UNIVERSAL GRAPHS BETWEEN A STRONG LIMIT SINGULAR AND ITS POWER

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ABSTRACT. The paper settles the problem of the consistency of the existence of a single universal graph between a strong limit singular and its power. Assuming that in a model of **GCH** κ is supercompact and the cardinals $\theta < \kappa$, $\lambda > \kappa$ are regular, as an application of a more general method we obtain a forcing extension in which $\mathrm{cf}(\kappa) = \theta$, the Singular Cardinal Hypothesis fails at κ and there exists a universal graph in cardinality $\lambda \in (\kappa, 2^\kappa)$.

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References like [Shea, Th0.2=Ly5] means the label of Th.0.2 is y5. The reader should note that the version in my website is usually more updated than the one in the mathematical archive.

Annotated Content

§ 0. Introduction

§ 0(A). **Background.**

The existence of universal graphs in infinite cardinalities has been widely investigated (where we mean that the graph G is universal in cardinality $|G|$ if each graph of the same cardinality is isomorphic to some induced subgraph of G). By the classical result [Rad64], the so called countable random graph is a universal graph on $\aleph_0$ (which is also unique up to isomorphism). A classical result (which is now a standard induction argument) yields that there is a κ^+-saturated graph on 2^κ[CK73], and so a graph on 2^κ into which each graph on κ^+ embeds (and we can replace κ^+, 2^κ, κ^+-saturated by κ, $2^{<\kappa}$, κ-special). Therefore under **GCH** in every uncountable cardinality there is a universal graph. (However, concerning certain proper class of graphs the situation is more intricate, even for the countable case, see [FK97], [Kom89], [KS95], [CS16], [KS19].) For the problem of universal objects in more complex theories (i.e. than that of the graphs), and the relevance of the present work in model theory see also the survey [She21b], or earlier [Dža05], see lately [She20], [Sheb].

On the other hand, without assuming **GCH** it is in general much more difficult to construct universal objects, while there are certainly no universal graphs after adding enough Cohen subsets see [KS92].

As for positive results, for regular cardinals $\kappa < \lambda$ consistently there is a universal graph on λ, while $2^\kappa > \lambda$ [She90]. While the argument in [She90] also gives a universal ω-edge colored graph on ω_1 with $\neg$**CH** (which feature will utilized in this paper), recently [SS21] proved that assuming $\neg$**CH** a universal graph on ω_1 does not imply that there is a universal ω-edge colored graph. (Again we remark that, if we restrict ourselves to specific classes of graphs there are both negative [Koj98], and positive results [Mek90], for weak universal families (see $(*)$ below) in the absence of **GCH** see [She93], [DS04]. In all the above the case $\lambda = \kappa^+$ was considerably easier.)

In the present paper we investigate universal graphs in the interval between a strong limit singular cardinal and its power. The question is also motivated by the following. Recall that for $\mu = \aleph_0$ its power $2^{\aleph_0}$ may be large, moreover a relevant forcing axiom (e.g. **MA**) possibly holds. Similarly for $\mu = \aleph_1 = 2^{\aleph_0}$, 2^μ large, or $\mu = \mu^{<\mu}$ parallel results hold for forcing notions which are e.g. $< \mu$-complete, satisfying a strong form of μ^+-cc (the strong form is necessary, see [Shear]). On the other hand for μ strong limit singular we know much less, therefore the existence of universals also serves as a central test problem regarding the consistency of forcing axioms at μ.

More directly we continue the work of Džamonja-Shelah in [DS03], which proved for the case $\mathrm{cf}(\mu) = \aleph_0$ (assuming a supercompact) the consistency of

- $(*)$ (a) μ is strong limit singular and $\mu^{++} < 2^\mu$,
  - (b) there is a graph G_* of cardinality μ^{++} which is universal for graphs of cardinality μ^+ (equivalently there is a sequence $\bar{G} = \langle G_\alpha : \alpha < \mu^{++} \rangle$ of graphs each of cardinality μ^+, universal for the family of such graphs).

see [DS03] for the case $\mathrm{cf}(\mu) = \aleph_0$, and Cummings-Džamonja-Magidor-Morgan-Shelah prove this for arbitrary cofinality in [CDM$^+$17]. Earlier Mekler-Shelah

[MS89] had proved such consistency results replacing (b) by uniformization results; also starting naturally with a supercompact cardinal. Later, $(*)$ was proved to be consistent for small singular μ's, see [CDM16], [Dav17].

Our aim is to solve the problem arising naturally there: First replacing weak universal by universal. Second, replacing $\lambda = \mu^+$ by $\lambda \in (\mu, 2^\mu)$, so formulating the following assertion the following assertion:

$(*)^+$ (a) μ is strong limit singular and $\mu^{++} < 2^\mu$,
(b) there is a universal graph G_* in μ^+, i.e. universal for graphs of cardinality μ^+, G_* itself of cardinality μ^+,
$(b)^+$ as (b), but replacing μ^+ for some cardinal in $(\mu, 2^\mu)$.

Our proof starts with a supercompact cardinal κ, and we show (as part of a more general axiomatic frame) that a stronger version of a universal on $\lambda > \kappa$ (e.g. $\lambda = \kappa^+$) is sufficient for the existence of a universal graph on λ even after forcing with some $\mathbb{P}$ satisfying the axiomatic requirements. Then we first build a general frame for the preparation, and then construct the strong universal as in [She90] suited to the present frame. (Here we remark that some large cardinal hypotheses are essential, as the failure of the Singular Cardinal Hypothesis itself implies that there is an inner model with the Mitchell order $o(\kappa) = \kappa^{++}$ for a measurable cardinal κ (in fact these are equiconsistent).)

The paper is organized as follows. In §1 we introduce the concept of $(\lambda, \kappa) - i$ $(i = 0, 1)$ systems, and in Claim 1.3 we prove that extending a ground model already admitting some strong version of universal using such a $(\lambda, \kappa) - i$ system results in a model with the desired universal object. In §2 we prove that Prikry forcing, Magidor forcing and Radin forcing give rise to a $(\lambda, \kappa) - 1$ system provided the relevant filters satisfy some reasonable assumptions. In §3(A) we prepare the ground, in Claim 3.2 build the frame to force $(\lambda, \kappa) - 1$ systems using a supercompact cardinal. In §3(B) we construct a forcing for obtaining the strong universal fitting in the frame in Claim 3.2.

In works in preparation we intend to replace graphs by more general classes; much of our for is not specific to graphs. Also for consistency of $(*)^+$ for a small singular μ, e.g. $\mu = \aleph_\omega = \beth_\omega$.

§ 0(B). **Preliminaries.** We are interested in universal objects in the class of graphs, i.e. models of the first order language admitting no functions, only a single symmetric, nonreflexive binary relation. Under ordinals we always mean von Neumann ordinals, and for a set X the symbol $|X|$ always refers to the smallest ordinal with the same cardinality. If f is a mapping with $\mathrm{dom}(f) \supseteq X$, then $f``X = \{f(x) : x \in X\}$, i.e. the pointwise image of X. For a set X the symbol $\mathscr{P}(X)$ denotes the power set of X, while if κ is an ordinal we use the standard notation $[X]^\kappa$ for $\{Y \in \mathscr{P}(X) : |Y| = \kappa\}$, similarly for $[X]^{<\kappa}$, $[X]^{<\kappa}$, etc. By a sequence we mean a function on an ordinal, where for a sequence $\bar{s} = \langle s_\alpha : \alpha < \mathrm{dom}(\bar{s})\rangle$ the length of $\bar{s}$ (in symbols $\ell g(\bar{s})$) denotes $\mathrm{dom}(\bar{s})$. Moreover, for sequences $\bar{s}, \bar{t}$ let $\bar{s} \frown \bar{t}$ denote the natural concatenation (of length $\ell g(\bar{s}) + \ell g(\bar{t})$). For a set X, and ordinal α we use ${}^\alpha X = \{\bar{s} : \ell g(\bar{s}) = \alpha, \ \mathrm{ran}(\bar{s}) \subseteq X\}$, and for cardinals λ, κ we use the symbol $\lambda^\kappa = |{}^\kappa\lambda|$ (that is, the least ordinal equivalent to it).

Regarding iterated forcing and quotient forcing we will mostly use the terminology of the survey [Bau76]. However we adhere to the following conventions.

Convention 0.1. Regarding forcing we follow the convention that "$p \leq q$" means that q is the stronger, i.e. giving more information.

Convention 0.2. A notion of forcing $\mathbb{P}$ is $<\mu$-directed closed ($<\mu$-closed, resp.), if for any directed (increasing, resp.) system $\{p_\alpha : \alpha < \nu < \mu\}$ there exists a common upper bound p_* in $\mathbb{P}$.

A filter $\mathcal{F} \subseteq \mathscr{P}(X)$ is κ-complete, if for each $\{F_\alpha : \alpha < \nu < \kappa\} \subseteq \mathcal{F}$ we have $\bigcap_{\alpha<\nu} F_\alpha \in \mathcal{F}$. A partial order T is μ-directed, if for each $\{t_\alpha : \alpha < \nu < \kappa\} \subseteq T$ there exists a common upper bound $t_* \in T$.

§ 1. The frame and deducing the consistency results

§ 1(A). What We Do.

In the present paper we introduce a more general framework and apply it for the class of graphs.

We shall start with a large cardinal, like a Laver indestructible supercompact, or with forcing a relative of it. We then have a two step forcing.

First, a forcing $\mathbb{P}$ with the following three properties:

(a) preserving the largeness of κ,
(b) moreover, in $\mathbf{V}^{\mathbb{P}}$
there is a normal κ-complete filter on κ such that $(D, ^* \supseteq)$ is λ^+-directed for a suitable cardinal $\lambda < 2^\kappa$,
(c) preparing the ground for the results we like to have on λ, e.g. has a strong version of "there is a universal graph in $\lambda, \lambda < 2^\kappa$".

Second, a forcing $\mathbb{Q}$ (in $\mathbf{V}^{\mathbb{P}}$) such that:

(d) makes κ singular,
(e) preserves κ is strong limit and 2^κ large,
(f) but to get the desired property of λ, we use $\mathbb{Q}$ that fits the frame in Definition 1.1 below,
(g) then prove the existence of a universal object using the frame

(or instead of $(\mathbf{V}^{\mathbb{P}})^{\mathbb{Q}}$ use $\mathbf{V}^{\mathbb{P}}[\underset{\sim}{X}]$ for a $\mathbb{Q}$-name $\underset{\sim}{X}$).

Now in §1, Definition 1.1 define the family of (λ, κ)-systems fitting (f), then we deduce the existence of universal graphs in λ (a case of (g)).

In §2 we shall prove that classical forcings for making κ singular fit our frame, i.e. satisfy (d)-(g).

In §3 we shall deal with finding $\mathbb{P}$ as in (a),(b),(c), so have to combine the specific forcing (say forcing a universal graph in λ, i.e. clause (c)) and guaranteeing the existence of e.g. a normal ultrafilter of which is λ^+-complete in a suitable sense (i.e. clause (b)).

Definition 1.1.
1) We say $\mathbf{r}$ is a $(\lambda, \kappa) - 1$-system when $\mathbf{r} = (\mathbb{R}, \underset{\sim}{X}, \leq_{\rm pr}, \mathscr{S}) = (\mathbb{R}_{\mathbf{r}}, \underset{\sim}{X}_{\mathbf{r}}, \leq_{\mathbf{r},\rm pr}, \mathscr{S}_{\mathbf{r}})$ satisfies the following

(a) κ is inaccessible,
(b) $\lambda \in [\kappa^+, 2^\kappa)$,
(c) $\mathbb{R}$ is a forcing notion preserving "κ is strong limit",
(d) $\underset{\sim}{X}$ is an $\mathbb{R}$-name of a subset of κ,
(e) $\leq_{\rm pr} \subseteq \leq_{\mathbb{R}}$ is a quasi-order,
(f) for each $p \in \mathbb{R}$ we have $\mathscr{S}_p \subseteq \{\bar{q} \in {}^\kappa\mathbb{R} : p \leq_{\rm pr} q_\varepsilon$ for every $\varepsilon < \kappa\}$,
(g) whenever $\gamma < \kappa$, $p \in \mathbb{R}$, $\underset{\sim}{\tau}$ are such that $p \Vdash$ "$\underset{\sim}{\tau}$ is an $\mathbb{R}$-name of a subset of γ or just of $\mathscr{H}^{\mathbf{V}}(\gamma)$" then:
    (∗) there are $\bar{q} \in \mathscr{S}_p, \bar{\gamma} = \langle \gamma_\varepsilon : \varepsilon < \kappa \rangle \in {}^\kappa\kappa$ and $\bar{c} = \langle c_\varepsilon : \varepsilon < \kappa \rangle$, where
        •₁ each $c_\varepsilon \in \mathbf{V}$ is a code for $|\mathscr{H}^{\mathbf{V}}(\gamma_\varepsilon)|$-Borel function F_{c_ε} from $\mathscr{P}(\mathscr{H}^{\mathbf{V}}(\gamma_\varepsilon))$ into $\mathscr{P}(\mathscr{H}^{\mathbf{V}}(\gamma))$,
        •₂ $q_\varepsilon \Vdash$ "$\underset{\sim}{\tau} = F_{c_\varepsilon}(\underset{\sim}{X} \cap \mathscr{H}(\gamma_\varepsilon))$", and so it belongs to $\mathscr{H}(\kappa)$;
(h) if $\bar{q}_\alpha \in \mathscr{S}_p$ for $\alpha < \lambda$, then for some $q_* \in \mathbb{R}$ for every $\alpha < \lambda$ there is $\varepsilon_\alpha < \kappa$ such that $q_{\alpha,\varepsilon_\alpha} \leq_{\mathbb{R}} q_*$.

2) We say $\mathbf{r}$ is a $(\lambda, \kappa) - 2$-system when above in clause (g) we restrict ourselves to $\underset{\sim}{\tau}$'s such that $\Vdash$ "$\underset{\sim}{\tau} \in \mathbf{V}[\underset{\sim}{X}]$";,
2A) We may omit the 1 in "1-system".
3) We say $\mathbf{r}$ is nice when the forcing $\mathbb{R}_{\mathbf{r}}$ collapses no cardinal.
4) It is enough to require in clause (g) that $(*)$ holds for every p, $\underset{\sim}{\tau}$ such that $p \Vdash$ "$\underset{\sim}{\tau} \in \{0,1\}$", as this formally weaker assumption easily implies clause (g).

Discussion 1.2.
1) Here we only deal with the question "when is there a universal graph in the cardinal λ?".
2) Of course, in Definition 1.1, we are interested in the case $\Vdash_{\mathbb{R}_{\mathbf{r}}}$ "κ is singular".
3) There are such $\mathbf{r}$'s: Prikry forcing, Magidor forcing, cases of Radin forcing, see 2.1 on.
name in a derived forcing.

Claim 1.3. *For a fixed $\iota \in \{0,1\}$ and cardinals κ, λ*
1) *In $\mathbf{V}_\iota$ there is a universal graph of cardinality λ when:*
    (a) $\mathbf{r} \in \mathbf{V}$ is $(\lambda, \kappa) - \iota$-system, we define $\mathbf{V}_\iota = \mathbf{V}^{\mathbb{R}_{\mathbf{r}}}$ if $\iota = 1$, and $\mathbf{V}_\iota = \mathbf{V}[X_{\mathbf{r}}]$ in case of $\iota = 2$,
    (b) $\kappa < \lambda < 2^\kappa$ (e.g. $\lambda = \kappa^+$),
    (c) in $\mathbf{V}$, there is a universal member of $(K_\kappa)_\lambda$, see below (Definition 1.4).
2) *Moreover, $\mathrm{univ}(K_\lambda) \leq \chi$ (where K_λ denotes the class of graphs on λ-many vertices) is true in $\mathbf{V}_\iota$, when:*
    (a),(b) as above
    (c) $\mathrm{univ}((K_\kappa)_\lambda) \leq \chi$ in $\mathbf{V}$.

Definition 1.4. $(K_\kappa)_\lambda$ is the class of edge colored graphs with κ colors, equivalently $M \in (K_\kappa)_\lambda$ iff
(a) $M = (|M|, R^M_\varepsilon)_{\varepsilon<\kappa}$,
(b) $\|M\| = \lambda$,
(c) R^M_ε is a symmetric irreflexive two-place relation on $|M|$,

(d) $\langle R^M_\varepsilon : \varepsilon < \kappa\rangle$ is a partition of $\{(a,b) : a \neq b \in M\}$.

Proof. (Claim 1.3) Let (in $\mathbf{V}$) $\langle (c_\varepsilon, \gamma_\varepsilon) : \ \varepsilon < \kappa\rangle$ list

$\{(c,\gamma) : \ c : \mathscr{P}(\mathscr{H}^{\mathbf{V}}(\gamma)) \to \{0,1\}$ is a code for an $|\mathscr{H}^{\mathbf{V}}(\gamma)|$ – Borel function$\}$,

Assume that

$(*)_1$ there is a sequence $\langle M^*_\delta : \ \delta < \chi\rangle$ in $(K_\kappa)_\lambda$ forming a universal sequence for $(K_\kappa)_\lambda$ (in the universe $\mathbf{V}$, of course; so $\chi = 1$ means that $M^*_0 \in (K_\kappa)_\lambda$ is universal),

where $M^*_\delta = (\lambda, \ldots, R^{M^*_\delta}_\varepsilon, \ldots)_{\varepsilon<\kappa}$, it is enough to prove that in $\mathbf{V}_\iota$ we have $\text{univ}(K_\lambda) \leq \chi$.

Now we define the sequence of $\mathbb{R}_{\mathbf{r}}$-names $\underset{\sim}{M}_\delta$ $(\delta < \chi)$ for graphs as follows.

$(*)_2$ (a) the set of nodes of $\underset{\sim}{M}_\delta$ is $\{\alpha : \ \alpha < \lambda\}$,
(b) for $\alpha \neq \beta < \mu$ let $(\alpha,\beta) \in R^{\underset{\sim}{M}_\delta}$ $\underline{\text{iff}}$ for the unique $\varepsilon < \kappa$ with $(\alpha,\beta) \in R^{M^*_\delta}_\varepsilon$ we have $F_{c_\varepsilon}(\underset{\sim}{X} \cap \mathscr{H}(\gamma_\varepsilon)) = 1$.

So clearly

$(*)_3$ for each $\delta < \chi$ $\underset{\sim}{M}_\delta$ is an $\mathbb{R}_{\mathbf{r}}$-name of a graph with set of nodes λ.

Hence it suffices to prove:

$(*)_4$ $\Vdash_{\mathbb{R}}$ "$\mathbf{V}_\iota \models \langle \underset{\sim}{M}_\delta : \ \delta < \chi\rangle$ is a universal sequence in K_λ".

So why does $(*)_4$ hold? Assume

$(*)_{4.1}$ $p \Vdash$ "$\underset{\sim}{N} \in \mathbf{V}_\iota$ is a graph with set of nodes λ".

Let $\langle (\alpha_\gamma, \beta_\gamma) : \gamma < \lambda\rangle \in \mathbf{V}$ list the set of pairs (α,β) such that $\alpha < \beta < \lambda$. For each $\gamma < \lambda$ (considering the $\mathbb{R}_{\mathbf{r}}$-names $\underset{\sim}{\tau}_\gamma$ for the truth value of $(\alpha_\gamma,\beta_\gamma) \in R^{\underset{\sim}{N}}$) clause (g) of Definition 1.1 1) gives $\bar{q}_\gamma = \langle q_{\gamma,\varepsilon} : \ \varepsilon < \kappa\rangle \in \mathscr{S}_p$, $\bar{\zeta}_\gamma = \langle \zeta_{\gamma,\varepsilon} : \ \varepsilon < \kappa\rangle \in {}^\kappa\kappa$ and $\bar{c}_\gamma = \langle c_{\gamma,\varepsilon} : \varepsilon < \kappa\rangle$ such that for each $\gamma < \lambda$

- $\bullet_1$ $c_{\gamma,\varepsilon}$ is a code for a $|\mathscr{H}(\zeta_{\gamma,\varepsilon})|$-Borel function $\mathscr{P}(\mathscr{H}(\zeta_{\gamma,\varepsilon})) \to \{0,1\}$ $(\varepsilon < \kappa)$
- $\bullet_2$ $q_{\gamma,\varepsilon} \Vdash_{\mathbb{R}}$ "$(\alpha_\gamma,\beta_\gamma) \in R^{\underset{\sim}{M}} \Leftrightarrow F_{c_{\gamma,\varepsilon}}(\underset{\sim}{X} \cap \mathscr{H}(\zeta_{\gamma,\varepsilon})) = 1$".

Now by clause (h) of Definition 1.1 1), there are q and $\langle \varepsilon_\gamma = \varepsilon(\gamma) : \gamma < \lambda\rangle \in {}^\lambda\kappa$ such that:

- $\bullet_3$ q is above $q_{\gamma,\varepsilon(\gamma)}$ for every $\gamma < \lambda$.

Now we define N_* as follows:

$(*)_{4.3}$ (a) $N_* = (\lambda, (R^{N_*}_\varepsilon)_{\varepsilon<\kappa})$, where
(b) $R^{N_*}_\varepsilon = \{(\alpha_\gamma,\beta_\gamma) : \gamma < \lambda \text{ and } \varepsilon_\gamma = \varepsilon\}$ $(\forall \varepsilon \in \kappa)$.

Clearly

$(*)_{4.4}$ $N_* \in (K_\kappa)_\lambda$ (with set of nodes λ) belongs to $\mathbf{V}$.

Now choose a suitable $\delta < \chi$ and a function f so that:

$(*)_{4.5}$ $f : N^* \to M^*_\delta$ is an embedding, $f \in \mathbf{V}$

[which exists by $(*)_1$.] Finally it is straightforward to check that

$(*)_{4.6}$ $q \Vdash$ "f is an embedding of $\underset{\sim}{N}$ into $\underset{\sim}{M}_\delta$".

[Recall how we defined N_* from N $(*)_{4.3}$, M_δ from M^*_δ $(*)_2$, and the choice of f $(*)_{4.5}$.]

$\square_{\text{Claim1.3}}$

Naturally we ask

Question 1.5.
1) What about $(K_\kappa)_\lambda$?
2) More seriously about the theory of triangle free graphs, or of T_{feq} (equivalently T_{ceq}, see [Sheb]). Note, on T_{feq} see [She93], or [DS04], and on the non-existence in case of T_{ceq} see [She21a].
3) Moreover, $(\text{Mod}_T, \prec), T$ simple? Or even NSOP$_2$? (of cardinality $< \kappa$). We have to be more careful because of, e.g. function symbols.

A work in preparation deals with 1.5 2), 3). Concerning 1.5 1) (note that this does not reflect on Claim 1.3):

Claim 1.6. *Assume κ is strong limit singular and $\kappa < \lambda < 2^\kappa$. Then in $(K_\kappa)_\lambda$ there is no universal member.*

Remark 1.7. It suffices to have $\beth_\omega(\text{cf}(\kappa)) < \kappa$, and $(\alpha < \kappa \ \rightarrow \ |\alpha|^{\text{cf}(\kappa)} < \kappa)$.

Proof. By [She06, Thm 1.13 and 1.14 (2) on RGCH]

$(*)_0$ there[1] is a regular $\sigma \in (\text{cf}(\kappa), \kappa)$ such that $\lambda^{[\sigma,\kappa]} = \lambda$, i.e. there is $\mathscr{P}' \subseteq \{u \subseteq \lambda : |u| \leq \kappa\}$ of cardinality λ such that every $u \subseteq \lambda$ of cardinality $\leq \kappa$ is the union $< \sigma$ members of $\mathscr{P}'$.

Therefore, as $\sigma = \text{cf}(\sigma) > \text{cf}(\kappa)$

$(*)_1$ there is $\mathscr{P} \subseteq \{u \subseteq \lambda : |u| < \kappa\}$ of cardinality λ such that every $u \subseteq \lambda$ of cardinality $< \kappa$ is the union $< \sigma$ members of $\mathscr{P}$.

Let $M_* \in (K_\kappa)_\lambda$ and we shall prove that it is not universal; without loss of generality the universe of M_* is λ. Now for each $u \in \mathscr{P}$ and $\alpha < \lambda$ let

$$v(\alpha, u, M_*) = \{\varepsilon < \kappa : \text{ for some } \beta \in u \text{ we have } (\alpha, \beta) \in R_\varepsilon^{M_*}\},$$

so $v(\alpha, u, M_*) \subseteq \kappa$ has cardinality $< \kappa$. Let

$$\mathscr{P}_1 = \{w \in [v(\alpha, u, M_*)]^{\leq \text{cf}(\kappa)} : \ u \in \mathscr{P}, \alpha \in \lambda\},$$

so

$(*)_2$ $\mathscr{P}_1 \subseteq [\kappa]^{\leq \text{cf}(\kappa)}$.

Now

$(*)_3$ $|\mathscr{P}_1| \leq |\mathscr{P}| + 2^{<\kappa} \leq \lambda < 2^\kappa = \kappa^{\text{cf}(\kappa)}$.

Hence

$(*)_4$ we can find $v \subseteq \kappa$ of cardinality $\text{cf}(\kappa)$ which is not in $\mathscr{P}_1$, moreover, $u \in \mathscr{P}_1 \Rightarrow |u \cap v| < \text{cf}(\kappa)$,

[1] In fact, instead "κ strong limit singular σ as above", it suffices to assume less.

which is justified by the following argument: Let $\langle v_\gamma : \gamma < 2^\kappa \rangle$ be a sequence of members of $[\kappa]^{\mathrm{cf}(\kappa)}$ with any two having intersection of cardinality $< \mathrm{cf}(\kappa)$, hence for every $u \in \mathscr{P}_1, \{\gamma < 2^\kappa : |u \cap v_\gamma| = \mathrm{cf}(\kappa)\}$ has cardinality $\leq 2^{\mathrm{cf}(\kappa)} < \kappa$, so all but $\leq \lambda$ sets of v_γ's are as required.

Now consider the following N:

$(*)_5$ (a) $N = (A \cup B, \ldots, R^N_\varepsilon, \ldots)_{\varepsilon < \kappa}$ belongs to $(K_\kappa)_{\sigma^{\mathrm{cf}(\kappa)}}$, where $|A| = \sigma$, $|B| = \sigma^{\mathrm{cf}(\kappa)}$, $A \cap B = \emptyset$,
(b) $R^N_\varepsilon \neq \emptyset$ <u>iff</u> $\varepsilon \in v$,
(c) letting $\langle \varepsilon_i : i < \mathrm{cf}(\kappa) \rangle$ list v (from $(*)_4$), for every sequence $\langle \alpha_i : i < \mathrm{cf}(\kappa) \rangle$ in A with no repetitions there is $\beta \in B$ such that $(\alpha_i, \beta) \in R^N_{\varepsilon_i}$ for $i < \mathrm{cf}(\kappa)$.

Now if g embeds N into M_* then for some $\{u_\varepsilon : \varepsilon < \partial < \sigma\} \subseteq \mathscr{P}$, we have $\mathrm{Rang}(g \restriction A) = \cup\{u_\varepsilon : \varepsilon < \partial\}$. Now as $|A| = \sigma = \mathrm{cf}(\sigma)$, there is $\varepsilon < \partial$ such that $|u_\varepsilon \cap \mathrm{Rang}(g \restriction A)| \geq \sigma \geq \mathrm{cf}(\kappa)$ so we can choose pairwise distinct $\alpha_i \in A$ $(i < \mathrm{cf}(\kappa))$ such that $\{g(\alpha_i) : i < \mathrm{cf}(\kappa)\} \subseteq u_\varepsilon$. Let $\beta \in B$ as in $(*)_5(c)$. So $g(\beta)$ is well defined and we get an easy contradiction by $(*)_4$.

