

Mad families and non-meager filters

Haim Horowitz and Saharon Shelah

Abstract

We prove the consistency of $ZF + DC +$ "there are no mad families" + "there exists a non-meager filter on ω " relative to ZFC , answering a question of Neeman and Norwood.

We also introduce a weaker version of madness, and we strengthen the result from [HwSh:1090] by showing that no such families exist in our model.¹

Introduction

This paper is a continuation of [HwSh:1090], which is part of the ongoing effort to investigate the possible non-existence and definability of mad families. In [HwSh:1090] we proved that $ZF+DC+$ "there are no mad families" is equiconsistent with ZFC (previous results by Mathias and Toernquist established the consistency of that statement relative to large cardinals, see [Ma1] and [To]). In this paper we extend our results from [HwSh:1090] to address the following question by Neeman and Norwood:

Question ([NN]): If there are no mad families, does it follow that every filter is meager?

By a result of Mathias ([Ma2]), if every set of reals has the Ramsey property, then every filter is meager.

We shall construct a model of $ZF+DC$ where there are no mad families, but there is a non-meager filter on ω . Our proof relies heavily on [HwSh:1090], the main change is that now we're dealing with a class K_2 consisting of pairs $(\mathbb{P}, \mathcal{A})$ such that \mathbb{P} is ccc and forces MA_{\aleph_1} , and in addition, \mathbb{P} forces that \mathcal{A} is independent (we shall require more, see definition 2). In order to imitate the proof from [HwSh:1090], we need to prove analogous amalgamation results for an appropriate subclass of K_2 . As in [HwSh:1090], our final model is obtained by forcing with \mathbb{P} where $(\mathbb{P}, \mathcal{A})$ is a "very large" object in a subclass of K_2 , and the non-meager filter will be constructed from \mathcal{A} , which should contain many Cohen reals.

Finally, we consider the notion of nearly mad families (see definition 14), which was also introduced in [NN]. We introduce the notion of a somewhat mad family, which includes both mad and nearly mad families, and we prove that no somewhat mad families exist in our model.

Remark: It was brought to our attention by Paul Larson and Jindra Zapletal that a model of "there are no mad families and there is a non-meager filter on ω " can also be constructed using the arguments from their paper [LZ]. However, they assume the existence of a proper class of Woodin cardinals, while in this paper we obtain the result from ZFC without large cardinals.

A non-meager filter without mad families

¹Date: January 11, 2017

2000 Mathematics Subject Classification: 03E35, 03E15, 03E25

Keywords: mad families, non-meager filters, amalgamation

Publication 1103 of the second author

Partially supported by European Research Council grant 338821.

Hypothesis 1: We fix μ and λ such that $\aleph_2 \leq \mu$, $\lambda = \lambda^{<\mu}$, $\mu = cf(\mu)$ and $\alpha < \mu \rightarrow |\alpha|^{\aleph_1} < \mu$.

Definition 2: A. Let K_2 be the class of \mathbf{k} such that:

- a. Each \mathbf{k} has the form $(\mathbb{P}, \mathcal{A}) = (\mathbb{P}_{\mathbf{k}}, \mathcal{A}_{\mathbf{k}})$.
- b. \mathbb{P} is a ccc forcing such that $\Vdash_{\mathbb{P}} MA_{\aleph_1}$.
- c. \mathcal{A} is a set of canonical \mathbb{P} -names of subsets of ω .
- d. $\Vdash_{\mathbb{P}}$ " \mathcal{A} is independent, i.e. every finite non-trivial Boolean combination of elements of \mathcal{A} is infinite".

B. For $\mathbf{k} \in K_2$ and a $\mathbb{P}_{\mathbf{k}}$ -name \tilde{b} , let $\Vdash_{\mathbb{P}_{\mathbf{k}}} "$ $\tilde{b} \in pos(\mathbf{k})$ " mean $\Vdash_{\mathbb{P}_{\mathbf{k}}} \tilde{b} \in [\omega]^\omega$ and there is no non-trivial Boolean combination of sets from $\mathcal{A}_{\mathbf{k}}$ that is almost disjoint to \tilde{b} ".

C. Let \leq_1 be the following partial order on K_2 : $\mathbf{k}_1 \leq_1 \mathbf{k}_2$ if and only if:

- a. $\mathbb{P}_{\mathbf{k}_1} \leq \mathbb{P}_{\mathbf{k}_2}$.
- b. $\mathcal{A}_{\mathbf{k}_1} \subseteq \mathcal{A}_{\mathbf{k}_2}$.

D. Let \leq_2 be the following partial order on K_2 :

$\mathbf{k}_1 \leq_2 \mathbf{k}_2$ if and only if:

- a. As in B(a).
- b. As in B(b).
- c. If \tilde{b} is a $\mathbb{P}_{\mathbf{k}_1}$ -name then $\Vdash_{\mathbb{P}_{\mathbf{k}_1}} "$ $\tilde{b} \in pos(\mathcal{A}_{\mathbf{k}_1})$ " implies $\Vdash_{\mathbb{P}_{\mathbf{k}_2}} "$ $\tilde{b} \in pos(\mathcal{A}_{\mathbf{k}_2})$ ".

E. Let K_2^+ be the class of $\mathbf{k} \in K_2$ such that $\Vdash_{\mathbb{P}_{\mathbf{k}}} "$ $\mathcal{A}_{\mathbf{k}}$ is a maximal independent set everywhere", where \mathcal{A} is a maximal independent set everywhere if for every $a_0, \dots, a_{n-1} \in \mathcal{A}$ without repetition, $b := \bigcap_{l < n} a_l^{\text{if } l \text{ is even}} \in [\omega]^\omega$ and $\{a \cap b : a \in \mathcal{A} \setminus \{a_0, \dots, a_{n-1}\}\}$ is a maximal independent set in $[b]^\omega$.

F. When we write " $a_0, \dots, a_{n-1} \in \mathcal{A}$ ", we mean that $(a_i : i < n)$ is without repetition, moreover, $i < j < n \rightarrow \Vdash_{\mathbb{P}} a_i \neq a_j$.

