# On the non-existence of mad families

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**Abstract**

We show that the non-existence of mad families is equiconsistent with $ZFC$, answering an old question of Mathias. We also consider the above result in the general context of maximal independent sets in Borel graphs, and we construct a Borel graph $G$ such that $ZF + DC +$ "there is no maximal independent set in $G$" is equiconsistent with $ZFC +$ "there exists an inaccessible cardinal".[1]

## 1. Introduction

We study the possibility of the non-existence of mad families in models of $ZF+DC$. Recall that $\mathcal{F} \subseteq [\omega]^\omega$ is mad if $A, B \in \mathcal{F} \to |A \cap B| < \aleph_0$, and $\mathcal{F}$ is maximal with respect to this property. Assuming the axiom of choice, it's easy to construct mad families, thus leading to natural investigations concerned with the definability of mad families. By a classical result of Mathias [Ma], mad families can't be analytic (as opposed to the classical regularity properties, there might be $\Pi^1_1$ mad families, which is the case when $V = L$ [Mi]). The possibility of the non-existence of mad families was demonstrated by Mathias who proved the following result:

**Theorem [Ma]:** Suppose there is a Mahlo cardinal, then there is a model of $ZF + DC +$ "There are no mad families".

The following natural question was asked by Mathias in [Ma]:

**Question [Ma]:** What's the consistency strength of $ZF + DC +$ "there are no mad families"?

For almost four decades, no progress was made on that problem. In 2015, Asger Toernquist was able to reduce the upper bound on the consistency strength with the following result:

**Theorem [To]:** There are no mad families in Solovay's model.

We shall settle the above problem in this paper by proving the following result:

**Theorem 1:** $ZF+DC+$"There are no mad families" is equiconsistent with $ZFC$.

In [HwSh:1089] we prove the existence of a Borel maximal eventually different family[2] in $ZF$, which is quite surprising when contrasted with the above mentioned results. One possible approach to explaining the surprising difference between mad and maximal eventually different families is via Borel combinatorics. It's easy to see that there are Borel graphs $G_{MAD}$ and $H_{MED}$ such that there exists a mad (maximal eventually different) family iff there exists a maximal independent set in $G_{MAD}$ ($H_{MED}$). Therefore, we may view the above problems and results as private cases of the more general problem of understanding maximal independent sets in Borel graphs.

We shall prove that large cardinals are necessarily involved in the understanding of the general problem, suggesting a much more complicated picture:

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[2]A family $\mathcal{F} \subseteq \omega^\omega$ is eventually different if for every $f, g \in \mathcal{F}$, if $f \neq g$ then $f(n) \neq g(n)$ for large enough $n$. $\mathcal{F}$ is a maximal eventually different (MED) family if $\mathcal{F}$ is an eventually different family and is not strictly contained in another eventually different family.

**Theorem 2:** There exists a Borel graph $G$ such that $ZF + DC +$ "there is no maximal independent set in $G$" is equiconsistent with $ZFC +$ "there exists an inaccessible cardinal".

Theorem 1 will be the main result of Section 2, Theorem 2 will be the main result of Section 3.

## 2. $ZF + DC +$ "there are no mad families" is equiconsistent with $ZFC$

### Background and previous results

Results on the consistency strength of regularity properties go back to the works of Solovay and Shelah ([So] and [Sh:176]), who proved that $ZF + DC +$ "all sets of reals are Lebesgue measurable" is equiconsistent with $ZFC +$ "there exists an inaccessible cardinal" while no large cardinals are needed for the Baire property.

As for the consistency strength of the non-existence of mad families, the problem, which first appeared in [Ma] in 1977, is discussed in [Br] and [Kan]. We shall prove in this section that the non-existence of mad families is equiconsistent with $ZFC$.

### A discussion of the proof

In order to prove the theorem, we will construct a sufficiently saturated forcing notion. We consider the class $K$ of ccc forcing notions that force $MA_{\aleph_1}$. Using the fact that ccc is equivalent to Knaster's condition under $MA_{\aleph_1}$ (and that Knaster's condition is preserved under products), we are able to prove that $K$ has the amalgamation property. By a bookkeeping argument, using the amalgmation property of $K$, we construct a forcing notion $\mathbb{P} \in K$ such that if $\mathbb{P}_1, \mathbb{P}_2 \in K$ are of a "small" cardinality, $\mathbb{P}_1 \lessdot \mathbb{P}_2$ and $f_1 : \mathbb{P}_1 \to \mathbb{P}$ is a complete embedding, then there exists a complete embedding $f_2 : \mathbb{P}_2 \to \mathbb{P}$ such that $f_1 \subseteq f_2$.

We then force with $\mathbb{P}$, and consider the inner model $HOD(\mathbb{R}^{<\mu})$ inside the generic extension (where $\mu$ is a cardinal that we fix in the beginning of the proof). In order to prove that there are no mad families in our model, we assume towards contradiction that $\underset{\sim}{\mathcal{F}}$ is a canonical $\mathbb{P}-$name of a mad family, and we find a forcing $\mathbb{Q}_* \lessdot \mathbb{P}$ of cardinality $< \mu$ such that $\underset{\sim}{\mathcal{F}}$ is defined using a canonical $\mathbb{Q}_*-$name. The next step is to find a pair $(\mathbb{Q}, \underset{\sim}{D})$ such that:

1. $\mathbb{Q}_* \lessdot \mathbb{Q} \lessdot \mathbb{P}$.
2. $|\mathbb{Q}| < \mu$.
3. $\underset{\sim}{F} \restriction \mathbb{Q} := \{\underset{\sim}{a} : \underset{\sim}{a}$ is a canonical $\mathbb{Q}$-name of a subset of $\omega$ such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{a} \in \underset{\sim}{\mathcal{F}}$"$\}$ is a canonical $\mathbb{Q}-$name of a mad family in $V^{\mathbb{Q}}$.
4. $\underset{\sim}{D}$ is a $\mathbb{Q}-$name of a Ramsey ultrafilter on $\omega$.
5. $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{D} \cap (\underset{\sim}{\mathcal{F}}) = \emptyset$".

Next we consider a forcing notion $\mathbb{Q}_1 \in K$ such that $|\mathbb{Q}_1| < \mu$ such that $\mathbb{Q} \star \mathbb{M}_{\underset{\sim}{D}} \lessdot \mathbb{Q}_1$, where $\mathbb{M}_{\underset{\sim}{D}}$ is the Mathias forcing restricted to the Ramsey ultrafilter $\underset{\sim}{D}$. Letting $\underset{\sim}{A_1}$ be the name for the generic of $\mathbb{M}_{\underset{\sim}{D}}$, there is a name $\underset{\sim}{a}_1$ of a member of the mad family restricted to $\mathbb{Q}_1$ (denoted by $\underset{\sim}{\mathcal{F}_1}$) witnessing the madness of the family with respect to $\underset{\sim}{a_1}$. Finally, a contradiction will be derived by considering an isomorphic copy $(\mathbb{Q}_2, \underset{\sim}{\mathbb{M}_D}, \underset{\sim}{a_2}, \underset{\sim}{A_2}, \underset{\sim}{\mathcal{F}_2})$ of $(\mathbb{Q}_1, \underset{\sim}{\mathbb{M}_D}, \underset{\sim}{a_1}, \underset{\sim}{A_1}, \underset{\sim}{\mathcal{F}_1})$ and amalgamating over $\mathbb{Q}$.

### Proof of Theorem 1

**Hypothesis 1:** 1. $\lambda = \lambda^{<\mu}$, $\mu = cf(\mu)$, $\alpha < \mu \to |\alpha|^{\aleph_1} < \mu$, $\aleph_0 < \theta = \theta^{\aleph_1} < \kappa = cf(\kappa) \leq \mu$ and $\alpha < \kappa \to |\alpha|^{\aleph_1} < \kappa$.

For example, assuming $GCH$, the hypothesis holds for $\mu = \aleph_3 = \kappa$, $\lambda = \aleph_4$ and $\theta = \aleph_2$.

2. For transparency, we may assume $CH$.

**Definition 2:** 1. Let $K = \{\mathbb{P} : \mathbb{P}$ is a $ccc$ forcing notion such that $\Vdash_{\mathbb{P}}$ "$MA_{\aleph_1}$"$\}$.

2. Let $\leq_K$ be the partial order $\lessdot$ on $K$.

3. We say that $(\mathbb{P}_\alpha : \alpha < \alpha_*)$ is $\leq_K$-increasing continuous if $\mathbb{P}_\alpha \in K$ for every $\alpha < \alpha_*$, $\alpha < \beta \to \mathbb{P}_\alpha \lessdot \mathbb{P}_\beta$ and if $\beta < \alpha_*$ is a limit ordinal then $\underset{\gamma<\beta}{\cup} \mathbb{P}_\gamma \lessdot \mathbb{P}_\beta$.

**Claim 3:** 1. $(K, \leq_K)$ has the amalgamation property.

2. If $\mathbb{P}_1$ is a ccc forcing notion, then there is $\mathbb{P}_2 \in K$ such that $\mathbb{P}_1 \lessdot \mathbb{P}_2$ and $|\mathbb{P}_2| \leq |\mathbb{P}_1|^{\aleph_1} + 2^{\aleph_1}$.