So N cannot be embedded into M_*, hence we are done. $\square_{1.6}$

§ 2. Proving known forcings fit the frame

§ 2(A). **Near a Large Singular.**

Here we do not collapse cardinals, just change cofinalities.

Claim 2.1. *There is a nice (λ, κ)-system $\mathbf{r}$ such that $\mathbb{R}_\mathbf{r} = \mathbb{P}$ <u>when</u>:*

(A) (a) $\kappa < \lambda < 2^\kappa$ are cardinals,
(b) D is a normal ultrafilter on κ,
(c) if $\mathscr{A} \subseteq D$ has cardinality $\leq \lambda$, <u>then</u> for some $B \in D$ we have $(\forall A \in \mathscr{A})(B \subseteq A \mod [\kappa]^{<\kappa})$, (e.g. D is generated by a $\subseteq^*_\kappa$-decreasing sequence of length of a regular cardinal $> \lambda$),
(d) $\mathbb{P}$ is Prikry forcing for D (so change the cofinality of κ to $\aleph_0$ and add no bounded subset of κ and satisfies the κ^+-c.c).

Proof. Recalling the definition of Prikry forcing for D:

$(*)_1$ (a) $p \in \mathbb{P}$ iff $p = (w, A) = (w_p, A_p)$, where $w_p \in [\kappa]^{<\aleph_0}$ and $A_p \in D$ and $[0, \max w_p] \cap A = \emptyset$,
(b) $p \leq_\mathbb{P} q$ <u>iff</u> $w_p \subseteq w_q \subseteq w_p \cup A_p$ and $A_p \supseteq A_q$.

We define the system $\mathbf{r}$ by letting:

$(*)_2$ (a) $\kappa_\mathbf{r} = \kappa$,
(b) $\lambda_\mathbf{r} = \lambda$,
(c) $\mathbb{R}_\mathbf{r} = \mathbb{P}$,
(d) $\underset{\sim}{X}_\mathbf{r} = $ the generic $= \cup\{w_p : p \in \mathbf{G}_\mathbb{P}\}$,

(e) $\leq_{\rm pr} = \leq_{\mathbf{r},\rm pr}$ is defined by $p \leq_{\rm pr} q$ iff $w_p = w_q \wedge A_p \supseteq A_q$ (and $p, q \in \mathbb{R}_{\mathbf{r}}$),

(f) for $p \in \mathbb{R}_{\mathbf{r}} = \mathbb{P}$ let $\mathscr{S}_p = \mathscr{S}_{\mathbf{r},p} := \{\bar{q} : \bar{q} = \langle q_\varepsilon : \varepsilon < \kappa\rangle$ and for some $B \in D$ we have $B \subseteq A_p$ and $\{A_{q_\varepsilon} : \varepsilon < \kappa\}$ list $\{A : A \subseteq A_p$ and $A = B$ mod $[\kappa]^{<\kappa}\}$.

We still have to prove that $\mathbf{r}$ is as required, namely, that $\mathbf{r}$ satisfies conditions listed in Definition 1.1 1).

Now in Definition 1.1 1), clauses (a)-(f) hold trivially. For clause (g) recall that by 4) from Definition 1.1 it is enough to check for only p, $\underset{\sim}{\tau}$, where "$\tau \in \{0,1\}$" is forced by p. Recall the following well-known fact:

$(*)_3$ if $p \in \mathbb{P}$, $p \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\tau} \in \{0,1\}$" <u>then</u> for some $A' \subseteq A_p$ from D we have: if $\alpha \in \kappa$ and $u \subseteq A_p \cap \alpha$ is finite then $(w_p \cup u, A' \backslash \alpha)$ forces a value for $\underset{\sim}{\tau}$.

[For the sake of completeness we prove $(*)_3$: by the Prikry-lemma, for each $s \in [A_p]^{<\aleph_0}$ there exists $A_s \subseteq A_p \setminus ((\max s)+1)$, $A_s \in D$, such that $(w \cup s, A_s)$ decides the value of $\underset{\sim}{\tau}$. Now let A' be the diagonal intersection of A_s's ($s \in [A_p]^{<\aleph_0}$), pedantically $\Delta_{\alpha<\kappa}(\bigcap_{s\in[\alpha+1]^{<\aleph_0}} A_s)$, it is straightforward to check that A' works.]

So given $p \in \mathbb{P}, \gamma$ and $\underset{\sim}{\tau}$ as in clause (g) from Definition 1.1, let $A' \subseteq A_p$ be as in $(*)_3$ and let $\bar{q} = \langle q_\varepsilon : \varepsilon < \kappa\rangle$ be defined by: $q_\varepsilon \in \mathbb{P}, w_{q_\varepsilon} = w_p$ and $\{A_{q_\varepsilon} : \varepsilon < \kappa\}$ list $\{A \subseteq A_p : A \equiv A' \mod [\kappa]^{<\kappa}\}$.

We still have to choose the $\gamma_\varepsilon, F_\varepsilon$. For each ε choose $\zeta_\varepsilon \in A_{q_\varepsilon}$ such that $A_{q_\varepsilon} \backslash \zeta_\varepsilon = A' \backslash \zeta_\varepsilon$. Clause $(*)_3$ ensures that there is a function $f : [A_p \cap \zeta_\varepsilon]^{<\aleph_0} \to \{0,1\}$ in $\mathbf{V}$ such that $q_\varepsilon \Vdash \underset{\sim}{\tau} = f(\underset{\sim}{X} \cap \zeta_\varepsilon)$.

Lastly, for clause (h), assume $p \in \mathbb{R}_{\mathbf{r}} = \mathbb{P}$ and $\bar{q} = \langle \bar{q}_\alpha : \alpha < \lambda\rangle$ satisfies $\bar{q}_\alpha \in \mathscr{S}_p$. So for each $\alpha < \lambda$ there exists $B_\alpha \subseteq A_p$ such that $\{A_{q_{\alpha,\varepsilon}} : \varepsilon < \kappa\}$ lists $\{A \in D : A \subseteq A_p,\ A \equiv B_\alpha \mod [\kappa]^{<\kappa}\}$, hence by clause (A)(c) of the assumption of the claim, there is $B \in D$, a subset of A_p such that $B \subseteq B_\alpha \mod [\kappa]^{<\kappa}$ for each $\alpha \in \lambda$ and let $q_* = (w_p, B)$ so clearly $p \leq_{\rm pr} q_*$. Also for each $\alpha < \lambda$, for some $\zeta < \kappa$ we have $B \backslash \zeta \subseteq B_\alpha$ hence because $\bar{q}_\alpha \in \mathscr{S}_p$ for some $\varepsilon < \kappa$ we have $A_{q_{\alpha,\varepsilon}} = (B_\alpha \backslash \zeta) \cup (A_p \cap \zeta) \supseteq B$ hence $q_{\alpha,\varepsilon} \leq q_*$.

We still have to prove that $\mathbf{r}$ is nice but as $\mathbb{P}$ satisfies the κ^+-c.c., and the Prikry lemma this is obvious. $\square_{2.1}$

Claim 2.2. *There is a* $(\lambda,\kappa) - 1$*-system* $\mathbb{R}_{\mathbf{r}}$ *with* $\mathbf{V}^{\mathbb{R}_{\mathbf{r}}} \models \mathrm{cf}(\kappa) = \theta$ <u>*when*</u>*:*

(B) (a) $\theta = \mathrm{cf}(\theta) < \theta_* < \kappa < \lambda < 2^\kappa$*,*

(b) $\bar{D} = \langle D_i : i < \theta\rangle$ *is a sequence of normal ultrafilters on* κ*, increasing in Mitchell order, i.e.* $i < j \Rightarrow D_i \in \mathrm{MosCol}({}^\kappa\mathbf{V}/D_j)$*,*

(c) each D_i $(i \leq \theta)$ *is* $< \lambda^+$*-directed mod* $[\kappa]^{<\kappa}$*, i.e. satisfies the condition in Claim* 2.1*(A)(c),*

(d) the forcing $\mathbb{R}_{\mathbf{r}}$ *changes the cofinality of* κ *to* θ*, preserves each cardinal and the function* $\mu \mapsto 2^\mu$*, satisfies the* κ^+*-c.c. Moreover, we can prescribe that in* $\mathbf{V}^{\mathbb{P}}$ *there is no new subset of* θ_**.*

Proof. Using [Kru07, Proposition 2.1], condition b implies the following.

Subclaim 2.3. *If* $\bar{D} = \langle D_i : i < \theta\rangle$ *is an increasing (w.r.t. the Mitchell order) sequence of normal ultrafilters on* κ*,* $\theta \leq \kappa$*, then there exists a coherent sequence*

$\langle \bar{U}_\varepsilon : \varepsilon < \kappa+1\rangle$, $\bar{U}_\varepsilon = \langle U_\varepsilon(\alpha) : \alpha < o^U(\varepsilon)\rangle$ *for some function* $o^U : \kappa+1 \to \kappa$ *such that* $\bar{D} = \bar{U}_\kappa$, *which means:*

$(\mathsf{T})_a$ *for each* $\varepsilon \leq \kappa$, $\alpha < o^U(\varepsilon)$ $U_\varepsilon(\alpha)$ *is an* ε*-complete normal ultrafilter on* ε,

$(\mathsf{T})_b$ *moreover, for each* $\varepsilon \leq \kappa$, $\alpha < o^U(\varepsilon)$ *letting* $\mathbf{j}_{\varepsilon,\alpha} : \mathbf{V} \to \mathrm{MosCol}({}^\varepsilon\mathbf{V}/U_{\varepsilon,\alpha})$ *be the associated elementary embedding, we have*

$$(\mathbf{j}_{\varepsilon,\alpha}(\bar{U} \restriction \varepsilon))_\varepsilon = \langle U_\varepsilon(\beta) : \beta < \alpha\rangle,$$

$(\mathsf{T})_c$ $\langle U_\kappa(\alpha) : \alpha < o^U(\kappa)\rangle = \langle D_\alpha : \alpha < \theta\rangle$.

Now we define the forcing $\mathbb{P}_{\overline{U}}$ to be the Magidor forcing associated to the sequence $\bar{D} = \bar{U}_\kappa = \langle U_\kappa(\alpha) : \alpha \leq \theta\rangle$, (see also [Mag78], or [Git10]), here we use the definition from [Git10, Definition 5.22]

Definition 2.4. Define $\mathbb{P}_{\overline{U}}$ to be the following (auxiliary) poset.

$(*_1)$ Let $p = \langle d_0, d_1, \ldots, d_n, d_{n+1} = \langle \kappa, A_\kappa\rangle\rangle \in \mathbb{P}_{\overline{U}}$, iff
  - (a) $A_\kappa \in \bigcap \bar{U}_\kappa = \bigcap_{\alpha<\theta} U_{\kappa,\alpha}$,
  - (b) each d_j $(j \leq n)$ is of the form
    - either $\langle \varepsilon, A_\varepsilon\rangle$ for some $\varepsilon < \kappa$, where $o^U(\varepsilon) > 0$, moreover,
    $$A_\varepsilon \in \bigcap \bar{U}_\varepsilon = \bigcap_{\gamma<o^U(\varepsilon)} U_{\varepsilon,\gamma},$$
    (this case we define $\kappa(d_j) = \varepsilon$),
    - or $d_j = \varepsilon$, when $o^U(\varepsilon) = 0$ (and we let $\kappa(d_j) = \varepsilon$).
  - (c) $\kappa(d_0) < \kappa(d_1) < \cdots < \kappa(d_n) < \kappa(d_{n+1}) = \kappa$,
  - (d) moreover, for each $j \leq n$ if d_{j+1} is a pair, then $\kappa(d_j) < \min A_{\kappa(d_{j+1})}$.

$(*_2)$ We define

$$p = \langle d_0, d_1, \ldots, d_n, d_{n+1} = \langle \kappa, A_\kappa\rangle\rangle \leq q = \langle e_0, e_1, \ldots, e_n, e_{m+1} = \langle \kappa, B_\kappa\rangle\rangle,$$

if
  - (a) $m \geq n$, and
  - (b) there exists a sequence $0 \leq i_0 < i_1 < \cdots < i_n < j_{n+1} = m$ such that for each $j \leq n+1$ we have
    - $\kappa(d_j) = \kappa(e_{i_j})$, and
    - $B_{\kappa(d_j)} \subseteq A_{\kappa(d_j)}$,
  - (c) moreover, for each $k \leq m$ not of the form i_j $(j \leq n+1)$, if $i_l = \min\{i_j : j \leq n+1,\ i_j > k\}$, then
  $$B_{\kappa(e_k)} \cup \{\kappa(e_k)\} \subseteq A_{\kappa(d_l)}.$$

$(*_3)$ Now there are pairwise disjoint sets Y_α $(\alpha < \theta)$ by $\delta \in Y_\alpha$ iff $o^U(\delta) = \alpha$, and

$$\{p \in \mathbb{P}_{\overline{U}} : p \geq \langle\langle \kappa, \bigcup_{\alpha<\theta} Y_\alpha\rangle\rangle\}$$

is the Magidor forcing changing the cofinality of κ to $\min\{\omega, \mathrm{cf}(\theta)\}$.

Definition 2.5. We define $p \leq_* q$ to be true iff $p \leq q$ and $\ell g(p) = \ell g(q)$.

We define the system $\mathbf{r}$ by letting:

$(*_3)$ (a) $\kappa_{\mathbf{r}} = \kappa$,
  (b) $\lambda_{\mathbf{r}} = \lambda$,

(c) $\mathbb{R}_{\mathbf{r}} = \{p \in \mathbb{P}_{\overline{U}} : \ p \geq \langle\langle \kappa, \bigcup_{\alpha<\theta} Y_\alpha \rangle\rangle\}$,

(d) let $\underset{\sim}{X}_{\mathbf{r}}$ be the generic sequence, i.e.

$$\underset{\sim}{X}_{\mathbf{r}} = \cup\{\{\kappa(d_j) : j < \ell g(p)\} : \ p = \langle d_0, d_1, \ldots, d_{\ell g(p)-1}\rangle \in \mathbf{G}_{\mathbb{P}}\} \setminus \{\kappa\},$$

(e) $\leq_{\rm pr} = \leq_{\mathbf{r},\rm pr}$ is defined by $p \leq_{\rm pr} q$ iff $p \leq_* q$,

(f) for $p = \langle d_0, d_1, \ldots, d_n, d_{n+1} = \langle \kappa, A_{p,\kappa}\rangle\rangle \in \mathbb{R}_{\mathbf{r}} = \mathbb{P}$ let

$$\mathscr{S}_p = \mathscr{S}_{\mathbf{r},p} := \left\{ \begin{array}{l} \bar{q} : \bar{q} = \langle q_\varepsilon : \varepsilon < \kappa\rangle, \text{ where} \\ (\bullet_1) q_\varepsilon = \langle d_0, d_1, \ldots, d_n, \langle \kappa, A_{q_\varepsilon,\kappa}\rangle\rangle, \text{ and} \\ \text{for some } B \in \bigcap \bar{U}_\kappa \text{ we have} \\ (\bullet_2) \ B \subseteq A_{p,\kappa}, \text{ and} \\ (\bullet_3) \ \{A_{q_\varepsilon,\kappa} : \varepsilon < \kappa\} \text{ lists } \{A_* : A_* \subseteq A_{p,\kappa} \ \wedge \ A_* = B \mod [\kappa]^{<\kappa}\} \end{array} \right\}.$$

It is known that $\underset{\sim}{X}$ is a club of κ of order type θ, moreover, if condition $\langle\langle\beta\rangle, \langle\kappa, A\rangle\rangle$ is in the generic filter (for some $\beta < \kappa$, $o^U(\beta) = 0$, then the forcing adds no new subset of β. This means that by $(\top)_b$ $\{\beta < \kappa : \ o^U(\beta) = 0\} \in U_{\kappa,0}$ is of cardinality κ, so there is no problem assuming that $\langle\langle\beta\rangle, \langle\kappa, A\rangle\rangle \in \mathbf{G}$ for some $\beta \geq \theta_*$. In order to finish the proof of Claim 2.2 it suffices to verify that the forcing defined in $*_3$ is a $(\lambda, \kappa) - 1$-system.

Subclaim 2.6. *If $\langle \bar{U}_\varepsilon : \ \varepsilon < \kappa + 1\rangle$ is a coherent sequence, where the ultrafilters $\{U_\kappa(\alpha) : \ o^{\bar{U}}(\kappa)\}$ are $< \lambda^+$-directed mod $[\kappa]^{<\kappa}$, then the forcing $\mathbb{P}_{\bar{U}}$ from Definition 2.4 is a $(\lambda, \kappa) - 1$-system.*

Proof. Now we have only to check the requirements of Definition 1.1 1). Recall the following properties of the Magidor forcing, see [Git10, Sec. 5.1 and 5.2].

Fact 2.7. *(Prikry Lemma) For each $p \in \mathbb{P}_{\overline{U}}$ and each formula $\sigma(\underset{\sim}{x}_0, \ldots, \underset{\sim}{x}_m)$ there exists $q \geq_* p$, $q \parallel \sigma(\underset{\sim}{x}_0, \ldots, \underset{\sim}{x}_m)$ (i.e. either $q \Vdash \sigma(\underset{\sim}{x}_0, \ldots, \underset{\sim}{x}_m)$, or $q \Vdash \neg\sigma(\underset{\sim}{x}_0, \ldots, \underset{\sim}{x}_m)$).*

Fact 2.8. *Suppose that $\mathbf{G} \subseteq \mathbb{P}_{\overline{U}}$ is generic over $\mathbf{V}$, $p = \langle d_0, d_1, \ldots, d_n, d_{n+1} = \langle \kappa, A_{p,\kappa}\rangle\rangle \in \mathbf{G}$, $d_i = \langle \kappa(d_i), A_{\kappa(d_i)}\rangle$, then the filter $\mathbf{G} \restriction (\kappa(d_i)+1) := \{q \restriction (\kappa(d_i)+1) : q \in \mathbf{G}\}$ is $\mathbf{V}$-generic over the Prikry forcing $\mathbb{P}_{\overline{U}\restriction(\kappa(d_i)+1)}$ associated to the coherent sequence $\langle \overline{U}_\delta = \langle U_\delta(\gamma) : \ \gamma < o^U(\delta)\rangle : \ \delta \leq \kappa(d_i)\rangle$.*

These two fact clearly imply the following.

Lemma 2.9. *For each formula $\sigma(\underset{\sim}{x}_0, \ldots, \underset{\sim}{x}_m)$ and δ with for some $p = \langle d_0, d_1, \ldots, d_n, d_{n+1}\rangle \in \mathbf{G}$ and $m \leq n$ $\delta = \kappa(d_m)$ (i.e. $\delta \in X_{\mathfrak{r}}$) there exists $q \in \mathbb{P}_{\overline{U}\restriction(\delta+1)} \geq p \restriction (\delta+1)$, such that $q = q^* \restriction (\delta + 1)$ for some $q^* \in \mathbf{G}$, and for some $A^* \in \bigcap \overline{U}_\kappa$*

$$q \frown \langle \kappa, A^*\rangle \parallel_{\mathbb{P}_{\overline{U}}} \sigma(\underset{\sim}{x}_0, \ldots, \underset{\sim}{x}_m). \tag{2.1}$$

Proof. By Fact 2.8 it is enough to show that there are densely many such q's in $\mathbb{P}_{\overline{U}\restriction(\delta+1)}$. But by the Prikry Lemma for $\mathbb{P}_{\overline{U}\restriction(\delta+1)}$ there are in fact $\geq_*$-densely many such $q's$. □

Similarly to the case of Prikry forcing, this has the following consequence.

Claim 2.10. *For each $p = \langle d_0, d_1, \ldots, d_n, \langle \kappa, A_{p,\kappa}\rangle \in \mathbb{P}$, $\underset{\sim}{\tau}$ (with $p \Vdash \underset{\sim}{\tau} \in \{0, 1\}$) there exists a set $A' \in \bigcap \bar{U}_\kappa$, $A' \subseteq A_{p,\kappa}$, such that whenever $q = \langle e_0, e_1, \ldots, e_m, \langle \kappa, A_{q,\kappa}\rangle\rangle \geq p' = \langle d_0, d_1, \ldots, d_n, \langle \kappa, A'\rangle\rangle$, $\alpha \in A_{p,\kappa}$ are given with $\kappa(e_m) \leq \alpha$, and q forces a value for $\underset{\sim}{\tau}$, then so does*

$$q' = \langle e_0, e_1, \ldots, e_m, \langle \kappa, A' \cap (\alpha, \kappa)\rangle\rangle .$$

Proof. For each $\alpha \in A_{p,\kappa}$ define $B_\alpha \subseteq A_{p,\kappa}$ so that whenever $q = \langle e_0, e_1, \dots, e_m = \langle \kappa, A_{q,\kappa}\rangle\rangle \geq p$ (with $\kappa(e_0), \kappa(e_1), \dots, \kappa(e_m) \leq \alpha$) decides the value of $\underset{\sim}{\tau}$, then so does $q' = \langle e_0, e_1, \dots, e_{m+1} = \langle \kappa, B_\alpha\rangle\rangle$. This can be done easily: first for each possible $e_0, e_1, \dots, e_m$ choose a set $B_{e_0,e_1,\dots,e_n} \subseteq (\alpha, \kappa)$ with $\langle e_0, e_1, \dots, e_n, \langle \kappa, B_{e_0,e_1,\dots,e_n}\rangle\rangle$ deciding the value of $\underset{\sim}{\tau}$ if such a $B_{e_0,e_1,\dots,e_n}$ exists, otherwise just let $B_{e_0,e_1,\dots,e_n} = A_{p,\kappa} \cap (\alpha, \kappa)$. Second, let $B_\alpha = \bigcap_{e_0,e_1,\dots,e_n} B_{e_0,e_1,\dots,e_n}$. Now it is easy to check that the diagonal intersection $A' = \Delta_{\alpha \in A_{p,\kappa}} B_\alpha \in \bigcap \bar{U}_\kappa$ works (note that the intersection of normal measures is a normal filter). $\square$

Claim 2.11. *For every $p \in \mathbb{P}$ and $\underset{\sim}{\tau}$, if $p \Vdash \underset{\sim}{\tau} \in \{0,1\}$, then we can choose $\bar{q} = \langle q_\varepsilon : \varepsilon < \kappa\rangle \in \mathscr{P}_p$, $\langle \gamma_\varepsilon : \varepsilon < \kappa\rangle$, $\langle c_\varepsilon : \varepsilon < \kappa\rangle$ where each c_ε is a code for a γ_ε-Borel function from $\mathscr{P}(\mathscr{H}^{\mathbf{V}}(\gamma_\varepsilon))$ to $\{0,1\}$ so that*

$$q_\varepsilon \Vdash \underset{\sim}{\tau} = f_{c_\varepsilon}(\underset{\sim}{X} \cap \gamma_\varepsilon).$$

Proof. First if $p = \langle d_0, d_1, \dots, d_n, \langle \kappa, A_{p,\kappa}\rangle\rangle \in \mathbb{P}$, $\underset{\sim}{\tau}$ are as in clause (g) from Definition 1.1, let $A' = A'(p, \underset{\sim}{\tau}) \subseteq A_{p,\kappa}$ be given by Claim 2.10 and

- $(*_4)$ let $\bar{q} = \langle q_\varepsilon : \varepsilon < \kappa\rangle \in \mathscr{S}_p$ be defined by: $q_\varepsilon \in \mathbb{P}, q_\varepsilon = \langle d_0, d_1, \dots, d_n, \langle \kappa, A_{q_\varepsilon,\kappa}\rangle\rangle$ where $\{A_{q_\varepsilon,\kappa} : \varepsilon < \kappa\}$ lists $\{A_* \subseteq A_{p,\kappa} : A_* \equiv A' \mod [\kappa]^{<\kappa}\}$.

We still have to choose the $\gamma_\varepsilon, c_\varepsilon$. For each ε choose $\zeta_\varepsilon \in A_{q_\varepsilon,\kappa}$ such that $A_{q_\varepsilon,\kappa} \cap (\zeta_\varepsilon, \kappa) = A' \cap (\zeta_\varepsilon, \kappa)$. Now this with Claim 2.10 imply that q_ε forces that $\underset{\sim}{\tau}$ only depends on $\mathbf{G} \restriction (\zeta_\varepsilon + 1)$, in the following way.

Subclaim 2.12. *If $q_\varepsilon \in \mathbf{G}$, then for some $q^* \in \mathbf{G}$, $\delta \leq \zeta_\varepsilon$*

$$q^* \restriction (\delta+1) \frown \langle \kappa, A_{q_\varepsilon,\kappa} \setminus [0, \zeta_\varepsilon]\rangle \parallel \text{“}\underset{\sim}{\tau} = 1\text{”}.$$

Proof. Indeed, if $q_\varepsilon \in \mathbf{G}$, then by genericity there is some $\delta \leq \zeta_\varepsilon$, and $q' = \langle e_0, e_1, \dots, e_m\rangle \geq q_\varepsilon$, $q' \in \mathbf{G}$, and $\kappa(e_k) = \delta$, $A_{q',\kappa(e_{k+1})} \cap [0, \zeta_\varepsilon] = \emptyset$ (i.e. q' forces $\max(\underset{\sim}{X} \cap [0, \zeta_\varepsilon]) = \delta$)), w.l.o.g. $q' \geq q_\varepsilon$. Now by Lemma 2.9 there is some $q^* \in \mathbf{G}$, $A^* \in \bigcap \overline{U}_\kappa$ with

$$q^* \restriction (\delta+1) \frown \langle \kappa, A^*\rangle \parallel \text{“}\underset{\sim}{\tau} = 1\text{”}, \tag{2.2}$$

w.l.o.g. $q^* \geq q' \geq q_\varepsilon$. But then by the construction of $A' = A(p, \underset{\sim}{\tau})$ we have

$$q^* \restriction (\delta+1) \frown \langle \kappa, A' \setminus [0, \delta]\rangle \parallel \text{“}\underset{\sim}{\tau} = 1\text{”}, \tag{2.3}$$

therefore as $A_{q_\varepsilon,\kappa} \cap (\zeta_\varepsilon, \kappa) = A' \cap (\zeta_\varepsilon, \kappa)$ (and $\delta \leq \zeta_\varepsilon$),

$$q^* \restriction (\delta+1) \frown \langle \kappa, A' \setminus [0, \delta]\rangle \leq q^* \restriction (\delta+1) \frown \langle \kappa, A_{\zeta_\varepsilon} \setminus [0, \zeta_\varepsilon]\rangle \parallel \text{“}\underset{\sim}{\tau} = 1\text{”}, \tag{2.4}$$

as desired. $\square_{\text{Subclaim2.12}}$

It is not difficult to check that this implies that for every $q^{**} = \langle e_0, e_1, \dots, e_m\rangle \in \bigcup_{\delta \leq \zeta_\varepsilon} \mathbb{P}_{\overline{U} \restriction (\delta+1)}$ with

$$q^{**} \frown \langle \kappa, A_{\zeta_\varepsilon} \setminus [0, \zeta_\varepsilon]\rangle \Vdash \underset{\sim}{\tau} = j_{q^{**}},$$

we have

$$q_\varepsilon \Vdash \big(\{\kappa(e_i) : i \leq m\} \subseteq \underset{\sim}{X} \cap [0, \zeta_\varepsilon] \subseteq \{\kappa(e_i) : \ i \leq m\} \cup (\cup\{A_{q^{**},\kappa(e_i)} : \ i \leq m\})\big) \to \underset{\sim}{\tau} = j_{q'}.$$

Now one can define a code c_ε for a partial $|2^{\zeta_\varepsilon}|$-Borel function from $\mathscr{P}([0, \zeta_\varepsilon])$ to $\{0,1\}$ requiring that

$$q_\varepsilon \Vdash \underset{\sim}{\tau} = f_{c_\varepsilon}(\underset{\sim}{X} \cap [0, \zeta_\varepsilon])$$

(in fact it is even a ζ_ε-Borel function).

