APPLYING SET THEORY
E88

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Abstract. We prove some results in set theory as applied to general topology and model theory. In particular, we study $\aleph_1$-collectionwise Hausdorff, Chang Conjecture for logics with Malitz-Magidor quantifiers and monadic logic of the real line by odd/even Cantor sets.

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§ 0. Introduction

In §1 we prove a result in general topology saying: if $\diamondsuit^*_{\aleph_1}$ then any normal space is $\aleph_1 - \mathrm{CWH}$ (= collectionwise Hausdorff); done independently of and in parallel to Fleisner and Alan D. Taylor.

In §2 we prove the Chang Conjecture for Magidor-Malitz Quantifiers. A recent related work is [HU17].

In §3 we prove the Monadic Theory of the tree ${}^{\omega>}2$ is complicated under a quite weak set theoretic assumption. Earlier [She75] proved this (i.e. the result on the monadic logic) assuming CH or at least a consequence of it.

The present note was circulated in the Spring of 1979 in a collection; it include each of the sections (as well as other preprints) but those three were not published. However, [She85]. have results related to section three; in particular it was conjectured there (in Remark 2.15) that there are two non-principal ultrafilters of ω with no common lower bound in the Rudin Keisler order; a conjecure which had been refuted in [BS87].

Later, Gurevich-Shelah [GS82] proved undecidability in ZFC, with further developments then more in Shelah [She88], still the older proof gives information not covered by them. For more see [BS87], [GKKS02], [GGK04].

The results are old, still in particular, §1 gives a direct proof of the result compared to others and §3 gives a considerably more transparent easier proof of the result of [GS82].

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§ 1. A note in general topology if $\diamondsuit^*_{\aleph_1}$ then any normal space is $\aleph_1 - \text{CWH}$ (= collectionwise Hausdorff)

The normal Moore space problem has been a major theme in general topology, see the recent survey Dow-Tall [DT18]. In this connection, Fleissner [Fle84, p.6] proved: ($\mathbf{V} = \mathbf{L}$) every normal first countable (topological) space is CWH (CWH means collectionwise Hausdorff). He used a strengthening of diamond. The author proved Fleissner strengthening (for $\aleph_1$) does not follow from ZFC$+\diamondsuit^+_{\aleph_1}$ (see [She81, Th.5,pg.31]). Here we prove nevertheless $\diamondsuit^*_{\aleph_1}$ implies every normal first countable space is $\aleph_1 - \text{CWH}$.

The central idea of the proof in §1 is inspired by one key idea in Fleissner [Fle84]. Fleissner implicitly used a stronger combinatorial principle $\diamondsuit_{\text{SS}}$. In 1979, the author and independently both Fleissner and Alan D. Taylor all saw (as mentioned in [Tay81], [SS00] that a weaker principle, $\diamondsuit^*_{\omega_1}$, would suffice. Later Smith and Szeptycki [SS00] derive better results. On more recent results on diamond and strong negation see [She10] and references there.

Convention 1.1. Below δ always denotes a limit ordinal ($< \omega_1$).

For transparency, below we refer to the following equivalent form of $\diamondsuit^*_{\omega_1}$.

Definition 1.2. Let $\diamondsuit^*_{\aleph_1}$ mean that there exist a sequence $\langle \mathbf{g}_\delta : \delta < \omega_1 \rangle$ where $\mathbf{g}_\delta = \langle \bar{g}^{\delta,k} : k < \omega \rangle$ is of the form $\bar{g}^{\delta,k} = \langle g^{\delta,k}_n : n < \omega \rangle$, where $g^{\delta,k}_n : \delta \to \omega$ has the property that, for any sequence $\bar{g} = \langle g_n : n < \omega \rangle$ with $g_n : \{\delta : \delta < \omega_1\} \to \omega$, there is a club (closed unbounded) set $C \subseteq \omega_1$ such that, for each $\gamma \in C$, there is $k = k(\gamma) \in \omega$ with

$$\bar{g} \restriction \gamma := \langle g_n \restriction \gamma : n < \omega \rangle = \bar{g}^{\gamma,k} = \langle g^{\gamma,k}_n : n < \omega \rangle.$$

Claim 1.3. *Assume $\diamondsuit^*_{\aleph_1}$. If X is Hausdorff first countable normal and $|X| = \aleph_1$ then X is* CWH*.*

Proof. Let $\langle \mathbf{g}_\delta : \delta < \omega_1 \rangle$ be as in 1.2.

