Sh:1088

Annals of Pure and Applied Logic 172 (2021) 102986

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Universal graphs and functions on ω_1

Saharon Shelah [a,b,1], Juris Steprāns [c,*,2]

[a] Department of Mathematics, Rutgers University, Hill Center, Piscataway, New Jersey 08854-8019, USA
[b] Institute of Mathematics, Hebrew University, Givat Ram, Jerusalem 91904, Israel
[c] Department of Mathematics, York University, 4700 Keele Street, Toronto, Ontario M3J 1P3, Canada

ARTICLE INFO

Article history:
Received 17 May 2016
Received in revised form 22 February 2021
Accepted 2 April 2021
Available online 16 April 2021

MSC:
03E17
03E35
03E65
03C30
03C50

Keywords:
Universal graphs
Forcing
Cardinal invariants

ABSTRACT

It is shown to be consistent with various values of $\mathfrak{b}$, $\mathfrak{d}$ and $2^{\aleph_1}$ that there is a universal graph on ω_1. Moreover, it is also shown that it is consistent that there is a universal graph on ω_1 — in other words, a universal symmetric function from ω_1^2 to 2 — but no such function from ω_1^2 to ω. The method used relies on iterating well known reals, such as Miller and Laver reals, and alternating this with the PID forcing which adds no new reals. The last sections examine the question of set valued universal functions.

1. Introduction

1.1. Background to the problem

It is well known that countably saturated models are universal for models of cardinality $\aleph_1$. However, in the case of graphs, the existence of a saturated model is equivalent to $2^{\aleph_0} = \aleph_1$. These two observations immediately raise the question of whether it is possible to have a universal graph of cardinality $\aleph_1$ in the absence of the Continuum Hypothesis. This provided the motivation for the articles [1], [2] and [3] which solved not only this question, but also others about the universality of different structures.

* Corresponding author.
E-mail addresses: shelah@math.rutgers.edu (S. Shelah), steprans@yorku.ca (J. Steprāns).

[1] The first author's research for this paper was partially supported by the United States-Israel Binational Science Foundation (grant No. 1838/19), and by the National Science Foundation (grant No. DMS 1833363). This is article has been assigned number 1088 in the first author's list of publications that can be found at http://shelah.logic.at.
[2] The second author's research for this paper was partially supported by NSERC of Canada.

https://doi.org/10.1016/j.apal.2021.102986
0168-0072/© 2021 Published by Elsevier B.V.

While the results to be presented here have their roots in this work, they are also motivated by considerations that are not model theoretic. A function $U : X^2 \to X$ is said to be *(Sierpiński) universal* if for any $G : X^2 \to X$ there is $e : X \to X$ such that $G(x, y) = U(e(x), e(y))$ for all x and y in X. The function e will be called an *embedding* of G into U. An early reference to this notion can be found in Problem 132 of the Scottish Book [4] in which Sierpiński asked if there is a Borel function which is universal in this sense, when X is the real line. He had already shown in [5] that there is a Borel universal function assuming the Continuum Hypothesis. This notion of universal function is also studied in Rado [6]. More recently, this notion and various generalizations of it were studied in [7], in which a restricted form of the following definition appears as Definition 7.5.

Definition 1.1. A function $U : \kappa^2 \to \lambda$ is *weakly universal* if for every $f : \kappa^2 \to \lambda$ there exist one-to-one functions $h : \kappa \to \kappa$ and $k : \lambda \to \lambda$ such that $k(f(\alpha, \beta)) = U(h(\alpha), h(\beta))$ for all α and β in κ. The pair (h, k) will be called a weak embedding.

Definition 7.4 of [7] defines a function $U : \kappa^2 \to \kappa$ to be *model theoretically universal* if it is weakly universal, as in Definition 1.1, but with $h = k$. Note that U is Sierpiński universal if it is weakly universal with k being the identity. Remark 7.7 of [7] claims that all three notions — Sierpiński universal, weakly universal, and model theoretically universal — are equivalent for maps into 2.

The following is Theorem 5.9 of [7] showing that there is no difference between asking about the existence of universal graphs — in other words, symmetric, irreflexive functions from ω_1^2 to 2 — and non-symmetric functions from ω_1^2 to 2.

Theorem 1.2. *For any infinite cardinal κ the following are equivalent:*

1. *For each $n \in \mathbb{N}$ there is a universal function from κ^2 to n.*
2. *For some $n \in \mathbb{N}$ with $n \geq 2$ there is a universal function from κ^2 to n.*
3. *There is a symmetric, irreflexive function from κ^2 to 2 universal for all symmetric, irreflexive functions from κ^2 to 2*
4. *There is a universal graph on κ.*

Given Theorem 1.2 it is reasonable to focus attention only on symmetric functions from ω_1^2 to ω and this will be done from now on. This justifies the following:

Convention: Any function with domain ω_1^2 will be assumed to be symmetric. In other words, $f : \omega_1^2 \to Y$ should be read as $f : [\omega_1]^2 \to Y$.

Moreover, given that there are only two possible values for k when $\lambda = 2$, the validity of Remark 7.7 of [7] should now be clear and, of course, assuming the Continuum Hypothesis there is a Sierpiński universal function from $\mathbb{R}^2$ to $\mathbb{R}$, and hence there is also a weakly universal function from ω_1^2 to 2. Problem 7.8 of [7] asks if the existence of one type of universal function implies the existence of the others in general. To provide a negative answer to the question, it is therefore necessary to consider models where the Continuum Hypothesis fails. This provides an other path to the questions considered in [1], [2] and [3].

When $\kappa = \omega_1$ and $\lambda = 2$ it was shown in [1] and [2] that it is consistent with the failure of the Continuum Hypothesis that there is a universal function $U : \omega_1^2 \to 2$ which satisfies all three universality properties. The methods used in the generalization of this result by Mekler to other theories in [3] provide examples of Sierpinski universal functions $U : \omega_1^2 \to \lambda$ for λ equal to 2, ω or ω_1. In the models of [1] and [2] the cardinal invariants $\mathfrak{b}$ and $\mathfrak{d}$ have the following values respectively: $\aleph_1$ and $\aleph_2$. One might, therefore ask, whether these values are needed for the existence of universal functions from $U : \omega_1^2 \to \lambda$ with the failure of the Continuum Hypothesis. This question becomes even more interesting in light of the positive results to be presented in Lemma 6.1 and Lemma 7.8 which rely on small values of $\mathfrak{b}$ and $\mathfrak{d}$ respectively. It will be shown in Corollary 3.10 that it is consistent with the existence of universal functions from $U : \omega_1^2 \to 2$ that $\mathfrak{b} = \mathfrak{d} = \aleph_2$ and in Corollary 4.19 that it is consistent with the existence universal functions from $U : \omega_1^2 \to 2$ that $\mathfrak{b} = \mathfrak{d} = \aleph_1$.

However, the main goal of this paper will be to answer Problem 5.10 of [7]. This is done in Corollary 6.2 which establishes that there is a Sierpiński universal function from ω_1^2 to 2 but no such function from ω_1^2 to ω. It is worth recalling that Theorem 5.9 of [7] asserts that if $2 \leq n < \omega$ then there is a Sierpiński universal function from ω_1^2 to 2 if and only if there is a Sierpiński universal function from ω_1^2 to n. A solution to Problem 7.8 of [7] will also be presented under the assumption that $\mathfrak{d}$ is small. This and related questions are discussed in §7. That section also contains some arguments extending the methods of [1] to some weak versions of embedding. Finally, if one is only interested in obtaining a model of set theory in which the Continuum Hypothesis fails, yet there is a universal graph of cardinality $\aleph_1$, then §2 presents an easier argument than the original of [1]. On the other hand, §9 presents a reformulation and simplification of arguments from [1].

1.2. The universal graphs

The graphs that will be shown to be universal in the arguments to follow will all come from some initial model of set theory in which the Continuum Hypothesis holds. For §2, §3 and §4 the following will be relevant.

Definition 1.3. Given any function $G : \omega_1^2 \to \omega$ and $\eta \in \omega_1$ define $G^\eta : \eta \to \omega$ by $G^\eta(\zeta) = G(\zeta, \eta)$ and then define $S^\eta(G) = \{G^\mu \restriction \eta\}_{\mu \geq \eta}$. A function $G : \omega_1^2 \to \omega$ such that $S^\eta(G)$ is everywhere non-meagre for each $\eta \geq \omega$ will be called *category saturated*.

Definition 1.4. Let ν be an atomic probability measure on ω and let ν^η be the Fubini product of this measure on ω^η for any $\eta \in \omega_1$. A function $G : \omega_1^2 \to \omega$ such that $S^\eta(G)$ has outer measure 1 for each $\eta \geq \omega$ will be called *ν-saturated*. The notion defined here will not be needed in this full generality, but is does find an application in [8]. For the purposes of this article a function $G : \omega_1^2 \to 2$ will be called *measure saturated* if it is ν-saturated where ν is the measure on 2 giving each point equal measure.

Lemma 1.5. *Assuming the Continuum Hypothesis, there is a symmetric function from ω_1^2 to ω that is category saturated and ν-saturated for every atomic probability measure ν.*

Indeed, using the Continuum Hypothesis it is easy to construct $G : \omega_1^2 \to \omega$ such that $S^\eta(G) = \omega^\eta$ for each $\eta \in \omega_1$. Note, however, that adding a real will destroy this stronger property; nevertheless, in certain generic extensions the weaker properties of being category saturated or ν-saturated may persist.

1.3. Notation and terminology

Since trees will play a central role in the following discussion, it may be worthwhile reviewing some notation and terminology, even though most of this is standard and almost all of the notation used will follow that of Sections 1.1.D and 7.3.D of [9]. By a tree T will be meant a subset $T \subseteq \omega^{<\omega} = \bigcup_{n\in\omega} \omega^n$ that is closed under initial segments — in other words, if $t \in T$ and $k \leq |t|$ then $t \restriction k \in T$. If T is a tree and $t \in T$ then $T[t]$ will denote the tree defined by

$$T[t] = \{s \in T \mid s \subseteq t \text{ or } t \subseteq s\}$$

and $\boldsymbol{succ}_T(t)$ will denote the set $\{s \in T \mid s \supseteq t \text{ and } |s| = |t| + 1\}$.

A tree T will be called *infinite splitting* if $|\boldsymbol{succ}_T(t)| \in \{1, \aleph_0\}$ for each $t \in T$. Define $\mathbf{split}(T) = \{t \in T \mid |\boldsymbol{succ}_T(t)| = \aleph_0\}$ and define

$$\mathbf{split}_n(T) = \{t \in \mathbf{split}(T) \mid |\{k \in |t| \mid t \restriction k \in \mathbf{split}(T)\}| = n\}.$$

If T is infinite splitting then let $\Psi_T : \omega^{<\omega} \to \mathbf{split}(T)$ be the unique bijection from $\omega^{<\omega}$ to $\mathbf{split}(T)$ preserving the lexicographic ordering. For $t \in \omega^{<\omega}$ let $T\langle t\rangle = T[\Psi_T(t)]$ — the reader is warned that this notation differs from [7]. To emphasize that only elements of a tree are being considered, the notation $s \sqsubseteq t$ will be used to indicate that $s = t \restriction k$ for some $k \leq |t|$.

Let $\{u_i\}_{i\in\omega}$ enumerate $\omega^{<\omega}$ in such a way that if $k < |u_i|$ then there is $j \in i$ such that $u_i \restriction k = u_j$. Then for infinite splitting trees T and S the ordering $\leq_n$ is defined by $T \leq_n S$ if $T \subseteq S$ and $\Psi_S(u_j) = \Psi_T(u_j)$ for all $j \leq n$. Define $\boldsymbol{stem}(T) = \Psi_T(\varnothing)$ and let $T_{\setminus \boldsymbol{stem}}$ denote $\{t \in T \mid \boldsymbol{stem}(T) \subsetneq t\}$.

Finally, recall that Miller forcing, or rational perfect set forcing, is denoted by **PT** in [9] and consists of all infinite splitting trees ordered by inclusion. Laver forcing, on the other hand, is denoted by **LT** in [9] and consists of all infinite splitting trees T such that $T_{\setminus \boldsymbol{stem}} = \mathbf{split}(T)$, also ordered by inclusion. In the case of Laver forcing the notion of a *front* is useful: If $T \in \mathbf{LT}$ then $F \subseteq T$ is a front if it consists of incomparable elements of T and every maximal branch of T contains an element of F.

For a tree $W \subseteq \omega^{<\omega}$ let $\max(W)$ denote the maximal elements of W with respect to inclusion, the natural ordering on the tree. As usual, and in contradiction to common sense, a tree with no infinite branches will be called *well founded*. If F is a front, then there is a well founded tree W such that $F = \max(W)$.

1.4. The ideal $\mathcal{S}(G)$

Definition 1.6. If $f : \omega \to \omega+1$ define $\mathbb{T}_f$ to be $\bigcup_{n\in\omega} \prod_{j\in n} f(j)$ with inclusion as a tree ordering; the special case when f has constant value ω will be denoted $\mathbb{T}_\omega$. If $f : \omega \to \omega + 1$ then a function $\psi : \mathbb{T}_f \to [\omega_1]^{<\aleph_0}$ satisfying that $\psi(s) \cap \psi(t) = \varnothing$ unless $s = t$ will be said to have *disjoint range*. In later sections additional requirements will be imposed on functions with disjoint range. If G is a filter of subtrees of $\mathbb{T}_f$ and ψ has disjoint range define

$$S(G, \psi) = \bigcup_{t\in\bigcap G} \psi(t)$$

noting that when G is sufficiently generic then $\bigcap G$ is a branch through $\mathbb{T}_f$. If G is a generic filter of trees over a model V define

$$\mathcal{S}(G) = \{S(G, \psi) \mid \psi \in V \text{ and } \psi \text{ has disjoint range}\}.$$

It will be shown that in various generic extensions $\mathcal{S}(G)$ is a P-ideal and this will be used in conjunction with the P-ideal dichotomy which is the following the statement $(*)$ from [10].

Definition 1.7. The statement $(*)$ says that for every P-ideal $\mathcal{I}$ on ω_1 one of the following two alternatives holds:

1. there is an uncountable $A \subseteq \omega_1$ such that $[A]^{\aleph_0} \subseteq \mathcal{I}$
2. ω_1 can be decomposed into countably many sets orthogonal to $\mathcal{I}$.

The applications of the P-ideal dichotomy in [10] and [11] all rely on simply finding an uncountable set all of whose countable subsets belong to a given ideal. The arguments to be presented here rely on a stronger version of the axiom, but one which is, nevertheless, true in the model constructed in [10]. The following theorem is implicit in Lemma 3.1 of [10]; the following argument simply verifies this assertion.

Theorem 1.8 *(Abraham & Todorcevic). Let $\mathcal{I}$ be a P-ideal on ω_1 that is generated by a family of $\aleph_1$ countable sets — in particular, this will hold if $2^{\aleph_0} = \aleph_1$ and $\mathcal{I} \subseteq [\omega_1]^{\leq \aleph_0}$. Then there is a proper partial order $\mathbb{P}_{\mathcal{I}}$, that adds no reals, even when iterated with countable support, such that there is a $\mathbb{P}_{\mathcal{I}}$-name $\dot{Z}$ for a subset of ω_1 such that for any $Y \subseteq \omega_1$ which is not the union of countably many sets orthogonal to $\mathcal{I}$*

$$\mathbb{1} \Vdash_{\mathbb{P}_{\mathcal{I}}} \text{“}\dot{Z} \cap Y \neq \varnothing\text{”} \tag{1.1}$$

$$\mathbb{1} \Vdash_{\mathbb{P}_{\mathcal{I}}} \text{“}(\forall \eta \in \omega_1)\ \dot{Z} \cap \eta \in \mathcal{I}\text{”}. \tag{1.2}$$

Proof. The proof is the same as that in [10]. To begin, given a non-principal P-ideal $\mathcal{I}$ on ω_1, fix A_ξ such that

- $A_\xi \subseteq \xi$
- $A_\xi \in \mathcal{I}$
- if $\xi \in \eta$ then $A_\xi \subseteq^* A_\eta$
- every member of $\mathcal{I}$ is almost included in some A_ξ.

The partial order $\mathbb{P}_{\mathcal{I}}$ is defined to consist of pairs $p = (x_p, \mathfrak{X}_p)$ such that:

- $x_p \in \mathcal{I}$ (The reader should not be confused by the claim in [10] that x_p can be any countable subset of ω_1.)
- $|\mathfrak{X}_p| \leq \aleph_0$
- $\mathfrak{X}_p \subseteq [\omega_1]^{\aleph_1}$.

The ordering on $\mathbb{P}_{\mathcal{I}}$ is defined by defining $p \leq q$ if:

- $x_p \cap \sup(x_q) = x_q$
- $\mathfrak{X}_p \supseteq \mathfrak{X}_q$
- $\{\xi \in X \mid x_p \setminus x_q \subseteq A_\xi\} \in \mathfrak{X}_p$ for every $X \in \mathfrak{X}_q$

and $\dot{Z}$ is the name for $\bigcup_{p \in G} x_p$ where $G \subseteq \mathbb{P}_{\mathcal{I}}$ is generic. Lemma 3.1 of [10] establishes that if ω_1 cannot be decomposed into countably many sets orthogonal to $\mathcal{I}$, then for each $\gamma \in \omega_1$ the set of $p \in \mathbb{P}_{\mathcal{I}}$ such that $x_p \setminus \gamma \neq \varnothing$ is a dense subset of $\mathbb{P}_{\mathcal{I}}$. It will now be verified that the same argument establishes that if $Y \subseteq \omega_1$ is not the union of countably many sets orthogonal to $\mathcal{I}$ then $D(Y) = \{p \in \mathbb{P}_{\mathcal{I}} \mid x_p \cap Y \neq \varnothing\}$ is a dense subset of $\mathbb{P}_{\mathcal{I}}$.

To see this suppose that there is no member of $D(Y)$ extending p. Then for each $\mu \in Y$ there is some $X \in \mathfrak{X}_p$ such that

$$X(\mu) = \{\xi \in X \mid \mu \in A_\xi\}$$

is countable. For $X \in \mathfrak{X}_p$ let

$$B(X) = \{\mu \in Y \mid |X(\mu)| = \aleph_0\}$$

and note that the hypothesis on Y implies that $\bigcup_{X \in \mathfrak{X}_p} B(X) \supseteq Y$. It therefore suffices to show that each $B(X)$ is orthogonal to $\mathcal{I}$. To see that this is so, suppose that b is an infinite subset of $B(X)$ and that $b \in \mathcal{I}$. Then $\{\xi \in X \mid b \subseteq^* A_\xi\}$ is clearly a co-countable subset of X because of the cofinality of $\{A_\xi\}_{\xi \in \omega_1}$. Moreover

$$\{\xi \in X \mid b \subseteq^* A_\xi\} = \bigcup_{F \in [b]^{<\aleph_0}} \{\xi \in X \mid (b \setminus F) \subseteq A_\xi\}$$

and hence there is some $F \in [b]^{<\aleph_0}$ such that $\{\xi \in X \mid (b \setminus F) \subseteq A_\xi\}$ is uncountable. Then if $\mu \in b \setminus F$ this contradicts that $X(\mu) \supseteq \{\xi \in X \mid (b \setminus F) \subseteq A_\xi\}$ and $|X(\mu)| = \aleph_0$ because $\mu \in B(X)$. □

2. Category saturated graphs are universal after adding Miller reals

Lemma 2.1. *If $T \Vdash_{\mathbf{PT}}$ "$\dot{S} \in \mathcal{S}(\dot{G})$ and $\dot{h} : \dot{S} \to 2$" then there is $T^* \subseteq T$ and $f^* : \omega_1 \to 2$ such that*

$$T^* \Vdash_{\mathbf{PT}} \text{"}f^* \restriction \dot{S} = \dot{h}\text{"}.$$

Proof. Given $T \in \mathbf{PT}$ find $\bar{T} \subseteq T$ and ψ with disjoint range such that $\bar{T} \Vdash_{\mathbf{PT}}$ "$\dot{S} = S(\dot{G}, \psi)$". Now construct T_n and f^*_n such that:

1. $T_0 = \bar{T}$
2. $T_{n+1} \leq_n T_n$
3. the domain of f^*_n is $\bigcup_{j \in n} \bigcup_{s \sqsubseteq \Psi_{T_n}(u_j)} \psi(\Psi_{T_n}(s))$
4. if $j \in n$ then $T_n\langle u_j \rangle \Vdash_{\mathbf{PT}}$ "$f^*_n \restriction \bigcup_{s \sqsubseteq \Psi_{T_n}(u_j)} \psi(\Psi_{T_n}(s)) \subseteq \dot{f}$".

Then if $T^* = \bigcap_{n \in \omega} T_n$ it is clear that if $f^* \supseteq \bigcup_{n \in \omega} f^*_n$ then ψ and f^* witness that T^* satisfies the lemma.

To complete the induction it suffices to note that $T_n\langle u_n \rangle \Vdash_{\mathbf{PT}}$ "$\bigcup_{s \sqsubseteq \Psi_{T_n}(u_j)} \psi(\Psi_{T_n}(s)) \subseteq \dot{S}$" and that there are $T^* \subseteq T_n\langle u_n \rangle$ such that $\boldsymbol{stem}(T^*) = \boldsymbol{stem}(T_n\langle u_n \rangle)$ and $f^* : \bigcup_{s \sqsubseteq \Psi_{T_n}(u_n)} \psi(s) \to 2$ such that

$$T^* \Vdash_{\mathbf{PT}} \text{"}f^* = \dot{f} \restriction \bigcup_{s \sqsubseteq \Psi_{T_n}(u_n)} \psi(s)\text{"}.$$

Let $f^*_{n+1} = f^*_n \cup f^*$ and note that letting

$$T_{n+1} = (T_n \setminus T_n\langle u_n \rangle_{\setminus \boldsymbol{stem}}) \cup T^*$$

satisfies (2). □

Corollary 2.2. $\mathbb{T}_\omega \Vdash_{\mathbf{PT}}$ *"$\mathcal{S}(\dot{G})$ is closed under subsets".*

Proof. Let $G \subseteq \mathbf{P}$ be generic over V and let $S(G, \psi) \in \mathcal{S}(G)$ with $\psi \in V$. Then if $X \subseteq S(G, \psi)$, by Lemma 2.1 there is $f : \omega_1 \to 2$ such that $f \in V$ such that $X = f^{-1}\{1\} \cap S(G, \psi)$. Then if ψ^* is defined by $\psi^*(t) = \psi(t) \cap f^{-1}\{1\}$ it follows that $\psi^* \in V$ and has disjoint range. Clearly $X = S(G, \psi^*)$. □

Notation 2.3. If $\psi : \mathbb{T}_\omega \to [\omega_1]^{<\aleph_0}$ and $T \subseteq \mathbb{T}_\omega$ is a subtree then, in an abuse of standard notation, the function $\psi \restriction T$ will be defined by

$$\psi \restriction T(t) = \begin{cases} \psi(t) & \text{if } t \in T \\ \varnothing & \text{otherwise.} \end{cases}$$

Lemma 2.4. $\mathbb{T}_\omega \Vdash_{\mathbf{PT}}$ *"$\mathcal{S}(\dot{G})$ is closed under finite unions."*

Proof. Given Corollary 2.2 it suffices to show that if

$$T \Vdash_{\mathbf{PT}} \text{"}\{\dot{X}, \dot{Y}\} \subseteq \mathcal{S}(\dot{G}) \text{ and } \dot{X} \cap \dot{Y} = \varnothing\text{"} \tag{2.1}$$

then there is ψ^* with disjoint range and $T^* \subseteq T$ such that $T^* \Vdash_{\mathbf{PT}}$ "$\dot{X} \cup \dot{Y} = S(\dot{G}, \psi^*)$". Begin by finding $\tilde{T} \subseteq T$ and ψ_X and ψ_Y such that $\tilde{T} \Vdash_{\mathbf{PT}}$ "$\dot{X} = S(\dot{G}, \psi_X)$ and $\dot{Y} = S(\dot{G}, \psi_Y)$".

Now let $\psi(t) = \psi_X(t) \cup \psi_Y(s)$ and construct $\{T_n\}_{n\in\omega}$ such that

1. $T_0 = \tilde{T}$
2. $T_{n+1} \leq_n T_n$
3. $\psi(s) \cap \psi(t) = \varnothing$ if $t \in T_n$ and $s \sqsubseteq \Psi_{T_n}(u_j)$ for some $j \leq n$ and $t \neq s$.

If this can be done then let $T^* = \bigcap_n T_n$ and note that $\psi^* = \psi \restriction T^*$ has disjoint range and that $T^* \Vdash_{\mathbf{PT}}$ "$\dot{X} \cup \dot{Y} = S(\dot{G}, \psi^*)$". To complete the induction note that Hypothesis 2.1 implies that if $t \not\sqsubseteq s \in \tilde{T}$ then $\psi_X(t) \cap \psi_Y(s) = \varnothing$ and hence (3) holds for $n = 0$. Given T_n let

$$B = \bigcup_{j\leq n} \bigcup_{s \sqsubseteq \Psi_{T_n}(u_j)} \psi(s)$$

and keep in mind that $B^* = \{t \in T_n \mid \psi(t) \cap B \neq \varnothing\}$ is finite and $B^* \subseteq \bigcup_{j\leq n} T_n\langle u_j\rangle_{\setminus \textit{\textbf{stem}}}$. It is therefore possible to find $T_{n+1} \leq_n T_n$ such that $B^* \cap T_{n+1} = \varnothing$ as required. □

Corollary 2.5. $\mathbb{T}_\omega \Vdash_{\mathbf{PT}}$ *"$\mathcal{S}(\dot{G})$ is an ideal."*

Proof. Use Corollary 2.2 and Lemma 2.4. □

Lemma 2.6. *If ψ_i have disjoint range and $T_i \in \mathbf{PT}$ for $i \in \omega$ then there are $\bar{T}_i \leq_0 T_i$ such that*

$$(\forall i < j < \omega)(\forall t \in (\bar{T}_i)_{\setminus \textit{\textbf{stem}}})(\forall s \in (\bar{T}_j)_{\setminus \textit{\textbf{stem}}})\ \psi_j(s) \cap \psi_i(t) = \varnothing \tag{2.2}$$

Proof. This is a dovetailed version of the argument for Lemma 2.4. Note that including the stems of the trees is impossible. □

Lemma 2.7. *If $\dot{S}$ is a $\mathbf{PT}$-name such that $T \Vdash_{\mathbf{PT}}$ "$\dot{S} \in \mathcal{S}(\dot{G})$" and $k \in \omega$ then there is $\bar{T} \leq_k T$ and ψ with disjoint range such that $\bar{T} \Vdash_{\mathbf{PT}}$ "$\dot{S} \equiv^* S(\dot{G}, \psi)$".*

Proof. Lemma 2.6 applied to the finite family $\{T\langle u_i\rangle\}_{i\in k}$ implies that the general case will follow easily from the case $k = 0$. For each $t \in \mathbf{split}_1(T)$ find $T_t \subseteq T[t]$ and ψ_t with disjoint range such that $T_t \Vdash_{\mathbf{PT}}$ "$\dot{S} = S(\dot{G}, \psi_t)$".