$\square_{\text{Claim2.11}}$

$\square_{\text{Subclaim2.6}}$

Finally it is left to verify clause (h) from Definition 1.1. Fix $p \in \mathbb{P}$ and $\bar{q}_\alpha = \langle q_{\alpha,\varepsilon} : \varepsilon < \kappa\rangle \in \mathscr{S}_p$ $(\alpha < \lambda)$. Now recall $(*_3)$ (f), and let $A'_\alpha \in \bigcap \bar{U}_\kappa = \bigcap_{\beta \leq \theta} U_{\kappa,\beta}$ the set corresponding to the sequence $\bar{q}_\alpha$, i.e. (if $d_0, d_1, \ldots, d_n, d_{n+1} = \langle \kappa, A_{p,\kappa}\rangle$ denote the components of p)

$$(2.5)\qquad \begin{array}{l} \bar{q}_\alpha = \langle q_{\alpha,\varepsilon} : \varepsilon < \kappa\rangle \text{ where } q_{\alpha,\varepsilon} = \langle d_0, d_1, \ldots, d_n, \langle \kappa, A_{q_{\alpha,\varepsilon},\kappa}\rangle\rangle \text{ and} \\ \{A_{q_{\alpha,\varepsilon},\kappa} : \varepsilon < \kappa\} \text{ lists } \{A_* : A_* \subseteq A_{p,\kappa} \text{ and } A_* = A'_\alpha \mod [\kappa]^{<\kappa}\}. \end{array}$$

Then for each fixed $\beta \leq \theta$ as $A'_\alpha \in U_{\kappa,\beta}$ $(\forall \alpha < \lambda)$, using (B) (c) there is $B_\beta \in U_{\kappa,\beta}$ such that $B_\beta \subseteq A_{p,\kappa}$, and

- $(*_5)$ for each $\alpha < \lambda$ $|B_\beta \setminus A'_\alpha| < \kappa$,

Let

- $(*_6)$ $B_* = \bigcup_{\beta \leq \theta} B_\beta \in \bigcap \bar{U}_\kappa$.

Therefore $(*_5)$ implies (recalling $\theta < \kappa$)

- $(*_7)$ for each $\alpha < \lambda$: $|B_* \setminus A'_\alpha| < \kappa$, therefore for some $\zeta_\alpha < \kappa$ we $B_* \cap (\zeta_\alpha, \kappa) \subseteq A'_\alpha$.

Defining q_* as $\langle d_0, d_1, \ldots, d_n, \langle \kappa, B_*\rangle\rangle$ clearly $p \leq q_*$ as $B_* \subseteq A_{p,\kappa}$. Moreover, for any fixed $\alpha < \lambda$ by (2.5) there exists some $\varepsilon < \kappa$ with the property that

- $(*_8)$ $(A_{q_{\alpha,\varepsilon}})_\kappa \cap (\zeta_\alpha, \kappa) = A'_\alpha \cap (\zeta_\alpha, \kappa) \supseteq B_* \cap (\zeta_\alpha, \kappa)$, and
- $(*_9)$ $(A_{q_{\alpha,\varepsilon}})_\kappa \cap [0, \zeta_\alpha] = B_* \cap [0, \zeta_\alpha]$,

so $B_* \subseteq (A_{q_{\alpha,\varepsilon}})_\kappa$, thus concluding $q_{\alpha,\varepsilon} \leq q_*$.

$\square_{2.2}$

Next we will give another example of a (λ, κ)-system, the Radin forcing, provided the measure sequence has some similar λ^+-directedness property.

Definition 2.13. In order to state the following claim we need to prepare with the notions below.

- (i) Let κ be a cardinal $\mathbf{j} : \mathbf{V} \to \mathbf{M}$ be an elementary embedding (into a transitive inner model $\mathbf{M}$) with $\text{crit}(\mathbf{j}) = \kappa$. We call the sequence $\overline{F} = \langle F(\alpha) : \alpha < \text{dom}(\overline{F})\rangle$ a $\mathbf{j}$-sequence of ultrafilters, if
  - (a) $F(0) = \kappa$,
  - (b) every $F(\alpha) \subseteq \mathscr{P}(\mathbf{V}_\kappa)$,
  - (c) and for each $0 < \alpha < \text{dom}(\overline{F})$, $\forall X \subseteq \mathbf{V}_\kappa$: $[X \in F(\alpha)$ iff $(\overline{F} \restriction \alpha) \in \mathbf{j}(X)]$.
- (ii) for each ultrafilter sequence $\overline{F}$ that are $\mathbf{j}$-sequence for some suitable $\mathbf{j}$ let $\kappa(\overline{F})$ denote the critical point of the witnessing $\mathbf{j}$, thus the F_α's are concentrated on $\mathbf{V}_{\kappa(\overline{F})}$, while for an ordinal α we mean α under $\kappa(\alpha)$.

Therefore for each $\alpha < \text{dom}(\overline{F})$ $F(\alpha)$ is a κ-complete normal ultrafilter on $\mathbf{V}_\kappa$, where under normality we mean that for each sequence $\langle X_\beta : \beta < \kappa\rangle$ in $F(\alpha)$ the diagonal intersection

$$\Delta_{\beta<\kappa} X_\beta = \{\overline{f} : \forall \gamma < \kappa(\overline{f}) : \overline{f} \in X_\gamma\} \in F(\alpha).$$

(iii) Let $A^{(n)}$ $(n \in \omega)$ be the following sequence of classes

$A^{(0)} = \{\overline{F} : \ \overline{F}$ is a $\mathbf{j}$-sequence of ultrafilters for some $\mathbf{j} : \mathbf{V} \to \mathbf{M}\}$,

and

$$A^{(n+1)} = \{\overline{F} \in A^{(n)} : \forall 0 < \alpha \in \mathrm{dom}(\overline{F}) \ V_{\kappa(\overline{F})} \cap A^{(n)} \in F(\alpha)\}.$$

Finally let

$$\mathbf{A} = \bigcap_{n \in \omega} A^{(n)}.$$

(iv) For any set $X \subseteq A^{(0)}$ and a set I of ordinals let

$$X \restriction I = \{\overline{F} \in X : \ \kappa(\overline{F}) \in I\}.$$

Claim 2.14. *There is a (λ, κ)-system such that $\mathbb{R}_{\mathbf{r}} = \mathbb{P}$ <u>when</u>:*

(C) (a) $\theta_* < \kappa < \lambda < 2^\kappa$,

(b) $\overline{F}_*$ *is an ultrafilter sequence consisting of κ-complete ultrafilters on $\mathbf{V}_\kappa$, $\overline{F}_* \in \mathbf{A}$.*

(c) *there exists $f : \kappa \to \kappa$ such that*

$$\{\overline{F} : \ \mathrm{dom}(\overline{F}) < f(\kappa(\overline{F}))\} \in \bigcap \overline{F}_* = \bigcap_{0<\alpha<\mathrm{dom}(\overline{F}_*)} F_*(\alpha),$$

(i.e. when for a witnessing $\mathbf{j}$ for $\overline{F}_*$ $\mathbf{j}(f)(\kappa) \geq \mathrm{dom}(\overline{F}_*)$, for example if $\mathrm{dom}(\overline{F}_*) \leq (2^{2^\kappa})^{\mathbf{M}}$),

(d) $\bigcap \overline{F}_* = \bigcap_{0<\alpha<\mathrm{dom}(\overline{F}_*)} F_*(\alpha)$ *is $<\lambda^+$-directed in the following sense. For every sequence $\langle X_\alpha : \alpha < \lambda\rangle$ in $\bigcap \overline{F}_*$ there exists $X_* \in \bigcap \overline{F}_*$ such that*

$$\forall \alpha < \lambda \ \exists \beta < \kappa : \ X_* \restriction (\beta, \kappa) \subseteq X_\alpha.$$

(e) $\mathbb{P} = \mathbb{P}_{\overline{F}_*}$ *is the Radin forcing for $\overline{F}_*$ (see Definition 2.15 below), so preserves the function $\mu \mapsto 2^\mu$, moreover, we can prescribe that in $\mathbf{V}^{\mathbb{P}}$ there is no new subset of θ_*, and $\mathbb{P}$ satisfies the κ^+-c.c.*

Proof. We will use the definition of the Radin forcing from [Git10, Definition 5.2]. Observe that the definition only depends on $\bigcap \overline{F}_*$.

Definition 2.15. For an ultrafilter sequence $\overline{F}_* \in \mathbf{A}$ we define the Radin forcing $\mathbb{P}$ to be the collection of finite sequences of the form $p = \langle d_0, d_1, \ldots, d_n, d_{n+1} = \langle \overline{F}_*, A_{p,\kappa}\rangle\rangle$, where

$(*_1)$ (a) $A_{p,\kappa} \in \bigcap \overline{F}_* = \bigcap_{0<\alpha<\mathrm{dom}(\overline{F}_*)} F_*(\alpha)$, $A_{p,\kappa} \in \mathbf{A}$,

(b) each d_j $(j \leq n)$ is either of the form

- $\langle \overline{F}_{d_j}, A_{d_j}\rangle$ where $\overline{F}_{d_j} \in \mathbf{A}$, $A_{d_j} \subseteq \mathbf{A}$, moreover,

$$A_{d_j} \in \bigcap \overline{F}_{d_j} = \bigcap_{0<\gamma<\mathrm{dom}(\overline{F}_{d_j})} F_{d_j}(\gamma).$$

  If $\varepsilon = \kappa(\overline{F}_{d_j})$ we may refer to $\langle \overline{F}_{d_j}, A_{d_j}\rangle$ as $\langle \overline{F}_{p,\varepsilon}, A_{p,\varepsilon}\rangle$, and we also define $\kappa(d_j) = \kappa(\overline{F}_{d_j})$.
- or $d_j = \varepsilon$ for some $\varepsilon < \kappa$ (when we let $\kappa(d_j) = \varepsilon$).

(c) $\kappa(d_0) < \kappa(d_1) < \cdots < \kappa(d_n) < \kappa(d_{n+1}) = \kappa$,

(d) moreover, for each $j \leq n$ if d_{j+1} is a triplet, then $A_{p,\kappa(d_{j+1})} \cap V_{\kappa(d_j)} = \emptyset$.

$(*_2)$ For the sequences

$$p = \langle d_0, d_1, \dots, d_n, d_{n+1} = \langle \overline{F}_*, A_{p,\kappa}\rangle\rangle,$$

$$q = \langle e_0, e_1, \dots, e_n, e_{m+1} = \langle \overline{F}_*, A_{q,\kappa}\rangle\rangle$$

we let $p \leq q$, if

(a) $m \geq n$, and

(b) there exists a sequence $0 \leq i_0 < i_1 < \dots < i_n < j_{n+1} = m$ such that for each $j \leq n+1$ we have
- $\kappa(d_j) = \kappa(e_{i_j})$,
- and

either $\overline{F}_{p,\kappa(d_j)} = \overline{F}_{q,\kappa(e_{i_j})}$ and $A_{q,\kappa(e_{i_j})} \subseteq A_{p,\kappa(d_j)}$,

or $d_j = e_{i_j} = \kappa(d_j) = \kappa(e_{i_j})$,

(c) moreover, for each $k \leq m$ not of the form i_j $(j \leq n+1)$, if $i_l = \min\{i_j : j \leq n+1,\ i_j > k\}$, then

$$A_{q,\kappa(e_k)} \cup \{\overline{F}_{q,\kappa(e_k)}\} \subseteq A_{p,\kappa(d_l)}.$$

Definition 2.16. We define $p \leq_* q$ to be true iff $p \leq q$ and $\ell g(p) = \ell g(q)$.

We define the system $\mathbf{r}$ by letting:

$(*_3)$ (a) $\kappa_{\mathbf{r}} = \kappa$,

(b) $\lambda_{\mathbf{r}} = \lambda$,

(c) $\mathbb{R}_{\mathbf{r}} = \mathbb{P}$,

(d) let $\underset{\sim}{X}_{\mathbf{r}}$ be the generic sequence, i.e.

$$\underset{\sim}{X}_{\mathbf{r}} = \cup\{\{\kappa(d_j) : j < \ell g(p)\} :\ p = \langle d_0, d_1, \dots, d_{\ell g(p)-1}\rangle \in \mathbf{G}_{\mathbb{P}}\} \setminus \{\kappa\},$$

(e) $\leq_{\rm pr} = \leq_{\mathbf{r},\rm pr}$ is defined by $p \leq_{\rm pr} q$ iff $p \leq_* q$,

(f) for $p = \langle d_0, d_1, \dots, d_n, d_{n+1} = \langle \overline{F}_*, A_{p,\kappa}\rangle\rangle \in \mathbb{R}_{\mathbf{r}} = \mathbb{P}$ let

$$\mathscr{S}_p = \mathscr{S}_{\mathbf{r},p} := \left\{ \begin{array}{l} \bar{q} : \bar{q} = \langle q_\varepsilon : \varepsilon < \kappa\rangle, \text{ where} \\ (\bullet_1) q_\varepsilon = \langle d_0, d_1, \dots, d_n, \langle \overline{F}_*, A_{q_\varepsilon,\kappa}\rangle\rangle, \text{ and} \\ \text{for some } B \in \bigcap \overline{F}_* \text{ we have} \\ (\bullet_2)\ B \subseteq A_{p,\kappa}, \text{ and} \\ (\bullet_3)\ \{A_{q_\varepsilon,\kappa} : \varepsilon < \kappa\} \text{ lists } \{A_* : A_* \subseteq A_{p,\kappa} \ \wedge\ A_* = B \mod [\kappa]^{<\kappa}\} \end{array} \right\}.$$

Now we check the requirements of Definition 1.1.

It is known that if a condition $\langle\langle\beta\rangle, \langle\overline{F}_*, A_\kappa\rangle\rangle$ is in the generic filter (for some $\beta < \kappa$) then the forcing adds no new subset of β. This implies that as $\bigcap \overline{F}_* \subseteq F_*(0)$, which is concentrated on the ordinals, i.e. on κ itself, w. l. o. g. we can assume that $\langle \beta, \langle \overline{F}_*, A\rangle\rangle \in \mathbf{G}$ for some $\beta \geq \theta_*$.

Now we have only to check the requirements of Definition 1.1. Recall the following properties of the Radin forcing, see [Git10, Sec. 5.1].

Fact 2.17. *(Prikry Lemma) For each $p \in \mathbb{P}$ and each formula $\sigma(\underset{\sim}{x}_0, \dots, \underset{\sim}{x}_m)$ there exists $q \geq_* p$, $q \parallel \sigma(\underset{\sim}{x}_0, \dots, \underset{\sim}{x}_m)$ (i.e. either $q \Vdash \sigma(\underset{\sim}{x}_0, \dots, \underset{\sim}{x}_m)$, or $q \Vdash \neg\sigma(\underset{\sim}{x}_0, \dots, \underset{\sim}{x}_m)$).*

The following claims, which complete the proof of Claim 2.14 have the same proofs as in the case of Magidor forcing. In Claim 2.18 condition $(C)/(c)$ is essential for the argument.

Claim 2.18. *For each* $p = \langle d_0, d_1, \ldots, d_{n+1} = \langle \overline{F}_*, A_{p,\kappa}\rangle\rangle \in \mathbb{P}$, $\underset{\sim}{\tau}$ *(with* $p \Vdash \underset{\sim}{\tau} \in \{0,1\}$*) there exists a set* $A' \in \bigcap \overline{F}_*$, $A' \subseteq A_{p,\kappa}$, *such that whenever* $q = \langle e_0, e_1, \ldots, e_m, \langle \overline{F}_*, A_{q,\kappa}\rangle\rangle \geq p' = \langle d_0, d_1, \ldots, d_n, \langle \overline{F}_*, A'\rangle\rangle$, $\alpha \geq \kappa(e_m)$ *are given and* q *forces a value for* $\underset{\sim}{\tau}$, *then so does*

$$q' = \langle e_0, e_1, \ldots, e_m, \langle \overline{F}_*, A' \restriction (\alpha, \kappa)\rangle\rangle .$$

Claim 2.19. *For every* $p \in \mathbb{P}$ *and* $\underset{\sim}{\tau}$, *if* $p \Vdash \tau \in \{0,1\}$, *then there exists* $\bar{q} = \langle q_\varepsilon : \varepsilon < \kappa\rangle \in \mathscr{P}_p$, $\langle \gamma_\varepsilon : \varepsilon < \kappa\rangle \in {}^\kappa\kappa$, $\langle c_\varepsilon : \varepsilon < \kappa\rangle$ *such that each* c_ε *is a code for a* γ_ε*-Borel function from* $\mathscr{P}(\gamma_\varepsilon)$ *to* $\{0,1\}$, *and*

$$q_\varepsilon \Vdash \underset{\sim}{\tau} = f_{c_\varepsilon}(\underset{\sim}{X}).$$

$\square_{\text{Claim2.14}}$

§ 3. The preparatory forcing

§ 3(A). The general frame. This subsection is dedicated for the preparation, in Claim 3.2 we provide a general frame to force a $(\lambda, \kappa) - 1$ system.

Definition 3.1. For D a $< \kappa$-directed system (i.e. generating a κ-complete filter D^*) on $\cup D \subseteq V_\kappa$ (so $D^* \subseteq \mathscr{P}(\cup D)$) let $\mathbb{Q} = \mathbb{Q}_D$ be the following forcing notion (with the notations from Definition 2.13, also applying to filters concentrated on κ):

- (A) $p \in \mathbb{Q}$ <u>iff</u>
  - (a) $p = (w, A) = (w_p, A_p)$, and for some $\sigma_p < \kappa$ we have
  - (b) $w_p \subseteq V_\kappa$, $w_p = w_p \restriction [0, \sigma_p)$,
  - (c) $A_p \subseteq \cup D$, $A_p \in D^*$ and $A_p = A_p \restriction [\sigma_p, \kappa)$.
- (B) $\mathbb{Q} \models p \leq q$ <u>iff</u>
  - (a) $p, q \in \mathbb{Q}$,
  - (b) $w_p \subseteq w_q \subseteq w_p \cup A_p$,
  - (c) $A_p \supseteq A_q$,
- (C) $\underset{\sim}{w} = \cup\{w_p : p \in \underset{\sim}{\mathbf{G}}\}$.

Claim 3.2. *If (A) and (B) hold, then (C) where:*

- *(A)* $\mathbf{v} = (\mathbf{V}_0, \kappa, \mathbf{h}, \mathbf{p}, \mathbf{G}_\kappa, \mathbf{V}_1)$ *satisfies:*
  - *(a)* $\mathbf{V}_0$ *is a universe of set theory,*
  - *(b) in* $\mathbf{V}_0$ κ *is supercompact and* $\mathbf{h} : \kappa \to \mathscr{H}(\kappa)$ *is a Laver diamond,*
  - *(c)* $\mathbf{p}$ *is the Easton support iteration* $\langle \mathbb{P}_{\mathbf{p},\alpha}, \underset{\sim}{\mathbb{Q}}_{\mathbf{p},\beta} : \alpha \leq \kappa, \beta < \kappa\rangle = \langle \mathbb{P}^0_\alpha, \underset{\sim}{\mathbb{Q}}^0_\beta : \alpha \leq \kappa, \beta < \kappa\rangle$ *built essentially as in Laver* [Lav78] *from* $\mathbf{h}$ *and let* $\mathbb{P}_\mathbf{p} = \mathbb{P}_{\mathbf{p},\kappa}$ *(hence for* $\alpha < \kappa$ *also* $\mathbb{P}^0_\alpha \in V_\kappa^{\mathbf{V}_0}$*), see Definition 3.4,*
  - *(d)* $\mathbf{G}_\kappa = \mathbf{G}_{\mathbf{p},\kappa} \subseteq \mathbb{P}_\mathbf{p}$ *is generic over* $\mathbf{V}_0$ *and* $\mathbf{V} = \mathbf{V}_1 = \mathbf{V}_0[\mathbf{G}_\kappa]$.
- *(B)* *(a)* $\kappa < \lambda < \chi = \chi^\lambda$,
  - *(b)* $\mathbb{P}^1_\chi = \langle \mathbb{P}^1_\alpha, \underset{\sim}{\mathbb{Q}}^1_\beta : \alpha \leq \chi, \beta < \chi\rangle \in \mathbf{V}_1$ *is an iteration with* $< \kappa$ *support such that* $\mathbb{P}^1_\chi$ *is* λ^+*-c.c. and* $<\kappa$*-directed closed, preserving cardinals,*

(c) *for each* $\alpha < \chi$

$$\mathbf{V}_1^{\mathbb{P}^1_\alpha} \models |\mathbb{Q}^1_\alpha| \leq \chi.$$

(d) *for the set* $S^* \subseteq \chi$ *there is a system* $\langle \underset{\sim}{D}_\delta : \delta \in S^* \rangle \in \mathbf{V}_1$, $\underset{\sim}{D}_\delta$ *is a* $\mathbb{P}^1_\delta$*-name of a subset of* $\mathscr{P}^{\mathbf{V}_1^{\mathbb{P}^1_\delta}}(V_\kappa)$, *and if* $\underset{\sim}{D}_\delta$ *generates a* κ*-complete filter satisfying* $(\forall \alpha < \kappa)\ |\cup D_\alpha) \restriction \alpha| < \kappa$, *then the forcing* $\underset{\sim}{\mathbb{Q}}^1_\delta$, $\delta \in S^*$ *is of the form* $Q_{\underset{\sim}{D}_\delta}$, *the forcing from Definition* 3.1. *Moreover,* $\langle \underset{\sim}{D}_\delta : \delta \in S^* \rangle$ *satisfies the following*

(#) $\mathbf{V}_1^{\mathbb{P}^1_\chi} \models \forall X \subseteq V_\kappa\ \forall D \in [\mathscr{P}(X)]^{\leq \lambda}$:
- $(\forall \alpha < \kappa\ |X \restriction \alpha| < \kappa) \wedge$
- *(D generates a proper* $< \kappa$*-complete filter)*

$\longrightarrow (D = D_\delta$ *for some* $\delta \in S^*)$.

(C) *in* $\mathbf{V}_1^{\mathbb{P}^1_\chi}$ *we have* 2^κ *is* χ, *and the following.*

(a) *There is a* κ*-complete normal ultrafilter* U, *which is* $< \lambda^+$*-directed mod* $[\kappa]^{<\kappa}$.

(b) *(Setting for Magidor forcing:) There is a sequence* $\bar{U} = \langle U_i : i < \kappa \rangle$ *of normal ultrafilters on* κ, *strictly increasing in the Mitchell order, i.e.* $i < j \Rightarrow U_i \in \mathrm{MosCol}({}^\kappa(\mathbf{V}^{\mathbb{P}^1_\chi})/U_j)$, *such that each* U_i *is* $< \lambda^+$*-directed mod* $[\kappa]^{<\kappa}$.

(c) *(Setting for Radin forcing:) For any* $\Upsilon \geq \kappa$ *and* η *there is a* κ*-complete fine normal ultrafilter* W *on* $[\Upsilon]^{<\kappa}$ *such that for the elementary embedding* $\mathbf{j}_W$ *of* $\mathbf{V}_1^{\mathbb{P}^1_\chi}$ *with critical point* κ *we have (letting* $\overline{U}$ *denote the measure sequence associated to* $\mathbf{j}_W$*):*

($\star$) *for every* $\sigma \leq \min(\mathrm{dom}(\overline{U}, \eta))$ *if the filter* $\bigcap(\overline{U} \restriction \sigma) = \bigcap_{\gamma < \sigma} U_\gamma$ *concentrates on a set* $X \subseteq V_\kappa$ *with* $(\forall \alpha < \kappa)\ |X \restriction \alpha| < \kappa$, *then* $\bigcap(\overline{U} \restriction \sigma)$ *is* $< \lambda^+$*-directed in the following sense: Whenever* $\langle A_i : i < \lambda \rangle$ $(\forall i < \lambda\ A_i \in \bigcap(\overline{U} \restriction \sigma))$ *is given, there exists* $A_* \in \bigcap(\overline{U} \restriction \sigma)$ *such that*

$$\forall i \in \lambda\ \exists \delta_i < \kappa :\ A_* \restriction [\delta_i, \kappa) \subseteq A_i. \tag{3.1}$$

In particular κ *is supercompact.*

Remark 3.3. This continues Džamonja-Shelah [DS03].

Proof. First we construct the iteration $\mathbb{P}^0$ using the Laver function $\mathbf{h} : \kappa \to \mathscr{H}(\kappa) \in \mathbf{V}_0$. The construction $\mathbb{P}^0 = \langle \mathbb{P}^0_\alpha, \underset{\sim}{\mathbb{Q}}^0_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ goes by induction, we follow [Lav78], only with a slight technical modification which we will need in the proof of ((C))(b).

Let $\mathbf{h}$ be as in [Lav78] (i.e.

$(\bullet)_1$ for each $\lambda \geq \kappa$, $x \in \mathscr{H}(\lambda^+)$ there exists a κ-complete fine normal ultrafilter U on $[\lambda]^{<\kappa}$ such that for the associated elementary embedding $\mathbf{j}_U$

$$\mathbf{j}_U(\mathbf{h})(\kappa) = x).$$

Definition 3.4. We define $\mathbb{P}^0 = \langle \mathbb{P}^0_\alpha, \underset{\sim}{\mathbb{Q}}^0_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ and $\langle \mu_\alpha : \alpha < \kappa \rangle$ by induction. If $\langle \mathbb{P}^0_\alpha, \underset{\sim}{\mathbb{Q}}^0_\beta : \alpha < \gamma, \beta < \gamma \rangle$ are already defined, then

$(\bullet)_I$ if γ is strongly inaccessible then $\mathbb{P}^0_\gamma$ is the direct limit (i.e. we use bounded support),

$(\bullet)_{II}$ otherwise let $\mathbb{P}^0_\gamma$ be the inverse limit of $\mathbb{P}^0_\beta$'s $(\beta < \gamma)$ (i.e. for a function p with $\mathrm{dom}(f) = \gamma$ $p \in \mathbb{P}^0_\gamma$ iff $(\forall \beta < \gamma)\ p \restriction \beta \in \mathbb{P}^0_\beta$). .

Second,

$(\bullet)_a$ if $\sup\{\mu_\alpha : \alpha < \gamma\} \leq \gamma$, γ is inaccessible, and $\mathbf{h}(\gamma) = \langle \underset{\sim}{Q}_*, \mu_*, \underset{\sim}{U}\rangle$, where $\underset{\sim}{Q}_*$ is a $\mathbb{P}^0_\gamma$-name for $<\gamma$-directed closed notion of forcing, μ_* is an ordinal, $\underset{\sim}{U}$ is a (possibly trivial) $\mathbb{P}^0_\gamma$-name, then let

$$\underset{\sim}{\mathbb{Q}}^0_\gamma = \underset{\sim}{\mathbb{Q}}_*, \quad \mu_\gamma = \mu_*.$$

$(\bullet)_b$ In the remaining case let $\underset{\sim}{\mathbb{Q}}^0_\gamma$ be the trivial forcing, $\mu_\gamma = \gamma$.

Recall $\mathbf{G}^0_\kappa \subseteq \mathbb{P}^0_\kappa$ is generic over $\mathbf{V}_0$ such that $\mathbf{V}_0[\mathbf{G}^0_\kappa] = \mathbf{V}_1$, and let $\mathbf{G}^1_\chi \subseteq \mathbb{P}^1_\chi$ be generic over $\mathbf{V}_1$, let $\mathbf{V}_2 = \mathbf{V}_1[\mathbf{G}^1_\chi] = \mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi]$. Note that as $|\mathbb{P}^0_\kappa| = \kappa$, $\kappa < \lambda$, $((B))$ implies that

$(\bowtie)_1$ $\mathbf{V}_1 = \mathbf{V}_0[\mathbf{G}^0_\kappa] \models \chi^\lambda = \chi$, thus $\mathrm{cf}(\chi) > \lambda$ holds too.

