Transcendence bases, well-orderings of the reals and the axiom of choice

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Abstract

We prove that $ZF + DC +$ "there exists a transcendence basis for the reals" $+$ "there is no well-ordering of the reals" is consistent relative to ZFC. This answers a question of Larson and Zapletal.[1]

Introduction

It's well-known that the axiom of choice has far-reaching consequences for the structure of the real line. Among them, to name a few, are the existence of non-measurable sets of reals, nonprincipal ultrafilters on ω, paradoxical decompositions of the unit sphere, mad families and more. As the aforementioned statements are consistently false over $ZF + DC$, it's natural to study the possible implications between them in the absence of choice. This direction of study has gained considerable interest in recent years, with many consistency results showing mostly the independence over $ZF + DC$ between various properties of the real line implied by the axiom of choice. We mention several such examples:

Theorem ([Sh:218]): It's consistent relative to an inaccessible cardinal that $ZF+DC$ holds, all set of reals are Lebesgue measurable and there is a set of reals without the Baire property.

Theorem ([HwSh:1113]): It's consistent relative to an inaccessible cardinal that $ZF+DC$ holds, all sets of reals are Lebesgue measurable and there is a mad family.

Theorem ([LaZa1]): It's consistent relative to a proper class of Woodin cardinals that there exists a mad family and there are no ω_1 sequences of reals, nonatomic measures on ω and total selectors for E_0.

Our current paper will focus on two consequences of the axiom of choice for the real line, namely the existence of a transcendence basis for the reals and the existence of a well-ordering of the reals. The following question was asked by Larson and Zapletal in their forthcoming book:

Question ([LaZa2]): Does the existence of a transcendence basis for the reals imply the existence of a well-ordering of the reals?

We shall prove that the answer is negative, namely:

Main result: $ZF+DC+$"there exists a transcendence basis for the reals"$+$"there is no well-ordering of the reals" is consistent relative to ZFC.

It should be noted that in the recent papers [BSWY] and [BCSWY], models of $ZF + DC$ were constructed where there exists a Hamel basis and there is no well-ordering of the reals. However, by [LaZa2], the existence of a Hamel basis (over $ZF+DC$) doesn't imply the existence of a transcendence basis (as explained there, the difference is related to certain model theoretic considerations involving the associated pre-geometries).

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The proof strategy will be similar to that of [Sh:218] and [HwSh:1113] (though no inaccessible cardinals will be used in the current proof). Our forcing $\mathbb{P}$ will consist of conditions $p = (u_p, \mathbb{Q}_p, \underset{\sim}{R}_p)$ where $\mathbb{Q}_p$ is a ccc forcing from some fixed $H(\lambda)$ that forces $MA_{\aleph_1}$ and $\underset{\sim}{R}_p$ is a set of $\mathbb{Q}_p$-names of reals that's forced by $\mathbb{Q}_p$ to be a transcendence basis for the reals. The order will be defined naturally. The sets of the form $\underset{\sim}{R}_p$ will approximate a transcendence basis in the final model, while the forcing notions $\mathbb{Q}_p$ will help us to prove the non-existence of a well-ordering of the reals using a standard amalgamation argument. The fact that each $\mathbb{Q}_p$ forces $MA_{\aleph_1}$ will guarantee that the relevant amalgamation will be ccc.

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The rest of the paper will be devoted to the proof of the main result mentioned above. We shall assume basic familiarity with amalgamation of forcing notions (see, e.g., [HwSh:1090]).

Proof of the main result

We will be forcing over a model of ZFC. The desired model will be obtained as an inner model of the generic extension.

Hypothesis 1: Throughout the paper, we fix infinite regular cardinals λ and κ and an infinite cardinal μ such that $\mu = \mu^{\aleph_1} < \lambda$, $\kappa = \mu^+$ or $\aleph_2 \leq cf(\kappa) \leq \kappa \leq \lambda$ and $(\forall \alpha < \kappa)([\alpha]^{\aleph_1} < \kappa)$ (note that this follows from $\mu = \mu^{\aleph_1} \wedge \kappa = \mu^+$).

Definition 2: We define the forcing notion $\mathbb{P}$ as follows:

A. $p \in \mathbb{P}$ iff $p = (u, \mathbb{Q}, \underset{\sim}{R}) = (u_p, \mathbb{Q}_p, \underset{\sim}{R}_p)$ where:

a. $u \in [\lambda]^{<\kappa}$.

b. $\mathbb{Q} \in H(\lambda)$ is a ccc forcing such that u is its underlying set of elements.

c. $\Vdash_{\mathbb{Q}} MA_{\aleph_1}$.

d. $\underset{\sim}{R}$ is a set of canonical $\mathbb{Q}$-names of reals that is forced by $\mathbb{Q}$ to be a transcendence basis of the reals. A canonical $\mathbb{Q}$-name of a real $\underset{\sim}{\tau}$ will be represented by $\{(\bar{p}_{q_1,q_2}, \eta_{q_1,q_2}) : q_1 < q_1$ are rationals$\}$ where for each $q_1 < q_2$, $\bar{p}_{q_1,q_2} = (p_{q_1,q_2,\alpha} : \alpha < \underset{\sim}{\lambda}_\tau)$ lists without repetition a maximal antichain of $\mathbb{Q}$, $\eta_{q_1,q_2} \in 2^{\underset{\sim}{\lambda_\tau}}$ and $p_{q_1,q_2,\alpha} \Vdash$ "$\underset{\sim}{\tau} \in [q_1, q_2]$ iff $\eta_{q_1,q_2}(\alpha) = 1$".