Now apply Lemma 2.6 to the infinite family $\{\psi_t\}_{t\in\mathbf{split}_1(T)}$ to find $\tilde{T}_t \subseteq T_t$ such that ψ defined by

$$\psi = \bigcup_{t\in\mathbf{split}_1(T)} \psi_t \restriction (\tilde{T}_t)_{\setminus \textit{\textbf{stem}}}$$

has disjoint range. Let $\bar{T} = \bigcup_{t\in\mathbf{split}_1(T)} \tilde{T}_t$. It is immediate that ψ and $\bar{T}$ satisfy the lemma. □

Lemma 2.8. *If $T \Vdash_{\mathbf{PT}}$ "$\{\dot{S}_n\}_{n\in\omega} \subseteq \mathcal{S}(\dot{G})$" then there are ψ_n with disjoint range and $T^* \subseteq T$ such that*

$$T^* \Vdash_{\mathbf{PT}} \text{"}(\forall n)\ \dot{S}_n \equiv^* S(\dot{G}, \psi_n)\text{"}$$

and $\psi_n(t) \cap \psi_m(s) = \varnothing$ for all n and m and $s \neq t$.

Proof. Construct by induction T_n and ψ_n such that:

1. $T_0 \leq_0 T$
2. $T_{n+1} \leq_{n+1} T_n$
3. $T_n \Vdash_{\mathbf{PT}}$ "$\dot{S} \equiv^* S(\dot{G}, \psi_n)$"
4. if i, j, k and ℓ are no greater than n and s and t are distinct elements of $T\backslash_{\boldsymbol{stem}}$ and $s \sqsubseteq \Psi_{T_n}(u_i)$ and $t \sqsubseteq \Psi_{T_n}(u_j)$ then $\psi_k(t) \cap \psi_\ell(s) = \varnothing$
5. if B_n is defined to be $\bigcup_{i\leq n}\bigcup_{k\leq n}\bigcup_{s\sqsubseteq\Psi_{T_n}(u_i)}\psi_k(s)$ and if $t \in \bigcup_{i\leq n} T_n\langle u_i\rangle\backslash_{\boldsymbol{stem}}$ and $k \leq n$ then $\psi_k(t) \cap B_n = \varnothing$.

If this can be done then simply let $T^* = \bigcap_n T_n$.

To carry out the induction, suppose that T_n and $\{\psi_i\}_{i\leq n}$ have been constructed. Use Lemma 2.7 to find $\bar{T}_{n+1} \leq_{n+1} T_n$ and $\bar{\psi}_{n+1}$ with disjoint range such that $\bar{T}_{n+1} \Vdash_{\mathbf{PT}}$ "$\dot{S} \equiv^* S(\dot{G}, \bar{\psi}_{n+1})$". To get Induction Hypothesis (5) to hold at $n + 1$ simply define ψ_{n+1} by

$$\psi_{n+1}(t) = \begin{cases} \varnothing & \text{if there is } j \leq n+1 \text{ such that } t \sqsubseteq \Psi_{T_{n+1}}(u_j) \\ \bar{\psi}_{n+1}(t) & \text{otherwise} \end{cases}$$

and then, by removing only finitely many successors of each $\Psi_{\bar{T}_{n+1}}(u_j)$ in $\bar{T}_{n+1}$ for $j \leq n+1$, it is possible to find define $T_{n+1} \leq_{n+1} \bar{T}_{n+1}$ such that if B_{n+1} is defined by Induction Hypothesis (5), then if $t \in \bigcup_{i\leq n+1} T_{n+1}\langle u_i\rangle\backslash_{\boldsymbol{stem}}$ and $k \leq n+1$ then $\psi_k(t) \cap B_{n+1} = \varnothing$.

To see that Induction Hypothesis (4) holds for $n+1$ suppose that i, j, k and ℓ are no greater than $n+1$ and s and t are distinct elements of $T\backslash_{\boldsymbol{stem}}$ and that $s \sqsubseteq \Psi_{T_{n+1}}(u_i)$ and $t \sqsubseteq \Psi_{T_{n+1}}(u_j)$. By the definition of ψ_{n+1} it may as well be assumed that k and ℓ are less than $n + 1$. Since $T_{n+1} \leq_{n+1} T_n$ it may as well be assumed that $i < j = n + 1$ and that $t \not\sqsubseteq \Psi_{T_{n+1}}(u_m)$ for any $m \leq n$. In other words,

$$t \in \bigcup_{i\leq n} T_n\langle u_i\rangle\backslash_{\boldsymbol{stem}}$$

and hence Induction Hypothesis (5) implies that $\psi_k(t) \cap \psi_\ell(s) = \varnothing$ as required. □

Lemma 2.9. $\mathbb{T}_\omega \Vdash_{\mathbf{PT}}$ *"$\mathcal{S}(\dot{G})$ is a P-ideal".*

Proof. Using Corollary 2.5 it suffices to show that if $T \Vdash_{\mathbf{PT}}$ "$\{\dot{S}_n\}_{n\in\omega} \subseteq \mathcal{S}(\dot{G})$" then there is $T^* \subseteq T$ and ψ with disjoint range such that $T^* \Vdash_{\mathbf{PT}}$ "$(\forall n \in \omega)\ \dot{S}_n \subseteq^* S(\dot{G}, \psi)$". Using Lemma 2.8 there are ψ_n with disjoint range and $T^* \subseteq T$ such that $T^* \Vdash_{\mathbf{PT}}$ "$(\forall n)\ \dot{S}_n \equiv^* S(\dot{G}, \psi_n)$" and $\psi_n(t) \cap \psi_m(s) = \varnothing$ for all n and m and $s \neq t$. Then define $\psi(t) = \bigcup_{j\leq|t|}\psi_j(t)$ and note that $T^* \Vdash_{\mathbf{PT}}$ "$(\forall n)\ \dot{S}_n \equiv^* S(\dot{G}, \psi_n) \subseteq^* S(\dot{G}, \psi)$". □

Lemma 2.10. *If G is* **PT** *generic over V and*

- $S \in \mathcal{S}(G)$,
- $S \subseteq \xi \in \omega_1$,
- $f : S \to 2$ *is a function in* $V[G]$
- $Z \subseteq 2^\xi$ *is nowhere meagre,*
- $Z \in V$

then there is $z \in Z$ such that $f \subseteq z$.

Proof. Let $\dot{S}$ and $\dot{h}$ be **PT**-names for S and f so that $T \Vdash_{\mathbf{PT}}$ "$\dot{S} \in \mathcal{S}(\dot{G})$ and $\dot{h} : \dot{S} \to 2$". Then use Lemma 2.1 to find $T^* \subseteq T$, ψ and $f^* : \omega \to 2$ such that $T^* \Vdash_{\mathbf{PT}}$ "$f^* \restriction \dot{S} = \dot{h}$ and $\dot{S} = S(\dot{G}, \psi)$". Note that it follows that if $t \in T^*$ then $T^*[t] \Vdash_{\mathbf{PT}}$ "$\dot{f} \restriction \psi(t) = f^* \restriction \psi(t)$".

Let $\bar{f} = \bigcup_{s \sqsubseteq \boldsymbol{stem}(T^*)} f^* \restriction \psi(s)$ and let $\mathcal{O}$ be the open set $\{h \in 2^\xi \mid h \supseteq \bar{f}\}$. Then $Z \cap \mathcal{O}$ is not meagre in $\mathcal{O}$. For $s \in T^*$ let s^* be the least element of $\mathbf{split}(T^*)$ such that $s \sqsubseteq s^*$ and define

$$\mathcal{O}_s = \left\{ x \in \mathcal{O} \;\middle|\; x \supseteq \bigcup_{s \sqsubseteq u \sqsubseteq s^*} f^* \restriction \psi(u) \right\}$$

letting $\mathcal{O}_s = \varnothing$ if $s \notin T^*$. If $t \in \mathbf{split}(T^*)$ then define

$$\mathcal{O}_t^+ = \bigcap_{j \in \omega} \left(\bigcup_{k > j} \mathcal{O}_{t^\frown k} \right)$$

and note that $\mathcal{O}_t^+$ is a dense G_δ in $\mathcal{O}$ for each $t \in \mathbf{split}(T^*)$.

Hence there is some $z \in Z \cap \bigcap_{t \in \mathbf{split}(T^*)} \mathcal{O}_t^+$ such that $\bar{f} \subseteq z$ and

$$(\forall t \in \mathbf{split}(T^*))(\exists^\infty s \in \boldsymbol{succ}_{T^*}(t)) \ \bigcup_{s \sqsubseteq u \sqsubseteq s^*} f^* \restriction \psi(u) \subseteq z.$$

It follows that there is $T^{**} \subseteq T^*$ such that $T^{**} \in \mathbf{PT}$ and such that if $t \in T^{**}$ then $f^* \restriction \psi(t) \subseteq z$ and hence $T^{**} \Vdash_{\mathbf{PT}}$ "$\dot{f} \subseteq z$". It follows that there is a dense set of conditions forcing the conclusion of the lemma. □

2.1. Applying the P-ideal dichotomy

Lemma 2.11. *If G is $\mathbf{PT}$ generic over V then in $V[G]$ no uncountable subset of ω_1 is orthogonal to $\mathcal{S}(G)$; in other words, if $Y \subseteq \omega_1$ is uncountable, then $[Y]^{\aleph_0} \cap \mathcal{S}(G) \neq \varnothing$.*

Proof. Suppose that $\dot{Z}$ is a $\mathbf{PT}$-name such that $T \Vdash_{\mathbf{PT}}$ "$\dot{Z} \in [\omega_1]^{\aleph_1}$". It suffices to construct a sequence of conditions $T_n \in \mathbf{PT}$ and ordinals ζ_n such that:

- $T_0 = T$,
- $T_{n+1} \leq_n T_n$ for each n
- $\zeta_n < \zeta_{n+1}$
- $T_n \langle u_j \rangle \Vdash_{\mathbf{PT}}$ "$\zeta_j \in \dot{Z}$" for each $j \in n$

because then it is possible to define $T_\omega = \bigcap_{n \in \omega} T_n$ and to define $\psi : \omega^{<\omega} \to \omega_1$ by

$$\psi(t) = \begin{cases} \{\zeta_j\} & \text{if } t = \Psi_{T_\omega}(u_j) \\ \varnothing & \text{otherwise.} \end{cases}$$

It is then immediate that $T_\omega \Vdash_{\mathbf{PT}}$ "$|S(\dot{G}, \psi) \cap \dot{Z}| = \aleph_0$".

To carry out the construction let T_n be given and let $\eta = \max_{j \in n} \zeta_{u_j}$ and let $m < n$ be maximal such that $u_m \sqsubseteq u_n$. Using that $T \Vdash_{\mathbf{PT}}$ "$\dot{Z} \setminus \eta \neq \varnothing$" it is possible to find $T^* \subseteq T_n \langle u_m \rangle \setminus \bigcup_{u_j \not\sqsubseteq u_m} T_n \langle u_j \rangle$ and $\zeta_n > \eta$ such that $T^* \Vdash_{\mathbf{PT}}$ "$\zeta_n \in \dot{Z}$". Let $t^* = \boldsymbol{stem}(T)$ and let $T_{n+1} = (T_n \setminus T_n[t^*]) \cup T^*$ and note that $T_{n+1} \langle u_n \rangle = T^*$. □

Theorem 2.12. *Let V be a model of the Continuum Hypothesis and suppose that $U : \omega_1^2 \to 2$ is a category saturated function in V and that $G \subseteq \mathbf{PT}$ is generic over V. In $V[G]$ let $H \subseteq \mathbb{P}_{\mathcal{S}(G)}$ be generic over $V[G]$. Then in $V[G][H]$ the function U is universal.*

Proof. In $V[G]$ let $R \subseteq \omega_1$ be uncountable satisfying the conclusion of Theorem 1.8. Given $W : \omega_1^2 \to 2$ in $V[G][H]$ construct by induction one-to-one embeddings $e_\eta : \eta \to R$ of $W \restriction \eta^2$ into U such that $e_\eta \subseteq e_\zeta$ if $\eta \leq \zeta$.

Since limit stages of the induction are trivial, it suffices to show that given e_η there is $e_{\eta+1}$ as required. Let S be the range of e_η and suppose that $S \subseteq \xi$. Then $S \in [R]^{\aleph_0} \subseteq \mathcal{S}(G)$. Let $f : S \to 2$ be defined by $f(\sigma) = W(e_\eta^{-1}(\sigma), \eta)$ and note that $f \in V[G]$ since $V[G]$ and $V[G][H]$ have the same reals. Recall that **PT** preserves non-meagre sets by Theorems 6.3.20 and 7.3.46 in [9]. By Lemma 2.10 it therefore follows that, using the notation of Definition 1.3, $\{\gamma \in \omega_1 \mid f \subseteq U^\gamma\}$ is an uncountable set in $V[G]$. By Lemma 2.11 and Conclusion 1.1 of Theorem 1.8 it follows that it is possible to find $\gamma \in R \setminus \xi$ such that $f \subseteq U^\gamma$ and, hence, $W(e_\eta^{-1}(\sigma), \eta) = f(\sigma) = U(\sigma, \gamma)$ for all $\sigma \in S$. Let $e_{\eta+1} = e_\eta \cup \{(\eta, \gamma)\}$. □

Corollary 2.13. *Given any regular $\kappa > \aleph_1$ it is consistent with set theory that*

- $\mathfrak{b} = \aleph_1$ *(indeed, $\mathfrak{a} = \aleph_1$)*
- $\mathfrak{d} = \aleph_2$
- $2^{\aleph_1} = \kappa$
- *there is a universal graph on ω_1.*

Proof. The required model is the one obtained by starting with a model of the Continuum Hypothesis in which $2^{\aleph_1} = \kappa$ and iterating, with countable support, ω_2 Miller reals at even coordinates and forcing with $\mathbb{P}_{\mathcal{S}(\dot{G})}$ at odd coordinates. Any category saturated graph in the initial model — and, in particular, any saturated graph — will be universal in the final extension. To see this, begin by observing that by Theorem 7.3.46 of [9] it follows that **PT** preserves $\sqsubseteq^{\mathbf{Cohen}}$ as defined in Definition 6.3.15 of [9]. Since $\mathbb{P}_{\mathcal{S}(\dot{G})}$ is proper and adds no new reals it is immediate that it also preserves $\sqsubseteq^{\mathbf{Cohen}}$. By Theorem 6.3.20 of [9] it follows that the entire countable support iteration preserves non-meagre sets and, hence, any category saturated graph in the initial model remains category saturated.

To see that all of these graphs are universal use Lemma 3.4 and Lemma 3.6 of [10] to conclude that each partial order in the ω_2 length iteration is proper and has the $\aleph_2$-pic of Definition 2.1 on page 409 of [12]. By Lemma 2.4 on page 410 of [12] it follows that the iteration has the $\aleph_2$ chain condition and, hence, that any graph on ω_1 appears at some stage. It is then routine to apply Theorem 2.12.

That $\mathfrak{d} = \aleph_2$ is a standard argument using that Miller forcing adds an unbounded real. The fact that $\mathfrak{b} = \aleph_1$ follows from the fact that $\mathfrak{b} \leq \boldsymbol{non}(\mathcal{M})$. To see that, in fact, the stronger result $\mathfrak{a} = \aleph_1$ holds, it is not possible to use the argument of Spinas or Eisworth mentioned in §11.9 of [13] because $\mathbb{P}_{\mathcal{S}(\dot{G})}$ is not a Souslin forcing, indeed, it does not even have cardinality $2^{\aleph_0}$. Instead, an earlier argument from [14] as exposited in §8 of [15] can be used. It is shown there, in the proof of Proposition 8.24, that is possible to construct a maximal almost disjoint family $\mathcal{A}$ of cardinality $\aleph_1$ under the assumption that $2^{\aleph_0} = \aleph_1$ that satisfies a property $\mathbf{ST}_\alpha$ for each $\alpha \in \omega_2$ and $\mathbf{ST}_\alpha$ implies that the maximality of $\mathcal{A}$ is preserved after an iteration of α Miller reals. The limit cases use only properness and the induction hypothesis that $\mathbf{ST}_\alpha$ is preserved. At successors it is shown in the proof of Proposition 8.24 that $\mathbf{ST}_\alpha$ is preserved by forcing with **PT**. Since $\mathbb{P}_{\mathcal{S}(\dot{G})}$ adds no new reals it is easily seen to have $\mathbf{ST}_\alpha$. Hence $\mathbf{ST}_\alpha$ is preserved throughout the iteration and so the maximality of $\mathcal{A}$ is preserved. □

3. Measure saturated graphs are universal after adding Laver reals

Many of the ideas developed in §2 an be extended to the context of the Laver model. One exception will be Lemma 3.7 that requires the following definition.

Definition 3.1. If ψ has disjoint range and $\{|\psi(s^\frown n)|\}_{n\in\omega}$ is bounded for each $s \in \mathbb{T}_\omega$ then ψ will be said to have *bounded disjoint range*. The definition of $S(G,\psi)$ for ψ with bounded disjoint range is the same as in Definition 1.6, however, if G is a generic filter of trees over a model V define

$$\mathcal{S}_b(G) = \{S(G,\psi) \mid \psi \in V \text{ and } \psi \text{ has bounded disjoint range}\}.$$

Lemma 3.2. *If $\dot{S}$ is a* **LT***-name such that $T \Vdash_{\mathbf{LT}}$ "$\dot{S} \in \mathcal{S}_b(\dot{G})$" and $k \in \omega$ then there is $\bar{T} \leq_k T$ and $\psi : \bar{T} \to [\omega_1]^{<\aleph_0}$ with bounded disjoint range such that $\bar{T} \Vdash_{\mathbf{LT}}$ "$\dot{S} \equiv^* S(\dot{G},\psi)$".*

Proof. Note that Lemma 2.6 remains true for Laver forcing. As in the proof of Lemma 2.7, Lemma 2.6 implies that the general case of the lemma will follow easily from the case $k = 0$. Using the standard rank argument for Laver forcing find $\tilde{T} \leq_0 T$ such that there is a front $F \subseteq \tilde{T}$ with the property that for each $t \in F$ there is $T_t \leq_0 \tilde{T}[t]$ and ψ_t with disjoint range such that $T_t \Vdash_{\mathbf{LT}}$ "$\dot{S} = S(\dot{G},\psi_t)$".

Now apply Lemma 2.6 to the family $\{\psi_t\}_{t\in F}$ to find $\tilde{T}_t \leq_0 T_t$ such that ψ defined by

$$\psi = \bigcup_{t\in F} \psi_t \restriction (\tilde{T}_t)_{\setminus \boldsymbol{stem}}$$

has disjoint range. Let $\bar{T} = \bigcup_{t\in F} \tilde{T}_t$. It is again immediate that ψ and $\bar{T}$ satisfy the lemma. □

Lemma 3.3. *If $T \Vdash_{\mathbf{LT}}$ "$\dot{S} \in \mathcal{S}_b(\dot{G})$ and $\dot{h} : \dot{S} \to 2$" then there is $T^* \subseteq T$ and $f^* : \omega_1 \to 2$ such that*

$$T^* \Vdash_{\mathbf{LT}} \text{“} f^* \restriction \dot{S} = \dot{h} \text{”}.$$

Proof. The proof is similar to that of Lemma 2.1 — indeed, the first paragraph applies without change. The last paragraph requires the following fact: If W is a well founded tree such that $|\boldsymbol{succ}_W(w)| = \aleph_0$ for each $w \in W \setminus \max(W)$ whose maximal nodes are coloured in finitely many colours then there is $\bar{W} \subseteq W$ such that

1. $|\boldsymbol{succ}_{\bar{W}}(w)| = \aleph_0$ for each $w \in \bar{W} \setminus \max(\bar{W})$
2. $\boldsymbol{stem}(\bar{W}) = \boldsymbol{stem}(W)$
3. $\mathbf{split}(\bar{W}) = \mathbf{split}(W) \cap (\bar{W} \setminus \max(\bar{W}))$
4. all the nodes in $\max(\bar{W})$ are coloured the same colour.

It follows from the third condition that $\max(\bar{W}) \subseteq \max(W)$.

Using the notation of the last paragraph of the proof of Lemma 2.1, keep in mind that $T_n\langle u_n\rangle \Vdash_{\mathbf{LT}}$ "$\psi(\Psi_{T_n}(u_n)) \subseteq \dot{S}$". (Also note that because Laver trees have the property that all nodes above the root are branching, there is no need for considering $\bigcup_{s\sqsubseteq\Psi_{T_n}(u_n)} \psi(s)$ as in Miller forcing.) There is then a well founded tree $W \subseteq T_n\langle u_n\rangle$ such that for each maximal $w \in W$ there is an extension $T_w \leq_0 T_n[w]$ and there is f_w such that $T_w \Vdash_{\mathbf{LT}}$ "$\dot{f} \restriction \psi(\Psi_{T_n}(u_n)) = f_w$". There is then a colouring of the maximal nodes of W by the f_w. This yields $\bar{W} \subseteq W$ such that this colouring has constant value f^* on the maximal nodes of $\bar{W}$ and Conditions (1) to (4) hold. To complete the induction let $T^* = \bigcup_{w\in\bar{W}} T_n[w]$ and note that $T^* \leq_0 T_n\langle u_n\rangle$. Moreover $T^* \Vdash_{\mathbf{P}}$ "$\dot{f} \restriction \psi(\Psi_{T_n}(u_n)) = f^*$". Let $f^*_{n+1} = f^*_n \cup f^*$ and note that letting

$$T_{n+1} = (T_n \setminus T_n\langle u_n\rangle_{\setminus \boldsymbol{stem}}) \cup T^*$$

satisfies the lemma. □

Lemma 3.4. $\mathbb{T}_\omega \Vdash_{\mathbf{LT}}$ *"$\mathcal{S}_b(\dot{G})$ is an ideal".*

Proof. Lemma 3.3 shows that $\mathbb{T}_\omega \Vdash_{\mathbf{LT}}$ "$\mathcal{S}_b(\dot G)$ is closed under subsets". The proof of Lemma 2.4 can now be used verbatim. $\square$

Lemma 3.5. *If $T \Vdash_{\mathbf{LT}}$ "$\{\dot S_n\}_{n\in\omega} \subseteq \mathcal{S}_b(\dot G)$" then there are ψ_n with bounded, disjoint range and $T^* \subseteq T$ such that $T^* \Vdash_{\mathbf{LT}}$ "$(\forall n)\ \dot S_n \equiv^* S(\dot G, \psi_n)$" and $\psi_n(t) \cap \psi_m(s) = \varnothing$ for all n and m and $s \neq t$.*

Proof. The proof is exactly the same as that of Lemma 2.8 except that the ψ_n are chosen to be bounded. $\square$

Lemma 3.6. $\mathbb{T}_\omega \Vdash_{\mathbf{LT}}$ *"$\mathcal{S}_b(\dot G)$ is a P-ideal".*

Proof. As in the proof of Lemma 2.9, using Lemma 3.4 it suffices to show that if $T \Vdash_{\mathbf{LT}}$ "$\{\dot S_n\}_{n\in\omega} \subseteq \mathcal{S}_b(\dot G)$" then there is $T^* \subseteq T$ and ψ with bounded, disjoint range such that $T^* \Vdash_{\mathbf{LT}}$ "$(\forall n \in \omega)\ \dot S_n \subseteq^* S(\dot G, \psi)$". Using Lemma 3.5 there are ψ_n with bounded, disjoint range and $T^* \subseteq T$ such that $T^* \Vdash_{\mathbf{LT}}$ "$(\forall n)\ \dot S_n \equiv^* S(\dot G, \psi_n)$" and $\psi_n(t) \cap \psi_m(s) = \varnothing$ for all n and m and $s \neq t$. Then define $\psi(t) = \bigcup_{k\leq |t|} \psi_k(t)$ and note that $T^* \Vdash_{\mathbf{LT}}$ "$(\forall n)\ \dot S_n \equiv^* S(\dot G, \psi_n) \subseteq^* S(\dot G, \psi)$". Note that ψ is bounded since each of the ψ_n is bounded and $|\psi(t^\frown j)| \leq \sum_{k\leq|t|+1} |\psi_k(t^\frown j)|$. $\square$

Lemma 3.7. *If G is* **LT** *generic over V and*

- *$S \in \mathcal{S}_b(G)$,*
- *$S \subseteq \xi \in \omega_1$,*
- *$f : S \to 2$ is a function in $V[G]$*
- *$Z \subseteq 2^\xi$ has full outer measure as defined in Definition 1.4,*
- *$Z \in V$*

then there is $z \in Z$ such that $f \subseteq z$.

Proof. Let $\dot S$ and $\dot f$ be **LT**-names for S and f so that $T \Vdash_{\mathbf{PT}}$ "$\dot S \in \mathcal{S}_b(\dot G)$ and $\dot f : \dot S \to 2$". Then use Lemma 3.3 to find $T^* \subseteq T$, ψ with bounded, disjoint range and $f^* : \omega \to 2$ such that $T^* \Vdash_{\mathbf{LT}}$ "$f^* \restriction \dot S = \dot f$ and $\dot S = S(\dot G, \psi)$". Note that it follows that if $t \in T^*$ then $T^*[t] \Vdash_{\mathbf{LT}}$ "$\dot f \restriction \psi(t) = f^* \restriction \psi(t)$".

Let $\bar f = \bigcup_{s \sqsubseteq \boldsymbol{stem}(T^*)} f^* \restriction \psi(s)$ and let $\mathcal{O}$ be the open set $\left\{h \in 2^\xi \;\middle|\; h \supseteq \bar f\right\}$. Then $Z \cap \mathcal{O}$ has positive measure. For $s \in T^*$ let $\mathcal{O}_s = \{x \in \mathcal{O} \mid x \restriction \psi(s) = f^* \restriction \psi(s)\}$ letting $\mathcal{O}_s = \varnothing$ if $s \notin T^*$. If $t \in \mathbf{split}(T^*)$ then define

$$\mathcal{O}_t^+ = \bigcap_{j\in\omega} \left(\bigcup_{k>j} \mathcal{O}_{t^\frown k} \right)$$

and note that $\mathcal{O} \setminus \mathcal{O}_t^+$ is a null set for each $t \in \mathbf{split}(T^*)$ because, since ψ is bounded, there is a non-zero lower bound for the measures of the $\mathcal{O}_{t^\frown k}$ which, because of disjointness, correspond to independent events.

