On the non-existence of κ-mad families

Haim Horowitz and Saharon Shelah

Abstract

Starting from a model with a Laver-indestructible supercompact cardinal κ, we construct a model of $ZF + DC_\kappa$ where there are no κ-mad families.[1]

Introduction

The study of the definability and possible non-existence of mad families has a long tradition, originating with the paper [Ma] of Mathias where it was proven that mad families can't be analytic and that there are no mad families in the Solovay model constructed from a Mahlo cardinal. It was later shown by Toernquist that an inaccessible cardinal suffices for the consistency of this statement ([To]), and it was then shown by the authors that the non-existence of mad families (in $ZF + DC$) is actually equiconsistent with ZFC ([HwSh:1090]).

The current paper can be seen as a continuation of the line of investigation of [HwSh:1090], as well as of [HwSh:1145], where the definability of κ-mad families was considered. Recall the following definition:

Definition 1: Let κ be an infinite regular cardinal. A family $\mathcal{A} \subseteq [\kappa]^\kappa$ is κ-almost disjoint if $|A \cap B| < \kappa$ for every $A \neq B \in \mathcal{A}$. $\mathcal{A}$ will be called κ-maximal almost disjoint (κ-mad) if $\mathcal{A}$ is κ-almost disjoint and can't be extended to a larger κ-almost disjoint family.

Assuming the existence of a Laver-indestructible supercompact cardinal κ, we constructed in [HwSh:1145] a generic extension where κ remained supercompact and there are no $\Sigma^1_1(\kappa) - \kappa-$mad families, thus obtaining a higher analog of Mathias' result.

Our current main goal is to obtain a higher analog of the main result of [HwSh:1090], i.e. for an uncountable cardinal $\theta > \aleph_0$, we would like to construct a model of $ZF + DC_\theta$ where there are no θ-mad families. As opposed to [HwSh:1090], we only achieve this goal assuming the existence of a supercompact cardinal. The main result of the paper is the following:

Theorem 2: a. Suppose that $\aleph_0 < cf(\theta) = \theta < cf(\kappa) = \kappa \leq \lambda = \lambda^{<\kappa}$ and θ is a Laver indestructible supercompact cardinal, then there is a model of $ZF + DC_{<\kappa} +$ "there exist no θ-mad families".

b. If we start from a universe V, then the final model V_1 will have the same cardinals and same $H(\theta)$ as V.

We shall force with a partial order $\mathbb{P}$ where the conditions themselves are forcing notions (this is somewhat similar to [Sh:218], [HwSh:1093] and [HwSh:1113], as well as to the recent work of Viale in [Vi], where a similar approach is applied to the study of generic absoluteness). Forcing with $\mathbb{P}$ will generically introduce the forcing notion $\mathbb{Q}$ that will give us the desired results. More specifically, we shall fix a Laver-indestructible supercompact cardinal θ. The conditions in $\mathbb{P}$ will be elements from a suitable $H(\lambda^+)$ that are $(< \theta)$-support iterations along wellfounded partial orders of $(< \theta)$-directed closed forcing notions satisfying a strong version of θ^+-cc. Given $\mathbf{q}_1, \mathbf{q}_2 \in \mathbb{P}$, we will have $\mathbf{q}_1 \leq_{\mathbb{P}} \mathbf{q}_2$ when the iteration given by $\mathbf{q}_1$ is an

[1]Date: June 12, 2019
2010 Mathematics Subject Classification: 03E15, 03E25, 03E35, 03E55
Keywords: Generalized descriptive set theory, mad families, supercompact cardinals
Publication 1168 of the second author

"initial segment" (in an adequate sense) of the iteration given by $\mathbf{q}_2$. Forcing with $\mathbb{P}$ will introduce a generic iteration $\mathbf{q}_G$ given by the union of $\mathbf{q} \in \mathbb{P}$ that belong to the generic set. In the further generic extension given by $\mathbf{q}_G$, we shall consider $V_1 = HOD(\mathcal{P}(\theta)^{<\kappa} \cup V)$ (for an adequate fixed κ). We shall then prove that there are no θ-mad families in V_1. In order to prove this fact, we shall consider towards contradiction a condition $(\mathbf{q}_0, \underset{\sim}{p_0})$ that forces a counterexample $\mathcal{A}$, where $\mathbf{q}_0$ will be "sufficiently closed". The filter that's dual to the ideal generated by $\mathcal{A}$ will then be extended to a θ-complete ultrafilter (using the Laver-indestructibility of θ), and we shall obtain a contradiction with the help of an amalgamation argument over $\mathbf{q}_0$ using a higher analog of Mathias forcing relative to this ultrafilter.

The rest of the paper will be devoted to the proof of Theorem 2.