Observation 3: a. \leq_1 and \leq_2 are partial orders, and if $\mathbf{k}_1, \mathbf{k}_2 \in K_2^+$ then $\mathbf{k}_1 \leq_1 \mathbf{k}_2 \rightarrow \mathbf{k}_1 \leq_2 \mathbf{k}_2$.

b. If $\mathbf{k}_1 \leq_2 \mathbf{k}_2$ and \tilde{b} is a $\mathbb{P}_{\mathbf{k}_1}$ -name, then $\Vdash_{\mathbb{P}_{\mathbf{k}_1}} "$ $\tilde{b} \in pos(\mathcal{A}_{\mathbf{k}_1})$ " iff $\Vdash_{\mathbb{P}_{\mathbf{k}_2}} "$ $\tilde{b} \in pos(\mathcal{A}_{\mathbf{k}_2})$ ".

Proof: We shall prove the second claim of 3(a), everything else should be clear. Suppose that $\Vdash_{\mathbb{P}_{\mathbf{k}_1}} "$ $\tilde{b} \in pos(\mathcal{A}_{\mathbf{k}_1})$ ", but for some $a_0, \dots, a_{n-1} \in \mathcal{A}_{\mathbf{k}_2}$ and $p \in \mathbb{P}_{\mathbf{k}_2}$, $p \Vdash_{\mathbb{P}_{\mathbf{k}_2}} "$ $\tilde{b} \cap (\bigcap_{l < n} a_l^{\text{if } l \text{ is even}})$ is finite". Let $G \subseteq \mathbb{P}_{\mathbf{k}_1}$ be generic over V such that $p \in G$ and we shall work over $V[G]$. WLOG there is $n_1 < n$ such that $a_l \in \mathcal{A}_{\mathbf{k}_1}$ iff $l < n_1$, and denote $a_* = \bigcap_{l < n_1} a_l^{\text{if } l \text{ is even}}$. As $\Vdash_{\mathbb{P}_{\mathbf{k}_1}} "$ $\tilde{b} \in pos(\mathcal{A}_{\mathbf{k}_1})$ ", it follows that $\tilde{b} \cap a_*$ is infinite. It's enough to show that for some Boolean combination a_{**} from $\mathcal{A}_{\mathbf{k}_1}$, $a_{**} \subseteq a_*$ and $a_{**} \subseteq^* \tilde{b}$, as then $a_{**} \cap (\bigcap_{n_1 \leq l < n} a_l^{\text{if } l \text{ is even}}) \subseteq^* \tilde{b} \cap (\bigcap_{l < n} a_l^{\text{if } l \text{ is even}})$, and therefore it's finite, contradicting the definition of $\mathcal{A}_{\mathbf{k}_2}$. As $\mathbf{k}_1 \in K_2^+$, it follows that $\{a_* \cap c : c \in \mathcal{A}_{\mathbf{k}_1} \setminus \{a_l : l < n_1\}\}$ is a maximal independent set in $[a_*]^\omega$, hence there

are $c_0, \dots, c_{m-1} \in \mathcal{A}_{\mathbf{k}_1} \setminus \{a_l : l < n_1\}$ such that $(\bigcap_{l < n_1} a_l^{\text{if } 1 \text{ is even}}) \cap (\bigcap_{k < m} c_m^{\text{if } m \text{ is even}}) \subseteq^* b$, so $a_{**} = (\bigcap_{l < n_1} a_l^{\text{if } 1 \text{ is even}}) \cap (\bigcap_{k < m} c_m^{\text{if } m \text{ is even}})$ is as required. \square

Observation 4: $\mathbf{k}_1 \leq_2 \mathbf{k}_2$ and $\mathbf{k}_1 \in K_2^+$ when the following hold for some κ :

- a. $\mathbf{k}_2 \in K_2^+$.
- b. $\mathbf{k}_2 \in H(\kappa)$.
- c. M is a model such that $\mathbf{k}_2 \in M \prec_{\mathcal{L}_{\aleph_2, \aleph_2}} (H(\kappa), \in)$.
- d. $\mathbf{k}_1 = \mathbf{k}_2^M$.

Proof: By observation 3, recalling that " $\mathbb{P} \models ccc$ " and " $\mathbb{P} \models MA_{\aleph_1}$ " are $\mathcal{L}_{\aleph_2, \aleph_2}$ -expressible.

Claim 5: For every $\mathbf{k} \in K_2$ there is $\mathbf{k}' \in K_2^+$ such that $\mathbf{k} \leq_1 \mathbf{k}'$. Moreover, if $|\mathbb{P}_{\mathbf{k}}| < \mu$ then we can find such \mathbf{k}' that satisfies $|\mathbb{P}_{\mathbf{k}'}| < \mu$.

Proof: If $|\mathbb{P}_{\mathbf{k}}| < \mu$, let $\lambda_* = \mu$, otherwise, let λ_* be a regular cardinal greater than $(2 + |\mathbb{P}_{\mathbf{k}}|)^{\aleph_1}$, such that $\alpha < \lambda_* \rightarrow |\alpha|^{\aleph_1} < \lambda_*$.

we try to choose a sequence $(\mathbf{k}_\alpha : \alpha < \lambda_*)$ by induction on $\alpha < \lambda_*$ such that:

1. $\mathbf{k}_0 = \mathbf{k}$.
2. $(\mathbf{k}_\beta : \beta \leq \alpha)$ is an increasing continuous sequence of members of K_2 (with respect to \leq_1).
3. $|\mathbb{P}_{\mathbf{k}_\alpha}| < \mu$.
4. For every $\alpha < \lambda_*$, if $\mathbf{k}_\alpha \notin K_2^+$, we choose a Boolean combination a_α from $\mathcal{A}_{\mathbf{k}_\alpha}$ and $b_\alpha \subseteq a_\alpha$ witnessing the failure of the condition from Definition 2(E). We then define $\mathbb{P}_{\mathbf{k}_{\alpha+1}}$ as an extension (with respect to \leq) of $\mathbb{P}_{\mathbf{k}_\alpha} \star \text{Cohen}$ to a ccc forcing that forces MA_{\aleph_1} , we let η_α be the relevant Cohen generic real and we let $\mathcal{A}_{\mathbf{k}_{\alpha+1}} = \mathcal{A}_{\mathbf{k}_\alpha} \cup \{b_\alpha \cup \eta_\alpha^{-1}(\{1\})\}$.
5. If $\alpha < \lambda_*$ is a limit ordinal, we define \mathbf{k}_α as in the proof of claim 7 below.