Hence there is some $z \in Z \cap \bigcap_{t\in\mathbf{split}(T^*)} \mathcal{O}_t^+$ such that $\bar f \subseteq z$ and

$$(\forall t \in \mathbf{split}(T^*))(\exists^\infty s \in \boldsymbol{succ}_{T^*}(t))\ f^* \restriction \psi(s) \subseteq z.$$

It follows that there is $T^{**} \leq_0 T^*$ such that $T^{**} \in \mathbf{LT}$ and such that if $t \in T^{**}$ then $f^* \restriction \psi(t) \subseteq z$ and hence $T^{**} \Vdash_{\mathbf{LT}}$ "$\dot f \subseteq z$". It follows that there is a dense set of conditions forcing the conclusion of the lemma. $\square$

Lemma 3.8. *If G is* **LT** *generic over V then no uncountable subset of ω_1 is orthogonal to $\mathcal{S}_b(G)$ in $V[G]$.*

Proof. Given $T \in \mathbf{LT}$ such that $T \Vdash_{\mathbf{LT}}$ "$\dot{Z} \in [\omega_1]^{\aleph_1}$" proceed by induction on n to find T_n, F_n and φ_n such that

1. $T_n \in \mathbf{LT}$
2. F_n is a front in T_m for $n \leq m$
3. for all $w \in F_{n+1}$ there is $w^* \in F_n$ such that $w^* \sqsubseteq w$
4. $\varphi_n : F_n \to \omega_1$ is one-to-one
5. $\max_{w \in F_n} \varphi_n(w) < \min_{w \in F_{n+1}} \varphi_{n+1}(w)$
6. if $w \in F_n$ then $T_n[w] \Vdash_{\mathbf{LT}}$ "$\varphi_n(w) \in \dot{Z}$".

If this can be done then let $T^* = \bigcap_n T_n$ and note that F_n is a front in T^* for each n. Moreover, if $w \in F_n$ then $T^*[w] \Vdash_{\mathbf{LT}}$ "$\varphi_n(w) \in \dot{Z}$". Then defining ψ by

$$\psi(w) = \begin{cases} \{\varphi_n(w)\} & \text{if } w \in F_n \\ \varnothing & \text{if } w \notin \bigcup_n F_n \end{cases}$$

obviously yields that ψ is bounded with disjoint range. Clearly $T^* \Vdash_{\mathbf{LT}}$ "$|S(\dot{G}, \psi) \cap \dot{Z}| = \aleph_0$" as required for the lemma.

Now suppose that T_n, F_n and φ_n are given. (To start, choose T_0 such that $T_0 \Vdash_{\mathbf{LT}}$ "$\zeta \in \dot{Z}$" for some ζ and define $F_0 = \{\boldsymbol{stem}(T_0)\}$ and $\varphi_0(\boldsymbol{stem}(T_0)) = \zeta$.) Let $\mu_n \in \omega_1$ contain the range of φ_n. Define D to be the set of all $t \in T_n$ such that there are uncountably many $\xi \in \omega_1$ for which there is $T^* \leq_0 T[t]$ such that $T^* \Vdash_{\mathbf{LT}}$ "$\xi \in \dot{Z} \setminus \mu_n$". A standard pigeonhole argument shows that $T_n \setminus D$ contains no element of $\mathbf{LT}$.

Hence, for each $t \in F_n$ there is $T^t \leq_0 T_n[t]$ and a front F^t of T^t such that $F^t \subseteq D$. Let $F_{n+1} = \bigcup_{t \in F_n} F^t$. Enumerating F_{n+1} as $\{w_k\}_{k \in \omega}$ it is possible to use the definition of D to inductively define $\varphi_{n+1}(w_j)$ so that φ is one-to-one and there is $S_j \leq_0 T_n[w_j]$ such that $S_j \Vdash_{\mathbf{LT}}$ "$\varphi_{n+1}(w_j) \in \dot{Z} \setminus \mu_n$". Then define $T_{n+1} = \bigcup_{j \in \omega} S_j$ thus completing the inductive construction. □

Theorem 3.9. *Let V be a model of set theory and suppose that $U : \omega_1^2 \to 2$ is a measure saturated function in V and that $G \subseteq \mathbf{LT}$ is generic over V. In $V[G]$ let $H \subseteq \mathbb{P}_{\mathcal{S}_b(G)}$ be generic over $V[G]$. Then in $V[G][H]$ the function U is universal.*

Proof. This is the same as the proof of Theorem 2.12 using measure in the place of category and the fact that $\mathbf{LT}$ preserves outer measure. □

Corollary 3.10. *It is consistent with $\mathfrak{b} = \mathfrak{d} = \aleph_2$ and with $2^{\aleph_1}$ assuming any reasonable value that there is a universal graph on ω_1.*

Proof. This the same as the proof of Corollary 2.13. □

4. Measure saturated graphs are universal after adding ω^ω-bounding reals

This section contains the key consistency result needed to establish the main result of this article, namely that the existence of a universal graph on ω_1 does not entail the existence of a universal function from ω_1^2 to ω. The key concepts are already contained in §3, the only difference being that the partial order $\mathbf{PT}_{f,g}$ of Definition 7.3.3 of [9] will be used in the place of $\mathbf{LT}$. As can be seen in the exposition of $\mathbf{PT}_{f,g}$ in [9], there are a great many analogies between this partial order and $\mathbf{LT}$ and these explain the similarities between §3 and this section. The partial order $\mathbf{PT}_{f,g}$ will be defined in Definition 4.2. This definition will rely on a particular pair of functions f and g defined in Definition 4.1.

Definition 4.1. First let $a_n \in (0,1)$ be such that $\prod_{n=j}^{\infty} a_n > 1 - 1/2^{j^2}$. Let $g(0,0) = 1$. If $g(n,n)$ has been defined let $f(n)$ be so large that $f(n) \geq g(n,n)$ and

$$f(n) > \sum_{m \leq n^2} 2^m. \tag{4.1}$$

Then let $g(n+1,0) = f(n) + 1$. If $g(n,k)$ has been defined then choose $g(n,k+1)$ to be so large that if

- $[X_{i,j}]_{i \in g(n,k+1), j \in n}$ is a matrix of independent 2-valued random variables with mean $1/2$
- $\varphi : g(n,k+1) \times n \to 2$

then the probability that

$$|\{i \in g(n,k+1) \mid (\forall j \in n)\ X_{i,j} = \varphi(i,j)\}| \geq g(n,k) \tag{4.2}$$

is greater than $a_n^{1/F(n)}$. Since there is no harm in increasing $g(n,k+1)$ it can also be assumed that

$$g(n,k+1) > g(n,k)\Xi(n) \tag{4.3}$$

where $\Xi(n)$ is greater than each of the following:

1. $2n$
2. $\max_{j \in n} f(j)$
3. $2F(n)^2 S(n) + 1$ where $F(n) = \prod_{j \in n} f(j)$ and $S(n) = \prod_{j \in n} jF(j)$.

Note that $F(n)$ represents the cardinality of $\mathbb{T}_f\{n\}$ defined in Definition 4.2 and $S(n)$ represents the maximum possible cardinality of $\bigcup_{t \in T \restriction n} \psi(t)$ where ψ has asymptotically small disjoint range as defined in Definition 4.3. The functions f and g are fixed throughout this section.

Definition 4.2. Recall that Definition 7.3.3 of [9] defines $\mathbf{PT}_{f,g}$ to consist of trees $T \subseteq \mathbb{T}_f$ (see Definition 1.6) such that there is a non-decreasing function $r : \omega \to \omega$ satisfying that $\lim_{n\to\infty} r(n) = \infty$ and such that

$$|\boldsymbol{succ}_T(t)| > g(|t|, r(|t|))$$

for all $t \in T$. For any $T \in \mathbf{PT}_{f,g}$ fix $r_T : \omega \to \omega$ witnessing that $T \in \mathbf{PT}_{f,g}$. The ordering on $\mathbf{PT}_{f,g}$ is inclusion. The ordering $\leq_n$ on $\mathbf{PT}_{f,g}$ is defined by $T \leq_n S$ if $T \subseteq S$ and for each $t \in T$ either $\boldsymbol{succ}_T(t) = \boldsymbol{succ}_S(t)$ or $|\boldsymbol{succ}_t(t)| \geq g(|t|, n)$. It will be useful in this section to use the notations $T \restriction k = \{t \in T \mid |t| \leq k\}$ and $T \restriction \{k\} = \{t \in T \mid |t| = k\}$ for a tree T and $k \in \omega$.

Definition 4.3. A function ψ with disjoint range will be said to have *asymptotically small disjoint range* if $|\psi(t)| < |t|$ for all $t \in \mathbb{T}_f$ and $\lim_m \max_{t \in \omega^m} |\psi(t)|/|t| = 0$. Then define

$$\mathcal{S}_a = \{S(G,\psi) \mid \psi \text{ is asymptotically small disjoint range and } \psi \in V\}$$

where $G \subseteq \mathbf{PT}_{f,g}$ is a generic filter over V.

Lemma 4.4. *(Lemma 7.3.4 of [9]) If $\{T_n\}_{n\in\omega} \subseteq \mathbf{PT}_{f,g}$ and $\{k_n\}_{n\in\omega}$ is an increasing sequence of positive integers such that $T_{n+1} \leq_{k_n} T_n$ for all n then there is $T \in \mathbf{PT}_{f,g}$ such that $T \leq_{k_n} T_n$ for all n.*

Lemma 4.5. *(Lemma 7.3.5 of [9]) $\{\leq_n\}_{n\in\omega}$ witnesses that $\mathbf{PT}_{f,g}$ satisfies Axiom A.*

Corollary 4.6. *(Corollary 7.3.10 of [9]) If $T \Vdash_{\mathbf{PT}_{f,g}}$ "$\dot{a} \in f(n)$" then there is $S \subseteq T$ such that:*

- *for each $s \in S \restriction \{n+1\}$ there is a_s such that $S[s] \Vdash_{\mathbf{PT}_{f,g}}$ "$\dot{a} = a_s$"*
- *$S \restriction n+1 = T \restriction n+1$*
- *$|\boldsymbol{succ}_S(s)| \geq |\boldsymbol{succ}_T(s)|/f(n)$ for all $s \in S$ such that $|s| > n$.*

Lemma 4.7. *If $\dot{G}$ is a name for the generic subset of $\mathbf{PT}_{f,g}$ and if $T \Vdash_{\mathbf{PT}_{f,g}}$ "$\dot{S} \in \mathcal{S}_a(\dot{G})$ and $\dot{h} : \dot{S} \to 2$" then there is $T^* \subseteq T$ and $h^* : \omega_1 \to 2$ such that*

$$T^* \Vdash_{\mathbf{PT}_{f,g}} \text{“}h^* \restriction \dot{S} = \dot{h}\text{”}.$$

Proof. Given $T \in \mathbf{PT}_{f,g}$ find $\bar{T} \subseteq T$ and ψ asymptotically small with disjoint range such that $\bar{T} \Vdash_{\mathbf{PT}_{f,g}}$ "$\dot{S} = S(\dot{G}, \psi)$". Now construct T_n, k_n and h^*_n such that:

1. $T_0 = \bar{T}$
2. $T_{n+1} \leq_n T_n$
3. the domain of h^*_n is $\bigcup_{s \in T_n \restriction k_n} \psi(s)$
4. if $s \in T_n \restriction \{k_n\}$ then $T_n[s] \Vdash_{\mathbf{PT}_{f,g}}$ "$\bigcup_{j \leq k_n} h^*_n \restriction \psi(s \restriction j) \subseteq \dot{h}$"
5. $r_{T_n}(k) \geq n$ for $k > k_n$.

If the construction can be completed, then by Lemma 4.4 there is T^* such that $T^* \leq_{k_n} T_n$ for all n. It is clear that if $h^* \supseteq \bigcup_{n \in \omega} h^*_n$ then ψ and h^* witness that T^* satisfies the lemma.

To carry out the construction suppose that T_n, h^*_n and k_n have been chosen and that, moreover, that Induction Hypothesis (5) is more than satisfied in the sense that $r_{T_n}(k) \geq n+1$ for $k > k_n$. Begin by choosing increasing k_{n+1} such that $r_{T_n}(k_{n+1}) \geq n+3$. Use Inequality (4.1) to let $\{z_{\ell,n}\}_{\ell \in f(n)}$ enumerate $\bigcup_{m \leq n^2} 2^m$. For each $s \in T_n \restriction \{k_{n+1}\}$ note that

$$\left| \bigcup_{k_n < j \leq k_{n+1}} \psi(s \restriction j) \right| \leq \sum_{j=k_n+1}^{k_{n+1}} j \leq k^2_{n+1}$$

and let

$$\sigma_s : \left| \bigcup_{k_n < j \leq k_{n+1}} \psi(s \restriction j) \right| \to \bigcup_{k_n < j \leq k_{n+1}} \psi(s \restriction j)$$

be the ordering preserving mapping. Then define $\dot{\varphi}_s(j) = \dot{h}(\sigma_s(j))$ and then, given T_n and h^*_n, define $\dot{a}$ to satisfy

$$\varphi_s = z_{\dot{a}, k_{n+1}} \text{ where } \{s\} = \dot{G} \cap T_n \restriction \{k_{n+1}\}.$$

Then use Corollary 4.6 to find $\tilde{T}_{n+1}$ such that:

6. for each $s \in \tilde{T}_{n+1} \restriction \{k_{n+1}\}$ there is a_s such that $\tilde{T}_{n+1}[s] \Vdash_{\mathbf{PT}_{f,g}}$ "$\dot{a} = a_s$"
7. $\tilde{T}_{n+1} \restriction k_{n+1} = T_n \restriction k_{n+1}$
8. $|\boldsymbol{succ}_{\tilde{T}_{n+1}}(s)| \geq |\boldsymbol{succ}_{T_n}(s)|/f(k_{n+1})$ for all $s \in \tilde{T}_{n+1}$ such that $|s| > k_{n+1}$.

Sh:1088

Then define h_s with domain $\bigcup_{j\leq k_{n+1}} \psi(s \restriction j)$ by $h_s(\eta) = z_{a_s,k_{n+1}}(\sigma_s^{-1}(\eta))$. The problem is that $\bigcup_{s\in T_{n+1}\restriction\{k_{n+1}\}} h_s$ may not be a function. In order to deal with this define $\hat{T}_{k_{n+1}-j}$ for $j \leq k_{n+1} - k_n$ as follows:

9. $\hat{T}_{k_{n+1}} = \tilde{T}_{n+1}$
10. for all $s \in \hat{T}_{k_{n+1}-j} \restriction \{k_{n+1} - j\}$ the mapping $\bar{s} \mapsto h_{\bar{s}}$ is constant on $\hat{T}_{k_{n+1}-j}[s] \restriction \{k_{n+1}\}$
11. if $s \in \hat{T}_{k_{n+1}-(j+1)}$ and $|s| \neq k_{n+1} - (j+1)$ then $\boldsymbol{succ}_{\hat{T}_{k_{n+1}-j}}(s) = \boldsymbol{succ}_{\hat{T}_{k_{n+1}-(j+1)}}(s)$
12. if $s \in \hat{T}_{k_{n+1}-(j+1)}$ and $|s| = k_{n+1} - (j+1)$ then $|\,\boldsymbol{succ}_{\hat{T}_{k_{n+1}-j}}(s)|/2^{k_{n+1}-j} \leq |\,\boldsymbol{succ}_{\hat{T}_{k_{n+1}-(j+1)}}(s)|$.

To obtain (10), (11), and (12) a simple pigeonhole argument suffices, keeping in mind that $|\psi(t)| \leq |t|$ to obtain Induction Hypothesis (12).

Then let $T_{n+1} = \hat{T}_{k_n}$. It follows that if $s \in T_{k_n} \restriction \{k_n\}$ then the mapping $\bar{s} \mapsto h_{\bar{s}}$ is constant on $\hat{T}_{k_n}[s] \restriction \{k_{n+1}\}$. Then the disjoint range of ψ makes it possible to define

$$h^*_{n+1} = h^*_n \cup \bigcup_{s\in T_{n+1}\restriction\{k_{n+1}\}} h_s$$

and to have h^*_{n+1} be a function. To see that $T_{n+1} \leq_n T_n$ start by noting that from Inequality ((4.3)(2)) it follows that if $j > k_{n+1}$ then $g(j, r_{T_n}(j)) \geq g(j, r_{T_n}(j) - 1)f(k_{n+1})$ and Inequality (8) implies that if $|s| > k_{n+1}$ then

$$|\boldsymbol{succ}_{\tilde{T}_{n+1}}(s)| \geq \frac{|\,\boldsymbol{succ}_{T_n}(s)|}{f(k_{n+1})} \geq \frac{g(|s|, r_{T_n}(|s|))}{f(k_{n+1})} \geq g(|s|, r_{T_n}(|s|) - 1) \geq g(|s|, n + 2)$$

and from Condition (11) it follows that $\boldsymbol{succ}_{T_{n+1}}(s) = \boldsymbol{succ}_{\tilde{T}_{n+1}}(s)$. Hence the stronger version of Induction Hypothesis (5) holds if $|s| > k_{n+1}$.

It must now be shown that Induction Hypothesis (5) holds if $|s| \leq k_{n+1}$. To begin, assume that $k_n < |s|$. Using Inequality (4.3(2)) and Conditions (11) and (12) it follows that $|\,\boldsymbol{succ}_{T_{n+1}}(s)| \geq g(|s|, r_{\tilde{T}_{n+1}}(|s|) - 1)$. But from Condition (7) it follows that $g(|s|, r_{\tilde{T}_{n+1}}(|s|) - 1) = g(|s|, r_{T_n}(|s|) - 1)$ and so $|\,\boldsymbol{succ}_{T_{n+1}}(s)| \geq g(|s|, r_{T_n}(|s|) - 1)$. Since $|s| > k_n$ and the stronger version of Induction Hypothesis (5) has been assumed, it follows that $r_{T_n}(|s|) \geq n + 1$ and so $|\,\boldsymbol{succ}_{T_{n+1}}(s)| \geq g(|s|, n)$ as required for the original Induction Hypothesis (5). In order to finish, note that Condition (11) implies that $T_{n+1} \restriction k_n = T_n \restriction k_n$ and hence the Induction hypothesis applies if $|s| \leq k_n$. □

Lemma 4.8. *Suppose that*

- $m \leq F(n)$
- $T_i \in \mathbf{PT}_{f,g}$ *for each* $i \in m$
- $r_{T_i}(\ell) > L + 1$ *for each* $i \in m$ *and* $\ell \geq n$
- $\psi_i : T_i \to [\omega_1]^{<\aleph_0}$ *is asymptotically small with disjoint range for each* $i \in m$.

Then there are $T^*_i \subseteq T_i$ *such that*

$$T^*_i \restriction n = T_i \restriction n \tag{4.4}$$

$$(\forall \ell \geq n)\ r_{T^*_i}(\ell) \geq L \tag{4.5}$$

$$(\forall i < j < m)(\forall t \in T^*_i)(\forall s \in T^*_j)\ \ \text{if } \max(|t|, |s|) > n \text{ then } \psi_i(t) \cap \psi_j(s) = \varnothing. \tag{4.6}$$

Proof. Let $r(j) = \min_{i\in m} r_{T_i}(j) > L+1$. For $k \geq n$ construct by induction on $i \in m$ sets $S_{i,t} \subseteq \boldsymbol{succ}_{T_i}(t)$ for each $t \in T_i \restriction \{k\}$ such that if

$$B_{k,i} = \left(\bigcup_{j\in m}\bigcup_{s\in T_j\restriction k}\psi_j(s)\right)\cup\left(\bigcup_{j\in i}\bigcup_{t\in T_j\restriction\{k\}}\bigcup_{s\in S_{j,t}}\psi_j(s)\right)$$

then $S_{i,t} \subseteq \{s \in \boldsymbol{succ}_{T_i}(t) \mid \psi_i(s)\cap B_{k,i} = \varnothing\}$ and $|S_{i,t}| = g(k, r(k)-1)$. To see that this is possible observe that

$$\left|\bigcup_{j\in m}\bigcup_{s\in T_j\restriction k}\psi_j(s)\right| \leq mS(k) \leq F(k)S(k)g(k,r(k)-1) \leq F(k)^2S(k)g(k,r(k)-1)$$

and that

$$\left|\bigcup_{j\in i}\bigcup_{t\in T_j\restriction\{k\}}\bigcup_{s\in S_{j,t}}\psi_j(s)\right| \leq iF(k)(k+1)g(k,r(k)-1) \leq F(k)^2(k+1)g(k,r(k)-1) \leq F(k)^2S(k)g(k,r(k)-1)$$

and so it follows that

$$|B_{k,i}| \leq 2S(k)F(k)^2g(k,r(k)-1)$$

and hence, by the disjoint range of ψ_i, it follows that

$$\left|\left\{s \in \underset{T_i}{\boldsymbol{succ}}(t) \mid \psi_i(s)\cap B_{k,i} = \varnothing\right\}\right| \leq |B_{k,i}| \leq 2S(k)F(k)^2g(k,r(k)-1).$$

By Inequality (4.3(3)) it follows that

$$\begin{aligned}\left|\underset{T_i}{\boldsymbol{succ}}(t) - \left\{s \in \underset{T_i}{\boldsymbol{succ}}(t) \mid \psi_i(s)\cap B_{k,i} = \varnothing\right\}\right| \geq g(k,r(k)) - F(k)^2S(k)g(k,r(k)-1) \geq \\ (F(k)^2S(k)+1)g(k,r(k)-1) - F(k)^2S(k)g(k,r(k)-1) = g(k,r(k)-1) \quad \square\end{aligned} \tag{4.7}$$

Corollary 4.9. *If $T \Vdash_{\mathbf{PT}_{f,g}}$ "$\dot{S} \in \mathcal{S}_a(\dot{G})$" and $n \in \omega$ then there is $T^* \leq_n T$ and ψ such that $T^* \Vdash_{\mathbf{PT}_{f,g}}$ "$\dot{S} \equiv^* S(\dot{G},\psi)$".*

Proof. Begin by finding n^* such that $r_T(\ell) > n+1$ for all $\ell > n^*$. Using a standard rank argument, find $\bar{T} \subseteq T$ such that

- $\bar{T} \restriction n^* = T \restriction n^*$
- $r_{\bar{T}}(\ell) > n$ for $\ell > n^*$
- there is some $n^{**} \geq n^*$ such that for each $s \in \bar{T} \restriction \{n^{**}\}$ there is some ψ_s such that $\bar{T}[s] \Vdash_{\mathbf{PT}_{f,g}}$ "$\dot{S} = S(\dot{G},\psi_s)$"
- $r_{\bar{T}[s]}(\ell) > n+1$ for each $s \in \bar{T} \restriction \{n^{**}\}$ and $\ell > n^{**}$.

Use Lemma 4.8 to find $T^*_s \subseteq \bar{T}[s]$ for each $s \in \bar{T} \restriction \{n^{**}\}$ such that

$$(\forall \ell \geq n^{**})\ r_{T^*_s}(\ell) \geq n$$
$$(\forall s \neq t \in \bar{T} \restriction \{n^{**}\})(\forall t^* \in T^*_t)(\forall s^* \in T^*_s)\ \text{ if } \max(|t|,|s|) > n^{**} \text{ then } \psi_t(t^*)\cap\psi_s(s^*) = \varnothing.$$

Then define $T^* = \bigcup_{s\in\bar{T}\restriction\{n^{**}\}} T^*_s$ and define ψ by

$$\psi(t) = \begin{cases} \varnothing & \text{if } |t| < n^{**} \text{ or } t \notin T^* \\ \psi_s(t) & \text{if } s \in \bar{T}\restriction\{n^{**}\} \text{ and } s \subsetneq t \text{ and } t \in T^*_s. \end{cases} \qquad \square$$

Lemma 4.10. *If ψ_0 and ψ_1 are asymptotically small with disjoint range, $T \in \mathbf{PT}_{f,g}$ and $n \in \omega$ then there is $T^* \leq_n T$ and J such that if ψ is defined by*

$$\psi(t) = \begin{cases} \varnothing & \text{if } |t| \leq J \\ \psi_0(t) \cup \psi_1(t) & \text{otherwise} \end{cases}$$

then $\psi \restriction T^*$ is asymptotically small with disjoint range.

Proof. Fix J such that $r_T(\ell) > n+3$ for $\ell \geq J$. The lemma will follow from the following three claims.