Without loss of generality $X_* = \{\delta : \delta < \omega_1\} \subseteq X$ and X_* is closed discrete in the space X. Let $U^\delta_n (n < \omega)$ be a basis of open neighborhoods of δ (for $\delta < \omega_1$). We shall define by induction on $\alpha < \omega_1$ a limit ordinal $\gamma_\alpha < \omega_1$ and $\langle f_n(\gamma) : n < \omega, \gamma < \gamma_\alpha \rangle$ such that γ_α is increasing continuous with α and $\gamma_0 = 0$. For $\alpha = 0$ choose $\gamma_\alpha = \omega$; $f_n(\gamma) = 0$. For limit α let γ_α be $\cup\{\gamma_\beta : \beta < \alpha\}$. For $\alpha = \beta + 1$ if $\gamma_\alpha > \alpha$ then we let $\gamma_\alpha = \gamma_\beta + \omega$ and let $f_n(\gamma) = 0$ for $\gamma \in [\gamma_\beta, \gamma_\alpha)$. Finally assume that $\alpha = \delta^*, \gamma_{\delta^*} = \delta^*$ so $\delta^* \in X_*$.

We have chosen above the functions $\langle g^{\delta^*,k}_n : n < \omega, k < \omega \rangle$ with $g^{\delta^*,k}_n : \delta^* \to \omega$; now for each $n, k < \omega$ let $A^{\delta^*,n,k}_\ell = \cup\{U^\delta_{g^{\delta^*,k}_n(\delta)} : \delta < \delta^*, f_n(\delta) = \ell\}$ (for $n < \omega, \ell < 2$). Call $k < \omega$ <u>good</u> for δ^* when for infinitely many (pairs) n, ℓ we have

$$B^{\delta^*,n,k}_\ell := \text{cl}(A^{\delta^*,n,k}_\ell) \cap (X_* \setminus \delta^*) \neq \emptyset.$$

We let $\gamma_\alpha = \gamma_{\delta^*+1} = \min\{\delta : \delta > \delta^*$ and if $\ell < 2$ and $n, k < \omega$ and $B^{\delta^*,n,k}_\ell \neq \emptyset$ then $(\delta^*, \delta) \cap B^{\delta^*,n,k}_\ell \neq \emptyset\}$.

Now we choose $f_n \restriction [\delta^*, \gamma_\alpha)$ for $n < \omega$ such that such that for any k good for δ^*, for some $n, \ell, \delta \geq \delta^*$ we have

$$f_n(\delta) = 1 - \ell(\text{ so } \delta \in \mathrm{cl}(A_\ell^{\delta^*,n,k})).$$

Then we complete arbitrarily the f_n so that its domain is γ_α.

Thus we have defined $f_n(n < \omega)$ with $f_n : \omega_1 \to 2$. For each n the sets $f_n^{-1}\{1\} \cap X_*, f_n^{-1}\{0\} \cap X_*$ form a partition of X_*, both are closed and discrete subsets of X. But X is normal. So there are functions $g_n : X_* \to \omega$ for $n < \omega$ so that letting for $\ell = 0, 1$

$$A_\ell^n = \cup\{U^\delta_{g_n(\delta)} : \delta \in X_*, f_n(\delta) = \ell\}$$