Why can we carry the induction at stage $\alpha + 1$ for α as in (4)? We shall prove that for each α , $\Vdash_{\mathbb{P}_{\mathbf{k}_{\alpha+1}}} \text{"}\mathcal{A}_{\mathbf{k}_{\alpha+1}} \text{ is independent"}$. Let $a_\alpha^* = b_\alpha \cup \eta_\alpha^{-1}(\{1\})$, note that $\Vdash_{\mathbb{P}_{\mathbf{k}_{\alpha+1}}} \text{"}a_\alpha^* \notin \mathcal{A}_{\mathbf{k}_\alpha}\text{"}$, as otherwise there are $p \in \mathbb{P}_{\mathbf{k}_{\alpha+1}}$, $n < \omega$ and $a' \in \mathcal{A}_{\mathbf{k}_\alpha}$ such that $p \Vdash_{\mathbb{P}_{\mathbf{k}_{\alpha+1}}} \text{"}a' \setminus n = a_\alpha^* \setminus n\text{"}$, and therefore $p \Vdash_{\mathbb{P}_{\mathbf{k}_{\alpha+1}}} \text{"}\eta_\alpha^{-1}(\{1\}) \upharpoonright (\omega \setminus b_\alpha \setminus n) = a_\alpha^* \setminus b_\alpha \setminus n = a' \setminus b_\alpha \setminus n \in V^{\mathbb{P}_{\mathbf{k}_\alpha}}\text{"}$, a contradiction (as η_α is Cohen and $\omega \setminus b_\alpha \setminus n$ is infinite). Now if $a_\alpha = \bigcap_{l < n} a_{\alpha, l}^{\text{if } 1 \text{ is even}}$ and $c = a_\alpha \cap (\bigcap_{l < m} d_l^{\text{if } 1 \text{ is even}})$ where $d_0, \dots, d_{m-1} \in \mathcal{A}_{\mathbf{k}_\alpha} \setminus \{a_{\alpha, 0}, \dots, a_{\alpha, n-1}\}$, then $c \setminus a_\alpha^*$ and $c \cap a_\alpha^*$ are infinite (as η_α is Cohen and $c \setminus b$ is infinite), so $\mathcal{A}_{\mathbf{k}_{\alpha+1}}$ is forced to be independent.

If for some $\alpha < \lambda_*$, $\mathbf{k}_\alpha \in K_2^+$, when we're done. Otherwise, by Fodor's lemma, there are $\alpha < \beta < \lambda_*$ such that $(a_\alpha, b_\alpha) = (a_\beta, b_\beta)$, a contradiction. \square

Definition 6: We say that $(\mathbf{k}_\alpha : \alpha < \beta)$ is increasing continuous if $\alpha_1 < \alpha_2 \rightarrow \mathbf{k}_{\alpha_1} \leq_2 \mathbf{k}_{\alpha_2}$, and for every limit $\delta < \beta$, $\bigcup_{i < \delta} \mathbb{P}_{\mathbf{k}_i} \subseteq \mathbb{P}_{\mathbf{k}_\delta}$.

Claim 7: Every increasing continuous sequence in (K_2^+, \leq_2) has an upper bound. Moreover, if the length of the sequence has cofinality $> \aleph_1$, then the union is an upper bound in K_2^+ .

Proof: Given an increasing continuous sequence $(\mathbf{k}_\alpha : \alpha < \beta)$, we choose $\mathbb{P}_{\mathbf{k}_\beta}$ as in [HwSh:1090] and we let $\mathcal{A}_{\mathbf{k}_\beta} = \bigcup_{\alpha < \beta} \mathcal{A}_{\mathbf{k}_\alpha}$. This is enough for \leq_1 , so by claim 5 we're done. \square

Claim 8: A. If $\mathbf{k}_1 \in K_2$ then there are \mathbf{k}_2 and \tilde{a} such that:

- a. $\mathbf{k}_1 \leq_1 \mathbf{k}_2$.
- b. $\mathcal{A}_{\mathbf{k}_1} \cup \{\tilde{a}\} \subseteq \mathcal{A}_{\mathbf{k}_2}$.
- c. \tilde{a} is Cohen over $V^{\mathbb{P}_{\mathbf{k}_1}}$.
- d. $|\mathbb{P}_{\mathbf{k}_2}| \leq (2 + |\mathbb{P}_{\mathbf{k}_1}|)^{\aleph_1}$.

B. Moreover, we may require that $\mathbf{k}_2 \in K_2^+$.

Proof: A. Let $\mathbb{P} = \mathbb{P}_{\mathbf{k}_1} \star \mathbb{C}$ where \mathbb{C} is Cohen forcing, now let $\mathbb{P}_{\mathbf{k}_2}$ be a ccc forcing such that $\mathbb{P} \triangleleft \mathbb{P}_{\mathbf{k}_2}$, $\Vdash_{\mathbb{P}_{\mathbf{k}_2}}$ "MA $_{\aleph_1}$ " and $|\mathbb{P}_{\mathbf{k}_2}| \leq (2 + |\mathbb{P}|)^{\aleph_1}$. Finally, let $\mathcal{A}_{\mathbf{k}_2} = \mathcal{A}_{\mathbf{k}_1} \cup \{\tilde{a}\}$ where \tilde{a} is a name for a Cohen real added by $\mathbb{P}_{\mathbf{k}_2}$, it's easy to see that $(\mathbb{P}_{\mathbf{k}_2}, \mathcal{A}_{\mathbf{k}_2})$ are as required.

B. By claim 5. \square

Definition 9: We define the amalgamation property in the context of K_2^+ as follows: K_2^+ has the amalgamation property if A implies B where:

- A. a. $\mathbf{k}_l \in K_2^+$ ($l = 0, 1, 2$).
- b. $\mathbf{k}_0 \leq_2 \mathbf{k}_l$ ($l = 1, 2$).
- c. $\mathbb{P}_{\mathbf{k}_1} \cap \mathbb{P}_{\mathbf{k}_2} = \mathbb{P}_{\mathbf{k}_0}$.
- B. There exists $\mathbf{k}_3 = (\mathbb{P}_{\mathbf{k}_3}, \mathcal{A}_{\mathbf{k}_3}) \in K_2^+$ such that $\mathbf{k}_l \leq_2 \mathbf{k}_3$ ($l = 1, 2$).