Claim 4.11. *There is $T_1 \subseteq T$ such that:*

- $T_1 \cap \omega^J = T \cap \omega^J$
- $r_{T_1}(k) \geq r_T(k) - 1$ *for all* $k \geq J$
- *for each $t \in T_1$ such that $|t| > J$ the family $\{\psi(s)) \mid s \in \boldsymbol{succ}_{T_1}(t)\}$ is pairwise disjoint.*

Proof. Keep in mind that $|\psi_i(t^\frown k)| \leq |t|$ for each $t \in T$ and $t^\frown k \in \boldsymbol{succ}_T(t)$. Hence for each k the set of j such that $\psi_0(t^\frown k) \cap \psi_1(t^\frown j) \neq \varnothing$ has cardinality less than $|t|$. It follows that for each k the set of j such that $\psi(t^\frown k) \cap \psi(t^\frown j) \neq \varnothing$ has cardinality less $2|t|$. Therefore there is $S \subseteq \boldsymbol{succ}_T(t)$ such that the family $\{\psi(s)\}_{s\in S}$ is pairwise disjoint and $|S| \geq |\boldsymbol{succ}_T(t)|/2|t|$. This yields T_1 and Inequality (4.3(1)) guarantees that $r_{T_1}(k) \geq r_T(k) - 1$. $\square$

Claim 4.12. *There is $T_2 \subseteq T_1$ such that:*

- $T_2 \cap \omega^J = T_1 \cap \omega^J$
- $r_{T_2}(k) \geq r_{T_1}(k) - 1$ *for all* $k \geq J$
- *for each $m \geq J$ the family $\{\psi(s) \mid s \in T_2\{m\}\}$ is pairwise disjoint.*

Proof. Begin by noting the following simple fact: If $\mathcal{F}_x \in [\omega_1]^{<k}$ is a pairwise disjoint family for each $x \in u$ then there are $\mathcal{G}_x \subseteq \mathcal{F}_x$ such that $\bigcup_{x\in u} \mathcal{G}_x$ is a pairwise disjoint family and $|\mathcal{G}_x| \geq |\mathcal{F}_x|/uk$. For each m and $s \in T_1 \restriction \{m\}$ let $\mathcal{F}_s = \{\psi(\bar{s}) \mid \bar{s} \in \boldsymbol{succ}_{T_1}(s)\}$. Since $|\psi(\bar{s})| \leq 2m$ for $s \in T_1 \restriction \{m\}$ and $\bar{s} \in \boldsymbol{succ}_{T_1}(s)$ it follows that there are $\mathcal{G}_s \subseteq \mathcal{F}_s$ such that $\bigcup_{s\in T_1\restriction\{m\}} \mathcal{G}_s$ is a pairwise disjoint family and

$$|\mathcal{G}_s| \geq \frac{|\mathcal{F}_s|}{2m|T_1 \restriction \{m\}|} \geq \frac{|\mathcal{F}_s|}{2mF(m)}.$$

The $\mathcal{G}_s$ can then be used to construct $T_2 \subseteq T_1$ such that if $s \in T_2$ then $\boldsymbol{succ}_{T_2}(s) = \mathcal{G}_s$. It follows that $\{\psi(s) \mid s \in T_2 \restriction \{m\}\}$ is pairwise disjoint and Inequality (4.3(3)) guarantees that $r_{T_2}(k) \geq r_{T_1}(k) - 1$ for all k since it is easily seen that $F(m)^2 S(m) \geq 2mF(m)$. $\square$

Claim 4.13. *There is $T_3 \subseteq T_2$ such that:*

- $T_3 \cap \omega^J = T_2 \cap \omega^J$

- $r_{T_3}(k) \geq r_{T_2}(k) - 1$ *for all* $k \geq J$
- *for each* $m \in \omega$ *the family* $\{\psi(s) \mid s \in T_2$ *and* $J \leq |s| \leq m\}$ *is pairwise disjoint.*

Proof. Note that $|\bigcup_{t\in T_2\restriction m} \psi(t)| \leq S(m)$. Hence if $t \in T_2 \restriction \{m\}$ then there are fewer than $S(m)$ successors of t that intersect $\bigcup_{t\in T_2\restriction\{m\}} \psi(t)$. Removing all of these yields the desired T_3 and Inequality (4.3(3)) guarantees that $r_{T_3}(k) \geq r_{T_2}(k) - 1$ for all k. □

Now apply the three claims in order. □

Corollary 4.14. *If* $G \subseteq \mathbf{PT}_{f,g}$ *is generic over* V *then* $\mathcal{S}_a(G)$ *is an ideal in* $V[G]$*.*

Proof. That $\mathcal{S}_a(G)$ is closed under subsets follows from Lemma 4.7. To see that $\mathcal{S}_a(G)$ is closed under unions suppose that $T \Vdash_{\mathbf{PT}_{f,g}}$ "$\dot{A}_i \in \mathcal{S}_a(G)$" for $i \in 2$. Use Corollary 4.9 to find ψ_i and T^* such that $T^* \Vdash_{\mathbf{PT}_{f,g}}$ "$\dot{A}_i = S(\dot{G}, \psi_i)$" for $i \in 2$. Then use Lemma 4.10 to find T^{**} such that $T^{**} \subseteq T^*$ and so that if $\psi(t)$ is defined to be $\psi_0(t) \cup \psi_1(t)$ for sufficiently large t then $\psi \restriction T^*$ is asymptotically small with disjoint range. Hence $T^{**} \Vdash_{\mathbf{PT}_{f,g}}$ "$\dot{A}_0 \cup \dot{A}_1 \equiv^* S(\dot{G}, \psi)$" and it is easy to adjust ψ to correct the finite error. □

Corollary 4.15. $\mathbb{T}_f \Vdash_{\mathbf{PT}_{f,g}}$ *"$\mathcal{S}_a(\dot{G})$ is a P-ideal".*

Proof. It suffices to show that if $T \in \mathbf{PT}_{f,g}$ and

$$T \Vdash_{\mathbf{PT}_{f,g}} \text{“}(\forall n \in \omega)\ \dot{S}_n \in S_a(\dot{G})\text{”}$$

then there is $T^* \subseteq T$ and $\psi : T^* \to [\omega_1]^{<\aleph_0}$ with asymptotically small disjoint such that

$$T^* \Vdash_{\mathbf{PT}_{f,g}} \text{“}(\forall n \in \omega)\ \dot{S}_n \subseteq^* S(\dot{G}, \psi)\text{”}.$$

Inductively find T_n, k_n and ψ_n such that

- $T_{n+1} \leq_n T_n$
- ψ_n has disjoint range and is asymptotically small
- $T_n \Vdash_{\mathbf{PT}_{f,g}}$ "$\dot{S}_n \subseteq^* S(\dot{G}, \psi_n)$"
- $|\psi_n(t)|/|t| < 1/n$ if $t \in T_n$ and $|t| > k_n$
- $\psi_{n+1}(t) = \psi_n(t)$ if $|t| \leq k_{n+1}$
- $\psi_{n+1}(t) \supseteq \psi_n(t)$ for all $t \in T_{n+1}$ such that $|t| > k_{n+1}$
- $k_{n+1} > k_n$

and then let $T^* = \bigcap_{n\in\omega} T_n$ and define $\psi(t) = \bigcup_{n\in\omega} \psi_n(t)$. Observe that $\lim_{t\in T^*} |\psi(t)|/|t| = 0$ and that ψ clearly has disjoint range since each ψ_n does. Also note that $T^* \Vdash_{\mathbf{PT}_{f,g}}$ "$\dot{S}_n \subseteq^* S(\dot{G}, \psi_n) \subseteq^* S(\dot{G}, \psi)$".

At stage n use Corollary 4.9 to find $\hat{T} \leq_n T_n$ and $\hat{\psi}$ that is asymptotically small with disjoint range such that $\hat{T} \Vdash_{\mathbf{PT}_{f,g}}$ "$\dot{S} \equiv^* S(\dot{G}, \hat{\psi})$". Then let $k > k_n$ be such that $|\hat{\psi}(t) \cup \psi_n(t)|/|t| < 1/(n+1)$ if $|t| > k_{n+1}$. Then use Corollary 4.10 to find $T_{n+1} \leq_n \hat{T}$ and $k_{n+1} > k$ such that if ψ_{n+1} is defined by

$$\psi_{n+1}(t) = \begin{cases} \psi_n(t) & \text{if } |t| \leq k_{n+1} \\ \hat{\psi}(t) \cup \psi_n(t) & \text{if } |t| > k_{n+1} \end{cases}$$

then $T_{n+1} \Vdash_{\mathbf{PT}_{f,g}}$ "$\dot{S}_n \subseteq^* S(\dot{G}, \hat{\psi}) \subseteq^* S(\dot{G}, \psi_{n+1})$" and ψ_{n+1} is asymptotically small with disjoint range. This satisfies all the induction requirements. □

Lemma 4.16. *If G is $\mathbf{PT}_{f,g}$ generic over V then in $V[G]$ every uncountable subset of ω_1 is orthogonal to $\mathcal{S}_a(\dot{G})$.*

Proof. This is similar to the proof of Lemma 3.8. □

Lemma 4.17. *If $G \subseteq \mathbf{PT}_{f,g}$ is generic over V and*

- $S \in \mathcal{S}_a(G)$,
- $S \subseteq \xi \in \omega_1$,
- $h : S \to 2$ *is a function in* $V[G]$
- $Z \subseteq 2^\xi$ *has full outer measure as defined in Definition 1.4,*
- $Z \in V$

then there is $z \in Z$ such that $h \subseteq z$.

Proof. Let $T_1 \in G$ and $\xi \in \omega_1$ be such that $T_1 \Vdash_{\mathbf{LT}}$ "$\dot{h} : \dot{S} \to 2$ and $\dot{S} \subseteq \xi$" where $\dot{h}$ is a $\mathbf{PT}_{f,g}$-name for h and $\dot{S}$ is a $\mathbf{PT}_{f,g}$-name for S. Use Lemma 4.9 to find $T_2 \in G$ and ψ such that $T_2 \Vdash_{\mathbf{LT}}$ "$\dot{S} = S(\dot{G}, \psi)$". Then use Lemma 4.7 to find $T_3 \in G$ and h^* such that $T_3 \Vdash_{\mathbf{LT}}$ "$\dot{h} = h^* \restriction S(\dot{G}, \psi)$". Let $T \in G$ be lower bound for T_1, T_2 and T_3 and let J be such that $r_T(\ell) > 1$ for all $\ell \geq J$. Let $t \in T \restriction \{J\}$ be such that $T[t] \in G$. There is no harm in assuming that $T = T[t]$.

Let $\bar{h} = h^* \restriction \bigcup_{j \leq J} \psi(t \restriction j)$. Let h be a random element of 2^ξ. Note that $|\bigcup_{j \leq J} \psi(t \restriction j)| < J^2$ and so the probability that $h \supseteq \bar{f}$ is at least $1/2^{J^2}$. On the other hand, note that Condition 4.2 of Definition 4.1 implies that for any $n > J$ and any $s \in T \restriction \{n\}$ the probability that

$$|\{s' \in \boldsymbol{succ}_T(s) \mid h^* \restriction \psi(s') \subseteq h\}| \geq g(n, r_T(n) - 1) = g(|s|, r_T(|s|) - 1)$$

is greater than $a_n^{1/F(n)}$ and $F(n) \geq |T \restriction \{n\}|$. Hence, for any n the probability that

$$(\forall s \in T \restriction \{n\}) \, |\{s' \in \boldsymbol{succ}_T(s) \mid h^* \restriction \psi(s') \subseteq h\}| \geq g(|s|, r_T(|s|) - 1)$$

is greater than a_n. It follows from the choice of the a_n that the probability that

$$(\forall n > J)(\forall s \in T \restriction \{n\}) \, |\{s' \in \boldsymbol{succ}_T(s) \mid h^* \restriction \psi(s') \subseteq h\}| \geq g(|s|, r_T(|s|) - 1) \tag{4.8}$$

is greater than $1 - 1/2^{J^2}$ and, moreover, the event (4.8) is independent from the event that $h \supseteq \bar{h}$ because ψ has disjoint range. It follows that there is some $z \in Z$ such that (4.8) holds for $z = h$ and $\bar{h} \subseteq z$. It follows that there is $T^* \subseteq T$ such that $T^* \in \mathbf{PT}_{f,g}$ and such that if $s \in T^*$ then $h^* \restriction \psi(s) \subseteq z$. Then $T^* \Vdash_{\mathbf{PT}_{f,g}}$ "$\dot{h} \subseteq z$". Since $T \in G$ was arbitrary, it follows by genericity that it is possible to find $T^* \in G$. □

Theorem 4.18. *Let V be a model of set theory and suppose that $U : \omega_1^2 \to 2$ is a measure saturated function in V and that $G \subseteq \mathbf{PT}_{f,g}$ is generic over V. In $V[G]$ let $H \subseteq \mathbb{P}_{\mathcal{S}_a(\dot{G})}$ be generic over $V[G]$. Then in $V[G][H]$ the graph U is universal.*

Proof. This is the same as the proof of Theorem 3.9. □

Corollary 4.19. *It is consistent that $\mathfrak{b} = \mathfrak{d} = \aleph_1$ and $2^{\aleph_1}$ is arbitrarily large and there is a universal graph on ω_1.*

Proof. This the same as the proof of Corollary 2.13 but using that $\mathbf{PT}_{f,g}$ is ω^ω bounding. □

5. Tree partial orders with additional structure

5.1. The general framework

The results of the preceding sections were originally obtained by a more complicated argument that was eventually replaced by the simpler arguments described in §2, §3 and §4. However, there is one result for which the simplified argument does not seem to be sufficient; this is the question of finding a universal function from ω_1^2 to ω rather than a universal function from ω_1^2 to 2. This section will describe an argument showing that it is consistent with $\mathfrak{b} < \mathfrak{d}$ — indeed, $\boldsymbol{non}(\mathcal{M}) = \aleph_1 < \mathfrak{d} = \aleph_2$ — that there is a universal function from ω_1^2 to ω; this can also be deduced from the results of [3]. The argument has wider applicability though, which motivated readers will be able to find on their own. For example, replacing Miller forcing with Laver forcing in this section will yield similar results but with $\mathfrak{b} = \aleph_2$.

Definition 5.1. Given $G_0 : \omega_1^2 \to \omega$ and $G_1 : \omega_1^2 \to \omega$ let $\mathcal{E}(G_0, G_1)$ denote the set of all finite, one-to-one functions e that are isomorphisms between $G_1 \restriction \boldsymbol{domain}(e)^2$ and $G_0 \restriction \boldsymbol{range}(e)^2$; in other words, $G_1(\eta, \zeta) = G_0(e(\eta), e(\zeta))$ all distinct η and ζ in the domain of e.

It is, of course, possible that $\mathcal{E}(G_0, G_1)$ is empty; for example, this would happen were it the case that G_0 and G_1 had disjoint ranges. However, the $\mathcal{E}(G_0, G_1)$ of interest in this section will be those for which G_0 is a candidate for a universal function.

Definition 5.2. If $G_0 : \omega_1^2 \to \omega$ and $G_1 : \omega_1^2 \to \omega$ and $T \subseteq \mathbb{T}_\omega$ is a tree then a function $E : T \to \mathcal{E}(G_0, G_1)$ will be called *good* provided that:

(a) if s and t belong to T and $s \sqsubseteq t$ then $E(s) \subseteq E(t)$
(b) if s and t belong to T then $\boldsymbol{range}(E(t)) \cap \boldsymbol{range}(E(s)) = \boldsymbol{range}(E(s \wedge t))$ where $s \wedge t$ represents the largest initial segment common to both s and t.

The following definitions will be used only in the context of $\mathbb{P} = \mathbf{PT}$ but it seems worth providing the more general context since the definitions are applicable for any partial order consisting of trees ordered by inclusion.

Definition 5.3. Let $\mathbb{P}$ be a tree partial order. If $G_0 : \omega_1^2 \to \omega$ and $G_1 : \omega_1^2 \to \omega$ then define $\mathbb{P}_{G_0,G_1}$ to consist of triples (T, E, η) such that

1. $T \in \mathbb{P}$
2. $E : T \to \mathcal{E}(G_0, G_1)$ is good
3. $\eta \in \omega_1$.

If $p = (T, E, \eta) \in \mathbb{P}_{G_0,G_1}$ the notation (T^p, E^p, η^p) will be used to denote (T, E, η). Define $p \leq q$ if and only if

4. $T^p \subseteq T^q$
5. $E^p(t) = E^q(t)$ if $t \sqsubseteq \boldsymbol{stem}(T^p)$ and $t \neq \boldsymbol{stem}(T^p)$
6. $E^p(t) \supseteq E^q(t)$ for each $t \in T^p \setminus_{\boldsymbol{stem}}$
7. $\boldsymbol{range}(E^p(t) \setminus E^q(t)) \cap \eta^q = \varnothing$ for all $t \in T^p$
8. $\eta^p \geq \eta^q$.

Definition 5.4. If $H \subseteq \mathbb{P}_{G_0,G_1}$ is generic define $E_H : \omega_1 \to \omega_1$ by $E_H = \bigcup_{p \in H} E(\boldsymbol{stem}(T^p))$.

It is immediate that E_H is a partial embedding of G_1 into G_0. However, some extra requirements will be needed to guarantee that E_H is a total embedding. The following arguments restrict attention to Miller forcing.

5.2. Modifying Miller forcing

The first thing to check is that $\mathbf{PT}_{G_0,G_1}$ is proper, indeed, that it satisfies Axiom A. This will rely on the partial orders $\prec_n$ defined in Definition 7.3.44 of [9]. However, the proof is not immediate as this is the place at which η^p, the third component of $p \in \mathbf{PT}_{G_0,G_1}$, is used.

Lemma 5.5. *If $G_0 : \omega_1^2 \to \omega$ and $G_1 : \omega_1^2 \to \omega$ then $\mathbf{PT}_{G_0,G_1}$ satisfies Axiom A.*

Proof. For p and q in $\mathbf{PT}_{G_0,G_1}$ define $p \prec_n q$ if

1. $p \leq q$
2. $\mathbf{split}_n(T^p) = \mathbf{split}_n(T^q)$
3. $E^p(t) = E^q(t)$ for all $t \in \mathbf{split}_n(T^p)$.

It needs to be verified that these partial orders witness that $\mathbf{PT}_{G_0,G_1}$ satisfies Axiom A. The only point that needs an argument is that given a dense $D \subseteq \mathbf{PT}_{G_0,G_1}$, $p \in \mathbf{PT}_{G_0,G_1}$ and $n \in \omega$ there is $q \prec_n p$ such that for each $t \in \mathbf{split}_n(T^p) = \mathbf{split}_n(T^q)$ the conditions $(T^q[s], E^q \restriction T^q[s], \eta^q)$ all belong to D if $s \in \boldsymbol{succ}_{T^q}(t)$.

To see that this is so, let $\{t_k\}_{k\in\omega}$ enumerate $\bigcup_{t\in \mathbf{split}_n(T^p)} \boldsymbol{succ}_{T^p}(t)$. Construct inductively $p_k \in \mathbf{PT}_{G_0,G_1}$ and $\eta_k \in \omega_1$ such that:

- $\eta_0 \geq \eta^p$ and $\eta_0 \supseteq \bigcup_{t\in T^p} \boldsymbol{range}(E^p(t))$
- $\eta_k \supseteq \bigcup_{i\in k} \bigcup_{s\in T^{p_i}} \boldsymbol{range}(E^{p_i}(s))$
- $\eta_k \geq \eta^{p_i}$ for $i \in k$
- $p_k \leq (T^p[t_k], E^p \restriction T^p[t_k], \eta_k)$
- $p_k \in D$.

Then let $T^q = \bigcup_{k\in\omega} T^{p_k}$ and $E^q = \bigcup_{k\in\omega} E^{p_k}$ and $\eta^q = \bigcup_{k\in\omega} \eta^{p_k}$ and set $q = (T^q, E^q, \eta^q)$. Note that Condition (b) of Definition 5.2 holds because of (7) in Definition 5.3. Note also that Condition (5) of Definition 5.3 is used to ensure that E^q is well defined. □

Lemma 5.6. *If $G_0 : \omega_1^2 \to \omega$ is category saturated and $\xi \in \omega_1$ then*

$$\{p \in \mathbf{PT}_{G_0,G_1} \mid \xi \in \boldsymbol{domain}(E^p(\boldsymbol{stem}(T^p)))\}$$

is dense in $\mathbf{PT}_{G_0,G_1}$.

Proof. Let $p \in \mathbf{PT}_{G_0,G_1}$. For $s \in \mathbf{split}_m(T^p)$ let

$$\mathcal{F}(s) = \left\{E^p(t) \setminus E^p(s) \;\middle|\; t \in \mathbf{split}_{m+1}(T) \text{ and } t \sqsupseteq s\right\}$$

and note that any two distinct functions in $\mathcal{F}(s)$ have disjoint ranges by Condition (b) of Definition 5.2. For $f \in \mathcal{F}(s)$ define $F_{\xi,f} : \boldsymbol{range}(f) \to 2$ by $F_{\xi,f}(f(\zeta)) = G_1(\zeta, \xi)$.

Let $\mu \in \omega_1$ be so large that the range of each $E^p(t)$ is contained in μ. Note that if

$$\mathcal{D}(s) = \{h \in 2^\mu \mid h \supseteq F_{\xi,f} \text{ for infinitely many } f \in \mathcal{F}(s)\}$$

then $\mathcal{D}(s)$ is dense a G_δ subset of 2^μ with the product topology. Since G_0 is category saturated it is possible to find $\xi^* > \eta^p$ such that $G_0^{\xi^*} \restriction \mu \in \mathcal{D}(s)$ for all $s \in \mathbf{split}(T^p)$ where $G_0^{\xi^*}$ is defined in Definition 1.3. Moreover, because $\{G_0^\xi\}_{\xi\in\omega_1}$ is everywhere of second category, it can be assumed that $G_0(E^p(\boldsymbol{stem}(T^p))(\zeta), \xi^*) = G_1(\zeta, \xi)$ for each $\zeta \in \boldsymbol{domain}(\boldsymbol{stem}(T^p))$.

Now let T be set of all initial segments of elements of

$$\left\{t \in T^p_{\setminus \boldsymbol{stem}} \;\middle|\; F_{\xi, E^p(t)} \subseteq G_0^{\xi^*}\right\}$$

and note that $T \in \mathbf{PT}$ because Condition (5) of Definition 5.3 allows $\psi(\boldsymbol{stem}(T^p))$ to change. Hence, that $(T, E^p \restriction T, \eta^p) \in \mathbf{PT}_{G_0,G_1}$. Moreover if E is defined by

$$E(t) = \begin{cases} E^p(t) \cup \{(\xi, \xi^*)\} & \text{if } t \sqsupseteq \boldsymbol{stem}(T) \\ E^p(t) & \text{otherwise} \end{cases}$$

then E is good and hence $(T, E, \eta^p) \in \mathbf{PT}_{G_0,G_1}$. □

The following is the version of Theorem 7.3.46 required for $\mathbf{PT}_{G_0,G_1}$.

Theorem 5.7. $\mathbf{PT}_{G_0,G_1}$ *preserves* $\sqsubseteq^{\mathbf{Cohen}}$ *as defined in Definition 6.3.15 on page 295 of [9].*

Proof. Given $p \in \mathbf{PT}_{G_0,G_1}$ and a countable elementary submodel $\mathfrak{N}$ such that

$$p \Vdash_{\mathbf{PT}_{G_0,G_1}} \text{“}\dot{c} \text{ is a Cohen real over } \mathfrak{N}\text{”}$$

let $T' \in \mathbf{PT}$ be the tree constructed in the proof of Theorem 7.3.46 in [9]. Observe that this proof does not require extending E^p or η^p. Hence $q = (T', E^p \restriction T', \eta^p) \in \mathbf{PT}_{G_0,G_1}$ and it is easy to check that

$$q \Vdash_{\mathbf{PT}_{G_0,G_1}} \text{“}\dot{c} \text{ is a Cohen real over } \mathfrak{N}[\dot{G}]\text{”}. \quad \square$$

Corollary 5.8. *The countable support iteration of partial orders of the form* $\mathbf{PT}_{G_0,G_1}$ *preserves the non-meagreness of non-meagre sets from the ground model.*

Proof. This follows from Theorem 6.3.20 in [9]. □

Theorem 5.9. *It is consistent with set theory that* $\mathfrak{b} = \aleph_1$, $\mathfrak{d} = \aleph_2 = \mathfrak{c}$ *and there is a universal function from* ω_1^2 *to* ω.

Proof. Let V be a model of the Continuum Hypothesis and, in V, let $\{\dot{G}_\xi\}_{\xi\in\omega_2}$ enumerate all $\mathbb{P}$ names for functions from ω_1^2 to ω where $\mathbb{P}$ is a partial order of cardinality $\aleph_1$. Also in V, using Lemma 1.5 let G be a category saturated function from ω_1^2 to ω. Let $\mathbb{P}_\xi$ be defined inductively so that if ξ is a limit ordinal then $\mathbb{P}_\xi$ is the countable support limit of $\{\mathbb{P}_\eta\}_{\eta\in\xi}$ and, if G_ξ is a $\mathbb{P}_\xi$ name then $\mathbb{P}_{\xi+1} = \mathbb{P}_\xi * \mathbf{PT}_{G,G_\xi}$ and let $\mathbb{P}_{\xi+1} = \mathbb{P}_\xi * \mathbf{PT}$ otherwise.

If $H \subseteq \mathbb{P}_{\omega_2}$ is generic over V and $\bar{G} : \omega_1^2 \to \omega$ belongs to $V[H]$ then, since $\mathbb{P}_{\omega_2}$ is proper by Lemma 5.5, it follows that $\bar{G} \in V[H \cap \mathbb{P}_\xi]$ for some $\xi \in \omega_2$. Hence there is some $\eta \in \omega_2$ such that G_η is a $\mathbb{P}_\eta$ name for $\bar{G}$. By Theorem 5.7, Lemma 6.3.19 of [9] and Corollary 5.8 it follows that $1 \Vdash_{\mathbb{P}_\eta}$ “G is category saturated ” and hence by Lemma 5.6 it follows that the domain of $E_{H\cap\mathbf{PT}_{G,G_\eta}}$ as defined in Definition 5.4 is all of ω_1 and so $E_{H\cap\mathbf{PT}_{G,G_\eta}}$ is an embedding of G_η into G as required. Since G is category saturated in $V[H]$ it follows that $\mathfrak{b} \leq \boldsymbol{non}(\mathcal{M}) = \aleph_1$. It is easy to see that each $\mathbf{PT}_{G,G_\xi}$ adds an unbounded real and so $\mathfrak{d} = \aleph_2$ in $V[H]$. □

Note that, unlike the corresponding arguments in §2, §3 and §4, the argument of Theorem 5.9 deals with only one function at each stage of the iteration. While in the earlier sections the saturated graph constructed using the Continuum Hypothesis in the ground model remains universal at all stages of the iteration, in Theorem 5.9 this is only achieved at the end of the iteration. Hence it is not possible to arrange to start with a ground model where $2^{\aleph_1}$ is arbitrarily large as in the earlier sections.

6. The key combinatorial lemma

This section describes the combinatorial reason why the existence of a universal graph does not imply the existence of a universal function with range ω.

Lemma 6.1. *If $\mathfrak{b} = \aleph_1$ and there is a sequence of pairs of natural numbers $\{(m_i, n_i)\}_{i\in\omega}$ such that $m_i < n_i < m_{i+1}$ for each $i \in \omega$ and*

$$\left(\forall \mathcal{F} \in \left[\prod_{i\in\omega} [n_i]^{m_i}\right]^{\aleph_1}\right)\left(\exists g \in \prod_{i\in\omega} n_i\right)(\forall f \in \mathcal{F})\,(\exists m \in \omega)(\forall k \geq m)\; g(k) \notin f(k) \tag{6.1}$$

then there is no universal $c : \omega_1^2 \to \omega$.

Proof. Let $B_\eta : \eta \to \omega$ be a bijection for each $\eta \in \omega_1$. Suppose that $c : \omega_1^2 \to \omega$ is a universal function. If $\eta \in \xi \in \omega_1$ and $j \in \omega$ let

$$f_{\eta,\xi}(j) = \left\{c(B_\eta^{-1}(k), \xi) \in n_j \;\mid k \in m_j\right\} \tag{6.2}$$

and use the hypothesis of the lemma to find a function $g_\eta \in \prod_{i\in\omega} n_i$ such that $g_\eta(j) \notin f_{\eta,\xi}(j)$ for every $\xi \in \omega_1$ and for all but finitely many $j \in \omega$.

Let $\mathcal{U}$ be a family of increasing functions from ω to ω that is $\leq^*$ unbounded and such that $|\mathcal{U}| = \aleph_1$. Let $\psi : \mathcal{U} \times \omega_1 \to \omega_1$ be a bijection and define

$$b : \omega \times \omega_1 \to \omega$$

by $b(j, \psi(u, \eta)) = g_\eta(u(j))$.