B. $p \leq_{\mathbb{P}} q$ iff

a. $u_p \subseteq u_q$.

b. $\mathbb{Q}_p \lessdot \mathbb{Q}_q$.

c. $\underset{\sim}{R}_p \subseteq \underset{\sim}{R}_q$.

Definition 3: We define the following $\mathbb{P}$ names:

a. $\underset{\sim}{\mathbb{Q}} = \cup\{\mathbb{Q}_p : p \in \underset{\sim}{G}_{\mathbb{P}}\}$.

b. $\underset{\sim}{R} = \cup\{\underset{\sim}{R}_p : p \in \underset{\sim}{G}_{\mathbb{P}}\}$.

Claim 4: a. $\mathbb{P}$ is a forcing notion of cardinality $\lambda^{<\kappa}$, preserving cardinals and cofinalities of cardinals $\leq \kappa$ and $> \lambda^{<\kappa}$.

b. If $\delta < \kappa$ is a limit ordinal and $\bar{p} = (p_\alpha : \alpha < \delta)$ is $\leq_{\mathbb{P}}$-increasing and satisfies $\alpha < \delta \to \bigcup_{\beta<\alpha} \mathbb{Q}_{p_{1+\beta}} \lessdot \mathbb{Q}_{p_\alpha}$, then $\bar{p}$ has an upper bound p_δ such that $\vec{p}(p_\delta)$ is $\leq_{\mathbb{P}}$-increasing continuous.

c. In clause (b), if $\aleph_2 \leq cf(\delta)$, then p_δ can be chosen as the union of the p_αs.

d. $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}}$ is ccc and λ is its underlying set of elements".

e. $\Vdash_{\mathbb{P}}$ " $\Vdash_{\underset{\sim}{\mathbb{Q}}}$ "$\underset{\sim}{R}$ is a transcendence basis for the reals.

f. Every permutation g of λ naturally induces an automorphism $\hat{g}$ of $\mathbb{P}$ and $\underset{\sim}{\mathbb{Q}}$ which maps $\underset{\sim}{R}$ to itself.

Remark: Recall that a condition in $\mathbb{P}$ is a triple $(u, \mathbb{Q}, \underset{\sim}{R})$ where $\mathbb{Q}$ is a forcing whose universe is $u \in [\lambda]^{<\kappa}$ and $\underset{\sim}{R}$ is a set of canonical $\mathbb{Q}$-names. If g is a permutation of λ, then we can let $\mathbb{Q}^*$ be the forcing isomorphic to $\mathbb{Q}$ whose universe is $u^* := g''u$. This isomorphism naturally maps $\mathbb{Q}$-names to $\mathbb{Q}^*$-names, so $\underset{\sim}{R}$ is mapped to a set $\underset{\sim}{R}^*$ with the same properties. The desired automorphism of $\mathbb{P}$ will thus be defined by $\hat{g}(u, \mathbb{Q}, \underset{\sim}{R}) = (u^*, \mathbb{Q}^*, \underset{\sim}{R}^*)$. We shall use the notation $\hat{g}$ for the function induced by g on $\mathbb{P}$, as well as on the $\mathbb{P}$-names and $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$. We also remind the reader of the standard fact that if $\hat{g}$ is an automorphism of a forcing $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ and $(p, r) \Vdash \phi(\underset{\sim}{\tau})$, then $\hat{g}(p, r) \Vdash \phi(\hat{g}(\underset{\sim}{\tau}))$.

Proof (of Claim 4): a. By clause (b), $\mathbb{P}$ is $(< \kappa)$-complete, hence it preserves cardinals and cofinalities $\leq \kappa$. The rest should be straightforward.

b. As $\bigcup_{\alpha<\delta} \mathbb{Q}_{p_\alpha}$ is ccc, it can be extended to a ccc forcing $\mathbb{Q}_{p_\delta}$ such that $\bigcup_{\alpha<\delta} \mathbb{Q}_{p_\alpha} \lessdot \mathbb{Q}_{p_\delta}$ and $\Vdash_{\mathbb{Q}_{p_\delta}} MA_{\aleph_1}$. As the union of the $\underset{\sim}{R}_{p_\alpha}$ is algebraically independent, we can extend it to a transcendence basis for the reals.

c. Letting $\mathbb{Q}_\delta = \bigcup_{\alpha<\delta} \mathbb{Q}_{p_\alpha}$, obviously $\mathbb{Q}_\delta$ is ccc. In order to show that $\Vdash_{\mathbb{Q}_\delta} MA_{\aleph_1}$, it's enough to show that for forcing notions of cardinality $\aleph_1$ in $V^{\mathbb{Q}_\delta}$. As $\aleph_2 \leq cf(\delta)$, the names for a given ccc forcing in $V^{\mathbb{Q}_\delta}$ and $\aleph_1$-many of its dense subsets are already $\mathbb{Q}_\alpha$-names for some $\alpha < \delta$, and as $\Vdash_{\mathbb{Q}_\alpha} MA_{\aleph_1}$, we're done. Similarly, every $\mathbb{Q}_\delta$-name for a real is already a $\mathbb{Q}_\alpha$-name for some $\alpha < \delta$, hence $\bigcup_{\alpha<\delta} \underset{\sim}{R}_{p_\alpha}$ is a $\mathbb{Q}_\delta$-name of a transcendence basis.

