A Borel maximal cofinitary group

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Abstract

We construct a Borel maximal cofinitary group.[1]

Introduction

The study of mad families and their relatives occupies a central place in modern set theory. As the straightforward way to construct such families involves the axiom of choice, questions on the definability of such families naturally arise. The following classical result is due to Mathias:

Theorem ([Ma]): There are no analytic mad families.

In recent years, there has been considerable interest in the definability of several relatives of mad families, such as maximal eventually different families and maximal cofinitary groups. A family $\mathcal{F} \subseteq \omega^\omega$ is a maximal eventually different family if $f \neq g \in \mathcal{F} \rightarrow f(n) \neq g(n)$ for large enough n, and $\mathcal{F}$ is maximal with respect to this property. The following result was recently discovered by the authors:

Theorem ([HwSh1089]): Assuming ZF, there exists a Borel maximal eventually different family.

As for maximal cofinitary groups (see definition 1 below), several consistency results were established on the definability of such groups, for example, the following results by Kastermans and by Fischer, Friedman and Toernquist:

Theorem ([Ka]): There is a Π^1_1-maximal cofinitary group in L.

Theorem ([FFT]): $\mathfrak{b} = \mathfrak{c} = \aleph_2$ is consistent with the existence of a maximal cofinitary group with a Π^1_2-definable set of generators.

Our main goal in this paper is to establish the existence of a Borel maximal cofinitary group in ZF. We intend to improve the current results in a subsequent paper, and prove the existence of closed MED families and MCGs.

The main theorem

Definition 1: $G \subseteq S_\infty$ is a maximal cofinitary group if G is a subgroup of S_∞, $|\{n : f(n) = n\}| < \aleph_0$ for every $Id \neq f \in G$, and G is maximal with respect to these properties.

Theorem 2 (ZF)**:** There exists a Borel maximal cofinitary group.

The rest of the paper will be dedicated for the proof of the above theorem. It will be enough to prove the existence of a Borel maximal cofinitary group in $Sym(U)$ where U is an arbitrary set of cardinality $\aleph_0$.

Convention: Given two sequences η and ν, we write $\eta \leq \nu$ when η is an initial segment of ν.

Definition 3: The following objects will remain fixed throughout the proof:

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a. $T = 2^{<\omega}$.

b. $\bar{u} = (u_\rho : \rho \in T)$ is a sequence of pairwise disjoint sets such that $U = \cup\{u_\rho : \rho \in T\} \subseteq H(\aleph_0)$ (will be chosen in claim 4).

c. $<_*$ is a linear order of $H(\aleph_0)$ of order type ω such that given $\eta, \nu \in T$, $\eta <_* \nu$ iff $lg(\eta) < lg(\nu)$ or $lg(\eta) = lg(\nu) \wedge \eta <_{lex} \nu$.

d. For every $\eta \in T$, $\Sigma\{|u_\nu| : \nu <_* \eta\} \ll |u_\eta|$.

e. Borel functions $\mathbf{B} = \mathbf{B}_0$ and $\mathbf{B}_{-1} = \mathbf{B}_0^{-1}$ such that $\mathbf{B} : Sym(U) \to 2^\omega$ is injective with a Borel image, and $\mathbf{B}_{-1} : 2^\omega \to Sym(U)$ satisfies $\mathbf{B}(f) = \eta \to \mathbf{B}_{-1}(\eta) = f$.

f. Let $\mathbf{A}_1 = \{f \in Sym(U) : f$ has a finite number of fixed points$\}$, $\mathbf{A}_1$ is obviously Borel.

g. $\{f_{\rho,\nu} : \nu \in 2^{lg(\rho)}\}$ generate the group K_ρ (defined below) considered as a subgroup of $Sym(u_\rho)$.

Claim 4: There exists a sequence $(u_\rho, \bar{f}_\rho, \bar{A}_\rho : \rho \in T)$ such that:

a. $\bar{f}_\rho = (f_{\rho,\nu} : \nu \in T_{lg(\rho)})$.

b. $f_{\rho,\nu} \in Sym(u_\rho)$ has no fixed points.

c. $\bar{A}_\rho = (A_{\rho,\nu} : \nu \in T_{lg(\rho)})$. We shall denote $\bigcup_{\nu \in T_{lg(\rho)}} A_{\rho,\nu}$ by A'_ρ.

d. $A_{\rho,\nu} \subseteq u_\rho \subseteq H(\aleph_0)$ and $\Sigma\{|u_\eta| : \eta <_* \rho\} \ll |A_{\rho,\nu}|$.

e. $\nu_1 \neq \nu_2 \in T_{lg(\rho)} \to A_{\rho,\nu_1} \cap A_{\rho,\nu_2} = \emptyset$.

f. If $\rho \in 2^n$ and $w = w(..., x_\nu, ...)_{\nu \in 2^n}$ is a non-trivial group term of length $\leq n$ then:

1. $w(..., f_{\rho,\nu}, ...)_{\nu \in 2^n} \in Sym(u_\rho)$ has no fixed points.
2. $(w(..., f_{\rho,\nu}, ...)_{\nu \in 2^n}(\bigcup_{\nu \in 2^n} A_{\rho,\nu})) \cap \bigcup_{\nu \in 2^n} A_{\rho,\nu} = \emptyset$.

g. $\{f_{\rho,\nu} : \nu \in T_{lg(\rho)}\}$ generate the group K_ρ (whose set of elements is u_ρ) which is considered as a group of permutations of u_ρ.

Proof: We choose $(u_\rho, \bar{f}_\rho, \bar{A}_\rho)$ by $<_*$-induction on ρ as follows: Arriving at ρ, we choose the following objects:

a. n^1_ρ such that $\Sigma\{|u_\nu| : \nu <_* \rho\} 2^{lg(\rho)+7} \ll n^1_\rho$ and let $n^0_\rho = \frac{n^1_\rho}{2^{lg(\rho)}}$.

b. Let H_ρ be the group generated freely by $\{x_{\rho,\nu} : \nu \in T_{lg(\rho)}\}$.

c. In H_ρ we can find $(y_{\rho,n} : n < \omega)$ which freely generate a subgroup (we can do it explicitly, for example, if a and b freely generate a group, then $(a^n b^n : n < \omega)$ are as required), wlog for w_1 and w_2 as in 4(f) and $n_1 < n_2$ we have $w_1 y_{\rho,n_1} \neq w_2 y_{\rho,n_2}$.

Now choose $A^1_{\rho,\nu} \subseteq \{y_{\rho,n} : n < \omega\}$ for $\nu \in 2^{lg(\rho)}$ such that $\nu_1 \neq \nu_2 \to A^1_{\rho,\nu_1} \cap A^1_{\rho,\nu_2} = \emptyset$ and $n^0_\rho \leq |A^1_{\rho,\nu}|$.

d. Let $\Lambda_\rho = \{w : w = w(..., x_{\rho,\nu}, ...)_{\nu \in T_{lg(\rho)}}$ is a group word of length $\leq lg(\rho)\}$.

As H_ρ is free, it's residually finite, hence there is a finite group K_ρ and an epimorphism $\phi_\rho : H_\rho \to K_\rho$ such that $\phi_\rho \restriction ((\bigcup_{\nu \in 2^{lg(\rho)}} A^1_{\rho,\nu}) \cup \Lambda_\rho \cup \{wa : w \in \Lambda_\rho \wedge a \in \bigcup_{\nu \in 2^{lg(\rho)}} A^1_{\rho,\nu}\})$ is injective (note that there is no use of the axiom of choice as we can argue in a model of the form $L[A]$). WLOG $K_\rho \subseteq H(\aleph_0)$ and K_ρ is disjoint to $\cup\{u_\nu : \nu <_* \rho\}$.

We now define the following objects:

a. $u_\rho = K_\rho$.

b. $A_{\rho,\nu} = \{\phi_\rho(a) : a \in A^1_{\rho,\nu}\}$.

c. For $\nu \in 2^{lg(\rho)}$, let $f_{\rho,\nu} : u_\rho \to u_\rho$ be multiplication by $\phi_\rho(x_{\rho,\nu})$ from the left.