Proof of the main result

Definition 3: A. Let K be the class of $\mathbf{q}$ that consist of the following objects with the following properties:

a. $U = U_{\mathbf{q}}$ a well-founded partial order whose elements are ordinals. We let $U^+ = U \cup \{\infty\}$ where ∞ is a new element above all elements from U, and for $\alpha \in U^+$, we let $U_{<\alpha} = \{\beta \in U : \beta <_U \alpha\}$.

b. An iteration $(\mathbb{P}_{\mathbf{q},\alpha}, \underset{\sim}{\mathbb{Q}}_{\mathbf{q},\beta} : \alpha \in U^+, \beta \in U) = (\mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \in U^+, \beta \in U)$. We shall often denote the iteration itself by $\mathbf{q}$.

c. $\mathbf{q}$ is a $(<\theta)$-support iteration, and in addition:

(α) Each $\underset{\sim}{\mathbb{Q}}_\beta$ is a $\mathbb{P}_\beta$-name of a forcing notion whose set of elements is an object X_β from V.

(β) Given $\alpha \in U^+$, $p \in \mathbb{P}_\alpha$ iff p is a function with domain $dom(p) \in [U_{<\alpha}]^{<\theta}$ such that $p(\beta)$ is a canonical $\mathbb{P}_\beta$-name for every $\beta \in dom(p)$.

(γ) $\leq_{\mathbb{P}_\alpha}$ is defined as usual.

(δ) If $w \subseteq U$ is downward closed (i.e. $\alpha <_U \beta \in w \to \alpha \in w$) and $\mathbb{P}_{\mathbf{q},w} = \mathbb{P}_w = \mathbb{P}_\infty \restriction w = \{p \in \mathbb{P}_\infty : dom(p) \subseteq w\}$, then $\mathbb{P}_w \lessdot \mathbb{P}_\infty$.

d. In $V^{\mathbb{P}_\beta}$, $\underset{\sim}{\mathbb{Q}}_\beta$ satisfies $*^\epsilon_\theta$ for a fixed limit $\epsilon < \theta$, namely, if $\{p_\alpha : \alpha < \theta^+\} \subseteq \underset{\sim}{\mathbb{Q}}_\beta$, then there is some club $E \subseteq \theta^+$ and a pressing down function $f : E \to \theta^+$ such that if $\delta_1, \delta_2 \in E$, $cf(\delta_1) = cf(\delta_2)$ and $f(\delta_1) = f(\delta_2)$, then p_{δ_1} and p_{δ_2} have a common least upper bound.

e. For $\beta \in U$, the following holds in $V^{\mathbb{P}_\beta}$: If I is a directed partial order of cardinality $<\theta$ and $(p_s : s \in I) \in \mathbb{Q}^I_\beta$ is $\leq_{\mathbb{Q}_\beta}$-increasing, then $\{p_s : s \in I\}$ has a $\leq_{\mathbb{Q}_\beta}$-least upper bound.

B. Let $\leq_K$ be the following partial order on K:

$\mathbf{q}_1 \leq_K \mathbf{q}_2$ iff the following conditions hold:

a. $U_{\mathbf{q}_1} \subseteq U_{\mathbf{q}_2}$ as partial orders.

b. If $U_{\mathbf{q}_2} \models \alpha < \beta$ and $\beta \in U_{\mathbf{q}_1}$, then $\alpha \in U_{\mathbf{q}_1}$.

c. If $w \subseteq U_{\mathbf{q}_1}$ is downward closed, then $\mathbb{P}_{\mathbf{q}_1,w} = \mathbb{P}_{\mathbf{q}_2,w}$.

d. If $\alpha \in U_{\mathbf{q}_1}$, then $\underset{\sim}{\mathbb{Q}}_{\mathbf{q}_1,\alpha} = \underset{\sim}{\mathbb{Q}}_{\mathbf{q}_2,\alpha}$ (this is well-defined recalling clause (b)).

C. Let K_{wf} be the class of U as in (A)(a), and let $\leq_{wf}$ be the partial order on K_{wf} defined as in clauses (B)(a) and (B)(b).