Also note that as $\mathbb{P}^1_\chi$ is $< \kappa$-closed

$(\bowtie)_2$ $V^{\mathbf{V}_2}_\kappa = V^{\mathbf{V}_1}_\kappa$, and $\mathbf{V}_2 \models$ "κ is still inaccessible."

First observe that because of our cardinal arithmetic assumptions $\chi^\kappa \leq \chi^\lambda = \chi$ $(((B))(\mathrm{a}))$, and as $|\mathbb{P}^0_\kappa| = \kappa$, we have $(\chi^\lambda)^{\mathbf{V}_1} = \chi^{\lambda\cdot\kappa} = \chi$, so by an easy induction (and by the λ^+-cc) $|\mathbb{P}^1_\chi|^{\mathbf{V}_1} = \chi$, so

$(\bowtie)_3$ $|\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi| = \chi$ up to equivalence.

Recalling $\chi^\lambda = \chi$ again, clearly

$(\bowtie)_4$ $\mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi] \models 2^\chi = (2^\chi)^{\mathbf{V}_0}$,
$(\bowtie)_5$ $\mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi] \models 2^\kappa = \chi$.

Definition 3.5. We have to introduce the following objects.

$(\bullet)_2$ Let $\mathbf{j} : \mathbf{V}_0 \to \mathbf{M}$ be an elementary embedding with critical point κ such that $(\mathbf{j}(\mathbf{h}))(\kappa) = \langle \underset{\sim}{\mathbb{P}}^1_\chi, \chi^+, \check{\emptyset}\rangle$ ($\check{\emptyset} = \emptyset$ is the canonical name for the empty set) and $\mathbf{j}(\kappa) > \chi$, ${}^\chi\mathbf{M} \subseteq \mathbf{M}$,

$(\bullet)_3$ Let $\langle \mathbb{P}^0_\alpha, \underset{\sim}{\mathbb{Q}}^0_\beta : \alpha \leq \mathbf{j}(\kappa), \beta < \mathbf{j}(\kappa)\rangle = \mathbf{j}(\langle \mathbb{P}^0_\alpha, \mathbb{Q}^0_\beta : \alpha \leq \kappa, \beta < \kappa\rangle)$ so $\underset{\sim}{\mathbb{Q}}^0_\kappa = \underset{\sim}{\mathbb{P}}^1_\chi$, and

$(\bullet)_4$ let $\underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\chi)} = \mathbf{j}(\underset{\sim}{\mathbb{P}}^1_\chi)$, i.e.

$$(\text{a } \mathbb{P}^0_{\mathbf{j}(\kappa)}\text{-name for a } < \mathbf{j}(\kappa)\text{-directed closed notion of forcing})^{\mathbf{M}}.$$

(Recall that $\underset{\sim}{\mathbb{P}}^1_\chi$ is a $\mathbb{P}^0_\kappa$-name for the iteration $\langle \underset{\sim}{\mathbb{P}}^1_\alpha, \underset{\sim}{\mathbb{Q}}^1_\beta : \alpha \leq \chi, \beta < \chi\rangle \in \mathbf{V}_0^{\mathbb{P}^0_\kappa}$.)

Similarly to $(\bowtie)_2$

$(\bowtie)_6$ $V^{\mathbf{M}[\mathbf{G}^0_{\kappa+1}]}_\kappa = V^{\mathbf{M}[\mathbf{G}^0_\kappa]}_\kappa = V^{\mathbf{V}_2}_\kappa$, ($\kappa$ is inaccessible$)^{\mathbf{M}[\mathbf{G}^0_{\kappa+1}]}$.

From now on we will identify $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi$ with the $(\kappa+1)$-step iteration $\mathbb{P}^0_{\kappa+1}$, and also

$(\bowtie)_7$ $\mathbf{G}^0_{\kappa+1} = \mathbf{G}^0_\kappa * \mathbf{G}^1_\chi$ is a generic subset of $\mathbb{P}^0_{\kappa+1} = \mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi$ (over $\mathbf{V}_0$).

Remark 3.6. Having completed the requirements of Claim 3.2 we remark that given a scheme for an iteration fitting all our assumptions other than $((B))(\mathrm{d})$, it is easy to adapt it to have $(\#)$ using $\chi^\lambda = \chi$ $(\bowtie)_1$.

Now we can prove the statements in 3.2(C).
Case 1: First we verify 3.2(C)(a).
We would like a suitable κ-complete ultrafilter in $\mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi]$, for which we will use the basic trick: using the elementary embedding $\mathbf{j} : \mathbf{V}_0 \to \mathbf{M}$, extending $\mathbf{V}_0$ with $\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi$, $\mathbf{M}$ with $\mathbf{G}^0_{\kappa+1} (= \mathbf{G}^0_\kappa * \mathbf{G}^1_\chi)$, and finding a single condition in $\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)}/\mathbf{G}^0_{\kappa+1}$ compatible with $\{\mathbf{j}(p \restriction \{\kappa\}) = \mathbf{j}(p) \restriction \{\mathbf{j}(\kappa)\} : p \in \mathbf{G}^0_\kappa * \mathbf{G}^1_\chi\}$ giving us sufficient information (just as if there existed some extension $\tilde{\mathbf{j}} : \mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi] \to \mathbf{M}[\mathbf{H}^0_{\mathbf{j}(\kappa)} * \mathbf{H}'_{\mathbf{j}(\chi)}])$.

We will need the following facts.

Fact 3.7. *The filter $\mathbf{G}^0_{\kappa+1}$ is generic over $\mathbf{M}$ as well, and the forcing notions $\mathbb{P}^0_{j(\kappa)}/\mathbf{G}^0_{\kappa+1}$ and $(\mathbb{P}^0_{j(\kappa)} * \underset{\sim}{\mathbb{P}}'_\gamma)/\mathbf{G}^0_{\kappa+1}$ ($\gamma \leq \mathbf{j}(\chi)$) are well defined and $< \chi^+$-directed closed in $\mathbf{M}[\mathbf{G}^0_{\kappa+1}]$.*

Proof. First recall that a pair $(p, \underset{\sim}{q}) \in (\mathbb{P}^0_{j(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}}(\chi))/\mathbf{G}^0_{\kappa+1}$ iff $p = p_0 \restriction (\kappa, \mathbf{j}(\kappa))$ for some $p_0 \in \mathbb{P}^0_{\mathbf{j}(\kappa)}$, and $(\Vdash_{\mathbb{P}^0_{\mathbf{j}(\kappa)}} \underset{\sim}{q} \in \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\chi)})^{\mathbf{M}}$. We only have to refer to the construction of the iteration Definition 3.4 i.e. recall that

- (i) we have $\Vdash_{\mathbb{P}^0_\kappa}$ "$\underset{\sim}{\mathbb{P}}^1_\chi$ is $< \kappa$-support iteration of $< \kappa$-directed closed forcing notions", and
- (ii) for each $\alpha \leq \beta < \kappa$ we have that $\Vdash_{\mathbb{P}^0_\beta}$ "$\underset{\sim}{\mathbb{Q}}^0_\beta$ is $< \beta$-directed closed", and is the trivial forcing if $\beta < \sup\{\mu_\varrho : \varrho < \beta\}$ (in particular, if $\beta < \sup\{\mu_\varrho : \varrho < \alpha\}$),
- (iii) for each $\alpha < \beta < \kappa$, where β is limit and $\mathrm{cf}(\beta) < \mu_\alpha$ the iteration $\mathbb{P}^0_\beta$ is the inverse limit of $\mathbb{P}^0_\delta$'s ($\delta < \beta$).

So using [Bau78, Thm. 5.5] for each $\alpha < \beta < \kappa$ the quotient $(\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi)/\mathbf{G}^0_\alpha$ (of the $\kappa+1$-long iteration $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi = \mathbb{P}^0_{\kappa+1}$) is $< \beta$-directed closed in $\mathbf{V}_0[\mathbf{G}^0_\alpha]$ if $\beta \leq \sup\{\mu_\varrho : \varrho < \alpha\}$, and $\mathbb{P}^0_\alpha$ has the β-cc. Thus by elementarity (letting $\alpha = \kappa+1$, $\beta = \chi^+ = \mu_\alpha$, recalling $(\chi^+)^{\mathbf{M}} = \chi^+$ by ${}^\chi\mathbf{M} \subseteq \mathbf{M}$):

$$\mathbf{M}[\mathbf{G}^0_{\kappa+1}] \models \text{"}(\mathbb{P}^0_{j(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}}(\chi))/\mathbf{G}^0_{\kappa+1} \text{ is } < \chi^+\text{-directed closed."}$$

$\square$

Fact 3.8. $\mathbf{V}_1 \models$ *"$(\mathbb{P}^0_{j(\kappa)} * \underset{\sim}{\mathbb{P}}'_\gamma)/\mathbf{G}^0_{\kappa+1}$ is $< \chi^+$-directed closed."*

Fact 3.8 follows from the fact below.

Fact 3.9. $\mathbf{V}[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi] \models$ "${}^\chi\mathbf{M}[\mathbf{G}^0_{\kappa+1}] \subseteq \mathbf{M}[\mathbf{G}^0_{\kappa+1}]$".

Proof. For 3.9 pick a name $\underset{\sim}{f}$ for a function $\underset{\sim}{f} : \chi \to \mathbf{M}[\mathbf{G}^0_{\kappa+1}]$, and observe that w.l.o.g. we can assume that $\underset{\sim}{f} : \chi \to \mathrm{ORD}$, i.e. for each $\alpha < \chi$ $f(\alpha)$ is an ordinal, in particular $\mathrm{ran}(f) \subseteq \mathbf{M}$. Now for each α there exists a maximal antichain $A_\alpha = \{a^\alpha_i : i < |A_\alpha|\} \subseteq \mathbb{P}^0_{\kappa+1}$, and $\{x^\alpha_i : i < |A_\alpha\} \subseteq \mathbf{M}$, s.t. $a^\alpha_i \Vdash \underset{\sim}{f}(\alpha) = x^\alpha_i$. Now as $\mathbb{P}^0_{\kappa+1} = \mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi$ is of power χ, we have $|A_\alpha| \leq \chi$, therefore as $\mathbf{M}$ is closed under sequences of length χ (($\bullet$)$_2$, Definition 3.5) there is indeed a name $\underset{\sim}{g} \in \mathbf{M}$, such that $\Vdash \underset{\sim}{f} = \underset{\sim}{g}$. $\square_{\text{Fact3.9}}$

Definition 3.10. In $\mathbf{V}_0[\mathbf{G}^0_{\kappa+1}]$, $\zeta \in S^*$ let

(1) $\varepsilon_\zeta \in \mathbf{V}[\mathbf{G}^0_\kappa * \underset{\sim}{\mathbf{G}}^1_{\zeta+1}]$ denote the generic subset of $V^{\mathbf{V}_1}_\kappa$ (or just κ) given by $\mathbb{Q}^1_\zeta$, i.e.

$$\Vdash_{\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_{\zeta+1}} \varepsilon_\zeta = \cup\{\underset{\sim}{\varepsilon} : \exists \underset{\sim}{A} : (\underset{\sim}{\varepsilon}, \underset{\sim}{A}) \in \mathbf{G}_{\underset{\sim}{\mathbb{Q}}^1_\zeta}\}$$

(after identifying $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_{\zeta+1} = \mathbb{P}^0_\kappa * (\underset{\sim}{\mathbb{P}}^1_\zeta * \underset{\sim}{\mathbb{Q}}^1_\zeta)$ with $(\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\zeta) * \underset{\sim}{\mathbb{Q}}^1_\zeta$).

(2) Define $\mathscr{N}_\zeta$ to be a set of $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\zeta$-names of subsets of V_κ containing exactly one name from each equivalence class, i.e. no $\underset{\sim}{A} \neq \underset{\sim}{B} \in \mathscr{N}_\zeta \Vdash_{\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\zeta} \underset{\sim}{A} = \underset{\sim}{B}$, but each set in the extension is represented.

Observe that by $(\bowtie)_6$ we can assume that

$(\bowtie)_8$ $\mathscr{N}_\zeta \subseteq \mathbf{M}$,

and as $|V_\kappa^{\mathbf{V}_2}| = \kappa$, and by the λ^+-cc (B)b

$(\bowtie)_9$ $|\mathscr{N}_\zeta| \le |\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\zeta|^\lambda = \chi$,

so by ${}^\chi\mathbf{M} \subseteq \mathbf{M}$ we can assume that

$(\bowtie)_{10}$ $\mathscr{N}_\zeta \in \mathbf{M}$, and $\mathbf{j} \restriction \mathscr{N}_\zeta \in \mathbf{M}$.

(3) Using the notation

$$\mathscr{A}_{\underset{\sim}{\mathbb{Q}}^1_\zeta} = \{\underset{\sim}{A} \in \mathscr{N}_\zeta : \ (\varepsilon, \underset{\sim}{A}) \in \mathbf{G}_{\underset{\sim}{\mathbb{Q}}^1_\zeta} \text{ for some } \varepsilon\},$$

note that $\mathscr{A}_{\underset{\sim}{\mathbb{Q}}^1_\zeta} \in \mathbf{M}[\mathbf{G}^0_\kappa * \mathbf{G}^1_{\zeta+1}]$ (so $\underset{\sim}{\mathscr{A}}_{\underset{\sim}{\mathbb{Q}}^1_\zeta}$ is a $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_{\zeta+1}$-name for a set of $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\zeta$-names). Now similarly $\mathbf{j}``\mathscr{A}_{\underset{\sim}{\mathbb{Q}}^1_\zeta} = \{\mathbf{j}(\underset{\sim}{A}) : \ \underset{\sim}{A} \in \mathscr{A}_{\underset{\sim}{\mathbb{Q}}^1_\zeta}\} \in \mathbf{M}[\mathbf{G}^0_\kappa * \mathbf{G}^1_{\zeta+1}]$ is a set of $\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\zeta)}$-names, and we can define the $\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\zeta)}$-name $\underset{\sim}{A}'_{\mathbf{j}(\zeta)} \in \mathbf{M}$ for a subset of $V_{\mathbf{j}(\kappa)}$ so that

$$\mathbf{M} \models \ \text{“} \Vdash_{\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\zeta)}} \ \underset{\sim}{A}'_{\mathbf{j}(\zeta)} = \cap\{\mathbf{j}(\underset{\sim}{A}) : \ \underset{\sim}{A} \in \mathscr{A}_{\underset{\sim}{\mathbb{Q}}^1_\zeta}\}\text{”}.$$

Claim 3.11. *There is a sequence $\langle q_\zeta : \ \zeta \le \chi\rangle \in \mathbf{V}[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi]$ such that:*

$(*)_{1.1}$ *(a) $q_\zeta \in (\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{j(\chi)})/\mathbf{G}^0_{\kappa+1}$, and if $\varepsilon < \zeta$ then $q_\varepsilon \le q_\zeta$,*

(b) $q_\zeta \in (\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{j(\zeta)})/\mathbf{G}^0_{\kappa+1}$, i.e. $q_\zeta \restriction \mathbf{j}(\kappa) \Vdash_{\mathbb{P}^0_{\mathbf{j}(\kappa)}} q_\zeta(\mathbf{j}(\kappa)) \in \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\zeta)}$),

(c) whenever $p \in \mathbf{G}^0_{\kappa+1} \cap (\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\zeta)$ then $\mathbf{j}(p \restriction \{\kappa\}) = \mathbf{j}(p) \restriction \{\mathbf{j}(\kappa)\} \le q_\zeta$ (in the quotient forcing $(\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{j(\chi)})/\mathbf{G}^0_{\kappa+1}$),

(d) whenever $\underset{\sim}{A}$ is a $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\zeta$-name of a subset of κ (so $\mathbf{j}(\underset{\sim}{A})$ is a $\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\zeta)}$-name for a subset of $\mathbf{j}(\kappa)$) then

$$q_\zeta \parallel_{(\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\zeta)})/\mathbf{G}^0_{\kappa+1}} \ \kappa \in \mathbf{j}(\underset{\sim}{A}).$$

(e) if $\zeta \in S^*$ for $D_\zeta = \underset{\sim}{D}_\zeta[\mathbf{G}^1_\zeta]$ (from (#) of d) we have: If D_ζ generates a κ-complete filter on V_κ (in $\mathbf{V}_1[\mathbf{G}^1_\zeta] = \mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\zeta]$) <u>then</u> we have (that q_ζ forces)

$$(q_{\zeta+1}(\mathbf{j}(\kappa)) (\mathbf{j}(\zeta)) \ge \left(\underset{\sim}{\varepsilon}_\zeta \cup \left(\underset{\sim}{A}'_{\mathbf{j}(\zeta)} \restriction \{\kappa\}\right), \underset{\sim}{A}'_{\mathbf{j}(\zeta)} \restriction (\kappa+1, \mathbf{j}(\kappa))\right). \tag{3.2}$$

(In the proof of only 3.2 D_ζ's for that $D_\zeta \subseteq cP(\kappa)$ are relevant, and it is enough to ensure that if for each $A \in D_\zeta$ we have $\kappa \in A$ (forced by q_ζ), then $q_{\zeta+1} \Vdash$ “$\kappa \in \mathbf{j}(\varepsilon_\zeta)$”.)

Proof. Working in $\mathbf{V}_2 = \mathbf{V}_0[\mathbf{G}_{\kappa+1}]$ we can define the q_η's ($\eta < \chi$, $q_\eta \in (\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\eta)})/\mathbf{G}^0_{\kappa+1}$) by induction on η. Assume that q_ξ's ($\xi < \eta$) are chosen and (a) – (e) hold. First we choose q'_ξ satisfying (a), (c), (e) which we will then further strengthen.

Let $q'_0 \in (\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(0)})/\mathbf{G}^0_{\kappa+1} = \mathbb{P}^0_{\mathbf{j}(\kappa)}/\mathbf{G}^0_{\kappa+1}$ be the empty condition. For η limit we choose $q'_\eta \in (\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}``\eta})/\mathbf{G}^0_{\kappa+1}$ to be an upper bound of the increasing sequence

$\langle q_\xi : \xi < \eta\rangle$ satisfying (c). Now it is easy to see that (c) holds for q'_η, even if $\mathbb{P}^0_\kappa * \mathbb{P}^1_\eta$ is bigger than the direct limit of $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\xi$'s $(\xi < \eta)$; also recall Fact 3.8.

If $\eta = \xi + 1$ is a successor and

- if $\xi \notin S^*$,

then using simply the $<(2^\chi)^+$-directed closedness of $\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)})/\mathbf{G}^0_{\kappa+1}$ (by Fact 3.8) define $q'_\eta \in (\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\xi+1)})/\mathbf{G}^0_{\kappa+1}$ to be an upper bound of $q_\xi \in \mathbb{P}^0_\kappa * \mathbb{P}^1_\xi$ and the set $\{\mathbf{j}(p) : p \in (\mathbb{P}^0_\kappa * \mathbb{P}^1_{\xi+1}) \cap \mathbf{G}^0_{\kappa+1}\}$.

Otherwise,

- if $\xi \in S^*$,

(where $\eta = \xi + 1$) then recall that by the definition of $\underset{\sim}{\mathbb{Q}}^1_{\xi+1}$ each $p \in (\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_{(\xi+1)})$ the coordinate $p(\xi+1)$ is a $(\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\xi)$-name for a pair (ε, A) with $\varepsilon = \varepsilon \restriction (0,\gamma)$ for some $\gamma < \kappa$ and $A \subseteq V_\kappa^{\mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\xi]}$, $A = A \restriction [\gamma, \kappa)$. As the fact that $\underset{\sim}{D}_\xi$ generates a $<\kappa$-closed filter on V_κ implies that $\mathbf{j}(\underset{\sim}{D}_\xi)$ generates a $<\mathbf{j}(\kappa)$-closed filter in $V_{\mathbf{j}(\kappa)}$, we can choose $q'_{\xi+1}$ so that $q'_{\xi+1}(\mathbf{j}(\xi))$ satisfies (3.2) (with $\zeta = \xi$), hence (e) (with $\zeta = \xi + 1 = \eta$) as well.

Finally, for (d), first note that we can assume $\underset{\sim}{A} \in \mathscr{N}_\eta$, so there are at most χ-many of them. Now choosing an increasing sequence of conditions $\langle q''_\gamma : \gamma < \chi\rangle$ in $(\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\eta)})/\mathbf{G}^0_{\kappa+1}$ with $q''_0 = q'_\eta$, we can decide for each name $\underset{\sim}{X}$ the statement $\kappa \in \mathbf{j}(\underset{\sim}{X})$. So using the $< \chi^+$-directed closedness of $(\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\eta)})/\mathbf{G}^0_{\kappa+1}$ in $\mathbf{V}_0[\mathbf{G}^0_{\kappa+1}]$ (Fact 3.8), we can choose q_η to be an upper bound of the sequence $\langle q''_\gamma : \gamma < \chi\rangle$, yielding (d) as desired.

Finally, q_χ is defined to be an upper bound of the q_η's $(\eta < \chi)$.

$\square_{\text{Claim3.11}}$

Fact 3.12. *By the definition of $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi$, and the way we q_χ we constructed we have:*

$(\bowtie)_{11}$ *For each $\delta \in S^*$ if D_δ generates a κ-complete ultrafilter on V_κ, then*

$$\Vdash_{\mathbb{P}^0_{\kappa+1}} \forall \underset{\sim}{A} \in \underset{\sim}{D}_\delta\ \exists \alpha < \kappa \text{ s.t. } (\underset{\sim}{\varepsilon}_\delta \restriction (\alpha,\kappa) \subseteq \underset{\sim}{A}),$$

$(\bowtie)_{12}$ *moreover, (in $\mathbf{M}[\mathbf{G}^0_{\kappa+1}]$)*

$$q_\chi \Vdash_{(\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\chi)})/\mathbf{G}^0_{\kappa+1}} \forall d \left(\kappa(d) = \kappa \ \wedge \ d \in \bigcap_{\underset{\sim}{A} \in \underset{\sim}{D}_\delta} \mathbf{j}(\underset{\sim}{A}) \right) \to (d \in \mathbf{j}(\underset{\sim}{\varepsilon}_\delta)),$$

in particular, this defines the normal ultrafilter

$(\bullet)_5$ $D^\bullet = \{\underset{\sim}{A}[\mathbf{G}^0_{\kappa+1}] : \Vdash_{\mathbb{P}^0_{\kappa+1}} \underset{\sim}{A} \subseteq \kappa,\ q_\chi \Vdash \text{“}\kappa \in \mathbf{j}(\underset{\sim}{A})\text{”}\}$,

$(\bowtie)_{13}$ *if $\delta \in S^*$ is such that $D_\delta \subseteq D^\bullet$, then ε_δ is the pseudointersection of D_δ, moreover, $\varepsilon_\delta \in D^\bullet$.*

This together with (#) complete the proof of $((C))$(a).

Case 2: For 3.2(C)(b) we proceed as follows. In $\mathbf{V}_1^{\mathbb{P}^1_\chi}$ we have to find a sequence $\bar{U} = \langle U_\alpha : \alpha < \kappa\rangle$ of normal measures on κ increasing in the Mitchell order, such that each U_α satisfies our closedness properties, namely, whenever $\langle X_\nu : \nu < \lambda\rangle$ is a sequence in U_α, there exists $X' \in U_\alpha$, $|X' \setminus X_\nu| < \kappa$ for each $\nu < \lambda$. Let U_0 be the normal ultrafilter using Case 1, i.e. $((C))$(a).

Working in $\mathbf{V}_1[\mathbf{G}^1_\chi] = \mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi]$ we will construct the sequence by induction, so fix $\alpha < \kappa$, and assume that U_β's are already defined for $\beta < \alpha$. So

$(\bullet)_6$ let $\overline{\underset{\sim}{U}}$ be a $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi = \mathbb{P}^0_{\kappa+1}$-name for $\langle U_\beta :\ \beta < \alpha\rangle \in \mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi]$, where $1_{\mathbb{P}^0_{\kappa+1}}$ forces that $\overline{\underset{\sim}{U}} = \langle \underset{\sim}{U}_\beta :\ \beta < \alpha\rangle$ is an increasing sequence of κ-complete normal ultrafilters w.r.t. the Mitchell-order of length α, each $\underset{\sim}{U}_\beta$ is $<\lambda^+$-directed modulo $[\kappa]^{<\kappa}$.

and fix an elementary embedding $\mathbf{j}_* : \mathbf{V}_0 \to M_*$ with critical point κ, ${}^\chi M_* \subseteq M_*$ with

$$\mathbf{j}_*(\mathbf{h})(\kappa) = \langle \underset{\sim}{\mathbb{P}}^1_\chi, \chi^+, \underset{\sim}{U}\rangle \tag{3.3}$$

(recall the definition of $\mathbf{h}$ $(\bullet)_1$, this is possible).

Defining $\underset{\sim}{\mathbb{P}}'_* = \mathbf{j}_*(\mathbb{P}^1)$, and letting $(\mathbb{P}^0_*)_{\mathbf{j}_*(\kappa)} = \mathbf{j}_*(\mathbb{P}^0_\kappa)$ observe that by the definition of $\mathbb{P}^0_\kappa$ (Definition 3.4)

$$\mathbf{j}_*(\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi) = (\mathbb{P}^0_*)_{j_*(\kappa)} * (\underset{\sim}{\mathbb{P}}'_*)_{\mathbf{j}_*(\chi)},$$

and

$$(\mathbb{P}^0_*)_{\kappa+1} = \mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi.$$

Now our fixed $\mathbf{G}^0_{\kappa+1} \subseteq \mathbb{P}^0_{\kappa+1}$ is generic over $\mathbf{V}_0$ and also over $\mathbf{M}_*$.

With a slight abuse of notation (in the proof of Case 2 from now on, in order to avoid notational awkwardness) we will refer to $(\mathbb{P}^0_*)_{\mathbf{j}_*(\kappa)}$ as $\mathbb{P}^0_{\mathbf{j}_*(\kappa)}$, and to $(\underset{\sim}{\mathbb{P}}'_*)_{\mathbf{j}_*(\chi)}$ as $\mathbb{P}'_{\mathbf{j}_*(\chi)}$; moreover, observe that all the preceding facts and claims hold in this setting, we only used that $\mathbf{j}(\mathbf{h}(\kappa)) = \langle \underset{\sim}{\mathbb{P}}^1_\chi, \chi^+, \underset{\sim}{x}\rangle$ for some name $\underset{\sim}{x}$, which obviously holds for $\mathbf{j}_*$ as well. In this new setting we appeal to Claim 3.11, obtaining the condition $q^*_\chi \in \mathbb{P}^0_{\mathbf{j}_*(\kappa)+1}/\mathbf{G}^0_{\kappa+1}$, and the κ-complete normal ultrafilter

$$D^\bullet_* = \{\underset{\sim}{A}[\mathbf{G}^0_{\kappa+1}] :\ \mathbf{M}_*[\mathbf{G}^0_{\kappa+1}] \models \text{“}q^*_\chi \Vdash_{\mathbb{P}^0_{\mathbf{j}_*(\kappa)} * \mathbb{P}'_{\mathbf{j}_*(\chi)}/\mathbf{G}^0_{\kappa+1}} \kappa \in \mathbf{j}_*(\underset{\sim}{A})\}\text{”} \tag{3.4}$$

(which is κ-complete normal ultrafilter over $\mathbf{V}_0[\mathbf{G}^0_{\kappa+1}]$, belonging to $\mathbf{V}_0[\mathbf{G}^0_{\kappa+1}]$) and $< \lambda^+$-directed w.r.t. $\supseteq^*$. We only need to prove the following claim, implying that the filter $D^\bullet_*$ dominates $\{U_\beta : \beta < \alpha\}$ in the Mitchell order.