It's now easy to verify that $(u_\rho, \bar{A}_\rho, \bar{f}_\rho)$ are as required, so $U = \cup\{u_\rho : \rho \in T\}$. $\square$

Definition and claim 5: A. a. Given $f \in Sym(U)$, let $g = F_1(f)$ be $g^*_{\mathbf{B}(f)}$, where for $\nu \in 2^\omega$, g^*_ν is the permutation of U defined by: $g^*_\nu \restriction u_\rho = f_{\rho,\nu\restriction lg(\rho)}$ (recall that $\bar{u}$ is a partition of U and each $f_{\rho,\nu}$ belongs to $Sym(u_\rho)$, therefore g is well-defined and belongs to $Sym(U)$).

b. Let G_1 be the subgroup of $Sym(U)$ generated by $\{g^*_\nu : \nu \in 2^\omega\}$ (which includes $\{F_1(f) : f \in Sym(U)\}$).

c. Let I_1 be the ideal on U generated by the sets $v \subseteq U$ satisfying the following property:

$(*)_v$ For some $\rho = \rho_v \in 2^\omega$, for every n, there is at most one pair (a, ν) such that $\nu \in T$, $a \in v \cap u_\nu$ and $\rho \cap \nu = \rho \restriction n$.

c(1). Note that I_1 is indeed a proper ideal: Suppose that $v_0, ..., v_n$ are as above and let $\rho_0, ..., \rho_n$ witness $(*)_{v_i}$ $(i = 0, ..., n)$. Choose k such that $2^k > n + 1$ and choose $\eta \in 2^k \setminus \{\rho_i \restriction k : i \leq n\}$. For each $i \leq n$, there is $k(i) \leq k$ such that $\eta \cap \rho_i = \rho_i \restriction k(i)$. For each $i \leq n$, let $n(i)$ be the length ν such that (a, ν) witness $(*)_{v_i}$ for $k(i)$. Choose η' above η such that $lg(\eta') > n(i)$ for every i, then $u_{\eta'} \cap (\bigcup\limits_{i \leq n} v_i) = \emptyset$.

d. Let $K_1 = \{f \in Sym(U) : fix(f) \in I_1\}$ where $fix(f) = \{x : f(x) = x\}$.

e. For $\eta \in T$, $a, b \in u_\eta$, $n = lg(\eta) < \omega$ and let $y_{a,b} = ((f_{\eta,\rho_{a,b,l}}, i_{a,b,l}) : l < l_{a,b} = l(*))$ such that:

1. $\rho_{a,b,l} \in 2^n$.

2. $i_{a,b,l} \in \{1, -1\}$.

3. $b = (f_{\eta,\rho_{a,b,0}})^{i_{a,b,0}} \cdots (f_{\eta,\rho_{a,b,l(*)-1}})^{i_{a,b,l(*)-1}}(a)$.

4. $l_{a,b} = l(*)$ is minimal under 1-3, $y_{a,b}$ is $<_*$-minimal under this requirement.

5. $i_l \neq i_{l+1} \to \rho_{a,b,l} \neq \rho_{a,b,l+1}$.

By claim 4(g) and definition 3(c), $y_{a,b}$ is always well-defined.

B. There are Borel functions $\mathbf{B}_{1,1}, \mathbf{B}_{1,2}$, etc with domain $Sym(U)$ such that:

a. $\mathbf{B}_{1,1}(f) \in \{0, 1\}$ and $\mathbf{B}_{1,1}(f) = 0$ iff $|fix(f)| < \aleph_0$.

b. Letting $\eta_1 = \mathbf{B}(f)$, $\mathbf{B}_{1,2}(f) \in \{0, 1\}$ and $\mathbf{B}_{1,2}(f) = 1$ iff $\mathbf{B}_{1,1}(f) = 0$ and for infinitely many n, $f''(A'_{\eta_1 \restriction n}) \nsubseteq \cup\{u_\rho : \rho \leq_* \eta_1 \restriction n\}$ (where $A'_{\eta_1\restriction n}$ is defined in 4(c).

c. $\mathbf{B}_{1,3}(f) \in \omega$ such that if $\mathbf{B}_{1,1}(f) = \mathbf{B}_{1,2}(f) = 0$ then for every $\mathbf{B}_{1,3}(f) \leq n$, $f''(A'_{\eta_1\restriction n}) \subseteq \cup\{u_\rho : \rho \leq_* \eta_1 \restriction n\}$.

d. $\mathbf{B}_{1,4}(f) \in \{0, 1\}$ and $\mathbf{B}_{1,4}(f) = 1$ iff $\mathbf{B}_{1,1}(f) = \mathbf{B}_{1,2}(f) = 0$ and $\{l_{a,f(a)} : a \in v_n$ and $\mathbf{B}_{1,3}(f) \leq n\}$ is unbounded, where $v_n := \{a \in A'_{\eta_1\restriction n} \subseteq u_{\eta_1\restriction n} : f(a) \in u_{\eta_1\restriction n}\}$.

e. $\mathbf{B}_{1,5}(f) \in \omega$ such that: If $\mathbf{B}_{1,4}(f) = \mathbf{B}_{1,2}(f) = \mathbf{B}_{1,1}(f) = 0$ then $\mathbf{B}_{1,5}(f)$ is a bound of $\{l_{a,f(a)} : a \in v_n$ and $\mathbf{B}_{1,3}(f) \leq n\}$.

f. $\mathbf{B}_{1,6}(f) \in \{0, 1\}$ such that: $\mathbf{B}_{1,6}(f) = 1$ iff $\mathbf{B}_{1,1}(f) = \mathbf{B}_{1,2}(f) = \mathbf{B}_{1,4}(f) = 0$ and for every m there exists $n > m$ such that: There are $a_1 \neq a_2 \in v_n$ such that for some l, $l < min\{l_{a_1,f(a_1)}, l_{a_2,f(a_2)}\}$, $\rho_{a_1,f(a_1),l} \neq \rho_{a_2,f(a_2),l}$ and $\rho_{a_1,f(a_1),l} \restriction m = \rho_{a_2,f(a_2),l} \restriction m$.

g. $\mathbf{B}_{1,7}(f)$ is a sequence $(a_n = a_n(f) : n \in \mathbf{B}_{1,8}(f))$ such that if $\mathbf{B}_{1,6}(f) = 1$ then:

1. $a_n \in v_n$

2. $\mathbf{B}_{1,8}(f) \in [\omega]^\omega$

3. $l_{a_n,f(a_n)} = l(*) = \mathbf{B}_{1,9}(f)$

4. $l_{**} = \mathbf{B}_{2,0}(f) < l_*$

5. $(\rho_{a_n,f(a_n),l_{**}} : n \in \mathbf{B}_{1,8}(f))$ are pairwise incomparable.

6. For every $l < l_*$, the following sequence is constant: $(TV(\rho_{a_n,f(a_n),l} \leq \rho_{a_k,f(a_k),l}) : n < k \in \mathbf{B}_{1,8}(f))$.

h. $\mathbf{B}_{2,1}(f)$ is a sequence $(A_n = A_n(f) : n \in \mathbf{B}_{2,2}(f))$ such that if $\mathbf{B}_{1,1}(f) = \mathbf{B}_{1,2}(f) = \mathbf{B}_{1,4}(f) = \mathbf{B}_{1,6}(f) = 0$ then:

1. $\mathbf{B}_{2,2}(f) \in [\omega]^\omega$

2. $A_n \subseteq A'_{\eta_1 \restriction n}$ (recalling that $\eta_1 = \mathbf{B}(f)$)

3. $l_{a,f(a)} = l_* = \mathbf{B}_{2,3}(f)$ for $n \in \mathbf{B}_{2,2}(f)$ and $a \in A_n$

4. $(i_l : l < l_*) = (i_{a,f(a),l} : l < l_*)$ (recalling definition 5(e)) for every $n \in \mathbf{B}_{2,2}(f)$ and $a \in A_n$.

5. $\frac{1}{\mathbf{B}'_{2,3}(f)(n)} \leq \frac{|A_n|}{|v_n|}$ where $\mathbf{B}'_{2,3}(f)(n) \in \omega \setminus \{0\}$, $\mathbf{B}_{2,3}(f)(n) \ll |v_n|$ and v_n is defined in 5(B)(d).