We shall now observe some easy basic properties of the objects defined above:

Observation 4: a. If $(U_\alpha : \alpha < \delta)$ is $\leq_{wf}$-increasing, then $\bigcup_{\alpha<\delta} U_\alpha$ is a $\leq_{wf}$-least upper bound for $(U_\alpha : \alpha < \delta)$.

b. $\leq_K$ is a partial order on K.

c. If $\mathbf{q}_2 \in K$ and $U_1 \subseteq U_{\mathbf{q}_2}$ is downward closed, then there is a unique $\mathbf{q}_1 \in K$ such that $\mathbf{q}_1 \leq_K \mathbf{q}_2$ and $U_{\mathbf{q}_1} = U_1$.

d. If $(\mathbf{q}_\alpha : \alpha < \delta)$ is $\leq_K$-increasing, then there is a unique $\mathbf{q}_\delta \in K$ such that $\alpha < \delta \to \mathbf{q}_\alpha \leq_K \mathbf{q}_\delta$ and $U_{\mathbf{q}_\delta} = \bigcup_{\alpha<\delta} U_{\mathbf{q}_\alpha}$.

e. If $U_0, U_1, U_2 \in K_{wf}$, $U_0 = U_1 \cap U_2$ and $U_0 \leq_{wf} U_l$ $(l = 1, 2)$, then there is a unique $U \in K_{wf}$ such that $\bigwedge_{l=1,2} U_l \leq_{wf} U$, $\alpha \in U$ iff $\alpha \in U_1 \vee \alpha \in U_2$ and $\leq_U = \leq_{U_1} \cup \leq_{U_2}$. We denote this U by $U_1 +_{U_0} U_2$.

f. If $\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2 \in K$, $\mathbf{q}_0 \leq_K \mathbf{q}_l$ $(l = 1, 2)$ and $U_{\mathbf{q}_0} = U_{\mathbf{q}_1} \cap U_{\mathbf{q}_2}$, then there is a unique $\mathbf{q} \in K$ such that $\bigwedge_{l=1,2} \mathbf{q}_l \leq_K \mathbf{q}$ and $U_\mathbf{q} = U_{\mathbf{q}_1} +_{U_{\mathbf{q}_0}} U_{\mathbf{q}_2}$. We shall denote this $\mathbf{q}$ by $\mathbf{q}_1 +_{\mathbf{q}_0} \mathbf{q}_2$.

g. If $\alpha \in U^+_\mathbf{q}$, then $\mathbb{P}_{\mathbf{q},\alpha}$ is a $(< \theta)$-complete forcing satisfying $*^\epsilon_\theta$ (hence θ^+-cc).

h. Suppose that $\mathbf{q} \in K$ and $\underset{\sim}{\mathbb{Q}}$ is a $\mathbb{P}_{\mathbf{q},\infty}$-name of a forcing notion whose universe is from V, such that the conditioncs of definitions 3(d) and 3(e) are satisfied, then there is $\mathbf{q}' \in K$ such that $\mathbf{q} \leq_K \mathbf{q}'$, $U_{\mathbf{q}'} = U_\mathbf{q} \cup \{\gamma\}$, $U_{\mathbf{q}'} \models \alpha < \gamma$ for every $\alpha \in U_\mathbf{q}$ and $\underset{\sim}{\mathbb{Q}}_{\mathbf{q}',\gamma} = \underset{\sim}{\mathbb{Q}}$. $\square$

Definition 5: The forcing notion $\mathbb{P}$ will be defined as follows:

a. The conditions of $\mathbb{P}$ are the elements $\mathbf{q}$ of $K \cap H(\lambda^+)$ such that $U_\mathbf{q} \subseteq \lambda^+$, and for every $\beta \in U_\mathbf{q}$, $\underset{\sim}{\mathbb{Q}}_\beta$ is a name for a forcing whose underlying set of conditions is some $X_\beta \subseteq \lambda^+$.

b. Given $\mathbf{q}_1, \mathbf{q}_2 \in \mathbb{P}$, $\mathbb{P} \models$ "$\mathbf{q}_1 \leq \mathbf{q}_2$" iff $\mathbf{q}_1 \leq_K \mathbf{q}_2$.

c. Given a generic set $G \subseteq \mathbb{P}$, we let $\mathbf{q}_G = \cup\{\mathbf{q} : \mathbf{q} \in G\}$.

Claim 6: a. $\mathbb{P}$ is $(< \kappa)$-strategically complete. Moreover, it's $(< \lambda^+)$-complete and $(< \theta)$-directed closed.

b. $\Vdash_\mathbb{P}$ "$\underset{\sim}{\mathbf{q}_G} \in K$", hence $\Vdash_\mathbb{P}$ "$\mathbb{P}_{\underset{\sim}{\mathbf{q}_G},\infty}$ is $(< \theta)$-directed closed and θ^+-cc".

c. If $\delta < \lambda^+$, $cf(\delta) > \theta$ and $(\mathbf{q}_\alpha : \alpha < \delta)$ is $\leq_\mathbb{P}$-increasing, then $\mathbf{q} := \bigcup_{\alpha<\delta} \mathbf{q}_\alpha$ belongs to $\mathbb{P}$ and $\mathbb{P}_\mathbf{q} = \bigcup_{\alpha<\delta} \mathbb{P}_{\mathbf{q}_\alpha}$. By θ^+-c.c., $\underset{\sim}{a}$ is a canonical $\mathbb{P}_\mathbf{q}$-name of a member of $[\theta]^\theta$ iff $\underset{\sim}{a}$ is a canonical $\mathbb{P}_{\mathbf{q}_\alpha}$-name of a member of $[\theta]^\theta$ for some $\alpha < \delta$.