3. If $(\mathbb{P}_\alpha : \alpha < \delta)$ is $\leq_K$-increasing continuous and $\delta$ is a limit ordinal, then $\underset{\alpha<\delta}{\cup} \mathbb{P}_\alpha \models ccc$, hence by (2) there is $\mathbb{P}_\delta \in K$ such that $(\mathbb{P}_\alpha : \alpha < \delta)^\frown(\mathbb{P}_\delta)$ is $\leq_K$-increasing continuous.

4. If $\mathbb{P} \in K$ and $X \subseteq \mathbb{P}$ such that $|X| < \mu$, then there exists $\mathbb{Q} \in K$ such that $X \subseteq \mathbb{Q}$, $\mathbb{Q} \leq_K \mathbb{P}$ and $|\mathbb{Q}| \leq 2^{\aleph_1} + |X|^{\aleph_1}$.

**Proof:** 1. Suppose that $\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2 \in K$ and $f_l : \mathbb{P}_0 \to \mathbb{P}_l$ $(l = 1, 2)$ are complete embeddings. Let $\mathbb{P}_1 \times_{f_1,f_2} \mathbb{P}_2$ be the amalgamation of $\mathbb{P}_1$ and $\mathbb{P}_2$ over $\mathbb{P}_0$ (as in [RoSh672]), i.e. $\{(p_1, p_2) \in \mathbb{P}_1 \times \mathbb{P}_2 : (\exists p \in \mathbb{P}_0)(p \Vdash_{\mathbb{P}}$ "$p_1 \in \mathbb{P}_1/f_1(\mathbb{P}_0) \wedge p_2 \in \mathbb{P}_2/f_2(\mathbb{P}_0)$"$)\}$. As $\mathbb{P}_0 \lessdot \mathbb{P}_1 \times_{f_1,f_2} \mathbb{P}_2$, $\Vdash_{\mathbb{P}_0}$ "$MA_{\aleph_1}$" and $MA_{\aleph_1}$ implies that every ccc forcing notion is Knaster (and recalling that being Knaster is preserved under products), it follows that $\mathbb{P}_1 \times_{f_1,f_2} \mathbb{P}_2 \models ccc$, and by (2) we're done.

2. $\mathbb{P}_2$ is obtained as thee composition of $\mathbb{P}_1$ with the ccc forcing notion of cardinality $|\mathbb{P}_1|^{\aleph_1} + 2^{\aleph_1}$ forcing $MA_{\aleph_1}$.

4. As in the proof of subclaim 1 in claim 6 (see next page). $\square$

**Claim 4:** There is a ccc forcing notion $\mathbb{P}$ of cardinality $\lambda$ such that:

1. For every $X \subseteq \mathbb{P}$, $|X| < \mu \to (\exists \mathbb{Q} \in K)(X \subseteq \mathbb{Q} \lessdot \mathbb{P} \wedge |\mathbb{Q}| < \mu)$.

2. If $\mathbb{P}_1, \mathbb{P}_2 \in K$ have cardinality $< \mu$, $\mathbb{P}_1 \lessdot \mathbb{P}_2$ and $f_1$ is a complete embedding of $RO(\mathbb{P}_1)$ into $RO(\mathbb{P})$, then there is $f_1 \subseteq f_2$ that is a complete embedding of $RO(\mathbb{P}_2)$ into $RO(\mathbb{P})$.

**Proof:** We choose $\mathbb{P}_\alpha \in K$ by induction on $\alpha < \lambda$, such that the sequence is $\leq_K -$increasing continuous and each $\mathbb{P}_\alpha$ has cardinality $\lambda$, as follows:

1. For limit $\alpha$ we choose $\mathbb{P}_\alpha \in K$ such that $\underset{\beta<\alpha}{\cup} \mathbb{P}_\beta \lessdot \mathbb{P}_\alpha$ . We can do this by claim 3(2) and the induction hypothesis.

2. For $\alpha = \beta + 1$, we let $((\mathbb{P}_1^\gamma, \mathbb{P}_2^\gamma, f_1^\gamma) : \gamma < \lambda)$ be an enumeration of all triples as in 4(2) for $\mathbb{P}_\beta$. We construct a $\leq_K -$increasing continuous sequence $(\mathbb{P}_\gamma^* : \gamma \leq \lambda)$ by induction as follows: $\mathbb{P}_0^* = \mathbb{P}_\beta$. $\mathbb{P}_{\gamma+1}^*$ is the result of a $K-$amalgamation for the $\gamma$th triple, and for limit $\gamma$ we define $\mathbb{P}_\gamma^*$ as in (1). Finally, we let $\mathbb{P}_\alpha = \mathbb{P}_\lambda^*$.

Note that by claim 3(4), requirement (1) is satisfied for every forcing notion from $K$, hence it's enough to guarantee that requirement (2) is satisfied. It's now easy to see that $\mathbb{P} = \underset{\alpha<\lambda}{\cup} \mathbb{P}_\alpha$ is as required. $\square$

**Definition/Claim 5:** Let $\mathbb{P}$ be the forcing notion from claim 4 and let $G \subseteq \mathbb{P}$ be generic over $V$. In $V[G]$, let $V_1 = HOD(\mathbb{R}^{<\kappa})$, then $V_1 \models ZF + DC_{<\kappa}$. $\square$

**Main claim 6:** There are no mad families in $V_1$.

**Proof:** Let $\underset{\sim}{\mathcal{F}}$ be a canonical $\mathbb{P}$-name of a mad family (i.e. a canonical $\mathbb{P}$-name of a family of subsets of $\omega$), and let $\underset{\sim}{\bar{\eta}}$ be a sequence of length $< \kappa$ of canonical $\mathbb{P}$-names of reals such that $\underset{\sim}{\mathcal{F}}$ is definable over $V$ using $\underset{\sim}{\bar{\eta}}$. Let $K_{\mathbb{P}} = \{\mathbb{Q} \in K : \mathbb{Q} \lessdot \mathbb{P} \wedge |\mathbb{Q}| < \kappa\}$. By claim 4(1), there is $\mathbb{Q}_* \in K_{\mathbb{P}}$ such that $\underset{\sim}{\bar{\eta}}$ is a canonical $\mathbb{Q}_*$-name. Let $K^+_{\mathbb{P}}$ be the set of $\mathbb{Q} \in K_{\mathbb{P}}$ such that $\mathbb{Q}_* \lessdot \mathbb{Q}$ and $\underset{\sim}{\mathcal{F}} \restriction \mathbb{Q}$ is a canonical $\mathbb{Q}$-name of a mad family in $V^{\mathbb{Q}}$, where $\underset{\sim}{\mathcal{F}} \restriction \mathbb{Q} = \{\underset{\sim}{a} : \underset{\sim}{a}$ is a canonical $\mathbb{Q}$-name of a subset of $\omega$ such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{a} \in \underset{\sim}{\mathcal{F}}$"$\}$.

**Subclaim 1**: $K^+_{\mathbb{P}}$ is $\lessdot$-dense in $K_{\mathbb{P}}$.

**Proof**: Let $\mathbb{Q} \in K_{\mathbb{P}}$ and let $\sigma = |\mathbb{Q}_* + 2|^{\aleph_1} < \kappa$. We choose $Z_i$ by induction on $i < \omega_2$ such that:

a. $Z_i \subseteq \mathbb{P}$ and $|Z_i| \leq \sigma$.

b. $j < i \rightarrow Z_j \subseteq Z_i$.

c. $Z_0 = \mathbb{Q}_* \cup \mathbb{Q}$.

d. If $i = 3j+1$, then for every canonical name using members of $Z_{3j}$ of an "$MA_{\aleph_1}$ problem" in $Z_i$ we have a name for a solution.

e. If $i = 3j+2$, then $Z_i \lessdot \mathbb{P}$.

f. If $i = 3j+3$, then for every canonical $Z_{3j+2}$-name $\underset{\sim}{a}$ of an infinite subset of $\omega$, there is a canonical $Z_i$-name $\underset{\sim}{b}$ such that $\Vdash_{\mathbb{P}}$ "$|\underset{\sim}{a} \cap \underset{\sim}{b}| = \aleph_0 \wedge \underset{\sim}{b} \in \underset{\sim}{\mathcal{F}}$".

It's now easy to verify that $Z_{\omega_2}$ is as required: By (c) and (e), $\mathbb{Q}_* \lessdot Z_{\omega_2} \lessdot \mathbb{P}$, hence also $Z_{\omega_2} \models ccc$. By (a), $|Z_{\omega_2}| < \kappa$. By (d), $\Vdash_{Z_{\omega_2}}$ "$MA_{\aleph_1}$" (given names for $\aleph_1$ dense sets, we have canonical names depending on $\aleph_1$ conditions, hence there is some $j < \omega_2$ such that they are $Z_{3j+2}$-names), hence $Z_{\omega_2} \in K$. By (f), $\underset{\sim}{\mathcal{F}} \restriction Z_{\omega_2}$ is a canonical $Z_{\omega_2}$-name of a mad family in $V^{Z_{\omega_2}}$.