we have

$$A_0^n \cap A_1^n = \emptyset.$$

Let g_n^+ be any function from ω_1 to ω extending g_n. For some closed unbounded set $C \subseteq X_*$ we have: $\delta^* \in C \Rightarrow (\exists k)(\langle g_n^+ {\restriction} \delta^* : n < \omega\rangle = \langle g_n^{\delta^*,k} : n < \omega\rangle)$. Let the first such k be denoted $k(\delta^*)$. Without loss of generality every $\delta^* \in C$ satisfy $\gamma_{\delta^*} = \gamma$ hence if $\delta^* \in C \wedge n < \omega \wedge k < \omega \wedge \ell < 2$ and $B_\ell^{\delta^*,n,k} = \mathrm{cl}(A_\ell^{\delta^*,n,k}) \cap (X_* \backslash \delta^*) \neq \emptyset$ then $\min(B_\ell^{\delta^*,n,k}) < \min(C \backslash (\delta^* + 1))$.

For $\delta^* \in C$ now $k(\delta^*)$ cannot be good for δ^*, (by the definition).

Now for at least one n (in fact, for infinitely many n-s) we have $\mathrm{cl}(A_\ell^n | \delta^*) \cap (X_* \setminus \delta^*) = \emptyset$ for $\ell \in \{0, 1\}$, let $n(\delta^*)$ be the first such n.

Define

$$B_n = \{\delta : \text{ for some } \delta^* \in C \cup \{0\} \text{ we have } \delta^* \leq \delta < \min(C \backslash \delta) \text{ and } n = \max\{n(\delta^*), n(\delta)\}$$

Now $\bigcup\limits_n (g_n {\restriction} B_n)$ almost exhibits X_* has the right sequence of neighborhoods. So we can deal with each B_n separately (just choose $\mathscr{U}_n$ by induction on n such that $\mathscr{U}_n$ is open, $\mathscr{U}_n \cap X_* = B_n$ and $\mathscr{U}_n \subseteq X \backslash c\ell(\bigcup\limits_{\ell < n} \mathscr{U}_\ell)$, possible by normality).

By dealing as follows with each interval $[\delta^*, \min(C \backslash (\delta^* + 1))$ for $\delta^* \in C \cup \{0\}$ we have $U^\delta_{g_n(\delta)} (\delta \in B_n)$ as required.

That is, for $\gamma \in C \cup \{0\}$ with γ^+ its successor in C, choose a (countable) family of pairwise disjoint open sets $\mathscr{U}_\gamma(\beta)$ for $\beta \in X_* \wedge \gamma \leq \beta < \gamma^+$, with $\beta \in \mathscr{U}_\gamma(\beta)$, this is possible as in the choice of the $\mathscr{U}_n$'s.

Now for $\beta \in X_*$ we let $W_\beta = \mathscr{U}_{n(\beta)} \cap \mathscr{U}_{\gamma(\beta)}(\beta) \cap \mathscr{U}^\beta_{g_{n(\beta)}(\beta)}$ where:

- $\gamma(\gamma) = \max(C \cap (\beta + 1))$
- $m(\beta) = \max\{n(\delta^*), n((\delta^*)^+) : \delta^* = \max(C \cap \beta) \leq \beta < (\delta^*)^+\}$

Finally $\langle W_\beta : \beta \in X_* \rangle$ is a sequence of pairwise disjoint open sets of X with $\beta \in X_* \Rightarrow \beta \in W_\beta$, so we are done. $\square_{1.3}$

Remark 1.4. As in [Fle84] it suffices to assume every point in the space has a neighborhood basis of cardinality $\aleph_1$.

§ 2. Chang Conjecture for Magidor-Malitz Quantifiers

Silver (see [Sil71]) had proved the consistency of Chang conjecture, i.e.

$\oplus$ any model M with universe $\aleph_2$ and countable signature (=vocabulary) τ, has an elementary submodel $N, \|N\| = \aleph_1$ and $|N| \cap \omega_1|$ is countable.

