The Slicing Axioms

Ziemowit Kostana*
Institute of Mathematics of the Czech Academy of Sciences, Czech Republic
University of Warsaw, Poland

Saharon Shelah†‡
Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Israel
Department of Mathematics, Rutgers University, Piscataway, USA

November 16, 2021

Abstract

We introduce a family of axioms, denoted Slice_κ, that claim the existence of nontrivial decompositions of the form

$$2^{<\kappa} = \bigcup_{\alpha<\kappa} 2^{<\kappa} \cap M_\alpha,$$

where $\{M_\alpha|\ \alpha < \kappa\}$ is a sequence of transitive models of set theory. We study compatibility of these axioms with versions of Martin's Axiom, and in particular show that $\mathrm{Slice}_{\omega_1}$ is compatible only with some very weak form of MA.

Keywords: Martin's Axiom, Suslin forcing, transitive models
MSC classification: 03E50 03E17 03E35

1 Introduction

1.1 How "compact" is the real line?

We introduce and study a family of axioms Slice_κ for cardinal numbers κ. The axiom Slice_κ basically claims that there exists an increasing sequence $\{M_\alpha|\ \alpha < \kappa\}$ of transitive models of ZFC, that decomposes the sets 2^δ, for $\delta < \kappa$, into an increasing unions

$$2^\delta = \bigcup_{\alpha<\kappa} 2^\delta \cap M_\alpha.$$

Our initial motivation was to find a single model of Martin's Axiom, which doesn't satisfy typical consequences of PFA. This was in turn motivated by the following intuition:

If the universe is sufficiently complete, in the sense that it has many generic filters, then any transitive submodel containing enough reals, contain all the reals.

This intuition is supported for example by the following result:

Theorem 1 (Thm. 8.6, [11]). *If MM holds, then any inner model with correct ω_2 contains all reals.*

The conclusion is quite strong, so it makes sense to ask what is left if we weaken MM to MA_{ω_1}. This motivated us to formulate the axiom $\mathrm{Slice}_{\omega_1}$, which turned out to be inconsistent with MA_{ω_1}. The main results of this paper are the following

Theorem 2 (Thm. 11). $\mathrm{Slice}_{\omega_1} \implies \neg\,\mathrm{MA}_{\omega_1}(\sigma\text{-}centered)$.

*Research of Z. Kostana was supported by the GAČR project EXPRO 20-31529X and RVO: 67985840.
†Research of second author partially supported by Israel Science Foundation (ISF) grant no: 1838/19. Paper number 1210 on the publication list.
‡Research of both authors partially supported by NSF grant no: DMS 1833363.

Theorem 3 (Thm. 18)**.** *If κ is a regular cardinal such that $\kappa^\omega = \kappa$, then the following theory is consistent*

$$ZFC + MA(\textit{Suslin}) + \text{Slice}_{\omega_1} + \text{“}2^\omega = \kappa\text{”}.$$

Theorem 4 (Thm. 25)**.** *Assume that $\omega < \kappa \leq \theta$ are regular cardinals, such that $\theta^{<\kappa} = \theta$. Then the following theory is consistent*

$$ZFC + MA_{<\kappa} + \text{Slice}_\kappa + \text{“}2^\omega = \theta\text{”}.$$

The first of these results provides another argument in favor of the informal claim from the beginning. The class of Suslin forcings is a class of c.c.c. forcings, which admit simple (analytic) definitions (see Definition 17). This class is more extensively described in [2]. Martin's Axiom for this class is a considerable weakening of the full MA.

Theorem 5 ([9])**.** *$MA(\textit{Suslin})$ implies each of the following:*

1. $\text{Add}(\mathcal{N}) = 2^\omega$,
2. $\text{Add}(\mathcal{SN}) = 2^\omega$,
3. 2^ω *is regular,*
4. *each MAD family of subsets of ω has size 2^ω.*

It follows from 1. that all cardinal characteristics in the Cichoń's diagram have value 2^ω. $\mathcal{SN}$ stands for the class of strong measure zero sets.

Theorem 6 ([9])**.** *$MA(\textit{Suslin})$ does not imply any of the following:*

1. $\mathfrak{t} = 2^\omega$,
2. $\mathfrak{s} = 2^\omega$,
3. $\forall \kappa < 2^\omega \quad 2^\kappa = 2^\omega$,
4. *there is no Suslin tree.*

For an elaborated discussion of cardinal invariants of the continuum we refer the reader to [4]. Finally, it should be noted that our axiom Slice_{ω_1} is very similar to the axiom $\diamondsuit^{\text{Cohen}}$, introduced recently in [6].

1.2 Preliminaries

All non-standard notions are introduced in the subsequent sections. By *reals* we mean elements of the sets ω^ω, 2^ω, or seldom $\mathbb{R}$. We take the liberty to freely identify Borel functions with their Borel codes, so whenever we claim that

$$f \in M,$$

for some Borel function $f \subseteq 2^\omega \times 2^\omega$, and $M \models ZFC$, it should be understood that it is the Borel code of f that belongs to M (so we don't bother if, for instance, $\text{dom}\, f \not\subseteq M$).
When we write $\mathbb{P} = \{\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha |\ \alpha < \theta\}$ for a finite-support iteration of forcings, we sometimes denote by $\mathbb{P}$ the final step of the iteration, that is $\mathbb{P} = \mathbb{P}_\theta$. When dealing with infinite iterations, we assume that $\mathbb{P}_0$ is the trivial forcing. A function $i : \mathbb{P}_0 \hookrightarrow \mathbb{P}_1$ is a *complete embedding* if the following assertions hold:

1. $\forall p, q \in \mathbb{P}_0 \quad p_0 \leq p_1 \implies i(p_0) \leq i(p_1)$,
2. $\forall p, q \in \mathbb{P}_0 \quad p_0 \perp p_1 \implies i(p_0) \perp i(p_1)$,
3. If $\mathcal{A} \subseteq \mathbb{P}_0$ is a maximal antichain, then $i[\mathcal{A}] \subseteq \mathbb{P}_1$ is a maximal antichain.

We write $\mathbb{P}_0 \lessdot \mathbb{P}_1$ if $\mathbb{P}_0 \subseteq \mathbb{P}_1$ and the inclusion is a complete embedding. The importance of complete embeddings comes from the closely related notion of *quotient forcing*. If $\mathbb{P}_0 \lessdot \mathbb{P}_1$, and $G \subseteq \mathbb{P}_0$ is a filter generic over V, then $V[G]$ contains the quotient forcing $\mathbb{P}_1/G \subseteq \mathbb{P}_1$, consisting of all conditions $p \in \mathbb{P}_1$ that are compatible with every element of G. The crucial property of this notion is that forcing with $\mathbb{P}_1$ is equivalent to forcing with a two-step iteration: $\mathbb{P}_0 * \mathbb{P}_1/\dot{G}$ (see [10], p.244).

We will be frequently using the following two observations.

Proposition 7. *If V is a countable transitive model of ZFC, $\mathbb{P}_0, \mathbb{P}_1 \in V$, and $\mathbb{P}_0 \subseteq \mathbb{P}_1$ is an inclusion of partial orders, then the following conditions are equivalent:*

1. $\mathbb{P}_0 \lessdot \mathbb{P}_1$,
2. *If a filter $G \subseteq \mathbb{P}_1$ is $\mathbb{P}_1$-generic over V then $G \cap \mathbb{P}_0$ is $\mathbb{P}_0$-generic over V.*

Proposition 8. *Let $\mathbb{P} \lessdot \mathbb{S}$ be any forcing notions, and fix $p \in \mathbb{P}$. Let $\dot{q}$, $\dot{r}$ be $\mathbb{P}$-names, and finally let $\sigma(-,-)$ be a formula with parameters in the ground model, which is also absolute between transitive models of set theory (for example a Σ^1_1 formula in the language of arithmetic, or a bounded formula in the language of set theory). Then*

$$p \Vdash_{\mathbb{S}} \sigma(\dot{q}, \dot{r}) \iff p \Vdash_{\mathbb{P}} \sigma(\dot{q}, \dot{r}).$$

Proof. In the direction from left to right, if $p \in G \subseteq \mathbb{P}$ is generic over the ground model V, then we can extend G to a generic filter $G' \subseteq \mathbb{S}$. Notice that $\dot{q}[G] = \dot{q}[G']$, and $\dot{r}[G] = \dot{r}[G']$. By the absoluteness of $\sigma(-,-)$ we have

$$V[G] \models \sigma(\dot{q}[G], \dot{r}[G]) \iff V[G'] \models \sigma(\dot{q}[G'], \dot{r}[G']).$$