Claim 3.13. *For each $\beta < \alpha$ there exists a sequence $\langle W_\gamma :\ \gamma < \kappa\rangle \in \mathbf{V}_0[\mathbf{G}^0_{\kappa+1}]$, where*

- *for $D^\bullet_*$-many $\gamma < \kappa$ the set W_γ is an ultrafilter over γ,*
- *for each $X \in \mathscr{P}(\kappa) \cap \mathbf{V}_0[\mathbf{G}^0_{\kappa+1}]$*
$$X \in U_\beta \iff \{\gamma < \kappa :\ (X \cap \gamma) \in W_\gamma\} \in D^\bullet_*.$$

Proof. Using (3.3)

$$\left\{\begin{array}{ll} \gamma < \kappa : & \mathbf{h}(\gamma) = \langle \underset{\sim}{x}_\gamma, \alpha_\gamma, \underset{\sim}{y}_\alpha\rangle, \text{ where } \underset{\sim}{y}_\alpha \text{ is a } \mathbb{P}^0_{\gamma+1}\text{-name} \\ & \text{for a sequence of subsets of } \mathscr{P}(\gamma) \text{ (of length } \alpha), \\ & \underset{\sim}{x}_\alpha = \underset{\sim}{\mathbb{Q}}^0_\alpha \end{array}\right\} \in D^\bullet_*,$$

so we can fix $Y \in D^\bullet_* \cap \mathbf{V}_0$, and the sequence $\langle \underset{\sim}{W}_\gamma :\ \gamma < \kappa\rangle$ such that

$(\blacktriangle_1)$ for each $\gamma \in Y$ $\underset{\sim}{W}_\gamma$ is a $\mathbb{P}^0_{\gamma+1}$-name for a subset of $\mathscr{P}(\gamma)$,
$(\blacktriangle_2)$ $\mathbf{j}_*(\langle \underset{\sim}{W}_\gamma :\ \gamma < \kappa\rangle)(\kappa) = \underset{\sim}{U}_\beta$.

We will prove that $W_\gamma = \underset{\sim}{W}_\gamma[\mathbf{G}^0_{\kappa+1}]$ $(\gamma < \kappa)$ works.

For a fixed $\mathbb{P}^0_\kappa * \mathbb{P}^1_\chi$-name $\underset{\sim}{X} \in \mathbf{V}_0$ (for a subset of κ) define the $\mathbb{P}^0_\kappa * \mathbb{P}^1_\chi$-name $\underset{\sim}{Z}_X \in \mathbf{V}_0$ as follows.

$$1_{\mathbb{P}^0_\kappa * \mathbb{P}^1_\chi} \Vdash \underset{\sim}{Z}_X = \{\gamma < \kappa :\ \underset{\sim}{X} \restriction \gamma \in \underset{\sim}{W}_\gamma\}, \tag{3.5}$$

We only have to verify that

$$\underset{\sim}{X}[\mathbf{G}^0_{\kappa+1}] \in \underset{\sim}{U}_\beta[\mathbf{G}^0_{\kappa+1}] \text{ iff } \underset{\sim}{Z}_X[\mathbf{G}^0_{\kappa+1}] \in D^\bullet_*. \tag{3.6}$$

But the latter is defined (by (3.4)) as

$$\begin{gathered}\underset{\sim}{Z}_X[\mathbf{G}^0_{\kappa+1}] \in D^\bullet_*, \\ \Updownarrow \\ (\text{in } \mathbf{M}_*[\mathbf{G}^0_{\kappa+1}])\ q^*_\chi \Vdash_{\mathbb{P}^0_{\mathbf{j}_*(\kappa)} * \mathbb{P}'_{\mathbf{j}_*(\chi)}/\mathbf{G}^0_{\kappa+1}} \kappa \in \mathbf{j}_*(\underset{\sim}{Z}_X),\end{gathered}$$

Therefore, as $\mathbf{j}_*(\underset{\sim}{W})(\kappa) = \underset{\sim}{U}_\beta$ by (3.5), for (3.6) it suffices to show

$$\underset{\sim}{X}[\mathbf{G}^0_{\kappa+1}] \in \underset{\sim}{U}_\beta[\mathbf{G}^0_{\kappa+1}] \text{ iff } q^*_\chi \Vdash \mathbf{j}_*(\underset{\sim}{X}) \restriction \kappa \in \underset{\sim}{U}_\beta. \tag{3.7}$$

But then by the elementarity of $\mathbf{j}_*$ (and $\mathrm{crit}(\mathbf{j}_*) = \kappa$)

$$\forall \alpha < \kappa, \forall p \in \mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi : \quad p \Vdash_{\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi} \check{\alpha} \in \underset{\sim}{X} \iff \mathbf{j}_*(p) \Vdash_{\mathbb{P}^0_{\mathbf{j}_*(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}_*(\mathbf{j}_*(\chi))}} \check{\alpha} \in \mathbf{j}_*(\underset{\sim}{X}),$$

and recalling $p \in \mathbf{G}^0_{\kappa+1}$ implies $q^*_\chi \geq \mathbf{j}_*(p)$) we get that

$(*)_1$ q^*_χ forces $\mathbf{j}_*(\underset{\sim}{X}) \restriction \kappa$ to be equal to $\underset{\sim}{X}[\mathbf{G}^0_{\kappa+1}]$.

This yields (3.7), completing the proof of Case 2.

□

Case 3: For 3.2(C)(c). First we redefine the elementary embedding $\mathbf{j}$ from Definition 3.5 (and as well $\mathbb{P}^0_{\mathbf{j}(\kappa)}$, $\underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\chi)}$):

Definition 3.14.

- $(\bullet)_2$ Let $\rho = |2^{(\Upsilon\cdot\chi)^\kappa} + \eta|$, and
- $(\bullet)_3$ define $\mathbf{j} : \mathbf{V}_0 \to \mathbf{M}$ be an elementary embedding with critical point κ such that $(\mathbf{j}(\mathbf{h}))(\kappa) = \langle \underset{\sim}{\mathbb{P}}^1_\chi, \rho^+, \check{\emptyset}\rangle$ ($\check{\emptyset} = \emptyset$ is the canonical name for the empty set) and $\mathbf{j}(\kappa) > \rho$, ${}^\rho\mathbf{M} \subseteq \mathbf{M}$,
- $(\bullet)_4$ Let $\langle \mathbb{P}^0_\alpha, \underset{\sim}{\mathbb{Q}}^0_\beta : \alpha \leq \mathbf{j}(\kappa), \beta < \mathbf{j}(\kappa)\rangle = \mathbf{j}(\langle \mathbb{P}^0_\alpha, \underset{\sim}{\mathbb{Q}}^0_\beta : \alpha \leq \kappa, \beta < \kappa\rangle)$ so $\underset{\sim}{\mathbb{Q}}^0_\kappa = \underset{\sim}{\mathbb{P}}^1_\chi$, and let $\underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\chi)} = \mathbf{j}(\underset{\sim}{\mathbb{P}}^1_\chi)$.

Similarly as in Facts 3.7, 3.8, 3.9 we can get the following.

Fact 3.15. *The filter $\mathbf{G}^0_{\kappa+1}$ is generic over $\mathbf{M}$ as well, and the forcing notions $\mathbb{P}^0_{j(\kappa)}/\mathbf{G}^0_{\kappa+1}$ and $(\mathbb{P}^0_{j(\kappa)} * \underset{\sim}{\mathbb{P}}'_\gamma)/\mathbf{G}^0_{\kappa+1}$ ($\gamma \leq \mathbf{j}(\chi)$) are well defined and $< |2^\Upsilon + \eta|^+$-directed closed in $\mathbf{M}[\mathbf{G}^0_{\kappa+1}]$.*

Fact 3.16. $\mathbf{V}_1 \models$ *"$(\mathbb{P}^0_{j(\kappa)} * \underset{\sim}{\mathbb{P}}'_\gamma)/\mathbf{G}^0_{\kappa+1}$ is $< |2^\Upsilon + \eta|^+$-directed closed."*

Fact 3.8 follows from the fact below.

Fact 3.17. $\mathbf{V}[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi] \models$ " ${}^{2^\Upsilon+\eta}\mathbf{M}[\mathbf{G}^0_{\kappa+1}] \subseteq \mathbf{M}[\mathbf{G}^0_{\kappa+1}]$".

Using this new $\mathbf{j}$ we will extract the ultrafilter $W \subseteq \mathscr{P}([\Upsilon]^{<\kappa})$ (in the sense of $\mathbf{V}_0[\mathbf{G}^0_{\kappa+1}]$), and the sequence of ultrafilters $\overline{U}$ as well from the information provided by $\mathbf{G}^0_{\kappa+1} = \mathbf{G}^0_\kappa * \mathbf{G}^1_\chi$, and $q_\chi \in (\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{j(\chi)})/\mathbf{G}^0_{\kappa+1}$ (given by Claim 3.11), and then we will prove that it is indeed a measure sequence corresponding to the elementary embedding $\mathbf{j}_W$. Obviously

$(\circledcirc_1)$ $\mathbf{j}(\kappa) > \chi$, ${}^\chi M \subseteq M$.

Observe that Claim 3.11 is true in this setting as and let the master condition $q_\chi \in (\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{j(\chi)})/\mathbf{G}^0_{\kappa+1}$ be given by that. First we claim that by possibly extending q_χ, we can assume that

($\circledcirc_2$) For each $A \in \mathscr{P}([\Upsilon]^{<\kappa}) \cap \mathbf{V}_2$ the condition $q_\chi \in (\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)})/\mathbf{G}^0_{\kappa+1}$ decides about "$\mathbf{j}$"$\Upsilon \in \mathbf{j}(A)$" (in $\mathbf{M}[\mathbf{G}^0_{\kappa+1}]$).

For this first we count the possible A's. Recall that $\mathbb{P}^1_\chi$ is $< \kappa$-closed $(((B))/$ (b)

$$[\chi]^{<\kappa} \cap \mathbf{V}_2 = [\chi]^{<\kappa} \cap \mathbf{V}_1 = [\chi]^{<\kappa} \cap \mathbf{V}_0[\mathbf{G}^0_\kappa],$$

and as $|\mathbb{P}^0_\kappa| = \kappa$,

$$|[\Upsilon]^{<\kappa} \cap \mathbf{V}_2| \leq (\Upsilon \cdot \chi)^\kappa. \tag{3.8}$$

Second, as $|\mathbb{P}^0_\kappa * \mathbb{P}^1_\chi| = \chi$, we have

$$\mathbf{V}_2 = \mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi] \models \mathscr{P}([\chi]^{<\kappa})| \leq (2^{(\chi \cdot \Upsilon)^\kappa})^{\mathbf{V}_0} \leq \rho. \tag{3.9}$$

Now using Fact 3.8 we can extend q_χ to another condition q_* in (at most) ρ-many steps (in $(\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{j(\zeta)})/\mathbf{G}^0_{\kappa+1}$) so that

($\circledcirc_3$) for each name $\underset{\sim}{A}$ for a subset of $[\chi]^{<\kappa}$

$$\mathbf{M}[\mathbf{G}^0_{\kappa+1}] \models \ q_* \parallel \ \mathbf{j}\text{``}\Upsilon \in \underset{\sim}{A},$$

and so (by possibly replacing q_χ by q_*) ($\circledcirc_2$) holds, indeed. Now we can define the κ-complete, fine, normal ultrafilter

$$W = \{\underset{\sim}{A}[\mathbf{G}^0_\kappa * \mathbf{G}^1_\kappa] \in [\Upsilon]^{<\kappa} : \ q_\chi \Vdash \mathbf{j}\text{``}\chi \in \mathbf{j}(A)\} \in \mathbf{V}_2, \tag{3.10}$$

Now let $\mathbf{j}_W : \mathbf{V}_2 \to \mathbf{M}_W = \mathrm{Mos}({}^{[\chi]^{<\kappa}}\mathbf{V}_2/W)$ be the corresponding elementary embedding, and let $\overline{U} = \langle U_\alpha : \ \alpha < \mathrm{dom}(\overline{U})\rangle$ be the ultrafilter sequence of maximal length associated to $\mathbf{j}_W$, that is, the following holds in $\mathbf{V}_2$.

($\boxminus_1$) $U_0 = \kappa$, and for each $\alpha \in \mathrm{dom}(\overline{U})$, $\alpha > 0$ the set $U_\alpha \subseteq \mathscr{P}(V_\kappa)$ is a κ-complete normal ultrafilter satisfying

$$\forall A \subseteq V_\kappa : \ A \in U_\alpha \iff U \restriction \alpha \in \mathbf{j}_W(A)$$

(therefore for each $\alpha < \mathrm{dom}(\overline{U})$ $U \restriction \alpha \in \mathbf{M}_W$),

($\boxminus_2$) $\overline{U} \notin \mathbf{M}_W$.

The following two claims complete the proof of 3.2$((C))$(c) as by our assumptions $\mathrm{dom}(\overline{U}) = \eta \leq \rho$.

Claim 3.18. *For every ultrafilter sequence $\overline{F} \in \mathbf{M}_W$ with $\kappa(\overline{F}) = \kappa$ there exists an ultrafilter sequence $\overline{F}' \in \mathbf{M}[\mathbf{G}^0_{\kappa+1}]$ with $\kappa(\overline{F}') = \kappa$ such that for each name $\underset{\sim}{A}$ for a subset of $V_\kappa^{\mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi]}$ we have*

$$\overline{F} \in \mathbf{j}_W(\underset{\sim}{A}[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi]) \iff \mathbf{M}[\mathbf{G}^0_{\kappa+1}] \models \ q_\chi \Vdash \overline{F}' \in \mathbf{j}(\underset{\sim}{A}).$$

Claim 3.19. *For every set of ($\mathbb{P}_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)}$-names for) ultrafilter sequences $\{\overline{\underset{\sim}{F}_i} : i < \sigma\} \subseteq \mathbf{M}$ with $\kappa(\underset{\sim}{F}_i) = \kappa$ $(i < \sigma)$ if the filter*

$$F_* = \bigcap_{i<\sigma} \{A \subseteq V_\kappa^{\mathbf{V}_2} : \ q_\chi \Vdash \overline{\underset{\sim}{F}_i} \in \mathbf{j}(A)\}$$

satisfies $(\forall \alpha < \kappa) : |\cup F_* \restriction \alpha| < \kappa$, then F_* is $< \lambda^+$-directed in the sense that for any system $\langle X_\alpha : \alpha < \lambda\rangle$ in F_* there is a set $X' \in F_*$ s.t. for each $\alpha < \lambda$ there exists $\delta < \kappa$ with $X' \restriction [\delta, \kappa) \subseteq X_\alpha$.

Proof. Instead of factoring through our elementary embeddings (after forcing) we provide a direct calculation. Fix the ultrafilter sequence $F \in \mathbf{M}_W$, and pick a function $f \in \mathbf{V}_2$, $\mathrm{dom}(f) = [\Upsilon]^{<\kappa}$, $\mathbf{j}_W(f)(\mathbf{j}_W``\Upsilon) = F$, where we can assume that

$$\forall x \in \mathrm{dom}(f)\ f(x) \text{ is an u.f. sequence with } \kappa(f(x)) = \mathrm{otp}(\kappa \cap x). \tag{3.11}$$

Now we can fix a $\mathbb{P}^0_\kappa * \mathbb{P}^1_\chi$-name $\underset{\sim}{f} \in \mathbf{V}_0$ of f, such that $1_{\mathbb{P}^0_\kappa * \mathbb{P}^1_\chi}$ forces (3.11). Now as in V_0 $\underset{\sim}{f}$ is a $\mathbb{P}^0_\kappa * \mathbb{P}^1_\chi$-name for a function with $\mathrm{dom}(f) = [\Upsilon]^{<\kappa}$, by elementarity $\mathbf{j}(\underset{\sim}{f})$ is a $\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)}$-name for a function with domain $[\mathbf{j}(\Upsilon)]^{<\mathbf{j}(\kappa)}$, thus there is name $\underset{\sim}{F}' \in \mathbf{M}$ such that

$$\mathbf{M} \models\ \Vdash_{\mathbb{P}^1_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)}} \mathbf{j}(\underset{\sim}{f})(\mathbf{j}``\Upsilon) = \underset{\sim}{F}'. \tag{3.12}$$

It is only left to check that for each $X \subseteq V^{\mathbf{V}_2}_\kappa$ the conditions "$F \in \mathbf{j}_W(X)$" and "$\underset{\sim}{F}' \in \mathbf{j}(X)$" are equivalent. More precisely, we prove the following.

($\circ$) For every fixed $\mathbb{P}^0_\kappa * \mathbb{P}^1_\chi$-name $\underset{\sim}{X}$ for a subset of $V^{\mathbf{V}_2}_\kappa$

$$F \in \mathbf{j}_W(\underset{\sim}{X}[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi]) \iff q_\chi \Vdash \underset{\sim}{F}' \in \mathbf{j}(\underset{\sim}{X}).$$

As $F = \mathbf{j}_W(f)(\mathbf{j}_W``\Upsilon)$ we can reformulate the lhs. as the statement

$$\mathbf{V}[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi] \models\ \{y \in [\Upsilon]^{<\kappa}:\ f(y) \in X\} \in W,$$

i.e. for some $p \in \mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi]$

$$p \Vdash_{\mathbb{P}^0_\kappa * \mathbb{P}^1_\chi} \{y \in [\Upsilon]^{<\kappa}:\ \underset{\sim}{f}(y) \in \underset{\sim}{X}\} \in \underset{\sim}{W}.$$

Now for the the $\mathbb{P}^0_\kappa * \mathbb{P}^1_\chi$-name $\underset{\sim}{C} := \{y \in [\Upsilon]^{<\kappa}:\ \underset{\sim}{f}(y) \in \underset{\sim}{X}\}$ we have (by ($\circledcirc_2$) and (3.10))

$$\underset{\sim}{C}[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi] \in W \iff q_\chi \Vdash \mathbf{j}``\Upsilon \in \mathbf{j}(\underset{\sim}{C}).$$

(Recall that q_χ decides this by ($\circledcirc_3$) as $\underset{\sim}{C}$ is a name for a subset of $[\Upsilon]^{<\kappa}$.) This latter is equivalent to

$$q_\chi \Vdash \mathbf{j}(\underset{\sim}{f})(\mathbf{j}``\Upsilon) \in \mathbf{j}(\underset{\sim}{X}),$$

so recalling $\Vdash \mathbf{j}(\underset{\sim}{f})(\mathbf{j}``\Upsilon) = F'$ by (3.12) this is clearly equivalent to $q_\chi \Vdash \underset{\sim}{F}' \in \mathbf{j}(\underset{\sim}{X})$, therefore ($\circ$) holds, as desired. $\square_{\text{Claim3.18}}$

Proof. Fix $\langle \underset{\sim}{F}'_i : i < \sigma\rangle$ given by Claim 3.18

We only have to recall how we constructed q_χ, which ensures the existence of the desired pseudointersection. Fix a sequence $\langle X_\alpha : \alpha < \lambda\rangle$ in the filter F_*. Now let $D' = \{X_\alpha :\ \alpha < \lambda\}$, which is equal to D_ζ for some $\zeta < \chi$ by (#) from our assumptions (B)/d. Finally, recalling Definition 3.10 and (3.2) from Claim 3.11 we get that for the generic sequence ε_ζ (which is a pseudointersection of the $D' = D_\zeta$)

$$q_{\zeta+1} \Vdash \mathbf{j}(\varepsilon_\zeta) \restriction (\kappa+1) = \varepsilon_\zeta \cup (A'_{\mathbf{j}(\zeta)} \restriction [\kappa, \kappa+1)),$$

which means that by the definition of $A'_{\mathbf{j}(\zeta)}$ (Definition 3.10)

$$\forall i < \sigma\ (\forall X \in D_\zeta\ q_\chi \Vdash \underset{\sim}{F}_i \in \mathbf{j}(X)) \Rightarrow (q_\chi \Vdash \overline{\underset{\sim}{F}_i} \in A'_{\mathbf{j}(\zeta)})$$

$\square_{\text{Claim3.19}}$

$\square_{\text{Lemma3.2}}$

§ 3(B). The preliminary forcing for obtaining $(\kappa, \lambda) - 1$ systems together with a universal in $(K_\kappa)_\lambda$.

This subsection deals with the application of Claim 3.2, we show that it is possible to force a universal object in $(K_\kappa)_\lambda$ with a notion of forcing satisfying requirements from Claim 3.2.

Conclusion 3.20. *Assume κ is supercompact $\kappa < \lambda < \chi = \chi^\lambda$, λ is regular, $(\forall\theta)(\theta \in \mathrm{Card} \wedge \kappa \leq \theta < \lambda \Rightarrow 2^\theta = \theta^+)$ and $\sigma = \mathrm{cf}(\sigma) < \kappa$.*
Then in some forcing extension $\mathbf{V}^{\mathbb{P}}$ preserving cardinals and preserving cofinalities $> \kappa$ and in $\mathbf{V}^{\mathbb{P}}, 2^\kappa = \chi, \kappa$ strong limit singular of cofinality σ and there is a universal graph in cardinality λ.

Proof. We shall use 1.1, but we have to justify it. That is, we need a forcing fitting in the scheme in Claim 3.2 with $\mathbf{V}_0 = \mathbf{V}$, specifying the $(< \kappa)$-directed-complete iteration $\mathbb{P}^1_\chi = \langle \mathbb{P}^1_\alpha, \underset{\sim}{\mathbb{Q}}^1_\beta : \alpha \leq \chi, \beta < \chi \rangle \in \mathbf{V}_1 = \mathbf{V}^{\mathbb{P}^0_\kappa}$ in which we are free to choose $\underset{\sim}{\mathbb{Q}}_\beta$'s on β's outside $S^* \subseteq \chi$. (And then conclusion $(C)/$ a or b with Claim 1.3 together with Claim 2.1 or 2.2 give the desired consistency result.) Our task is to construct (in $\mathbf{V}_1$) a suitable iteration $\mathbb{P}^1_\chi$, and checking that $\mathbb{P}^1_\chi$ is

- $(\mathsf{T})_1$ $<\kappa$-directed closed,
- $(\mathsf{T})_2$ is of cardinality χ (up to equivalence),
- $(\mathsf{T})_3$ has the λ^+-c.c.,
- $(\mathsf{T})_4$ not collapsing cardinals, and
- $(\mathsf{T})_5$ $\mathbf{V}_1 \models \Vdash_{\mathbb{P}^1_\chi}$ "there is a universal graph in $(K_\kappa)_\lambda$",
- $(\mathsf{T})_6$ and we can choose $S^* \in [\chi \setminus \{0,1\}]^\chi$, $S^* \in \mathbf{V}_1$, $|\chi \setminus S^*| = \chi$, and the $\mathbb{P}^1_\delta$-names $\underset{\sim}{D}_\delta$ $(\delta \in S^*)$ satisfying $((B))$(d) from Claim 3.2.

We will do the same as in [She90], we define (in $\mathbf{V}_1$)

1. $\mathbb{Q}^1_0$ to be the forcing of χ-many stationary sets of λ, any two intersecting in a set of size smaller than κ,
2. $\underset{\sim}{\mathbb{Q}}^1_\beta$ for $\beta \in \chi \setminus (S^* \cup \{0\})$ the main iteration from [She90] just with κ-many colors: forcing a generic random graph, and the embeddings into it with $< \kappa$-support partial functions.

We need to check that the iteration is indeed λ^+-cc, which will be ensured by showing that (in $\mathbf{V}_1$) $\mathbb{Q}^1_0$ is λ^+-cc, and in $(\mathbf{V}_1)^{\mathbb{Q}^1_0}$ the iteration of $\mathbb{Q}^1_\alpha$'s $(0 < \alpha < \chi)$, i.e. $\mathbb{P}^1_\chi / \mathbf{G}^1_1$ has the κ^+-cc .
First for future reference we we have to remark that

- $(*)_1$ in $\mathbf{V}_1 = \mathbf{V}_0^{\mathbb{P}^0_\kappa}$ κ is still inaccessible as according to Claim 3.2 $((A))$ $\mathbb{P}^0_\kappa$ is an Easton-support iteration with $\mathbb{P}^0_\alpha \in V_\kappa^{\mathbf{V}_0}$ for each $\alpha < \kappa$. As $|\mathbb{P}^0_\kappa| = \kappa$ our cardinal arithmetic assumptions above κ are also preserved.

Working in $\mathbf{V}_1$ we define the first step $\mathbb{Q}^1_0$ to be $Q(\lambda, \chi, \kappa)$ as in [Bau76, Sec. 6.], see below $(\mathsf{T})_2$ in Definition 3.22.

Lemma 3.21. *In $\mathbf{V}_1$ there exists a forcing poset $\mathbb{Q}^1_0$ that is $<\kappa$-directed closed, of power χ, having λ^+-cc, preserving cardinals from $(\kappa, \lambda]$, and*

$$\mathbf{V}_1^{\mathbb{Q}^1_0} \models \exists \{S_\alpha : \alpha < \chi\} \subseteq \mathscr{P}(\lambda), \text{ a system of stationary sets s.t. } \forall \alpha < \beta < \chi : |S_\alpha \cap S_\beta| < \kappa.$$

Proof.

Definition 3.22. First we define the following auxiliary posets.

$(\mathsf{T})_1$ For a regular cardinal μ we let $Q'(\lambda,\chi,\mu)$ be the set of functions f satisfying
  (i) $\mathrm{dom}(f) \in [\chi]^{<\mu}$,
  (ii) for each $\alpha \in \mathrm{dom}(f)$ $f(\alpha) \in [\lambda]^{<\mu}$,
  with $f \leq g$, iff
  (iii) $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$,
  (iv) $\forall \alpha \in \mathrm{dom}(f)$: $f(\alpha) \subseteq g(\alpha)$,
  (v) for each $\alpha \neq \beta \in \mathrm{dom}(f)$ $f(\alpha) \cap f(\beta) = g(\alpha) \cap g(\beta)$.

$(\mathsf{T})_2$ Let $Q(\lambda,\chi,\kappa) \subseteq \prod_{\mu \in \mathrm{Reg} \cap [\kappa,\lambda]} Q'(\lambda,\chi,\mu)$ be the collection of the following functions f
  (i) $\forall \mu < \nu \in \mathrm{Reg} \cap [\kappa,\lambda]$, $\forall \alpha \in \mathrm{dom}(f_\mu)$: $f_\mu(\alpha) \subseteq f_\nu(\alpha)$
  with the pointwise ordering inherited from the full product $\prod_{\mu \in \mathrm{Reg} \cap [\kappa,\lambda]} Q'(\lambda,\chi,\mu)$.

Definition 3.23. We let $\mathbb{Q}^1_0 = Q(\lambda,\chi,\kappa) \in \mathbf{V}_1$.

For later reference we note the following. Recall that $\chi^\lambda = \chi$ by our assumptions.

Observation 3.24. *For each $\mu \in \mathrm{Reg} \cap [\kappa,\lambda]$ $|Q'(\lambda,\chi,\mu)| \leq \chi^{<\mu} \cdot \lambda^{<\mu} = \chi$. Therefore $|\mathbb{Q}^1_0| = \chi$.*

By [Bau76, Lemma 6.3], recalling $(\sigma \in \mathrm{Card} \cap [\kappa,\lambda)) \to (2^\sigma = \sigma^+)$, so $\lambda^{<\lambda} = \lambda$ we have the following.

Claim 3.25. *$Q(\lambda,\chi,\kappa)$ is λ^+-cc, $<\kappa$-directed closed, preserving cofinalities and cardinals.*

Clearly

$(\ddagger)_1$ every directed subset of power less than κ in $\mathbb{Q}^1_0 = Q(\lambda,\chi,\kappa)$ has a least upper bound.