d. Let $G \subseteq \mathbb{P}$ be generic over V, we shall argue in $V[G]$. Given $I = \{q_\alpha : \alpha < \omega_1\} \subseteq \mathbb{Q}$, as $\mathbb{P}$ is $(< \kappa)$-complete, it doesn't add new sequences of ordinals of length ω_1, hence $I \in V$. For every $p \in \mathbb{P}$, there is some $q \in \mathbb{P}$ above p such that $I \subseteq \mathbb{Q}_q$. Therefore, there is some $p \in G$ such that $I \subseteq \mathbb{Q}_p$. As $\mathbb{Q}_p$ is ccc, there are two elements of I that are compatible in $\mathbb{Q}_p$ and hence they're compatible in $\mathbb{Q}$. It follows that $\mathbb{Q}$ is ccc. By a similar density argument, for every $\alpha < \lambda$, there is some $p \in G$ such that $\alpha \in \mathbb{Q}_p$, hence λ is the underlying set of elements of $\mathbb{Q}$.

e. As before, we shall argue in $V[G]$ where $G \subseteq \mathbb{P}$ is generic over V. The algebraic independence of $\underset{\sim}{R}$ follows from G being directed. As for the maximality of $\underset{\sim}{R}$, as before, suppose that $\underset{\sim}{r}$ is a $\mathbb{Q}$-name for a real, then by a similar argument as in clause (d), there is $p \in G$ such that $\underset{\sim}{r}$ is a $\mathbb{Q}_p$-name. As $\underset{\sim}{R}_p$ is a $\mathbb{Q}_p$-name of a transcendence basis, we're done.

f. This is straightforward. Note that the claim is that $\hat{g}$ maps the name $\underset{\sim}{R}$ to itself, that is, $p \Vdash$ "$\underset{\sim}{\tau} \in \underset{\sim}{R}$" iff $\hat{g}(p) \Vdash$ "$\hat{g}(\underset{\sim}{\tau}) \in \underset{\sim}{R}$". In fact, for $p \in \mathbb{P}$ and $\underset{\sim}{\tau}$ we have that $\underset{\sim}{\tau}$ is a member of $\underset{\sim}{R_p}$ iff $\hat{g}(\underset{\sim}{\tau})$ is a member of $\underset{\sim}{R_{\hat{g}(p)}}$. $\square$

Definition/Observation 5: Let V_1 be the model $HOD(\mathbb{R}^{<\kappa} \cup \{\underset{\sim}{R}\} \cup V)$ inside $V^{\underset{\sim}{\mathbb{P}*\mathbb{Q}}}$ (note that this means that if $G \subseteq \mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is generic over V, then $\underset{\sim}{R}$ above is interpreted as $\underset{\sim}{R}[G]$), then V_1 is a model of $ZF + DC_{<\kappa}$ with the same reals as $V^{\underset{\sim}{\mathbb{P}*\mathbb{Q}}}$. In particular, V_1 contains a transcendence basis for the reals (using Claim 4(e)). $\square$

We shall obtain the desired result by proving that there is no well ordering of the reals in V_1. Before that, we shall prove our main amalgamation claim, towards which we mention some basic definitions and facts regarding amalgamation: Suppose that $\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2$ are forcing notions and $f_l : \mathbb{P}_0 \to \mathbb{P}_l$ $(l = 1, 2)$ are complete embeddings. The amalgamation of $\mathbb{P}_1$ and $\mathbb{P}_2$ over $\mathbb{P}_0$, denoted $\mathbb{P}_1 \times_{f_1,f_2} \mathbb{P}_2$ is the set $\{(p_1, p_2) \in \mathbb{P}_1 \times \mathbb{P}_2 : (\exists p \in \mathbb{P}_0)(p \Vdash_{\mathbb{P}}$ "$p_1 \in \mathbb{P}_1/f_1(\mathbb{P}_0) \wedge p_2 \in \mathbb{P}_2/f_2(\mathbb{P}_0)$")$\}$ ordered in the natural way. If f_1 and f_2 are the identity mappings, we shall denote this by $\mathbb{P}_1 \times_{\mathbb{P}_0} \mathbb{P}_2$. We shall use the fact that forcing with $\mathbb{P}_1 \times_{\mathbb{P}_0} \mathbb{P}_2$ is the same as forcing with $\mathbb{P}_0 * ((\mathbb{P}_1/\mathbb{P}_0) \times (\mathbb{P}_2/\mathbb{P}_0))$. We shall also use the fact that $MA_{\aleph_1}$ implies that every ccc forcing is Knaster and that being Knaster is preserved under products. As a corollary, if $\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2$ are ccc and $\Vdash_{\mathbb{P}_0}$ "$MA_{\aleph_1}$", then the amalgamation $\mathbb{P}_1 \times_{\mathbb{P}_0} \mathbb{P}_2$ is ccc. We refer the reader to [RoSh672] for more information on this subject. We shall now turn to the proof of the main amalgamation claim:

Main amalgamation claim 6: (A) implies (B) where:

A. a. $\mathbb{Q}_0 \lessdot \mathbb{Q}_l$ $(l = 1, 2)$.

b. $\Vdash_{\mathbb{Q}_l}$ "$\underset{\sim}{B}_l = \{\underset{\sim}{r_{l,i}} : i < n_l\}$ is algebraically independent over $\mathbb{R}^{V^{\mathbb{Q}_0}}$.

c. $\mathbb{Q} = \mathbb{Q}_1 \times_{\mathbb{Q}_0} \mathbb{Q}_2$.

B. $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{B_1} \cup \underset{\sim}{B_2}$ is algebraically independent over $\mathbb{R}^{V^{\mathbb{Q}_0}}$.

Proof: Assume towards contradiction that there is a counterexample to the claim. As forcing with $\mathbb{Q}$ is the same as forcing with $\mathbb{Q}_0 * ((\mathbb{Q}_1/\mathbb{Q}_0) \times (\mathbb{Q}_2/\mathbb{Q}_0))$, if there is a counterexample to the claim, then by working in $V^{\mathbb{Q}_0}$ we obtain a counterexample where $\mathbb{Q}_0$ is trivial and $\mathbb{Q} = \mathbb{Q}_1 \times \mathbb{Q}_2$. Therefore, we may assume wlog that $\mathbb{Q} = \mathbb{Q}_1 \times \mathbb{Q}_2$ and $\mathbb{Q}_0$ is trivial. We may also assume wlog that it's forced by $\mathbb{Q}$ that $\underset{\sim}{\bar{r_1}}$ and $\underset{\sim}{\bar{r_2}}$ form a counterexample (if $(q_1, q_2) \in \mathbb{Q}_1 \times \mathbb{Q}_2$ forces that $\underset{\sim}{\bar{r_1}}$ and $\underset{\sim}{\bar{r_2}}$ form a counterexample, then we can replace $\mathbb{Q}_l$ by $\mathbb{Q}_l \restriction q_l$ for $l = 1, 2$).

Subclaim: We may assume wlog that $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are Cohen forcing.

Proof of Subclaim: Suppose that $\bar{x} = (\mathbb{Q}_1, \mathbb{Q}_2, \underset{\sim}{\bar{r_1}}, \underset{\sim}{\bar{r_2}})$ form a counter example to the amalgamation claim, we shall construct a counter example $\bar{x}' = (\mathbb{Q}_1', \mathbb{Q}_2', \underset{\sim}{\bar{r_1'}}, \underset{\sim}{\bar{r_2'}})$ where $\mathbb{Q}_1', \mathbb{Q}_2'$ are Cohen forcing. As $\bar{x}$ is a counter example to the claim, there is a nontrivial polynomial $P = P(x_0, ..., x_{n_1-1}, y_0, ..., y_{n_2-1})$ with coeficients in $\mathbb{R}^V$ and a condition $(p_1, p_2) \in \mathbb{Q}_1 \times \mathbb{Q}_2$ such that $(p_1, p_2) \Vdash_{\mathbb{Q}_1 \times \mathbb{Q}_2}$ "$P(\underset{\sim}{\bar{r_1}}, \underset{\sim}{\bar{r_2}}) = 0$". It's now possible to choose $(\bar{p_{1,n}}, \bar{p_{2,n}}, \bar{a_{1,n}}, \bar{a_{2,n}})$ by induction on $n < \omega$ such that the following conditions hold:

a. $\bar{p_{l,n}} = (p_{l,n,\nu} : \nu \in \omega^n)$ $(l = 1, 2)$.

b. Each $p_{l,n,\nu}$ is a condition in $\mathbb{Q}_l$ $(l = 1, 2)$.

c. If $n = m + 1$, $l \in \{1, 2\}$ and $\nu \in \omega^n$ then $p_{l,m,\nu \restriction m} \leq p_{l,n,\nu}$.

d. $\bar{a_{l,n}} = (a^-_{l,n,\eta,i}, a^+_{l,n,\eta,i} : \eta \in \omega^n, i < n_l)$.

e. $a^-_{l,n,\eta,i}$ and $a^+_{l,n,\eta,i}$ are rationals such that $a^+_{l,n,\eta,i} - a^-_{l,n,\eta,i} < \frac{1}{2^n}$.

f. $p_{l,n,\eta} \Vdash_{\mathbb{Q}_l}$ "$\underset{i<n_l}{\wedge} a^-_{l,n,\eta,i} < \underset{\sim}{r_{l,i}} < a^+_{l,n,\eta,i}$".