In the other direction, we proceed in a similar way, using the fact that for any generic filter $G \subseteq \mathbb{S}$, the intersection $G \cap \mathbb{P}$ is $\mathbb{P}$-generic. □

2 The Slicing Axioms

Definition 9. Let κ be any uncountable cardinal. We will say that Slice_κ holds if there exists a sequence of transitive classes (not necessarily proper) $\{M_\alpha |\ \alpha < \kappa\}$, such that the following conditions are satisfied

- $\forall\, \alpha < \kappa \quad (M_\alpha, \in) \models ZFC$,
- $\forall\, \alpha < \kappa \quad \omega_1^{M_\alpha} = \omega_1$,
- $\forall\, \delta \in [\omega, \kappa) \quad 2^\delta = \bigcup_{\alpha<\kappa} 2^\delta \cap M_\alpha$,
- $\forall\, \alpha < \beta < \kappa\, \forall\, \delta \in [\omega, \kappa) \quad 2^\delta \cap M_\alpha \subsetneq 2^\delta \cap M_\beta$.

Notice, that the last requirement is equivalent to a seemingly weaker

$$\forall\, \alpha < \beta < \kappa \quad 2^\omega \cap M_\alpha \subsetneq 2^\omega \cap M_\beta.$$

We will say that the sequence $\{M_\alpha |\ \alpha < \kappa\}$ *preserves cardinals* if $\kappa \in M_0$ and for each cardinal $\lambda \in M_0$ and each $\alpha < \kappa$, λ^{M_α} is a cardinal.

Let us observe that the axiom Slice_κ is outright false for any singular κ.

Proposition 10. *If* Slice_κ *is true, than κ is regular.*

Proof. If $\delta = \operatorname{cof} \kappa < \kappa$, then Slice_κ gives us a nontrivial decomposition of the form

$$2^\delta = \bigcup_{\alpha<\kappa} 2^\delta \cap M_\alpha,$$

and after passing to a cofinal sequence, we also have a decomposition

$$2^\delta = \bigcup_{\alpha<\delta} 2^\delta \cap M_{\gamma_\alpha}.$$

For each $\alpha < \delta$ we pick $s_\alpha \in 2^\delta \setminus M_{\gamma_\alpha}$. Using a definable bijection between δ and $\delta \times \delta$, we see that the sequence $(s_\alpha)_{\alpha<\delta}$ must belong so some model M_α. But this contradicts the choice of s_α. □

The most important of the slicing axioms is perhaps $\mathrm{Slice}_{\omega_1}$, since it claims that the real line can be decomposed into an increasing union of ω_1 many sets, which belong to bigger and bigger models. The fact that MA_{ω_1} is inconsistent with $\mathrm{Slice}_{\omega_1}$ shows, that the Martin's Axiom on ω_1 imposes certain compactness on the real line.

3 Slicing the real line

We begin with showing that Martin's Axiom on ω_1 is not compatible with $\mathrm{Slice}_{\omega_1}$.

Theorem 11. $\mathrm{Slice}_{\omega_1} \implies \neg\,\mathrm{MA}_{\omega_1}(\sigma\text{-}centered)$.

In the proof we will utilize a known result from [7]. Recall that a set $A \subseteq \mathbb{R}$ is a *Q-set*, if each subset of A is a relative F_σ, i.e. for each $B \subseteq A$ there exists an F_σ subset $F \subseteq \mathbb{R}$ such that $A \cap F = B$.

Theorem 12 ([7]). *$\mathrm{MA}_{\omega_1}(\sigma$-centered) implies that each set of cardinality ω_1 is a Q-set.*

Proof of Theorem 11. Assume that MA_{ω_1} holds, and $(M_\alpha)_{\alpha<\omega_1}$ is a sequence of models witnessing $\mathrm{Slice}_{\omega_1}$. $M_0 \models$ "2^ω is uncountable", so there exists a sequence of pairwise distinct reals $X = \{x_\alpha |\ \alpha < \omega_1\} \in M_0$ (note that this sequence is *really* of the length ω_1). Let $f : \omega_1 \hookrightarrow 2^\omega$ be a function such that $\forall \alpha < \omega_1\ f(\alpha) \notin M_\alpha$. We will obtain a contradiction, by showing that there exists some $\eta < \omega_1$, for which $\mathrm{rg}(f) \subseteq M_\eta$.

For every natural number m, let $A_m = \{x_\alpha |\ f(\alpha)(m) = 1\} = X \cap F_m$, where F_m is an F_σ subset of reals. Since the sequence $(F_m)_{m<\omega}$ can be coded by a real, clearly it belongs to some model M_η. It is enough to show that using this sequence and X we can give a definition of $\mathrm{rg}(f)$. But

$$\mathrm{rg}(f) = \{x \in 2^\omega |\ \exists \alpha < \omega_1\ \forall m < \omega \quad x_\alpha \in F_m \iff x(m) = 1\}.$$

□

It is compatible with any value of 2^ω that $\mathrm{Slice}_{\omega_1}$ holds and is witnessed by a cardinal preserving sequence.

Proposition 13. *Let $\mathbb{P}$ be any finite-support product of c.c.c. forcings adding reals, and of the length at least ω_1. Then $\mathbb{P} \Vdash \mathrm{Slice}_{\omega_1}$, and the corresponding sequence of models is cardinal preserving.*

Proof. Let us consider a finite-support product of c.c.c. forcings

$$\mathbb{P} = \prod_{i\in I} \mathbb{P}_i,$$

where each $\mathbb{P}_i$ adds some real number, and $|I| \geq \omega_1$. We can decompose I into a strictly increasing union $I = \bigcup_{\gamma<\omega_1} I_\gamma$. For each $\alpha < \omega_1$ the product $\prod_{i\in I_\alpha} \mathbb{P}_i$ can be identified with a complete suborder of $\mathbb{P}$.

If $G \subseteq \mathbb{P}$ is generic over some model V, then $\mathrm{Slice}_{\omega_1}$ is witnessed by the sequence

$$M_\alpha = V[G \cap \prod_{i\in I_\alpha} \mathbb{P}_i].$$

□

Recall that a set of reals is called *ω_1-dense*, if each nonempty open interval in this set has size ω_1. The following was proved by Baumgartner in [3]. The following was proved by Baumgartner in [3].

Theorem 14 ([3]). *It is consistent with MA_{ω_1}, that all ω_1-dense subsets of reals are order-isomorphic. In particular, each ω_1-dense set of reals has a non-trivial order-automorphism.*

The natural question whether this assertion follows from MA_{ω_1} was resolved by Avraham and the second author in [1].

Theorem 15 ([1]). *It is consistent with MA_{ω_1}, that there exists a rigid ω_1-dense real order type.*

This is also an easy consequence of $\mathrm{Slice}_{\omega_1}$.

Theorem 16. *$\mathrm{Slice}_{\omega_1}$ implies that there is an ω_1-dense rigid subset of the real line.*

Proof. Let $(M_\alpha)_{\alpha<\omega_1}$ be a sequence witnessing $\mathrm{Slice}_{\omega_1}$. For each α, we choose

$$x_\alpha \in \mathbb{R} \cap (M_\alpha \setminus \bigcup_{\beta<\alpha} M_\beta).$$

We can easily arrange the construction, so that we hit each open interval ω_1-many times. The set $X = \{x_\alpha | \ \alpha < \omega_1\}$ is ω_1-dense, and it remains to prove, that it is also rigid. Suppose that $f : X \to X$ is an order isomorphism. f extends uniquely to a continuous function $f' : \mathbb{R} \to \mathbb{R}$, and each such function can be coded by a real number. Therefore there is some $\eta < \omega_1$, such that $f' \in M_\eta$. Now, for any $\xi > \eta$, it is not possible that $f(x_\eta) = x_\xi$, because it would mean $x_\xi \in M_\eta$, contrary to the choice of x_ξ. But, likewise, it is not possible that $f^{-1}(x_\eta) = x_\xi$. The conclusion is that for all $\xi > \eta$, $f(x_\xi) = x_\xi$. But this means that f is identity on a dense set, and therefore everywhere. □

4 Slicing the real line while preserving MA(Suslin)

We are going to show that $\mathrm{Slice}_{\omega_1}$ is consistent with a version of Martin's Axiom which takes into account only partial orders representable as analytic sets (see [2], Ch. 3.6, or [9]).