We shall now prove that such $Z_i$ can be constructed for $i \leq \omega_2$: For $i = 0$ it's given by (c) and for limit ordinals we simply take the union. For $i = 3j+1$ and $i = 3j+3$ we enumerate the canonical names for either the $MA_{\aleph_1}$ problem or the infinite subsets of $\omega$ (depending on the stage of the induction), there are $\leq \sigma$ such names. At stage $3j+1$ we use the fact that $\mathbb{P}$ forces $MA_{\aleph_1}$ in order to extend $Z_{3j}$ using $\mathbb{P}$-names for the solutions of the $MA_{\aleph_1}$-problems. At stage $3j+3$, we extend the forcing similarly, using the fact that $\underset{\sim}{\mathcal{F}}$ is a name of a mad family. For $i = 3j+2$, we let $Z_{3j+2}$ be the closure of $Z_{3j+1}$ under the functions $f_1 : \mathbb{P} \times \mathbb{P} \rightarrow \mathbb{P}$ and $f_2 : [\mathbb{P}]^{[\leq \aleph_0]} \rightarrow \mathbb{P}$ where: $f_1(p,q)$ is a common upper bound of $p$ and $q$ if they're compatible, and $f_2(X)$ is incompatible with all members of $X$ provided that $X$ is countable and not predense.

**Subclaim 2**: If $\mathbb{Q} \in K^+_{\mathbb{P}}$ and $F : \mathbb{Q} \rightarrow \mathbb{P}$ is a complete embedding over $\mathbb{Q}_*$, then $F$ maps $\underset{\sim}{\mathcal{F}} \restriction \mathbb{Q}$ to $\underset{\sim}{\mathcal{F}} \restriction F(\mathbb{Q})$.

**Proof**: As $F$ is the identity over $\mathbb{Q}_*$ and $\underset{\sim}{\mathcal{F}}$ is definable using a $\mathbb{Q}_*$-name.

We now arrive at the two main subclaims:

**Subclaim 3**: There is a pair $(\mathbb{Q}, \underset{\sim}{D})$ such that:

a. $\mathbb{Q}_* \lessdot \mathbb{Q} \in K^+_{\mathbb{P}}$.

b. $\underset{\sim}{D}$ is a name of a Ramsey ultrafilter on $\omega$.

c. $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{D} \cap (\underset{\sim}{F} \restriction \mathbb{Q}) = \emptyset$".

**Subclaim 4**: Subclaim 3 implies claim 6.

**Proof of subclaim 4**: Let $\underset{\sim}{\mathbb{M}_D}$ be the $\mathbb{Q}$-name for the Mathias forcing restricted to the ultrafilter $\underset{\sim}{D}$. Let $\mathbb{Q}_1 \in K$ such that $\mathbb{Q} \star \underset{\sim}{\mathbb{M}_D} \lessdot \mathbb{Q}_1$ and $|\mathbb{Q}_1| < \kappa$ (such forcing notion exists by 3(2)), and let $\underset{\sim}{A_1}$ be the $\mathbb{Q}_1$-name for the $\underset{\sim}{\mathbb{M}_D}$-generic real.

Let $F_1 : \mathbb{Q}_1 \to \mathbb{P}$ be a complete embedding such that $F_1$ is the identity on $\mathbb{Q}$ (such an embedding exists by claim 4(2). There is $\mathbb{Q}_1' \in K_{\mathbb{P}}^{+}$ such that $F_1(\mathbb{Q}_1) \lessdot \mathbb{Q}_1'$ by subclaim 1. There is a pair $(\mathbb{Q}_1'', F_1')$ such that $\mathbb{Q}_1 \lessdot \mathbb{Q}_1''$ and $F_1' : \mathbb{Q}_1'' \to \mathbb{Q}_1'$ is an isomorphism extending $F_1'$. WLOG $(\mathbb{Q}_1'', F_1') = (\mathbb{Q}_1, F_1)$, so $F_1(\mathbb{Q}_1) \in K_{\mathbb{P}}^{+}$.

Let $\underset{\sim}{\mathcal{F}_1} = F_1^{-1}(\underset{\sim}{\mathcal{F}} \restriction F_1(\mathbb{Q}_1))$.

As $\Vdash_{F_1(\mathbb{Q}_1)}$ "$\underset{\sim}{\mathcal{F}} \restriction F_1(\mathbb{Q}_1)$ is mad", it follows that $\Vdash_{\mathbb{Q}_1}$ "$\underset{\sim}{\mathcal{F}_1}$ is mad", hence there is some $\underset{\sim}{a_1}$ such that $\underset{\sim}{a_1}$ is a canonical $\mathbb{Q}_1$-name for a subset of $\omega$, $\Vdash_{\mathbb{Q}_1}$ "$\underset{\sim}{a_1} \in \underset{\sim}{\mathcal{F}_1}$" and $\Vdash_{\mathbb{Q}_1}$ "$\underset{\sim}{a_1} \cap \underset{\sim}{A_1}$ is infinite". Recalling the basic property of the forcing $\underset{\sim}{\mathbb{M}_D}$, every infinite subset of $\underset{\sim}{A_1}$ is generic, therefore, by considering $\underset{\sim}{A_1} \cap \underset{\sim}{a_1}$ instead of $\underset{\sim}{A_1}$, WLOG $\Vdash_{\mathbb{Q}_1}$ "$\underset{\sim}{A_1} \subseteq \underset{\sim}{a_1}$".

Now let $(\mathbb{Q}_2, \underset{\sim}{\mathbb{M}_D}, \underset{\sim}{a_2}, \underset{\sim}{A_2}, \underset{\sim}{\mathcal{F}_2})$ be an isomorphic copy of $(\mathbb{Q}_1, \underset{\sim}{\mathbb{M}_D}, \underset{\sim}{a_1}, \underset{\sim}{A_1}, \underset{\sim}{\mathcal{F}_1})$ such that the isomorphism is over $\mathbb{Q}$. Consider the amalgamation $\mathbb{Q}_3 = \mathbb{Q}_1 \times_{\mathbb{Q}} \mathbb{Q}_2$. By the basic properties of $\mathbb{P}$, there is a complete embedding $F_3 : \mathbb{Q}_3 \to \mathbb{P}$ over $\mathbb{Q}$. By the density of $K_{\mathbb{P}}^{+}$, there is $\mathbb{Q}_4' \in K_{\mathbb{P}}^{+}$ such that $F_3(\mathbb{Q}_3) \lessdot \mathbb{Q}_4'$. As before, choose $(\mathbb{Q}_4, F_4)$ such that $\mathbb{Q}_3 \lessdot \mathbb{Q}_4$ and $F_4 : \mathbb{Q}_4 \to \mathbb{Q}_4'$ is an isomorphism extending $F_3$.

Now observe that $\Vdash_{\mathbb{Q}_4}$ "$\underset{\sim}{A_1} \cap \underset{\sim}{A_2}$ is infinite": Let $G \subseteq \mathbb{Q}$ be generic, then in $V[G]$ we have: $\mathbb{Q}_3/G = (\mathbb{Q}_1/G) \times (\mathbb{Q}_2/G) \lessdot \mathbb{Q}_4/G$. As $\underset{\sim}{\mathbb{M}_D}[G] \lessdot \mathbb{Q}_l/\mathbb{Q}$ $(l = 1, 2)$, we have $\underset{\sim}{\mathbb{M}_D}[G] \times \underset{\sim}{\mathbb{M}_D}[G] \lessdot \mathbb{Q}_3/G$, so it's enough to show that $\Vdash_{\underset{\sim}{\mathbb{M}_D}[G] \times \underset{\sim}{\mathbb{M}_D}[G]}$ "$|\underset{\sim}{A_1} \cap \underset{\sim}{A_2}| = \aleph_0$": Let $((w_1, B_1), (w_2, B_2)) \in \underset{\sim}{\mathbb{M}_D}[G] \times \underset{\sim}{\mathbb{M}_D}[G]$ and $n < \omega$, so $B_1 \cap B_2 \in \underset{\sim}{D}[G]$ is infinite, therefore, there is $n_1 > n, sup(w_1 \cup w_2)$ such that $n_1 \in B_1 \cap B_2$. Let $q = ((w_1 \cup \{n_1\}, B_1 \setminus (n_1 + 1)), (w_2 \cup \{n_1\}, B_2 \setminus (n_1 + 1)))$, then $p \leq q$ and $q \Vdash$ "$n_1 \in \underset{\sim}{A_1} \cap \underset{\sim}{A_2}$".

Therefore, $\Vdash_{\mathbb{Q}_4}$ "$\underset{\sim}{a_1} \cap \underset{\sim}{a_2}$ is infinite" (as the intersection contains $\underset{\sim}{A_1} \cap \underset{\sim}{A_2}$).

It now follows that $\Vdash_{\mathbb{Q}_4}$ "$\underset{\sim}{a_1} = \underset{\sim}{a_2}$": First note that $\Vdash_{F_4(\mathbb{Q}_4)}$ "$F_4(\underset{\sim}{a_1}), F_4(\underset{\sim}{a_2}) \in \underset{\sim}{\mathcal{F}} \restriction F_4(\mathbb{Q}_4)$". Now $F_4(\mathbb{Q}_4) = \mathbb{Q}_4' \in K_{\mathbb{P}}^{+}$, so $\underset{\sim}{\mathcal{F}} \restriction F_4(\mathbb{Q}_4)$ is a canonical $F_4(\mathbb{Q}_4)$-name of a mad family, therefore $\Vdash_{F_4(\mathbb{Q}_4)}$ "$F_4(\underset{\sim}{a_1}) = F_4(\underset{\sim}{a_2})$", hence $\Vdash_{\mathbb{Q}_4}$ "$\underset{\sim 1}{a} = \underset{\sim}{a_2}$".