6. $\mathbf{B}_{2,4,n}(f) = \bar{\rho_n^*} = (\rho_l^n : l < l_*) = (\rho_{a,f(a),l} : l < l_*)$ for every $n \in \mathbf{B}_{2,2}(f)$ and $a \in A_n$.

7. $(TV(\rho_l^n \leq \rho_l^m) : n < m \in \mathbf{B}_{2,2}(f))$ is constantly $\mathbf{B}_{2,5,l}(f)$

i. $\mathbf{B}_{2,6}(f) \in \{0,1\}$ is 1 iff $\mathbf{B}_{1,1}(f) = \mathbf{B}_{1,2}(f) = \mathbf{B}_{1,4}(f) = \mathbf{B}_{1,6}(f) = 0$ and in $(h)(7)$, $\mathbf{B}_{2,5,l} = false$ for some $l < l_*$.

j. $\mathbf{B}'_{2,6}(f) \in \{0,1\}$ is 0 iff $\mathbf{B}_{1,1}(f) = \mathbf{B}_{1,2}(f) = \mathbf{B}_{1,4}(f) = \mathbf{B}_{1,6}(f) = 0$ and $\mathbf{B}_{2,6}(f) = 0$

Proof: By the proof of Ramsey's theorem and the arguments which are implicit in the proof of claim 7 below. Note that while the statement "there exists an infinite homogeneous set" is analytic, we can Borel-compute that homogeneous set. See the proof of claim 6 in [HwSh:1089] for more details. $\square$

Definition and claim 6: a. 1. Let H_3 be the set of $f \in Sym(U)$ such that:

α. $\mathbf{B}_{1,1}(f) = 0$

β. If $\mathbf{B}_{1,2}(f) = \mathbf{B}_{1,4}(f) = \mathbf{B}_{1,6}(f) = 0$ then $\mathbf{B}'_{2,6}(f) = 1$

a. 2. H_3 is Borel.

b. For $f \in Sym(U)$ let G_f be the set of $g \in Sym(U)$ such that:

1. If $f \notin H_3$ then $G_f = \{F_1(f)\}$.

2. If $f \in H_3$ then G_f be the set of g such that for some $(B, \eta_1, \eta_2, \bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{\nu})$ we have:

A. $B \subseteq \omega$ is infinite.

B. $\eta_1 = \mathbf{B}(f) \in 2^\omega$ and $\eta_2 \in 2^\omega$.

C. $\bar{a} = (a_n : n \in B)$.

D. If $n \in B$ then $a_n \in A'_{\eta_1 \restriction n} = \bigcup\limits_{\rho \in 2^n} A_{\eta_1 \restriction n, \rho} \subseteq u_{\eta_1 \restriction n}$ (recall that we denote $\bigcup\limits_{\rho \in 2^n} A_{\eta_1 \restriction n, \rho}$ by $A'_{\eta_1 \restriction n}$).

E. $\bar{b} = (b_n : n \in B)$ and $\bar{\nu} = (\nu_n : n \in B)$, $\nu_n \in T$, such that for each $n \in B$, $b_n = f(a_n)$ and $b_n \in u_{\nu_n}$. $\bar{c} = (c_n : n \in B)$, $\bar{d} = (d_n : n \in B)$ and $\bar{e} = (e_n : n \in B)$ are such that $b_n, c_n \in u_{\nu_n}$ and $e_n \in u_{\eta_1 \restriction n}$.

F. For every $n \in B$, $g(a_n) = f(a_n) = b_n$.

G. For every $n \in B$, $g(b_n) = F_1(f)(a_n) = e_n$.

H. For every $n \in B$ we have $c_n = F_1(f)^{-1}(f(a_n))$ and $g(c_n) = F_1(f)(f(a_n)) = d_n$.

I. If $b \in U$ is not covered by clauses F-H, then $g(b) = F_1(f)(b)$.

J. g has no fixed points.

K. One of the following holds:

a. For every $n \in B$, $\eta_1 \restriction n <_* \nu_n$, $lg(\eta_2 \cap \nu_n) > max\{lg(\nu_m) : m \in B \cap n\}$ hence $(\nu_n \cap \eta_2 : n \in B)$ is $\leq -$increasing.

b. For every $n \in B$, $\nu_n = \eta_1 \restriction n$ and $l(a_n, f(a_n), n)$ is increasing (see definition 5(e)).

c. For every $n \in B$, $\nu_n = \eta_1 \restriction n$ and in addition, $l(a_n, f(a_n), n) = l_*$ for every n, $i_{a_n, f(a_n), l} = i_l$ for $l < l_*$ and for some $l_{**} < l_*$, the elemnts of $(\rho_{a_n, f(a_n), l_{**}} : l_{**} < l_*)$ are pairwise incomparable. $\square$

Claim 7: If $f \in H_3$ then there exists $g \in Sym(U)$ such that for some $(B, \eta_1, \eta_2, \bar{a}, \bar{b}, \bar{\nu})$, g and $(B, \eta_1, \eta_2, \bar{a}, \bar{b}, \bar{\nu})$ are as required in claim 6(c)(2) (and therefore, there are also $(\bar{c}, \bar{d}, \bar{e})$ as required there). Moreover, g is unique once $(B, \eta_1, \eta_2, \bar{a}, \bar{b}, \bar{\nu})$ is fixed.

Remark 7A: In claim 9 we need g to be Borel-computable from f, which is indeed the case by the discussion in the proof of claim 5 and by the proof of claim 6 in [HwSh:1089].

Proof: $f \in H_3$, so $\mathbf{B}_{1,1}(f) = 0$.

We shall first observe that if g is defined as above, then g is a permutation of U with no fixed points. It's also easy to see that g is unique once $(B, \eta_1, \eta_2, \bar{a}, \bar{b}, \bar{\nu})$ has been chosen. Therefore, it's enough to find $(B, \eta_1, \eta_2, \bar{a}, \bar{b}, \bar{\nu})$ as required.

Case I ($\mathbf{B}_{1,2}(f) = 1$): For infinitely many n, $f''(A'_{\eta_1 \restriction n}) \nsubseteq \cup\{u_\rho : \rho \leq_* \eta_1 \restriction n\}$. In this case, let $B_0 = \{n :$ there is $a \in A'_{\eta_1 \restriction n}$ such that $f(a) \notin \cup\{u_\rho : \rho \leq_* \eta_1 \restriction n\}$, and for every $n \in B_0$, let a_n be the $<_*$-first element in $A'_{\eta_1 \restriction n}$ witnessing that $n \in B_0$. Let $b_n = f(a_n)$ and let $\nu_n \in T$ be the sequence for which $b_n \in u_{\nu_n}$. Apply Ramsey's theorem (we don't need the axiom of choice, as we can argue in some $L[A]$) to get an infinite set $B \subseteq B_0$ such that $c_{k,l} \restriction [B]^k$ is constant for every $(k, l) \in \{(2,1), (2,2), (2,4), (3,1), (3,3)\}$, where for $n_1 < n_2 < n_3$:

a) $c_{2,1}(n_1, n_2) = TV(lg(\nu_1) < lg(\nu_2))$.

b) $c_{2,2}(n_1, n_2) = TV(\nu_{n_2} \in \{\nu_n : n \leq n_1\})$.

c) $c_{3,1}(n_1, n_2, n_3) = TV(lg(\nu_{n_2} \cap \nu_{n_3}) > \nu_{n_1})$.

d) $c_{3,3}(n_1, n_2) = \nu_{n_2}(lg(\nu_{n_1} \cap \nu_{n_2})) \in \{0, 1, \text{undefined}\}$.

We shall prove now that $(lg(\nu_n) : n \in B)$ has an infinite increasing subsequence: Choose an increasing sequence $n(i) \in B$ by induction on i such that $j < i \to lg(\nu_{n(j)}) < lg(\nu_{n(i)})$. Arriving at stage $i = j + 1$, suppose that there is no such $n(i)$, then $\{f(a_n) : n \in B \setminus n(j)\} \subseteq \cup\{u_\rho : lg(\rho) \leq lg(\nu_{n(j)})\}$, hence $\{f(a_n) : n \in B \setminus n(j)\}$ is finite. Similarly, $\{\nu_n : n \in B \setminus n(j)\}$ is finite, and therefore, there are $n_1 < n_2 \in B \setminus n(j)$ such that $\nu_{n_1} = \nu_{n_2}$ and $f(a_{n_1}) = f(a_{n_2})$. As f is injective, $a_{n_1} = a_{n_2}$, and by the choice of the a_n, $a_{n_1} \in u_{\eta_1 \restriction n_1}$ and $a_{n_2} \in u_{\eta_1 \restriction n_2}$.