Definition 17. A partial order $(\mathbb{P}, \le)$ has a *Suslin definition* if $\mathbb{P} \in \Sigma^1_1(\omega^\omega)$, and both ordering and incompatibility relations in $\mathbb{P}$ are analytic relations on ω^ω. $\mathbb{P}$ is *Suslin* if it has a Suslin definition and is c.c.c.

The following is the main result of this Section.

Theorem 18. *If κ is a regular cardinal such that $\kappa^\omega = \kappa$, then the following theory is consistent*

$$ZFC + MA(\textit{Suslin}) + \mathrm{Slice}_{\omega_1} + \text{“}2^\omega = \kappa\text{”}.$$

Let $\psi(-,-,-,-)$ be a universal analytic formula, i.e. a Σ^1_1 formula with the property that for each analytic set $P \subseteq \omega^\omega \times \omega^\omega \times \omega^\omega$ there exists $r \in \omega^\omega$ such that

$$P = \{x \in \omega^\omega \times \omega^\omega \times \omega^\omega | \ \psi(x, r)\}.$$

We want to use ψ to add generic filters to all possible Suslin forcings. We will say that $\psi(-,-,-,\dot{r}_\alpha)$ *defines* $\dot{\mathbb{Q}}_\alpha$ if $\dot{r}_\alpha$ is a $\mathbb{P}_\alpha$-name for a real and $\mathbb{P}_\alpha$ forces each of the following

$$\dot{\mathbb{Q}}_\alpha \text{ is a separative partial order with the greatest element } 0,$$
$$\psi(x, 1, 1, \dot{r}_\alpha) \iff x \in \dot{\mathbb{Q}}_\alpha,$$
$$\psi(x, y, 2, \dot{r}_\alpha) \iff x \le_{\dot{\mathbb{Q}}_\alpha} y,$$
$$\psi(x, y, 3, \dot{r}_\alpha) \iff x \perp_{\dot{\mathbb{Q}}_\alpha} y.$$

We will write $\psi^\in(x, z)$ for $\psi(x, 1, 1, z)$, $\psi^\perp(x, y, z)$ for $\psi(x, y, 3, z)$, and $\psi^\le(x, y, z)$ for $\psi(x, y, 2, z)$.

We are going to iterate all Suslin forcings, each of them cofinally many times. More precisely, we define by induction a finite-support iteration $\{\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha | \ \alpha < \kappa\}$:

- $\mathbb{P}_0 = \{0\}$,
- $\mathbb{P}_\alpha \Vdash$ “$\dot{\mathbb{Q}}_\alpha = \{x \in \omega^\omega | \ \psi(x, \dot{r}_\alpha)\}$ if this formula defines a Suslin forcing; else $\dot{\mathbb{Q}} = \{0\}$”,

The variable $\dot{r}_\alpha$ ranges over all reals, and all possible names for reals, each of them cofinally many times. In order to iterate through all possible parameters using a suitable bookkeeping, we introduce the class of *simple* conditions, following [2].

Definition 19. By induction on α we define *simple* conditions in $\mathbb{P}_\alpha$.

- $\alpha = 0$. $\mathbb{P}_0 = \{0\}$, and we declare 0 to be simple.
- $\alpha + 1$. $(p, \dot{q}) \in \mathbb{P}_{\alpha+1}$ is simple if $p \in \mathbb{P}_\alpha$ is simple and

  $$\dot{q} = \{(m, n, p^m_n) | \ m, n < \omega, \ p^m_n \in \mathbb{P}_\alpha\},$$

  where each p^m_n is a simple condition in $\mathbb{P}_\alpha$. (for each $m \in \omega$, the set $\{p^m_n | \ n < \omega\}$ is a maximal antichain deciding $\dot{q}(m)$, i.e. $p^m_n \Vdash \dot{q}(m) = n$)

- $\lim \alpha$. $p \in \mathbb{P}_\alpha$ is simple if for each $\beta < \alpha$, $p \restriction \beta \in \mathbb{P}_\beta$ is simple.

It is straightforward to check by induction, that the set of simple $\mathbb{P}_\alpha$-conditions is dense in $\mathbb{P}_\alpha$, and that each $\mathbb{P}_\alpha$ has at most κ many names for reals (if we restrict to names with simple conditions). We declare the forcings $\mathbb{P}_\alpha$ to consist only of simple conditions, so formally we write

$$\mathbb{P}_{\alpha+1} = \{(p, \dot{q}) \in \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha | \ (p, \dot{q}) \text{ is simple}\}.$$

Proposition 20. *If κ is an uncountable regular cardinal such that $\kappa^\omega = \kappa$, then*

$$\mathbb{P}_\kappa \Vdash MA(\textit{Suslin}) + \text{“}2^\omega = \kappa\text{”}.$$

Proof. Let us denote by W_α the corresponding extensions of V by $\mathbb{P}_\alpha$. Let $(S, \leq)$ be a Suslin forcing in W_κ. Assume S is defined by the formula $\psi(-, r)$. We fix a family $\{A_\gamma | \ \gamma < \lambda\}$ of maximal antichains in S, where $\lambda < \kappa$. Of course the formula $\psi(-, r)$ defines different sets in different models of set theory, so following the common custom we will denote by S^N the interpretation of S in the model N, i.e.

$$S^N = \{x \in \omega^\omega \cap N | \ N \models \psi^\in(x, r)\}.$$

Notice that the family $\{A_\gamma | \ \gamma < \lambda\}$ is a function from λ to $[\omega^\omega]^\omega$, and so is added in some intermediate step if the iteration. Let us fix an ordinal $\delta < \kappa$ such that $\{A_\gamma | \ \gamma < \lambda\} \in W_\delta$, and $\mathbb{P}_\delta \Vdash \dot{r}_\delta = r$. Now $\mathbb{P}_\delta \Vdash \dot{\mathbb{Q}}_\delta = S^{W_\delta}$, so $W_{\delta+1}$ contains a filter $G_0 \subseteq S^{W_\delta}$ intersecting all A_γ's (note that by absoluteness of $\psi^\in$ and $\psi^\perp$, the sets A_γ are maximal antichains in S^{W^δ}). The filter generated by G_0 in S^{W_κ} is the required generic filter. □

If N is a transitive class containing κ, we can define by induction the relativized iteration $\mathbb{P}^N_\kappa \subseteq \mathbb{P}_\kappa$, taking into account only names from N.

- $\mathbb{P}^N_0 = \{0\}$,
- $\mathbb{P}^N_\alpha \Vdash$ “$\dot{\mathbb{Q}}^N_\alpha = \{x \in \omega^\omega | \ \psi^\in(x, \dot{r}_\alpha)\}$ if this formula defines a Suslin forcing, $\dot{r}_\alpha \in N$, and $\dot{r}_\alpha$ is a $\mathbb{P}^N_\alpha$-name; else $\dot{\mathbb{Q}}^N_\alpha = \{0\}$”,
- $\mathbb{P}^N_{\alpha+1} = \mathbb{P}^N_\alpha * \dot{\mathbb{Q}}^N_\alpha$.

We take direct limits in the limit step, so that $\mathbb{P}^N_\alpha$ is really a subset of $\mathbb{P}_\alpha$. Note, that we do not define names $\dot{r}_\alpha$ inductively along the way, since they have already been defined in the construction of $\mathbb{P}_\kappa$, which we take as granted. This construction is inspired by the lemmas 1.4 and 1.5 from [9], and conceptually is very similar. In order for it to work as desired, we prove by induction some properties of $\mathbb{P}^N_\alpha$.

Theorem 21. *If N is a transitive class containing κ, then for all $\alpha \leq \kappa$*

$$\mathbb{P}^N_\alpha \lessdot \mathbb{P}_\alpha.$$

Specifically:

1. *If $p_0 \perp p_1$ in $\mathbb{P}^N_\alpha$, then $p_0 \perp p_1$ in $\mathbb{P}_\alpha$.*
2. *If $p_0 \leq p_1$ in $\mathbb{P}^N_\alpha$, then $p_0 \leq p_1$ in $\mathbb{P}_\alpha$.*
3. *If $\mathcal{A} \subseteq \mathbb{P}^N_\alpha$ is a maximal antichain, then $\mathcal{A}$ is maximal in $\mathbb{P}_\alpha$.*

Proof. We proceed by induction on α.
1.