Now obviously in $\mathbf{V}_1^{\mathbb{Q}^1_0}$

$(\ddagger)_2$ the generic subsets S_α $(\alpha < \chi)$ defined by $\Vdash_{\mathbb{Q}^1_0} \underset{\sim}{S}_\alpha = \cup\{f_\kappa(\alpha) : f \in \underset{\sim}{\mathbf{G}}\}$ form a κ-almost disjoint system, i.e. if $\alpha < \beta$, then $\Vdash |\underset{\sim}{S}_\alpha \cap \underset{\sim}{S}_\beta| < \kappa$,

we only need to verify that

$(\ddagger)_3$ for each $\alpha < \chi$ the subset

$$S_\alpha \text{ is stationary subset of } \lambda,$$

which is a standard argument, but for the sake of completeness we elaborate. (In fact, recalling [Bau76, Lemmas 6.3-6.5.] with the aid of the following it is easy to argue that $(S_\alpha \cap E^\lambda_{\geq\kappa})$ i.e. restricting S_α to points of cofinality at least κ is stationary.)

Claim 3.26. *The notion of forcing $Q(\lambda,\chi,\kappa)$ is equivalent to the two-step iteration $Q(\lambda,\chi,\kappa^+) * \underset{\sim}{Q}'(\lambda,\chi,\kappa,\underset{\sim}{F})$ where*

$$\mathbf{V}_1^{Q(\lambda,\chi,\kappa^+)} \models \quad \begin{array}{l} \bullet\ F_\alpha\ (\alpha \in \chi) \textit{ is the generic sequence in } [\lambda]^\lambda, \\ \bullet\ Q'(\lambda,\chi,\kappa,F) \subseteq Q'(\lambda,\chi,\kappa) \textit{ defined by} \\ \quad [f \in \underset{\sim}{Q}'(\lambda,\chi,\kappa,\underset{\sim}{F}) \iff \forall \alpha \in \mathrm{dom}(f)\ f(\alpha) \subseteq \underset{\sim}{F}_\alpha]. \end{array}$$

Moreover, $Q(\lambda,\chi,\kappa^+)$ is $<\kappa^+$-closed, (in $\mathbf{V}_1^{Q(\lambda,\chi,\kappa^+)}$), and $Q'(\lambda,\chi,\kappa,F)$ has the $\kappa^+ - cc$).

Looking at the definition of the forcing $Q(\lambda,\chi,\kappa)$ if we are given a condition p, and a $Q(\lambda,\chi,\kappa)$-name $\underset{\sim}{C}_*$ for a club set in λ first recall that $Q(\lambda,\chi,\kappa)$ is $<\kappa$-closed (Claim 3.26), in particular $<\omega_1$-closed, as κ is inaccessible. We can define an increasing sequence p^j $(j<\omega)$ in $Q(\lambda,\chi,\kappa)$ with $p^0=p$, and an increasing sequence of ordinals ϱ_j $(j<\kappa)$ satisfying $p^j \Vdash \varrho_j \in \underset{\sim}{C}_*$, and if $j<k$, then $\sup\cup\{p^j_\lambda(\beta):\ \beta\in\mathrm{dom}(p^j_\lambda)\}<\varrho_k$. This is possible, as $|\mathrm{dom}(p_j)|<\lambda$, as well as $|p^j_\lambda(\beta)|<\lambda$, and λ is regular. Then clearly any upper bound of the p^j's forces $\varrho_\omega:=\sup\{\varrho_j:\ j<\omega\}\in\underset{\sim}{C}_*$, but as the least upper bound p^ω does not say anything about the statements $\varrho_\omega\in\underset{\sim}{S}_\beta$ $(\beta<\chi)$ we can extend it to a condition $(p^\omega)'$ with $\varrho_\omega\in(p^\omega)'_\mu(\alpha)$ for each $\mu\in\mathrm{Reg}\cap[\kappa,\lambda]$ (thus $(p^\omega)'\Vdash\varrho_\omega\in\underset{\sim}{S}_\alpha\cap\underset{\sim}{C}_*$). This completes the proof of Lemma 3.21. $\square_{\text{Lemma3.21}}$

As $\mathbb{Q}^1_0$ as already defined in Definition 3.23 we can define the iteration $\langle\mathbb{P}^1_\alpha,\ \underset{\sim}{\mathbb{Q}}^1_\beta:\ \alpha\le\chi,\beta<\chi\rangle$ for which we have to define S^*.

Definition 3.27. We let $0,1\notin S^*\subseteq\chi$ be such that $|S^*|=\chi$, $|\chi\setminus S^*|=\chi$.

Definition 3.28. We let $\langle\mathbb{P}^1_\alpha,\ \underset{\sim}{\mathbb{Q}}^1_\beta:\ \alpha\le\chi,\beta<\chi\rangle$ be the following $<\kappa$-support iteration. The definition of the $\mathbb{P}^1_\beta$-name $\underset{\sim}{\mathbb{Q}}^1_\beta$ goes by induction on β as follows, distinguishing three cases. But first

- ⊛ we have to remark that in steps with $\beta\in S^*$ we will only assume that $\underset{\sim}{D}_\beta$ is a $\mathbb{P}^1_\beta$-name for a system of subsets if $V^{\mathbf{V}_1}_\kappa$, where

$$\Vdash_{\mathbb{P}^1_\beta}\ \underset{\sim}{D}_\beta\in[\mathscr{P}(V^{\mathbf{V}_1}_\kappa)]^{\le\lambda},$$

  first we will deduce some properties of $\mathbb{P}^1_\chi$ based on only this weak assumption up until the end of the proof of Lemmas 3.35 and 3.34 and then we will verify that the $\underset{\sim}{D}_\beta$'s $(\beta\in S^*)$ can be suitably chosen (while defining the iteration $\mathbb{P}^1_\chi$) so that the iteration fulfills all our remaining demands from $(\mathsf{T})_1$-$(\mathsf{T})_6$. Similarly, for steps in $\chi\setminus S^*\setminus\{0,1\}$ up until the end of the proof of Lemmas 3.35 and 3.34 we only assume that $\Vdash_{\mathbb{P}^1_\beta}\underset{\sim}{M}_\beta\in(K_\kappa)_\lambda$, i.e. is a $\mathbb{P}^1_\beta$-name for a κ-colored graph on λ.
- For every $M=\langle|M|,R^M_\alpha:\ \alpha<\kappa\rangle\in(K_\kappa)_\lambda$ we will use the notation $c_M:[\lambda]^2\to\kappa$ denoting the color of the edge between i and j, i.e.

$$c_M(i,j)=\alpha\iff(i,j)\in R^M_\alpha.$$

Case (1): $\beta=1$.

Let $\mathbb{Q}^1_1\in\mathbf{V}_1^{\mathbb{Q}^1_0}$ be the forcing for obtaining a random κ-colored graph on λ with conditions of power $<\kappa$, i.e. $q\in\mathbb{Q}^1_1$ iff

- (i) $q\subseteq\{[i\ R_\gamma\ j],\ [i\ \neg R_\gamma\ j]:\ i\neq j<\lambda,\ \gamma<\kappa\}$,
- (ii) $\forall i\neq j<\lambda$ we have

$$([i\ R_\gamma\ j],[i\ R_{\gamma'}\ j]\in q)\ \longrightarrow\ (\gamma=\gamma'),$$

- (iii) $|q|<\kappa$,

with the usual ordering. Then

- $(\diamond)_1$ the generic object $\underset{\sim}{M}_*=\langle\lambda,\underset{\sim}{R}^{M_*}_\alpha:\ \alpha<\kappa\rangle$ satisfies

$$\Vdash_{\mathbb{P}^1_2}\ \langle\underset{\sim}{R}^{M_*}_\alpha:\ \alpha<\kappa\rangle\text{ is a partition of }[\lambda]^2.$$

Case (2): $\beta\in\chi\setminus S^*\setminus\{0,1\}$.

In order to define $\mathbb{Q}^1_\beta \in \mathbf{V}_1^{\mathbb{P}^1_\beta}$ (formally a $\mathbb{P}^1_\beta$-name $\underset{\sim}{\mathbb{Q}}^1_\beta \in \mathbf{V}_1$) we first need to work in $\mathbf{V}_1^{\mathbb{P}^1_1}$ as preparation. Let Υ be large enough regular cardinal, and define the continuous increasing chain $\overline{N}_\beta = \langle N_{\beta,\gamma} : \gamma < \lambda \rangle \in \mathbf{V}_1^{\mathbb{Q}^1_0}$ so that

- β, $\mathbb{P}^1_\beta$, $\langle \overline{N}_\gamma : \gamma \in \beta \setminus S^* \setminus \{0,1\} \rangle$, $\mathbf{G}^1_0 \in N_{\beta,0}$,
- $\kappa + 1 \subseteq N_{\beta,0}$,
- for each $\gamma < \lambda$:
  - $(\bullet)_a$ $N_{\beta,\gamma} \prec \left(\mathscr{H}(\Upsilon)^{\mathbf{V}_1^{\mathbb{P}^1_1}}, \in \right)$,
  - $(\bullet)_b$ $|N_{\beta,\gamma}| < \lambda$,
  - $(\bullet)_c$ $N_{\beta,\gamma} \cap \lambda$ is an initial segment of λ
  - $(\bullet)_d$ $N_{\beta,\gamma} \cap \lambda < N_{\beta,\gamma+1} \cap \lambda$,
  - $(\bullet)_e$ for $\varepsilon < \lambda$ limit $N_{\beta,\varepsilon} = \bigcup_{\gamma<\varepsilon} N_{\beta,\gamma}$,

and

$(\diamond)_2$ let $\xi_\beta(\gamma) = N_{\beta,\gamma} \cap \lambda$ $(\gamma < \lambda)$.

So the set $\{\xi_\beta(\gamma) : \gamma < \lambda\}$ is a club subset of λ, and as S_β is stationary (Lemma 3.21) the set $C_\beta = \mathrm{cl}(S_\beta \cap \{\xi_\beta(\gamma) : \gamma < \lambda\})$ (i.e. the smallest closed set containing $S_\beta \cap \{\xi_\beta(\gamma) : \gamma < \lambda\}$) is a club. Therefore the system $\langle N_{\beta,\gamma} : \gamma < \lambda \wedge \xi_\beta(\gamma) \in C_\beta \rangle$ clearly satisfies our requirements, hence (after reparametrization) we can assume that

$(\diamond)_3$ $\{\xi_\beta(\gamma+1) : \gamma \in \lambda\} \subseteq S_\beta$,

and we let

$(\diamond)_4$ $N^*_\beta = \{\xi_\beta(\gamma) : \delta \in \lambda\}$.

For later reference we remark that the κ-almost disjointness of the S_α's and $(\diamond)_3$ together implies

$(\diamond)_5$ if $\beta \neq \gamma < \chi$ then $|\{\xi_\beta(\delta+1) : \delta \in \lambda\} \cap \{\xi_\gamma(\delta+1) : \delta \in \lambda\}| < \kappa$.

Now the forcing $\mathbb{Q}^1_\beta \in \mathbf{V}_1^{\mathbb{P}^1_\beta}$ is defined so that it shall give an embedding f_β of the κ-colored graph $M_\beta \in \mathbf{V}_1^{\mathbb{P}^1_\beta}$ into M_*, formally defined by

$(\diamond)_6$ $q \in \mathbb{Q}^1_\beta$, iff

- (i) q is a set of elementary conditions of the following form
  - $[f_\beta(i) = j]$, where $j \in \{\xi_\beta(\nu+1) : \kappa i \leq \nu < \kappa(i+1)\}$ (so necessarily $i < j$),
  - $[j \notin \mathrm{ran}(f_\beta)]$ for some $j < \lambda$,
- (ii) the collection q corresponds to a partial injection, and free of any explicitly contradictory subset of terms, under which we mean that
  - (a) there are no $i, j \in \lambda$ s.t. $[f_\beta(i) = j]$, $[j \notin \mathrm{dom}(f_\beta)] \in q$,
  - (b) there are no $i, j_0 \neq j_1 \in \lambda$ s.t. $[f_\beta(i) = j_0]$, $[f_\beta(i) = j_1] \in q$,
  - (c) there are no $[f_\beta(i_0) = j_0], [f_\beta(i_1) = j_1] \in q$ s.t. $c_{M_\beta}(i_0, i_1) \neq c_{M_*}(j_0, j_1)$.

  Note that f_β's are automatically injective by (i).
- (iii) $|q| < \kappa$.

Case (3): $\beta \in S^*$.

As $\underset{\sim}{D}_\beta$ is a $\mathbb{P}^1_\beta$-name for a system of subsets of $V_\kappa^{\mathbf{V}_1}$, if additionally for each $\alpha < \kappa$ $|(\cup \underset{\sim}{D}_\beta) \restriction \alpha| < \kappa$ holds (and if $\underset{\sim}{D}_\beta$ generates a proper κ-complete filter), then we define $\mathbb{Q}^1_\beta$ to be the Mathias forcing $\mathbb{Q}_{D_\beta}$ from Definition 3.1, otherwise we can let $\mathbb{Q}^1_\beta$ to be the trivial forcing. Note that this requirement ensures that

$(\diamond)_7$ if $(w, A) \in \mathbb{Q}^1_\beta$, then $|w| < \kappa$.

This completes Definition 3.28.

Now as $\mathbb{P}^1_\chi$ is a $< \kappa$-support iteration of $< \kappa$-directed closed posets, $\mathbb{P}^1_\chi$ itself is $< \kappa$-directed closed by [Bau78, Thm 2.7], in particular not adding any new sequence of length $< \kappa$, we have:

Observation 3.29. *For each $\beta \in \chi \setminus S^* \setminus \{0, 1\}$ forcing with $\mathbb{Q}^1_\beta$ over $\mathbf{V}_1^{\mathbb{P}^1_\beta}$ adds an embedding $f_\beta : M_\beta \to M_*$.*

We already saw that $\mathbb{P}^1_1 = \mathbb{Q}^1_0$ is λ^+-cc (Lemma 3.21), now we prove that in $\mathbf{V}_1^{\mathbb{P}^1_1}$ the quotient forcing $\mathbb{P}^1_\chi/\mathbf{G}^1_1$ has the κ^+-cc (no matter how we choose the $\mathbb{P}^1_\beta$-name $\underset{\sim}{D}_\beta$, or $\underset{\sim}{M}_\beta$ only satisfying $\circledast$ for $2 \le \beta < \chi$) after which not only the λ^+-ccness of $\mathbb{P}^1_\chi$ follows, but some easy calculation will be sufficient for $(\top)_2$-$(\top)_6$. In order to prove the antichain property we will need some technical preparation, the same way as in [She90]. Recalling that each $\mathbb{P}^1_\alpha$ is $< \kappa$-closed (and $(\diamond)_7$) is straightforward to prove (by induction on α) that

$(*)_2$ The set

$$D^\bullet_\alpha = \{p \in \mathbb{P}^1_\alpha : \ \forall \gamma \in \operatorname{dom}(p) \quad \begin{array}{ll} \beta \in S^* \to \exists w_\gamma \in \mathbf{V}_1 \text{ s.t.} & \Vdash_{\mathbb{P}^1_\gamma} p(\gamma) = (\check{w}_\gamma, \underset{\sim}{A}_\gamma) \\ \text{otherwise: } \exists s_\gamma \in \mathbf{V}_1 \text{ s.t.} & \Vdash_{\mathbb{P}^1_\gamma} p(\gamma) = \check{s}_\gamma\} \end{array}$$

is a dense subset of $\mathbb{P}^1_\alpha$.

$(*)_3$ Therefore, in the quotient forcing $\mathbb{P}^1_\alpha/\mathbf{G}^1_1$ (as defined in [Bau78], or see below) the set

$$D^0_\alpha = \{p \in \mathbb{P}^1_\alpha/\mathbf{G}^1_1 : \ \exists q_0 \in \mathbf{G}^1_1 : \ \langle q_0 \rangle \cup p \in D^\bullet_\alpha\} \ \in \mathbf{V}_1^{\mathbb{P}^1_1}$$

is dense (where $\mathbb{P}^1_\alpha/\mathbf{G}^1_1 = \{p \restriction (\operatorname{dom}(p) \setminus \{0\}) : \ p \in \mathbb{P}^1_\alpha\} \in \mathbf{V}_1^{\mathbb{P}^1_1}$, and $p \le_{\mathbb{P}^1_\alpha/\mathbf{G}^1_1} q$, iff for some $r_0 \in \mathbf{G}^1_1 \subseteq \mathbb{P}^1_1$ $\langle r_0 \rangle \cup p \le_{\mathbb{P}^1_\alpha} \langle r_0 \rangle \cup q$).

$(*)_4$ With a slight abuse of notation (in order to avoid further notational awkwardness) we will identify each condition $p \in D^0_\alpha \subseteq \mathbb{P}^1_\alpha/\mathbf{G}^1_1$ with the function on the same domain, but for each $\gamma \in \operatorname{dom}(p)$

- if $\beta \in S^*$ then writing $p(\beta) = (w, \underset{\sim}{A})$ (instead of some $\mathbb{P}^1_\beta$-name satisfying $\langle q_0 \rangle \cup p \restriction \gamma \Vdash_{\mathbb{P}^1_\beta} p(\beta) = (\check{w}, \underset{\sim}{A})$ for some $q_0 \in \mathbf{G}^1_1$),
- or $p(\beta) = s$, where s is a set of symbols as in Case (1), (2) in Definition 3.28 (instead of $\langle q_0 \rangle \cup p \restriction \beta \Vdash_{\mathbb{P}^1_\beta} p(\beta) = \check{s}$ for some $q_0 \in \mathbf{G}^1_1$).

Note as $\mathbb{P}^1_1$ is $< \kappa$-closed (recall that $D^0_\alpha \subseteq \mathbf{V}_1$) that

$(*)_5$ (in $\mathbf{V}_1^{\mathbb{P}^1_1}$) for any $\alpha \le \chi$, and increasing sequence $\overline{p} = \langle p_\zeta : \ \zeta < \varepsilon < \kappa \rangle$ in D^0_α has a least upper bound in $\mathbb{P}^1_\alpha$, which we will denote by $\lim_{\zeta < \varepsilon} p_\zeta$, and this limit is in D^0_α. For the sake of completeness we include the formal definition of $\lim_{\zeta < \varepsilon} p_\zeta$. The limit of $\overline{p} = \langle p_\zeta : \ \zeta < \varepsilon < \kappa \rangle$ is the function p^*, where

(a) $\operatorname{dom}(p^*) = \bigcup_{\zeta < \varepsilon} \operatorname{dom}(p_\zeta)$,

(b) for $\beta \in S^* \cap \operatorname{dom}(p^*)$ $p^*(\beta) = (\bigcup_{\zeta < \varepsilon} w_{p_\zeta(\beta)}, \underset{\sim}{A}_\beta)$, where $p_\zeta(\beta) = (w_{p_\zeta(\beta)}, A_{p_\zeta(\beta)})$, and $\underset{\sim}{A}_\beta$ is the $\mathbb{P}^1_\beta$-name defined so that $\Vdash_{\mathbb{P}^1_\beta} \underset{\sim}{A}_\beta = \bigcap_{\zeta < \varepsilon} \underset{\sim}{A}_{p_\zeta(\beta)}$ holds,

(c) for $\beta \in \chi \setminus S^* \setminus \{0,1\}$ set $p^*(\beta) = \bigcup_{\zeta<\varepsilon} p_\zeta(\beta)$.

Definition 3.30. In $\mathbf{V}_1^{\mathbb{P}^1_1}$ for $\alpha \leq \chi$, $\delta \leq \lambda$ for each condition $p \in D^0_\alpha$ we define $p^{[\delta]}$ to be the function with $\mathrm{dom}(p^{[\delta]}) = \mathrm{dom}(p)$,

(a) if $1 \in \mathrm{dom}(p)$, then $p^{[\delta]}(1) = \{[i\ R_\gamma\ j] \in p(1) :\ i, j < \delta\}$,
(b) for $1 < \beta \in \mathrm{dom}(p) \cap S^*$ we let $p^{[\delta]}(\beta) = p(\beta)$,
(c) otherwise (for $1 < \beta \in \mathrm{dom}(p) \setminus S^*$) we let

$$\begin{aligned} p^{[\delta]}(\beta) = \quad & \{[f_\beta(i) = j] \in p(\beta) :\ i, j < \max\{\xi_\beta(\gamma) :\ \gamma < \lambda,\ \xi_\beta(\gamma) \leq \delta\}\} \\ & \qquad\qquad\qquad\qquad \cup \\ & \{[j \notin \mathrm{ran}(f_\beta)] \in p(\beta) :\ j < \max\{\xi_\beta(\gamma) :\ \gamma < \lambda,\ \xi_\beta(\gamma) \leq \delta\}\}. \end{aligned}$$

Observe that, because of each p and each $p(\beta)$ ($\beta \in \mathrm{dom}(p)$) has support of size $< \kappa$, and $\lambda > \kappa$ is regular,

$(*)_6$ for each $\alpha \leq \chi$, $p \in D^0_\alpha \subseteq (\mathbb{P}^1_\alpha/\mathbf{G}^1_1)$ we have $p^{[\delta]} = p$ for every large enough δ, and
$(*)_7$ clearly $p^{[\delta]} \restriction \beta = (p \restriction \beta)^{[\delta]}$ (for $\beta < \alpha$).
$(*)_8$ for $p \leq q \in D^0_\alpha$ with $p^{[\delta]}, q^{[\delta]} \in D^0_\alpha$ we obviously have $p^{[\delta]} \leq q^{[\delta]}$.

Note that for $p \in D^0_\alpha \subseteq \mathbb{P}^1_\alpha/\mathbf{G}^1_1$ the reduced function $p^{[\delta]}$ is in $\mathbf{V}_1^{\mathbb{P}^1_1}$ (even in $\mathbf{V}_1$), but is not necessarily a condition in $\mathbb{P}^1_\alpha/\mathbf{G}^1_1$. It is straightforward to check by induction on α the following.

Observation 3.31. *For each $\alpha \leq \chi$, $p \in D^0_\alpha$ and $\delta < \lambda$*

a) *$p^{[\delta]}$ is an actual condition (i.e. belongs to $D^0_\alpha \subseteq \mathbb{P}^1_\alpha/\mathbf{G}^1_1$), iff for every $\beta \in \mathrm{dom}(p)$*

$$p^{[\delta]} \restriction \beta \in \mathbb{P}^1_\beta,$$

and (letting $\delta^-_\beta = \max(N^*_\beta \cap (\delta+1))$

$$\begin{aligned} & \forall [f_\beta(i_0) = j_0], [f_\beta(i_1) = j_1] \in p(\beta) : \\ & j_0, j_1 < \delta^-_\beta \ \longrightarrow\ p^{[\delta]} \restriction \beta \Vdash_{\mathbb{P}^1_\beta/\mathbf{G}^1_1} c_{M_\beta}(i_0, i_1) = c_{M_*}(j_0, j_1). \end{aligned} \tag{3.13}$$

b) *In particular, for limit α*

$$p^{[\delta]} \in \mathbb{P}^1_\alpha/\mathbf{G}^1_1 \iff (\textit{for cofinally many } \varepsilon < \alpha :)\ p^{[\delta]} \restriction \varepsilon \in \mathbb{P}^1_\varepsilon,$$

c) *while for $\alpha = \beta + 1$*

$$p^{[\delta]} \in \mathbb{P}^1_\alpha/\mathbf{G}^1_1 \iff p^{[\delta]} \restriction \beta \in \mathbb{P}^1_\beta/\mathbf{G}^1_1 \textit{ and (3.13) holds.}$$

The following notion and lemma is of central importance.

Definition 3.32. In $\mathbf{V}_1^{\mathbb{P}^1_1}$ for $\alpha \leq \chi$ define

$$D^*_\alpha = \{p \in D^0_\alpha : \forall \delta < \lambda\ p^{[\delta]} \in \mathbb{P}^1_\alpha/\mathbf{G}^1_1\}.$$

Having Observation 3.31 in our mind it is easy to check the following.

$(*)_9$ Whenever $\langle p_\zeta :\ \zeta < \varepsilon < \kappa\rangle$ is an increasing sequence in D^*_α, then $\lim_{\zeta<\varepsilon} p_\zeta \in D^*_\alpha$.

This leads us to note how the statements $p \in D^*_\alpha$ and $p \restriction \beta \in D^*_\beta$ ($\beta < \alpha$) relate to each other.

Observation 3.33. *For each $\alpha \leq \chi$, $p \in D^0_\alpha$*

a) *$p \in D^*_\alpha$, iff for every $\beta \in \mathrm{dom}(p)$ and for every $\delta < \lambda$*

$$p \restriction \beta \in D^*_\beta,$$

and (letting $\delta^-_\beta = \max(N^*_\beta \cap (\delta + 1))$

$$\begin{array}{l} \forall [f_\beta(i_0) = j_0], [f_\beta(i_1) = j_1] \in p(\beta) : \\ j_0, j_1 < \delta^-_\beta \longrightarrow p^{[\delta]} \restriction \beta \Vdash_{\mathbb{P}^1_\beta / \mathbf{G}^1_1} c_{M_\beta}(i_0, i_1) = c_{M_*}(j_0, j_1). \end{array} \tag{3.14}$$

b) *In particular, for limit α*

$$p \in D^*_\alpha \iff (\text{for cofinally many } \varepsilon < \alpha) : \ p \restriction \varepsilon \in D^*_\varepsilon,$$

c) *while for $\alpha = \beta + 1$*

$$p \in D^*_\alpha \iff [p \restriction \beta \in D^*_\beta] \text{ and } [\text{for each } \delta < \lambda \text{ (3.14) holds.}]$$

Lemma 3.34. *For $\alpha \leq \chi$*

$(\blacksquare)^1_\alpha$

$$\mathbf{V}_1^{\mathbb{P}^1_1} \models D^*_\alpha \text{ is dense in } \mathbb{P}^1_\alpha / \mathbf{G}^1_1.$$

Lemma 3.35. *For every $\alpha \leq \chi$*

$(\blacksquare)^2_\alpha$

$$\mathbf{V}_1^{\mathbb{P}^1_1} \models \mathbb{P}^1_\alpha / \mathbf{G}^1_1 \text{ has the } \kappa^+\text{-cc.}$$

Proof. We proceed by induction, and prove Lemmas 3.34 and 3.35 simultaneously: More exactly we prove Lemma 3.34 for α provided that both Lemmas holds for β's less than α, and we verify the κ^+-cc property for $\mathbb{P}^1_\alpha$ assuming that D^*_α is a dense subset of $\mathbb{P}^1_\alpha / \mathbf{G}^1_1$. For $\alpha \leq 2$ (when $\mathbb{P}^1_2 / \mathbf{G}^1_1$ is essentially the forcing $\mathbb{Q}^1_1$ of the random graph Case (1) of Definition 3.28) the statement $(\blacksquare)^1_\alpha$ clearly holds.

Suppose that that we know that for each $\varepsilon < \alpha$ $(\blacksquare)^1_\alpha$ and $(\blacksquare)^2_\alpha$ hold . Assume first that α is limit. If $\mathrm{cf}(\alpha) \geq \kappa$, then $\mathbb{P}^1_\alpha = \bigcup_{\varepsilon < \alpha} \mathbb{P}^1_\varepsilon$, $D^*_\alpha = \bigcup_{\varepsilon < \alpha} D^*_\varepsilon$, so the latter is dense, we are done.