Claim 10: a. (K_2^+, \leq_2) has the amalgamation property.

b. Suppose that $\mathbf{k}_0, \mathbf{k}_1$ and $\mathbf{k}_2 \in K_2^+$, $g : \mathbb{P}_{\mathbf{k}_0} \rightarrow \mathbb{P}_{\mathbf{k}_1}$ is an embedding such that $(g(\mathbb{P}_{\mathbf{k}_0}), g(\mathcal{A}_{\mathbf{k}_0})) \leq_2 \mathbf{k}_1$ and $\mathbf{k}_0 \leq_2 \mathbf{k}_2$, then there exist $\mathbf{k}, \mathbf{k}' \in K_2^+$ and f such that:

1. $\mathbf{k}_1 \leq_2 \mathbf{k}$ and $|\mathbb{P}_{\mathbf{k}}| \leq (2 + |\mathbb{P}_{\mathbf{k}_1}| + |\mathbb{P}_{\mathbf{k}_2}|)^{\aleph_1}$.
2. $(g(\mathbb{P}_{\mathbf{k}_0}), g(\mathcal{A}_{\mathbf{k}_0})) \leq_2 \mathbf{k}' \leq_2 \mathbf{k}$.
3. $f : \mathbb{P}_{\mathbf{k}_2} \rightarrow \mathbb{P}_{\mathbf{k}'}$ is an isomorphism mapping $\mathcal{A}_{\mathbf{k}_2}$ to $\mathcal{A}_{\mathbf{k}'}$.
4. $g \subseteq f$.

Proof: a. We shall first prove that $\mathcal{A}_{\mathbf{k}_1} \cap \mathcal{A}_{\mathbf{k}_2} = \mathcal{A}_{\mathbf{k}_0}$. Note that $\mathcal{A}_{\mathbf{k}_0} \subseteq \mathcal{A}_{\mathbf{k}_1} \cap \mathcal{A}_{\mathbf{k}_2}$ is true by the definition of \leq_2 , so suppose that $\tilde{a} \in \mathcal{A}_{\mathbf{k}_1} \setminus \mathcal{A}_{\mathbf{k}_0}$, we need to show that $\tilde{a} \notin \mathcal{A}_{\mathbf{k}_2}$. As $\mathbf{k}_0 \leq_2 \mathbf{k}_1$, \tilde{a} is not a $\mathbb{P}_{\mathbf{k}_0}$ -name. Therefore, it's not a $\mathbb{P}_{\mathbf{k}_2}$ -name, hence $\tilde{a} \notin \mathcal{A}_{\mathbf{k}_2}$.

Now construct \mathbb{P} as in [HwSh:1090], i.e. we take the amalgamation $\mathbb{P}' = \mathbb{P}_{\mathbf{k}_1} \times_{\mathbb{P}_{\mathbf{k}_0}} \mathbb{P}_{\mathbf{k}_2}$

and then we take $\mathbb{P} \in K$ such that $\mathbb{P}' \triangleleft \mathbb{P}$ and $|\mathbb{P}| \leq (2 + |\mathbb{P}'|)^{\aleph_1}$. Now let $\mathcal{A} := \mathcal{A}_{\mathbf{k}_1} \cup \mathcal{A}_{\mathbf{k}_2}$. We need to show that \mathcal{A} is as required, i.e. we need to prove that $(\mathbb{P}, \mathcal{A})$ satisfy requirements (A)(d) and (D)(c) in Definition 2 (in the end, we will use claim 5 for the requirement in Definition (2)(D)(c)). By symmetry, it's enough to show that if $\Vdash_{\mathbb{P}_{\mathbf{k}_1}}$ " $\tilde{b} \in \text{pos}(\mathcal{A}_{\mathbf{k}_1})$ " and $\tilde{a}_0, \dots, \tilde{a}_{n-1} \in \mathcal{A}$, then $\Vdash_{\mathbb{P}}$ " $\tilde{b} \cap (\bigcap_{l < n} \tilde{a}_l^{\text{if } (l \text{ is even})}) \in$ "

$[\omega]^\omega$. Let $n_2 := n$, wlog there are $n_0 < n_1 < n_2$ such that $a \in \mathcal{A}_{\mathbf{k}_0} \iff l < n_0$, $a \in \mathcal{A}_{\mathbf{k}_1} \iff l \in [0, n_1)$ and $a \in \mathcal{A}_{\mathbf{k}_2} \iff l \in [0, n_0) \cup [n_1, n_2)$. It's enough to show that the last statement is forced by \mathbb{P}' , so let $k < \omega$ and $p = (p_1, p_2) \in \mathbb{P}'$, we shall find $q \in \mathbb{P}'$ and $m < \omega$ such that $p \leq q$, $k \leq m$ and $q \Vdash_{\mathbb{P}'}$ " $m \in \tilde{b} \cap (\bigcap_{l < n} a_l^{\text{if } (l \text{ is even})})$ ". Let $p_0 \in \mathbb{P}_{\mathbf{k}_0}$ witness " $(p_1, p_2) \in \mathbb{P}'$ ", i.e. $p_0 \Vdash_{\mathbb{P}_{\mathbf{k}_0}}$ " $\bigwedge_{l=1,2} p_l \in \mathbb{P}_{\mathbf{k}_l}/\mathbb{P}_{\mathbf{k}_0}$ ". Let $\tilde{b}^* = \{m : p_1 \not\Vdash_{\mathbb{P}_{\mathbf{k}_1}/\mathbb{P}_{\mathbf{k}_0}} "m \notin \tilde{b} \cap (\bigcap_{l \text{ if } l \text{ is even}} : l \in [n_0, n_1])\}$ " (so \tilde{b}^* is a $\mathbb{P}_{\mathbf{k}_0}$ -name) and let $p_0 \in G_0 \subseteq \mathbb{P}_{\mathbf{k}_0}$ be generic over V , then $\tilde{b}^* = \tilde{b}^*[G_0] \in V[G_0]$ and as $p_1 \Vdash_{\mathbb{P}_{\mathbf{k}_1}}$ " $\tilde{b} \in \text{pos}(\mathcal{A}_{\mathbf{k}_1})$ ", it follows that $p_0 \Vdash_{\mathbb{P}_{\mathbf{k}_0}}$ " $\tilde{b}^* \in [\omega]^\omega$ ", moreover, $p_0 \Vdash_{\mathbb{P}_{\mathbf{k}_0}}$ " $\tilde{b}^* \in \text{pos}(\mathcal{A}_{\mathbf{k}_0})$ ". Let \tilde{b}^{**} be the $\mathbb{P}_{\mathbf{k}_0}$ -name defined as \tilde{b}^* if p_0 is in the generic set, and as ω otherwise. As $\mathbf{k}_0 \leq_2 \mathbf{k}_2$, it follows that $p_2 \Vdash_{\mathbb{P}_{\mathbf{k}_2}/G_0}$ " $\tilde{b}^{**} \cap (\bigcap_{l \in [0, n_0) \cup [n_1, n_2)} a_l^{\text{if } (l \text{ is even})}) \in [\omega]^\omega$ ".