- $\alpha = 0$. Clear.
- $\alpha + 1$. We can assume that $\dot{\mathbb{Q}}^N_\alpha$ is defined by the formula $\psi(-, \dot{r}_\alpha)$, for otherwise $\mathbb{P}^N_{\alpha+1} = \mathbb{P}^N_\alpha$, and we are done by the induction hypothesis. Fix two incompatible conditions $p_0, p_1 \in \mathbb{P}^N_{\alpha+1}$. Then $p_0 = (p'_0, \dot{q}_0)$, $p_1 = (p'_1, \dot{q}_1)$, where $p'_0, p'_1 \in \mathbb{P}^N_\alpha$, and

$$p'_0 \Vdash \psi^\in(\dot{q}_0, \dot{r}_\alpha),$$

$$p'_1 \Vdash \psi^\in(\dot{q}_1, \dot{r}_\alpha).$$

The forcing relation used above is a relation from $\mathbb{P}^N_\alpha$, however since $\dot{r}_\alpha$, $\dot{q}_0$ and $\dot{q}_1$ are $\mathbb{P}^N_\alpha$-names, this is the same relation as coming from $\mathbb{P}_\alpha$ (see Proposition 8). We aim to show that $p_0 \perp p_1$ in $\mathbb{P}_{\alpha+1}$.

If $p'_0 \perp p'_1$ in $\mathbb{P}_\alpha$, then clearly $p_0 \perp p_1$ in $\mathbb{P}_{\alpha+1}$, so assume otherwise, and fix $p \leq p'_0, p'_1$ (in $\mathbb{P}_\alpha$). Let $p \in G \subseteq \mathbb{P}_\alpha$ be a filter generic over V. Conditions p_0 and p_1 were incompatible in $\mathbb{P}^N_{\alpha+1}$ and, by the induction hypothesis, $G \cap \mathbb{P}^N_\alpha \subseteq \mathbb{P}^N_\alpha$ is generic over V, therefore

$$V[G \cap \mathbb{P}^N_\alpha] \models \psi^\perp(\dot{q}_0[G], \dot{q}_1[G], \dot{r}_\alpha[G]).$$

By absoluteness

$$V[G] \models \psi^\perp(\dot{q}_0[G], \dot{q}_1[G], \dot{r}_\alpha[G]).$$

Since p was arbitrary, it follows that $p_0 \perp p_1$ in $\mathbb{P}_{\alpha+1}$.

- $\lim \alpha$. Follows from the induction hypothesis, since conditions have finite supports.

2.

- $\alpha = 0$. Clear.
- $\alpha + 1$. Again, we can assume that $\dot{\mathbb{Q}}^N_\alpha$ is defined by the formula $\psi(-, \dot{r}_\alpha)$. Fix two conditions $p_0 \leq p_1 \in \mathbb{P}^N_{\alpha+1}$. Then $p_0 = (p'_0, \dot{q}_0)$, $p_1 = (p'_1, \dot{q}_1)$, where $p'_0, p'_1 \in \mathbb{P}^N_\alpha$, and
$$p'_0 \Vdash \psi^\in(\dot{q}_0, \dot{r}_\alpha),$$
$$p'_1 \Vdash \psi^\in(\dot{q}_1, \dot{r}_\alpha).$$
By the induction hypothesis $p'_0 \leq p'_1$ in $\mathbb{P}_\alpha$. Moreover $\dot{r}_\alpha$, $\dot{q}_0$ and $\dot{q}_1$ are $\mathbb{P}^N_\alpha$-names, so – in the light of Proposition 8 – the forcing relation
$$p'_0 \Vdash \dot{q}_0 \leq \dot{q}_1$$
holds in $\mathbb{P}^N_\alpha$ as well as in $\mathbb{P}_\alpha$.
- $\lim \alpha$. Follows from the induction hypothesis, since conditions have finite supports.

3.

- $\alpha = 0$. Clear.
- $\lim \alpha$. Let $\mathcal{A}$ be a maximal antichain in $\mathbb{P}^N_\alpha$, and $\overline{p} \in \mathcal{A}$. Given that $\mathbb{P}^N_\alpha$ is a finite-support iteration of c.c.c. forcings, it satisfies the countable chains condition, therefore we can assume that $\mathcal{A} = \{\overline{p}_n |\ n < \omega\}$. There is some $\gamma < \alpha$ such that $\overline{p} \in \mathbb{P}_\gamma$. $\{\overline{p}_n \restriction \gamma |\ n < \omega\}$ might not be an antichain in $\mathbb{P}^N_\gamma$, however each condition in $\mathbb{P}^N_\gamma$ is compatible with some $p_n \restriction \gamma$. We can refine $\{\overline{p}_n \restriction \gamma |\ n < \omega\}$ to an antichain in $\mathbb{P}^N_\gamma$, and this antichain will remain maximal in $\mathbb{P}_\gamma$ by the induction hypothesis. Therefore $\{\overline{p}_n \restriction \gamma |\ n < \omega\}$ meets every condition in $\mathbb{P}_\gamma$, and in particular some $\overline{p}_n \restriction \gamma$ is compatible with $\overline{p}$ in $\mathbb{P}_\gamma$. But then $\overline{p}_n$ is compatible with $\overline{p}$ in $\mathbb{P}_\alpha$.
- $\alpha + 1$. In the light of Proposition 7, it is sufficient to show that for any $G \subseteq \mathbb{P}_{\alpha+1}$ generic over V, $G \cap \mathbb{P}^N_{\alpha+1}$ is also generic over V.

Lemma 22. *If $G \subseteq \mathbb{P}_\alpha$ is generic over V, and $H \subseteq \dot{\mathbb{Q}}_\alpha[G]$ is generic over $V[G]$, then $H \cap \dot{\mathbb{Q}}^N_\alpha[G] \subseteq \dot{\mathbb{Q}}^N_\alpha[G]$ is generic over $V[G \cap \mathbb{P}^N_\alpha]$.*

Why is this sufficient? Let $\overline{G} \subseteq \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$ be a filter generic over V. Recalling the notation from [10],

$$\overline{G} = G * H = \{(p, \dot{q}) |\ p \in G,\ \dot{q}[G] \in H\},$$

where

$$G = \{p \in \mathbb{P}_\alpha |\ \exists \dot{q} \ \in \dot{\mathbb{Q}} \quad (p, \dot{q}) \in \overline{G}\},$$

and

$$H = \{\dot{q}[G] |\ \exists p \in G \quad (p, \dot{q}) \ \in \overline{G}\}.$$

It is known that for any iteration $\mathbb{P} * \dot{\mathbb{Q}}$, if $G \subseteq \mathbb{P}$ is generic over V and $H \subseteq \dot{\mathbb{Q}}[G]$ is generic over $V[G]$, then $G * H$ is generic for $\mathbb{P} * \dot{\mathbb{Q}}$ over V (for details consult for example [10], Section 5, Chapter VIII). Let $G' = G \cap \mathbb{P}^N_\alpha$. It is generic for $\mathbb{P}^N_\alpha$ over V by the induction hypothesis. Now for filters G and H defined above

$$(G * H) \cap (\mathbb{P}^N_\alpha * \dot{\mathbb{Q}}^N_\alpha) = \{(p, \dot{q}) |\, p \in G',\ \dot{q}[G] \in H,\ \dot{q} \in \dot{\mathbb{Q}}^N_\alpha\} =$$
$$\{(p, \dot{q}) \in \mathbb{P}^N_\alpha * \dot{\mathbb{Q}}^N_\alpha |\, p \in G',\ \dot{q}[G'] \in H\} = G' * (H \cap \dot{\mathbb{Q}}^N_\alpha[G']).$$

But if the conclusion of Lemma 22 holds, this is a $\mathbb{P}^N_\alpha * \dot{\mathbb{Q}}^N_\alpha$-generic filter over V.

We turn to the proof of Lemma 22.

Proof. Fix a maximal antichain $\mathcal{A} \subseteq \dot{\mathbb{Q}}^N_\alpha[G] = \dot{\mathbb{Q}}^N_\alpha[G']$, belonging to $V[G']$. As $\mathcal{A}$ is a countable set of reals, it can be coded using a single real $z \in \omega^\omega$. Recall that $\dot{\mathbb{Q}}^N_\alpha[G']$ is defined in $V[G']$ by the formula ψ with the parameter $\dot{r}_\alpha[G'] = \dot{r}_\alpha[G]$. It is standard to check, that the following claim can be written as a Π^1_1 formula.