It's now enough to show that $\Vdash_{\mathbb{Q}_4}$ "$\underset{\sim}{a_1} = \underset{\sim}{a_2} \in V^{\mathbb{Q}}$": Work in $V[G]$. First note that $\Vdash_{\mathbb{Q}_l/G}$ "$\underset{\sim}{A_l}$ is almost contained in every member of $\underset{\sim}{D}[G]$, hence (by subclaim 3) it's almost disjoint to every member of $\underset{\sim}{F} \restriction \mathbb{Q}$", and also $\Vdash_{\mathbb{Q}_l/G}$ "$\underset{\sim}{a_l} \in V^{\mathbb{Q}}$, hence $\underset{\sim}{a_l} \in \underset{\sim}{F} \restriction \mathbb{Q}$". Now recall that $\Vdash_{\mathbb{Q}_l}$ "$\underset{\sim}{A_l} \subseteq \underset{\sim}{a_l}$", together we get a contradiction. Therefore, it remains to show that $\Vdash_{\mathbb{Q}_4}$ "$\underset{\sim}{a_1} = \underset{\sim}{a_2} \in V^{\mathbb{Q}}$": By the claim above, $\Vdash_{\mathbb{Q}_3}$

"$\underset{\sim}{a}_1 = \underset{\sim}{a}_2$". Work in $V[G]$, so $\underset{\sim}{a}_l$ is a $\mathbb{Q}_l/G$-name ($l = 1, 2$). Suppose that the claim doesn"t hold, then there are $q_1, r_1 \in \mathbb{Q}_1/G$ and $n < \omega$ such that $q_1 \Vdash$ "$n \in \underset{\sim}{a}_1$" and $r_1 \Vdash$ "$n \notin \underset{\sim}{a}_1$". Let $q_2, r_2 \in \mathbb{Q}_2/G$ be the "conjugates" of $(q_1, r_1)$ (i.e. their images under the isomorphism that was previously mentioned), then $(q_1, r_2) \in \mathbb{Q}_3/G$ forces that $n \in \underset{\sim}{a}_1$ and $n \notin \underset{\sim}{a}_2$, contradicting the fact that $\Vdash_{\mathbb{Q}_3}$ "$\underset{\sim}{a}_1 = \underset{\sim}{a}_2$". This completes the proof of subclaim 4.

**Proof of subclaim 3**: Let $\sigma = |\mathbb{Q}_*|^{\aleph_1} < \kappa$. We choose $(\mathbb{Q}_\epsilon, \underset{\sim}{A}_\epsilon)$ by induction on $\epsilon < \sigma^+$ such that:

a. $\mathbb{Q}_\epsilon \in K^+_{\mathbb{P}}$ and $|\mathbb{Q}_\epsilon| \leq \sigma$.

b. $\underset{\sim}{A}_\epsilon$ is a canonical $\mathbb{Q}_\epsilon$-name of a subset of $\omega$.

c. $\Vdash_{\mathbb{Q}_\epsilon}$ "$\underset{\sim}{A}_\epsilon$ is not almost included in a finite union of elements of $\underset{\sim}{\mathcal{F}} \restriction \mathbb{Q}_\epsilon$.

d. $(\mathbb{Q}_0, \underset{\sim}{A}_0) = (\mathbb{Q}_*, \omega)$. WLOG $\mathbb{Q}_* \in K^+_{\mathbb{P}}$, as $K^+_{\mathbb{P}}$ is $\lessdot$-dense in $K_{\mathbb{P}}$.

e. $(\mathbb{Q}_\zeta : \zeta < \epsilon)$ is $\lessdot$-increasing.

f. $\Vdash_{\mathbb{Q}_\epsilon}$ "$(\underset{\sim}{A}_\zeta : \zeta < \epsilon)$ is $\subseteq^*$-decreasing".

g. If $\epsilon = 2\xi + 1$ and $\Lambda_\epsilon \neq \emptyset$ where $\Lambda_\epsilon = \{(\zeta, \underset{\sim}{a}) : \zeta \leq \xi, \underset{\sim}{a}$ is a canonical $\mathbb{Q}_\zeta$-name of a subset of $\omega$ such that $\nVdash_{\mathbb{Q}_{2\xi}}$ "$\underset{\sim}{A}_{2\xi} \subseteq^* \underset{\sim}{a}$ or $\underset{\sim}{A}_{2\xi} \subseteq^* \omega \setminus \underset{\sim}{a}$"$\}$, then letting $\Gamma_\epsilon = \{\zeta : (\zeta, \underset{\sim}{a}) \in \Lambda_\epsilon\}$ and $\zeta_\epsilon = min(\Gamma)$, for some $\underset{\sim}{a}_\epsilon$, $(\zeta_\epsilon, \underset{\sim}{a}_\epsilon) \in \Lambda_\epsilon$ and $\Vdash_{\mathbb{Q}_\epsilon}$ "$\underset{\sim}{A}_\epsilon \subseteq^* \underset{\sim}{a}_\epsilon$ or $\underset{\sim}{A}_\epsilon \subseteq^* (\omega \setminus \underset{\sim}{a}_\epsilon)$".

h. If $\epsilon = 2\xi + 2$ and $\mathcal{F}_\epsilon \neq \emptyset$ where $\mathcal{F}_\epsilon = \{(\zeta, \underset{\sim}{f}) : \zeta \leq \xi$ and $\underset{\sim}{f}$ is a canonical $\mathbb{Q}_\zeta$-name of a function from $[\omega]^2$ to $\{0, 1\}$ such that $\Vdash_{\mathbb{Q}_{2\xi+1}}$ "$\neg(\exists n)\underset{\sim}{f} \restriction [\underset{\sim}{A}_\xi \setminus n]^2$ is constant", $\bigwedge\limits_{n<\omega} \bigvee\limits_{l<2} \underset{\sim}{A}_{\epsilon-1} \subseteq^* \{i : \underset{\sim}{f}(i, n) = l\}$ and $\bigvee\limits_{l<2} \underset{\sim}{A}_{\epsilon-1} \subseteq^* \{n : (\forall^\infty i \in \underset{\sim}{A}_{\epsilon-1})\underset{\sim}{f}(i, n) = l\}$ $\}$, then letting $\Gamma_\epsilon = \{\zeta : (\zeta, \underset{\sim}{f}) \in \mathcal{F}_\epsilon\}$ and $\zeta_\epsilon = min(\Gamma_\epsilon)$, for some $\underset{\sim}{f}_\epsilon$, $(\zeta_\epsilon, \underset{\sim}{f}_\epsilon) \in \mathcal{F}_\epsilon$ and $\Vdash_{\mathbb{Q}_\epsilon}$ "$\underset{\sim}{f}_\epsilon \restriction [\underset{\sim}{A}_\epsilon]^2$ is constant".

**Subclaim 3a**: The above induction can be carried for every $\epsilon < \sigma^+$.

**Subclaim 3b**: Subclaim 3 is implied by subclaim 3a.

**Proof of Subclaim 3b**: First we consider the case where $\sigma^+ < \kappa$. Let $\mathbb{Q} = \bigcup\limits_{\epsilon<\sigma^+} \mathbb{Q}_\epsilon$, note that as $\aleph_2 \leq cf(\sigma^+)$, $\mathbb{Q} \in K^+_{\mathbb{P}}$. By the choice of $\mathbb{Q}_0$, $\mathbb{Q}_* \lessdot \mathbb{Q}$. Now define a $\mathbb{Q}$-name $\underset{\sim}{D} := \{\underset{\sim}{B} : \underset{\sim}{B}$ is a canonical $\mathbb{Q}$-name of a subset of $\omega$ such that $\Vdash_{\mathbb{Q}}$ "$(\exists \epsilon < \sigma^+)(\underset{\sim}{A}_\epsilon \subseteq^* \underset{\sim}{B})$"$\}$. By (g), $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{D}$ is an ultrafilter": For example, in order to see that $\underset{\sim}{D}$ is forced to be upwards closed, suppose that $p_1 \Vdash$ "$\underset{\sim}{B} \subseteq^* \underset{\sim}{A} \subseteq \omega$ and $\underset{\sim}{B} \in \underset{\sim}{D}$", then there are $p_1 \leq p_2$, $n < \omega$ and $\epsilon < \sigma^+$ such that $p_2 \Vdash$ "$\underset{\sim}{B} \setminus n \subseteq \underset{\sim}{A}$ and $\underset{\sim}{A}_\epsilon \setminus n \subseteq \underset{\sim}{B}$". There is a condition $p_3$ and a canonical name $\underset{\sim}{A}_3$ such that $p_2 \leq p_3$ and $p_3 \Vdash$ "$\underset{\sim}{A} = \underset{\sim}{A}_3$". Let $\{p_{3,i} : i < \omega\}$ be a maximal antichain in $\mathbb{Q}$ such that $p_3 = p_{3,0}$ and let $\underset{\sim}{A}_4$ be the $\mathbb{Q}$-name defined as:

1. $\underset{\sim}{A}_4[G_{\mathbb{Q}}] = \underset{\sim}{A}_3[G_{\mathbb{Q}}]$ if $p_{3,0} \in G_{\mathbb{Q}}$

2. $\underset{\sim}{A}_4[G_{\mathbb{Q}}] = \underset{\sim}{B}[G_{\mathbb{Q}}]$ if $p_{3,0} \notin G_{\mathbb{Q}}$.