Proof: The claim follows directly from the definitions. The fact that $\Vdash_\mathbb{P}$ "$\underset{\sim}{\mathbf{q}_G} \in K$" follows from the general fact that if I is a directed set, $\{\mathbf{q}_t : t \in I\} \subseteq K$ and $s \leq_I t \to \mathbf{q}_s \leq_K \mathbf{q}_t$, then $\cup\{\mathbf{q}_t : t \in I\}$ is well-defined and belongs to K. This also shows that $\mathbb{P}$ is $(< \theta)$-directed closed. $\square$

We shall now define our desired model:

Definition 7: a. In $V^\mathbb{P}$, let $\mathbb{Q} = \mathbb{P}_{\underset{\sim}{\mathbf{q}_G},\infty}$.

b. Let $V_2 = V^{\mathbb{P} \star \underset{\sim}{\mathbb{Q}}}$.

c. Let V_1 be $HOD(\mathcal{P}(\theta)^{<\kappa} \cup V)$ inside V_2.

Claim 8: a. $V_1 \models ZF + DC_{<\kappa}$.

b. $(Ord^{<\kappa})^{V_1} = (Ord^{<\kappa})^{V_2}$, hence $\mathcal{P}(\theta)^{V_1} = \mathcal{P}(\theta)^{V_2}$.

Proof: We shall prove the first part of clause (b), the rest should be clear. Clearly, $(Ord^{<\kappa})^{V_1} \subseteq (Ord^{<\kappa})^{V_2}$. Now let $\eta \in (Ord^{\gamma})^{V_2}$ for some $\gamma < \kappa$, then $\eta = \underset{\sim}{\eta}[G]$ for some name $\underset{\sim}{\eta}$ of a member of Ord^{γ}, where $G \subseteq \mathbb{P} \star \underset{\sim}{\mathbb{Q}}$ is generic. $G = G_1 \star G_2$ where $G_1 \subseteq \mathbb{P}$ is generic and $G_2 \subseteq \underset{\sim}{\mathbb{Q}}[G_1]$ is generic. Working in $V[G_1]$, $\underset{\sim}{\eta}/G_1$ is a $\underset{\sim}{\mathbb{Q}}[G_1]$-name. As $\underset{\sim}{\mathbb{Q}}[G_1]$ is θ^+-cc, for every $\beta < \gamma$ there is a maximal antichain $\{p_{\beta,i} : i < \theta\} \subseteq \underset{\sim}{\mathbb{Q}}[G_1]$ of conditions that force a value to $\underset{\sim}{\eta}/G_1(\beta)$. Let $\{\zeta_{\beta,i} : i < \theta\}$ be the set corresponding values forced by the above conditions. Let $\Gamma = \{\underset{\sim}{p}_{\beta,i}, \underset{\sim}{\zeta}_{\beta,i} : \beta < \gamma, i < \theta\}$ be the corresponding $\mathbb{P}$-names for the above objects (so we can regard them as $\mathbb{P}$-names for ordinals). As there are $< \kappa$ such names and $\mathbb{P}$ is $(< \kappa)$-strategically complete, there is a dense set of $\mathbf{q} \in \mathbb{P}$ that force values to all elements of Γ. Therefore, there is some $\mathbf{q} \in \mathbb{P} \cap G_1$ that forces values to all elements of Γ (and the values forced are necessarily $\{p_{\beta,i}, \zeta_{\beta,i} : \beta < \gamma, i < \theta\}$). It follows that $\{p_{\beta,i}, \zeta_{\beta,i} : \beta < \gamma, i < \theta\} \in V$. In V_2, there is a function $f : \gamma \to \theta$ such that for every $\beta < \gamma$, $\eta(\beta) = \zeta_{\beta,f(\beta)}$. As $f \in \mathcal{P}(\theta)^{<\kappa}$ and $\{p_{\beta,i}, \zeta_{\beta,i} : \beta < \gamma, i < \theta\} \in V$, it follows that $\eta \in V_1$. $\square$

Main Claim 9: There are no θ-mad families in V_1.

The rest of the paper will be devoted to the proof of Claim 9.