Silver did this by starting with a model $\mathbf{V}$ with κ Ramsey (in fact, something weaker suffices), forcing MA and then collapsing κ to $\aleph_2$ by $\mathbb{P}^\kappa_{\text{Set}} = \{f : \text{Dom}(f) \subseteq \{\mu : \aleph_1 < \mu < \kappa, \mu \text{ a cardinal}\}$ has cardinality $\leq \aleph_1$, and for some $\alpha < \omega_1$, $(\forall \mu \in \text{Dom} f)(f(\mu)$ is a function from α to $\mu)\}$. See also Koszmider [Kos05] for a topological application.

We can ask whether this submodel N can inherit more properties from M.

Definition 2.1. Let us define a (technical variant of) Magidor-Malitz quantifiers.

$M \models (Q^n \bar{x})\varphi(x_1, \ldots, x_n)$ means that there is a set $A \subseteq M, A$ is of cardinality $\|M\|$ such that $(\forall a_1 \ldots a_n \in A)\varphi(a_1 \ldots a_n)$.

The result is that:

Claim 2.2. *In $\oplus$ above, we can have N an elementary submodel of M even for the logic $\mathbb{L}(Q^0, Q^1, \ldots)_{n<\omega}$. So e.g. Suslinity of trees is preserved.*

For this we need the following.

Definition 2.3. Call a forcing $\mathbb{P}$ suitable <u>when</u> for any sequence $\langle p_i : i < \omega_1\rangle$ of members of $\mathbb{P}$ there is a set $\mathscr{U} \subseteq \omega_1$ of cardinality $\aleph_1$ such that: for any finite $u \subseteq \mathscr{U}$ there is $q \in \mathbb{P}$ such that $\bigwedge\limits_{i \in u} q \geq p_i$.

Claim 2.4. *Forcing by suitable forcing preserves satisfaction of sentences of Magidor-Malitz quantifiers for models of power $\aleph_1$.*

Proof. See [BJ95, 1.5-13,pg.34]. $\square_{2.4}$

Claim 2.5. *There is a suitable forcing $\mathbb{P}, |\mathbb{P}| = 2^{\aleph_1}$, such that in $\mathbf{V}^{\mathbb{P}}$: if $\mathbb{Q}$ is a suitable forcing of power $\aleph_1, \underset{\sim}{M}$ a $\mathbb{Q}$-name of a model of power $\aleph_1$, in a language $L \in \mathbf{V}$, universe $\aleph_1$, <u>then</u> there is a directed $\mathbf{G} \subseteq \mathbb{P}$, which determines $\underset{\sim}{M}$ as M and* such that *for any sentence ψ from the $\mathbb{L}(Q^0, Q^1, \ldots)$ (the variant of Magidor-Malitz logic from Definition 2.1)*

$$\Vdash_{\mathbb{Q}} \text{“}\underset{\sim}{M} \models \psi\text{” implies } M \models \psi.$$

Proof. Just iterate the required forcing notions with direct limit (i.e. finite support) and remembering it is known that suitability is preserved under iteration.

<u>Proof of Main result 2.2</u>:

Do as Silver, start with $\mathbf{V} \models$ “κ Ramsey”, force by $\mathbb{P}$ <u>from Claim 2.5</u>, and then use $\mathbb{P}^\kappa_{\text{Set}}$. The rest is as in his proof.