Therefore, in $V[G_0]$ there are (p'_2, m) such that:

- $p_2 \leq p'_2 \in \mathbb{P}_{\mathbf{k}_2}/G_0$.
- $m > k$.
- $p'_2 \Vdash_{\mathbb{P}_{\mathbf{k}_2}/G_0}$ " $m \in \tilde{b}^{**} \cap (\bigcap_{l \in [0, n_0) \cup [n_1, n_2)} a_l^{\text{if } (l \text{ is even})})$ ".

Note that as $\tilde{b}^{**} \in V[G_0]$, $V[G_0] \models "m \in \tilde{b}^{**}[G_0] = \tilde{b}^*[G_0]"$. Therefore, by the definitions of \tilde{b} and \tilde{b}^* , there is $p'_1 \in \mathbb{P}_{\mathbf{k}_1}/G_0$ above p_1 such that $p'_1 \Vdash_{\mathbb{P}_{\mathbf{k}_1}/G_0}$ " $m \in \tilde{b} \cap (\bigcap_{l \text{ if } l \text{ is even}} : l \in [n_0, n_1])$ ". Therefore, there is $p_0 \leq p'_0 \in G_0$ forcing (in $\mathbb{P}_{\mathbf{k}_0}$) all of the aforementioned statements about (p'_1, p'_2) in $V[G_0]$. Now it's easy to check that $q = (p'_1, p'_2)$ is as required. Finally, extend $(\mathbb{P}, \mathcal{A})$ (with respect to \leq_1) to a member of K_2^+ . By observation 3, we're done.

b. Follows from (a) by changing names. \square

Claim 11: There exists $\mathbf{k} = (\mathbb{P}_{\mathbf{k}}, \mathcal{A}_{\mathbf{k}}) = (\mathbb{P}, \mathcal{A}) \in K_2^+$ such that $|\mathbb{P}_{\mathbf{k}}| = \lambda$ and:

- For every $X \subseteq \mathbb{P}$ of cardinality $< \mu$, there exists $\mathbf{k}' = (\mathbb{Q}, \mathcal{A}') \in K_2^+$ such that $X \subseteq \mathbb{Q}$, $\mathbf{k}' \leq_2 \mathbf{k}$ and $|\mathbb{Q}| < \mu$.
- If $\mathbf{k}_1, \mathbf{k}_2 \in K_2^+$, $|\mathbb{P}_{\mathbf{k}_1}|, |\mathbb{P}_{\mathbf{k}_2}| < \mu$, $\mathbf{k}_1 \leq_2 \mathbf{k}_2$ and $f_1 : \mathbb{P}_{\mathbf{k}_1} \rightarrow \mathbb{P}$ is a complete embedding such that $(f_1(\mathbb{P}_{\mathbf{k}_1}), f_1(\mathcal{A}_{\mathbf{k}_1})) \leq_2 (\mathbb{P}, \mathcal{A})$, then there is a complete embedding f_2 such that $f_1 \subseteq f_2$ and $(f_2(\mathbb{P}_{\mathbf{k}_2}), f_2(\mathcal{A}_{\mathbf{k}_2})) \leq_2 (\mathbb{P}, \mathcal{A})$.

Proof: The first property is satisfied by every $\mathbf{k} \in K_2^+$ by observation 4. The proof of (2) is as in [HwSh:1090]. \square

Claim 12: A implies B where:

- a. $\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2 \in K_2^+$, $\mathbf{k}_0 \leq_2 \mathbf{k}_l$ ($l = 1, 2$) and $\mathbb{P}_{\mathbf{k}_0} = \mathbb{P}_{\mathbf{k}_1} \cap \mathbb{P}_{\mathbf{k}_2}$.
- \tilde{D} is a $\mathbb{P}_{\mathbf{k}_0}$ -name of a nonprincipal ultrafilter on ω .
- For $l = 1, 2$, \tilde{a}_l and \tilde{b}_l are canonical $\mathbb{P}_{\mathbf{k}_l}$ -names of a member of $[\omega]^\omega$.
- For $l = 1, 2$, $\Vdash_{\mathbb{P}_{\mathbf{k}_l}}$ " $\tilde{a}_l \cap \tilde{b}_l$ is infinite and \tilde{a}_l contains no members of \tilde{D} from $V^{\mathbb{P}_{\mathbf{k}_0}}$ ".
- $\mathbb{P}_{\mathbf{k}_1} \cap \mathbb{P}_{\mathbf{k}_2} = \mathbb{P}_{\mathbf{k}_0}$.
- For $l = 1, 2$, $\Vdash_{\mathbb{P}_{\mathbf{k}_l}}$ " \tilde{b}_l is a pseudo intersection of \tilde{D} ".