$\phi(x, y) =$ "x is a real coding a maximal antichain in the partial ordering defined by the formula $\psi(-, -, -, y)$".

Now
$$V[G'] \models \phi(z, \dot{r}_\alpha[G']),$$
and so by absoluteness
$$V[G] \models \phi(z, \dot{r}_\alpha[G]).$$
But $\psi(-, \dot{r}_\alpha[G])$ is the formula defining $\dot{\mathbb{Q}}_\alpha[G]$ in $V[G]$. Therefore $\mathcal{A}$ remains maximal in $\dot{\mathbb{Q}}_\alpha[G]$, and the conclusion of the Lemma easily follows. □

This concludes the proof.

□

Let us note that even if N is an inner model of ZFC, usually $\mathbb{P}^N_\kappa \notin N$. Definition of $\mathbb{P}^N_\kappa$ makes use of a list of $\mathbb{P}^N_\alpha$-names, for all $\alpha < \kappa$, and although *some* such enumeration belongs to N (as it is a model of AC), this particular might not. In what sense is $\mathbb{P}^N_\kappa$ a *relativized* version of $\mathbb{P}_\kappa$, is explained by the next lemma.

Lemma 23. *For each $\alpha \leq \kappa$, if $p \in \mathbb{P}_\alpha$ is simple then p is definable (in the language of set theory) with a parameter from κ^ω.*

Proof. We proceed by induction on α.

- $\alpha = 0$. Clear, since each real is definable with a real parameter.
- $\alpha + 1$. Let $r = (p, \dot{q})$ be simple. We can write
$$\dot{q} = \{(m, n, p^m_n) |\, m, n \in \omega,\ p^m_n \in \mathbb{P}_\alpha\},$$
where each p^m_n is simple. By the induction hypothesis each p^m_n is definable with a parameter from κ^ω, and so is p. Clearly r can be defined from them, and so r is definable with countably many parameters from κ^ω. We can easily code them as a single parameter.
- $\lim \alpha$. Fix $r \in \mathbb{P}_\alpha$. r has finite support, so there exists $\beta < \alpha$ containing the support of r. By the induction hypothesis $p \restriction \beta$ is definable with a parameter from κ^ω, and p is definable with parameters $p \restriction \beta$, β, and α.

□

From this point, we fix a list $\{\sigma_\alpha |\, \alpha < \kappa\}$ of sequences from κ^ω, such that each name $\dot{r}_\alpha$ is definable from σ_α.

Lemma 24. *Let N be any transitive model of ZFC containing κ, and let $M \prec H((2^\kappa)^+)$ be a countable elementary submodel such that $\{\sigma_\alpha |\, \alpha < \kappa\} \in M$, and $M \cap \kappa^\omega \subseteq N$. Then for any $\alpha \leq \kappa$, $\mathbb{P}_\alpha \cap M \subseteq \mathbb{P}^N_\alpha$.*

Proof. We proceed by induction.

- $\alpha = 0$. Clear.
- $\lim \alpha$. Fix $r \in \mathbb{P}_\alpha \cap M$. By the elementarity of M, there exists $\gamma \in \alpha \cap M$ that contains the support of r. From the induction hypothesis it follows that $r \upharpoonright \gamma \in \mathbb{P}^N_\gamma$. It is routine to verify by induction that for all $\gamma \leq \delta \leq \alpha$, $r \upharpoonright \delta \in \mathbb{P}^N_\delta$.
- $\alpha + 1$. Fix $r = (p, \dot{q}) \in M \cap (\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha)$. Without loss of generality we can assume that $\dot{\mathbb{Q}}_\alpha$ is defined by the formula $\psi(-, \dot{r}_\alpha)$. We need to check that $\dot{q}$ is a $\mathbb{P}^N_\alpha$-name for an element of $\dot{\mathbb{Q}}^N_\alpha$, which means that in particular $\dot{\mathbb{Q}}^N_\alpha$ needs to be defined by the same formula as $\dot{\mathbb{Q}}_\alpha$. In summary, our task is to verify the following three claims:
  - $\dot{q}$ is a (simple) $\mathbb{P}^N_\alpha$-name,
  - $\dot{r}_\alpha$ is a (simple) $\mathbb{P}^N_\alpha$-name,
  - $\dot{r}_\alpha \in N$.

  The condition $\dot{q}$ is of the form $\dot{q} = \{(m, n, p^m_n) |\, m, n < \omega,\ p^m_n \in \mathbb{P}_\alpha\}$. Given that all conditions p^m_n belong to M, they also belong to $\mathbb{P}^N_\alpha$ by the induction hypothesis. This shows that $\dot{q}$ is a $\mathbb{P}^N_\alpha$-name. For the same reason, $\dot{r}_\alpha$ is a $\mathbb{P}^N_\alpha$-name, once we show that $\dot{r}_\alpha \in M$. But the model M contains the list $\{\sigma_\alpha |\, \alpha < \kappa\}$ and the ordinal α. Therefore $\sigma_\alpha \in M$, and $\dot{r}_\alpha \in M$. Finally, given that $\sigma_\alpha \in \kappa^\omega \cap M \subseteq N$, we conclude that $\dot{r}_\alpha \in N$.

□

Proof of Theorem 18. We start with a model $V \models \text{Slice}_{\omega_1} +$"$2^\omega = \kappa$", and we assume moreover that the sequence $\{M_\alpha |\, \alpha < \omega_1\}$ witnessing Slice_{ω_1} satisfies the following stronger property:

$$\kappa^\omega = \bigcup_{\alpha<\omega_1} \kappa^\omega \cap M_\alpha.$$

Such a model is easy to get, for example by adding κ many Cohen reals to a model of CH using the finite support-iteration, and proceeding like in the proof of Proposition 13. We also assume that $\kappa \in M_0$.

Let $\mathbb{P} = \{\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha |\, \alpha < \kappa\}$ be the described iteration, which forces

$$MA(\text{Suslin}) + \text{“}2^\omega = \kappa\text{”}.$$

We claim that if $G \subseteq \mathbb{P}$ is generic over V, then the sequence $V[G \cap \mathbb{P}^{M_\alpha}]$ witnesses Slice_{ω_1} in $V[G]$. For this we need to show two things:

1. If $r \in \omega^\omega \cap V[G]$, then $r \in V[G \cap \mathbb{P}^{M_\alpha}]$, for some $\alpha < \omega_1$.
2. None of the models $V[G \cap \mathbb{P}^{M_\alpha}]$ contains all reals from $V[G]$.

Concerning 1. suppose that $\mathbb{P}_\kappa \Vdash \dot{r} \in \omega^\omega$. We can assume that

$$\dot{r} = \{(m, n, p^m_n) |\, m, n < \omega\},$$

and all conditions p^m_n are simple. Fix a countable elementary submodel $\overline{M} \prec H((2^\kappa)^+)$, that contains the list $\{\sigma_\alpha |\, \alpha < \kappa\}$, and the name $\dot{r}$. We pick α big enough, so that $\overline{M} \cap \kappa^\omega \subseteq M_\alpha$. Applying Lemma 24 with $M = \overline{M}$, and $N = M_\alpha$, we see that $\dot{r}$ is a $\mathbb{P}^{M_\alpha}$-name. Therefore $\dot{r}$ is a $\mathbb{P}^{M_\alpha}$-name, and so

$$\dot{r}[G] = \dot{r}[G \cap \mathbb{P}^{M_\alpha}] \in V[G \cap \mathbb{P}^{M_\alpha}].$$

Concerning 2. fix a real $r \in \omega^\omega \setminus M_\alpha$. There exists a representation of the Cohen forcing as a Borel subset of ω^ω, from which the real r is definable. For concreteness, let us put

$$\mathbb{C}_r = \omega^{<\omega} \cup \{r\} \subseteq \omega^\omega,$$

where $\omega^{<\omega}$ is identified with the set of sequences from ω^ω that are eventually equal zero. We order $\omega^{<\omega}$ by the end-extension and we declare that

$$\forall s \in \omega^{<\omega}\ s \perp r.$$

Since $\mathbb{C}_r$ is clearly Suslin, there exists a real r' such that

$$\mathbb{C}_r = \{x \in \omega^\omega |\ \psi^\in(x, r')\}.$$

We claim that $r' \notin M_\alpha$. Suppose otherwise. Let $\sigma(x)$ stand for the formula

$$\psi^\in(x, r') \wedge x \notin \omega^{<\omega}.$$

Note that

$$V \models \exists\, x \in \omega^\omega\ \sigma(x),$$

and so by absoluteness the same holds in M_α. Fix $r'' \in \omega^\omega \cap M_\alpha$, such that

$$M_\alpha \models \sigma(r'').$$

Again $V \models \sigma(r'')$ by absoluteness. But this shows that $r = r''$, and therefore $r \in M_\alpha$, contradicting the choice of r.