Therefore, $\underset{\sim}{A}_4$ is a canonical name for a subset of $\omega$, $\Vdash$ "$\underset{\sim}{A}_4 \in \underset{\sim}{D}$" and $p_3 \Vdash$ "$\underset{\sim}{A}_4 = \underset{\sim}{A}$".

In order to see that for every $\mathbb{Q}$-name $\underset{\sim}{a} \subseteq \omega$, it's forced that $\underset{\sim}{a} \in \underset{\sim}{D} \vee \omega \setminus \underset{\sim}{a} \in \underset{\sim}{D}$, we have to show that every such name is being handled by clause (g) at some stage of the induction. Suppose that for some name $\underset{\sim}{a}$ it's not the case. Each such name is a $\mathbb{Q}_\zeta$-name for some $\zeta < \sigma^+$, so pick a minimal $\zeta$ for which there is such a $\mathbb{Q}_\zeta$-name. Therefore, for every $\epsilon = 2\xi + 1$ such that $\zeta \leq \xi$, $\zeta_\epsilon \leq \zeta$, so at each such stage we're handling a $\mathbb{Q}_\zeta$-name. As $|\mathbb{Q}_\zeta|^{\aleph_0} \leq \sigma$, the number of $\mathbb{Q}_\zeta$-names is at most $\sigma$ and the number of induction steps is larger, we get a contradiction. Similarly, it follows by (h) that $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{D}$ is a Ramsey ultrafilter": Let $\underset{\sim}{f}$ be a $\mathbb{Q}-$name of a function from $[\omega]^2$ to $\{0,1\}$ (wlog $\underset{\sim}{f}$ is a canonical name). As $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{D}$ is an ultrafilter", for every $n < \omega$, $\{\{i : \underset{\sim}{f}(i,n) = l\} : l < 2\}$ is a $\mathbb{Q}-$name of a partition of $\omega$ in $V^{\mathbb{Q}}$, hence for some $\underset{\sim}{l}_{\underset{\sim}{f},n}$, $V^{\mathbb{Q}} \models$ "$\{i : \underset{\sim}{f}(i,n) = \underset{\sim}{l}_{\underset{\sim}{f},n}\} \in \underset{\sim}{D}$", and therefore, for some $\underset{\sim}{\xi} = \underset{\sim}{\xi}_{\underset{\sim}{f},n}$, $V^{\mathbb{Q}} \models$ "$\underset{\sim}{A}_{\underset{\sim}{\xi}} \subseteq^* \{i : \underset{\sim}{f}(i,n) = \underset{\sim}{l}_{\underset{\sim}{f},n}\}$". Now $\{\{n : \underset{\sim}{l}_{\underset{\sim}{f},n} = k\} : k < 2\}$ is a canonical $\mathbb{Q}-$name of a partition of $\omega$, so again, there is $\underset{\sim}{k}_f$ such that $\{n : \underset{\sim}{l}_{\underset{\sim}{f},n} = \underset{\sim}{k}_f\} \in \underset{\sim}{D}$, and there is $\underset{\sim}{\xi}_1$ such that $\underset{\sim}{A}_{\xi_1} \subseteq^* \{n : \underset{\sim}{l}_{\underset{\sim}{f},n} = \underset{\sim}{k}_f\}$. As $\mathbb{Q} \models ccc$, there is $\xi < \sigma^+$ such that all of the above names are $\mathbb{Q}_\xi-$names and $\Vdash_{\mathbb{Q}_\xi}$ "$\underset{\sim}{\xi}_2, \underset{\sim}{\xi}_{\underset{\sim}{f},n} \leq \xi$ for every $n < \omega$". As the sequence of the $\underset{\sim}{A}_\zeta$ is $\subseteq^*$-decreasing, $\underset{\sim}{f}$ has the form of the functions appearing in requirement (h) of the induction, hence by (h) there is a large homogeneous set for $\underset{\sim}{f}$.

By (c), it follows that $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{D} \cap \underset{\sim}{\mathcal{F}} \restriction \mathbb{Q} = \emptyset$"

We now consider the case where $\sigma^+ = \kappa$. In this case we add a slight modification to our inductive construction: The induction is now on $\epsilon < \sigma$. We fix a partition $(S_\xi : \xi < \sigma)$ of $\sigma$ such that $|S_\xi| = \sigma$ and $S_\xi \cap \xi = \emptyset$ for each $\xi < \sigma$. At stage $\xi$ of the induction we fix enumertions $(\underset{\sim}{a}^\xi_i : i \in S_\xi)$ and $(\underset{\sim}{f}^\xi_i : i \in S_\xi)$ of the canonical $\mathbb{Q}_\xi$-names for the subsets of $\omega$ and the 2-colorings of $[\omega]^2$ such that for some $\zeta < \xi$, $\underset{\sim}{A}_\zeta$ satisfies the condition from (h) with respect to $\underset{\sim}{f}^\xi_i$.

We now replace the original (g) and (h) by (g)' and (h)' as follows:

(g)' If $\epsilon = 2i + 1$ and $i \in S_\xi$ then $\Vdash_{\mathbb{Q}_\epsilon}$ "$\underset{\sim}{A}_{2\xi} \subseteq^* \underset{\sim}{a}^\xi_i \vee \underset{\sim}{A}_{2\xi} \subseteq^* \omega \setminus \underset{\sim}{a}^\xi_i$".

(h)' If $\epsilon = 2i + 2$ and $i \in S_\xi$ then $\Vdash_{\mathbb{Q}_\epsilon}$ "$\underset{\sim}{f}^\xi_i[\underset{\sim}{A}_\epsilon]$ is constant".

Note that $\xi \leq i$ in the clauses above, as $S_\xi \cap \xi = \emptyset$, therefore, at stage $\epsilon = 2i + l$ $(l = 1, 2)$, the names $\underset{\sim}{a}^\xi_i$ and $\underset{\sim}{f}^\xi_i$ are well-defined when $i \in S_\xi$.

As $\aleph_2 \leq cf(\sigma)$, then as before, letting $\mathbb{Q} = \bigcup_{\epsilon<\sigma} \mathbb{Q}_\epsilon$, $\mathbb{Q}_* \lessdot \mathbb{Q} \in K^+_{\mathbb{P}}$. As before, $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{D}$ is a filter", and by clause (c), $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{D} \cap \underset{\sim}{\mathcal{F}} \restriction \mathbb{Q} = \emptyset$", and by (g)', $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{D}$ is an ultrafilter". By (h)', $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{D}$ is a Ramsey ultrafilter" (the argument is the same as in the case of $\sigma^+ < \kappa$), so we're done.

**Proof of subclaim 3a**:

We give the argument for the case $\sigma^+ < \kappa$. The case $\sigma^+ = \kappa$ is essentially the same.

**Case I** ($\epsilon = 0$): Trivial.

**Case II** ($\epsilon = 2\xi + 1$): We let $\mathbb{Q}_\epsilon = \mathbb{Q}_{2\xi}$. Pick some $(\zeta_\epsilon, \underset{\sim}{a}_\epsilon) \in \Lambda_\epsilon$, the $\mathbb{Q}_\epsilon$-name $\underset{\sim}{A}_\epsilon$ will be defined as follows: If $\underset{\sim}{A}_{2\xi} \cap \underset{\sim}{a}_\epsilon$ satisfies clause (c) of the induction, then we let $\underset{\sim}{A}_\epsilon = \underset{\sim}{A}_{2\xi} \cap \underset{\sim}{a}_\epsilon$. Otherwise, let $\underset{\sim}{A}_\epsilon = \underset{\sim}{A}_{2\xi} \setminus \underset{\sim}{a}_\epsilon$. We need to show that $\underset{\sim}{A}_\epsilon$ satifies clause (c). Suppose not, then both $\underset{\sim}{A}_{2\xi} \cap \underset{\sim}{a}_\epsilon$ and $\underset{\sim}{A}_{2\xi} \setminus \underset{\sim}{a}_\epsilon$ don't satisfy clause (c), but then $\underset{\sim}{A}_{2\xi}$ is almost included in a finite union of elements of $\underset{\sim}{\mathcal{F}} \restriction \mathbb{Q}_{2\xi}$, a contradiction.

**Case III** ($\epsilon = 2\xi + 2$): Pick some $(\zeta_\epsilon, \underset{\sim}{f}_\epsilon) \in \mathcal{F}_\epsilon$. By the definition of $\mathcal{F}_\epsilon$, in $V^{\mathbb{Q}_{\epsilon-1}}$, for every $n < \omega$ there are $l_n^\epsilon < 2$ and $k_n^\epsilon < \omega$ such that for every $k \in \underset{\sim}{A}_{\epsilon-1}$, if $k_n^\epsilon \leq k$ then $f(k, n) = l_n^\epsilon$. In addition, there are $k_\epsilon, l_\epsilon$ such that $k_\epsilon \leq n \in \underset{\sim}{A}_{\epsilon-1} \to l_n^\epsilon = l_\epsilon$.

