.8. *Suppose $s_*, t_* \in S$ and $(s_*, t_*) \notin R$. Then in $V[\mathbf{G}_*]$ the exact sequence $\mathbf{x}_{s_*,t_*}$ does not split. In particular,* $\operatorname{Ext}(\mathbb{G}_{s_*}, \mathbb{G}_{t_*}) \neq 0$.

Proof. Suppose towards contradiction that the exact sequence $\mathbf{x}_{s_*,t_*}$ splits and let $p \in \mathbb{P}_*$ and $\underset{\sim}{h}$ be such that

$$p \Vdash \text{“}\underset{\sim}{h} : \mathbb{G}_s \to \underset{\sim}{\mathbb{H}}_{s,t} \text{ is such that } \underset{\sim}{h} \circ \underset{\sim}{g}_{s,t} = \mathrm{id}_{\mathbb{G}_s}\text{”}.$$

As in the proof of Lemma 5.6, we can find an extension $q \leq p$ such that

(1) $\alpha_q \in E_q \cap S^{\lambda}_{\aleph_0}$,
(2) q decides $\underset{\sim}{h} \restriction \mathbb{G}^q_{s,\alpha_q}$, say

$$q \Vdash \text{“}\underset{\sim}{h} \restriction \mathbb{G}^q_{s,\alpha_q} = h_*\text{”}.$$

We will need the following claim.

Claim 5.9. *Adopt the above notation. Then there exists an exact sequence* $\mathbf{x}$ *fits in the following commutative diagram*

$$\begin{array}{cccccccccc}
\mathbf{x} := 0 & \longrightarrow & \mathbb{G}'_t & \xrightarrow{f'} & \mathbb{H}'_{s,t} & \xrightarrow{g'} & \mathbb{G}'_s & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
\mathbf{x}_{s,t,\alpha_q} := 0 & \longrightarrow & \mathbb{G}^q_{t,\alpha_q} & \xrightarrow{f^q_{s,t,\alpha_q}} & \mathbb{H}^q_{s,t,\alpha_q} & \xrightarrow{g_{s,t,\alpha_q}} & \mathbb{G}^q_{s,\alpha_q} & \longrightarrow & 0
\end{array}$$

Also, there exists no splitting h' *of* g' *extending* h_*.

Proof. Recall that $\mathbf{x}_{s,t,\alpha_q}$ is given, and we need to introduce $\mathbf{x}$ in a way that it fits in the above display commutative digram. We look at the free Abelian group

$$\mathbb{G}'_t := \mathbb{G}^q_{t,\alpha_q} \oplus g_{s,t,\alpha_q}(\mathbb{H}^q_{s,t,\alpha_q}) = \mathbb{G}^q_{t,\alpha_q} \oplus \mathbb{G}^q_{s,\alpha_q},$$

and let $\{e_j\}_{j\in J}$ be a base of the free group $\mathbb{G}^q_{s,\alpha_q}$. Now, we look at the subgroup $\mathbb{G}'_s := \bigoplus_{j\in J} \mathbb{Z}\frac{e_j}{2}$ of the $\mathbb{Q}$-vector space $\bigoplus_{j\in J} \mathbb{Q}e_j$. This is a free Abelian group. Finally, we set $\mathbb{H}'_{s,t} := \mathbb{G}'_t \oplus \mathbb{G}'_s$. From this, the maps $f' : \mathbb{G}'_t \to \mathbb{H}'_{s,t}$ and $g' : \mathbb{H}'_{s,t} \to \mathbb{G}'_s$ are the natural injection and projection. These are as required. $\square$

Let r be an extension of q by adding the exact sequence $\mathbf{x}$ at its top height. Then $r \in \mathbb{P}_*$ and

$$r \Vdash \text{“there is no splitting map for } \underset{\sim}{g}_{s,t} \text{ extending } h_*\text{”},$$

which contradicts the choice of p and $\underset{\sim}{h}$. $\square$

Let us now work in the generic extension $V[\mathbf{G}_*]$. We define a λ-support iteration

$$\mathbb{P} = \langle\langle \mathbb{P}_\alpha : \alpha \leq \lambda^+\rangle, \langle \dot{\mathbb{Q}}_\beta : \beta < \lambda^+\rangle\rangle$$

of forcing notions which forces $\mathrm{Ext}(\mathbb{G}_s, \mathbb{G}_t) = 0$ for all $s, t \in S$ with sRt. We first define the building blocks of this iteration.

Suppose $W \supseteq V[\mathbf{G}_*]$ is a forcing extension of $V[\mathbf{G}_*]$, $s, t \in S$ are such that sRt and suppose that

$$\mathbf{x} := 0 \longrightarrow \mathbb{G}_s \xrightarrow{f} \mathbb{H} \xrightarrow{g} \mathbb{G}_t \longrightarrow 0$$

is an exact sequence in W.