Once we know that $r' \notin M_\alpha$, let us fix $\gamma < \kappa$ such that $\mathbb{P}_\gamma \Vdash \dot{r}_\gamma = r'$. It follows that

$$\mathbb{P}_\gamma^{M_\alpha} \Vdash \dot{\mathbb{Q}}_\gamma^{M_\alpha} = \{0\},$$

and

$$\mathbb{P}_\gamma \Vdash \dot{\mathbb{Q}}_\gamma = \mathbb{C}_r.$$

The mapping from $\mathbb{C}_r$ into $\mathbb{P}_{\gamma+1}$ given by the formula

$$x \mapsto (1_{\mathbb{P}_\gamma}, x)$$

is a complete embedding, and it extends to a complete embedding

$$f : \mathbb{C}_r \hookrightarrow \mathbb{P}_\kappa.$$

Let $\dot{H}$ be a $\mathbb{P}_\kappa^{M_\alpha}$-name for a generic filter. In the generic extension $V[H]$, the function f remains a complete embedding of a Cohen forcing into $\mathbb{P}_\kappa$, and by the choice of γ, f is in fact a a complete embedding into $\mathbb{P}_\kappa/H$ (since $H \subseteq \mathbb{P}_\kappa^{M_\alpha}$, each condition of the form $(1_{\mathbb{P}_\gamma}, x) \in \mathbb{P}_{\gamma+1}$ is compatible with every condition from H). In conclusion, forcing with $\mathbb{P}_\kappa$ adds a Cohen real over $V[G \cap \mathbb{P}_\kappa^{M_\alpha}]$. $\square$

4.1 Computation of cardinal invariants

For any given group G, one can study an associated cardinal invariant $\mathrm{c(Sym(G))}$ that stands for the minimal cardinality κ, for which the group G can be represented as a union of a chain of the length κ, consisting of proper subgroups of G. A substantial amount of literature is devoted to study this cardinal invariant for symmetric groups of infinite sets (for example [12], [15], [14], [8]). It is known that

$$\mathfrak{g} \leq \mathrm{c}(\mathrm{Sym}(\omega)) \leq \mathfrak{d},$$

where $\mathfrak{g}$ is the *groupwise density number*. The lower bound was proved by Brendle and Losada [5], and the upper bound is due to Sharp and Thomas [13].

It is easy to observe that $\mathrm{Slice}_{\omega_1} \implies$ "$\mathrm{c}(\mathrm{Sym}(\omega)) = \omega_1$": if $\mathrm{Slice}_{\omega_1}$ is witnessed by a sequence $(M_\alpha)_{\alpha<\omega_1}$, then the equality $\mathrm{c}(\mathrm{Sym}(\omega)) = \omega_1$ is witnessed by the sequence of groups $(M_\alpha \cap \mathrm{Sym}(\omega))_{\alpha<\omega_1}$. As a matter of fact, this observation shows that Theorem 18 generalizes Lemma 2.6 from [16], which claims that the equality $\mathrm{c}(\mathrm{Sym}(\omega)) = \omega_1$ is preserved under finite-support iterations of Suslin forcings.

Together with some well-known results (consult [4]), we have the following series of inequalities:

$$\mathrm{c}(\mathrm{Sym}(\omega)) \geq \mathfrak{g} \geq \mathfrak{h} \geq \mathfrak{t} \geq \mathfrak{m} \geq \omega_1.$$

It follows that in our model all these invariants are equal ω_1. Together with the fact that MA(Suslin) decides all cardinal characteristics from the Cichoń's diagram to be equal 2^ω, we have computed all of the classical cardinal invariants of the continuum, except $\mathfrak{s}$. But the proof of Theorem 18 shows that $\mathrm{Slice}_{\omega_1}$ is witnessed by some sequence $(M_\alpha)_{\alpha<\omega_1}$, for which there exists a Cohen real over each of the models. Given that a Cohen real splits all reals from the ground model, we conclude that $\mathfrak{s} = \omega_1$.

5 Slicing $2^{<\kappa}$

Although MA_{ω_1} is inconsistent with $\mathrm{Slice}_{\omega_1}$, it is consistent with Slice_κ for any $\kappa > \omega_1$. The idea of the proof is very much like that of Theorem 18, and actually even simpler, because we don't need to code the steps of the iteration as analytic sets.

Theorem 25. *Assume that $\omega < \kappa \leq \theta$ are regular cardinals, and $\theta^{<\kappa} = \theta$. Then the following theory is consistent for any cardinal $\lambda < \kappa$:*

$$ZFC + MA_\lambda + \mathrm{Slice}_\kappa + \text{“}2^\omega = \theta\text{”}.$$

We are going to apply a finite-support iteration of the form

$$\mathbb{P} = \{\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha |\ \alpha < \theta\},$$

where for each $\alpha < \theta$

$$\mathbb{P}_\alpha \Vdash \dot{\mathbb{Q}}_\alpha = (\lambda, \dot{\leq}_\alpha).$$

We also assume that $0 \in \lambda$ is always the largest element in $\dot{\mathbb{Q}}_\alpha$. We want to arrange the iteration so that each c.c.c. partial order of size λ will appear cofinally many times (see [10], p. 278), and for this reason, we will be considering only names of the form

$$\dot{\leq}_\alpha = \{(\phi(\beta), p_i^\beta)|\ i < \omega, \beta < \lambda\},$$

where $\phi : \lambda \to \lambda \times \lambda$ is a fixed bijection, definable from λ. A standard induction shows that for any $\alpha \leq \theta$ there exists at most θ-many such names, and $|\mathbb{P}_\alpha| \leq \theta$. Using an appropriate bookkeeping, we can include all c.c.c. partial orders of size λ in the iteration, and therefore we obtain:

Theorem 26. *Under the assumptions of Theorem 25*

$$\mathbb{P}_\theta \Vdash MA_\lambda + \text{“}2^\omega = \theta\text{”}.$$

Definition 27. By induction on α, we define the class of *simple* $\mathbb{P}_\alpha$-conditions.

- $\alpha = 0$. $\mathbb{P}_0 = \{0\}$, and we declare 0 to be simple.
- $\alpha + 1$. $(p, \dot{q}) \in \mathbb{P}_{\alpha+1}$ is simple if $p \in \mathbb{P}_\alpha$ is simple, $\dot{q} = \{(\gamma_n, p_n)|\ n < \omega\}$, and conditions p_n are simple.
- $\lim \alpha$. $p \in \mathbb{P}_\alpha$ is simple if for each $\beta < \alpha$, $p \restriction \beta \in \mathbb{P}_\beta$ is simple.

Like in the previous section, it is easy to check that the set of simple conditions is always dense.

Lemma 28. *For each $\alpha \leq \theta$, if $p \in \mathbb{P}_\alpha$ is simple then p is definable (in the language of set theory) with a parameter from θ^ω.*

Proof.

- $\alpha = 0$. Clear.
- $\alpha+1$. Let $r = (p, \dot{q})$ be simple. We can write $\dot{q} = \{(\gamma_n, p_n)|\ n < \omega\}$, where conditions p_n are simple. By the induction hypothesis, each p_n is definable with a parameter from θ^ω, and so is p. Clearly r can be defined from them, and so r is definable with countably many parameters, which we can code as one.
- $\lim \alpha$. Fix $r \in \mathbb{P}_\alpha$. r has finite support, so there exists $\beta < \alpha$ containing the support of r. By the induction hypothesis, $p \restriction \beta$ is definable with a parameter from θ^ω, and so p is definable with the parameters $p \restriction \beta$, β, and α.