B. There is $\mathbf{k} \in K_2^+$ such that:

- a. $|\mathbb{P}_{\mathbf{k}}| \leq (2 + |\mathbb{P}_{\mathbf{k}_1}| + |\mathbb{P}_{\mathbf{k}_2}|)^{\aleph_1}$.
- b. $\mathbf{k}_l \leq_2 \mathbf{k}$ ($l = 1, 2$).
- c. $\Vdash_{\mathbb{P}_{\mathbf{k}}} \text{"} \underset{\sim}{a_2} \setminus \underset{\sim}{a_1} \text{ and } \underset{\sim}{a_1} \setminus \underset{\sim}{a_2} \text{ are infinite"}$.

Proof: Let $\mathbf{k} \in K_2^+$ be the object constructed by the proof of claim 10. We need to prove that \mathbf{k} satisfies clause (B)(c). For $l = 0, 1, 2$, let $\mathbb{P}_l = \mathbb{P}_{\mathbf{k}_l}$ and let \mathbb{P}' be as in the proof of claim 10, so it suffices to prove that $\Vdash_{\mathbb{P}'} \text{"} \underset{\sim}{a_2} \setminus \underset{\sim}{a_1} \text{ and } \underset{\sim}{a_1} \setminus \underset{\sim}{a_2} \text{ are infinite"}$.

Let $p = (p_1, p_2) \in \mathbb{P}'$, $k < \omega$ and let $p_0 \in \mathbb{P}_0$ be a witness of " $\underset{\sim}{p} = (p_1, p_2) \in \mathbb{P}'$ ". Now let $G_0 \subseteq \mathbb{P}'$ be generic over V such that $p_0 \in G_0$ and let $D = D[G_0]$. By the assumptions, for $l = 1, 2$, $p_l \Vdash_{\mathbb{P}_l/G_0} \text{"} \underset{\sim}{a_l} \cap \underset{\sim}{b_l} \text{ is an infinite pseudo intersection of } D \text{"}$. In $V[G_0]$, let $b_l^* = \{m : p_l \not\Vdash_{\mathbb{P}_l/G_0} \text{"} m \notin \underset{\sim}{a_l} \cap \underset{\sim}{b_l} \text{"}\}$, then $p_l \Vdash_{\mathbb{P}_l/G_0} \text{"} \underset{\sim}{a_l} \cap \underset{\sim}{b_l} \subseteq b_l^* \text{, hence } b_l^* \text{ infinite"}$. As $p_l \Vdash_{\mathbb{P}_l/G_0} \text{"} \underset{\sim}{a_l} \cap \underset{\sim}{b_l} \text{ is a pseudo intersection of } D \text{"}$, necessarily $V[G_0] \models b_l^* \in D$.

Let $a_l^* = \{m : p_l \Vdash_{\mathbb{P}_l/G_0} m \in \underset{\sim}{a_l}\}$, so $a_l^* \in V[G_0]$ and $p_l \Vdash_{\mathbb{P}_l/G_0} \text{"} a_l^* \subseteq \underset{\sim}{a_l} \text{"}$. Now recall that $\Vdash_{\mathbb{P}_{\mathbf{k}_l}} \text{"} \underset{\sim}{a_l} \text{ contains no member of } D \text{ from } V^{\mathbb{P}_{\mathbf{k}_0}} \text{"}$, therefore $p_l \Vdash_{\mathbb{P}_l/G_0} \text{"} a_l^* \notin \underset{\sim}{D} \text{"}$.

Hence in $V[G_0]$ (recalling D is an ultrafilter), $b := (b_1^* \cap b_2^*) \setminus (a_1^* \cup a_2^*) \in D$. Let $m \in b$ be such that $k < m$. By the definition of b_l^* , there is $p'_l \in \mathbb{P}_l/G_0$ above p_l such that $p'_l \Vdash_{\mathbb{P}_l/G_0} \text{"} m \in \underset{\sim}{a_l} \cap \underset{\sim}{b_l} \text{"}$. By the definition of a_l^* , there is $p''_l \in \mathbb{P}_l/G_0$ above p_l such that $p''_l \Vdash_{\mathbb{P}_l/G_0} \text{"} m \notin \underset{\sim}{a_l} \text{"}$. Let $p'_0 \in G_0$ be a condition above p_0 forcing the above statements, so p'_0 is witnessing the fact that $(p'_1, p'_2), (p''_1, p''_2) \in \mathbb{P}'$ are above $p = (p_1, p_2)$. Now $m > k$, $(p'_1, p'_2) \Vdash \text{"} m \in \underset{\sim}{a_1} \setminus \underset{\sim}{a_2} \text{"}$ and $(p''_1, p''_2) \Vdash \text{"} m \in \underset{\sim}{a_2} \setminus \underset{\sim}{a_1} \text{"}$, which completes the proof. \square

Definition 13: Let $\mathbb{P} = \mathbb{P}_{\mathbf{k}}$ be the forcing from claim 11, let $G \subseteq \mathbb{P}$ be generic over V and in $V[G]$, let $V_1 = HOD(\mathbb{R}^{<\mu} \cup \{\mathcal{A}_{\mathbf{k}}\})$.

Definition 14 ([NN]): A family $\mathcal{F} \subseteq [\omega]^\omega$ is nearly mad if $|A \cap B| < \aleph_0$ or $|A \Delta B| < \aleph_0$ for every $A \neq B \in \mathcal{F}$, and \mathcal{F} is maximal with respect to this property.

Theorem 15: $V_1 \models ZF + DC_{<\mu} + \text{"there are no mad families"} + \text{"there are no nearly mad families"} + \text{"there exists a non-meager filter on } \omega \text{"}$.

Proof: 1. In order to see that there exists a non-meager filter in V_1 , let $\underset{\sim}{D}$ be the filter generated by $\mathcal{A}_{\mathbf{k}}$ and the cofinite sets. By claim 8 and the choice of \mathbf{k} , $\underset{\sim}{D}$ contains many Cohen reals and therefore is non-meager.

2. The proof of the non-existence of mad families is exactly as in [HwSh:1090], where (K_2^+, \leq_2) here replaces $(K, <)$ there, and claim 10 is used for the amalgamation arguments. Alternatively, see the proof of (3) below.