WLOG $k_n^\epsilon < k_{n+1}^\epsilon$ for every $n < \omega$. By the induction hypothesis, as $\Vdash_{\mathbb{Q}_{\epsilon-1}}$ "$\underset{\sim}{\mathcal{F}} \restriction \mathbb{Q}_{\epsilon-1}$ is mad" and as $\underset{\sim}{A}_\epsilon$ satisfies clause (c), there are pairwise distinct $\underset{\sim}{a}_{\epsilon,n} \in \underset{\sim}{\mathcal{F}} \restriction \mathbb{Q}_{\epsilon-1}$ such that $\underset{\sim}{b}_{\epsilon,n} = \underset{\sim}{a}_{\epsilon,n} \cap \underset{\sim}{A}_{\epsilon-1}$ is infinite for every $n < \omega$. We now choose $n_i$ by induction on $i$ such that:

a. $n_i \in \underset{\sim}{A}_{\epsilon-1} \setminus k_\epsilon$.

b. If $i = j + 1$ then $n_i > n_j$ and $n_i > k_{n_j}^\epsilon$.

c. If $i \in (j^2, (j+1)^2)$ then $n_i \in b_{\epsilon, i-j^2}$.

This should suffice: By (a)+(b), $\underset{\sim}{f} \restriction \{n_i : i < \omega\}$ is constantly $l_\epsilon$. By (c), $\{n_i : i < \omega\}$ is not almost included in a finite union of elements of $\underset{\sim}{\mathcal{F}} \restriction \mathbb{Q}_{\epsilon-1}$: This follows from the fact that for each $n < \omega$, $\{n_i : i < \omega\}$ contains infinitely many members of $\underset{\sim}{b}_{\epsilon,n}$, hence of $\underset{\sim}{a}_{\epsilon,n}$. As $\{n_i : i < \omega\}$ has infinite intersection with an infinite number of members of $\underset{\sim}{\mathcal{F}} \restriction \mathbb{Q}_{\epsilon-1}$, it can't be covered by a finite number of members of $\underset{\sim}{\mathcal{F}} \restriction \mathbb{Q}_{\epsilon-1}$.

Therefore, $\mathbb{Q}_\epsilon := \mathbb{Q}_{\epsilon-1}$ and $\underset{\sim}{A}_\epsilon := \{n_i : i < \omega\}$ are as required.

Why is it possible to carry the induction? As each $\underset{\sim}{b}_{\epsilon,n}$ is infinite, and requirements (a)+(b) only exclude a finite number of elements, this is obviously possible.

**Case IV** ($\epsilon$ is a limit ordinal): We choose $(\mathbb{Q}_{\epsilon,n}, \underset{\sim}{a}_{\epsilon,n}, \underset{\sim}{b}_{\epsilon,n})$ by induction on $n < \omega$ such that:

a. $\bigcup\limits_{\xi < \epsilon} \mathbb{Q}_\xi \subseteq \mathbb{Q}_{\epsilon,n} \in K_{\mathbb{P}}^+$.

b. If $n = m + 1$ then $\mathbb{Q}_{\epsilon,m} \lessdot \mathbb{Q}_{\epsilon,n}$.

If $n > 0$ then we also require:

c. $\underset{\sim}{a}_{\epsilon,n}$ is a $\mathbb{Q}_{\epsilon,n}$-name of a member of $\underset{\sim}{\mathcal{F}} \restriction \mathbb{Q}_{\epsilon,n}$.

d. $\underset{\sim}{b}_{\epsilon,n}$ is a $\mathbb{Q}_{\epsilon,n}$-name of an infinite subset of $\omega$.

e. $\Vdash_{\mathbb{Q}_{\epsilon,n}}$ "$\underset{\sim}{b}_{\epsilon,n} \subseteq \underset{\sim}{a}_{\epsilon,n} \wedge \bigwedge\limits_{\zeta<\epsilon} \underset{\sim}{b}_{\epsilon,n} \subseteq^* \underset{\sim}{A}_\zeta$".

f. $\Vdash_{\mathbb{Q}_{\epsilon,n}}$ "$\underset{\sim}{a}_{\epsilon,l} \neq \underset{\sim}{a}_{\epsilon,n}$ for $l < n$".

Why can we carry the induction? By the properties of $\mathbb{P}$, there is $\mathbb{Q}_{\epsilon,0} \in K^+_{\mathbb{P}}$ such that $\bigcup\limits_{\zeta<\epsilon} \mathbb{Q}_\zeta \subseteq \mathbb{Q}_{\epsilon,0}$. Let $\underset{\sim}{D}_{\epsilon,0}$ be a $\mathbb{Q}_{\epsilon,0}$-name of an ultrafilter containing $\{\underset{\sim}{A}_\epsilon : \epsilon < \zeta\}$, let $\mathbb{M}_{\underset{\sim}{D}_{\epsilon,0}}$ be the $\mathbb{Q}_{\epsilon,0}$-name for the corresponding Mathias forcing and let $\underset{\sim}{w}$ be the name for the generic set of natural numbers added by it. By the properties of $\mathbb{P}$, there is $\mathbb{Q}_{\epsilon,1} \in K^+_{\mathbb{P}}$ such that $\mathbb{Q}_{\epsilon,0} \lessdot \mathbb{Q}_{\epsilon,1}$ and $\mathbb{Q}_{\epsilon,1}$ adds a pseudo-intersection $\underset{\sim}{w}$ to $\underset{\sim}{D}_{\epsilon,0}$.

There is a $\mathbb{Q}_{\epsilon,1}$-name $\underset{\sim}{a}_{\epsilon,1}$ such that $\Vdash_{\mathbb{Q}_{\epsilon,1}}$ "$\underset{\sim}{a}_{\epsilon,1} \in \underset{\sim}{\mathcal{F}} \restriction \mathbb{Q}_{\epsilon,1} \wedge |\underset{\sim}{a}_{\epsilon,1} \cap \underset{\sim}{w}| = \aleph_0$". Let $\underset{\sim}{b}_{\epsilon,1} = \underset{\sim}{w} \cap \underset{\sim}{a}_{\epsilon,1}$, then clearly $(\mathbb{Q}_{\epsilon,1}, \underset{\sim}{a}_{\epsilon,1}, \underset{\sim}{b}_{\epsilon,1})$ are as required. Suppose now that $(\mathbb{Q}_{\epsilon,l}, \underset{\sim}{a}_{\epsilon,l}, \underset{\sim}{b}_{\epsilon,l})$ were chosen for $l \leq k$. Note that $\Vdash_{\mathbb{Q}_{\epsilon,k}}$ "$\{\omega \setminus \bigcup\limits_{l\leq k} \underset{\sim}{a}_{\epsilon,l}\} \cup \{\underset{\sim}{A}_\zeta : \zeta < \epsilon\}$ have the FIP". Suppose not, then there is $\zeta < \epsilon$ such that $\Vdash$ "$\underset{\sim}{A}_\zeta \subseteq^* \bigcup\limits_{l\leq k} a_{\epsilon,l}$", as $\Vdash_{\mathbb{P}}$ "$\bigwedge\limits_{l\leq k} \underset{\sim}{a}_{\epsilon,l} \in \underset{\sim}{\mathcal{F}}$", this is a contradiction: It's enough to show that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{A}_\zeta$ is not almost contained in a finite union of members of $\underset{\sim}{F}$". Suppose that $p \Vdash_{\mathbb{P}}$ "$\underset{\sim}{A}_\zeta \subseteq^* \bigcup\limits_{l\leq k} \underset{\sim}{b}_l$" where $\underset{\sim}{b}_l$ are elements of $\underset{\sim}{\mathcal{F}}$. Let $G \subseteq \mathbb{P}$ be a generic set containing $p$, then $V[G] \models$ "$\underset{\sim}{A}_\zeta[G] \subseteq \bigcup\limits_{l\leq k} \underset{\sim}{b}_l[G]$". $G \cap \mathbb{Q}_\zeta$ is generic, $\{b \in \underset{\sim}{\mathcal{F}} \restriction \mathbb{Q}_\zeta[G \cap \mathbb{Q}_\zeta] : |b \cap \underset{\sim}{A}_\zeta[G \cap \mathbb{Q}_\zeta]| = \aleph_0\}$ is infinite. Therefore, in $V[G]$ there are $b_i \in \underset{\sim}{\mathcal{F}}[G]$ $(i < \omega)$ such that $|\underset{\sim}{A}_\zeta[G] \cap b_i| = \aleph_0$ for each $i < \omega$, so $\underset{\sim}{A}_\zeta[G]$ can't be almost covered by a finite number of members of $\underset{\sim}{\mathcal{F}}[G]$, which is a contradiction.