g. If $n = m + 1$, $\rho \in \omega^m$, $l \in \{1, 2\}$, $((a_i, b_i) : i < n_l)$ is a sequence of pairs of rationals such that $a_i < b_i$ for $i < n_l$ and $p_{l,m,\rho} \nVdash_{\mathbb{Q}_l}$ "$\neg(\underset{i<n_l}{\wedge} a_i < \underset{\sim}{r_{l,i}} < b_i)$", then for some $k < \omega$, $p_{l,n,\hat{\rho(k)}} \Vdash_{\mathbb{Q}_l}$ "$\underset{i<n_l}{\wedge} a_i < \underset{\sim}{r_{l,i}} < b_i$".

h. Moreover, we have $a_i < a^-_{l,n,\hat{\rho(k)},i} < a^+_{l,n,\hat{\rho(k)},i} < b_i$.

i. Moreover, if $n = m + 1$ and $\nu_1, \nu_2 \in \omega^m$, then for some k_1 and k_2, letting $\rho_l = \hat{\nu_l}(k_l)$ $(l = 1, 2)$ we have: For all $x_1, ..., x_{n_1}, y_1, ..., y_{n_2}$, if $\underset{i<n_1}{\wedge} a^-_{l,n,\rho_1,i} < x_i < a^+_{l,n,\rho_1,i}$ and $\underset{j<n_2}{\wedge} a^-_{l,n,\rho_2,j} < y_j < a^+_{l,n,\rho_2,j}$ then $-\frac{1}{2^n} < P(x_1, ..., x_{n_1-1}, y_1, ..., y_{n_2-1}) < \frac{1}{2^n}$.

j. The $a^-_{l,n,\eta,i}$ are increasing with η and the $a^+_{l,n,\eta,i}$ are decreasing with η.

The induction is straightforward where for clause (i) we use the fact that $(p_1, p_2) \Vdash_{\mathbb{Q}_1 \times \mathbb{Q}_2}$ "$P(\underset{\sim}{\bar{r_1}}, \underset{\sim}{\bar{r_2}}) = 0$".

For $l = 1, 2$ we define the following objects:

a. $\mathbb{Q}'_l = (\omega^{<\omega}, \leq)$ (where $\leq$ is the usual inclusion for functions).

b. $\underset{\sim}{\eta_l}$ is the name for the generic real of $\mathbb{Q}'_l$.

c. For $i < n_l$, $\underset{\sim}{r'_{l,i}}$ is the unique real in $\underset{n<\omega}{\cap} (a^-_{l,n,\underset{\sim}{\eta_l} \restriction n,i}, a^+_{l,n,\underset{\sim}{\eta_l} \restriction n,i})$.

Now $\mathbb{Q}'_l$ are equivalent to Cohen forcing, and by clause (i) of the induction, $\Vdash_{\mathbb{Q}'_1 \times \mathbb{Q}'_2}$ "$P(\underset{\sim}{r'_1}, \underset{\sim}{r'_2}) = 0$". Therefore, in order to prove the subclaim, it suffices to show that $\Vdash_{\mathbb{Q}'_l}$ "$\underset{\sim}{r'_{l,1}}, ..., \underset{\sim}{r'_{l,n_1-1}}$ are algebraically independent over $\mathbb{R}^V$". Assume towards contradiction that there is some $\eta \in \mathbb{Q}'_l$ and a nontrivial polynomial $P'_l(x_0, ..., x_{n_l-1})$ such that $\eta \Vdash_{\mathbb{Q}'_l}$ "$P'_l(\underset{\sim}{r'_l}) = 0$". By the assumption on $(\mathbb{Q}_l, \underset{\sim}{r_l})$, letting $n = lg(\eta)$, $p_{l,n,\eta} \Vdash_{\mathbb{Q}_l}$ "$P'_l(\underset{\sim}{r_l}) \neq 0$". Let $G_l \subseteq \mathbb{Q}_l$ be generic over V such that $p_{l,n,\eta} \in G_l$, so wlog $P'_l(\underset{\sim}{r_l}[G_l]) > 0$. By continuity, there are rationals $a_i < b_i$ $(i < n_l)$ such that $V[G] \models$ "for every $x_0, ..., x_{n_l-1}$, $\underset{i<n_l}{\wedge} a_i < x_i < b_i \rightarrow P'_l(x_0, ..., x_{n_l-1}) > 0$ and $\underset{\sim}{r_{l,i}}[G_l] \in (a_i, b_i)$". Therefore, the first part of the statement holds in V and there is some $q \in G_l$ such that $p_{l,n,\eta} \leq q$ and q forces the second part of the statement. In particular, $p_{l,n,\eta} \nVdash_{\mathbb{Q}_l}$ "$\neg(\underset{i<n_l}{\wedge} a_i < \underset{\sim}{r_{l,i}} < b_i)$". By clause (g) of the induction, there is some $k < \omega$ such that $p_{l,n+1,\hat{\eta}(k)} \Vdash_{\mathbb{Q}_l}$ "$\underset{i<n_l}{\wedge} a_i < \underset{\sim}{r_{l,i}} < b_i$" and $a_i < a^-_{l,n+1,\hat{\eta}(k),i} < a^+_{l,n+1,\hat{\eta}(k),i} < b_i$. Now $\hat{\eta}(k)$ is a condition in $\mathbb{Q}'_l$ that forces in $\mathbb{Q}'_l$ that $\underset{\sim}{r_{l,i}}' \in (a_i, b_i)$ for all $i < n_l$. It follows that $\hat{\eta}(k)$ forces in $\mathbb{Q}'_l$ that $P'_l(\underset{\sim}{r_{l,0}}', ..., \underset{\sim}{r_{l,n_l-1}}') > 0$, contradicting the choise of η and $P - l'$. It follows that $\Vdash_{\mathbb{Q}'_l}$ "$\underset{\sim}{r'_{l,1}}, ..., \underset{\sim}{r'_{l,n_1-1}}$ are algebraically independent over $\mathbb{R}^V$", which completes the proof of the subclaim.