Now suppose that $e : \omega_1 \to \omega_1$ is an embedding of the partial function b into c. Let η be such that $e(j) \in \eta$ for all $j \in \omega$ and let $u \in \mathcal{U}$ be such that there are infinitely many k such that $B_\eta(e(k)) \in m_{u(k)}$. Choose j so large that $g_\eta(u(j)) \notin f_{\eta,e(\psi(u,\eta))}(u(j))$ and such that $B_\eta(e(j)) \in m_{u(j)}$. Then

$$b(j, \psi(u, \eta)) = g_\eta(u(j)) \neq c(B_\eta^{-1}(B_\eta(e(j))), e(\psi(u, \eta))) = c(e(j), e(\psi(u, \eta))) \tag{6.3}$$

contradicting that e is an embedding. □

Corollary 6.2. *The existence of a universal graph on ω_1 does not imply the existence of a universal function from ω_1^2 to ω.*

Proof. In the model establishing Corollary 4.19 there is a universal graph. This model is obtained from an ω_2 length iteration with the partial orders $\mathbf{PT}_{f,g}$ as cofinally many iterands. Using $n_i = f(i)$ and $m_i = g(i, 0)$ it follows that given $\mathcal{F}$ in an intermediate model at stage ζ, any $\mathbf{PT}_{f,g}$-generic function in $\prod_{i\in\omega} n_i$ added at stage after ζ witnesses that Condition (6.1) holds in this model. This is true because for any $T \in \mathbf{PT}_{f,g}$ and for $t \in T$ such that $|t|$ is sufficiently large it will be the case $|\,\boldsymbol{succ}_T(t)| - m_{|t|} \geq g(|t|, r_T(|t|) - 1)$. Now apply Lemma 6.1 to get that there is no universal function from ω_1^2 to ω. □

7. Allowing embeddings to permute the range

There are various ways of generalizing the notions of embeddings discussed in the introduction. The following two are singled out because something can be said about them.

Definition 7.1. A function $U : \omega_1^2 \to \lambda$ will be called (ρ_0, ρ_1)-*weakly universal* if for every $f : \omega_1^2 \to \lambda$ there exists a one-to-one function $h : \omega_1 \to \omega_1$ and functions $e_0 : \lambda \to \lambda$ and $e_1 : \lambda \to \lambda$ such that

$$(\forall \alpha \in \omega_1)(\forall \beta \in \omega_1)\ e_0(f(\alpha, \beta)) = e_1(U(h(\alpha), h(\beta))) \tag{7.1}$$

$$\text{if } \rho_i > 1 \text{ then } (\forall \xi \in \lambda)\ |e_i^{-1}(\xi)| < \rho_i \tag{7.2}$$

$$\text{if } \rho_i = 1 \text{ then } (\forall \xi \in \lambda)\ e_i(\xi) = \xi \tag{7.3}$$

The triple (h, e_0, e_1) will be called a (ρ_0, ρ_1)-weak embedding and, of course, in the case that $\rho_i = 1$ the e_i can be ignored. Note that a $(1, 1)$ weak embedding is just an embedding.

While asking for the function $U : \omega_1^2 \to \lambda$ to be $(\lambda^+, 1)$-weakly universal is trivial (just let U be constant) this is not so clear for the notion of $(1, \lambda^+)$-weakly universal.

Proposition 7.2. *A function $U : \omega_1^2 \to \lambda$ is $(1, \lambda^+)$-weakly universal if and only if for every $f : \omega_1^2 \to \lambda$ there exists a one-to-one function $h : \omega_1 \to \omega_1$ such that*

$$\textit{if } f(\alpha, \beta) \neq f(\alpha^*, \beta^*) \textit{ then } U(h(\alpha), h(\beta)) \neq U(h(\alpha^*), h(\beta^*))$$

for all α, β, α^* and β^* in ω_1.

Proof. Given the property define $e_1 : \lambda \to \lambda$ by $e_1(U(h(\alpha), h(\beta))) = f(\alpha, \beta)$. The other implication is even more trivial. □

Trivial as it may seem, it is worth noting that Proposition 7.2 implies that it is easy to find a $(1, \omega_2)$-weakly universal function $U : \omega_1^2 \to \omega_1$ (simply make U one-to-one). It is shown in unpublished notes of Tanmay Inamdar [16] that Martin's Axiom for partial orders with Knaster's Property implies that there is a $(1, \aleph_1)$-weakly universal function from ω_1^2 to ω. That argument is reproduced in the following. Recall that a partial order has Knaster's Property (Property K) if every uncountable subset contains an uncountable subset, any two of whose elements are compatible.

Proposition 7.3. *Martin's Axiom for partial orders with Property K implies that there is a $(1, \aleph_1)$-weakly universal function from ω_1^2 to ω.*

Proof. Begin by constructing $\Psi_\alpha : \alpha \to \omega$ by induction on $\alpha \in \omega_1$ such that for each α and $Z \in [\omega_1 \setminus \alpha]^{<\aleph_0}$ there is $S \in [\alpha]^{<\aleph_0}$ such that for all distinct ξ and η in $\alpha \setminus S$ and ζ and μ in Z

$$\Psi_\zeta(\xi) \neq \Psi_\mu(\eta).$$

If $\beta \in \alpha$ let $U(\alpha, \beta) = \Psi_\alpha(\beta)$.

Given $F : \omega_1^2 \to \omega$ let $\{\mathfrak{M}_\xi\}_{\xi \in \omega_1}$ be a continuous, increasing sequence of countable elementary submodels of $(H(\aleph_2), U, F, \in)$ and let $\mathbb{P}$ consist of all finite functions e such that

- $e : \Gamma \to \omega_1$ is a strictly increasing function with $\Gamma \in [\omega_1]^{<\aleph_0}$
- if $\{\alpha, \beta\}, \{\alpha', \beta'\} \in [\Gamma]^2$ and $F(\alpha, \beta) \neq F(\alpha', \beta')$ then $U(\alpha, \beta) \neq U(\alpha', \beta')$
- if $\gamma \in \Gamma \cap \mathfrak{M}_\xi$ then $e(\gamma) \in \mathfrak{M}_\xi$.

Now if $\{e_\xi\}_{\xi\in\omega_1} \subseteq \mathbb{P}$ then let $A \subseteq \omega_1$ be uncountable and $k \in \omega$ be such that $|e_\alpha| = k$ for each $\alpha \in A$ and for each α and β in A if $\psi : \boldsymbol{domain}(e_\alpha) \to \boldsymbol{domain}(e_\beta)$ is the order preserving bijection

$$U(e_\alpha(\xi), e_\alpha(\eta)) = U(e_\beta(\psi(\xi)), e_\beta(\psi(\eta)))$$
$$F(\xi,\eta) = F(\psi(\xi),\psi(\eta))$$

for all ξ and η in the domain of e_α. Moreover, it can be assumed that $\{\boldsymbol{range}(e_\alpha)\}_{\alpha\in A}$ form a Δ-system with root E and that if $\alpha \in \beta$ then

$$\max(E) < \min(\boldsymbol{range}(e_\alpha) \setminus E) < \max(\boldsymbol{range}(e_\alpha) \setminus E) < \min(\boldsymbol{range}(e_\beta) \setminus E).$$

Now suppose that $\{\alpha_n\}_{n\leq\omega}$ is a subset of A enumerated in increasing order. Let $\alpha = \min(\boldsymbol{range}(e_{\alpha_\omega}) \setminus E)$ and let $Z = \boldsymbol{range}(e_{\alpha_\omega}) \setminus E$. Then apply the property of U to find $S \in [\alpha]^{<\aleph_0}$ such that for all distinct ξ and η in $\alpha \setminus S$ and ζ and μ in Z

$$\Psi_\zeta(\xi) \neq \Psi_\mu(\eta).$$

Then let $n \in \omega$ be so large that $S \cap \boldsymbol{range}(e_{\alpha_n}) = \varnothing$. It is then easy to check that $e_{\alpha_n} \cap e_{\alpha_\omega} \in \mathbb{P}$. An application of the Dushnik–Miller partition relation $\omega_1 \to (\omega_1, \omega + 1)$ then yields that $\mathbb{P}$ has Property K. The necessary density can be proved by an argument similar to that of the chain condition. □

On the other hand, it will follow from Proposition 8.10 that it is consistent with Martin's Axiom for partial orders with Property K of cardinality $\aleph_1$ that there is no universal function from ω_1^2 to ω; Corollary 5.22 of [7] establishes that it is consistent with Martin's Axiom for ccc partial orders of cardinality $\aleph_1$ that there is not even a universal function ω_1^2 to 2. However, Osvaldo Guzmán has shown in [17] that the hypothesis of Proposition 7.3 cannot be eliminated.

Theorem 7.4 *(Guzmán). It is consistent that there is no $(1, \aleph_1)$-weakly universal function from ω_1^2 to ω.*

In the same paper Guzmán shows that there are $(1, \aleph_1)$-weakly universal functions in the Sacks model. A key motivating question from the introduction was Problem 7.8 of [7].

Question 7.5. Does the existence of a $(2, 1)$-weakly universal function from ω_1^2 to ω imply the existence of a universal function from ω_1^2 to ω?

As will be seen by the next result, this question has an elementary solution not requiring the forcing methods developed in earlier sections if an assumption about the cardinal invariant $\mathfrak{d}$ is made.

Proposition 7.6. *If $\mathfrak{d} = \aleph_1$ then the existence of a $(1, 2)$-weakly universal function from ω_1^2 to ω implies the existence of a universal function from ω_1^2 to ω.*

Proof. Let $U : \omega_1^2 \to \omega$ be a $(1, 2)$-weakly universal function. It will first be shown that there is some $D : \omega \to \omega$ such that for every $f : \omega_1^2 \to \omega$ there is an embedding $h : \omega_1 \to \omega_1$ and a bijection $e : \omega \to \omega$ such that

1. $f(\alpha, \beta) = e(U(h(\alpha), h(\beta))$ for all α and β
2. $e(n) < D(n)$ for all n
3. $e^{-1}(n) < D(n)$ for all n.

If this fails then let $\mathfrak{D}$ be a dominating family in ω^ω of cardinality $\aleph_1$ and for each $D \in \mathfrak{D}$ let f_D be a function be such that there are no h and e satisfying Conditions (1), (2) and (3). There is no harm in assuming that the domain of f_D is X_D^2 and that the $\{X_D\}_{D\in\mathfrak{D}}$ are pairwise disjoint. Then let $f = \bigcup_{D\in\mathfrak{D}} f_D$.

Using that U is a $(1,2)$-weakly universal function let $h : \omega_1 \to \omega_1$ and $e : \omega \to \omega$ be such that Condition (1) is satisfied for f. Since $\mathfrak{D}$ is dominating there must be some $D \in \mathfrak{D}$ such that Conditions (2) and (3) are satisfied for e and D. Restricting h to X_D^2 yields a contradiction.

Now fix D satisfying Conditions (1), (2) and (3). Then let $\{J_i\}_{i\in\omega} \subseteq \omega$ be an increasing sequence such that if $n \le J_i$ then $D(n) < J_{i+1}$. Then define $U^*(\alpha,\beta) = i$ if $J_{2i-1} \le U(\alpha,\beta) < J_{2i+1}$. To see that U^* is a universal function from ω_1^2 to ω let $f^* : \omega_1^2 \to \omega$. Define $f(\alpha,\beta) = J_{2f^*(\alpha,\beta)}$. Using the choice of D, let $h : \omega_1 \to \omega_1$ and $e : \omega \to \omega$ be such that Conditions (1), (2) and (3) are satisfied. Then to verify that $U^*(h(\alpha),h(\beta)) = f^*(\alpha,\beta)$ it must be shown that $J_{2f^*(\alpha,\beta)-1} \le U(h(\alpha),h(\beta)) < J_{2f^*(\alpha,\beta)+1}$.

To see that this is so, assume first that $U(h(\alpha),h(\beta)) < J_{2f^*(\alpha,\beta)-1}$. Then from Condition (1) it follows that $e(U(h(\alpha),h(\beta))) = f(\alpha,\beta) = J_{2f^*(\alpha,\beta)}$. However, by the choice of $J_{2f^*(\alpha,\beta)}$ it follows that $D(U(h(\alpha),h(\beta))) < J_{2f^*(\alpha,\beta)}$ and Condition (2) then yields the contradiction that $e(U(h(\alpha),h(\beta))) < D(U(h(\alpha),h(\beta))) < J_{2f^*(\alpha,\beta)}$. On the other hand, it again follows from Condition (1) that $e(U(h(\alpha),h(\beta))) = f(\alpha,\beta) = J_{2f^*(\alpha,\beta)}$ and hence, by Condition (3), that $U(h(\alpha),h(\beta)) = e^{-1}(J_{2f^*(\alpha,\beta)}) < D(J_{2f^*(\alpha,\beta)}) < J_{2f^*(\alpha,\beta)+1}$. □

Corollary 7.7. *If $\mathfrak{d} = \aleph_1$ then the existence of a $(2,1)$-weakly universal function from ω_1^2 to ω implies the existence of a universal function from ω_1^2 to ω.*

Proof. The existence of a $(1,2)$-weakly universal function from ω_1^2 to ω is equivalent to the existence of a $(2,1)$-weakly universal function from ω_1^2 to ω. □

Proposition 7.8. *If $\mathfrak{d} = \aleph_1$ and there is an $(\aleph_0,k)$-weakly universal function from ω_1^2 to ω then there is a $(2,k)$-weakly universal function from ω_1^2 to ω.*

Proof. Suppose there is an $(\aleph_0,k)$-weakly universal function $U : \omega_1^2 \to \omega$ but no $(2,k)$-weakly universal function from ω_1^2 to ω. Let $\mathfrak{D} \subseteq \omega^\omega$ be a dominating family of cardinality $\mathfrak{d}$ consisting of strictly increasing functions with non-zero values. For $d \in \mathfrak{D}$ let

$$K_d^n = \underbrace{d \circ d \circ \ldots \circ d}_{n \text{ iterations}}(0)$$

and let $F_d : \omega \to \omega$ be defined by $F_d(i) = K_d^{4j}$ for all i such that $K_d^{4j-2} \le i < K_d^{4j+2}$. Let k_d be the function defined by $k_d(j) = K_d^{4j}$ and note that k_d is one-to-one.

Then $F_d \circ U : \omega_1^2 \to \{K_d^{4j}\}_{j\in\omega}$ is not a $(2,k)$-weakly universal function and so there is a witness to this, namely a function W_d that is not a $(2,k)$-weakly embeddable into $F_d \circ U$. There is no harm in assuming that there is a partition $\{P_d\}_{d\in\mathfrak{D}}$ of ω_1 such that $W_d : P_d^2 \to \omega$. Let

$$W = \bigcup_{d\in\mathfrak{D}} k_d \circ W_d.$$

Using that U is $(\aleph_0,k)$-weakly universal let $h : \omega_1 \to \omega_1$ be one-to-one, let $e_0 : \omega \to \omega$ be finite-to-one and let $e_1 : \omega \to \omega$ have fibres smaller than k such that $e_0(W(\alpha,\beta)) = e_1(U(h(\alpha),h(\beta)))$ for all α and β. Now let $d \in \mathfrak{D}$ be such that

$$d(i) \supseteq \bigcup_{j\le i} \{e_0^{-1} \circ e_1(j), e_1^{-1} \circ e_0(j)\} \tag{7.4}$$

and note that $(h \restriction P_d, e_0, e_1)$ is an $(\aleph_0, k)$-weak embedding of $k_d \circ W_d$ to U. It will be shown that $(h \restriction P_d, k_d, e_1)$ is actually a $(2, k)$-weak embedding of W_d into $F_d \circ U$; in other words, that $k_d \circ W_d(\alpha, \beta) = e_1 \circ F_d \circ U(h(\alpha), h(\beta))$ for α and β in P_d.

To see that this is the case, let $(\alpha, \beta) \in P_d^2$. Then $k_d(W_d(\alpha, \beta)) = K_d^{2j}$ for $j = W_d(\alpha, \beta)$ and hence, using that $e_1(K_d^{4j-1}, K_d^{4j+1}) \subseteq (K_d^{4j-2}, K_d^{4j+2})$, it suffices to show that $K_d^{4j-1} \leq U(h(\alpha), h(\beta)) < K_d^{4j+1}$. But

$$e_0(K_d^{4j}) = e_0(k_d(W_d(\alpha, \beta))) = e_0(W(\alpha, \beta)) = e_1(U(h(\alpha), h(\beta)))$$

and so it suffices to show that $K_d^{4j-1} \leq e_1^{-1} e_0(K_d^{4j}) < K_d^{4j+1} = d(K_d^{4j})$. This follows immediately from (7.4). □

Corollary 7.9. *If $\mathfrak{d} = \aleph_1$ and there is an $(\aleph_0, 1)$-weakly universal function from ω_1^2 to ω then there is a universal function from ω_1^2 to ω.*

Proof. Use Proposition 7.8 and Corollary 7.7. □

8. Set valued universality

This section will consider weakening the notion of universality for functions by attaching a set of potential values to each edge of a graph. The following definition provides the main idea.

Definition 8.1. Given a family $\mathcal{A} \subseteq \mathcal{P}(\lambda)$ function $U : \omega_1^2 \to \mathcal{A}$ is $\mathcal{A}$-*weakly universal* if for every $f : \omega_1^2 \to \lambda$ there exists a one-to-one function $h : \omega_1 \to \omega_1$ such that $f(\alpha, \beta) \in U(h(\alpha), h(\beta))$ for all α and β in ω_1. The function h will be called and $\mathcal{A}$-embedding of f.

Definition 8.2. Consider the following assertions about the existence of certain types of $\mathcal{A}$-weakly universal functions:

(a) There is a $[\omega]^{<\aleph_0}$-weakly universal function.
(b) There is a $[m]^k$-weakly universal function for positive integers m and k. (In this case the λ of Definition 8.1 is m.)
(c) There is a $\{A \subseteq \omega \mid |\omega \setminus A| = \aleph_0\}$-weakly universal function.
(d) There is a $\{\omega \setminus \{j\} \mid j \in \omega\}$-weakly universal function.
(e) There is a $[\omega]^k$-weakly universal function for some positive integer k.

Proposition 8.3. *Assertion (c) of Definition 8.2 is equivalent to Assertion (d).*

Proof. It is immediate that Assertion (c) implies Assertion (d). To see the converse let $\{P_i\}_{i\in\omega}$ be a partition of ω into infinite sets. Given U witnessing that Assertion (d) holds, define $U^*(\alpha, \beta) = \bigcup_{j\in U(\alpha,\beta)} P_j$. Now if $F : [\omega_1]^2 \to \omega$ let $F^*(\alpha, \beta)$ satisfy that $F(\alpha, \beta) \in P_{F^*(\alpha,\beta)}$. If $e : \omega_1 \to \omega_1$ is an $\{\omega \setminus \{j\} \mid j \in \omega\}$-embedding of F^* into U then it is also a $\{A \subseteq \omega \mid |\omega \setminus A| = \aleph_0\}$-embedding of F into U^*. □

The following monotonicity is immediate.

Proposition 8.4. *If $a \leq b \leq c \leq d$ and Assertion (b) holds for $[d]^a$ then it also holds for $[c]^b$.*

Proposition 8.5. *If $k < m$ and Assertion (b) holds for $[k+1]^k$ then it also holds for $[m]^k$.*

Proof. Let $J = \binom{m}{k+1}$. It is then possible to choose functions $t_i : J \to k+1$ for $i \in m$ such that for each $x \in [m]^{k+1}$ there is $j \in J$ such that $\{t_i(j) \mid i \in x\} = k+1$. It then follows that if $\psi : (k+1)^J \to \mathcal{P}(J)$ is defined by

$$\psi(s) = \{i \in m \mid (\forall j \in J)\ s(j) \neq t_i(j)\}$$

then $|\psi(s)| \leq k$ for each $s \in (k+1)^J$.

Now, given a function U with domain ω_1^2 that is $[k+1]^k$-weakly universal let the domain of U^* be $\omega_1^J \times \omega_1^J$ and let $U^*(f,g) = \psi(f * g)$ where the function $f * g$ is defined by $(k+1) \setminus \{f * g(j)\}) = U(f(j), g(j))$. To see that U^* is $[m]^k$-weakly universal let $F : \omega_1^J \times \omega_1^J \to m$. Let $\{g_\xi\}_{\xi \in \omega_1}$ enumerate ω_1^J and define

$$F^* : (\omega_1 \times J) \times (\omega_1 \times J) \to k+1$$

to be any function such that $F^*((\zeta, j), (\eta, j)) = t_{F(g_\zeta, g_\eta)}(j)$. Let $e : \omega_1^J \to \omega_1$ be a $[k+1]^k$-embedding of F^* into U. Then define $\bar{e} : \omega_1^J \to \omega_1^J$ by $\bar{e}(g_\xi)(j) = e(\xi, j)$.

To see that $\bar{e}$ is a $[m]^k$-embedding of F into U^* let $(\eta, \zeta) \in \omega_1^2$. It must be shown that $F(g_\eta, g_\zeta) \in U^*(\bar{e}(g_\eta), \bar{e}(g_\zeta))$ or, in other words, that $F(g_\eta, g_\zeta) \in \psi(\bar{e}(g_\eta) * \bar{e}(g_\zeta))$. So it needs to be shown that

$$(\forall j \in J)\ \bar{e}(g_\eta) * \bar{e}(g_\zeta) \neq t_{F(g_\eta, g_\zeta)}(j)$$

and this is equivalent to

$$(\forall j \in J)\ t_{F(g_\eta, g_\zeta)}(j) \in U(\bar{e}(g_\eta)(j), \bar{e}(g_\zeta)(j)) = U(e(\eta, j), e(\zeta, j)).$$

But $U(e(\eta, j), e(\zeta, j))$ contains $F^*((\eta, j), (\zeta, j)) = t_{F(g_\eta, g_\zeta)}(j)$. □

Corollary 8.6. *If Assertion (b) of Definition 8.2 holds then there is a unique k such that*

- *there is a $[k+1]^k$-weakly universal function,*
- *there are $[m]^n$-weakly universal functions for all $m \geq n \geq k$*
- *there is no $[m]^n$-weakly universal function if $n < k$ and $m > n$.*

Proof. This is an immediate consequence of Propositions 8.4 and 8.5. □

Definition 8.7. Define

$$\mathfrak{K} = \sup(\{k \in \omega \mid \text{there is no } [k+1]^k\text{-weakly universal function}\})$$

and define $\mathfrak{K}_\omega = \sup(\{k \in \omega \mid \text{there is no } [\omega]^k\text{-weakly universal function}\})$.

It now follows that of the assertions listed in Definition 8.2 only the following need be considered:

(A) $\mathfrak{K} < \omega$.
(B) There is a $[\omega]^{<\aleph_0}$-weakly universal function.
(C) There is a $\{\omega \setminus \{j\} \mid j \in \omega\}$-weakly universal function.
(D) $\mathfrak{K}_\omega < \omega$.

Observe that Assertion (A) implies Assertion (B) which, in turn, implies Assertion (C). Moreover, Assertion (D) easily implies Assertion (A). The fact that none of these implications can be reversed will follow from a collection of propositions.

- **Assertion (C) does not imply Assertion (B).** By Corollary 3.10 it follows from Proposition 8.9 that there is a $\{\omega \setminus \{j\} \mid j \in \omega\}$-weakly universal function in the model of that corollary. However, since $\mathfrak{b} > \aleph_1$ in that model, Proposition 8.8 implies that there is no $[\omega]^{<\aleph_0}$-weakly universal function in that model.
- **Assertion (B) does not imply Assertion (A).** Consider the model obtained by adding $\aleph_2$ random reals to a model of the Continuum Hypothesis. Then $\mathfrak{d} = \aleph_1$ in that model and so Proposition 8.14 implies that Assertion (B) holds. On the other hand, Corollary 8.13 says that Assertion (A) fails.
- **Assertion (A) does not imply Assertion (D).** This is Corollary 8.16.
- **Even Assertion (C) can fail.** This is the point of Corollary 8.11.

Proposition 8.8. *If $\mathfrak{b} > \aleph_1$ then there is no $[\omega]^{<\aleph_0}$-weakly universal function.*

Proof. Suppose that $U : \omega_1^2 \to [\omega]^{<\aleph_0}$ is an $[\omega]^{<\aleph_0}$-weakly universal function. For $\xi \in \eta \in \omega_1$ let $F_{\xi,\eta} : \xi \to \omega$ be chosen such that $F_{\xi,\eta}(\zeta) > \max(U(\eta, \zeta))$. Using the hypothesis that $\mathfrak{b} > \aleph_1$ let $F_\xi : \xi \to \omega$ be such that $F_\xi(\zeta) > F_{\xi,\eta}(\zeta)$ for all $\eta > \xi$ and for all but finitely many $\zeta \in \xi$. Let $F : \omega_1^2 \to \omega$ be defined by $F(\xi, \zeta) = F_\xi(\zeta)$ for $\zeta < \xi$.