This is a contradiction, as $u_{\eta_1 \restriction n_1} \cap u_{\eta_1 \restriction n_2} = \emptyset$.

Therefore, there is an infinite $B' \subseteq B$ such that $(lg(\nu_n) : n \in B')$ is increasing, and wlog $B' = B$.

Now we shall note that if $n_1 < n_2 < n_3$ are from B, then $lg(\nu_{n_2} \cap \nu_{n_3}) > lg(\nu_{n_1})$:

By the choice of B, $c_{3,1}(n_1, n_2, n_3)$ is constant for $n_1 < n_2 < n_3$, so it suffices to show that $c_{3,1} \restriction [B]^3 = true$. Let $n_1 = min(B)$ and $k = lg(\nu_{n_1}) + 1$. The sequence $(\nu_n \restriction k : n \in B \setminus \{n_1\})$ is infinite, hence there are $n_2 < n_3 \in B \setminus \{n_1\}$ such that $\nu_{n_2} \restriction k = \nu_{n_3} \restriction k$. Therefore, $lg(\nu_{n_1}) < k \leq lg(\nu_{n_2} \cap \nu_{n_3})$, and as $c_{3,1}$ is constant on $[B]^3$, we're done.

For $n < k \in B$ such that k is the successor of n in B, let $\eta_n = \nu_n \cap \nu_k$. Suppose now that $n < k < l$ are successor elements in B, then $lg(\eta_k) = lg(\nu_k \cap \nu_l) > lg(\nu_n) \geq lg(\nu_n \cap \nu_k) = lg(\eta_n)$, and $\eta_n, \eta_k \leq \nu_k$, therefore, η_n is a proper initial segment of η_k and $\eta_2 := \bigcup\limits_{n<\omega} \eta_n \in 2^\omega$. If $n < k \in B$ are successor elements, then $lg(\eta_k) > lg(\nu_n)$ (by a previous claim), therefore, $\nu_n \cap (T \setminus \nu_k)$ is disjoint to η_2, hence $\nu_n \cap \eta_2 = \nu_n \cap \nu_k = \eta_n$. Therefore, if $n < k \in B$ then $\nu_n \cap \eta_2$ is a proper initial segment of $\nu_k \cap \eta_2$ and $\eta_2 = \bigcup\limits_{n<\omega} (\nu_n \cap \eta_2)$.

It's now easy to verify that $(B, \eta_1, \eta_2, \bar{a}, \bar{b}, \bar{\nu})$ and g are as required.

Case II ($\mathbf{B}_{1,2}(f) = 0$ and n_1 stands for $\mathbf{B}_{1,3}(f)$): There is n_1 such that for every $n_1 \leq n$, $f''(A'_{\eta_1 \restriction n}) \subseteq \cup\{u_\rho : \rho \leq_* \eta_1 \restriction n\}$.

For each n, recall that $v_n = \{a \in A'_{\eta_1 \restriction n} \subseteq u_{\eta_1 \restriction n} : f(a) \in u_{\eta_1 \restriction n}\}$. v_n satisfies $|A'_{\eta_1 \restriction n} \setminus v_n| \leq \Sigma\{|u_\nu| : \nu <_* \eta_1 \restriction n\}$, and as $\Sigma\{|u_\nu| : \nu <_* \eta_1 \restriction n\} \ll |A'_{\eta_1 \restriction n}|$, it follows that $\Sigma\{|u_\nu| : \nu <_* \eta_1 \restriction n\} \ll |v_n|$. Recall also that for $a \in v_n$, as $f(a) \in u_{\eta_1 \restriction n}$, by definition 5(e), $y_{a,f(a)}$ is well-defined.

We now consider three subcases:

Case IIA ($\mathbf{B}_{1,4}(f) = 1$): The set of $l_{a,f(a),n}$ for $a \in v_n$ and $n_1 \leq n$ is unbounded. In this case, we find an infinite $B \subseteq [n_1, \omega)$ and $a_n \in v_n$ for each $n \in B$ such that $(l_{a_n, f(a_n), n} : n \in B)$ is increasing. Now let $\eta_2 := \eta_1$ and define $\bar{b}$, $\bar{\nu}$ and g as described in Definition 6. It's easy to verify that $(B, \eta_1, \eta_2, \bar{a}, \bar{b}, \bar{\nu})$ are as required.

Case IIB ($\mathbf{B}_{1,4}(f) = 0$ and $\mathbf{B}_{1,6}(f) = 1$): Case IIA doesn't hold, but $\mathbf{B}_{1,6}(f) = 1$ and there is an infinite $B \subseteq [n_1, \omega)$, $l_{**} < l_*$ (see below) and $(a_n \in v_n : n \in B)$ (given by $\mathbf{B}_{1,8}(f)$, $\mathbf{B}_{2,0}(f)$ and $\mathbf{B}_{1,7}(f)$, respectively) such that:

a. $l_{a_n, f(a_n)} = l_*$ **and** $l_{**} = \mathbf{B}_{2,0}(f) < l_*$**.**

b. $i_{a_n, f(a_n), l} = i^*_l$ **for** $l < l_*$**.**

c. If $n \in B$ **and** $m \in B \cap n$**, then** $\rho_{a_m, f(a_m), l_{**}} \not\subseteq \rho_{a_n, f(a_n), l_{**}}$**.**

In this case we define $\bar{b}$, $\bar{\nu}$ and g as in Definition 6 and we let $\eta_2 := \eta_1$. It's easy to see that $(B, \eta_1, \eta_2, \bar{a}, \bar{b}\bar{\nu})$ are as required.

Remark: By a routine Ramsey-type argument, it's easy to prove that if $\mathbf{B}_{1,6}(f) = 1$ then the values of $\mathbf{B}_{1,7}(f), \mathbf{B}_{2,0}(f)$ are well-defined and Borel-computable so the above conitions hold.

Case IIC ($\mathbf{B}_{1,4}(f) = \mathbf{B}_{1,6}(f) = 0$): $\neg IIA \wedge \neg IIB$. We shall first prove that $\mathbf{B}_{2,1}(f)$, $\mathbf{B}_{2,2}(f)$, $\mathbf{B}_{2,3}(f)$, $(\mathbf{B}_{2,4,n}(f) : n \in \mathbf{B}_{2,2}(f))$ and $(\mathbf{B}_{2,5,l}(f) : l < \mathbf{B}_{2,3}(f))$ are well-define and Borel computable.

Let $l(*)$ be the supremum of the $l(a, f(a))$ where $n_1 \leq n$ and $a \in v_n$ ($l(*) < \omega$ by $\neg 2A$). We can find $l(**) \leq l(*)$ such that $B_1 := \{n \in B : v_{n,1} = \{a \in$

$v_n : l(a, f(a)) = l(**)\}$ has at least $\frac{v_n}{l(*)}$ elements$\}$ is infinite. Next, we can find $i_*(l) \in \{1, -1\}$ for $l < l(**)$ such that $B_2 : \{n \in B_1 : v_{n,2} = \{a \in v_{n,1} : \bigwedge_{l<l(**)} i_{a,f(a),l} = i_*(l)\}$ has at least $\frac{|v_n|}{2^{l(**)}l(*)}$ elements$\}$ is infinite. For each $n \in B_2$, there are $\rho_{n,0}, ..., \rho_{n,l(**)-1} \in T_n$ such that $v_{n,3} = \{a \in v_{n,2} : \bigwedge_{l<l(**)} \rho_{a,f(a),l} = \rho_{n,l}\}$ has at least $\frac{|v_n|}{l(**)2^{l(**)}2^{nl(**)}}$ elements. By Ramsey's theorem, there is an infinite subset $B_3 \subseteq B_2$ such that for each $l < l(**)$, the sequence $(TV(\rho_{m,l} \leq \rho_{n,l}) : m < n \in B_3)$ is constant. Therefore, we're done showing that the above Borel functions are well-defined.