We shall now return to the proof of the main amalgamation claim:

Let $\chi \geq \aleph_1$ be large enough and let N be a countable elementary submodel of $(H(\chi), \in)$ such that $\mathbb{Q}_l, \underset{\sim}{\bar{r_l}} \in N$ $(l = 1, 2)$. As $\mathbb{Q}_l$ is Cohen, there is a $\mathbb{Q}_l$-name $\underset{\sim}{\eta_l}$ for a Cohen real over V that generates the generic for $\mathbb{Q}_l$. For each $l \in \{1, 2\}$ and $i < n_l$ there is a Borel function $\mathbf{B}_{l,i}$ such that $\underset{\sim}{r_{l,i}} = \mathbf{B}_{l,i}(\underset{\sim}{\eta_l})$, we may assume that the $\mathbf{B}_{l,i}$s belong to N as well. Let $\eta'_1 \in V$ be Cohen over N, let $G_2 \subseteq \mathbb{Q}_2$ be generic over V and let $\eta_2 = \underset{\sim}{\eta_2}[G_2]$. η_2 is Cohen over V and is also generic over $N[\eta'_1]$. Therefore, (η'_1, η_2) is generic for $\mathbb{Q}_1 \times \mathbb{Q}_2$ over N. As it's forced by $\mathbb{Q}_1 \times \mathbb{Q}_2$ over V that $\underset{\sim}{\bar{r_1}}\hat{\ }\underset{\sim}{\bar{r_2}}$ is a counterexample, there is a polynomial P witnessing this, i.e. $V \models$ " $\Vdash_{\mathbb{Q}_1 \times \mathbb{Q}_2}$ "$P(..., \mathbf{B}_{1,l}(\underset{\sim}{\eta'_1}), ..., ..., \mathbf{B}_{2,l}(\underset{\sim}{\eta_2}), ...) = 0$"". By absoluteness, the same stetement holds in N. By the genericity over N of (η'_1, η_2), $N[\eta'_1, \eta_2] \models P(..., \mathbf{B}_{1,l}(\eta'_1), ..., ..., \mathbf{B}_{2,l}(\eta_2), ...) = 0$. Therefore, there is $p_2 \in G_2 \subseteq \mathbb{Q}_2$ such that $N[\eta'_1] \models$ "$p_2 \Vdash_{\mathbb{Q}_2}$ "$\underset{\sim}{\bar{r_2}}$ is not algebraically independent over $\mathbb{R}^V$, as witnessed by $(\mathbf{B}_{1,l}(\eta'_1) : l < n_1)$"", and by absoluteness, the same holds in V. This contradicts assumption (A)(b) and completes the proof of the claim. $\square$

Before proving the relevant conclusion for $\mathbb{P}$, we need the following algebraic observation:

Observation 7: Let $p_1, p_2 \in \mathbb{P}$ and suppose that $p_1 \leq p_2$. Denote $\mathbb{Q}_{p_l}$ by $\mathbb{Q}_l$ and $\underset{\sim}{R_{p_l}}$ by $\underset{\sim}{R_l}$ $(l = 1, 2)$. Then $\Vdash_{\mathbb{Q}_2}$ "$\underset{\sim}{R_2} \setminus \underset{\sim}{R_1}$ is algebraically independent over $\mathbb{R}^{V^{\mathbb{Q}_1}}$".