But we have to choose G as in Claim 2.5, and notice that more is reflected to the submodel he uses, (just check the definition carefully) and work a little, and remember that $\aleph_1$-complete forcing preserves satisfaction of sentences in $\mathbb{L}(Q^0, \ldots)$ (and $\mathbb{P}^\kappa_{\text{Set}}$ is $\aleph_1$-complete). $\square_{2.5}$

§ 3. A Remark on the Monadic Theory of Order

In [She75] we prove the undecidability of the monadic theory of (the order) R, assuming CH, or the weaker Baire-like hypothesis that $\mathbb{R}$ is not the union of fewer than continuum sets of first category sets. This condition is weaken below to "$\neg$(St) at least for T where a closely related theory is the monadic theory T of $M = ({}^{\omega\geq}2, \triangleleft)$ where ${}^{\omega\geq}2$ is the set of sequences of zeros and ones of length $\leq \omega, \triangleleft$ is the (partial) order of being initial segment. T is closely related to Rabin's monadic theory of $({}^{\omega>}2, \triangleleft)$ which he proved decidable [Rab69]. We prove here that the statement "$\neg$(St)" implies the undecidability of T (and all results on its complexity, see [She75] and the paper of Gurevich on the subject) but it was not clear (at that time) whether (St) is consistent with ZFC.

Definition 3.1. A Cantor [set] C is a non-empty subset of ${}^{\omega\geq}2$ with the properties

(a) C is closed under initial segments,
(b) if η has length ω then $\eta \in C \equiv (\forall n)(\eta\restriction n \in C)$,
(c) $\eta \in C \cap {}^{\omega>}2$ implies $\eta \frown \langle 0\rangle \in C$ or $\eta \frown \langle 1\rangle \in C$,
(d) for every $\eta \in C \cap {}^{\omega>}2$, there is $\nu \in C \cap {}^{\omega>}2$ such that $\eta \triangleleft \nu$ and $\nu \frown \langle 0\rangle \in C, \nu \frown \langle 1\rangle \in C$.

Definition 3.2. 1) For a Cantor C, the set of its splitting points is $\mathrm{Sp}(C) = \{\eta \in {}^{\omega>}2 : \eta \frown \langle 0\rangle \in C \text{ and } \eta \frown \langle 1\rangle \in C\}$.
2) For a set $A \subseteq {}^{\omega>}2, C$ is an A-Cantor, if $\mathrm{Sp}(C) \subseteq A$.
3) For a set $S \subseteq \omega, C$ is called an S-Cantor, if

$$\mathrm{Sp}(C) \subseteq \bigcup_{n\in S} {}^n2.$$

4) An odd Cantor is one that is an $\{2n+1 : n < \omega\}$-Cantor. An even Cantor is one that is an $\{2n : n < \omega\}$-Cantor.

Now the statement we speak about is

Definition 3.3. Let (St) mean: the set ${}^{\omega}2$ is the union of less than $2^{\aleph_0}$ Cantors each of them odd or even.

Problem 3.4. Is $\neg$(St) consistent with ZFC? (solved in [BS87])

Claim 3.5. *Let $\{C_i : i < \alpha\}$ be a family of odd and even Cantors, ${}^{\omega\geq}2 = \bigcup\limits_{i<\alpha} C_i$. Then $2^{\aleph_0} \leq |\alpha|^+$.*

Proof. Let for $\eta, \nu \in {}^{\omega}2, \rho = p(\eta, \nu)$ be defined by $\rho(2n) = \eta(n), \rho(2n+1) = \nu(n)$, and then let $\eta = \mathrm{pr}_1(\rho), \nu = \mathrm{pr}_2(\rho)$.

Now for any even C, and η there is at most one ν such that $p(\eta,\nu) \in C$; why? if ν_0, ν_1 are such ν's, $\rho_\ell = p(\eta, \nu_\ell)$, then, by the definition of $p(-,-)$, for some $m < \omega, \rho_0\restriction m = \rho_1\restriction m, \rho_0(m) \neq \rho_1(m)$. If $m = 2n$ then $\rho_\ell(m) = \rho_\ell(2n) = \eta(n)$ so they are equal, contradiction. If $m = 2n+1$, then ($\rho_0(m) \neq \rho_1(m)$ and) $\rho_0\restriction m = \rho_1\restriction m$ is a splitting point of C, however m is odd and C is an even Cantor, a contradiction. So really there is at most one ν, and let $\varrho(\eta, C)$ be the unique ν such that $p(\eta,\nu) \in C$ if there is one and η otherwise.