3. The non-existence of a nearly mad family in V_1 will follow from the proofs below. \square

Somewhat mad families

Definition 16: A family $\mathcal{F} \subseteq [\omega]^\omega$ is somewhat mad if:

- a. For every $a_1, a_2 \in \mathcal{F}$, $|a_1 \cap a_2| < \aleph_0$ or $a_1 \subseteq^* a_2$ or $a_2 \subseteq^* a_1$.
- b. If $b \in [\omega]^\omega$ then for some $a \in \mathcal{F}$, $|a \cap b| = \aleph_0$.

Observation 17: Nearly mad families are somewhat mad. \square

Definition 18: Let $Pr(\mathbf{k}_1, \mathbf{k}_2, \tilde{D}, \tilde{b}_2)$ mean:

- a. $\mathbf{k}_1 \leq_2 \mathbf{k}_2$, \tilde{b}_2 is a $\mathbb{P}_{\mathbf{k}_2}$ -name and \tilde{D} is a $\mathbb{P}_{\mathbf{k}_1}$ -name such that $\Vdash_{\mathbb{P}_{\mathbf{k}_2}} \tilde{b}_2 \in [\omega]^\omega$ and $\Vdash_{\mathbb{P}_{\mathbf{k}_1}} \tilde{D}$ is a nonprincipal ultrafilter on ω .
- b. If $G_1 \subseteq \mathbb{P}_{\mathbf{k}_1}$ is generic over V , $p_1 \in \mathbb{P}_{\mathbf{k}_2}/G_1$, $b_0^* = \{n : p_1 \Vdash_{\mathbb{P}_{\mathbf{k}_1} G_1} n \in \tilde{b}_2\}$ and $b_1^* = \{n : p_1 \not\Vdash_{\mathbb{P}_{\mathbf{k}_1}/G_1} n \notin \tilde{b}_2\}$, then $V[G_1] \models b_1^* \setminus b_0^* \in \tilde{D}$.

Claim 19: (A) implies (B) where:

- A. a. $\mathbf{k}_1 \in K_2^+$.
- b. $\Vdash_{\mathbb{P}_{\mathbf{k}_1}} \tilde{D}$ is a nonprincipal ultrafilter on ω .
- c. $\Vdash_{\mathbb{P}_{\mathbf{k}_1}} \tilde{S}_1$ is somewhat mad.
- d. $\Vdash_{\mathbb{P}_{\mathbf{k}_1}} \tilde{S}_1 \cap \tilde{D} = \emptyset$.

B. There is \mathbf{k}_2 such that:

- a. $\mathbf{k}_2 \in K_2^+$.
- b. $\mathbf{k}_1 \leq_2 \mathbf{k}_2$.
- c. $|\mathbb{P}_{\mathbf{k}_2}| \leq (2 + |\mathbb{P}_{\mathbf{k}_1}|)^{\aleph_1}$.
- d. (α) implies (β) where:

α . (\mathbf{k}_3, S_2) satisfy the following properties:

1. $\mathbf{k}_2 \leq_2 \mathbf{k}_3 \in K_2^+$.
2. $\Vdash_{\mathbb{P}_{\mathbf{k}_3}} \tilde{S}_2$ is somewhat mad and $\tilde{S}_1 \subseteq \tilde{S}_2$.
3. $\Vdash_{\mathbb{P}_{\mathbf{k}_3}}$ "no member of $\tilde{S}_2 \setminus \tilde{S}_1$ contains a member of \tilde{D} ".

β . For some $\mathbb{P}_{\mathbf{k}_3}$ -name \tilde{a} , $\Vdash_{\mathbb{P}_{\mathbf{k}_3}} \tilde{a} \in \tilde{S}_2$ and $Pr(\mathbf{k}_1, \mathbf{k}_3, \tilde{D}, \tilde{a})$.

Proof: Using Mathias forcing restricted to \tilde{D} , it's easy to see that there is \mathbf{k}_2 and a $\mathbb{P}_{\mathbf{k}_2}$ name \tilde{b} such that $\mathbf{k}_1 \leq_2 \mathbf{k}_2 \in K_2^+$, $|\mathbb{P}_{\mathbf{k}_2}| \leq (2 + |\mathbb{P}_{\mathbf{k}_1}|)^{\aleph_1}$ and $\Vdash_{\mathbb{P}_{\mathbf{k}_2}} \tilde{b}$ is a pseudo intersection of \tilde{D} . Therefore, $\Vdash_{\mathbb{P}_{\mathbf{k}_2}} \tilde{b} \in [\omega]^\omega$ is almost disjoint to every $\tilde{a} \in \tilde{S}_1$ (by the fact that $\Vdash_{\mathbb{P}_{\mathbf{k}_1}} \tilde{S}_1 \cap \tilde{D} = \emptyset$).

We shall now prove that \mathbf{k}_2 satisfies (B)(d). Suppose that (\mathbf{k}_3, S_2) are as there. By the somewhat madness of \tilde{S}_2 , there is \tilde{a} such that $\Vdash_{\mathbb{P}_{\mathbf{k}_3}} \tilde{a} \in \tilde{S}_2$ and $|\tilde{a} \cap \tilde{b}| = \aleph_0$. Therefore, $\Vdash_{\mathbb{P}_{\mathbf{k}_3}} \tilde{a} \cap \tilde{b} \in [\omega]^\omega$ is a pseudo intersection of \tilde{D} . Now let $G_1 \subseteq \mathbb{P}_{\mathbf{k}_1}$ be generic over V . If $p_1 \in \mathbb{P}_{\mathbf{k}_3}/G_1$ then $b^* = \{n : p_1 \not\Vdash_{\mathbb{P}_{\mathbf{k}_3}/G_1} n \notin \tilde{a} \cap \tilde{b}\} \in \tilde{D}[G_1]$ by the fact that $\tilde{a} \cap \tilde{b}$ is a pseudo intersection of \tilde{D} . Now let $a^* = \{n : p_1 \Vdash_{\mathbb{P}_{\mathbf{k}_3}/G_1} n \in \tilde{a}\}$, then $p_1 \Vdash_{\mathbb{P}_{\mathbf{k}_3}/G_1} a^* \subseteq \tilde{a}$ is infinite. If $a^* \in \tilde{D}[G_1]$, then p_1 forces that \tilde{a} (which belongs to $\tilde{S}_2 \setminus \tilde{S}_1$) contains a member of $\tilde{D}[G_1]$, contradicting $(\alpha)(3)$. Therefore, $a^* \notin \tilde{D}[G_1]$, and \tilde{a} is as required in the definition of $Pr(\mathbf{k}_1, \mathbf{k}_3, \tilde{D}, \tilde{a})$. \square

Claim 20: There is no somewhat mad family in V_1 .