Proof: Suppose towards contradiction that there is some $q \in \mathbb{Q}_2$ and $\underset{\sim}{r_0}, ..., \underset{\sim}{r_{n_2-1}}$ (with no repetition) such that $q \Vdash_{\mathbb{Q}_2}$ "$\underset{\sim}{r_0}, ..., \underset{\sim}{r_{n_2-1}} \in \underset{\sim}{R_2} \setminus \underset{\sim}{R_1}$ are not algebraically independent over $\mathbb{R}^{V^{\mathbb{Q}_1}}$". By increasing q if necessary, we may assume wlog that there is a non-trivial polynomial $P(x_0, ..., x_{n_2-1})$ over $\mathbb{R}^{V^{\mathbb{Q}_1}}$ such that $q \Vdash_{\mathbb{Q}_2}$ "$P(\underset{\sim}{r_0}, ..., \underset{\sim}{r_{n_2-1}}) = 0$". Therefore, there are $\mathbb{Q}_1$-names of reals $\underset{\sim}{s_0}, ..., \underset{\sim}{s_{n_1-1}}$ and a polynomial $Q(x_0, ..., x_{n_2-1}, y_0, ..., y_{n_1-1})$ over the rationals such that $q \Vdash_{\mathbb{Q}_2}$ "$Q(x_0, ..., x_{n_2-1}, \underset{\sim}{s_0}, ..., \underset{\sim}{s_{n_1-1}}) = P(x_0, ..., x_{n_2-1})$". Recalling that $\underset{\sim}{R_1}$ is a $\mathbb{Q}_1$-name of a transcendence basis over the rationals, then by increasing q if necessary, there are $\mathbb{Q}_1$-names of reals $\underset{\sim}{t_0}, ..., \underset{\sim}{t_{n_0-1}}$ such that $q \Vdash_{\mathbb{Q}_2}$ "$\underset{\sim}{t_0}, ..., \underset{\sim}{t_{n_0-1}} \in \underset{\sim}{R_1}$ (with no repetition)" and $q \Vdash_{\mathbb{Q}_2}$ "$\underset{\sim}{s_0}, ..., \underset{\sim}{s_{n_1-1}}$ are algebraic over $\mathbb{Q}[\underset{\sim}{t_0}, ..., \underset{\sim}{t_{n_0-1}}]$" (here $\mathbb{Q}$ denotes the field of rational numbers). It follows that $q \Vdash_{\mathbb{Q}_2}$ "$\{\underset{\sim}{t_0}, ..., \underset{\sim}{t_{n_0-1}}, \underset{\sim}{r_0}, ...\underset{\sim}{r_{n_2-1}}\} \subseteq \underset{\sim}{R_2}$ is not algebraically independent over the rationals". By the choice of the $\underset{\sim}{t_i}$s and the $\underset{\sim}{r_i}$s, $q \Vdash_{\mathbb{Q}_2}$ "$\underset{\sim}{t_0}, ..., \underset{\sim}{t_{n_0-1}}, \underset{\sim}{r_0}, ..., \underset{\sim}{r_{n_2-1}}$ are without repetition". Together, we get a contradiction to the definition of the conditions in $\mathbb{P}$ and the fact that $p_2 \in \mathbb{P}$. $\square$

Conclusion 8: Suppose that $p_1, p_2 \in \mathbb{P}$ such that $p_1 \leq p_2$. Let g be a permutation of λ such that $g \restriction u_{p_1} = id$ and $g''(u_{p_2}) \cap u_{p_2} = u_{p_1}$, and let $p_3 = \hat{g}(p_2)$. Then there is $q \in \mathbb{P}$ such that $p_2, p_3 \leq q$ and $\mathbb{Q}_{p_2} \times_{\mathbb{Q}_{p_1}} \mathbb{Q}_{p_3} \lessdot \mathbb{Q}_q$.

Proof: Let $\mathbb{Q} = \mathbb{Q}_{p_2} \times_{\mathbb{Q}_{p_1}} \mathbb{Q}_{p_3}$. As $\mathbb{Q}_{p_1}$ is ccc and $\Vdash_{\mathbb{Q}_{p_1}}$ "$MA_{\aleph_1} + \mathbb{Q}_{p_2}/\mathbb{Q}_{p_1} \models ccc + \mathbb{Q}_{p_3}/\mathbb{Q}_{p_1} \models ccc$", it follows that $\mathbb{Q}$ is ccc (see e.g. [HwSh:1090] for details). By the previous observation, for $l = 2, 3$, $\Vdash_{\mathbb{Q}_{p_l}}$ "$\underset{\sim}{R_{p_l}} \setminus \underset{\sim}{R_{p_1}}$ is algebraically independent over

$\mathbb{R}^{V^{\mathbb{Q}_{p_1}}}$". Therefore, by Claim 6, $\Vdash_{\mathbb{Q}}$ "$(\underset{\sim}{R}_{p_2} \setminus \underset{\sim}{R}_{p_1}) \cup (\underset{\sim}{R}_{p_3} \setminus \underset{\sim}{R}_{p_1})$ is algebraically independent over $\mathbb{R}^{V^{\mathbb{Q}_{p_1}}}$". It follows that $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{R}_{p_2} \cup \underset{\sim}{R}_{p_3} = \underset{\sim}{R}_{p_1} \cup (\underset{\sim}{R}_{p_2} \setminus \underset{\sim}{R}_{p_1}) \cup (\underset{\sim}{R}_{p_3} \setminus \underset{\sim}{R}_{p_1})$ is algebraically independent over the rationals" (recall that if $\{\alpha_0, ..., \alpha_{n-1}\}$ are algebraically independent over the rationals and $\{\beta_0, ..., \beta_{m-1}\}$ are algebraically independent over a field $\mathbb{F}$ containing $\mathbb{Q} \cup \{\alpha_0, ..., \alpha_{n-1}\}$, then $\{\alpha_0, ..., \alpha_{n-1}, \beta_0, ..., \beta_{m-1}\}$ are algebraically independent over the rationals). By Hypothesis 1, there is a ccc forcing $\mathbb{Q}_q$ such that $\mathbb{Q} \lessdot \mathbb{Q}_q$, $\Vdash_{\mathbb{Q}_q} MA_{\aleph_1}$ and $|\mathbb{Q}_q| = u_q$ for some $u_q \in [\lambda]^{<\kappa}$. As $\Vdash_{\mathbb{Q}_q}$ "$\underset{\sim}{R}_{p_2} \cup \underset{\sim}{R}_{p_3}$ are algebraically independent over the rationals", there is a set $\underset{\sim}{R}_q$ of $\mathbb{Q}_q$-names of reals such that $\underset{\sim}{R}_{p_2} \cup \underset{\sim}{R}_{p_3} \subseteq \underset{\sim}{R}_q$ and $\Vdash_{\mathbb{Q}_q}$ "$\underset{\sim}{R}_q$ is a transcendence basis for the reals". Now let $q = (u_q, \mathbb{Q}_q, \underset{\sim}{R}_q)$, it's easy to verify that q is as required. $\square$

Recalling Observation 5, we shall complete the proof of the main result of the paper by proving the following claim:

Claim 9: There is no well-ordering of the reals in V_1.