Similarly if C is odd and $\eta \in {}^{\omega}2$, then for at most one $\nu, p(\nu,\eta) \in C$ and let $\varrho(\eta, C)$ be ν for this η, and let $\varrho(\eta, C) = \eta$ otherwise. Our definition of the function ϱ does not contradict, because no Cantor is both odd and even.

Let for $\eta \in {}^{\omega}2, \mathrm{Dp}(\eta) = \{\varrho(\eta, C_i) : i < \alpha\}$. So clearly $\mathrm{Dp}(\eta)$ is a subset of ${}^{\omega}2$ of cardinality $\leq |\alpha|$.

Now if $\eta, \nu \in {}^{\omega}2$, by hypothesis $\rho = p(\eta, \nu)$ belongs to some C_i. If C_i is odd this implies $\nu = \varrho(\eta, C_i) \in \mathrm{Dp}(\eta)$ and if C_i is even this implies $\eta = \varrho(C_i, \nu) \in \mathrm{Dp}(\nu)$.

If $|\alpha|^+ < 2^{\aleph_0}$ we can easily find a counterexample. $\square_{3.5}$

Claim 3.6. *Assume* $\neg$(St).

1) If $S_n \subseteq \omega$ are infinite pairwise almost disjoint (for $n \in \{0,1,2\}$), $C_i (i < \alpha < 2^{\aleph_0})$ are Cantors, each an S_n-Cantor for some n (or just an $S_n \cup S_2$-Contor for some n), C is a Cantor such that *for every $\eta \in C \cap {}^{\omega>}2, \ell \in \{0,1\}$, there is ν, such that $\eta \lhd \nu \in \mathrm{Sp}(C), \nu \in \bigcup\limits_{k \in S_\ell} {}^{k}2$.*

__Then__ there is $\eta \in C \setminus \bigcup\limits_{i<\alpha} C_i \setminus {}^{\omega>}2$.

2) Similarly for $S_n \subseteq {}^{\omega>}2$

Proof. 1) We can find a Cantor $C' \subseteq C$, and $0 = k(0) < k(1) < \ldots < k(n) < \ldots < \omega$ such that:

(∗) if $\eta \in {}^{k(n)}2$, then there are exactly two $\nu \in {}^{k(n+1)}2 \cap C', \eta \lhd \nu$, and if they are ν_1, ν_2 and $m := \min\{m : \nu_1(m) \neq \nu_2(m)\}$ then $m \in S_0 \cup S_1$ but $\notin S_2 \cup (S_0, \cap S_1)$. Moreover $m \in S_0$ __iff__ n is even.

Let $A = \{\eta \restriction k(n) : n < \omega, \eta \in C'\}$, so $A \subseteq C'$. Clearly there is an isomorphism f, of the models $({}^{\omega\geq}2, \lhd), (C', \lhd)$.

Let $C'_i = \{f(\eta) : \eta \in C', \eta \in C_i\}$, it is easy to check that each C'_i is countable, or the union of a countable set and a Cantor which is odd or is even.

We can find odd Cantor $C'_i (\alpha \leq i < \alpha\omega)$ such that all countable sets we mentioned are covered by them. Now by - "$\neg$(St)" there is $\eta \in {}^{\omega\geq}2$ such that $\eta \notin \bigcup\limits_{i<\alpha\omega} C'_i$ (as $\alpha\omega < 2^{\aleph_0}$) and $f^{-1}(\eta)$ is the required elements.

2) Similarly. $\square_{3.6}$

Now[1]

Claim 3.7. *Assume* $\neg$(St).
The monadic theory T is undecidable.

Proof. Below let P vary on Cantors and note that we can repeat the proof of [She75] with small adaptation (and prove T is undecidable). That is, the change needed is in [She75, 7.4] which has a set-theoretic hypothesis (CH or the Baire-like hypothesis mentioned above), so we repeat it with the needed changes below.