Now if $\mathbf{B}'_{2,6}(f) = 1$ then we finish as in the previous case (this time we're in the situation of 6(b)(2)(K)(c)). If $\mathbf{B}'_{2,6}(f) = 0$, then we get a contradiction to the assumption that $f \in H_3$, therefore we're done. $\square$

Claim 8: If $w(x_0, ..., x_{k_*-1})$ is a reduced non-trivial group word, $f_0, ..., f_{k_{**}-1} \in H_3$ are pairwise distinct, $g_l \in G_{f_l}$ $(l \in \{0, ..., k_{**} - 1\})$, $g_l = g^*_{\nu_l}$ where $\nu_l \in 2^\omega \setminus \{\mathbf{B}(f) : f \in H_3\}$, $l = k_{**}, ..., k_* - 1$ and $(\nu_l : k_{**} \leq l < k_*)$ is without repetition, then $w(g_0, ...g_{k_*-1}) \in Sym(U)$ has a finite number of fixed points.

Notation: For $l < k_{**}$, let $\nu_l := \mathbf{B}(f_l)$.

Proof: Assume towards contradiction that $w(x_0, ..., x_{k_*-1}) = x^{i(m-1)}_{k(m-1)} \cdot ... \cdot x^{i(0)}_{k(0)}$, $\{f_0, ..., f_{k_*-1}\}$ and $\{g_0, ..., g_{k_*-1}\}$ form a counterexample, where $i(l) \in \{-1, 1\}$, $k(l) < k_*$ and $k(l) = k(l+1) \to \neg(i(l) = -i(l+1))$ for every $l < m$. WLOG $m = lg(w)$ is minimal among the various countrexamples. Let $C = \{a \in U : w(g_0, ..., g_{k_*-1})(a) = a\}$, this set is infinite by our present assumption. For $c \in C$, define $b_{c,l}$ by induction on $l < m$ as follows:

1. $b_{c,0} = c$.
2. $b_{c,l+1} = g^{i(l)}_{k(l)}(b_{c,l})$.

Notational warning: The letter c with additional indices will be used to denote the elements of sequences of the form $\bar{c}$ from Definition 6(b)(2).

For all but finitely many $c \in C$, $(b_{c,l} : l < m)$ is without repetition by the minimality of m, so wlog this is true for every $c \in C$.

For every $c \in C$, let $\rho_{c,l} \in T$ be such that $b_{c,l} \in u_{\rho_{c,l}}$, and let $l_1[c]$ be such that $\rho_{c,l_1[c]} \leq_* \rho_{c,l}$ for every $l < m$. We can choose $l_1[c]$ such that one of the following holds:

1. $l_1[c] > 0$ and $\rho_{c,l_1[c]-1} \neq \rho_{c,l_1[c]}$
2. $l_1[c] = 0$ and $\rho_{c,m-1} \neq \rho_{c,0}$
3. $\rho_{c,0} = ... = \rho_{c,m-1}$

We may assume wlog that $(l_1[c] : c \in C)$ is constant and that actually $l_1[c] = 0$ for every $c \in C$. In order to see that we can assume the second part, for $j < m$ let $w_j(x_0, ..., x_{k_*-1}) = x^{i(j-1)}_{k(j-1)} \cdots x^{i(0)}_{k(0)} x^{i(m-1)}_{k(m-1)} \cdots x^{i(j)}_{k(j)}$, then $w_j(g_0, ..., g_{k_*-1}) \in Sym(U)$ is a conjugate of $w(g_0, ..., g_{k_*-1})$. The set of fixed points of $w_j(g_0, ..., g_{k_*-1})$ includes $\{b_{c,j} : c \in C\}$, and therefore it's infinite. For $c \in C$, $(b_{c,j}, b_{c,j+1}, ..., b_{c,m-1}, b_{c,0}, ..., b_{c,j-1})$ and $w_j(g_0, ..., g_{k_*-1})$ satisfy the same properties that $(b_{c,0}, ..., b_{c,m-1})$ and $w(g_0, ..., g_{k_*-1})$ satisfy. Therefore, if $(l_1[c] : c \in C)$ is constantly $j > 0$, then by conjugating and moving to $w_j(g_0, ..., g_{k_*-1})$, we may assume that $(l_1[c] : c \in C)$ is contantly 0.

Let $l_2[c] < m$ be the maximal such that $\rho_{c,0} = ... = \rho_{c,l_2[c]}$, so wlog $l_2[c] = l_*$ for every $c \in C$. For $l < k_*$ let $\eta_{1,l}$ be $\mathbf{B}(f_l)$ if $l < k_{**}$ and ν_l if $l \in [k_{**}, k_*)$ (we might also denote it by ρ_l in this case). As $f_l \in H_3$ for $l <_{k_**}$, and $\rho_l \notin \{\mathbf{B}(f) : f \in H_3\}$

for $l \in [k_{**}, k_*)$, it follows that $l_1 < k_{**} \leq l_2 < k_* \to \eta_{1,l_1} \neq \eta_{1,l_2}$. Therefore, $(\eta_{1,l} : l < k_*)$ is without repetition.

Now let $\eta_{2,l}$ be defined as follows:

1. If $l < k_{**}$, let $\eta_{2,l}$ be η_2 from definition 6(b)(2) for f_l and g_l.

2. If $l \in [k_{**}, k_*)$, let $\eta_{2,l} = \eta_{1,l}$.

Let $j(*) < \omega$ be such that:

a. $(\eta_{1,l} \restriction j(*) : l < k_*)$ is without repetition.

b. If $\eta_{1,l_1} \neq \eta_{2,l_2}$ then $\eta_{1,l_1} \restriction j(*) \neq \eta_{2,l_2} \restriction j(*)$ $(l_1, l_2 < k_*)$.

c. If $\eta_{2,l_1} \neq \eta_{2,l_2}$ then $\eta_{2,l_1} \restriction j(*) \neq \eta_{2,l_2} \restriction j(*)$ $(l_1, l_2 < k_*)$.

d. $j(*) > 3m, k_*$.

e. $j(*) > n(l_1, l_2)$ for every $l_1 < l_2 < k_*$, where $n(l_1, l_2)$ is defined as follows:

1. If $k_{**} \leq l_1, l_2$, let $n(l_1, l_2) = 0$.

2. If $l_1 < k_{**}$ or $l_2 < k_{**}$, let $(\nu_n^1 : n \in B_1)$ and $(\nu_n^2 : n \in B_2)$ be as in definition 6(b)(2) for (f_{l_1}, η_{1,l_1}) and (f_{l_2}, η_{1,l_2}), respectively. If there is no ν_n^1 such that $\nu_n^1 \nleq \eta_{1,l_1}$ and no ν_n^2 such that $\nu_n^2 \nleq \eta_{1,l_2}$, let $n(l_1, l_2) = 0$. Otherwise, there is at most one $n \in B_1$ such that $\nu_n^1 \nleq \eta_{1,l_1}$ and $\nu_n^1 \leq \eta_{1,l_2}$ and there is at most one $m \in B_2$ such that $\nu_m^2 \nleq \eta_{1,l_2}$ and $\nu_m^2 \leq \eta_{1,l_1}$. If there are ν_n^1 and ν_m^2 as above, let $n(l_1, l_2) = lg(\nu_n^1) + lg(\nu_m^2) + 1$. If there is ν_n^1 as above but no ν_m^2 as above, let $n(l_1, l_2) = lg(\nu_n^1) + 1$, and similarly for the dual case.

f. $j(*) > m(l_1, l_2)$ for every $l_1 < l_2 < k_{**}$ where $m(l_1, l_2)$ is defined as follows: Let $(\nu_n^1 : n \in B_1)$ and $(\nu_m^2 : m \in B_2)$ be as in definition 6(b)(2) for (f_{l_1}, η_{1,l_1}) and (f_{l_2}, η_{1,l_2}), respectively. As $\eta_{1,l_1} \neq \eta_{1,l_2}$, $|\{\nu_n^1 : n \in B_1\} \cap \{\nu_m^2 : m \in B_2\}| < \aleph_0$, let $s(l_1, l_2)$ be the supremum of the length of members in this intersection and let $m(l_1, l_2) := s(l_1, l_2) + 1$.