If $e : \omega_1 \to \omega_1$ were a $[\omega]^{<\aleph_0}$-embedding of F into U then it would be possible to find a strictly increasing family $\{\rho_\xi\}_{\xi\in\omega_1}$ such that $e(\rho_\xi) < e(\rho_\eta)$ whenever $\xi \in \eta$. It would then be possible to find $n \in \omega$ such that

$$F_{e(\rho_\omega)}(e(\rho_n)) > F_{e(\rho_\omega),e(\rho_{\omega+1})}(e(\rho_n)) > \max(U(e(\rho_{\omega+1}), e(\rho_n)))$$

contradicting that $F(e(\rho_{\omega+1}), e(\rho_n)) \in U(e(\rho_{\omega+1}), e(\rho_n))$. □

Proposition 8.9. *If there is a universal graph on ω_1 then there is a $\{\omega \setminus \{j\} \mid j \in \omega\}$-weakly universal function.*

Proof. Using Theorem 1.2 let $U : \omega_1^2 \to 2$ be a universal function. Then let $U^*(\eta, \zeta) = \omega \setminus \{U(\eta, \zeta)\}$. Now if $F : \omega_1^2 \to \omega$ define $F^*\omega_1^2 \to 2$ by

$$F^*(\eta, \zeta) = \begin{cases} 0 & \text{if } F(\eta, \zeta) \neq 0 \\ 1 & \text{if } F(\eta, \zeta) = 0 \end{cases}$$

An embedding of F^* into U is also a $\{\omega \setminus \{j\} \mid j \in \omega\}$-embedding of F into U^*. □

Proposition 8.10. *Suppose that:*

- $U : \omega_1^2 \to \omega$
- $\mathbb{C}$ *is Cohen forcing*
- $1 \Vdash_{\mathbb{C}}$ *"$\mathbb{P}$ has Property K".*

Then there is a $\mathbb{C}$-name $\dot{F}$ such that $\mathbb{1} \Vdash_{\mathbb{C}}$ "$\dot{F} : \omega_1^2 \to 2$" and

$$\mathbb{1} \Vdash_{\mathbb{C}*\mathbb{P}} \text{“}(\forall h \in \omega_1^{\omega_1}) \textit{ if } h \textit{ is one-to-one then } (\exists\eta)(\exists\zeta \neq \eta)\ \dot{F}(\eta, \zeta) = U(h(\eta), h(\zeta))\text{”}.$$

Proof. Let $\tau : \omega_1^2 \to \omega$ witness that $\aleph_1 \not\to [\aleph_1]^2_{\aleph_0}$. (See [18] for Todorcevic's construction of such a function.) Let $\dot{c}$ be a Cohen name for the element of ω^ω added by the Cohen real and define $\dot{F} = \dot{c} \circ \tau$. If

$$\mathbb{1} \Vdash_{\mathbb{C}*\mathbb{P}} \text{“}\dot{h} : \omega_1 \to \omega_1 \text{ is one-to-one”}$$

then for each $\eta \in \omega_1$ let $(g_\eta, p_\eta) \in \mathbb{C} * \mathbb{P}$ and η^* be such that $(g_\eta, p_\eta) \Vdash_{\mathbb{C}*\mathbb{P}}$ "$\dot{h}(\eta) = \eta^*$". There is then some Cohen condition $g \in \mathbb{C}$ and an uncountable $\Gamma \subseteq \omega_1$ such that $g_\eta = g$ for all $\eta \in \Gamma$ and such that

$g \Vdash_{\mathbb{C}}$ "$p_\eta \not\perp p_\zeta$" if η and ζ are in Γ. Let $k \in \omega \setminus \boldsymbol{domain}(g)$. There are then η and ζ in Γ such that $\tau(\eta, \zeta) = k$. Extend g to g^* such that $g^*(k) = U(\eta^*, \zeta^*)$. and let p be such that $g^* \Vdash_{\mathbb{C}}$ "$p \leq p_\eta$ and $p \leq p_\zeta$". Then

$$g^* \Vdash_{\mathbb{C}} \text{“}\dot{F}(\eta, \zeta) = \dot{c} \circ \tau(\eta, \zeta) = \dot{c}(k) = g^*(k) = U(\eta^*, \zeta^*) = U(\dot{h}(\eta), \dot{h}(\zeta))\text{”}$$

as required. □

Corollary 8.11. *In any model obtained by adding $\aleph_2$ Cohen reals with finite support there is no $\{\omega \setminus \{j\} \mid j \in \omega\}$-weakly universal function.*

Proof. This is standard, but see the proof of Corollary 8.13. □

Proposition 8.12. *Suppose that $2 \leq k < \omega$ and:*

- $U : \omega_1^2 \to k$
- *$\mathbb{S}$ is Solovay's random forcing*
- $1 \Vdash_{\mathbb{S}}$ *"$\mathbb{P}$ has Property K".*

Then there is an $\mathbb{S}$-name $\dot{F}$ such that $\mathbb{1} \Vdash_{\mathbb{S}}$ "$\dot{F} : \omega_1^2 \to 2$" and

$$\mathbb{1} \Vdash_{\mathbb{S}*\mathbb{P}} \text{“}(\forall h \in \omega_1^{\omega_1}) \text{ if } h \text{ is one-to-one then } (\exists \eta)(\exists \zeta \neq \eta)\ \dot{F}(\eta, \zeta) = U(h(\eta), h(\zeta))\text{”.}$$

Proof. Let $\tau : [\omega_1]^2 \to \omega$ witness that $\aleph_1 \not\to [\aleph_1]^2_{\aleph_0}$. Think of $\mathbb{S}$ as forcing with subsets of k^ω with positive measure with respect to λ, the product of uniform distributions on k. Let $r : \omega \to k$ be a random real and define $\dot{F} = \dot{r} \circ \tau$. If

$$\mathbb{1} \Vdash_{\mathbb{S}*\mathbb{P}} \text{“}\dot{h} : \omega_1 \to \omega_1 \text{ is one-to-one”}$$

then for each $\eta \in \omega_1$ let $B_\eta \subseteq k^\omega$ be a compact set of positive measure and p_η and η^* be such that $(B_\eta, p_\eta) \in \mathbb{S} * \mathbb{P}$ and such that $(B_\eta, p_\eta) \Vdash_{\mathbb{S}*\mathbb{P}}$ "$\dot{h}(\eta) = \eta^*$". Let $\dot{\Omega}$ be a name such that $\mathbb{1} \Vdash_{\mathbb{S}}$ "$|\dot{\Omega}| = \aleph_1$ and $\{p_\eta\}_{\eta \in \dot{\Omega}}$ is linked". Let Γ be an uncountable subset of ω_1 and $\mu > 0$ be such that there are $\bar{B}_\eta \subseteq B_\eta$ such that $\lambda(\bar{B}_\eta) > \mu$ and $\bar{B}_\eta \Vdash_{\mathbb{S}}$ "$\eta \in \dot{\Omega}$" for each $\eta \in \Gamma$. Then, for each $\eta \in \Gamma$ let C_η be a clopen set in k^ω such that

$$\lambda(C_\eta \triangle \bar{B}_\eta) < \mu/2(k+1).$$

Then let Γ^* be an uncountable subset of Γ for which there is a clopen set C such that $C_\eta = C$ for every $\eta \in \Gamma^*$.

For $j \in \omega$ and $i \in k$ let $S(j, i) \subseteq k^\omega$ be the Boolean value of the statement $r(j) = i$. It is then possible to find some $j \in \omega$ such that $\lambda(S(j, i) \cap C) = \lambda(C)/k$ for each $i \in k$. There are then η and ζ in Γ^* such that $\tau(\eta, \zeta) = j$. But $\lambda(C \setminus \bar{B}_\eta \cap \bar{B}_\zeta) < 1/(k+1)$ and hence $S(j, i) \cap \bar{B}_\eta \cap \bar{B}_\zeta$ has positive measure for each $i \in k$. In particular, $\lambda(S(j, U(\eta^*, \zeta^*)) \cap \bar{B}_\eta \cap \bar{B}_\zeta) > 0$ and so it is possible to find $B \subseteq S(j, U(\eta^*, \zeta^*)) \cap \bar{B}_\eta \cap \bar{B}_\zeta$ of positive measure and p such that $B \Vdash_{\mathbb{S}}$ "$p \leq p_\eta$ and $p \leq p_\zeta$". Then

$$(B, p) \Vdash_{\mathbb{S}*\mathbb{P}} \text{“}\dot{F}(\eta, \zeta) = \dot{r} \circ \tau(\eta, \zeta) = \dot{r}(j) = B(j) = U(\eta^*, \zeta^*) = U(\dot{h}(\eta), \dot{h}(\zeta))\text{”}$$

as required. □

Corollary 8.13. *If V is a model of the continuum hypothesis and G is generic over V for the measure algebra on 2^{ω_2} then $\mathfrak{K} = \omega$ in $V[G]$.*

Proof. Begin by noting that the measure algebra $\mathbb{M}(2^{\omega_2})$ on the Baire subsets of 2^{ω_2} is forcing equivalent to the measure algebra $\mathbb{M}(k^{\omega_2})$ for any finite k greater than 1. Given $U : \omega_1^2 \to [k+1]^k$ there is $\alpha \in \omega_2$ such that $U \in V[G \cap \mathbb{M}(2^\alpha)]$. Then $G \cap \mathbb{M}(2^{[\alpha,\alpha+\omega)})$ is a random real over $V[G \cap \mathbb{M}(2^\alpha)]$ and $\mathbb{1} \Vdash_{\mathbb{M}(2^{[\alpha,\alpha+\omega)})}$ "$\mathbb{M}(2^{[\alpha+\omega,\omega_2)})$ has Property K". Hence Proposition 8.12 can be applied to U^* defined by $U(\alpha,\beta) \cup \{U^*(\alpha,\beta)\} = k+1$ in order to find $\dot{F}$ witnessing that $\mathfrak{K} > k$. □

Proposition 8.14. *If $\mathfrak{d} = \aleph_1$ then there is a $[\omega]^{<\aleph_0}$-weakly universal function.*

Proof. For each $\eta \in \omega_1$ let $\mathcal{D}_\eta$ be a dominating family on ω^η or cardinality $\aleph_1$ — in other words, for each $f : \eta \to \omega$ there is $d \in \mathcal{D}_\eta$ such that $d(\xi) > f(\xi)$ for all $\xi \in \eta$. Let $\{S_\eta\}_{\eta\in\omega_1}$ be a partition of ω_1 into uncountable sets. Then let $\mathcal{D}_\eta$ be enumerated as $\{d_{\eta,\zeta}\}_{\zeta\in S_\eta\setminus\eta}$. Then for $\xi < \eta < \zeta < \omega_1$ and $\zeta \in S_\eta$ define $U(\xi,\zeta) = d_{\eta,\zeta}(\xi)$ (thinking of $d_{\eta,\zeta}(\xi)$ as an initial segment of natural numbers) noting that there is no ambiguity in this definition.

Now given $F : \omega_1^2 \to \omega$ construct the embedding h by induction. If $h \restriction \alpha$ has been defined let η be so large that it contains the image of α under h. Let $f : \eta \to \omega$ be any function such that $f(h(\xi)) \supseteq F(\xi,\alpha)$ for all $\xi \in \alpha$. There is then $\zeta \in S_\eta$ such that $d_{\eta,\zeta}(\xi) > f(\xi)$ for all $\xi \in \eta$. Let $h(\alpha) = \zeta$ and note that if $\xi \in \alpha$ then

$$U(h(\xi),h(\alpha)) = U(h(\xi),\zeta) = d_{\eta,\zeta}(h(\xi)) > f(h(\xi)) \supseteq F(\xi,\alpha)$$

as required. □

Lemma 8.15. *If $\mathfrak{b} = \aleph_1$, k is a positive integer and there is a sequence of pairs of natural numbers $\{(m_i,n_i)\}_{i\in\omega}$ such that $m_i < n_i < m_{i+1}$ and*

$$\left(\forall \mathcal{F} \in \left[\prod_{i\in\omega} [n_i]^{km_i}\right]^{\aleph_1}\right)\left(\exists g \in \prod_{i\in\omega} n_i\right)(\forall f \in \mathcal{F})\,(\exists m \in \omega)(\forall k \geq m)\ g(k) \notin f(k) \tag{8.1}$$

then there is no $[\omega]^k$-weakly universal function.

Proof. This is exactly the proof of Lemma 6.1 except assuming that c is a $[\omega]^k$-weakly universal function and replacing Equation (6.2) with

$$f_{\eta,\xi}(j) = n_j \cap \left(\bigcup_{k\in m_j} c(B_\eta^{-1}(k),\xi)\right) \tag{8.2}$$

and replacing Equation (6.3) with

$$b(j,\psi(u,\eta)) = g_\eta(u(j)) \notin c(B_\eta^{-1}(B_\eta(e(j))), e(\psi(u,\eta))) = c(e(j), e(\psi(u,\eta))) \tag{8.3}$$

in order to contradict that e is a $[\omega]^k$-embedding. □

Corollary 8.16. *It is consistent that $\mathfrak{K}_\omega = \omega$ and $\mathfrak{K} = 0$.*

Sh:1088

Proof. In the model establishing Corollary 4.19 there is a universal graph and so $\mathfrak{K} = 0$. This model is obtained from an ω_2 length iteration with the partial orders $\mathbf{PT}_{f,g}$ as cofinally many iterands. Using $n_i = f(i)$ and $m_i = g(i, 0)$ it follows that given $\mathcal{F}$ in an intermediate model at stage ζ, any $\mathbf{PT}_{f,g}$-generic function in $\prod_{i\in\omega} n_i$ added at stage after ζ witnesses that Condition (8.1) holds in this model for any positive k. This is true because for any $T \in \mathbf{PT}_{f,g}$ and for $t \in T$ such that $|t|$ is sufficiently large it will be the case $|\boldsymbol{succ}_T(t)| - km_{|t|} \geq g(|t|, r_T(|t|) - 1)$. Now apply Lemma 8.15 to get that $\mathfrak{K}_\omega = \omega$. □

The final result of this section provides an alternate argument for part of Corollary 8.16. The following definition is well known, but is worth restating since it has many variants.

Definition 8.17. The statement ♣ is that there is a family $\{s_\xi\}_{\xi\in\omega_1}$ such that

- if ξ is a limit ordinal then $s_\xi \subseteq \xi$ is cofinal in ξ
- if ξ is a limit ordinal then the order type of s_ξ is ω
- for any uncountable $X \subseteq \omega_1$ there is a stationary set of ξ such that $s_\xi \subseteq^* X$.

Of course the s_ξ for non-limit ordinals ξ are irrelevant.

Proposition 8.18. *If $2^{\aleph_0} > \aleph_1$ and ♣ holds then $\mathfrak{K} = \omega$.*

Proof. Let $\{s_\eta\}_{\eta\in\omega_1}$ witness that ♣ holds and let s_ξ be enumerated as $\{s_{\xi,j}\}_{j\in\omega}$. Let $\psi : \omega_1 \to \omega_1^2$ be a bijection and let $\psi(\xi) = (\psi_0(\xi), \psi_1(\xi))$. Then let $C \subseteq \omega_1$ be a club such that $\psi \restriction \eta$ is a bijection of η and η^2 for each $\eta \in C$. There is then no harm in assuming that if η is a limit point of C and $\eta_0 < \eta_1$ are successive elements of $C \cap \eta$ then there is at most one j such that $\eta_0 \leq s_{\eta,j} < \eta_1$.

Given $U : \omega_1^2 \to [k+1]^k$ construct $F : \omega_1^2 \to k+1$ by induction. Suppose that $F \restriction \xi^2$ has been constructed. For each $\zeta \geq \xi$ let $f_{\xi,\zeta} : \omega \to k+1$ be the function defined by $\{f_{\xi,\zeta}(j)\} \cup U(\psi_1(s_{\xi,j}), \zeta) = k+1$. Since $2^{\aleph_0} > \aleph_1$ there is some function $g : \omega \to k+1$ such that for each $\zeta \geq \xi$ there are infinitely many j such that $g(j) = f_{\xi,\zeta}(j)$. Define $F(\psi_0(s_{\xi,j}), \xi) = g(j)$ and define $F(\mu, \eta)$ arbitrarily for all other $(\mu, \eta) \in (\xi+1)^2$. Note that F is well defined by the choice of C and the assumption on the s_ξ.

Now suppose that $e : \omega_1 \to \omega_1$ is an embedding of F into U. There is then a club $C^* \subseteq C$ such that $e(\eta) \geq \eta$ for all $\eta \in C$. There is then some $\eta \in C^*$ such that $s_\eta \subseteq^* \psi^{-1}(e)$. Then $e(\eta) \geq \eta$ and there are infinitely many j such that $F(\psi_0(s_{\eta,j}), \eta) = f_{\eta,e(\eta)}(j)$ and hence that

$$\{F(\psi_0(s_{\eta,j}), \eta)\} \cup U(\psi_1(s_{\eta,j}), e(\eta)) = \{f_{\eta,e(\eta)}(j)\} \cup U(\psi_1(s_{\eta,j}), e(\eta)) = k+1.$$

But recall that $(\psi_0(s_{\eta,j}), \psi_1(s_{\eta,j})) \in e$ and hence there is some $\zeta \in \eta$ such that $(\psi_0(s_{\eta,j}), \psi_1(s_{\eta,j})) = (\zeta, e(\zeta))$. It follows that $\{F(\zeta, \eta)\} \cup U(e(\zeta), e(\eta)) = k+1$ contradicting that e is a $[k+1]^k$-embedding. □

9. Consistency of $\mathfrak{K}_\omega = k + 1$ for arbitrary k

All the results up to this point have only established that $\mathfrak{K}_\omega$ can consistently take on the values 0 and ω. The goal of this section is to correct this and prove the statement of the title, namely that $\mathfrak{K}_\omega$ can have any intermediate value. This will be Theorem 9.1. The proof of Theorem 9.1 is ultimately based on arguments in [1] and [3]. However, while some familiarity with those articles may be useful, the main argument will be presented in detail here since various modifications will need. Before presenting the argument, some definitions will need to be presented.

Theorem 9.1. *For any positive integer k it is consistent that $\mathfrak{K}_\omega = k+1$ and $\mathfrak{K} \geq k$ or, in other words, there is no $[k+1]^k$-weakly universal function.*

Definition 9.2. Given $U : \omega_1^2 \to [\omega]^{<\aleph_0}$ and $R : \omega_1^2 \to \omega$ and a continuous, increasing sequence $\mathfrak{M} = \{\mathfrak{M}_\xi\}_{\xi\in\omega_1}$ of countable, elementary submodels of some $H(\kappa)$ define $\mathbb{E}(U, R, \mathfrak{M})$ to consist of pairs (e, X) such that

1. $X \in [\omega_1]^{<\aleph_0}$
2. e is a finite, partial, one-to-one function from ω_1 to $\omega_1 \setminus X$
3. if $\{\eta, \zeta\} \in [\boldsymbol{domain}(e)]^2)$ then $R(\eta, \zeta) \in U(e(\eta), e(\zeta))$
4. if $\xi \in \boldsymbol{domain}(e) \cap (\mathfrak{M}_{\eta+1} \setminus \mathfrak{M}_\eta)$ then $e(\xi) \in \mathfrak{M}_{\eta+2} \setminus \mathfrak{M}_{\eta+1}$.

The partial order $\leq$ on $\mathbb{E}(U, R, \mathfrak{M})$ is coordinatewise inclusion. For $Y \in [\omega_1]^{<\aleph_0}$ and $p = (e, X) \in \mathbb{E}(U, R, \mathfrak{M})$ define $p + Y = (e, X \cup (Y \setminus \boldsymbol{range}(e)))$.

Definition 9.3. For $k \in \mathbb{N}$ let $\mathbb{U}(k)$ be the partial order for adding a function with name $\dot{U}_k$ from ω_1^2 to $[\omega]^k$ with finite approximations — in other words, $\mathbb{U}(k)$ adds $\aleph_1$ Cohen reals. To be a bit more precise, $\mathbb{U}(k)$ consist of finite partial functions from ω_1^2 to $[\omega]^k$ ordered by inclusion. Define $\mathbb{U}(\aleph_0)$ to consist of finite partial functions from ω_1^2 to $[\omega]^{<\aleph_0}$ ordered by inclusion.

Lemma 9.4. *If $\mathfrak{M}$ is a continuous, increasing sequence of countable elementary submodels of some $H(\kappa)$, $\mathbb{P}$ is a $\mathbb{U}(k)$ name for a partial order and $\dot{R}$ is a $\mathbb{U}(k) * \mathbb{P}$ name such that $\mathbb{1} \Vdash_{\mathbb{U}(k)*\mathbb{P}}$ "$\dot{R} : \omega_1^2 \to \omega$" then*

$$\mathbb{1} \Vdash_{\mathbb{U}(k)*\mathbb{P}*\mathbb{E}(\dot{U}_k,\dot{R},\mathfrak{M})} \text{“}\dot{E} \textit{ is a } [\omega]^k\textit{-embedding of } \dot{R} \textit{ into } \dot{U}_k\text{”}$$

where $\dot{E} = \bigcup_{(e,X)\in\dot{G}} e$ and $\dot{G}$ is a name for the generic subset of $\mathbb{E}(\dot{U}_k, \dot{R}, \mathfrak{M})$.

Proof. It should be clear that $\dot{E}$ is a name for a $[\omega]^k$-embedding. To see that its domain is all of ω_1 let $\xi \in \omega_1$ and $(u, p, (e, X)) \in \mathbb{U}(k) * \mathbb{P} * \mathbb{E}(\dot{U}_k, \dot{R}, \mathfrak{M})$ and note that it can be assumed that e and X are actual sets from the ground model. Since $\mathfrak{M}$ is in the ground model, it is possible to find $\eta \in \omega_1$ such that $\xi \in \mathfrak{M}_{\eta+1} \setminus \mathfrak{M}_\eta$. Let $(u', p') \leq (u, p)$ decide the value of $\dot{R}(\delta, \xi)$ to be v_δ for $\delta \in \boldsymbol{domain}(e)$. Then find $\zeta \in \omega_1 \cap (M_{\eta+2} \setminus \mathfrak{M}_{\eta+1})$ such that $\zeta \notin X \cup \boldsymbol{range}(e)$ and such that $(\rho, \zeta) \notin \boldsymbol{domain}(u')$ for all $\rho \in \boldsymbol{range}(e)$. Then let $e' = e \cup \{(\xi, \zeta)\}$ and extend u' to u'' so that Condition (3) of Definition 9.2 is satisfied; in other words, so that $u''(e(\delta), \zeta) = \{v_\delta\}$ for all $\delta \in \boldsymbol{domain}(e)$. Then (u'', p') forces that $(e', X) \in \mathbb{E}(\dot{U}_k, \dot{R}$ and $(u'', p', (e', X))$ forces that ξ belongs to the domain of $\dot{E}$. □

Definition 9.5. Assume that $2^{\aleph_1} = \aleph_2$ and let $\{\dot{R}_\alpha\}_{\alpha\in\omega_2}$ be an enumeration, each element occurring cofinally often, of all the potential forcing names for functions from ω_1^2 to ω in partial orders that are hereditarily of cardinality no greater than $\aleph_1$. Let $\psi_\alpha : |\alpha| \to \alpha$ be a bijection for $\alpha \in \omega_2$. For each $\alpha \in \omega_2$ let $\mathfrak{M}_\alpha = \{\mathfrak{M}_{\alpha,\xi}\}_{\xi\in\omega_1}$ be a continuous, increasing family of countable elementary submodels of $H(\aleph_3, \in)$ such that for each $\alpha \in \omega_2$ the following hold:

1. $\alpha \in \mathfrak{M}_{\alpha,0}$
2. $\{\mathfrak{M}_\beta\}_{\beta\in\alpha} \in \mathfrak{M}_{\alpha,0}$
3. $\{(\dot{R}_\beta, \psi_\beta)\}_{\beta\in\omega_2} \in \mathfrak{M}_{\alpha,0}$

For each $\alpha \in \omega_2$ let $C_\alpha = \{\lambda \in \omega_1 \mid \mathfrak{M}_{\alpha,\lambda} \cap \omega_1 = \lambda\}$. These are fixed for the rest of this section.

Lemma 9.6. *If $\beta \in \mathfrak{M}_{\alpha,\theta} \cap \alpha$ and $\theta^* = \mathfrak{M}_{\alpha,\theta} \cap \omega_1$ then $\theta^* \in C_\beta$.*

Proof. To see that $\theta^* \subseteq \mathfrak{M}_{\beta,\theta^*}$ note that $\gamma \subseteq \mathfrak{M}_{\beta,\gamma} \subseteq \mathfrak{M}_{\beta,\theta^*}$ for all $\gamma \in \theta^*$. On the other hand, if $\gamma \in \mathfrak{M}_{\beta,\theta^*} \cap \omega_1$ then $\gamma \in \mathfrak{M}_{\beta,\lambda}$ for some $\lambda \in \theta^*$ since θ^* is a limit. Then $\lambda \in \mathfrak{M}_{\alpha,\theta}$. Hence $\mathfrak{M}_{\beta,\lambda} \in \mathfrak{M}_{\alpha,\theta}$ and so $\gamma \in \mathfrak{M}_{\beta,\lambda} \cap \omega_1 \subseteq \mathfrak{M}_{\alpha,\theta} \cap \omega_1 = \theta^*$. □

Lemma 9.7. *If $\lambda \in C_\alpha$ and $\beta \in \mathfrak{M}_{\alpha,\lambda} \cap \alpha$ then $\mathfrak{M}_{\alpha,\lambda} \cap \beta = \mathfrak{M}_{\beta,\lambda} \cap \beta$.*

Proof. First note that since $\beta \in \mathfrak{M}_{\alpha,\lambda} \cap \alpha$ it follows by Lemma 9.6 and the assumption that $\lambda \in C_\alpha$ that $\lambda \in C_\beta$ and so $\mathfrak{M}_{\alpha,\lambda} \cap \omega_1 = \lambda = \mathfrak{M}_{\beta,\lambda} \cap \omega_1$. Clearly $\mathfrak{M}_{\alpha,\lambda} \cap \beta \supseteq \mathfrak{M}_{\beta,\lambda} \cap \beta$ because $\mathfrak{M}_\beta \in \mathfrak{M}_{\alpha,0}$ so suppose that $\xi \in \mathfrak{M}_{\alpha,\lambda} \cap \beta$. Then, using that $\beta \in \mathfrak{M}_{\alpha,\lambda}$, it follows that

$$\psi_\beta^{-1}(\xi) \in \mathfrak{M}_{\alpha,\lambda} \cap \omega_1 = \mathfrak{M}_{\beta,\lambda} \cap \omega_1$$

and hence $\xi = \psi_\beta(\psi_\beta^{-1}(\xi)) \in \mathfrak{M}_{\beta,\lambda}$ because $\beta \in \mathfrak{M}_{\beta,\lambda}$. $\square$

Definition 9.8. For a cardinal k such that $1 \leq k \leq \aleph_0$ finite support iterations $\mathbb{P}_\alpha$ (the k will be suppressed) for $\alpha \leq \omega_2$ will be constructed. In all cases $\mathbb{P}_0$ is trivial and $\mathbb{P}_1 = \mathbb{P}_0 * \mathbb{U}_k$ Using Definition 9.5, for each $\alpha \in \omega_2$ let $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{Q}_\alpha$ where $\mathbb{Q}_\alpha$ is

- $\mathbb{E}(\dot{U}_k, \dot{R}_\alpha, \mathfrak{M}_\alpha)$ if $\dot{R}_\alpha$ is a $\mathbb{P}_\alpha$ name and $1 \Vdash_{\mathbb{P}_\alpha}$ "$\dot{R}_\alpha : [\omega_1]^2 \to \omega$"
- $\mathbb{C}_k$ the Cohen partial order of all finite partial functions from ω to k otherwise.

Let $W \subseteq \omega_2$ be the set of ordinals ξ such that $\mathbb{Q}_\alpha = \mathbb{E}(\dot{U}_k, \dot{R}_\alpha, \mathfrak{M}_\alpha)$. Note that the definition of $\mathbb{P}_\alpha$ implies that $\mathbb{P}_\alpha \in \mathfrak{M}_{\alpha,0}$.