$\square_{3.7}$

Lemma 3.8. *Assume* $\neg$(St) *and let J be an index-set of cardinality at most $2^{\aleph_0}$,*
1) Assume the $D_i (i \in J)$ countable dense subsets of ${}^{\omega>}2$ and $D = \bigcup\limits_{i \in J} D_i$ and $\bar{D} = \langle D_i : i \in J \rangle$[2]. __Then__ there is $Q \subseteq {}^{\omega}2 \backslash D, Q = Q[\bar{D}]$ such that for every Cantor P :

[1] We have added 3.7 and 3.8 in 2019

[2] The main case is that the D_i-s are pairwise almost disjoint

(A) *if* $P \cap D \subseteq D_i (i \in J)$ *and* D_i *is dense in* P *then* $|P \cap Q| < 2^{\aleph_0}$

(B) *if for some* $i \in J$ *the sets* $P \cap D_i, P \setminus D_i$ *are dense in* P *then* $P \cap Q \neq \emptyset$.

2) For some such $\bar{D}$ *we can strengthen clause (B) above to*

(B) *if* P *is a Cantor and for every* $i \in J$ *the set* $D_i \cap P$ *is nowhere-dense in* P *then for every , dense subsets* D_1^*, D_2^* *of* $P \cap D$ *we can find* $D_1^\bullet \subseteq D_1^*, D_2^\bullet \subseteq D_2^*$ *satisfying for any* P *we have: is* $P \cap D_1^\bullet, P \cap D_2^\bullet$ *are dense in* P *then* $P \cap Q \neq \emptyset$.

Proof. 1) Let $\{P_\alpha : 0 < \alpha < 2^{\aleph_0}\}$ be any enumeration of the Cantor sets. We define $x_\alpha, \alpha < 2^{\aleph_0}$ by induction on α.

For $\alpha = 0, x_\alpha \in \mathbb{R}$ is arbitrary.

For any $\alpha > 0$, if P_α does not satisfy the assumptions of (B) then let $x_\alpha = x_0$ and if P satisfies the assumptions of (B) (hence in particular D is dense in P) let $x_\alpha \in P_\alpha - \bigcup\{P_\beta : \beta < \alpha, (\exists i \in J)(P_\beta \cap D \subseteq D_i \text{ and } D \text{ is dense in } P_\beta)\} = D$.

This is possible; to prove this let $\mathscr{U} = \{\beta < \alpha : \text{there is } i \in J \text{ such that } P_\beta \cap D \subseteq D_i\}$ and for $\beta \in \mathscr{U}$ let $i_\beta \in J$ be such that $P_\beta \subseteq D_{i_\beta}$ Let $i(*) \in J$ be such that $P \cap D_{i(*)}, P \setminus D_{i(*)}$ are dense in P. Now we apply 3.6(2), (or 3.6(1) if we restrict the D_i-s, does not matter).

So by (St) and the hypothesis $|P_\alpha \cap D| < 2^{\aleph_0}$ there exists such x_α.

Now let $Q = \{x_\alpha : \alpha < 2^{\aleph_0}\}$. If P satisfies the assumptions of (A), then $P \in \{P_\alpha : 0 < \alpha < 2^{\aleph_0}\}$. So for some $\alpha, P = P_\alpha$, hence $P \cap D \subseteq \{x_\beta : \beta < \alpha\}$, so $|P \cap D| < 2^{\aleph_0}$. If $P = P_\alpha$ satisfies the assumption of (B) then $x_\alpha \in P_\alpha, x_\alpha \in Q$, hence $P_\alpha \cap Q \neq \emptyset$.

2) Similarly.

So we have proved the lemma. $\square_{3.8}$

Remark 3.9. We can interpret the monadic theory of $(R, <)$ in T, but the converse was not clear at the time, but looking at it again probably we can carry the proof for $\mathbb{R}$.

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