Let $\underset{\sim}{D}_{\epsilon,k}$ be a $\mathbb{Q}_{\epsilon,k}$-name for a nonprincipal ultrafilter containing $\{\omega \setminus \bigcup\limits_{l\leq k} a_{\epsilon,l}\} \cup \{\underset{\sim}{A}_\zeta : \zeta < \epsilon\}$, as before, let $\mathbb{Q}_{\epsilon,k+1} \in K^+_{\mathbb{P}}$ such that $\mathbb{Q}_{\epsilon,k} \lessdot \mathbb{Q}_{\epsilon,k+1}$ and $\mathbb{Q}_{\epsilon,k+1}$ adds a pseudo-intersection $\underset{\sim}{w}_{k+1}$ to $\underset{\sim}{D}_{\epsilon,k}$. Again, $\Vdash_{\mathbb{Q}_{\epsilon,k+1}}$ "There is $\underset{\sim}{a}_{\epsilon,k+1} \in \underset{\sim}{\mathcal{F}} \restriction \mathbb{Q}_{\epsilon,k+1}$ such that $|\underset{\sim}{w}_{k+1} \cap \underset{\sim}{a}_{\epsilon,k+1}| = \aleph_0$", now let $\underset{\sim}{b}_{\epsilon,k+1} = \underset{\sim}{a}_{\epsilon,k+1} \cap \underset{\sim}{w}_{k+1}$. It's easy to see that $(\mathbb{Q}_{\epsilon,k+1}, \underset{\sim}{a}_{\epsilon,k+1}, \underset{\sim}{b}_{\epsilon,k+1})$ are as required.

We shall now prove that there is a forcing notion $\mathbb{Q}_\epsilon \in K^+_{\mathbb{P}}$ and a $\mathbb{Q}_\epsilon$-name $\underset{\sim}{A}_\epsilon$ such that $\bigcup\limits_{n<\omega} \mathbb{Q}_{\epsilon,n} \subseteq \mathbb{Q}_\epsilon$ and $\Vdash_{\mathbb{Q}_\epsilon}$ "$\bigwedge\limits_{\zeta<\epsilon} \underset{\sim}{A}_\epsilon \subseteq^* \underset{\sim}{A}_\zeta \wedge (\bigwedge\limits_{n<\omega} |\underset{\sim}{A}_\epsilon \cap \underset{\sim}{b}_{\epsilon,n}| = \aleph_0)$":

Let $\mathbb{Q}' = \bigcup\limits_{n<\omega} \mathbb{Q}_{\epsilon,n}$, we shall prove that there is a $\mathbb{Q}'$-name for a ccc forcing $\underset{\sim}{\mathbb{Q}}''$ that forces the existence of $\underset{\sim}{A}_\epsilon$ as above, such that $|\mathbb{Q}' * \underset{\sim}{\mathbb{Q}}''| < \kappa$:

Let $\underset{\sim}{\mathbb{Q}}''$ be the $\mathbb{Q}'$–name for the Mathias forcing $\mathbb{M}_{D'}$, restricted to the filter $\underset{\sim}{D}'$ generated by $\{\underset{\sim}{A}_\zeta : \zeta < \epsilon\} \cup \{[n, \omega) : n < \omega\}$, so there is a name $\underset{\sim}{A}^\epsilon$ such that $\Vdash_{\mathbb{Q}' * \underset{\sim}{\mathbb{Q}}''}$ "$\underset{\sim}{A}^\epsilon \in [\omega]^\omega$, $\bigwedge\limits_{\zeta<\epsilon} \underset{\sim}{A}^\epsilon \subseteq^* \underset{\sim}{A}_\zeta$ and $\bigwedge\limits_{n<\omega} |\underset{\sim}{A}^\epsilon \cap \underset{\sim}{b}_{\epsilon,n}| = \aleph_0$". Letting $\underset{\sim}{A}^\epsilon$ be the generic set added by $\mathbb{M}_{\underset{\sim}{D}'}$, in order to show that the last condition holds, we need to show that (in $V^{\mathbb{Q}'}$) if $p \in \mathbb{M}_{\underset{\sim}{D}'}$ and $k < \omega$, then there exists a stronger condition $q$ forcing that $k' \in \underset{\sim}{A}^\epsilon \cap \underset{\sim}{b}_{\epsilon,n}$ for some $k' > k$. Let $p = (w, S)$, as $S \in \underset{\sim}{D}'$, there is $\zeta < \epsilon$ and $l_* < \omega$

such that $\underset{\sim}{A}_\zeta \setminus l_* \subseteq S$. As $\underset{\sim}{b}_{\epsilon,n} \subseteq^* \underset{\sim}{A}_\zeta$, there is $sup(w)+k<k' \in \underset{\sim}{b}_{\epsilon,n} \cap \underset{\sim}{A}_\zeta \setminus l_* \cap S$, so we can obviously extend $p$ to a condition $q$ forcing that $k' \in \underset{\sim}{A}^\epsilon \cap \underset{\sim}{b}_{\epsilon,n}$.

By claim 3, there is $\mathbb{Q}^3 \in K$ such that $\mathbb{Q}' * \underset{\sim}{\mathbb{Q}}'' \lessdot \mathbb{Q}^3$ and $|\mathbb{Q}^3| \leq \sigma$. By the properties of $\mathbb{P}$, there is a complete embedding $f^3 : \mathbb{Q}^3 \to \mathbb{P}$ such that $f^3$ is the identity over $\mathbb{Q}_{\epsilon,0}$ (hence over $\mathbb{Q}_*$). Therefore, $\Vdash_{\mathbb{P}}$ " $\bigwedge_{n<\omega} f^3(\underset{\sim}{a}_{\epsilon,n}) \in \underset{\sim}{\mathcal{F}}$". By the (proof of the) density of $K^+_{\mathbb{P}}$, there is $\mathbb{Q}^4 \in K^+_{\mathbb{P}}$ such that $f^3(\mathbb{Q}^3) \lessdot \mathbb{Q}^4$ and $|\mathbb{Q}^4| \leq \sigma$. Let $\mathbb{Q}_\epsilon = \mathbb{Q}^4$, $\underset{\sim}{A}_\epsilon = f^3(\underset{\sim}{A}^\epsilon)$, we shall prove that $(\mathbb{Q}_\epsilon, \underset{\sim}{A}_\epsilon)$ are as required. Obviously, $\Vdash_{\mathbb{Q}_\epsilon}$ "$\underset{\sim}{A}_\epsilon \in [\omega]^\omega$", and as $f^3$ is the identity over each $\mathbb{Q}_\zeta$ $(\zeta<\epsilon)$, $\Vdash_{\mathbb{Q}_\epsilon}$ " $\bigwedge_{\zeta<\epsilon} \underset{\sim}{A}_\epsilon \subseteq^* \underset{\sim}{A}_\zeta$". The other requirements for $\mathbb{Q}_\epsilon$ and $\underset{\sim}{A}_\epsilon$ are trivial. It remains to show that $\Vdash_{\mathbb{Q}_\epsilon}$ "$\underset{\sim}{A}_\epsilon$ is not almost covered by a finite union of elements of $\underset{\sim}{\mathcal{F}} \restriction \mathbb{Q}_\epsilon$". As $\Vdash_{\mathbb{Q}_\epsilon}$ " $\bigwedge_{n<\omega} f^3(\underset{\sim}{a}_{\epsilon,n}) \in \underset{\sim}{\mathcal{F}} \restriction \mathbb{Q}_\epsilon$ and $\bigwedge_{n \neq m} f^3(\underset{\sim}{a}_{\epsilon,n}) \neq f^3(\underset{\sim}{a}_{\epsilon,m})$, it's enough to show that $\Vdash_{\mathbb{Q}_\epsilon}$ " $\bigwedge_{n<\omega} |\underset{\sim}{A}_\epsilon \cap f^3(\underset{\sim}{a}_{\epsilon,n})| = \aleph_0$", which follows from the fact that $\Vdash_{\mathbb{Q}' * \mathbb{Q}''}$ " $\bigwedge_{n<\omega} |\underset{\sim}{A}^\epsilon \cap \underset{\sim}{b}_{\epsilon,n}| = \aleph_0$" and the fact that $\Vdash_{\mathbb{Q}_{\epsilon,n}}$ "$\underset{\sim}{b}_{\epsilon,n} \subseteq \underset{\sim}{a}_{\epsilon,n}$". This completes the proof of the induction.

Remark: By the proof of the density of $K^+_{\mathbb{P}}$ in $K_{\mathbb{P}}$, whenever we have $\mathbb{Q} \in K_{\mathbb{P}}$ of cardinality $\leq \sigma$, we can construct $\mathbb{Q}' \in K^+_{\mathbb{P}}$ such that $\mathbb{Q} \lessdot \mathbb{Q}'$ and $|\mathbb{Q}'| \leq \sigma$. Therefore, at each of the steps in the limit case, it's possible to guarantee that the cardinality of the forcing is $\leq \sigma$. $\square$

## 3. Borel graphs and large cardinals

**Background**

The study of Borel and analytic graphs was initiated by Kechris, Solecki and Todorcevic in [KST], and has been a source of fruitful research ever since (see [KM] for a survey of recent results). Following the result from the previous section and the discussion in the introduction of the paper, one would hope to explain the discrepancy between mad families and maximal eventually different families in a more systematic manner. As mentioned in the introduction, mad and maximal eventually different families are simply maximal independent sets in the appropriate Borel graphs. Therefore, one might hope to obtain a dichotomy result, e.g. a result saying that for some Borel graphs it's consistent relative to $ZFC$ that no maximal independent sets exist, while for the others, the existence of such sets is provable in $ZF+DC$.

We shall prove in this section that there is a Borel graph $G$ for which $ZF+DC+$ "there is no maximal independent set in $G$" is equiconsistent with $ZFC+$ "there exists an inaccessible cardinal", so large cardinals are necessarily involved in the study of maximal independent sets in Borel graphs (and a dichotomy result as above is impossible).