Definition 9.9. As usual, define a condition $p \in \mathbb{P}_{\omega_2}$ to be determined if:

$$(\forall \sigma \in \boldsymbol{domain}(p) \cap W)(\exists (e_{p,\sigma}, X_{p,\sigma}))\ p \restriction \sigma \Vdash_{\mathbb{P}_\sigma} \text{"}p(\sigma) = (e_{p,\sigma}, X_{p,\sigma})\text{"} \tag{9.1}$$

$$(\forall \sigma \in \boldsymbol{domain}(p) \cap W)\ \ \boldsymbol{range}(e_{p,\sigma})^2 \subseteq \boldsymbol{domain}(p(0)) \tag{9.2}$$

$$(\forall \sigma \in \boldsymbol{domain}(p) \setminus W)(\exists f_{p,\sigma})\ p \restriction \sigma \Vdash_{\mathbb{P}_\sigma} \text{"}p(\sigma) = f_{p,\sigma}\text{"} \tag{9.3}$$

Definition 9.10. If $\alpha \in \omega_2$ and $p \in \mathbb{P}_\alpha$ is determined and $\eta \in \omega_1$ define $p[\alpha, \eta]$ as follows:

- $\boldsymbol{domain}(p[\alpha, \eta]) = \boldsymbol{domain}(p) \cap \mathfrak{M}_{\alpha,\eta}$
- $p[\alpha, \eta](0) = p(0) \cap \mathfrak{M}_{\alpha,\eta}$
- if $\sigma \in \boldsymbol{domain}(p[\alpha, \eta]) \setminus W$ then $p[\alpha, \eta](\sigma) = f_{p,\sigma}$
- if $\sigma \in \boldsymbol{domain}(p[\alpha, \eta]) \cap W$ then $p[\alpha, \eta](\sigma) = (e_{p,\sigma} \cap \mathfrak{M}_{\alpha,\eta}, X_{p,\sigma} \cap \mathfrak{M}_{\alpha,\eta}) + Z$ where

$$Z = \{\xi \in \omega_1 \cap \mathfrak{M}_{\alpha,\eta} \mid (\exists \zeta \in \omega_1 \setminus \mathfrak{M}_{\alpha,\eta})\ (\xi, \zeta) \in \boldsymbol{domain}(p(0))\}.$$

Note that if $\sigma \in W \cap \boldsymbol{domain}(p) \cap \mathfrak{M}_{\alpha,\eta}$ then even if $e_{p,\sigma} \cap \mathfrak{M}_{\alpha,\eta} = X_{p,\sigma} \cap \mathfrak{M}_{\alpha,\eta} = \varnothing$ it is still required that σ belong to the domain of $p[\alpha, \eta]$ and, in this case, $p[\alpha, \eta] = (\varnothing, \varnothing)$.

The fact that in some circumstances $p[\alpha, \eta] \in \mathbb{P}_\alpha$ will only be established in Lemma 9.12 so it must appear as an assumption in the next lemma.

Lemma 9.11. *If $\eta < \zeta$ belong to C_α and $p \in \mathbb{P}_\alpha$ is determined and $p[\alpha, \zeta] \in \mathbb{P}_\alpha$ then $(p[\alpha, \zeta])[\alpha, \eta] = p[\alpha, \eta]$.*

Proof. Proceed by induction on α. If $\alpha = 1$ then $p \in \mathbb{U}_k$ and $p[1, \eta] = p \cap \mathfrak{M}_{1,\eta}$. If α is a limit and $p \in \mathbb{P}_\alpha$ let μ be the successor of the maximal element of $\boldsymbol{domain}(p) \cap \mathfrak{M}_{\alpha,\zeta}$. Then by Lemma 9.7 and since $\zeta \in C_\alpha$ and $\mu \in \mathfrak{M}_{\alpha,\zeta}$ it follows that then $\mathfrak{M}_{\alpha,\zeta} \cap \mu = \mathfrak{M}_{\mu,\zeta} \cap \mu$ and so $(p \restriction \mu)[\mu, \zeta]$ and $p[\alpha, \zeta]$ have the same domain. Furthermore, $(p \restriction \mu)[\mu, \zeta] = p[\alpha, \zeta]$ because $\mathfrak{M}_{\alpha,\zeta} \cap \omega_1 = \mathfrak{M}_{\mu,\zeta} \cap \omega_1$. Hence, using that $p[\alpha, \zeta] \in \mathbb{P}_\alpha$ and the induction hypothesis,

$$(p[\alpha,\zeta])[\alpha,\eta] = (p \restriction \mu)[\mu,\zeta][\mu,\eta] = (p \restriction \mu)[\mu,\eta].$$

Again using Lemma 9.7 and the fact $\eta \in C_\alpha$ it follows that $(p \restriction \mu)[\mu,\eta] = p[\alpha,\eta]$ as required.

So assume that $\alpha = \beta + 1$. If $\beta \notin W$ the result also easily follows from the induction hypothesis, Lemma 9.7 and the fact that $\beta \in \mathfrak{M}_{\alpha,0}$. So now consider $\beta \in W$. Since $p[\alpha,\zeta] \in \mathbb{P}_\alpha$ it follows that $(p \restriction \beta)[\beta,\zeta] = p[\alpha,\zeta] \restriction \beta \in \mathbb{P}_\beta$ and hence $(p \restriction \beta)[\beta,\zeta])[\beta,\eta] = (p \restriction \beta)[\beta,\eta]$ by the induction hypothesis.

Then $e_{p[\beta,\mu],\beta} = e_{p,\beta} \cap \mathfrak{M}_{\alpha,\mu}$ for all μ so it is clear that $e_{(p[\alpha,\zeta])[\alpha,\eta],\beta} = e_{p[\alpha,\eta],\beta}$ and similarly $X_{(p[\alpha,\zeta])[\alpha,\eta],\beta} = X_{p[\alpha,\eta],\beta}$. So all that remains to be shown is that the Z of Definition 9.10 is the same in both cases. But Z depends only on the first coordinates of $p[\alpha,\zeta])[\alpha,\eta]$ and $p[\alpha,\eta]$ and these have already been shown to be equal at stage $\alpha = 1$. □

Lemma 9.12. *If $\alpha \in \omega_2$ and $p \in \mathbb{P}_\alpha$ and $\eta \in C_\alpha$ then*

1. *$p[\alpha,\eta] \in \mathbb{P}_\alpha$*
2. *for all $r \in \mathfrak{M}_{\alpha,\eta}$ if $r \leq p[\alpha,\eta]$ then $r \not\perp_{\mathbb{P}_\alpha} p$.*

Proof. Proceed by induction on α. The case $\alpha = 0$ is vacuous and $\alpha = 1$ is easy since $\mathbb{P}_1 = \mathbb{U}_k$ is Cohen forcing. The limit cases are worth dealing with first, so suppose that α is a limit ordinal. If $p \in \mathbb{P}_\alpha$ let μ be successor of the maximal element of $\boldsymbol{domain}(p) \cap \mathfrak{M}_{\alpha,\eta}$. Then by Lemma 9.7 and since $\eta \in C_\alpha$ and $\mu \in \mathfrak{M}_{\alpha,\eta}$ it follows that $\mathfrak{M}_{\alpha,\eta} \cap \mu = \mathfrak{M}_{\mu,\eta} \cap \mu$ and so $(p \restriction \mu)[\mu,\eta]$ and $p[\alpha,\eta]$ have the same domain. Furthermore, $(p \restriction \mu)[\mu,\eta] = p[\alpha,\eta]$ because $\mathfrak{M}_{\alpha,\eta} \cap \omega_1 = \mathfrak{M}_{\mu,\eta} \cap \omega_1$. By the induction hypothesis it follows that $(p \restriction \mu)[\mu,\eta] \in \mathbb{P}_\mu$ and hence $p[\alpha,\eta] \in \mathbb{P}_\mu \subseteq \mathbb{P}_\alpha$. Now suppose that $r \in \mathfrak{M}_{\alpha,\eta}$ and $r \leq p[\alpha,\eta] = (p \restriction \mu)[\mu,\eta]$. By the induction hypothesis it follows that $r \restriction \mu \not\perp_{\mathbb{P}_\mu} p \restriction \mu$. So there is $q \in \mathbb{P}_\mu$ such that $q \leq r \restriction \mu$ and $q \leq p \restriction \mu$. Note that $\boldsymbol{domain}(r) \setminus \mu \subseteq \mathfrak{M}_{\alpha,\eta}$ but $\boldsymbol{domain}(p) \setminus \mu$ is disjoint from $\mathfrak{M}_{\alpha,\eta}$. Hence $q \cup (r \restriction [\mu,\alpha)) \cup (p \restriction [\mu,\alpha))$ is a lower bound for r and p.

So assume that the lemma has been proven for β and that $\alpha = \beta + 1$. It will first be shown that if $p \in \mathbb{P}_\alpha$ and $\eta \in C_\alpha$ then $p[\alpha,\eta] \in \mathbb{P}_\alpha$. Of course the domain of $p[\alpha,\eta] \restriction \beta$ is a subset of $\mathfrak{M}_{\alpha,\eta}$ and so, by Lemma 9.7, it follows that the domain of $p[\alpha,\eta] \restriction \beta$ is also a subset of $\mathfrak{M}_{\beta,\eta}$ and hence $p[\alpha,\eta] \restriction \beta = (p \restriction \beta)[\beta,\eta]$. If $\beta \notin W$ then

$$p[\alpha,\eta] = (p \restriction \beta)[\beta,\eta]^\frown f_{p,\beta} \in \mathbb{P}_\beta * \mathbb{C}_k = \mathbb{P}_\alpha.$$

On the other hand, if $\beta \in W$ then, as in the previous case,

$$p[\alpha,\eta] = (p \restriction \beta)[\beta,\eta]^\frown((e_{p,\beta}, X_{p,\beta}) + Z) \in \mathbb{P}_\beta * \mathbb{Q}_\beta = \mathbb{P}_\alpha$$

where

$$Z = \{\xi \in \omega_1 \cap \mathfrak{M}_{\alpha,\eta} \mid (\exists \zeta \in \omega_1 \setminus \mathfrak{M}_{\alpha,\eta})\ (\xi,\zeta) \in \boldsymbol{domain}(p(0))\}. \tag{9.4}$$

Since p is a condition it of course follows that

$$(\forall (\xi,\zeta) \in \boldsymbol{domain}(e_{p,\beta}))\ p \restriction \beta \Vdash_{\mathbb{P}_\beta} \text{“}\dot{R}_\beta(\xi,\zeta) \in p(0)(e_{p,\beta}(\xi), e_{p,\beta}(\zeta))\text{”} \tag{9.5}$$

However, for $p[\alpha,\eta]$ to belong to $\mathbb{P}_\alpha$ it must be the case that

$$(\forall (\xi,\zeta) \in \boldsymbol{domain}(e_{p,\beta}))\ p[\alpha,\eta] \restriction \beta \Vdash_{\mathbb{P}_\beta} \text{“}\dot{R}_\beta(\xi,\zeta) \in p(0)(e_{p,\beta}(\xi), e_{p,\beta}(\zeta))\text{”}. \tag{9.6}$$

To see that Condition (9.6) actually does hold suppose not. Since $p[\alpha,\eta] \restriction \beta = (p \restriction \beta)[\beta,\eta]$ there must be $r \leq (p \restriction \beta)[\beta,\eta]$ and ζ^* and ξ^* in the domain of $e_{p,\beta}$ such that $r \Vdash_{\mathbb{P}_\beta}$ “$\dot{R}_\beta(\xi^*,\zeta^*) \notin p(0)(e_{p,\beta}(\xi^*), e_{p,\beta}(\zeta^*))$”

and, by elementarity and the fact that $\dot{R}_\beta$, $\mathbb{P}_\beta$ and $p[\alpha,\eta] \restriction \beta$ all belong to $\mathfrak{M}_{\alpha,\eta}$, it may be assumed that $r \in \mathfrak{M}_{\alpha,\eta}$. Then, by the induction hypothesis, it follows that $r \not\perp_{\mathbb{P}_\beta} p \restriction \beta$. But if $q \leq r$ and $q \leq p \restriction \beta$ then

$$q \Vdash_{\mathbb{P}_\beta} \text{“}\dot{R}_\beta(\xi^*, \zeta^*) \notin p(0)(e_{p,\beta}(\xi^*), e_{p,\beta}(\zeta^*))\text{”}$$

contradicting that Condition (9.5) holds.

It must now be shown that if $r \in \mathfrak{M}_{\alpha,\eta}$ and $r \leq p[\alpha,\eta]$ then $r \not\perp_{\mathbb{P}_\alpha} p$. It can, of course, be assumed that r is determined. If $\beta \notin W$ then it has already been observed that $p[\alpha,\eta] = (p \restriction \beta)[\beta,\eta]^\frown f_{p,\beta} \in \mathbb{P}_\alpha$. Then $r \restriction \beta \Vdash_{\mathbb{P}_\beta}$ “$f_{r,\beta} \supseteq f_{p,\beta}$” and, by the induction hypothesis $(r \restriction \beta) \not\perp_{\mathbb{P}_\beta} (p \restriction \beta)[\beta,\eta]$ yielding the required conclusion. The other possibility is that $p[\alpha,\eta] = (p \restriction \beta)[\beta,\eta]^\frown(e_{p,\beta}, X_{p,\beta}) + Z$ where Z is given by Equation (9.4). Let $\{\theta_i\}_{i=0}^j$ be ordinals in ω_1 such that:

- $\theta_i < \theta_{i+1}$
- for each $\xi \in \boldsymbol{domain}(e_{p,\beta}) \setminus \mathfrak{M}_{\alpha,\eta}$ there is some $i \in j$ such that $\xi \in \mathfrak{M}_{\alpha,\theta_i}$ and $e_{p,\beta}(\xi) \in \mathfrak{M}_{\alpha,\theta_i+1} \setminus \mathfrak{M}_{\alpha,\theta_i}$
- $\theta_0 = \eta = \theta_0^*$
- $p \in \mathfrak{M}_{\alpha,\theta_j^*}$

using that $\bigcup_{\rho\in\omega_1} \mathfrak{M}_{\alpha,\rho} \supseteq \alpha$ to obtain θ_j is so large that the last condition holds. Note that while θ_i may not be in C_α it is the case that if $\theta_i^* = \mathfrak{M}_{\alpha,\theta_i} \cap \omega_1$ then $\theta_i^* \in C_\beta$ by Lemma 9.6.

Hence the induction hypothesis can be applied to $\mathfrak{M}_{\beta,\theta_i^*}$ and $p \restriction \beta$. With this in mind, construct inductively conditions r_i such that:

1. $r_i \in \mathfrak{M}_{\beta,\theta_i^*}$
2. $r_{i+1} \leq r_i \leq r \restriction \beta$
3. $r_i \leq (p \restriction \beta)[\beta,\theta_i^*]$
4. for all $\gamma \in \boldsymbol{domain}(e_{r,\beta} \setminus e_{p,\beta})$ and $\delta \in e_{p,\beta}^{-1}(\mathfrak{M}_{\alpha,\theta_i+1} \setminus \mathfrak{M}_{\alpha,\theta_i})$ there is some $v_{\gamma,\delta}$ such that $r_i \Vdash_{\mathbb{P}_\beta}$ “$\dot{R}_\beta(\gamma,\delta) = v_{\gamma,\delta}$”
5. $v_{\gamma,\delta} \in r_{i+1}(0)(e_{r,\beta}(\gamma), e_{p,\beta}(\delta))$ for all $\gamma \in \boldsymbol{domain}(e_{r,\beta} \setminus e_{p,\beta})$ and $\delta \in e_{p,\beta}^{-1}(\mathfrak{M}_{\alpha,\theta_i} \setminus \mathfrak{M}_{\alpha,\eta})$.

If this can be done then $r_j \leq (p \restriction \beta)[\beta,\theta_j^*] = p \restriction \beta$ and $r_j \leq r \restriction \beta$. Let $r^* = r_j^\frown((e_{r,\beta}, X_{r,\beta}) + Z)$ where Z is defined in Equation (9.4). To see that $r^* \not\perp_{\mathbb{P}_\alpha} p$ it suffices to show that $r_j \Vdash_{\mathbb{P}_\beta}$ “$r(\beta) \not\perp_{\mathbb{E}(\dot{U}_k,\dot{R}_\beta,\mathfrak{M}_\alpha)} p(\beta)$”. This is the point of Induction Hypothesis (5). To be precise, let $e = e_{r,\beta} \cup e_{p,\beta}$ and note that e is a function. It must be shown that if γ and δ belong to $\boldsymbol{domain}(e)$ then

$$r_j \Vdash_{\mathbb{P}_\beta} \text{“}\dot{R}_\beta(\gamma,\delta) \in r_j(0)(e(\gamma), e(\delta))\text{”}.$$

If γ and δ both belong to $\boldsymbol{domain}(e_r)$ or both belong to $\boldsymbol{domain}(e_p)$ then there is nothing to do since both r and p are conditions in $\mathbb{P}_\alpha$. In every other case Hypothesis (5) applies by the choice of the θ_i.

To see that the induction can be completed start with $r_0 = r \restriction \beta \in \mathfrak{M}_{\alpha,\eta} = \mathfrak{M}_{\beta,\theta_0^*}$ and note that Induction Hypothesis (4) is vacuously satisfied because $\theta_0 \in C_\alpha$ and hence $e_{p,\beta} \restriction \theta_0 = e_{p,\beta} \cap \mathfrak{M}_{\alpha,\theta_0}$. Now suppose that r_i is given satisfying the induction hypotheses. Begin by letting $p^* \in \mathbb{P}_\beta$ be defined by

$$p^*(\xi) = \begin{cases} p^*(0) & \text{if } \xi = 0 \\ (p \restriction \beta)[\beta,\theta_{i+1}^*](\xi) & \text{otherwise} \end{cases}$$

where $p^*(0)$ is defined by

$p^*(0)(\rho,\sigma)$

$$= \begin{cases} \{v_{\gamma,\delta}\} & \text{if } (\rho,\sigma) = (e_{r,\beta}(\gamma), e_{p,\beta}(\delta)) \text{ and } \gamma \in \boldsymbol{domain}(e_{r,\beta} \setminus e_{p,\beta}) \text{ and } \delta \in e^{-1}_{p,\beta}(\mathfrak{M}_{\alpha,\theta_i+1} \setminus \mathfrak{M}_{\alpha,\theta_i}) \\ p(0)(\rho,\sigma) & \text{otherwise.} \end{cases}$$

The key point to note that there is no contradiction in this construction since $(e_{r,\beta}(\gamma), e_{p,\beta}(\delta))$ does not belong to the domain of $p(0)$ because $e_{r,\beta}(\gamma) \notin Z$, as defined in Equation (9.4), and $\gamma \notin \boldsymbol{domain}(e_{p,\beta})$.

Note also that $p^* \le (p \restriction \beta)[\beta, \theta^*_{i+1}]$ and note that $p^*[\beta,\theta^*_i] = (p \restriction \beta)[\beta,\theta^*_{i+1}][\beta,\theta^*_i]$ because if $v_{\gamma,\delta}$ is defined then $\delta \notin \mathfrak{M}_{\beta,\theta^*_i}$. By Lemma 9.11 it follows that $p^*[\beta,\theta^*_i] = (p \restriction \beta)[\beta,\theta^*_i]$. Then $r_i \le p^*[\beta,\theta^*_i]$ and hence, by the induction hypothesis, keeping in mind that $\theta^*_i \in C_\beta$, there is some $\bar{r} \in \mathbb{P}_\beta$ such that $\bar{r} \le r_i$ and $\bar{r} \le p^*$. Since both r_i and p^* belong to $\mathfrak{M}_{\beta,\theta^*_{i+1}}$ it can be assumed that so does $\bar{r}$. It is then an easy matter to extend $\bar{r}$ to $r_{i+1} \in \mathfrak{M}_{\beta,\theta^*_{i+1}}$ such that Induction Hypothesis (4) is also satisfied. Of course, the choice of p^* guarantees that Induction Hypothesis (5) is also satisfied. $\square$

Definition 9.13. Given $\alpha \in \omega_2 \setminus W$ define $\mathbf{r}$ to belong to $\mathbb{P}_{\alpha,j,\beta}$ if $\mathbf{r} = \{\mathbf{r}\langle i\rangle\}_{i\in k}$ such that for each $i \in k$:

1. $\mathbf{r}\langle i\rangle \in \mathbb{P}_\beta$ is determined
2. $\mathbf{r}\langle i\rangle \restriction \alpha = \mathbf{r}\langle 0\rangle \restriction \alpha$
3. $f_{\mathbf{r}\langle i\rangle,\alpha}(j) = i$.

Observe that the dependence on k is being suppressed in the notation because it should be clear from the context. If $\mathbf{r}$ and $\mathbf{p}$ are both in $\mathbb{P}_{\alpha,j,\beta}$ and $p \in \mathbb{P}_\alpha$ define

- $\mathbf{r} \le p$ if $\mathbf{r}\langle i\rangle \le p$ for each i
- $\mathbf{r} \le \mathbf{p}$ if $\mathbf{r}\langle i\rangle \le p\langle i\rangle$ for each i
- $\mathbf{r} \restriction \gamma$ to be $\{\mathbf{r}\langle i\rangle \restriction \gamma\}_{i\in k}$
- $\mathbf{r}[\gamma,\delta]$ to be $\{\mathbf{r}\langle i\rangle[\gamma,\delta]\}_{i\in k}$.

Lemma 9.14. *If $\alpha \in \omega_2$ and the following hold:*

1. $\eta \in C_\alpha$
2. $\bar{\alpha} \in (\alpha \setminus W) \cap \mathfrak{M}_{\alpha,\eta}$
3. $p \in \mathbb{P}_\alpha$
4. $\mathbf{r} \in \mathbb{P}_{\bar{\alpha},J,\alpha} \cap \mathfrak{M}_{\alpha,\eta}$
5. $\mathbf{r} \le p[\alpha,\eta]$ *(note this implies that $J \notin \boldsymbol{domain}(f_{p,\bar{\alpha}})$*

then there is $\mathbf{q} \in \mathbb{P}_{\bar{\alpha},J,\alpha}$ and such that $\mathbf{q} \le \mathbf{r}$ and $\mathbf{q} \le p$.

Proof. The argument uses the same ideas as the proof of Lemma 9.12 but the k in $\mathbb{U}(k)$ will now play a crucial role. Proceed by induction on α noting that the initial case and the limit cases can be handled as in Lemma 9.12. So assume the result has been proven for β and that $\alpha = \beta + 1$. It must now be shown that if $\mathbf{r} \in \mathfrak{M}_{\alpha,\eta} \cap \mathbb{P}_{\bar{\alpha},J,\alpha}$ and $\mathbf{r} \le p[\alpha,\eta]$ then there is $\mathbf{q} \in \mathbb{P}_{\bar{\alpha},J,\alpha}$ such that $\mathbf{q} \le p$ and $\mathbf{q} \le \mathbf{r}$.

If $\beta \notin W$ then $p[\alpha,\eta] = (p \restriction \beta)[\beta,\eta]^\frown f_{p,\beta} \in \mathbb{P}_\alpha$ and $\mathbf{r}\langle i\rangle \restriction \beta \Vdash_{\mathbb{P}_\beta}$ "$f_{\mathbf{r}\langle i\rangle,\beta} \supseteq f_{p,\beta}$". By the induction hypothesis there is $\mathbf{q} \in \mathbb{P}_{\bar{\alpha},J,\beta}$ such that $\mathbf{q} \le \mathbf{r}$ and $\mathbf{q} \le p \restriction \beta$. Then the family $\{q\langle i\rangle^\frown f_{\mathbf{r}\langle i\rangle,\beta}\}_{i\in k}$ yields the required conclusion. The other possibility is that $p[\alpha,\eta] = (p \restriction \beta)[\beta,\eta]^\frown((e_{p,\beta}, X_{p,\beta}) + Z)$ where Z is given by Equation (9.4). Let $\{\theta_i\}_{i=0}^j$ be ordinals chosen exactly as in the proof of Lemma 9.12.

Construct inductively conditions $\mathbf{r}_i \in \mathbb{P}_{\bar{\alpha},J,\alpha}$ such that:

1. $\mathbf{r}_i \in \mathfrak{M}_{\beta,\theta^*_i}$

2. $\mathbf{r}_{i+1} \leq \mathbf{r}_i \leq \mathbf{r} \restriction \beta$
3. $\mathbf{r}_i \leq (p \restriction \beta)[\beta, \theta_i^*]$
4. for all $\ell \in k$ and all $\gamma \in \boldsymbol{domain}(e_{\mathbf{r}\langle\ell\rangle,\beta} \setminus e_{p,\beta})$ and $\delta \in e_{p,\beta}^{-1}(\mathfrak{M}_{\alpha,\theta_i+1} \setminus \mathfrak{M}_{\alpha,\theta_i})$ there is some $v_{\gamma,\delta,\ell}$ such that $\mathbf{r}_i\langle\ell\rangle \Vdash_{\mathbb{P}_\beta}$ "$\dot{R}_\beta(\gamma,\delta) = v_{\gamma,\delta,\ell}$"
5. $v_{\gamma,\delta,\ell} \in \mathbf{r}_{i+1}\langle\ell\rangle(0)(e_{\mathbf{r}\langle\ell\rangle,\beta}(\gamma), e_{p,\beta}(\delta))$ for all $\gamma \in \boldsymbol{domain}(e_{\mathbf{r}\langle\ell\rangle,\beta} \setminus e_{p,\beta})$ and $\delta \in e_{p,\beta}^{-1}(\mathfrak{M}_{\alpha,\theta_i+1} \setminus \mathfrak{M}_{\alpha,\theta_i})$ and $\ell \in k$.

If this can be done then $\mathbf{r}_j \leq (p \restriction \beta)[\beta, \theta_j^*] = p \restriction \beta$. Let

$$\mathbf{q} = \{\mathbf{r}_j\langle\ell\rangle^\frown(e_{\mathbf{r}\langle\ell\rangle,\beta} \cup e_{p,\beta}, X_{\mathbf{r}\langle\ell\rangle,\beta} \cup X_{p,\beta}) + Z\}_{\ell\in k}$$

where Z is as in (9.4). To see that $\mathbf{q} \leq p$ and $\mathbf{q} \leq \mathbf{r}$ use Induction Hypothesis (5) as in Lemma 9.12 for each pair $\mathbf{r}\langle\ell\rangle$ and $\mathbf{q}\langle\ell\rangle$ individually.

To see that the induction can be completed start with $\mathbf{r}_0 = \mathbf{r} \restriction \beta \in \mathfrak{M}_{\alpha,\eta} = \mathfrak{M}_{\beta,\theta_0^*}$ and note that Induction Hypothesis (4) is vacuously satisfied. Now suppose that $\mathbf{r}_i$ is given satisfying the induction hypotheses. Begin by letting $p^* \in \mathbb{P}_\beta$ be defined by

$$p^*(\xi) = \begin{cases} p^*(0) & \text{if } \xi = 0 \\ (p \restriction \beta)[\beta, \theta_{i+1}^*](\xi) & \text{otherwise} \end{cases}$$

where $p^*(0)$ is defined by

$$p^*(0)(\rho,\sigma)$$
$$= \begin{cases} \{v_{\gamma,\delta,\ell}\}_{\ell\in k} & \text{if } (\rho,\sigma) = (e_{\mathbf{r}\langle\ell\rangle,\beta}(\gamma), e_{p,\beta}(\delta)) \text{ and } \gamma \in \boldsymbol{domain}(e_{\mathbf{r}\langle\ell\rangle,\beta} \setminus e_{p,\beta}) \text{ and } \delta \in e_{p,\beta}^{-1}(\mathfrak{M}_{\alpha,\theta_i+1} \setminus \mathfrak{M}_{\alpha,\theta_i}) \\ p(0)(\rho,\sigma) & \text{otherwise.} \end{cases}$$

Of course, some of the $v_{\gamma,\delta,\ell}$ of $\{v_{\gamma,\delta,\ell}\}_{\ell\in k}$ may not be defined but certainly $|\{v_{\gamma,\delta,\ell}\}_{\ell\in k}| \leq k$. Note that there is no contradiction in this construction as in the proof of Lemma 9.12 and that $p^*(0) \in \mathbb{U}_k$.