We may assume wlog that $lg(\rho_{c,l_1[c]}) > j(*)$ for every $c \in C$. We now consider two possible cases (wlog $TV((\rho_{c,l} : l < m)$ is constant) is the same for all $c \in C$):

Case I: For every $c \in C$, $(\rho_{c,l} : l < m)$ is not constant.

In this case, for each such $c \in C$, $l_2[c] < m - 1$ and $b_{c,l_2[c]} \in u_{\rho_{c,0}} = ... = u_{\rho_{c,l_2[c]}}$. Now $(b_{c,l_2[c]}, b_{c,l_2[c]+1}) \in g_{k(l_2[c])}^{i(l_2[c])}$, and as $\rho_{c,l_2[c]} \neq \rho_{c,l_2[c]+1}$, necessarily $k(l_2[c]) < k_{**}$. By the definition of $l_1[c]$ and the fact that $\rho_{c,l_1[c]} = \rho_{c,l_2[c]}$, necessarily $\rho_{c,l_2[c]} <_* \rho_{c,l_2[c]+1}$.

For each $l < m-1$, if $lg(\rho_{c,l}) < lg(\rho_{c,l+1})$, then either $g_{k(l)}$ or $g_{k(l)}^{-1}$ is as in definition 6(2)(b), so letting $n = lg(\rho_{c,l})$, $(\rho_{c,l}, \rho_{c,l+1})$ here correspond to $(\eta_1 \restriction n, \nu_n)$ there, and there are $(a^l, b^l, c^l, d^l, e^l) = (a_c^l, b_c^l, c_c^l, d_c^l, e_c^l)$ in our case that correspond to $(a_n, b_n, c_n, d_n, e_n)$ in 6(2)(b). In the rest of the proof we shall denote those sequences by $(a^l, b^l, c^l, d^l, e^l)$, as the identity of the relevant $c \in C$ should be clear. In addition, one of the following holds:

1. $i(l) = 1$ and $g_{k(l)}^{i(l)}(a^l) = b^l$.

2. $i(l) = -1$ and $g_{k(l)}^{i(l)}(b^l) = a^l$.

Similarly, for $l < m - 1$, if $lg(\rho_{c,l}) > lg(\rho_{c,l+1})$ then the above is true modulo the fact that now $(\rho_{c,l}, \rho_{c,l+1})$ correspond to $(\nu_n, \eta_1 \restriction n)$ and one of the following holds:

1. $i(l) = 1$ and $g_{k(l)}^{i(l)}(b^l) = e^l$.

2. $i(l) = -1$ and $g_{k(l)}^{i(l)}(b^l) = a^l$.

Therefore, if $l = l_2[c]$ then $lg(\rho_{c,l}) < lg(\rho_{c,l+1})$, so the first option above holds, and therefore $\rho_{c,l}$ is an initial segment of $\eta_{1,k(l)}$.

If $l = m-1$, then $lg(\rho_{c,m-1}) > lg(\rho_{c,0}) = lg(\rho_{c,m})$ and therefore $\rho_{c,0}$ is an initial segment of $\eta_{1,k(m-1)}$. It follows that $\rho_{c,0} = \rho_{c,l_2[c]}$ is an initial segment of $\eta_{1,k(l_2[c])} \cap \eta_{1,k(m-1)}$. Recalling that $lg(\rho_{c,0}) = lg(\rho_{c,l_1[c]}) > j(*)$ and that $(\eta_{1,l} \restriction j(*) : l < k_*)$ is without repetition, it follows that $k(m-1) = k(l_2[c])$.

We shall now prove that if $l_2[c] < m-1$ then $l_2[c] = m-2$. Let $(a^{l_2[c]}, b^{l_2[c]}, ...)$ be as above for $l = l_2[c]$, so $b_{c,l_2[c]+1} = b^{l_2[c]}$, and as $k(l_2[c]) = k(m-1)$, we get $b_{c,l_2[c]+1} = b^{l_2[c]} = b^{m-1}$. In order to show that $l_2[c] = m-2$, it suffices to show that $b^{m-1} = b_{c,m-1}$ (as the sequence of the $b_{c,l}$s is without repetition), which follows from the fact that $\rho_{c,0} <_* \rho_{c,m-1}$.

As we assume that the word w is reduced, and as $k(m-2) = k(l_2[c]) = k(m-1)$, necessarily $i(m-2) = i(m-1)$. We may assume wlog that $i(m-2) = i(m-1) = 1$ (the proof for $i(m-2) = i(m-1) = -1$ is similar, as we can replace w by a conjugate of its inverse).

Let $w' = w'(g_0, ..., g_{k_*-1}) := g_{k(m-3)}^{i(m-3)} \cdots g_{k(0)}^{i(0)}$, by the above considerations and as $b^{m-1} = b_{c,m-1}$, it follows that $e^{m-1} = g_{k(m-1)}(b^{m-1}) = g_{k(m-1)}^{i(m-1)}(b^{m-1}) = g_{k(m-1)}^{i(m-1)}(b_{c,m-1}) = b_{c,0}$. We also know that $g_{k(m-2)}^{i(m-2)}(w'(b_{c,0})) = g_{k(m-2)}(w'(b_{c,0}))$ is "higher" than $w'(b_{c,0})$. Therefore, $g_{k(m-2)}^{i(m-2)}(w'(b_{c,0})) = g_{k(m-2)}(w'(b_{c,0})) = b^{m-2} = b^{m-1}$. It also follows that $w'(b_{c,0}) = a^{m-2} = a^{m-1}$. Therefore, $w'(e^{m-1}) = a^{m-1}$.

We shall now prove that if $l < m-2$ then $b_{c,l+1} = (g^*_{\nu_{k(l)}})^{i(l)}(b_{c,l})$. Assume that for some $l < m-2$, $g_{k(l)}(b_{c,l}) \neq g^*_{\nu_{k(l)}}(b_{c,l})$ and we shall derive a contradiction. Let $(a^{m-2}, b^{m-2}, ...)$ be as before for $g_{k(m-2)}$, so $(b_{c,m-2}, b_{c,0}) = (a^{m-2}, e^{m-2})$.

Case I (a): $k(l) = k(m-2) = k(m-1)$. As the $b_{c,i}$s are without repetition, if $0 < l < m-2$, then $b_{c,l} \notin \{b_{c,m-2}, b_{c,0}\} = \{a^{m-2}, e^{m-2}\} = \{a^l, e^l\}$, and of course, $b_{c,l} \notin \{b^l, c^l, d^l\}$ (as it is a "lower" element). Therefore, $g_{k(l)}(b_{c,l}) = g^*_{\nu_{k(l)}}(b_{c,l})$, a contradiction. If $l = 0$, then $g_{k(m-1)}^{i(m-1)}(b_{c,m-1}) = b_{c,0}$ and $(b_{c,m-1}, b_{c,0}) = (b^{m-1}, e^{m-1})$. If $i(0) = -i(m-1)$, then by conjugating $g_{k(0)}$, we get a shorter word with infinitely many fixed points, contradicting our assumption on the minimality of m.

If $i(0) = i(m-1) = 1$, then we derive a contradiction as in the case of $0 < l$.

Case I (b): $k(l) \neq k(m-2) = k(m-1)$. In this case, we know that $g_{k(l)}$ almost coincides with $g^*_{\nu_{k(l)}}$, with the exception of at most $\{a^l, b^l, c^l, d^l, e^l\}$. Let $\rho_c := \rho_{c,0} = ... = \rho_{c,m-2}$, then necessarily $\rho_c \leq \nu_{k(m-1)} = \eta_{1,k(m-1)}$ (as $g_{k(m-1)}$ moves $b_{c,m-1}$ to a lower u_ρ (namely u_{ρ_c}), ρ_c plays the role of $\eta_1 \restriction n$ in Definition 6 for $g_{k(m-1)}$). By our assumption, $lg(\rho_c) > j(*)$ and $(\eta_{1,l} \restriction j(*) : l < k(*))$ is without repetition, therefore $\eta_{1,k(l)} \restriction lg(\rho_c) \neq \rho_c$, so $\rho_c \not\leq \eta_{1,k(l)}$. Therefore, when we consider $f_{k(l)}$ and $\eta_{1,k(l)}$ in definition $6(b)(2)$, then ρ_c has the form ν_n for some n. By the choice of $j(*)$, it's then impossible to have $\rho_c \leq \eta_{1,k(m-1)}$, a contradiction.