Proof: Assume towards contradiction that there are $(p_1, r_1) \in \mathbb{P} * \underset{\sim}{\mathbb{Q}}$ such that, over V, $(p_1, r_1) \Vdash_{\mathbb{P} * \mathbb{Q}}$ "$\underset{\sim}{f}$ is a one-to-one function from $\mathbb{R}$ to Ord" and such that $\underset{\sim}{f}$ is definable via a formula ϕ from $\underset{\sim}{R}$ and a sequence $(\underset{\sim}{\eta}_\epsilon : \epsilon < \epsilon(*))$ where $\epsilon(*) < \kappa$ and wlog each $\underset{\sim}{\eta}_\epsilon$ is a $\mathbb{Q}_{p_1}$ name for a real (by a similar argument as in claims 4(d) and 4(e), we can always extend p_1 to make this true). We shall apply Claim 4(f) and the remark following it throughout the proof. Choose $(p_2, r_2) \geq (p_1, r_1)$ and a name of a real $\underset{\sim}{r}$ such that $(p_2, r_2) \Vdash_{\mathbb{P} * \underset{\sim}{\mathbb{Q}}}$ "$\underset{\sim}{r} \in \mathbb{R}^{V^{\mathbb{Q}_{p_2}}} \setminus \mathbb{R}^{V^{\mathbb{Q}_{p_1}}}$", wlog $r_2 \in \mathbb{Q}_{p_2}$, and by extending the condition if necessary, we may assume wlog that (p_2, r_2) forces a value γ to $\underset{\sim}{f}(\underset{\sim}{r})$.

Let g be a permutation of λ such that $g \restriction u_{p_1} = id$ and $g''(u_{p_2}) \cap u_{p_2} = u_{p_1}$. We shall denote both of the induced automorphisms on $\mathbb{P}$ and $\mathbb{Q}$ by $\hat{g}$. Clearly, $\hat{g}(p_1) = p_1$. Let $p_3 = \hat{g}(p_2)$ and $r_3 = \hat{g}(r_2)$. By the previous claims, there is $q \in \mathbb{P}$ such that $p_2, p_3 \leq q$ and $\mathbb{Q}_{p_2} \times_{\mathbb{Q}_{p_1}} \mathbb{Q}_{p_3} \lessdot \mathbb{Q}_q$, and by the construction of the amalgamation, there is $r \in \mathbb{Q}_q$ above r_2 and r_3. As $\Vdash_{\mathbb{P} * \underset{\sim}{\mathbb{Q}}}$ "$\mathbb{R}^{V^{\mathbb{Q}_{p_2}}} \cap \mathbb{R}^{V^{\mathbb{Q}_{p_3}}} = \mathbb{R}^{V^{\mathbb{Q}_{p_1}}}$", it follows that $(q, r) \Vdash_{\mathbb{P} * \mathbb{Q}}$ "$\underset{\sim}{r} \neq g(\underset{\sim}{r})$". As $(p_2, r_2) \leq (q, r)$, $(q, r) \Vdash_{\mathbb{P} * \mathbb{Q}}$ "$\underset{\sim}{f}(\underset{\sim}{r}) = \gamma$". Recalling that $\underset{\sim}{f}$ is forced to be injective, we shall arrive at a contradiction by showing that $(q, r) \Vdash_{\mathbb{P} * Q}$ "$\underset{\sim}{f}(\hat{g}(\underset{\sim}{r})) = \gamma$". It's enough to show that the statement is forced by $(p_3, r_3) = (\hat{g}(p_2), \hat{g}(r_2))$, and in order to show that, it suffices to show that $\underset{\sim}{f} = \hat{g}(\underset{\sim}{f})$. Recalling that each $\underset{\sim}{\eta}_\epsilon$ in the definition of $\underset{\sim}{f}$ is a $\mathbb{Q}_{p_1}$-name and that g is the identity on u_{p_1}, it follows that $\hat{g}(\underset{\sim}{\eta}_\epsilon) = \underset{\sim}{\eta}_\epsilon$. By Claim 4(f), $\underset{\sim}{R}$ is preserved by $\hat{g}$. As $\underset{\sim}{f}$ is definable from $\underset{\sim}{R}$ and $(\underset{\sim}{\eta}_\epsilon : \epsilon < \epsilon(*))$, it follows that $\hat{g}(\underset{\sim}{f}) = \underset{\sim}{f}$. This completes the proof of the claim. $\square$

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