Second, if α is limit, but $\mathrm{cf}(\alpha) < \kappa$, then let $\langle \eta_\theta : \ \theta < \mathrm{cf}(\alpha) \rangle$ be a continuous increasing sequence with limit α, let $p_{-1} \in D^0_\alpha$ be arbitrary. We will choose the increasing sequence $\langle p_\theta : \ \theta < \mathrm{cf}(\alpha) \rangle$ in D^0_α with $p_0 \geq p_{-1}$, and $p_\theta \restriction \eta_\theta \in D^*_{\eta_\theta}$. This would suffice as for each $\theta < \mathrm{cf}(\kappa)$ the sequence $p_\varrho \restriction \eta_\theta$ $(\varrho \in \mathrm{cf}(\alpha))$ is eventually in $D^*_{\eta_\theta}$, so for $p^* = \lim_{\varrho < \mathrm{cf}(\alpha)} p_\varrho$ using $(*)_9$ we have $p^* \restriction \eta_\theta \in D^*_{\eta_\theta}$, leading to

$$(\forall \theta < \mathrm{cf}(\alpha)) \ p^* \restriction \eta_\theta \in D^*_{\eta_\theta},$$

so by *b*) we are done. For the construction of the p_θ's, as D^0_α and $D^*_{\eta_\theta}$'s are $< \kappa$-closed we only have to ensure that $p_\theta \in D^0_\alpha$ can be chosen so that not only $p_\theta \geq p_\varrho$ $(\varrho < \theta)$, but $p_\theta \restriction \eta_\theta \in D^*_{\eta_\theta}$. Now applying the induction hypothesis, and extending $(\lim_{\varrho < \theta} p_\varrho) \restriction \eta_\theta \leq p^*_\theta \in D^*_{\eta_\theta}$ we can choose p_θ to be the least upper bound of p^*_θ and $(\lim_{\varrho < \theta} p_\varrho)$ (in fact for θ limit we did not even have to appeal to the induction hypothesis).

Third, if $\alpha = \beta + 1$, let $p_{-1} \in D^0_\alpha$, and we will extend $p_{-1} \restriction \beta \leq p^* \in D^*_\beta$ (using $(\blacksquare)^1_\beta$) so that the right hand side of Observation 3.33 *c*) holds for $p = p^* \cup \langle p_{-1}(\beta) \rangle$.

For this, let $\{ j_\theta : \ \theta < \nu \}$ enumerate $\{ j < \lambda : \ [f_\beta(i) = j] \in p_{-1}(\beta)$ for some $i < \lambda \}$ in increasing order, and we can fix the system $\{ i_\theta : \ \theta < \nu \}$ so that

$(\odot)_1$ $\{ i_\theta : \ \theta < \nu \}$ is such that for each θ $[f_\beta(i_\theta) = j_\theta] \in p_{-1}(\beta)$.

Note that by Definition 3.28/Case (2)/ (i)

$(\odot)_2$ for each θ: $i_\theta < j_\theta$, and also we can choose γ_θ for each $\theta < \nu$ such that $\xi_\beta(\gamma_\theta) = j_\theta$, thus

$(\odot)_3$ we have

$$\{j < \lambda :\ \exists i < \lambda\ [f_\beta(i) = j] \in p_{-1}(\beta)\} = \{j_\theta :\ \theta < \nu\} = \{\xi_\beta(\gamma_\theta) :\ \theta < \nu\}.$$

Now we construct the increasing sequence $\langle p_\theta :\ \theta < \nu\rangle$ in D^*_β with the properties

(α) $p_{-1} \restriction \beta \leq p_0$,

(β) for each $\theta < \nu$, for each $\varepsilon_0 < \varepsilon_1 < \theta$

$$p_\theta^{[\xi_\beta(\gamma_{\varepsilon_1}+1)]} \Vdash_{\mathbb{P}^1_\beta/\mathbf{G}^1_1} c_{M_\beta}(i_{\varepsilon_0}, i_{\varepsilon_1}) = c_{M_*}(j_{\varepsilon_0}, j_{\varepsilon_1}).$$

This clearly suffices, as we can let $p^* = \lim_{\theta<\nu} p_\theta \in D^*_\beta$, and then $p^* \cup \langle p_{-1}(\beta)\rangle$ belongs to D^*_α: as $j_{\varepsilon_1} = \xi_\beta(\gamma_{\varepsilon_1})$ so $\xi_\beta(\gamma_{\varepsilon_1}+1)$ is the minimal $\delta < \lambda$ s.t. $(p^* \cup \langle p_{-1}(\beta)\rangle)^{[\xi_\beta(\gamma_{\varepsilon_1}+1)]}(\beta)$ contains the symbol $[f_\beta(i_{\varepsilon_1}) = j_{\varepsilon_1}]$, therefore by Observation 3.33 c) we are done, $(\blacksquare)^1_\alpha$ follows, indeed.

Appealing to the induction hypothesis let $p_0 \in D^*_\beta$, $p_0 \geq p_{-1}$. Using the $< \kappa$-closedness of D^*_β ($(*)_9$) it is enough to deal with the successor case, that is, for each θ choose $p_{\theta+1}$ so that $p_{\theta+1}^{[\xi_\beta(\gamma_\theta+1)]}$ forces that the partial function $i_\varepsilon \mapsto j_\varepsilon$ $(\varepsilon \leq \theta)$ is an embedding of $\underset{\sim}{M}_\beta \restriction \{i_\varepsilon :\ \varepsilon \leq \theta\}$ into $\underset{\sim}{M}_* \restriction \{j_\varepsilon :\ \varepsilon \leq \theta\}$. Using again $(*)_9$

$(\odot)_6$ it suffices to show that for each $\varepsilon < \theta$ and $q \geq p_{-1} \restriction \beta$, $q \in D^*_\beta$ there exists $q' \in D^*_\beta$, $q' \geq q$

$$q'^{[\xi_\beta(\gamma_\theta+1)]} \Vdash_{\mathbb{P}^1_\beta/\mathbf{G}^1_1} c_{M_\beta}(i_\varepsilon, i_\theta) = c_{M_*}(j_\varepsilon, j_\theta).$$

We will see that this follows from the following (formally) more general lemma, stated here for later reference.

Lemma 3.36. *For every $\beta \leq \chi$, $q \in D^*_\beta$, $\delta < \lambda$, i', $i'' \in \max(N^*_\beta \cap (\delta+1))$ there exists $q' \in D^*_\beta$, $q' \geq q$ such that*

$$q'^{[\delta]} \text{ forces a value for } c_{M_\beta}(i', i'').$$

Moreover, if $(\forall \gamma \in \mathrm{dom}(q) \setminus S^*)$ $([f_\beta(i) = j)] \in q(\gamma) \setminus q^{[\delta]}(\gamma)) \longrightarrow (j = \max(N^*_\gamma \cap (\delta+1))\ \wedge j < \delta)$ (hence $\delta \notin N^*_\gamma$ and $q(1) = q^{[\delta]}(1)$), then there exists q' with

$$(\forall \gamma \in \mathrm{dom}(q') \setminus S^*) : \quad q'(\gamma) \setminus q'^{[\delta]}(\gamma) = q(\gamma) \setminus q^{[\delta]}(\gamma)\ .$$

(Here we remark that the proof of the lemma uses the κ^+-cc property of $\mathbb{P}^1_\beta/\mathbf{G}^1_1$, but we will only use it for proving $(\odot)_6$, that is to complete the proof of $((\blacksquare)^1_\beta \wedge (\blacksquare)^1_\beta) \rightarrow (\blacksquare)^1_\alpha$.)

Proof. So fix $q \in D^*_\beta$, let ϱ be chosen so that $\xi_\beta(\varrho) = \max(N^*_\beta \cap (\delta+1))$, so $i', i'' < \xi_\beta(\varrho) \leq \delta$, and recall that for the model $N_{\beta,\varrho} \prec (\mathscr{H}^{\mathbf{V}_1^{\mathbb{P}^1_1}}(\Upsilon), \in)$ we know that $i', i'', \underset{\sim}{M}_\beta, \mathbb{P}^1_\beta, \mathbf{G}^1_1 \in N_{\beta,\varrho}$ (and thus $\mathbb{P}^1_\beta/\mathbf{G}^1_1 \in N_{\beta,\varrho}$). So we can find $A \in N_{\beta,\varrho}$ such that A is a maximal antichain in $N^0_\beta \subseteq \mathbb{P}^1_\beta/\mathbf{G}^1_1$, each $p \in A$ decides the value of $c_{M_\beta}(i', i'')$. But as $\mathbb{P}^1_\beta/\mathbf{G}^1_1$ has the κ^+-cc, and $\kappa+1 \subseteq N_{\beta,\varrho}$ we have that $A \subseteq N_{\beta,\varrho}$. So

$(\boxplus)_1$ let $q' \in D^*_\beta$ be a common upper bound of q and some $q'' \in A$.

We have to argue that not only $q' \Vdash_{\mathbb{P}^1_\beta/\mathbf{G}^1_1} c_{M_\beta}(i', i'') = c_*$ (for some $c_* < \kappa$) but

$$q'^{[\delta]} \Vdash_{\mathbb{P}^1_\beta/\mathbf{G}^1_1} c_{M_\beta}(i', i'') = c_*. \tag{3.15}$$

For (3.15) it is enough to prove that $q''^{[\delta]} = q''$, because then $q'^{[\delta]} \geq q''^{[\delta]} = q''$ (by $(*)_8$), yielding (3.15), as we wanted. But as $q'' \in N_{\beta,\varrho}$, and $\lambda \cap N_{\beta,\varrho} = \xi_\beta(\varrho) \leq \delta$ for each $\zeta \in \mathrm{dom}(q'') \setminus S^* \setminus \{0,1\}$ we have $\langle N_{\zeta,\iota} : \iota < \lambda\rangle \in N_{\beta,\varrho}$ (recall Case (2) from Definition 3.28), so $\xi_\beta(\varrho)$ is an accumulation point of the $\xi_\zeta(\iota)$'s. Hence we get that

$(\boxplus)_2$ for each $\zeta \in \mathrm{dom}(q'') \setminus S^* \setminus \{0,1\}$ $\xi_\beta(\varrho) = \xi_\zeta(\iota)$ for some $\iota < \lambda$ (in fact, for $\iota = \xi_\beta(\varrho)$),

so $q''^{[\xi_\beta(\varrho)]} = q''^{[\delta]} = q''$, we are done.

Finally, for the moreover part let $\delta^-_\gamma = \max(N_\gamma \cap (\delta+1))$), and define i^-_γ to be the unique ordinal s.t.

$$[f_\gamma(i^-_\gamma) = \delta^-_\gamma] \in q(\gamma) \tag{3.16}$$

(if there exists). Note that our conditions on q imply that if i^-_γ is defined, then $i^-_\gamma < \delta^-_\gamma$. Now by induction and by the first part define $q'' \geq q$ such that for every $\gamma \in \mathrm{dom}(q'') \setminus S^*$ with i^-_γ defined

$$([f_\gamma(i) = j] \in q''^{[\delta]}(\gamma)) \;\rightarrow\; q''^{[\delta]} \restriction \gamma \text{ decides the value } c_{\underset{\sim}{M}_\gamma}(i, i^-_\gamma),$$

and

$$([f_\gamma(i) = j] \in q''^{[\delta]}(\gamma)) \;\rightarrow\; q''^{[\delta]}(1) \text{ decides the value } c_{\underset{\sim}{M}_*}(j, \delta^-_\gamma)$$

(in fact this latter follows from $j, \delta^-_\gamma < \delta$ and (3.16)). Now clearly $q''^{[\delta]} \geq q^{[\delta]}$, and we want to define the condition q' to be the least upper bound of $q''^{[\delta]}$ and q, which is possible, as for every γ with i^-_γ defined we have that $q''^{[\delta]} \restriction \gamma$ forces that $q''^{[\delta]}(\gamma) \cup \{[f_\gamma(i^-_\gamma) = \delta^-_\gamma]\}$ is indeed a partial embedding.

$\square_{\text{Lemma3.36}}$

Turning back to the statement from $(\odot)_6$, as $j_\varepsilon < j_\theta = \xi_\beta(\gamma_\theta) < \xi_\beta(\gamma_\theta+1)$ we also have $i_\varepsilon, i_\theta < \xi_\beta(\gamma_\theta)$ (thus obviously $i_\varepsilon, i_\theta < \xi_\beta(\gamma_\theta+1)$). Apply the lemma with $\delta = \xi_\beta(\gamma_\theta+1)$, $i' = i_\varepsilon$, $i'' = i_\theta$,

$(\odot)_7$ let $q' \in D^*_\beta$ be given by the lemma, so $q' \Vdash_{\mathbb{P}^1_\beta/\mathbf{G}^1_1} c_{M_\beta}(i_\varepsilon, i_\theta) = c_{M_*}(j_\varepsilon, j_\theta)$

(which is obvious, as

$(\odot)_8$ $q' \geq p_{-1} \restriction \beta$, and p_{-1} is a proper condition in D^0_α with $[f_\beta(i_\theta) = j_\theta]$, $[f_\beta(i_\varepsilon) = j_\varepsilon] \in p_{-1}(\beta)$, hence $q' \frown \langle p_{-1}(\beta)\rangle$, too)

we have to argue that

$$q'^{[\xi_\beta(\gamma_\theta+1)]} \Vdash_{\mathbb{P}^1_\beta/\mathbf{G}^1_1} c_{M_\beta}(i_\varepsilon, i_\theta) = c_{M_*}(j_\varepsilon, j_\theta). \tag{3.17}$$

But $q'^{[\xi_\beta(\gamma_\theta+1)]} \Vdash_{\mathbb{P}^1_\beta/\mathbf{G}^1_1} c_{M_\beta}(i_\varepsilon, i_\theta) = c_*$, and if $[j_\varepsilon\ R_{c_*}\ j_\theta] \notin q'^{[\xi_\beta(\gamma_\theta+1)]}(1)$, so not in $q'(1)$, then adding $[j_\varepsilon\ R_{c_*+1}\ j_\theta]$ to the first coordinate of $q' \frown \langle p_{-1}(\beta)\rangle \in D^0_\alpha$ would lead to a contradiction. This verifies that assuming the induction hypotheses for β the set $D^*_{\beta+1}$ is dense in $\mathbb{P}^1_\beta/\mathbf{G}^1_1$.

Now assuming that D^*_α is dense we are ready to prove that $\mathbb{P}^1_\alpha/\mathbf{G}^1_1$ has the κ^+-cc. So let $\langle p_\gamma :\ \gamma < \kappa^+\rangle$ be an antichain in D^*_α. By extending each p_γ

$(\odot)_9$ we can assume that for each $\gamma < \kappa^+$

- (i) for each $\beta \in \mathrm{dom}(p_\gamma)$, for each $i_0, i_1, j_0 < j_1$ with $[f_\beta(i_0) = j_0]$, $[f_\beta(i_1) = j_1] \in p_\gamma(\beta)$ the condition $p^{[j_1]} \restriction \beta$ decides the value $c_{\underset{\sim}{M}_\beta}(i_0, i_1)$, (*ii*) for each $\gamma < \kappa^+$ the condition $p_\gamma(1)$ is a complete graph on the set L_γ with its edges colored, i.e.

$$L_\gamma = \{i < \lambda : \exists i' < \lambda \ \exists \varepsilon < \kappa \ [i \ R_\varepsilon \ i'] \in p_\gamma(1)\},$$
$$\text{so } (\forall i, j \in L_\gamma) \ (\exists \delta < \kappa) : \ [i \ R_\delta \ j] \in p_\gamma(1).$$

(*iii*) for each $\gamma < \kappa^+$ and $\beta \neq \beta' \in \mathrm{dom}(p_\gamma) \setminus S^* \setminus \{0, 1\}$ we have $\{\xi_\beta(\rho+1) : \rho < \lambda\} \cap \{\xi_{\beta'}(\rho + 1) : \rho < \lambda\} \subseteq L_\gamma$ (recall that $|\{\xi_\beta(\rho + 1) : \rho < \lambda\} \cap \{\xi_{\beta'}(\rho + 1) : \rho < \lambda\}| < \kappa$ by $(\diamond)_5$),
(*iv*) for each $\gamma < \kappa^+$ and $\beta \in \mathrm{dom}(p_\gamma) \setminus S^* \setminus \{0, 1\}$ and for each $j < \lambda$ if $[j \notin \mathrm{ran}(f_\beta)] \in p_\gamma(\beta)$, or $[f_\beta(i) = j] \in p_\gamma(\beta)$ for some $i < \lambda$, then $j \in L_\gamma$,
(*v*) for each $\gamma < \kappa^+$, $\beta \in \mathrm{dom}(p_\gamma)$ and $j < \lambda$, if $j \in L_\gamma$, then either $[j \notin \mathrm{ran}(f_\beta)] \in p_\gamma(\beta)$, or (for some i) $[f_\beta(i) = j] \in p_\gamma(\beta)$,
(*vi*) the set $L_\gamma \subseteq \lambda$ is closed, of limit order type,

[This is possible, a simple induction using Lemma 3.36, the fact

$$[f_\beta(i) = j] \in p_\gamma(\beta) \ \rightarrow \ j \in N^*_\beta$$

and $(*)_9$ yields that there is $p'_\gamma \geq p_\gamma$ in D^*_α, with $(p'_\gamma \restriction \beta)^{[j_1]}$ determining the value $c_{\underset{\sim}{M}_\beta}(i_0, i_1)$ whenever $[f_\beta(i_0) = j_0], [f_\beta(i_1) = j_1] \in p_\gamma(\beta)$, $j_0 < j_1$. Now repeating this ω-many times we get a condition satisfying (*i*). Then we can obtain an even stronger condition satisfying (*ii*)-(*vi*) by only adding symbols of the form $[j \notin \mathrm{ran}(f_\beta)]$ at coordinates $1 < \beta \in \chi \setminus S^*$ and extending also $p'_\gamma(1)$.] As κ is inaccessible in $\mathbf{V}_1$ by $(*)_1$, and in $\mathbf{V}_1^{\mathbb{P}^1_1}$ as $\mathbb{P}^1_1$ is $< \kappa$-closed we can apply the delta system lemma, so w.l.o.g. $\langle \mathrm{dom}(p_\gamma) : \gamma < \kappa^+\rangle$ forms a delta system. By applying the delta system lemma again we can assume that for each $\beta \in \chi \setminus S^*$ each of the following systems of sets forms a delta system:

- L_γ $(\gamma < \kappa^+)$,
- $I_\gamma(\beta) = \left\{ \begin{array}{ll} i : & [f_\beta(i) = j] \in p_\gamma(\beta) \vee \exists j \in [\xi_\beta(\kappa i), \xi_\beta(\kappa(i + 1))) \\ & [j \notin \mathrm{ran}(f_\beta)] \in p_\gamma(\beta) \end{array} \right\} (\gamma < \kappa^+).$

Therefore (recalling that each $i < \lambda$ has κ-many possible images) there are $\xi \neq \zeta < \kappa^+$, such that p_ξ and p_ζ has no explicitly contradictory terms on the coordinates concerning the κ-colored graphs, and agreeing in the first part of the condition on the coordinates dedicated to Mathias forcing, under which we mean the following (w.l.o.g. we can assume that $\xi = 0$, $\zeta = 1$):

$(\odot)_{10}$ for $\beta = 1$ for each $i, j \in L_0(1) \cap L_1(1)$ there exists some $\varepsilon < \kappa$ s.t. $[i \ R_\varepsilon \ j] \in p_0(1) \cap p_1(1)$,
$(\odot)_{11}$ for $\beta \in \chi \setminus S^* \setminus \{0, 1\}$ (if $\beta \in \mathrm{dom}(p_0) \cap \mathrm{dom}(p_1)$) the set $p_0(\beta) \cup p_1(\beta)$ determines a partial injection from a subset of λ to a subset of λ, i.e. satisfies (*ii*) (*a*), (*b*) (from Definition 3.28 Case (2)),
$(\odot)_{12}$ for $\beta \in S^* \cap \mathrm{dom}(p_0) \cap \mathrm{dom}(p_1)$ $p_0(\beta) = (w_\beta, \underset{\sim}{A}_{0,\beta})$, $p_1(\beta) = (w_\beta, \underset{\sim}{A}_{1,\beta})$ for some $w_\beta \in [V_\kappa^{\mathbf{V}_1}]^{<\kappa}$, and $\mathbb{P}^1_\beta$-names $\underset{\sim}{A}_{0.\beta}$, $\underset{\sim}{A}_{1,\beta}$.

Now p_0 and p_1 seem good candidates for a compatible pair in our supposed antichain, but we cannot take just the upper bound coordinatewise, as for coordinates $\beta > 1$ outside S^* it will not necessarily force that $p_0(\beta) \cup p_1(\beta)$ is an embedding of $\underset{\sim}{M}_\beta$ to $\underset{\sim}{M}_*$. Although it is not immediate, the following claim shows that we can construct a common upper bound, completing the proof of $(\blacksquare)^2_\alpha$ for α.

Claim 3.37. *There exists a condition $q \in D^*_\alpha$ extending both p_0 and p_1.*

Proof. Let $\{j_\varepsilon : \varepsilon < \varrho\}$ enumerate $L_0 \cup L_1 = \{j : [j \; R_\nu \; j'] \in p_0(1) \cup p_1(1) \text{ for some } j' < \lambda, \; \nu < \kappa\}$ (in increasing order).

$(\bullet)_1$ As L_0, L_1 are closed sets of ordinals without maximal element ((vi)) obviously so is $\{j_\varepsilon : \varepsilon < \varrho\}$, let j_ϱ be its supremum.

$(\bullet)_2$ By adding symbols of the form $[j \notin \operatorname{ran}(f_\beta)]$ to $p_0(\beta)$, $p_1(\beta)$ we can assume the following (not harming $(\odot)_{11}$)

- $(\bullet)_{2a}$ for $1 < \beta \in \operatorname{dom}(p_0) \cup \operatorname{dom}(p_1)$ if $[f_\beta(i) = j_\theta] \in p_0(\beta) \cup p_1(\beta)$ holds for no i then $[j_\theta \notin \operatorname{ran}(f_\beta)] \in p_0(\beta) \cap p_1(\beta)$,
- $(\bullet)_{2b}$ whenever $\beta \neq \beta' \in \operatorname{dom}(p_0) \cup \operatorname{dom}(p_1)$, $j^* \in \{\xi_\beta(\rho+1) : \rho < \lambda\} \cap \{\xi_{\beta'}(\rho+1) : \rho < \lambda\} \cap j_\varrho$ and there is no i with $[f_\beta(i) = j^*] \in p_0(\beta) \cup p_0(\beta)$ then $[j^* \notin \operatorname{ran}(f_\beta)] \in p_0(\beta) \cap p_1(\beta)$,
- $(\bullet)_{2c}$ observe that (recalling $(\odot)_9$) whenever $\beta \in \operatorname{dom}(p_0) \cup \operatorname{dom}(p_1) \setminus S^*$, and j is such that either $[j \notin \operatorname{ran}(f_\beta)] \in p_0(\beta) \cup p_1(\beta)$, or $[f_\beta(i) = j] \in p_0(\beta) \cup p_1(\beta)$ for some i, then $j < j_\varrho$,
- $(\bullet)_{2d}$ also observe that $[f_\beta(i) = j] \in p_0(\beta) \cup p_1(\beta)$ implies that $j = j_\varepsilon$ for some $\varepsilon < \varrho$.

$(\bullet)_3$ We construct the increasing sequence $\langle q_\varepsilon : \varepsilon < \varrho \rangle$ in D^*_α satisfying

$$q_\varepsilon^{[j_\varepsilon]} \geq p_0^{[j_\varepsilon]}, p_1^{[j_\varepsilon]},$$

$(\bullet)_4$ and also for each $\varepsilon < \varrho$ the strict inequality $q_\varepsilon(\beta) \gneq q_\varepsilon^{[j_\varepsilon]}(\beta)$ is only possible if $\beta \in \operatorname{dom}(p_0) \cup \operatorname{dom}(p_1) \setminus \{1\}$ and $(\delta_\varepsilon^\beta)^- = \max(N^*_\beta \cap (j_\varepsilon + 1)) < j_\varepsilon$ hold and for each such β the difference

$$q_\varepsilon(\beta) \setminus q_\varepsilon^{[j_\varepsilon]}(\beta) = \begin{cases} \{[f_\beta(i) = (\delta_\varepsilon^\beta)^-]\}, & \text{if} \quad [f_\beta(i) = (\delta_\varepsilon^\beta)^-] \in p_0(\beta) \cup p_1(\beta), \\ \{[(\delta_\varepsilon^\beta)^- \notin \operatorname{ran}(f_\beta)]\}, & \text{if} \quad [(\delta_\varepsilon^\beta)^- \notin \operatorname{ran}(f_\beta)] \in p_0(\beta) \cup p_1(\beta), \end{cases}$$

otherwise, if $[(\delta_\varepsilon^\beta)^- \notin \operatorname{ran}(f_\beta)] \notin p_0(\beta) \cup p_1(\beta)$ and for no i we have $[f_\beta(i) = (\delta_\varepsilon^\beta)^-] \in p_0(\beta) \cup p_1(\beta)$, then $q_\varepsilon(\beta) = q_\varepsilon^{[j_\varepsilon]}(\beta)$. (Since for the generic embedding f_β $\operatorname{ran}(f_\beta) \subseteq N^*_\beta$ must hold, roughly speaking q_ε contains all the information from p_0 and p_1 below j_ε.)

Now we claim that provided the sequence $\langle q_\varepsilon : \varepsilon < \varrho \rangle$ exists there is a common upper bound of p_0 and p_1.

Claim 3.38. *The least upper bound of $\langle q_\varepsilon : \varepsilon < \varrho \rangle$ (denoted by $q_\varrho \in D^*_\alpha$) can be extended to an upper bound of p_0 and p_1.*

Proof. As the sequence $\langle j_\varepsilon : \varepsilon < \varrho \rangle$ has no maximal element, and $q_\varrho \geq q_\varepsilon \geq q_\varepsilon^{[j_\varepsilon]} \geq p_0^{[j_\varepsilon]}, p_1^{[j_\varepsilon]}$ by $(\bullet)_3$, $(\bullet)_4$ clearly $q_\varrho(1) \geq p_0(1), p_1(1)$, and similarly $q_\varrho(\beta) \geq q_0(\beta) \geq p_0(\beta), p_1(\beta)$ for $\beta \in S^*$.

Now fix $1 < \beta \in \operatorname{dom}(p_0) \cup \operatorname{dom}(p_1) \setminus S^*$, let $\iota_\beta^- = \sup(N^*_\beta \cap j_\varrho)$. Note that if $\iota_\beta^- = j_\varrho$, then by Definition 3.30 $(\lim_{\varepsilon<\varrho} p_0^{[j_\varepsilon]})(\beta) = p_0^{[j_\varrho]}(\beta)$, which is equal to $p_0(\beta)$ by $(\bullet)_{2c}$, and similarly for p_1. This implies (recalling $q_\varepsilon \geq p_0^{[j_\varepsilon]}, p_1^{[j_\varepsilon]}$ by $(\bullet)_3$, $(\bullet)_4$) that $q_\varrho(\beta) \supseteq p_0(\beta) \cup p_1(\beta)$, as desired.

Now suppose that $\iota_\beta^- < j_\varrho$. Then clearly $p^{[\delta]}(\beta) = p^{[\iota_\beta^-]}(\beta)$ for every condition p and $\delta \in [\iota_\beta^-, j_\varrho)$. Observe that (by Definition 3.30 (c), and by the fact that $[f_\beta(i) = j] \in p(\beta)$ implies $j \in N^*_\beta$) the set

($\oint$) $p_0(\beta) \cup p_1(\beta) \setminus (p_0^{[\iota_\beta^-]}(\beta) \cup p_1^{[\iota_\beta^-]}(\beta))$ consists of only symbols of the form $[j \notin \mathrm{ran}(f_\beta)]$, except maybe $[f_\beta(i) = \iota_\beta^-]$ for a unique i (and then necessarily $\iota_\beta^- = j_\varepsilon$ for some $\varepsilon < \varrho$).

Then recalling $(\bullet)_4$ whenever $\varepsilon < \varrho$ is such that $j_\varepsilon > \iota_\beta^-$, then

$$q_\varepsilon(\beta) \supseteq \{[f_\beta(i) = j] \in p_0(\beta) \cup p_1(\beta) : \ j \leq \iota_\beta^-\},$$

similarly

$$q_\varepsilon(\beta) \supseteq \{[j \notin \mathrm{ran}(f_\beta)] \in p_0(\beta) \cup p_1(\beta) : \ j \leq \iota_\beta^-\}.$$

This together with ($\oint$) mean that we only have to add $\{[j \notin \mathrm{ran}(f_\beta)] \in p_0(\beta) \cup p_1(\beta) : \ j > \iota_\beta^-\}$ to q_ϱ, which is possible, since

$$q_\varrho(\beta) \subseteq q_\varrho^{[\iota_\beta^-]}(\beta) \cup \{[\iota_\beta^- \notin \mathrm{ran}(f_\beta)], [f_\beta(i) = \iota_\beta^-] : \ i \in \delta\}.$$

(In fact using $(\iota_\beta^-, j_\varrho) \cap N_\beta^* = \emptyset$ we could have argued that on coordinate β j's not belonging to N_β^* are irrelevant in terms of the generic embedding f_β and the generic filter.)