Therefore, $a^{m-2} = w'(g_0, ..., g_{k_*-1})(e^{m-2}) = w'(g_0, ..., g_{k_*-1})(b_{c,0}) = w'(g^*_{\nu_0}, ..., g^*_{\nu_{k_*-1}})(b_{c,0}) = w'(g^*_{\nu_0}, ..., g^*_{\nu_{k_*-1}})(e^{m-2})$. In the notation of the claim and definition 6(b)(2), $F_1(f_{k(m-2)})(a^{m-2}) = e^{m-2}$, therefore, by composing with w', we obtain a word composed of permutation of $u_{\rho_{c,m-2}}$ (in the sense of claim 4(f)) that fixes $e^{m-2} \in u_{\rho_{c,m-2}}$, therefore, $m-3 = 0$ (or else we get a contradiction by claim 4(f)).

It follows that $w(g_0,...,g_{k_*-1}) = g_{k(m-2)}g_{k(m-2)}g_{k(0)}^{i(0)}$ and $g_{k(0)}^{i(0)}(e^{m-2}) = a^{m-2}$. Now, obviously $\rho_{c,m-2} \leq \eta_{1,k(m-2)}$, so $\rho_{c,m-2} \nleq \eta_{1,k(0)} = \nu_{k(0)}$. By the definition, $g^*_{\nu_{k(0)}} \restriction u_{\rho_{c,m-2}} = f_{\rho_{c,m-2},\nu_{k(0)}\restriction lg(\rho_{c,m-2})} \neq f_{\rho_{c,m-2},\rho_{c,m-2}}$. Also $F_1(f_{k(m-2)}) \restriction u_{\rho_{c,m-2}} = g^*_{\nu_{k(m-2)}} \restriction u_{\rho_{c,m-2}} = f_{\rho_{c,m-2},\rho_{c,m-2}}$. Therefore, we get the following: $a^{m-2} = g_{k(0)}^{i(0)}(e^{m-2}) = (g^*_{\nu_{k(0)}})^{i(0)}(e^{m-2}) = (f_{\rho_{c,m-2},\nu_{k(0)}\restriction lg(\rho_{c,m-2})})^{i(0)}(e^{m-2})$ and $e^{m-2} = F_1(f_{k(m-2)})(a^{m-2}) = f_{\rho_{c,m-2},\rho_{c,m-2}}(a^{m-2})$. In conclusion, we get a contrdiction to claim 4(f), as we have a short non-trivial word that fixes e^{m-2}.

Case II: $(\rho_{c,l} : l < m)$ **is constant for every** $c \in C$ **(so** $l_2[c] = m-1$**).** Let $\rho_c := \rho_{c,0} = ... = \rho_{c,m-1}$. If $g_{k(l)}^{i(l)}(b_{c,l}) = (g^*_{\nu_{k(l)}})^{i(l)}(b_{c,l})$ for every $l < m$, then we get a contradiction to claim 4(f). Therefore, for every $c \in C$, the set $v_c = \{l < m : g_{k(l)}^{i(l)}(b_{c,l}) \neq (g^*_{\nu_{k(l)}})^{i(l)}(b_{c,l})\}$ is nonempty. Without loss of generality, v_c doesn't depend on c, and we shall denote it by v. For every $l \in v$, if $i(l) = 1$ then $(b_{c,l}, b_{c,l+1}) \in \{(a^l, b^l), (b^l, e^l), (c^l, d^l)\}$, if $i(l) = -1$ then $(b_{c,l}, b_{c,l+1}) \in \{(b^l, a^l), (e^l, b^l), (d^l, c^l)\}$.

We shall now prove that for some $k < k_{**}$, $k(l) = k$ for every $l \in v$. Suppose not, then for some $l_1 < l_2 \in v$, $k(l_1) \neq k(l_2)$. By the choice of $j(*)$, each of the following options in impossible: $\rho_{c,l_1} \leq \eta_{1,l_1} \wedge \rho_{c,l_2} \leq \eta_{1,l_2}$, $\rho_{c,l_1} \leq \eta_{1,l_1} \wedge \rho_{c,l_2} \nleq \eta_{1,l_2}$, $\rho_{c,l_1} \nleq \eta_{1,l_1} \wedge \rho_{c,l_2} \leq \eta_{1,l_2}$ or $\rho_{c,l_1} \nleq \eta_{1,l_1} \wedge \rho_{c,l_2} \nleq \eta_{1,l_2}$. Therefore we get a contradiction. It follows that $\{k(l) : l \in v\}$ is singelton, and we shall denote its only member by $k < k_{**}$.

Note that if $l_1 \in v$, $l_2 \in v$ is the successor of l_1 in v, $l_1 + 1 < l_2$ and $c \in C$ then $b_{c,l_1+1} \neq b_{c,l_2}$ (recall that $(b_{c,l} : l < m)$ is withut repetition). We shall now arrive at a contradiction by examining the following three possible cases (in the rest of the proof, we refer to $l(*)$ from Definition 5(A)(e) as "the distance between a and b", and similarly for any pair of members from some u_η):

Case II (a): g_k is as in definition 6(b)(2)(K)(a). In this case, for every $l \in v$, the only possibilities for $(b_{c,l}, b_{c,l+1})$ are either of the form (c, d) or (d, c) (and not both, as we don't allow repetition). As the distance between c and d is at most 2, we get a word made of $f_{\rho,\nu}$s of length $\leq m+1$ that fixes c, contradicting claim 4(f).

Case II (b): g_k is as in definition 6(b)(2)(K)(c). Pick $c \in C$ such that $lg(\rho_c)$ is alseo greater than $m + l_*$ where l_* is as in definition 6(b)(2)(K)(c) for g_k. As the sequence $(b_{c,l} : l < m)$ is without repetition, necessarily $1 \leq |v| \leq 3$.

If $|v| = 3$, then necessarily the sequences (a, b, e) or (e, b, a) occur in $(b_{c,l} : l < m)$, as well as (c, d) or (d, c). As the distance between a and e is 1 and the distance between c and d is ≤ 2, we get a contradiction as before.

Suppose that $|v| = 2$. If the sequence (a, b, e) appears in $(b_{c,l} : l < m)$, we get a contradiction as above. If (a, e) (or (e, a)) and (c, d) (or (d, c)) appear, we also get a contradiction as above. If $(a, b)/(b, a)$ and $(c, d)/(d, c)$ appear, as the distance between a and b is l_*, we get a word made of $f_{\rho,\nu}$s of length $\leq m + l(*)$ fixing c, a contradiction to claim 4(f). Finally, if $|v| = 1$ we get a contradiction similarly.

Case II (c): g_k is as in definition 6(b)(2)(K)(b). As in the previous case, where the only non-trivial difference is when either $|v| \in \{1, 2\}$ and the sequence $(a, b)/(b, a)$ appears in $(b_{c,l} : l < m)$, but not as a subsequence of $(a, b, e)/(e, b, a)$. If for some c this is not the case, then we finish as before, so suppose that it's the case for every $c \in C$. As the distance between c and d is ≤ 2, suppose wlog that $|v| = 1$, $k = k(m-1)$ (by conjugating) and the sequence $(b_{c,l} : l < m)$ ends with a and starts with b or vice versa. Therefore, every $c \in C$ is of the form a^n or b^n (where $n \in B$ and B is as in definition 6(b)(2) for g_k) and either $g_{k(0)}^{i(0)} \cdots g_{k_{m-2}}^{i(m-2)}(a^n) = b^n$

or $g_{k(0)}^{i(0)} \cdots g_{k_{m-2}}^{i(m-2)}(b^n) = a^n$, so the distance between a^n and b^n is $\leq m-1$. As C is infinite, the distance between a^n and b^n is $\leq m-1$ for infinitely many $n \in B$. This is a contradiction to the assumption from definition 6(b)(2)(K)(b) that the distance between a^n and b^n is increasing.

This completes the proof of claim 8. $\square$

Claim 9: There exists a Borel function $\mathbf{B}_4 : U^U \to U^U$ such that for every $f \in \mathbf{B}_1$, $\mathbf{B}_4(f) \in G_f$.