Suppose towards contradiction that there is a θ-mad family in V_1, so there is some $(\underset{\sim}{\mathbf{q}}_0, \underset{\sim}{p}_0) \in \mathbb{P} \star \underset{\sim}{\mathbb{Q}}$ forcing this statement about $\underset{\sim}{\mathcal{A}}$ where $\underset{\sim}{\mathcal{A}}$ is a canonical $\mathbb{P} \star \underset{\sim}{\mathbb{Q}}$-name of a θ-mad family definable using $\underset{\sim}{\eta}$, and $\underset{\sim}{\eta}$ is a canonical $\mathbb{P} \star \underset{\sim}{\mathbb{Q}}$-name of a parameter (so $\underset{\sim}{\eta} = ((\underset{\sim}{a}_\epsilon : \epsilon < \underset{\sim}{\epsilon}(*)), \underset{\sim}{x})$, where $\Vdash$ "$\underset{\sim}{\epsilon}(*) < \kappa$", each $\underset{\sim}{a}_\epsilon$ is a $\mathbb{P} \star \underset{\sim}{\mathbb{Q}}$-name of a subset of θ and $\Vdash$ "$\underset{\sim}{x} \in V$"). Let $G_0 \subseteq \mathbb{P}$ be generic over V such that $\mathbf{q}_0 \in G_0$. In $V[G_0]$, $\underset{\sim}{\eta}$ is a $\mathbb{P}_{\mathbf{q}_{G_0},\infty}$-name, and by increasing $\mathbf{q}_0$, we may assume wlog that $p_0 := \underset{\sim}{p}_0[G_0] \in \mathbb{P}_{\mathbf{q}_0}$, $x = \underset{\sim}{x}[G_0] \in V$, $\epsilon(*) = \underset{\sim}{\epsilon}(*)[G_0] \in \kappa$ and that each $\underset{\sim}{a}_\epsilon$ $(\epsilon < \epsilon(*))$ is a canonical $\mathbb{P}_{\mathbf{q}_0}$-name of a subset of θ. Given $\mathbf{q} \in \mathbb{P}$ above $\mathbf{q}_0$, let $\mathcal{A}_{\mathbf{q}}$ be the set of canonical $\mathbb{P}_{\mathbf{q}}$-names $\underset{\sim}{a}$ such that $(\mathbf{q}, \underset{\sim}{p}_0) \Vdash_{\mathbb{P} \times \mathbb{Q}}$ "$\underset{\sim}{a} \in \underset{\sim}{\mathcal{A}}$", so $\mathbf{q}_0 \leq \mathbf{q}_1 \leq \mathbf{q}_2 \to \mathcal{A}_{\mathbf{q}_1} \subseteq \mathcal{A}_{\mathbf{q}_2}$. Note that if $\mathbf{q}_0 \leq \mathbf{q}_1$, $\mathbb{P}_{\mathbf{q}_1,\infty} \models$ "$p_0 \leq p_1$" and $(\mathbf{q}_1, p_1) \Vdash$ "$\underset{\sim}{b} \in [\theta]^{\theta}$", then for some $(\mathbf{q}_2, \underset{\sim}{a})$ we have $\mathbf{q}_1 \leq_{\mathbb{P}} \mathbf{q}_2$, $\underset{\sim}{a} \in \mathcal{A}_{\mathbf{q}_2}$ and $(\mathbf{q}_2, p_0) \Vdash$ "$\underset{\sim}{b} \cap \underset{\sim}{a} \in [\theta]^{\theta}$". By extending any given $\mathbf{q}_1 \in \mathbb{P}$ above $\mathbf{q}_0$ in this way sufficiently many times to add witnesses for madness, and recalling Claim 6(c), we establish that the set $\{\mathbf{q}_1 : \mathbf{q}_0 \leq_{\mathbb{P}} \mathbf{q}_1$ and $\Vdash_{\mathbb{P}_{\mathbf{q}_1}}$ "$\mathcal{A}_{\mathbf{q}_1}$ is θ-mad"$\}$ is dense in $\mathbb{P}$ above $\mathbf{q}_0$.

Now, in V_2, let $I = \{A \subseteq \theta : A$ is contained in a union of $< \theta$ members of $\mathcal{A}\}$, then I is a θ-complete ideal and $\theta \notin I$. Let F be the dual filter of I, then F is θ-complete, and as θ is supercompact in V_2 (recalling that θ is Laver indestructible and that $\mathbb{P} \star \underset{\sim}{\mathbb{Q}}$ is $(< \theta)$-directed closed), there is a $\mathbb{P} \star \underset{\sim}{\mathbb{Q}}$-name $\underset{\sim}{D}$ such that $(\mathbf{q}_0, p_0) \Vdash_{\mathbb{P} \star \mathbb{Q}}$ "$\underset{\sim}{D}$ is a θ-complete ultrafilter on θ that extends $\underset{\sim}{F}$, and hence is disjoint to $\underset{\sim}{\mathcal{A}}$". By Claim 6 and a previous observation, we may assume wlog that $\mathbf{q}_0 \Vdash_{\mathbb{P}}$ "$\mathcal{A}_{\mathbf{q}_0}$ is θ-mad and $\underset{\sim}{D}_{\mathbf{q}_0} := \underset{\sim}{D} \cap \mathcal{P}(\theta)^{V^{\mathbb{P}_{\mathbf{q}_0,\infty}}}$ is a $\mathbb{P}_{\mathbf{q}_0,\infty}$-name of an ultrafilter on θ".