$\square_{\text{Claim3.38}}$

Claim 3.39. *There exists a sequence $\langle q_\varepsilon : \ \varepsilon < \varrho\rangle$ satisfying $(\bullet)_3$, $(\bullet)_4$.*

Proof. We define q_0 to be the upper bound of $p_0^{[j_0]}$ and $p_1^{[j_0]}$ to satisfy $(\bullet)_{2a}$, $(\bullet)_{2b}$: For $\beta \in S^*$ if $p_0(\beta) = (w_\beta, \underset{\sim}{A}_{0,\beta})$, $p_1(\beta) = (w_\beta, \underset{\sim}{A}_{1,\beta})$ then we let $s_0(\beta) = (w, \underset{\sim}{B}_\beta)$ (where $\underset{\sim}{B}_\beta$ is the $\mathbb{P}^1_\beta$-name satisfying $\Vdash_{\mathbb{P}^1_\beta} \underset{\sim}{B}_\beta = \underset{\sim}{A}_{0,\beta} \cap \underset{\sim}{A}_{1,\beta}$). Because of $q_0 = q_0^{[j_\varepsilon]}$ (by $(\bullet)_3$) and recalling $(\odot)_9/(iv)$ for $\gamma = 0, 1$ $q_0(1)$ can only be the empty condition. Furthermore, for $\beta \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1) \setminus S^*$, $\beta > 1$ we let

($\triangle$)$_1$ $q_0(\beta) = \{[j^* \notin \mathrm{ran}(f_\beta)] \in p_0(\beta) \cup p_1(\beta) : \ j < j_0 \ \wedge \ j \leq \sup(N_\beta^* \cap j_0)\}$.

So q_0, $q_0^+ \in D_\alpha^0$ in fact belong to D_α^*, and we obviously have $(\bullet)_3$, $(\bullet)_4$.

Now suppose that q_θ's are already defined for $\theta < \varepsilon$, and we shall construct q_ε, but we need to deal with limit and successor ε's differently.

Case A: ε is limit.

Let $s_\varepsilon = \lim_{\theta<\varepsilon} q_\theta \in D_\alpha^*$, we argue that we can choose a suitable extension of s_ε to be q_ε. For q_ε we only extend s_ε on coordinates $\beta \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1) \setminus (\{1\} \cup S^*)$. So fix such a β. First, if $j_\varepsilon \notin N_\beta^*$ (hence N_β^* is bounded in j_ε) then we let $q_\varepsilon(\beta) = s_\varepsilon(\beta)$. Second, if $j_\varepsilon \in N_\beta^*$, and it is an accumulation point of N_β^*, then again we do nothing, just let $q_\varepsilon(\beta) = s_\varepsilon(\beta)$. But if j_ε is a successor of $(j_\varepsilon^\beta)^- = \max(N_\beta^* \cap j_\varepsilon)$ in N_β^*, then first note that

($\triangle$)$_2$ $p_0^{[j_\varepsilon]}(\beta) \cup p_1^{[j_\varepsilon]}(\beta) \subseteq p_0^{[(j_\varepsilon^\beta)^-]}(\beta) \cup p_1^{[(j_\varepsilon^\beta)^-]}(\beta) \cup \{[j \notin \mathrm{ran}(f_\beta)] : \ j \geq (j_\varepsilon^\beta)^-\} \cup \{[f_\beta(i) = (j_\varepsilon^\beta)^-] : \ i < (j_\varepsilon^\beta)^-\}$

(in fact j's between two consecutive element of N_β^* are irrelevant in terms of the forcing and the embedding f_β). Moreover, as ε is limit (and $\langle j_\theta : \ \theta < \varrho\rangle$ is closed by $(\bullet)_1$) there is $\theta \in \varepsilon$ with $j_\theta \in ((j_\varepsilon^\beta)^-, j_\varepsilon)$, and by $(\bullet)_3$, $(\bullet)_4$ we have

($\triangle$)$_3$ $q_\theta(\beta) \subseteq r_\varepsilon(\beta) \subseteq r_\varepsilon^{[(j_\varepsilon^\beta)^-]}(\beta) \cup \{[(j_\varepsilon^\beta)^- \notin \mathrm{ran}(f_\beta)], [f_\beta(i) = (j_\varepsilon^\beta)^-] : \ i < (j_\varepsilon^\beta)^-\}$.

Again

($\triangle$)$_4$ $r_\varepsilon(\beta) \supseteq p_0^{[(j_\varepsilon^\beta)^-]}(\beta) \cup p_1^{[(j_\varepsilon^\beta)^-]}(\beta)$, and

($\triangle$)$_5$ $r_\varepsilon(\beta) \supseteq (p_0(\beta) \cup p_1(\beta)) \cap \{[(j_\varepsilon^\beta)^- \notin \mathrm{ran}(f_\beta)], [f_\beta(i) = (j_\varepsilon^\beta)^-] : \ i < (j_\varepsilon^\beta)^-\}$.

so there is no problem adding $\{[j \notin \mathrm{ran}(f_\beta)] \in p_0(\beta) \cup p_1(\beta) : \ (j^\beta_\varepsilon)^- < j < j_\varepsilon\}$ to $s_\varepsilon(\beta)$ obtaining $q_\varepsilon(\beta)$. In each of the cases it is also easy to check $(\bullet)_4$.

Case B: $\varepsilon = \theta + 1$.

We summarize first which symbols would the $q_\varepsilon(\beta)$'s ($\beta \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$) have to include in order for q_ε to satisfy $q_\varepsilon^{[j_\varepsilon]} \geq p_0^{[j_\varepsilon]}, p_1^{[j_\varepsilon]}$, and $(\bullet)_4$. Of course only the case $\beta \notin S^*$ is relevant.

$(\triangle)_6$ for $\beta = 1$ the set to cover is

$$p_0^{[j_\varepsilon]}(1) \cup p_1^{[j_\varepsilon]}(1) \setminus q_\theta(1) = \{[j_\theta \ R_\tau \ j] \in p_0(0) \cup p_1(0) : \ j < j_\theta, \ \tau < \kappa\}. \tag{3.18}$$

By $(\bullet)_{2d}$

$(\triangle)_7$ for $1 < \beta \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1) \setminus S^*$ the set $q_\varepsilon(\beta)$ has to include the set

$$\{[f_\beta(i) = j_\theta] \in p_0(\beta) \cup p_1(\beta) : \ i \in \lambda\} \tag{3.19}$$

(which is actually either a singleton, or the empty set) and

$$\{[j \notin \mathrm{ran}(f_\beta)] \in p_0(\beta) \cup p_1(\beta) : \ j \in \Big((\delta^\beta_\theta)^-, \delta^\beta_\varepsilon)^-\Big] \cup \{j_\theta\} \setminus \{j_\varepsilon\} \tag{3.20}$$

(where $(\delta^\beta_\theta)^- = \sup(N^*_\beta \cap (j_\theta + 1))$, $(\delta^\beta_\varepsilon)^- = \sup(N^*_\beta \cap (j_\varepsilon + 1))$, possibly $(\delta^\beta_\theta)^- = (\delta^\beta_\varepsilon)^-$). Recall that if $[f_\beta(i) = j_\theta] \in p_0(\beta) \cup p_1(\beta)$ for some i, then necessarily $j_\theta \in N^*_\beta$, hence $(\delta^\beta_\theta)^- = j_\theta$.

First we extend q_θ to a condition q^+_θ with $q^+_\theta(1)$ including the set in (3.18), and for $\beta \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1) \setminus S^*$ $q^+_\theta(\beta)$ including the set in (3.19).

Subclaim 3.40. *There exists $q^+_\theta \geq q_\theta$ in D^*_α with*

$(*)_a$ $q^+_\theta(1) \supseteq \{[j_\theta \ R_\tau \ j] \in p_0(0) \cup p_1(0) : \ j < j_\theta, \ \tau < \kappa\}$,

$(*)_b$ *for each* $0 < \beta \notin S^*$

$$q^+_\varepsilon(\beta) \supseteq \{[j \notin \mathrm{ran}(f_\beta)] \in p_0(\beta) \cup p_1(\beta) : \ j = (j^\beta_\theta)^-\},$$
$$q^+_\varepsilon(\beta) \supseteq \{[f_\beta(i) = (j^\beta_\theta)^-] \in p_0(\beta) \cup p_1(\beta) : \ i < (j^\beta_\varepsilon)^-\},$$

while

$(*)_c$ $q^+_\varepsilon(1) \subseteq q_\varepsilon^{+[j_\theta]}(1) \cup \{[j \ R_\nu \ j_\theta] : \ j < j_\theta, \ \nu < \kappa\}$,

$(*)_d$ *and for each* $1 < \beta \notin S^*$

$$q^+_\varepsilon(\beta) \subseteq q_\varepsilon^{+[(j^\beta_\theta)^-]}(\beta) \cup \{[f_\beta(i) = j_\theta] : \ i < j_\theta\} \cup \{[j_\theta \notin \mathrm{ran}(f_\beta)]\}.$$

Assuming the subclaim (which guarantees that q^+_θ satisfies $(\bullet)_4$) we only have to add symbols of the form $[j \notin \mathrm{ran}(f_\beta)]$ (sets in (3.20)) to the $q^+_\theta(\beta)$'s to obtain the condition $q_{\theta+1} = q_\varepsilon$ satisfying $(\bullet)_3$ and $(\bullet)_4$, therefore Subclaim 3.40 will finish the proof of Claim 3.39

Proof. (Subclaim 3.40)

$(\blacktriangle)_1$ For each fixed β where $\beta \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$ with $[f_\beta(i) = (\delta^\beta_\theta)^-] \in p_0(\beta) \cup p_1(\beta)$ for some i let i^β_θ denote this unique i.

Now observe that

$(\blacktriangle)_2$ for each β with i^β_θ defined, for each $j' < j_\varepsilon$ with $[f_\beta(i') = j'] \in q_\varepsilon(\beta)$ for some i' note that $i' < j' \leq (\delta^\beta_\theta)^- \leq j_\varepsilon$ and $i^\beta_\varepsilon < (\delta^\beta_\theta)^- \leq j_\varepsilon \in N^*_\beta$, so we can apply Lemma 3.36, and thus each condition q in D^*_α can be extended to $q' \in D^*_\alpha$ with $q'^{[j_\varepsilon]}$ deciding the color $c_{\underset{\sim}{M}_\beta}(i', i^\beta_\varepsilon)$.

So enumerating all possible pairs (β, i') (that are as in $(\blacktriangle)_2$) and recalling $(*)_9$ we infer that

$(\blacktriangle)_3$ for some $q^* \geq q_\theta$ the condition $q^{*[j_\varepsilon]} \restriction \beta \in D^*_\alpha$ decides the color $c_{\underset{\sim}{M}_\beta}(i', i^\beta_\varepsilon)$ for all such pairs from $\{(\beta, i') : \beta \in \operatorname{dom}(p_0) \cup \operatorname{dom}(p_1),\ \exists j\ [f_\beta(i') = j] \in q_\theta\}$,

$(\blacktriangle)_4$ repeat this for pairs in $\{(\beta, i') : \exists j\ [f_\beta(i') = j] \in q^{*[j_\varepsilon]}\}$, and let $q^{**} \in D^*$ be the condition obtained after countable many such steps,

so

$(\blacktriangle)_5$ the condition $q^{**} \in D^*_\alpha$, $q^{**} \geq q_\theta$ with $q^{**[j_\theta]} \restriction \beta$ deciding the color $c_{\underset{\sim}{M}_\beta}(i', i^\beta_\varepsilon)$ for all $(\beta, i') \in \{(\beta, i') : \beta \in \operatorname{dom}(p_0) \cup \operatorname{dom}(p_1),\ \exists j\ [f_\beta(i') = j] \in q^{**[j_\varepsilon]}(\beta)$,

Finally recall that by $(\bullet)_4$ $q_\theta(1) = q_\theta^{[j_\theta]}(1)$, and for each $\beta \in \operatorname{dom}(q_\theta) \setminus S^*$ if $q_\theta(\beta) = q_\theta^{[j_\theta]}(\beta)$ can only be non-empty if $\beta \in \operatorname{dom}(p_0) \cup \operatorname{dom}(p_1)$ (and if it is indeed non-empty then it is a singleton $[(j^\beta_\theta)^- \notin \operatorname{ran}(f_\beta)]$ or $[f_\beta(i) = (j^\beta_\theta)^-]$).

$(\blacktriangle)_6$ This means that after possibly replacing $q^{**}_\theta(\beta)$ by $q^{**[j_\theta]}(\beta) \cup q_\theta(\beta)$ using $(\blacktriangle)_5$ it is easy to see that we get a condition $q^{**} \in D^*_\alpha$ (which still satisfies both $(\bullet)_4$ and $(\blacktriangle)_5$).

Now we are at the position to construct the desired q^+_θ as an extension of q^{**}. (In order to include the symbols listed in $(*)_a$, and $(*)_b$ for β's with $(j^\beta_\theta)^- = j_\theta$, but constructing a proper condition in D^*_α), our task is to determine the color $\nu(j^*, j_\theta) = c_{M_*}(j^*, j_\theta)$ (i.e. add $[j^*\ R_{\nu(j^*, j_\theta)}\ j_\theta]$ to $q^{**}(1)$) for each j^* and β such that

- $[f_\beta(i^\beta_\theta) = j_\theta] \in p_0(\beta) \cup p_1(\beta)$,
- and for some i^* $[f_\beta(i^*) = j^*] \in q^{**[j_\theta]}(\beta)$,

so that $\nu(j^*, j_\theta) = c_{\underset{\sim}{M}_\beta}(i^*, i^\beta_\theta)$ (this latter value is the color forced by $q^{**[j_\theta]} \restriction \beta$ by $(\blacktriangle)_5$). Then adding also the symbols $[f_\beta(i^\beta_\theta) = j_\theta] \in p_0(\beta) \cup p_1(\beta)$ will work.

So fix a pair j^*, j_θ as above. Now we will make use of the preparations $(\odot)_9$ and $(\bullet)_2$ and show that there are no contradicting demands concerning the value of $\nu(j^*, j_\theta)$. We distinguish the following cases.

Case (1): for some $\nu^* < \kappa$ we have $[j^*\ R_{\nu^*}\ j_\theta] \in p_0(1) \cup p_1(1)$.

Then necessarily $j^* = j_\eta$ for some $\eta < \theta$ (and $j_\eta, j_\theta \in L_0$), and the only option is to

$$\text{put } [j_\eta\ R_{\nu^*}\ j_\theta] \in q^+_\varepsilon(1), \tag{3.21}$$

i.e. define $\nu(j_\eta, j_\theta) = \nu^*$. W.l.o.g. we can assume that $[j_\eta\ R_{\nu^*}\ j_\theta] \in p_0(1)$. Pick an arbitrary $\beta \in \operatorname{dom}(p_0) \cup \operatorname{dom}(p_1)$ satisfying $[f_\beta(i^\beta_\theta) = j_\theta] \in p_0(\beta) \cup p_1(\beta)$ and for some i^* $[f_\beta(i^*) = j_\eta] \in q^{**}(\beta)$.

If $\beta \in \operatorname{dom}(p_0)$, then by $(\odot)_9/(v)$ we have fact $j_\eta, j_\theta \in L_0$, which implies that both $[f_\beta(i^\beta_\theta) = j_\theta], [f_\beta(i^*) = j_\eta] \in p_0(\beta)$, so by $(\odot)_9/(i)$ $p_0^{[j_\theta]} \restriction \beta$ forces a value for $c_{\underset{\sim}{M}_\beta}(i^*, i^\beta_\theta)$. Hence, $q^{**[j_\theta]} \restriction \beta \geq q_\theta^{[j_\theta]} \restriction \beta \geq p_0^{[j_\theta]} \restriction \beta$ forces the same value for $c_{\underset{\sim}{M}_\beta}(i^*, i^\beta_\theta)$ (by our hypothesis on q_θ $(\bullet)_3$), which is ν^*.

Otherwise, assume that $\beta \notin \operatorname{dom}(p_0)$ (so necessarily $\beta \in \operatorname{dom}(p_1)$ and $[f_\beta(i^\beta_\theta) = j_\theta] \in p_1(\beta)$, and $j_\theta \in L_1$). Then again by $(\bullet)_2/(\bullet)_{2a}$ the only way that $[f_\beta(i^*) = j_\eta] \in q_\theta$ can happen for some i^* is when $[f_\beta(i^*) = j_\eta] \in p_1(\beta)$, but then $(\odot)_9/\ (iv)$

implies that $j_\eta \in L_1$, so $[j_\eta\ R_{\nu^*}\ j_\theta] \in p_1(\beta)$ is a member of $p_1(\beta)$, too, and then we can proceed as in the case above (i.e. arguing that $p_1^{[j_\theta]} \restriction \beta \Vdash c_{\underset{\sim}{M}_\beta}(i^*, i^\beta_\theta) = \nu^*$).

Case (2): for no $\nu^* < \kappa$ have we $[j^*\ R_{\nu^*}\ j_\varepsilon] \in p_0(1) \cup p_1(1)$.

Case (2A): $j^* = j_\eta$ for some $\eta < \theta$ (so by (ii) necessarily $|\{j_\eta, j_\theta\} \cap (L_0 \setminus L_1)| = |\{j_\eta, j_\theta\} \cap (L_1 \setminus L_0)| = 1$).

We can assume, that $j_\eta \in L_0 \setminus L_1$, $j_\theta \in L_1 \setminus L_0$. This means that

$(\blacktriangle)_7$ for no β there exists i such that $[f_\beta(i) = j_\eta] \in p_1(\beta)$, and similarly, $[f_\beta(i) = j_\theta] \in p_0(\beta)$ is impossible

by our assumption $(\odot)_9/(iv)$ on p_0 and p_1. So by $(\bullet)_2/(\bullet)_{2a}$ $[f_\beta(i) = j_\eta] \in q_\theta(\beta)$ is only possible for any $\beta \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$ if $[f_\beta(i) = j_\eta] \in p_0(\beta) \cup p_1(\beta)$, so this case necessarily $[f_\beta(i) = j_\eta] \in p_0(\beta)$. Summing up, for each β with the prospective q^+_θ forcing $j_\eta \in L_0 \setminus L_1$, $j_\theta \in L_1 \setminus L_0$ to be in the range of f_β the only possibility is that

$$[f_\beta(i^\beta_\theta) = j_\theta] \in p_1(\beta), \text{ and} \tag{3.22}$$

$$\text{for some } i^*\ [f_\beta(i^*) = j_\eta] \in p_0(\beta). \tag{3.23}$$

Now we argue that at most one such $\beta \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$ may exist (then by $(\blacktriangle)_5$ we can put $[j^*\ R_{\nu^*}\ j_\varepsilon] \in q^+_\theta(\beta)$ with $\nu^* < \kappa$ defined by $q^{**[j_\theta]} \restriction \beta \Vdash c_{\underset{\sim}{M}_\beta}(i^*, i^\beta_\theta) = \nu^*$, and we are done).

So assume on the contrary, let $\beta' \neq \beta''$ be such that (3.22) (3.23) hold. Then clearly $\beta', \beta'' \in \mathrm{dom}(p_0) \cap \mathrm{dom}(p_1)$, and $j_\theta, j_\eta \in \{\xi_{\beta'}(\rho+1) : \rho < \lambda\} \cap \{\xi_{\beta''}(\rho+1) : \rho < \lambda\}$, then by our assumption (on all the p_γ's) $(\odot)_9/(iii)$ contradicts $(\blacktriangle)_7$.

Case (2B): j^* is not of the form j_θ for any $\theta < \varepsilon$.

This case we argue that at most one $\beta \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$ could exist with $[f_\beta(i^\beta_\theta) = j_\theta] \in p_0(\beta) \cup p_1(\beta)$ satisfying that for some i^* $[f_\beta(i^*) = j^*] \in q^{**}(\beta)$. (Then again by $(\blacktriangle)_5$ we can put $[j^*\ R_{\nu^*}\ j_\theta] \in q^+_\theta(\beta)$ with $\nu^* < \kappa$, $q^{**[j_\theta]} \restriction \beta \Vdash c_{\underset{\sim}{M}_\beta}(i^*, i^\beta_\theta) = \nu^*$.)

So if there are $\beta' \neq \beta'' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$ with

- $[f_{\beta'}(i^*) = j^*] \in q_\varepsilon(\beta')$ for some i^*,
- $[f_{\beta''}(i^{**}) = j^*] \in q_\varepsilon(\beta'')$ for some i^{**},
- $[f_{\beta'}(i^{\beta'}_\varepsilon) = j_\varepsilon] \in p_0(\beta') \cup p_1(\beta')$,
- $[f_{\beta''}(i^{\beta''}_\varepsilon) = j_\varepsilon] \in p_0(\beta'') \cup p_1(\beta'')$,

then again as in Case (2A) we can get to an easy contradiction (i.e. $\beta', \beta'' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$, and $j^* \in \{\xi_{\beta'}(\rho+1) : \rho < \lambda\} \cap \{\xi_{\beta''}(\rho+1) : \rho < \lambda\}$, hence $(\odot)_9/(iv)$ and $(\bullet)_2/(\bullet)_{2a}$ imply $[j^* \notin \mathrm{ran}(f_\beta)] \in p_0(\beta') \cap p_1(\beta')$, similarly for β''. Now recall $q^{**} \geq q_\theta$ and $(\bullet)_4$).

$\square_{\text{Subclaim3.40}}$

$\square_{\text{Claim3.39}}$

$\square_{\text{Claim3.37}}$

$\square_{\text{Lemmas3.34and3.35}}$

Having proven that $\mathbb{P}^1_\chi$ (and each $\mathbb{P}^1_\alpha$, $\alpha \leq \chi$) is the composition of a λ^+-cc and a κ^+-cc forcing, so itself λ^+-cc, we have $(\top)_3$. Moreover, recall Claim 3.25 and that

$\mathbb{Q}^1_0 = Q(\lambda, \chi, \kappa)$, so $\mathbb{Q}^1_0$ does not collapse any cardinal, while $\mathbb{P}^1_\chi / \mathbf{G}^1_1$ is κ^+-cc, $< \kappa$-closed, so $\mathbb{P}^1_\chi$ being the composition of the forcings preserving cardinals itself does not collapse any cardinal, we get $(\top)_4$. An easy calculation yields the following.

Claim 3.41. *For each $\alpha < \chi$ we have $\mathbf{V}_1^{\mathbb{P}^1_\alpha} \models |\mathbb{Q}^1_\alpha| \leq \chi$. Therefore, up to equivalence $\mathbb{P}^1_\chi$ is of power χ.*

Proof. For $\mathbb{P}^1_1 = \mathbb{Q}^1_0$ we already know $|\mathbb{Q}^1_1|$ by Observation (3.24). We have to prove the two statements simultaneously by induction on α. As $\mathbb{P}^1_\chi$ is a $< \kappa$-support iteration, and $\chi^{<\kappa} \leq \chi^\lambda = \chi$, by our premises it is enough to prove for the successor case. So for each $\alpha < \chi$ it is enough to show that $\mathbf{V}_1^{\mathbb{P}^1_\alpha} \models |\mathbb{Q}^1_\alpha| \leq \chi$. For $\alpha = 1$ as $\mathbb{Q}^1_1$ is a forcing of a κ-colored random graph on λ with conditions of size $< \kappa$ we get that $|\mathbb{Q}^1_1| = \lambda^{<\kappa} \leq \chi$ (in fact $|\mathbb{Q}^1_1| = \lambda$).

For α with $1 < \alpha \notin S^*$ (so Definition 3.28 Case (2)). Again, each condition in $\mathbb{Q}^1_\alpha$ can be coded by a partial function of size $< \kappa$ on λ to $\lambda+1$, so $|\mathbb{Q}^1_\alpha| = \lambda^{<\kappa} \leq \chi$.

Finally, for $\alpha \in S^*$ (Definition 3.28 Case (3)), $\mathbb{Q}^1_\alpha = \mathbb{Q}_{D_\alpha}$ is the Mathias type forcing from Definition 3.1, where D_α is a system of subsets of $V_\kappa^{\mathbf{V}_1}$ generating a κ-complete filter, so $|\mathbb{Q}^1_\alpha| \leq (2^{|V_\kappa|})^{\mathbf{V}_1^{\mathbb{P}^1_\alpha}} = (2^\kappa)^{\mathbf{V}_1^{\mathbb{P}^1_\alpha}} \leq \chi$ (because $|\mathbb{P}^1_\alpha| = \chi$, $\mathbb{P}^1_\alpha$ is λ^+-cc, and we assumed $(\chi^\lambda)^{\mathbf{V}_1} = \chi$).

$\square_{\text{Lemma3.41}}$

So now we are ready to complete the definition of $\mathbb{P}^1_\chi$ by prescribing the names $\underset{\sim}{D}_\delta$ ($\delta \in S^*$) and $\underset{\sim}{M}_\delta$ ($1 < \delta \notin S^*$), which are standard easy bookkeeping arguments (using $|\mathbb{P}^1_\chi| = \chi$ and the λ^+-cc), but for the sake of completeness we elaborate. This will prove $(\top)_5$ and $(\top)_6$, so complete the proof of Conclusion 3.20.

Claim 3.42. *The system of $\underset{\sim}{D}_\delta$'s can be chosen so that for every $\mathbb{P}^1_\chi$-name $\underset{\sim}{D}$ with $\mathbf{V}_1 \Vdash_{\mathbb{P}^1_\chi} \underset{\sim}{D} \in [\mathscr{P}(V_\kappa)]^{\leq\lambda}$ there exists a $\delta \in S^*$, such that for the $\mathbb{P}^1_\delta$-name $\underset{\sim}{D}_\delta$ we have $\Vdash_{\mathbb{P}^1_\chi} \underset{\sim}{D} = \underset{\sim}{D}_\delta$*

Proof. It is obvious that by $\chi^\lambda = \chi$ (so $\mathrm{cf}(\chi) > \lambda$) and the λ^+-cc for every such $\underset{\sim}{D}$ there is a nice $\mathbb{P}^1_\delta$-name for some $\delta < \chi$. As forcing with the $< \kappa$-closed $\mathbb{P}^1_\chi$ does not add new elements to V_κ we get that for each δ there are $\chi^{\kappa\cdot\lambda} = \chi$-many such nice names. Also, as $|S^*| = \chi$ we can partition $S^* = \bigcup_{\alpha<\chi} S^*_\alpha$ with $S^*_\alpha \cap \alpha = \emptyset$, $|S^*_\alpha| = \chi$, we can let $\langle \underset{\sim}{D}_\delta : \delta \in S^*_\alpha \rangle$ list the nice names for subsets of $\mathscr{P}(V_\kappa)$. $\square_{\text{Claim3.42}}$

A similar calculation yields the following.

Claim 3.43. *The system of $\underset{\sim}{M}_\delta$'s can be chosen so that for every $\mathbb{P}^1_\chi$-name for a κ-colored graph $\underset{\sim}{M}$ on λ there exists a $1 < \delta \notin S^*$, such that for the $\mathbb{P}^1_\delta$-name $\underset{\sim}{M}_\delta$ we have $\Vdash_{\mathbb{P}^1_\chi} \underset{\sim}{M} = \underset{\sim}{M}_\delta$.*

Proof. Easy. $\square_{\text{Claim3.43}}$

$\square_{3.20}$

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