Proof: Suppose towards contradiction that \tilde{S} is a \mathbb{P} -name of a somewhat mad family. As in [HwSh:1090], let \tilde{D} be a \mathbb{P} -name of a Ramsey ultrafilter on ω such

that $\Vdash_{\mathbb{P}} \text{''}\tilde{S} \cap \tilde{D} = \emptyset\text{'}$. By claim 11(a), there is $\mathbf{k}_1 \leq_2 \mathbf{k}$ such that $\mathbf{k}_1 \in K_2^+$, $|\mathbb{P}_{\mathbf{k}_1}| < \mu$ and \tilde{S} is definable using a $\mathbb{P}_{\mathbf{k}_1}$ -name. Let $K_{\mathbb{P}}^+$ be the set of $\mathbf{k}' \in K_2^+$ such that $\mathbf{k}' \leq_2 \mathbf{k}$, $|\mathbb{P}_{\mathbf{k}'}| < \mu$, $\tilde{S} \upharpoonright \mathbb{P}_{\mathbf{k}'}$ is a canonical $\mathbb{P}_{\mathbf{k}'}$ -name of a somewhat mad family in $V^{\mathbb{P}_{\mathbf{k}'}}$ and $\tilde{D} \upharpoonright \mathbb{P}_{\mathbf{k}'}$ is a $\mathbb{P}_{\mathbf{k}'}$ -name of a Ramsey ultrafilter on ω . As in [HwSh:1090], $K_{\mathbb{P}}^+$ is \leq_2 -dense in K_2^+ , so there exists $\mathbf{k}_2 \in K_{\mathbb{P}}^+$ such that $\mathbf{k}_1 \leq \mathbf{k}_2$. Let $\mathbf{k}_3 \in K_2^+$ be as in claim 19 for $(\mathbf{k}_2, \tilde{S} \upharpoonright \mathbb{P}_{\mathbf{k}_2})$, wlog $\mathbf{k}_3 \leq_2 \mathbf{k}$ (see claim 11). Choose $\mathbf{k}_4 \in K_{\mathbb{P}}^+$ such that $\mathbf{k}_3 \leq_2 \mathbf{k}_4$ and let $\tilde{S}_4 := \tilde{S} \upharpoonright \mathbb{P}_{\mathbf{k}_4}$. Let \tilde{a} be a $\mathbb{P}_{\mathbf{k}_4}$ -name such that $Pr(\mathbf{k}_3, \mathbf{k}_4, \tilde{D}, \tilde{a})$ holds, as guaranteed by claim 19.

As in [HwSh:1090], there are $\mathbf{k}_5, \mathbf{k}_6 \in K_{\mathbb{P}}^+$ and an isomorphism f from \mathbf{k}_4 to \mathbf{k}_5 over \mathbf{k}_2 such that $\mathbb{P}_{\mathbf{k}_5}$ adds a generic for $M_{\tilde{D} \upharpoonright \mathbb{P}_{\mathbf{k}_4}}$ (Mathias forcing restricted to the ultrafilter $\tilde{D} \upharpoonright \mathbb{P}_{\mathbf{k}_4}$) and $(\mathbf{k}_2, \mathbf{k}_4, \mathbf{k}_5, \mathbf{k}_6)$ here are as $(\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ in claim 10, and wlog $\mathbf{k}_6 \leq_2 \mathbf{k}$. By the choice of f , $\Vdash_{\mathbb{P}} \text{''}\tilde{a}, f(\tilde{a}) \in \tilde{S}\text{'}$.

By claim 12, with $(\mathbf{k}_2, \mathbf{k}_4, \mathbf{k}_5, \mathbf{k}_6, \tilde{a}, f(\tilde{a}))$ standing for $(\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}, \tilde{a}_{\sim_1}, \tilde{a}_{\sim_2})$ there, it's forced by $\mathbb{P}_{\mathbf{k}_5}$, and hence by \mathbb{P} , that $\tilde{a} \setminus f(\tilde{a})$ and $f(\tilde{a}) \setminus \tilde{a}$ are infinite. As in [HwSh:1090], $\Vdash_{\mathbb{P}} \text{''}|\tilde{a} \cap f(\tilde{a})| = \aleph_0\text{'}$. As $\Vdash_{\mathbb{P}} \text{''}\tilde{a}, f(\tilde{a}) \in \tilde{S}\text{'}$, we get a contradiction. \square

References

- [HwSh:1090] Haim Horowitz and Saharon Shelah, Can you take Toernquist's inaccessible away? arXiv:1605.02419
- [LZ] Paul Larson and Jindrich Zapletal, Canonical models for fragments of the axiom of choice, users.miamioh.edu/larsonpb/lru5.pdf
- [Ma1] A. R. D. Mathias, Happy families, Ann. Math. Logic **12** (1977), no. 1, 59-111.
- [Ma2] A. R. D.. Mathias, A remark on rare filters.
- [NN] Itay Neeman and Zach Norwood, Happy and mad families in $L(\mathbb{R})$, preprint.
- [To] Asger Toernquist, Definability and almost disjoint families, arXiv:1503.07577.

(Haim Horowitz) Einstein Institute of Mathematics
 Edmond J. Safra campus,
 The Hebrew University of Jerusalem.
 Givat Ram, Jerusalem, 91904, Israel.
 E-mail address: haim.horowitz@mail.huji.ac.il

(Saharon Shelah) Einstein Institute of Mathematics
 Edmond J. Safra campus,
 The Hebrew University of Jerusalem.
 Givat Ram, Jerusalem, 91904, Israel.
 Department of Mathematics

Hill Center - Busch Campus,
Rutgers, The State University of New Jersey.
110 Frelinghuysen road, Piscataway, NJ 08854-8019 USA
E-mail address: shelah@math.huji.ac.il