Given an ultrafilter U on θ, the forcing $\mathbb{Q}_U$ is defined as follows: the conditions of $\mathbb{Q}_U$ have the form (u, A) where $u \in [\theta]^{<\theta}$ and $A \in U$. the order is defined naturally, i.e. $(u_1, A_1) \leq (u_2, A_2)$ iff $u_1 \subseteq u_2$, $u_2 \setminus u_1 \subseteq A_1$ and $A_2 \subseteq A_1$.

We may assume wlog that $\mathbb{P}_{\mathbf{q}_0,\infty}$ forces $2^\theta=\lambda$, hence there is a canonical $\mathbb{P}_{\mathbf{q}_0,\infty}$-name $\underset{\sim}{f}$ of a bijection from $\mathbb{Q}_{\underset{\sim}{D}_{\mathbf{q}_0}}$ onto λ. Let $\underset{\sim}{\mathbb{Q}}'$ be a name for the forcing such that $\Vdash_{\mathbb{P}_{\mathbf{q}_0}}$ "$\underset{\sim}{f}$ is an isomorphism from $\mathbb{Q}_{\underset{\sim}{D}_{\mathbf{q}_0}}$ onto $\underset{\sim}{\mathbb{Q}}'$". Let $\underset{\sim}{B}=\underset{\sim}{B}_{\underset{\sim}{D}_{\mathbf{q}_0}}$ be the $\mathbb{Q}_{\underset{\sim}{D}_{\mathbf{q}_0}}$-name $\cup\{u:(u,A)\in G_{\mathbb{Q}_{\underset{\sim}{D}_{\mathbf{q}_0}}}\}$, so $\Vdash_{\mathbb{P}_{\mathbf{q}_0,\infty}\star\mathbb{Q}_{\underset{\sim}{D}_{\mathbf{q}_0}}}$ "$\underset{\sim}{B}\in[\theta]^\theta$ is θ-almost disjoint to $\mathcal{A}_{\mathbf{q}_0}$". Let $\underset{\sim}{B}'$ be the canonical $\mathbb{P}_{\mathbf{q}_0,\infty}\star\mathbb{Q}_{\underset{\sim}{D}_{\mathbf{q}_0}}$-name for the image of $\underset{\sim}{B}$ under $\underset{\sim}{f}$.

Now observe that there is $\mathbf{q}'\in\mathbb{P}$ such that $\mathbf{q}_0\le_{\mathbb{P}}\mathbf{q}'$, $U_{\mathbf{q}'}=U_{\mathbf{q}_0}\cup\{\gamma\}$, $\alpha<_{U_{\mathbf{q}'}}\gamma$ for every $\alpha\in U_{\mathbf{q}_0}$ and $\underset{\sim}{\mathbb{Q}}_{\mathbf{q}',\gamma}=\underset{\sim}{\mathbb{Q}}'$. As before, there is $\mathbf{q}''\in\mathbb{P}$ above $\mathbf{q}'$ such that $p_0\Vdash_{\mathbb{P}_{\mathbf{q}'',\infty}}$ "$\mathcal{A}_{\mathbf{q}''}$ is θ-mad". Therefore, there is some canonical $\mathbb{P}_{\mathbf{q}''}$-name $\underset{\sim}{A}\in\mathcal{A}_{\mathbf{q}''}$ such that $p_0\Vdash_{\mathbb{P}_{\mathbf{q}'',\infty}}$ "$\underset{\sim}{A}\cap\underset{\sim}{B}'\in[\theta]^\theta$, so $\underset{\sim}{A}$ has intersection of size θ with every member of $\underset{\sim}{D}_{\mathbf{q}_0}$ and $\underset{\sim}{A}\notin\mathcal{A}_{\mathbf{q}_0}$".