Definition 5.10. The forcing notion $\mathbb{Q}^W_{s,t,\mathbf{x}}$ consists of partial functions $q : \mathrm{dom}(q) \to \mathbb{H}$ such that:

(1) $\mathrm{dom}(q) = \mathbb{G}_{s,\gamma}$, where $\gamma \notin E$,
(2) $g \circ q = \mathrm{id}_{\mathrm{dom}(q)}$.

$\mathbb{Q}^W_{s,t,\mathbf{x}}$ is ordered by inclusion.

Thus the forcing notion $\mathbb{Q}^W_{s,t,\mathbf{x}}$ aims to add a splitter for the exact sequence $\mathbf{x}$. The next lemma shows that this is indeed possible.

Lemma 5.11. *The forcing notion $\mathbb{Q}^W_{s,t,\mathbf{x}}$ is λ-closed, λ^+-c.c., and forcing with it adds a function $h : \mathbb{G}_t \to \mathbb{H}$ such that $g \circ h = \mathrm{id}_{\mathbb{G}_t}$.*

Proof. The fact that $\mathbb{Q}^W_{s,t,\mathbf{x}}$ is λ-closed follows from definitions 5.10(1) and 5.4(4), which allows us to take unions (or direct limits) at limit stages and still have a condition. The forcing is λ^+-c.c., by a simple Δ-system lemma. □

We are finally ready to define our iteration. Let $\beta < \lambda^+$ and suppose that $\mathbb{P}_\beta$ is defined. If $\Phi(\beta)$ is a $\mathbb{P}_* * \dot{\mathbb{P}}_\beta$-name for a triple $(s, t, \dot{\mathbf{x}})$, where $s, t \in S$, sRt and $\dot{\mathbf{x}}$ is a name for an exact sequence

$$\mathbf{x} := 0 \longrightarrow \mathbb{G}_s \xrightarrow{\dot{f}} \dot{\mathbb{H}} \xrightarrow{\dot{g}} \mathbb{G}_t \longrightarrow 0,$$

in $\mathscr{H}(\lambda^+)$, then

$$\Vdash_{\mathbb{P}_\beta} \text{“}\dot{\mathbb{Q}}_\beta = \dot{\mathbb{Q}}^{V^{\mathbb{P}_* * \dot{\mathbb{P}}_\beta}}_{s,t,\dot{\mathbf{x}}}\text{”}.$$

Otherwise, let $\dot{\mathbb{Q}}_\beta$ be forced to be the trivial forcing notion.

The next lemma follows from [12].

Lemma 5.12. *Work in $V[\mathbf{G}_*]$. The forcing notion $\mathbb{P} = \mathbb{P}_{\lambda^+}$ is λ-complete and λ^+-c.c.*

It follows from the above lemma that forcing with $\mathbb{P}$ preserves all cardinals and adds no new sequences of ordinals of length less than λ. Suppose

$$\mathbf{G} = \langle\langle \mathbf{G}_\alpha : \alpha \leq \lambda^+\rangle, \langle \mathbf{H}_\beta : \beta < \lambda^+\rangle\rangle$$

is $\mathbb{P}$-generic over $V[\mathbf{G}_*]$.

Lemma 5.13. *Work in $V[\mathbf{G}_* * \mathbf{G}]$. If $s, t \in S$ and sRt, then* $\mathrm{Ext}(\mathbb{G}_s, \mathbb{G}_t) = 0$.

Proof. It suffices to show that any exact sequence

$$\mathbf{x} := 0 \longrightarrow \mathbb{G}_s \xrightarrow{f} \mathbb{H} \xrightarrow{g} \mathbb{G}_t \longrightarrow 0$$

with $\mathbb{H}$ of size λ splits. Let $\dot{\mathbf{x}}$ be a $\mathbb{P}_* * \mathbb{P}$-name for $\mathbf{x}$ which we may assume that $\dot{\mathbf{x}} \in \mathscr{H}(\lambda^+)$. Furthermore, we can find some $\alpha < \lambda^+$ such that $\dot{\mathbf{x}}$ is a $\mathbb{P}_* * \mathbb{P}_\alpha$-name, and then by the choice of Φ, we may also assume that $\Phi(\alpha) = \dot{\mathbf{x}}$. By definition of forcing notion, we know $\Vdash_{\mathbb{P}_* * \mathbb{P}_{\alpha+1}}$ "$\dot{\mathbf{x}}$ splits ", and consequently

$$\Vdash_{\mathbb{P}_* * \mathbb{P}} \text{"}\dot{\mathbf{x}} \text{ splits "}.$$

This completes the proof. □

The proof of the next lemma is similar to the proof of Lemma 5.8.

Lemma 5.14. *Suppose $(s, t) \in (S \times S) \setminus R$. Then the exact sequence*

$$\mathbf{x}_{s,t} := 0 \longrightarrow \mathbb{G}_s \xrightarrow{f_{s,t}} \mathbb{H}_{s,t} \xrightarrow{g_{s,t}} \mathbb{G}_t \longrightarrow 0$$

does not split in $V[\mathbf{G}_* * \mathbf{G}]$.

This completes the proof of Theorem 5.1.

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