**A discussion of the proof**

Our goal is to construct a Borel graph such that the statement "$\omega_1$ is inaccessible by reals" will be translated to the non-existence of a maximal independent set in the graph. The graph will consist of reals such that each real codes a linear order $I$ of $\omega$ and a sequence $(r_i : i \in I)$ of distinct reals. The edge relation will be defined such that two reals are **not** connected by an edge iff one is embeddable into the other as an initial segment in a natural way. We will show that if $\omega_1 = \omega_1^{L[a]}$ for some real $a$, then by picking in $L[a]$ vertices that code the countable ordinals, we get a maximal independent set in $V$. The other half of the equiconsistency result will be obtained by showing that our graph has no maximal independent set in Solovay's model.

**Proof of Theorem 2**

**Definition 1:** We shall define a Borel graph $G = (V, E)$ as follows:

a. $V$ is the set of reals $r$ that code the following objects:

1. A linear order $I_r$ of the element of $\omega$ or some $n < \omega$.

2. A sequence $(s_{r,\alpha} : \alpha \in I_r)$ of pairwise distinct reals.

b. Given $r_1 \neq r_2 \in V$ and $b \in I_{r_2}$, let $X_{r_1,r_2,b}$ be the set of pairs $(a_1, a_2) \in I_{r_1} \times I_{r_2,<b}$ such that $s_{r_1,a_1} = s_{r_2,a_2}$.

c. Given $r_1 \neq r_2 \in V$, $\neg(r_1 E r_2)$ holds iff one of the following holds:

1. There exists $b \in I_{r_2}$ such that $X_{r_1,r_2,b}$ is an isomorphism from $I_{r_1}$ to $I_{r_2,<b}$.

2. There exists $b \in I_{r_1}$ such that $X_{r_2,r_1,b}$ is an isomorphism from $I_{r_2}$ to $I_{r_1,<b}$.

**Definition 2:** Given $r_1 \neq r_2 \in V$, we say that $r_2$ extends $r_1$, and denote this by $r_1 <_G r_2$, when $\neg(r_1 E r_2)$ and clause (1) holds in definition 1(c).

**Claim 3** $(ZF + DC)$**:** Let $X \subseteq V$ be an independent set.

a. $X$ is linearly ordered by $<_G$.

b. If $X$ is countable then $X$ is not a maximal independent set.

**Proof:** a. Obvious.

b. By clause (a), there is a linear order $I$ such that $X = \{r_i : i \in I\}$ and $i <_I j$ iff $r_i <_G r_j$. For every $i < j \in I$, let $F_{i,j}$ be the isomorphism from $I_{r_i}$ to a proper initial segment of $I_{r_j}$ witnessing $r_i <_G r_j$. Let $I_r$ be the direct limit of the system $(I_{r_i}, F_{j,k} : i, j, k \in I, j < k)$. For $a \in I_r$, let $s_{r,a}$ be $s_{r_i,a'}$ where $a' \in I_{r_i}$ is some representative of $a$. Let $r \in V$ be a real coding $I_r$ and $(s_{r,a} : a \in I_r)$, then $\neg(r E r_i)$ for every $r_i \in X$. $\square$

**Claim 4:** $ZF + DC +$"There is no maximal independent set in $G$" is equiconsistent with $ZFC +$ "There exists an inaccessible cardinal".

Claim 4 will follow from the following claims:

**Claim 5** $(ZF + DC)$**:** If there exists $a \in \omega^\omega$ such that $\aleph_1 = \aleph_1^{L[a]}$, then there exists a maximal independent set in $G$.

**Claim 6:** There is no maximal independent set in $G$ in Levy's model (aka Solovay's model).

Remark: While the set of vertices of $G$ is denoted by $V$, the set-theoretic universe will be denoted by $\mathbf{V}$.

**Proof of claim 5:** Let $(s_\alpha : \alpha < \omega_1^{L[a]}) \in L[a]$ be a sequence of pairwise distinct reals. For each $\alpha < \omega_1^{L[a]}$, let $r_\alpha \in (\omega^\omega)^{L[a]}$ be the $<_{L[a]}$ −first real that codes $(\alpha, (s_\beta : \beta < \alpha))$. The sequences $(s_\alpha : \alpha < \omega_1^{L[a]})$ and $(r_\alpha : \alpha < \omega_1^{L[a]})$ belong to $\mathbf{V}$, and as $\omega_1 = \omega_1^{L[a]}$, their length is $\omega_1$.

It's easy to see that $\{r_\alpha : \alpha < \omega_1^{L[a]}\}$ is a well-defined set and is an independent subset of $V$, we shall prove that it's a maximal independent set. Let $r \in V \setminus \{r_\alpha : \alpha < \omega_1^{L[a]}\}$ and suppose towards contradiction that $\neg(r E r_\alpha)$ for every $\alpha < \omega_1^{L[a]}$. There are two possible cases:

**Case I:** $r_\alpha <_G r$ for every $\alpha < \omega_1^{L[a]}$. In this case, $I_r$ is a linear order, and each $\alpha < \omega_1^{L[a]}$ embeds into $I_r$ as an initial segment. Therefore, $\omega_1 = \omega_1^{L[a]}$ embeds into $I_r$ as an initial segment, a contradiction.

**Case II:** $r <_G r_\alpha$ for some $\alpha < \omega_1^{L[a]}$. Let $\alpha$ be the minimal ordinal with this property, then $\alpha$ necessarily has the form $\beta+1$. If $r = r_\beta$, then we get a contradiction to the choice of $r$. If $r \neq r_\beta$, then it's easy to see that $rEr_\beta$, contradicting our assumption. $\square$

**Proof of claim 6:** Let $\kappa$ be an inaccessible cardinal and let $\mathbb{P} = Coll(\aleph_0, <\kappa)$, we shall prove that $\Vdash_{\mathbb{P}}$ "There is no maximal independent set in $G$ from $HOD(\mathbb{R})$". Suppose towards contradiction that $p \in \mathbb{P}$ forces that $\underset{\sim}{X}$ is such a set. Let $\mathbb{Q}$ be a forcing notion such that $\mathbb{Q} \lessdot \mathbb{P}$, $|\mathbb{Q}| < \kappa$, $p \in \mathbb{Q}$ and $\underset{\sim}{X}$ is definable using a parameter from $\mathbb{R}^{\mathbf{V}^{\mathbb{Q}}}$. By the properties of the Levy collapse, we may assume wlog that $\mathbb{Q} = \{0\}$ and $p = 0$. If $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{X} \subseteq (\omega^\omega)^{\mathbf{V}}$", then $\Vdash_{\mathbb{P}}$ "$|\underset{\sim}{X}| = \aleph_0$", and by claim 3, $\underset{\sim}{X}$ is not a maximal independent set in $\mathbf{V}^{\mathbb{P}}$, a contradiction. Therefore, there exist $p_1 \in \mathbb{P}$ and $\underset{\sim}{r_1}$ such that $p_1 \Vdash_{\mathbb{P}}$ "$\underset{\sim}{r_1} \in \underset{\sim}{X} \wedge \underset{\sim}{r_1} \notin \mathbf{V}$". Let $\mathbb{Q}_1 \lessdot \mathbb{P}$ be a forcing of cardinality $<\kappa$ such that $p_1 \in \mathbb{Q}_1$ and $\underset{\sim}{r_1}$ is a $\mathbb{Q}_1$-name. For $l = 2, 3$ let $(\mathbb{Q}_l, p_l, \underset{\sim}{r_l})$ be isomorphic copies of $(\mathbb{Q}_1, p_1, \underset{\sim}{r_1})$ such that $\prod\limits_{n=1,2,3} \mathbb{Q}_n \lessdot \mathbb{P}$ (identifying $\mathbb{Q}_1$ with its canonical image in the product). Choose $(p_1, p_2) \leq (q_1, q_2)$ such that $(q_1, q_2) \Vdash_{\mathbb{Q}_1 \times \mathbb{Q}_2}$ "$\underset{\sim}{r_1} \neq \underset{\sim}{r_2}$". As $(q_1, q_2) \Vdash_{\mathbb{Q}_1 \times \mathbb{Q}_2}$ "$\underset{\sim}{r_1}, \underset{\sim}{r_2} \in \underset{\sim}{X}$", then wlog $(q_1, q_2)$ forces that $\underset{\sim}{r_1} <_G \underset{\sim}{r_2}$ as witnessed by an isomorphism from $I_{\underset{\sim}{r_1}}$ to $I_{\underset{\sim}{r_2}, <s}$ for some $s \in I_{\underset{\sim}{r_2}}$. Let $q_3 \in \mathbb{Q}_3$ be the conjugate of $q_1$, then $(q_2, q_3)$ forces (in $\mathbb{Q}_2 \times \mathbb{Q}_3$) that $\underset{\sim}{r_2}, \underset{\sim}{r_3} \in \underset{\sim}{X}$ and $\underset{\sim}{r_3} <_G \underset{\sim}{r_2}$ as witnessed by an isomorphism from $I_{\underset{\sim}{r_3}}$ to $I_{\underset{\sim}{r_2}, <s}$. Now pick $(q_1, q_2, q_3) \leq (q_1', q_2', q_3')$ that forces in addition that $\underset{\sim}{r_1} \neq \underset{\sim}{r_3}$, then necessarily it forces that $\underset{\sim}{r_1} E \underset{\sim}{r_3}$, a contradiction. $\square$

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