□

An immediate consequence is that each of the names $\dot{\leq}_\alpha$ is definable with some parameter $\sigma_\alpha \in \theta^\lambda$. Like previously, we fix a list of such parameters $\{\sigma_\alpha |\ \alpha < \theta\} \subseteq \theta^\lambda$. We define by induction the relativized forcings $\mathbb{P}_\kappa^N \subseteq \mathbb{P}_\kappa$, taking into account only names from some transitive class N.

- $\mathbb{P}_0^N = \{0\}$,
- Assume $\mathbb{P}_\alpha^N$ is defined. We define a $\mathbb{P}_\alpha^N$-name $\dot{\mathbb{Q}}_\alpha^N$ as follows
  - $\dot{\mathbb{Q}}_\alpha^N = \dot{\mathbb{Q}}_\alpha$ if $\dot{\mathbb{Q}}_\alpha \in N$, and $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha^N$-name,
  - $\dot{\mathbb{Q}}_\alpha^N = \{0\}$ otherwise.
- $\mathbb{P}_{\alpha+1}^N = \mathbb{P}_\alpha^N * \dot{\mathbb{Q}}_\alpha^N$.

In limit steps we take direct limits, so $\mathbb{P}_\kappa^N \subseteq \mathbb{P}_\kappa$.

Lemma 29. *Let N be a transitive model of ZFC, containing θ. Let $M \prec H((2^\theta)^+)$ be an elementary submodel, such that $\lambda + 1 \subseteq M$, and $\{\sigma_\alpha | \ \alpha < \theta\} \in M$ (see the remark after Lemma 28). We assume moreover, that $\theta^\lambda \cap M \subseteq N$. Then for each $\alpha \leq \theta$, $\mathbb{P}_\alpha \cap M \subseteq \mathbb{P}_\alpha^N$.*

Proof. We proceed by induction.

- $\alpha = 0$. Clear.
- $\lim \alpha$. Fix $r \in \mathbb{P}_\alpha \cap M$. By the elementarity of M, there exists $\gamma \in \alpha \cap M$ that contains the support of r. From the induction hypothesis, it follows that $r \restriction \gamma \in \mathbb{P}_\gamma^N$. It is routine to verify by induction that for all $\gamma \leq \delta \leq \alpha$, $r \restriction \delta \in \mathbb{P}_\delta^N$.
- $\alpha + 1$. Fix $r = (p, \dot{q}) \in M \cap (\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha)$. Clearly $p \in \mathbb{P}_\alpha^N$ by the induction hypothesis. The name $\dot{q}$ is of the form
$$\dot{q} = \{(\gamma_n, p_n) | \ n < \omega\},$$
and for each $n < \omega$, $p_n \in \mathbb{P}_\alpha \cap M \subseteq \mathbb{P}_\alpha^N$. This shows that $\dot{q}$ is a $\mathbb{P}_\alpha^N$-name. It remains to show that $\mathbb{P}_\alpha^N \Vdash \dot{\mathbb{Q}}_\alpha^N = \dot{\mathbb{Q}}_\alpha$, and this in turn reduces to showing that $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha^N$-name belonging to N. To see this, let us note that since $(p, \dot{q}) \in M$, also $\alpha \in M$, and so $\sigma_\alpha \in \theta^\lambda \cap M \subseteq N$. It follows that $\dot{\mathbb{Q}}_\alpha \in M \cap N$. Recall, that $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a partial ordering of the form
$$\dot{\leq}_\alpha = \{(\phi(\beta), p_\beta^i) | \ \beta < \lambda, i < \omega\}.$$
Given that $\lambda + 1 \subseteq M$, we conclude that each of the conditions p_β^i belongs to M, and by the induction hypothesis, also to $\mathbb{P}_\alpha^N$. This shows that $\dot{\leq}_\alpha$, and in turn also $\dot{\mathbb{Q}}_\alpha$, are $\mathbb{P}_\alpha^N$-names, and concludes the proof.

□

Lemma 30. *If N is a transitive class, then for all $\alpha \leq \theta$*
$$\mathbb{P}_\alpha^N \lessdot \mathbb{P}_\alpha.$$

Specifically:

1. *If $p_0 \perp p_1$ in $\mathbb{P}_\alpha^N$, then $p_0 \perp p_1$ in $\mathbb{P}_\alpha$.*
2. *If $p_0 \leq p_1$ in $\mathbb{P}_\alpha^N$, then $p_0 \leq p_1$ in $\mathbb{P}_\alpha$.*
3. *If $\mathcal{A} \subseteq \mathbb{P}_\alpha^N$ is a maximal antichain, then $\mathcal{A}$ is maximal in $\mathbb{P}_\alpha$.*

Proof. We proceed by induction on α.
1.

- $\alpha = 0$. Clear.
- $\alpha + 1$. Assume $(p_0, \dot{q}_0) \perp (p_1, \dot{q}_1)$ in $\mathbb{P}_{\alpha+1}^N$. If $p_0 \perp p_1$ in $\mathbb{P}_\alpha^N$, then by the induction hypothesis $p_0 \perp p_1$ in $\mathbb{P}_\alpha$ and we are done. Suppose otherwise, and fix a condition $p \leq p_0, p_1$ in $\mathbb{P}_\alpha$. Let $G \subseteq \mathbb{P}_\alpha$ be any filter generic over V, containing p. Since $p_0, p_1 \in G \cap \mathbb{P}_\alpha^N$, we see that
$$\dot{q}_0[G \cap \mathbb{P}_\alpha^N] \perp \dot{q}_1[G \cap \mathbb{P}_\alpha^N]$$
in the model $V[G \cap \mathbb{P}_\alpha^N]$, and so in $V[G]$ as well (see Proposition 8). Since p and G were arbitrary, it follows that $(p_0, \dot{q}_0) \perp (p_1, \dot{q}_1)$ in $\mathbb{P}_{\alpha+1}$.

- $\lim \alpha$. Follows from the induction hypothesis, since the supports are finite.

2.

- $\alpha = 0$. Clear.
- $\alpha + 1$. Assume $(p_0, \dot{q}_0) \leq (p_1, \dot{q}_1)$ in $\mathbb{P}^N_{\alpha+1}$. From the induction hypothesis, we know that $p_0 \leq p_1$ in $\mathbb{P}_\alpha$, and $p_0 \Vdash \dot{q}_0 \leq \dot{q}_1$ in $\mathbb{P}^N_\alpha$. We must show that the assertion

$$p_0 \Vdash \dot{q}_0 \leq \dot{q}_1$$

holds also in $\mathbb{P}_\alpha$. If $\dot{\mathbb{Q}}^N_\alpha = \{0\}$ it is trivial. Otherwise $\dot{\mathbb{Q}}^N_\alpha = \dot{\mathbb{Q}}_\alpha$. In that case $\dot{q}_0$ and $\dot{q}_1$ are $\mathbb{P}^N_\alpha$-names, and the $\Vdash$ relation for them is the same in $\mathbb{P}^N_\alpha$ as in $\mathbb{P}_\alpha$, due to Proposition 8.

- $\lim \alpha$. Follows from the induction hypothesis, since the supports are finite.

3.

- $\alpha = 0$. Clear.
- $\alpha + 1$. The proof is exactly the same as in the paragraph after Lemma 22, so we need to prove the conclusion of Lemma 22 in the current setting. But this is trivial, once we recall that

$$\mathbb{P}^N_\alpha \Vdash \dot{\mathbb{Q}}^N_\alpha = \{0\},$$

or

$$\mathbb{P}^N_\alpha \Vdash \dot{\mathbb{Q}}^N_\alpha = \dot{\mathbb{Q}}_\alpha.$$

- $\lim \alpha$. Let $\{\overline{p}_n |\ n < \omega\}$ be a maximal antichain in $\mathbb{P}^N_\alpha$, and fix $\overline{p} \in \mathbb{P}_\alpha$. There is some $\gamma < \alpha$ such that $\overline{p} \in \mathbb{P}_\gamma$. The set $\{\overline{p}_n \restriction \gamma |\ n < \omega\}$ might not be an antichain in $\mathbb{P}^N_\gamma$, however each condition in $\mathbb{P}^N_\gamma$ is compatible with some $p_n \restriction \gamma$. We can refine $\{\overline{p}_n \restriction \gamma |\ n < \omega\}$ to an antichain in $\mathbb{P}^N_\gamma$, and this antichain will remain maximal in $\mathbb{P}_\gamma$ by the induction hypothesis. Therefore $\{\overline{p}_n \restriction \gamma |\ n < \omega\}$ meets every condition from $\mathbb{P}_\gamma$, and in particular some $\overline{p}_n \restriction \gamma$ is compatible with $\overline{p}$ in $\mathbb{P}_\gamma$. But then $\overline{p}_n$ is compatible with $\overline{p}$ in $\mathbb{P}_\alpha$.