Now let $(\mathbf{q}_1,\underset{\sim}{B}_1,\underset{\sim}{A}_1)=(\mathbf{q}'',\underset{\sim}{B}',\underset{\sim}{A})$ and let $(\mathbf{q}_2,\underset{\sim}{B}_2,\underset{\sim}{A}_2)$ be an isomorphic copy of $(\mathbf{q}_1,\underset{\sim}{B}_1,\underset{\sim}{A}_1)$ over $\mathbf{q}_0$ such that $U_{\mathbf{q}_1}\cap U_{\mathbf{q}_2}=U_{\mathbf{q}_0}$ and $\mathbf{q}_2\in\mathbb{P}$.

Claim 10: Let $\mathbf{q}_0$, $(\mathbf{q}_1,\underset{\sim}{B}_1,\underset{\sim}{A}_1)$ and $(\mathbf{q}_2,\underset{\sim}{B}_2,\underset{\sim}{A}_2)$ be as above (so $\mathbf{q}_0\le_K\mathbf{q}_l$ $(l=1,2)$, $U_{\mathbf{q}_1}\cap U_{\mathbf{q}_2}=U_{\mathbf{q}_0}$ and $\bigwedge\limits_{l=1,2}\Vdash_{\mathbb{P}_{\mathbf{q}_l,\infty}}$ "$\underset{\sim}{A}_l\in\underset{\sim}{\mathcal{A}}\setminus\mathcal{A}_{\mathbf{q}_0}$") and let $G\subseteq\mathbb{P}_{\mathbf{q}_0,\infty}$ be generic over V, then $\Vdash_{\mathbb{P}_{\mathbf{q}_1,\infty}/G\times\mathbb{P}_{\mathbf{q}_2,\infty}/G}$ "$\underset{\sim}{A}_2\setminus\underset{\sim}{A}_1,\underset{\sim}{A}_1\setminus\underset{\sim}{A}_2\in[\theta]^\theta$".

Proof: We shall prove the claim for $A_2\setminus A_1$, the other case is similar. Suppose towards contradiction that (p_1,p_2) forces that $\underset{\sim}{A}_2\setminus\underset{\sim}{A}_1\subseteq\gamma<\theta$. For $l\in\{1,2\}$, let $B_l=\{\epsilon<\theta:p_l\nVdash_{\mathbb{P}_{\mathbf{q}_l,\infty}/G}$ "$\epsilon\notin\underset{\sim}{A}_l$"$\}\in V[G]$. By the assumption of the claim, $B_l\in[\theta]^\theta$. By the θ-madness of $\underset{\sim}{\mathcal{A}}_0[G]$ in $V[G]$, there is some $Y\in\underset{\sim}{\mathcal{A}}_0[G]$ such that $|Y\cap B_2|=\theta$. As $p_1\Vdash_{\mathbb{P}_{\mathbf{q}_1,\infty}/G}$ "$|\underset{\sim}{A}_1\cap Y|<\theta$", there are q_1 and $\beta_1<\theta$ such that $p_1\le q_1\in\mathbb{P}_{\mathbf{q}_1,\infty}/G$ and $q_1\Vdash_{\mathbb{P}_{\mathbf{q}_1,\infty}/G}$ "$\underset{\sim}{A}_1\cap Y\subseteq\beta_1$". Let $\beta_2\in Y\cap B_2$ such that $max\{\gamma,\beta_1\}<\beta_2$ (recalling that $|Y\cap B_2|=\theta$). By the definition of B_2, there is $q_2\in\mathbb{P}_{\mathbf{q}_2,\infty}/G$ above p_2 that forces "$\beta_2\in\underset{\sim}{A}_2$". Therefore, $(p_1,p_2)\le(q_1,q_2)\in\mathbb{P}_{\mathbf{q}_1,\infty}/G\times\mathbb{P}_{\mathbf{q}_2,\infty}/G$ and $(q_1,q_2)\Vdash_{\mathbb{P}_{\mathbf{q}_1,\infty}/G\times\mathbb{P}_{\mathbf{q}_2,\infty}/G}$ "$\beta_2\in\underset{\sim}{A}_2\setminus\underset{\sim}{A}_1$", a contradiction. It follows that $\Vdash_{\mathbb{P}_{\mathbf{q}_1,\infty}/G\times\mathbb{P}_{\mathbf{q}_2,\infty}/G}$ "$\underset{\sim}{A}_2\setminus\underset{\sim}{A}_1\in[\theta]^\theta$". $\square$

Claim 11: Under the assumptions of Claim 10 (recalling that $\Vdash_{\mathbb{P}_{\mathbf{q}_l,\infty}}$ "$\underset{\sim}{A}_l\cap B\neq\emptyset$ for every $B\in\underset{\sim}{D}_{\mathbf{q}_0}$" $(l=1,2)$), we have $\Vdash_{\mathbb{P}_{\mathbf{q}_1,\infty}/G\times\mathbb{P}_{\mathbf{q}_2,\infty}/G}$ "$\underset{\sim}{A}_1\cap\underset{\sim}{A}_2\in[\theta]^\theta$".