Note also that $p^* \leq (p \restriction \beta)[\beta, \theta_{i+1}^*]$ and that $p^*[\beta, \theta_i^*] = (p \restriction \beta)[\beta, \theta_i^*]$ as in the proof of Lemma 9.12 because if $v_{\gamma,\delta,\ell}$ is defined then $\delta \notin \mathfrak{M}_{\beta,\theta_i^*}$. Then $\mathbf{r}_i \leq p^*[\beta, \theta_i^*]$ and hence, by the induction hypothesis, there is some $\bar{\mathbf{r}} \in \mathbb{P}_\beta$ such that $\bar{\mathbf{r}} \leq \mathbf{r}_i$ and $\bar{\mathbf{r}} \leq p^*[\beta, \theta_{i+1}^*]$. Since both $\mathbf{r}_i$ and p^* belong to $\mathfrak{M}_{\beta,\theta_{i+1}^*}$ it can be assumed that so does $\bar{\mathbf{r}}$. Then construct inductively s_ℓ such that

1. $s_0 \leq \bar{\mathbf{r}}\langle 0\rangle$
2. $s_\ell \in \mathfrak{M}_{\alpha,\theta_{i+1}^*}$
3. $s_{\ell+1} \leq (s_\ell \restriction \bar{\alpha}) \cup (\bar{\mathbf{r}}\langle\ell+1\rangle \restriction [\bar{\alpha}, \beta))$
4. $f_{s_\ell,\bar{\alpha}}(J) = \ell$
5. for all $\gamma \in \boldsymbol{domain}(e_{\mathbf{r}\langle\ell\rangle,\beta})$ and $\delta \in e_{p,\beta}^{-1}(\mathfrak{M}_{\alpha,\theta_i+1} \setminus \mathfrak{M}_{\alpha,\theta_i})$ there is some $v_{\gamma,\delta,\ell}$ such that $s_\ell \Vdash_{\mathbb{P}_\beta}$ "$\dot{R}_\beta(\gamma,\delta) = v_{\gamma,\delta,\ell}$"

Then define $\mathbf{r_{i+1}}\langle\ell\rangle = (s_{k-1} \restriction \bar{\alpha}) \cup s_\ell \restriction [\bar{\alpha}, \beta)$ for each $\ell \in k$ and note that $\mathbf{r}_{i+1} \in \mathfrak{M}_{\beta,\theta_{i+1}^*}$ and that $\mathbf{r}_{i+1} \leq \mathbf{r}_i$. Induction Hypothesis (4) is satisfied by construction and, of course, the choice of p^* guarantees that Induction Hypothesis (5) is also satisfied. □

Corollary 9.15. *Suppose that $\alpha \in \omega_2$ and that:*

1. $\eta < \zeta$ are in C_α

2. p_0 *and* p_1 *are determined conditions in* $\mathbb{P}_\alpha$
3. $p_0 \in \mathfrak{M}_{\alpha,\zeta}$
4. $\bar{\alpha} \in \alpha \setminus W \cap \mathfrak{M}_{\alpha,\eta}$
5. $f_{p_0,\bar{\alpha}} = f_{p_1,\bar{\alpha}} = f$
6. $J \notin \boldsymbol{domain}(f)$
7. $p_0[\alpha,\eta] = p_1[\alpha,\zeta]$

then there is $\mathbf{q} \in \mathbb{P}_{\bar{\alpha},J,\alpha}$ *such that* $\mathbf{q} \leq p_i$ *for* $i \in 2$.

Proof. Let $\mathbf{r} = \{\mathbf{r}\langle i\rangle\}_{i\in k}$ where

$$\mathbf{r}\langle i\rangle(\xi) = \begin{cases} p_0(\xi) & \text{if } \xi \neq \bar{\alpha} \\ f_{p_0,\bar{\alpha}} \cup \{(J,i)\} & \text{if } \xi = \bar{\alpha} \end{cases}$$

for each i and note that $\mathbf{r} \in \mathbb{P}_{\bar{\alpha},J,\alpha}$. Moreover, since $\mathbf{r} \in \mathfrak{M}_{\alpha,\zeta}$ and $\mathbf{r} \leq p_0[\alpha,\eta] = p_1[\alpha,\zeta]$ it follows that $\mathbf{r}$, p_1 and $\mathfrak{M}_{\alpha,\zeta}$ satisfy the hypotheses of Lemma 9.14. Hence there is $\mathbf{q} \in \mathbb{P}_{P\bar{\alpha},J,\alpha}$ such that $\mathbf{q} \leq \mathbf{r}$ and $\mathbf{q} \leq p_1$. Since $\mathbf{q} \leq \mathbf{r}$ implies that $\mathbf{q} \leq p_0$ this is as required. □

Corollary 9.16. *The partial order* $\mathbb{P}_{\omega_2}$ *satisfies the countable chain condition.*

Proof. Suppose that $\{p_\xi\}_{\xi\in\omega_1}$ is an uncountable subset of $\mathbb{P}_{\omega_2}$ consisting of determined conditions. Let α be sufficiently large that $\{p_\xi\}_{\xi\in\omega_1}$ is a subset of $\mathbb{P}_\alpha$. Then, using that $\mathfrak{M}_\alpha$ is continuous and that $p_\xi[\alpha,\xi] \in \mathfrak{M}_{\alpha,\xi}$, it is possible to find a stationary $S \subseteq C_\alpha$ and p such that $p_\xi[\alpha,\xi] = p$ for $\xi \in S$. Letting $\bar{\alpha} \in \mathfrak{M}_{\alpha,0} \setminus W$ of Corollary 9.15 be arbitrary, let η and ζ be arbitrary elements of S and let $\mathbf{q}$ satisfy the conclusion of Corollary 9.15. Then any $\mathbf{q}\langle i\rangle$ witnesses that $p_\eta \not\perp_{\mathbb{P}_{\omega_2}} p_\zeta$. □

Proof of Theorem 9.1. Use the iteration of Definition 9.8 with $\mathbb{U}_{k+1}$ as the first iterand. From Corollary 9.16 it follows that if G is $\mathbb{P}_{\omega_2}$ generic over V then $\aleph_1^V = \aleph_1^{V[G]}$. Using this and Lemma 9.4 it follows that $\dot{U}_{k+1}[G \cap \mathbb{U}_{k+1}]$ is a $[\omega]^{k+1}$-weakly universal and hence $\mathfrak{K}_\omega \leq k+1$.

So all that remains to be shown is that in $V[G]$ there is no $[k+1]^k$-weakly universal function from ω_1^2 to $[k+1]^k$. So suppose that $\dot{U}$ is a name for a counterexample. There is then some $\bar{\alpha} \in \omega_2$ such that $\dot{U}$ is a $\mathbb{P}_{\bar{\alpha}}$ name and, there is no harm in assuming that $\bar{\alpha} \notin W$. Let $\dot{c}_{\bar{\alpha}} : \omega \to k+1$ be a name for the Cohen generic real. Let $\tau : \omega_1^2 \to \omega$ witness that $\aleph_1 \not\to [\aleph_1]^2_{\aleph_0}$. Then suppose that $\alpha \in \omega_2$ and that $\dot{E}$ is a $\mathbb{P}_\alpha$-name such that

$$\mathbb{1} \Vdash_{\mathbb{P}_\alpha} \text{“}\dot{E} : \omega_1 \to \omega_1 \text{ is a } [k+1]^k\text{-embedding of } c_{\bar{\alpha}} \circ \tau \text{ into } \dot{U}\text{”}.$$

Then for each $\xi \in \omega_1$ it is possible to choose $p_\xi \in \mathbb{P}_\alpha$ and ξ^* such that $p_\xi \Vdash_{\mathbb{P}_\alpha}$ “$\dot{E}(\xi) = \xi^*$”. It is then possible to find a stationary set S and p such that $p_\eta[\alpha,\eta] = p$ for each $\eta \in S$. Let $\eta < \zeta$ be ordinals in $S \cap C_\alpha$ such that $\bar{\alpha} \in \mathfrak{M}_{\alpha,\eta}$ and $\tau(\eta,\zeta) = J \in \omega \setminus \boldsymbol{domain}(f_{p,\bar{\alpha}})$. Then use Corollary 9.15 to find $\mathbf{q} \in \mathbb{P}_{\bar{\alpha},J,\alpha}$ such that $\mathbf{q} \leq p_\eta$ and $\mathbf{q} \leq p_\zeta$.

It is then possible to find a condition $p^* \in \mathbb{P}_{\bar{\alpha}}$ such that $p^* \leq \mathbf{q}\langle 0\rangle \restriction \bar{\alpha}$ and such that $p^* \Vdash_{\mathbb{P}_{\bar{\alpha}}}$ “$\dot{U}(\eta^*,\zeta^*) = a$” for some $a \in [k+1]^k$. Let $m \in k+1 \setminus a$ and define $\bar{p} = p^* \cup (\mathbf{q}\langle m\rangle \restriction [\bar{\alpha},\alpha))$. Then $\bar{p} \leq p_\eta$ and $\bar{p} \leq p_\zeta$ and so

$$\bar{p} \Vdash_{\mathbb{P}_\beta} \text{“}\dot{E}(\eta) = \eta^*\text{”}$$
$$\bar{p} \Vdash_{\mathbb{P}_\beta} \text{“}\dot{E}(\eta) = \zeta^*\text{”}$$
$$\bar{p} \Vdash_{\mathbb{P}_{\bar{\alpha}+1}} \text{“}\dot{c}_{\bar{\alpha}} \circ \tau(\eta,\zeta) = \dot{c}_{\bar{\alpha}}(J) = m\text{”}$$

contradicting that $\mathbb{1} \Vdash_{\mathbb{P}_{\bar{\alpha}+1}}$ “$\dot{c}_{\bar{\alpha}} \circ \tau(\eta,\zeta) \in \dot{U}(\eta^*,\zeta^*)$”. □

10. A $[\omega]^{<\aleph_0}$-weakly universal function without any $(2,2)$-universal function

Note that Theorem 9.1 for the case $k = 1$ yields a model of set theory in which there is no universal function from ω_1^2 to ω yet there is a $[\omega]^2$-weakly universal function. The main theorem of this section provides a companion result by showing that there is a model of set theory in which there is an $[\omega]^2$-weakly universal function but there is no (a,b)-universal function for any finite a and b. The proof will require some modifications of §9. To begin, Definition 9.5 needs to be augmented.

Definition 10.1. Assume that $2^{\aleph_2} = \aleph_3$ and let $\{\dot{R}_\alpha\}_{\alpha\in\omega_3}$ be an enumeration, each element occurring cofinally often, of all the potential forcing names for functions from ω_1^2 to ω in partial orders that are hereditarily of cardinality no greater than $\aleph_2$. Let $\psi_\alpha : |\alpha| \to \alpha$ be a bijection for $\alpha \in \omega_3$. Let $\{S_\xi\}_{\xi\in\omega_3}$ be pairwise disjoint stationary subsets ω_3 consisting of ordinals of cofinality ω_2 and let $\{y_\xi\}_{\xi\in\omega_3}$ enumerate $H(\aleph_3)$. For each $\alpha \in \omega_3$ let $\mathfrak{M}_\alpha = \{\mathfrak{M}_{\alpha,\xi}\}_{\xi\in\omega_1}$ be a continuous, increasing family of countable elementary submodels of $H(\aleph_4)$ such that for each $\alpha \in \omega_3$ the following hold:

1. $\alpha \in \mathfrak{M}_{\alpha,0}$
2. $\{\mathfrak{M}_\beta\}_{\beta\in\alpha} \in \mathfrak{M}_{\alpha,0}$
3. $\{(\dot{R}_\beta, \psi_\beta, S_\beta, y_\beta)\}_{\beta\in\omega_3} \in \mathfrak{M}_{\alpha,0}$
4. for each $\sigma \in \omega_3$ there are unboundedly many α such that $\sigma \in \mathfrak{M}_{\alpha,0}$.

For each $\alpha \in \omega_3$ let $C_\alpha = \{\lambda \in \omega_1 \mid \mathfrak{M}_{\alpha,\lambda} \cap \omega_1 = \lambda\}$.

The necessary model will be obtained by forcing over a model of the Generalized Continuum Hypothesis. Let $\mathbb{P}_{\omega_3}$ be the iteration of Definition 9.8 with ω_3 in the place of ω_2 and $\mathbb{P}_1 = \mathbb{U}(2)$ and assume that it has been arranged that $\omega_3 \setminus W$ is the set of ordinals of cofinality ω_2. If $\alpha \notin W$ then define $\mathbb{Q}_\alpha = \mathbb{C}_2$. Corollary 9.15 needs to be modified for an iteration of length ω_3 and to apply to finitely many conditions. Keep in mind that, as in Definition 9.13, the dependence on k is still being suppressed but that $k = 2$ in this section.

Corollary 10.2. *Suppose that* $\alpha \in \omega_3$ *and that:*

1. $\{\eta_i\}_{i\in m}$ *is an increasing sequence of ordinals in* C_α
2. p_i *are determined conditions in* $\mathbb{P}_\alpha$ *for* $i \in m$
3. $p_i \in \mathfrak{M}_{\alpha,\eta_{i+1}}$ *for* $i \in m-1$
4. $\bar{\alpha} \in \alpha \setminus W \cap \mathfrak{M}_{\alpha,0}$
5. $f_{p_i,\bar{\alpha}} = f$ *for all* $i \in m$
6. $J \supseteq \boldsymbol{domain}(f)$
7. $p_i[\alpha,\eta_i] = p$ *for all* $i \in m$

then there is $\mathbf{q} \in \mathbb{P}_{\bar{\alpha},J,\alpha}$ *such that* $\mathbf{q} \le p_i$ *for all* $i \in m$.

Proof. This is essentially an iterated version of the proof of Corollary 9.15 but some changes are needed. Lemma 9.7 needs to be changed to include $\psi_\beta : \omega_2 \to \beta$ for $\beta \in \omega_3$ — indeed, the same argument will work for all ω_n with $n \in \omega$.

The other change needed is in the proof of Lemma 9.12. Recall that when choosing the increasing the set of θ_i it was shown that it can be assumed that θ_j is so large that $p \in \mathfrak{M}_{\alpha,\theta_j^*}$, the reason is that $\bigcup_{\rho\in\omega_1} \mathfrak{M}_{\alpha,\rho} \supseteq \alpha$. However, if $\alpha \in \omega_3$, this is no longer true and the best that can be done is that if $M = \bigcup_{\rho\in\omega_1} \mathfrak{M}_{\alpha,\rho} \cap \alpha$ then it can be assumed that θ_j is so large that $p \restriction M = p[\alpha,\theta_j]$. This suffices though, if r is required to belong to $\bigcup_{\rho\in\omega_1} \mathfrak{M}_{\alpha,\rho}$ and this is all that is needed for proof of Lemma 9.12. □

Theorem 10.3. *It is consistent that there is an* $[\omega]^2$*-weakly universal function but there is no* (a,b)*-universal function for any* a *and* b *in* $\mathbb{N}$.

Proof. Using the arguments of §9 it follows that $\mathbb{U}(2)$ adds an $[\omega]^2$-weakly universal function. To see that there is no (a,b)-universal function suppose that $U : \omega_1^2 \to \omega$ is such a function. As in the proof of Theorem 9.1 let $\beta \in \omega_3$ be such that there is $\mathbb{P}_\beta$-name $\dot{U}$ for U. Then $(\dot{U}, \beta) = y_\nu$ for some $\nu \in \omega_3$.

For each $\alpha \in S_\nu$ let $\mathfrak{D}_\alpha$ be an elementary submodel of $H(\aleph_4, \in)$ such that:

- $|\mathfrak{D}_\alpha| = \aleph_1$
- $[\mathfrak{D}_\alpha]^{\aleph_0} \subseteq \mathfrak{D}_\alpha$
- $\alpha \in \mathfrak{D}_\alpha$
- $\{(\dot{R}_\beta, \psi_\beta, S_\beta, y_\beta)\}_{\beta\in\omega_3} \in \mathfrak{D}_\alpha$ (and, hence, $\mathbb{P}_{\omega_3} \in \mathfrak{D}_\alpha$)
- $\dot{U}$ and β belong to $\mathfrak{D}_\alpha$.

Then let D and $S \subseteq S_\nu$ be stationary such that $\mathfrak{D}_\alpha \cap \alpha = D$ for all $\alpha \in S$ and such that if α and β are in S then there is an isomorphism $\Psi_{\alpha,\beta} : \mathfrak{D}_\alpha \to \mathfrak{D}_\beta$ which, of course, is constant on D. Note also that since $\{S_\beta\}_{\beta\in\omega_3} \in \mathfrak{M}_{\alpha,0}$, $\{y_\beta\}_{\beta\in\omega_3} \in \mathfrak{M}_{\alpha,0}$, $\alpha \in \mathfrak{D}_\alpha$ and $\alpha \in S_\nu$ it follows that $\nu \in \mathfrak{D}_\alpha$ and, hence, that $(\dot{U}, \beta) = y_\nu \in \mathfrak{D}_\alpha$. Fix σ_0 and σ_1 in $S \setminus \beta$ such that

$$\max(\mathfrak{D}_{\sigma_0} \cap \omega_3) < \min(\mathfrak{D}_{\sigma_1} \cap (\omega_3 \setminus D)). \tag{10.1}$$

Letting $\dot{c}_{\sigma_i}$ be a name for the Cohen real added by $\mathbb{C}_2$ at stage σ_i define $\dot{B}_{\sigma_i}$ be the $\mathbb{C}_2$-name for a function from ω to ω defined by

$$\dot{B}_{\sigma_i}(j) = \begin{cases} j & \text{if } \dot{c}_{\sigma_0}(k) = 1 \text{ and } k^2 \le j < (k+1)^2 \\ 0 & \text{otherwise.} \end{cases}$$

Following the proof of Theorem 9.1 and arguing in $\mathfrak{D}_{\sigma_0}$ for the moment, let γ be sufficiently large that:

- $\sigma_0 \in \mathfrak{M}_{\gamma,0}$ (see Definition 10.1)
- there is a $\mathbb{P}_\gamma$-name $\dot{E}$ such that $\mathbb{1} \Vdash_{\mathbb{P}_\gamma}$ "$\dot{E}$ is an embedding of $\dot{B}_{\sigma_0} \circ \tau$ into $\dot{U}$"

recalling that $\tau : \omega_1^2 \to \omega$ witnesses that $\aleph_1 \not\twoheadrightarrow [\aleph_1]^2_{\aleph_0}$. Then let $\{p_\eta\}_{\eta\in\omega_1}$ be such that $p_\eta \Vdash_{\mathbb{P}_\gamma}$ "$\dot{E}(\eta) = \eta^*$" for some η^*. Then find a stationary $A \subseteq C_\gamma$ and p such that $p_\eta[\gamma, \eta] = p$ for each $\eta \in A$. Let $\boldsymbol{domain}(f_{p,\alpha}) \subseteq J$.

Then let $k = ab$. Then let $\{\eta_i\}_{i\in 2k} \subseteq A$ be such that there is some $m > J$ such that $m^2 < \tau(\eta_{2i}, \eta_{2i+1}) < (m+1)^2$ for $i \in k$ and the $\tau(\eta_{2i}, \eta_{2i+1})$ are all distinct. Use Corollary 10.2 to find $\mathbf{q} \in \mathbb{P}_{\alpha,m,\gamma}$ such that $\mathbf{q} \le p_{\eta_i}$ for all $i \in 2k$ and, of course, it can be assumed that $\mathbf{q} \in \mathfrak{D}_{\sigma_0}$. Let $\bar{\mathbf{q}} = \Psi_{\sigma_0,\sigma_1}(\mathbf{q})$ and note that

$$\mathbf{q}\langle 0\rangle \restriction D = \mathbf{q}\langle 1\rangle \restriction D \tag{10.2}$$

because $D \subseteq \sigma_0$. Then let $q = \mathbf{q}\langle 0\rangle \cup \bar{\mathbf{q}}\langle 1\rangle$ and note that $q \in \mathbb{P}_{\omega_3}$ because of Equation (10.2) and the choice of σ_0 and σ_1 satisfying Condition (10.1). Then the following hold

$$q \Vdash_{\mathbb{P}_{\omega_3}} \text{"}(\forall i \in 2k)\ \dot{E}(\eta_i) = \Psi_{\sigma_0,\sigma_1}(\dot{E})(\eta_i)\text{"}$$
$$q \Vdash_{\mathbb{P}_{\omega_3}} \text{"}(\exists e_0, e_1)\ (\dot{E}, e_0, e_1) \text{ is an } (a,b)\text{-weak embedding of } \dot{B}_{\sigma_0} \circ \tau \text{ into } \dot{U}\text{"}$$
$$q \Vdash_{\mathbb{P}_{\omega_3}} \text{"}(\exists e_0, e_1)\ (\Psi_{\sigma_0,\sigma_1}(\dot{E}), e_0, e_1) \text{ is an } (a,b)\text{-weak embedding of } \dot{B}_{\sigma_1} \circ \tau \text{ into } \dot{U}\text{"}$$

recalling that the notion of an (a,b)-weak embedding can be found in Definition 8.1. To obtain a contradiction from this note that whatever values $\dot{U}$ takes on the pairs (η_i^*, η_j^*) it must be the case that there are functions

e_0 and $\bar{e}_0$, with preimages of cardinality less than a, and e_1 and $\bar{e}_1$, with preimages of cardinality less than b witnessing (a, b)-weak embedding.

The first observation is that, since $q \Vdash_{\mathbb{P}_{\omega_3}}$ "$\dot{B}_{\sigma_0}$ is constant on $\{\tau(\eta_{2i}, \eta_{2i+1})\}_{i\in k}$" it follows that

$$q \Vdash_{\mathbb{P}_{\omega_3}} \text{"}e_0 \circ \dot{B}_{\sigma_0} \circ \tau \text{ is constant on } \{\tau(\eta_{2i}, \eta_{2i+1})\}_{i\in k}\text{"}.$$

Hence the cardinality of the range of U on $\{(\eta^*_{2i}, \eta^*_{2i+1})\}_{i\in k}$ must be smaller than b because otherwise $e_1 \circ U$ would not be constant on $\{(\eta^*_{2i}, \eta^*_{2i+1})\}_{i\in k}$. On the other hand, for each $j \in k$ it must be that

$$\left|\left\{i \in k \;\middle|\; \bar{e}_0(J + i) = \bar{e}_0(\dot{B}_{\sigma_1}(\tau(\eta^*_{2i}, \eta^*_{2i+1})) = U(\eta^*_{2j}, \eta^*_{2j+1})\right\}\right| < a$$

contradicting that $k = ab$ because $q \Vdash_{\mathbb{P}_{\omega_3}}$ "$\dot{B}_{\sigma_1}$ is one-to-one on $[m^2, (m + 1)^2)$". □

11. Open questions

Question 11.1. Is $\mathfrak{K} = k$ consistent for all k?

Question 11.2. For which a, b, c and d in $\mathbb{N}$ does the existence of an (a, b)-universal function imply the existence of an (c, d)-universal function?

Question 11.3. Does the existence of a $(1, 2)$-weakly universal function imply that there is a universal function? See Corollary 7.7.

References

[1] S. Shelah, On universal graphs without instances of CH, Ann. Pure Appl. Log. 26 (1) (1984) 75–87, https://doi.org/10.1016/0168-0072(84)90042-3.
[2] S. Shelah, Universal graphs without instances of CH: revisited, Isr. J. Math. 70 (1) (1990) 69–81, https://doi.org/10.1007/BF02807219.
[3] A.H. Mekler, Universal structures in power $\aleph_1$, J. Symb. Log. 55 (2) (1990) 466–477, https://doi.org/10.2307/2274640.
[4] R.D. Mauldin (Ed.), The Scottish Book, Mathematics from the Scottish Café, Including Selected Papers Presented at the Scottish Book Conference Held at North Texas State University, Denton, Tex., May 1979, Birkhäuser, Boston, Mass., 1981.
[5] W. Sierpiński, Sur une fonction universelle de deux variables réelles, Bull. Int. Acad. Polon. Sci. A (1936) 8–12.
[6] R. Rado, Universal graphs and universal functions, Acta Arith. 9 (1964) 331–340.
[7] P.B. Larson, A.W. Miller, J. Steprāns, W.A.R. Weiss, Universal functions, Fundam. Math. 227 (3) (2014) 197–246, https://doi.org/10.4064/fm227-3-1.
[8] W. Chen-Mertens, M. Kojman, J. Steprans, Strong colorings over partitions, arXiv:2002.06705.
[9] T. Bartoszyński, H. Judah, Set Theory, A.K. Peters, Ltd., Wellesley, MA, 1995, on the structure of the real line.
[10] U. Abraham, S. Todorčević, Partition properties of ω_1 compatible with CH, Fundam. Math. 152 (2) (1997) 165–181.
[11] S. Todorčević, A dichotomy for P-ideals of countable sets, Fundam. Math. 166 (3) (2000) 251–267.
[12] S. Shelah, Proper and Improper Forcing, 2nd edition, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1998.
[13] A. Blass, Combinatorial cardinal characteristics of the continuum, in: Handbook of Set Theory, vols. 1, 2, 3, Springer, Dordrecht, 2010, pp. 395–489.
[14] S. Shelah, On cardinal invariants of the continuum, in: Axiomatic Set Theory, Boulder, Colo., 1983, in: Contemp. Math., vol. 31, Amer. Math. Soc., Providence, RI, 1984, pp. 183–207.
[15] A. Dow, More set-theory for topologists, Topol. Appl. 64 (3) (1995) 243–300, https://doi.org/10.1016/0166-8641(95)00034-E.
[16] T. Inamdar, Universal Functions, Notes Written During HIF Semester, Cambridge, October 2015.
[17] O. Guzmán, On $(1, \omega_1)$-weakly universal functions, Fundam. Math. 247 (1) (2019) 87–98, https://doi.org/10.4064/fm630-9-2018.
[18] S. Todorčević, Partitioning pairs of countable ordinals, Acta Math. 159 (3–4) (1987) 261–294, https://doi.org/10.1007/BF02392561.