□

Proof of Theorem 25. Let us fix a model

$$V \models ZFC + GCH + \text{Slice}_\kappa,$$

and let $\mathbb{P} = \mathbb{P}_\theta$ be the forcing defined in the beginning of the Section. Suppose that a sequence $\{M_\alpha |\ \alpha < \kappa\}$ witnesses Slice_κ in V, and $G \subseteq \mathbb{P}$ is generic over V. We aim to show that the sequence $V[G \cap \mathbb{P}^{M_\alpha}]$ witnesses Slice_κ in $V[G]$. For this we need to show two things:

1. If $F \in 2^{<\kappa} \cap V[G]$, then $F \in V[G \cap \mathbb{P}^{M_\alpha}]$ for some $\alpha < \kappa$.
2. None of the models $V[G \cap \mathbb{P}^{M_\alpha}]$ contains all reals from $V[G]$.

Concerning 1. assume that $\mathbb{P}_\theta \Vdash \dot{F} \in 2^\delta$, for some ordinal $\delta < \kappa$. Without loss of generality $\delta = |\delta| \geq \lambda$. We can also assume that

$$\dot{F} = \{(\alpha, \alpha_n, p^\alpha_n) |\ \alpha < \delta,\ n < \omega\},$$

and all conditions p^α_n are simple. We fix some elementary submodel $\overline{M} \prec H((2^\theta)^+)$ of size δ, of which we assume that $\delta + 1 \subseteq \overline{M}$, and $\{\sigma_\alpha |\ \alpha < \theta\}, \dot{F} \in \overline{M}$. Notice, that $\delta + 1 \subseteq \overline{M}$ guarantees that each of the conditions p^α_n is in $\overline{M}$. We pick $\alpha < \kappa$ big enough, so that $\overline{M} \cap \theta^\lambda \subseteq M_\alpha$. Now Lemma 29 shows that $p^\alpha_n \in \mathbb{P}^n_\alpha$, for all $\alpha < \delta, n < \omega$. This shows that $\dot{F}$ is a $\mathbb{P}^{M_\alpha}_\theta$-name, and it follows that

$$\dot{F}[G] = \dot{F}[G \cap \mathbb{P}^{M_\alpha}] \in V[G \cap \mathbb{P}^{M_\alpha}].$$

Concerning 2. fix a sequence $F \in 2^\omega \setminus M_\alpha$. Let $\mathbb{C}_F$ be any representation of the Cohen forcing, from which the sequence F is definable, and $\mathbb{C}_F$ is of the form

$$\mathbb{C}_F = (\lambda, \leq_F).$$

This of course leaves plenty of space for what specifically $\mathbb{C}_F$ might be, but for the sake of concreteness let us define $\leq_F$ as the transitive closure of the union of the following three relations:

1. $\leq_F \restriction \omega \times \omega$ is isomorphic to the countable atomless Boolean algebra,
2. $\forall\, 1 \leq \alpha < \omega \quad \alpha \cdot \omega <_F \alpha \cdot \omega + 1 \iff F(\alpha) = 1$,
3. $\forall\, 1 \leq \alpha < \omega \quad \alpha \cdot \omega >_F \alpha \cdot \omega + 1 \iff F(\alpha) = 0$.

We pick $\gamma < \theta$ for which
$$\mathbb{P}_\gamma \Vdash \dot{\mathbb{Q}}_\gamma = \mathbb{C}_F.$$
In this case, we also have
$$\mathbb{P}_\gamma^{M_\alpha} \Vdash \dot{\mathbb{Q}}_\gamma^{M_\alpha} = \{0\},$$
since $F \notin M_\alpha$.

The mapping from $\mathbb{C}_F$ into $\mathbb{P}_{\gamma+1}$ given by the formula
$$x \mapsto (1_{\mathbb{P}_\gamma}, x)$$
is a complete embedding, and it extends to a complete embedding
$$f : \mathbb{C}_F \hookrightarrow \mathbb{P}_\theta.$$

Let $\dot{H}$ be a $\mathbb{P}_\theta^{M_\alpha}$-name for a generic filter. In the generic extension $V[H]$, the function f remains a complete embedding of a Cohen forcing into $\mathbb{P}_\theta$, and by the choice of γ, f is in fact a complete embedding into $\mathbb{P}_\theta/H$. In conclusion, forcing with $\mathbb{P}_\theta$ adds a Cohen real over $V[G \cap \mathbb{P}_\theta^{M_\alpha}]$. □

Corollary 31. *The following theories are consistent*
$$ZFC + MA_{\omega_1} + \mathrm{Slice}_{\omega_2} + \text{“}2^\omega = \omega_2\text{”},$$
$$ZFC + MA_{\omega_1} + \mathrm{Slice}_{\omega_2} + \text{“}2^\omega = \omega_3\text{”},$$
$$ZFC + MA_{\omega_2} + \mathrm{Slice}_{\omega_3} + \text{“}2^\omega = \omega_{29}\text{”}.$$

6 Final comments

It is easy to see, that all sequences witnessing Slice_κ that we built are cardinal preserving. Moreover, we proved that MA_{ω_1} and $\mathrm{Slice}_{\omega_1}$ are not compatible. It looks reasonable to expect that for any regular cardinal κ
$$MA_\kappa \implies \neg\, \mathrm{Slice}_\kappa\,.$$

I thank prof. Piotr Zakrzewski for giving the idea of looking at the cardinal invariant $\mathrm{c}(\mathrm{Sym}(\omega))$.

References

[1] U. Avraham, S. Shelah, *Martin's Axiom does not imply that every two $\aleph_1$-dense sets of reals are isomorphic*, Israel Journal of Mathematics, vol. 38, Nos. 1-2 (1981)

[2] T. Bartoszyński, H. Judah, *Set Theory: On The Structure of The Real Line*, A.K. Peters Ltd. 1995

[3] J.E. Baumgartner, *All $\aleph_1$-dense sets of reals can be isomorphic*, Fund. Math. 79 (1973), 101-106

[4] A. Blass, *Combinatorial cardinal characteristics of the continuum*, Handbook of set theory. Springer, Dordrecht, 2010, 395-489

[5] J. Brendle, M. Losada, *The cofinality of the infinite symmetric group and groupwise density*, The Journal of Symbolic Logic, vol. 68, No.4 (2003): 1354-1361

[6] D. Calderón, I. Farah. *Can you take Akemann–Weaver's $\diamondsuit$ away?*, arXiv preprint arXiv:2006.06886 (2020)

[7] W. G. Fleissner, A.W. Miller, *On Q-sets*. Proc. Am. Math. Soc. vol. 78, n. 2 (1980)

[8] S. D. Friedman, L. Zdomskyy, *Measurable cardinals and the cofinality of the symmetric group*, Fundamenta Mathematicae 207.2 (2010): 101-122

[9] J. Ihoda, S. Shelah, *Souslin Forcing*, The Journal of Symbolic Logic, vol. 53, No. 4 (1988)

[10] K. Kunen, *Set Theory. Introduction to Independence Proofs*, 1980 Elsevier Science B.V.

[11] J. T. Moore, *What makes the continuum* $\aleph_2$, Foundations of mathematics, essays in honor of W. Hugh Woodin's 60th birthday, Harvard University (Caicedo et al., eds.) (2017), 259-287

[12] H. D. Macpherson, P. M. Neumann, *Subgroups of infinite symmetric groups.*, Journal of the London Mathematical Society 2.1 (1990): 64-84

[13] J. D. Sharp, S. Thomas, *Unbounded families and the cofinality of the infinite symmetric group*, Archive for Mathematical Logic 34.1 (1995): 33-45

[14] S. Shelah, S. Thomas, *The cofinality spectrum of the infinite symmetric group*, The Journal of Symbolic Logic, vol. 62, No.3 (1997): 902-916

[15] S. Thomas, *Groupwise density and the cofinality of the infinite symmetric group*, Archive for Mathematical Logic 37.7 (1998): 483-493.

[16] Y. Zhang, *Permutation groups and covering properties*, Journal of the London Mathematical Society 63.1 (2001): 1-15