Proof: As in [HwSh:1089], and we comment on the main point in the proof of claim 7. $\square$

Definition 10: Let G be the subgroup of $Sym(U)$ generated by $\{\mathbf{B}_4(f) : f \in H_3\} \cup \{g^*_\nu : \nu \in 2^\omega \setminus \{\mathbf{B}(f) : f \in H_3\}\}$.

Claim 11: G is a maximal cofinitary group.

Proof: G is cofinitary by claim 8, so it's enough to prove maximality. Assume towards contradiction that H is a counterexample and let $f_* \in H \backslash G$, so $\mathbf{B}_4(f_*) \in G$, and we shall denote $f^* = \mathbf{B}_4(f_*)$.

Case I: $f_* \in H_3$. In this case, by Definition 6, $\{a_n : n \in B\} \subseteq eq(f_*, f^*) := X$ (see thee relevant notation in definition 6), hence it's infinite. Therefore, $f_*^{-1} f^* \restriction X$ is the ientity, but $f_*^{-1} f^* \in H$ and H is cofinitary, therefore $f_*^{-1} f^* = Id$ so $f_* = f^* \in G$, a contradiction.

Case II: $f_* \notin H_3$. By the definition of H_3, $\mathbf{B}'_{2,6}(f_*) = 0$, so the sequences $\mathbf{B}_{2,1}(f_*) = (A_n = A_n(f_*) : n \in \mathbf{B}_{2,2}(f_*))$, $\bar{\rho_*^n} = (\rho_{n,i} : i < l_*) = (\rho_{a,f_*(a),i} : i < l_*)$ and $\bar{i} = (i_l : l < l_*) = (i_{a,f(a),l} : l < l_*)$ $(n \in \mathbf{B}_{2,2}(f_*), a \in A_n)$ are well-defined, and for every $l < l_*$, $(\rho_{n,l} : n \in \mathbf{B}_{2,2}(f_*))$ is $\leq$-increasing, so $\nu_l := \bigcup\limits_{n \in \mathbf{B}_{2,2}(f_*)} \rho_{n,l} \in 2^\omega$ is well-defined. Let $g = (g^*_{\nu_0})^{i_0} \cdots (g^*_{\nu_{l_*-1}})^{i_{l_*-1}} \in G_1$(we may assume that it's a reduced product). Let $w_1 = \{l < l_* : (\exists f_l \in H_3)(\nu_l = \mathbf{B}(f_l))\}$ and $w_2 = l_* \setminus w_1$. For $l < l_*$, define g_l as follows:

1. If $l \in w_1$, let $g_l = \mathbf{B}_4(f_l)$.

2. If $l \in w_2$, let $g_l = g^*_{\nu_l}$.

Let $g' = g_0^{i(0)} \cdots g_{l_*-1}^{i(l_*-1)}$. By the definition of G, $g_0, ..., g_{l_*-1} \in G$, hence $g' \in G$.

Again by Definition 6, if $l \in w_1$ then $g_l = F_1(f_l)$ mod I_1 and $g_l^{-1} = F_1(f_l)^{-1}$ mod I_1. Now suppose that $g(a) \neq g'(a)$, then there is a minimal $l < l_*$ such that $(g^*_{\nu_0})^{i(0)} \cdots (g^*_{\nu_l})^{i(l)}(a) \neq (g_0)^{i(0)} \cdots (g_l)^{i(l)}(a)$. Let $v_l = dif(g^*_{\nu_l}, g_l)$, then $a \in (g_0^{i0} \cdots g_{l-1}^{i(l-1)})^{-1}(v_l)$. In order to show that $(g_0^{i0} \cdots g_{l-1}^{i(l-1)})^{-1}(v_l) \in I_1$ it suffices to observe that for $i \in w_1$, functions of the form g_i, g_i^{-1} map elements of I_1 to elements of I_1, therefore it follow that $g = g'$ mod I_1. It suffices to show that $eq(f_*, g) \notin I_1$, as it will then follow that $eq(f_*, g') \notin I_1$, so $f_*^{-1} g' = Id$ on an I_1-positive set, hence on an infinite set. As $f_*^{-1} g' \in H$ and H is cofinitary, $f_*^{-1} g' = Id$, an therefore $f_* = g' \in G$, a contradiction.

So let $n \in \mathbf{B}_{2,2}(f_*)$ and $a \in A_n = A_n(f_*)$, and observe that $f_*(a) = g(a)$. Indeed, by the definition of $\mathbf{B}_{2,1}(f_*)$, for every such a, $f_*(a) = ((f^{i_0}_{\eta_1 \restriction n \rho_{n,0}}) \cdots (f^{i_{l_*-1}}_{\eta_1 \restriction n \rho_{n,l_*-1}}))(a)$ (where η_1 is as in the definition of $\mathbf{B}_{2,1}(f_*)$). It's now easy to verify that the last expression equals $g(a)$. It's also easy to verify that $\bigcup\limits_{n \in \mathbf{B}_{2,2}(f_*)} A_n \notin I_1$, therefore we're done.

$\square$

Claim 12: G is Borel.

Proof: It suffices to prove the following subclaim:

Subclaim: There exists a Borel function $\mathbf{B}_5$ with domain $Sym(U)$ such that if $g \in G$ then $\mathbf{B}_5(g) = (g_0, g_1, ..., g_m)$ such that $G \models$ "$g = g_0^{i_0} g_1^{i_1} \cdots g_m^{i_m}$" for some $(i_0, ..., i_m) \in \{-1, 1\}^{m+1}$.

Proof: By the definition of G, if $g \in G$ then there are m, $f_0, ..., f_m \in \mathbf{A}_1$ (possibly with repetition) and $i_0, ..., i_m \in \{-1, 1\}$ such that $g = g_0^{i_0} \cdots g_m^{i_m}$ where each g_i is either of the form $\mathbf{B}_4(f_i)$ for $f_i \in H_3$ (in this case, let $\nu_i := \mathbf{B}(f_i)$) or $g^*_{\nu_i}$ for $\nu_i \in 2^\omega \setminus \{\mathbf{B}(f) : f \in H_3\}$.

Now if n is greater than $m!$, then for some $u \subseteq 2^n$ such that $|u| \leq m! < \frac{2^n}{2}$, for every $\rho \in 2^n \setminus u$ we have:

a. For every $l \leq m$, $g_l \restriction u_\rho = f_{\rho, \nu_l \restriction lg(\rho)}$.

b. $g \restriction u_\rho$ can be represented as $f^{i_0}_{\rho, \nu_0 \restriction lg(\rho)} \cdots f^{i_m}_{\rho, \nu_m \restriction lg(\rho)} \in Sym(u_\rho)$.

d. By claim 4(f), the above representation of $g \restriction u_\rho$ is unique.

Therefore, from g we can Borel-compute $((\nu_i \restriction n : n < \omega) : i < m)$ hence $(\nu_i : i < m)$.

As H_3 is Borel and $\mathbf{B}$ is injective, the sets $\{\mathbf{B}(f) : f \in H_3\}$ and $2^\omega \setminus \{\mathbf{B}(f) : f \in H_3\}$ are Borel. Now if $\nu_i \in 2^\omega \setminus \{\mathbf{B}(f) : f \in H_3\}$, we can Borel compute $g_i = g^*_{\nu_i}$. If $\nu_i \in \{\mathbf{B}(f) : f \in H_3\}$, then $\nu_i = \mathbf{B}(f_i)$ and we can Borel-compute f_i (by applying $\mathbf{B}_{-1}$ from definition 3(e)) hence $\mathbf{B}_4(f_i)$.

□

References

[HwSh1089] Haim Horowitz and Saharon Shelah, A Borel maximal eventuallly different family, arXiv:1605.07123.

[FFT] Vera Fischer, Sy David Friedman, Asger Toernquist, Definable maximal cofinitary groups, arXiv:1603.02942.

[Ka] Bart Kastermans, The complexity of maximal cofinitary groups, Proc. of the Amer. Math. Soc., Vol. 137(1), 2009, 307-316

[Ma] A. R. D. Mathias, Happy families, Ann. Math. Logic **12** (1977), no. 1, 59-111. MR 0491197.

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