Proof: Assume towards contradiction that $(p_1,p_2)\in\mathbb{P}_{\mathbf{q}_1,\infty}/G\times\mathbb{P}_{\mathbf{q}_2,\infty}/G$ forces that $\underset{\sim}{A}_1\cap\underset{\sim}{A}_2\subseteq\gamma$ for some $\gamma<\theta$. It's forced by (p_1,p_2) that $\underset{\sim}{A}_l\subseteq B_l$ $(l=1,2)$ where B_l is as in the proof of the previous claim, hence it's forced by (p_1,p_2) that each B_l intersects each member of $\underset{\sim}{D}_{\mathbf{q}_0}$. As $B_1,B_2\in V[G]$, it follows that $B_1,B_2\in\underset{\sim}{D}_{\mathbf{q}_0}[G]$. Therefore, there is some $\beta\in(B_1\cap B_2)\setminus\gamma$, hence there is $q_l\in\mathbb{P}_{\mathbf{q}_l,\infty}/G$ above p_l that forces "$\beta\in\underset{\sim}{A}_l$" $(l=1,2)$. It follows that $(p_1,p_2)\le(q_1,q_2)\in\mathbb{P}_{\mathbf{q}_1,\infty}/G\times\mathbb{P}_{\mathbf{q}_2,\infty}/G$ and $(q_1,q_2)\Vdash_{\mathbb{P}_{\mathbf{q}_1,\infty}/G\times\mathbb{P}_{\mathbf{q}_2,\infty}/G}$ "$\beta\in\underset{\sim}{A}_1\cap\underset{\sim}{A}_2$", contradicting the choice of γ and (p_1,p_2). It follows that $\Vdash_{\mathbb{P}_{\mathbf{q}_1,\infty}/G\times\mathbb{P}_{\mathbf{q}_2,\infty}/G}$ "$\underset{\sim}{A}_1\cap\underset{\sim}{A}_2\in[\theta]^\theta$". $\square$

Now given $\mathbf{q}_0$, $(\mathbf{q}_1, \underset{\sim}{B}_1, \underset{\sim}{A}_1)$ and $(\mathbf{q}_2, \underset{\sim}{B}_2, \underset{\sim}{A}_2)$ as above, let $\mathbf{q}_3 = \mathbf{q}_1 +_{\mathbf{q}_0} \mathbf{q}_2$. Then $\mathbf{q}_3 \in \mathbb{P}$, $\mathbf{q}_1, \mathbf{q}_2 \leq_K \mathbf{q}_3$, and by claims 10 and 11, we get a contradiction. This completes the proof of Main Claim 9 and hence of Theorem 2. $\square$

We conclude with the following natural question:

Question: What's the consistency strength of $ZF + DC_\theta +$ "there are no θ-mad families" for some $\theta > \aleph_0$?

References

[HwSh:1090] Haim Horowitz and Saharon Shelah, Can you take Toernquist's inaccessible away? arXiv:1605.02419

[HwSh:1093] Haim Horowitz and Saharon Shelah, Transcendence bases, well-orderings of the reals and the axiom of choice, arXiv:1901.01508

[HwSh:1113] Haim Horowitz and Saharon Shelah, Madness and regularity properties, arXiv:1704.08327

[HwSh:1145] Haim Horowitz and Saharon Shelah, κ-Madness and Definability, arXiv:1805.07048

[Ma] A. R. D. Mathias, Happy families, Ann. Math. Logic **12** (1977), no. 1, 59-111

[Sh:218] Saharon Shelah, On measure and category, Israel J. Math. 52 (1985) 110-114

[To] Asger Toernquist, Definability and almost disjoint families, Advances in Mathematics 330, 61-73, 2018

[Vi] Matteo Viale, Category forcings, MM^{+++}, and generic absoluteness for the theory of strong forcing axioms, J. Amer. Math. Soc. 29 (2016), no. 3, 675-728

(Haim Horowitz) Department of Mathematics

University of Toronto

Bahen Centre, 40 St. George St., Room 6290

Toronto, Ontario, Canada M5S 2E4

E-mail address: haim@math.toronto.edu

(Saharon Shelah) Einstein Institute of Mathematics

Edmond J. Safra Campus,

The Hebrew University of Jerusalem.

Givat Ram, Jerusalem 91904, Israel.

Department of Mathematics

Hill Center - Busch Campus,

Rutgers, The State University of New Jersey.

110 Frelinghuysen Road, Piscataway, NJ 08854-8019 USA
E-mail address: shelah@math.huji.ac.il