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On the definability of mad families of vector spaces ☆

Haim Horowitz [a,*], Saharon Shelah [b,c,1]

[a] *Department of Mathematics, University of Toronto, Bahen Centre, 40 St. George St., Room 6290, Toronto, Ontario, M5S 2E4 Canada*
[b] *Einstein Institute of Mathematics, Edmond J. Safra Campus, The Hebrew University of Jerusalem, Givat Ram, Jerusalem 91904, Israel*
[c] *Department of Mathematics, Hill Center - Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA*

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We consider the definability of mad families in vector spaces of the form $\bigoplus\limits_{n<\omega} F$ where F is a field of cardinality $\leq \aleph_0$. We show that there is no analytic mad family of subspaces when $F = \mathbb{F}_2$, partially answering a question of Smythe. Our proof relies on a variant of Mathias forcing restricted to a certain idempotent ultrafilter whose existence follows from Glazer's proof of Hindman's theorem.

1. Introduction

Assuming the axiom of choice, one can easily construct sets of reals exhibiting certain maximality properties. One classical example of such sets is provided by maximal almost disjoint (mad) families. Recall that $\mathcal{A} \subseteq [\omega]^\omega$ is mad if, for all $x, y \in \mathcal{A}$, $x \neq y \rightarrow |x \cap y| < \aleph_0$ and $\mathcal{A}$ is maximal with respect to this property. The study of the definability of mad families goes back to Mathias in the 1970s. As with other regularity

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* Corresponding author.
E-mail addresses: haim@math.toronto.edu (H. Horowitz), shelah@math.huji.ac.il (S. Shelah).
[1] Publication 1152 of the second author.

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properties, such as Lebesgue measurability and the Baire property, it turned out that mad families can't be too nicely definable:

Theorem *([5]). There are no analytic mad families.*

The study of the definability of relatives of mad families has attracted significant attention in recent years. In a surprising development, contrary to the pattern described above, it was shown in [2] that there exists a Borel maximal eventually different family (where $\mathcal{F} \subseteq \omega^\omega$ is maximal eventually different if $f \neq g \in \mathcal{F} \to f(n) \neq g(n)$ for large enough n, and $\mathcal{F}$ is maximal with respect to this property). In a subsequent work ([4]), the definability of another type of relatives of mad families - known as maximal cofinitary groups - was studied, and it was shown that there exists a Borel maximal cofinitary group.

The current paper studies the definability of a new variant of mad families recently introduced by Iian Smythe in [7]. Given an $\aleph_0$-dimensional vector space $V = \bigoplus_{n<\omega} F$ over a field F of cardinality $\leq \aleph_0$, we can regard 2^V as a Polish space and consider the definability of families of subsets of V. Mad families of subspaces of V will be defined in the natural way, see Definition 1 below. Our main goal is to provide a partial answer to the following question:

Question *([7]).* For V as above, is there an analytic mad family of subspaces of V?

We shall prove that for $F = \mathbb{F}_2$, the answer is negative, i.e. we have an analog of Mathias' theorem. We shall assume towards contradiction that $\mathcal{A}$ is an analytic mad family of subspaces of V. A main ingredient in our proof will be the existence of a nonprincipal ultrafilter D on V that is disjoint to $\mathcal{A}$, contains all subspaces of finite codimension and has the property that if $A \in D$, then $v + A \in D$ for D-almost all v. Such an ultrafilter will be provided using Glazer's argument for the existence of idempotent ultrafilters in $\beta(V)$. We shall then consider the forcing $\mathbb{Q}_D$, a variant of Mathias forcing restricted to the ultrafilter D. $\mathbb{Q}_D$ will introduce a generic subset $\{\underset{\sim}{y}_k : k < \omega\}$ of V whose span is almost contained in every element of D. The above invariance property of $\underset{\sim}{D}$ will be used to show that $\{\underset{\sim}{y}_k : k < \omega\}$ is infinite using a standard density argument. As $\mathcal{A}$ is analytic, it remains mad in $\mathbf{V}^{\mathbb{Q}_D}$, and we can find a name $\underset{\sim}{B}$ of a new element of $\mathcal{A}$ that has an infinite intersection with $span\{\underset{\sim}{y}_k : k < \omega\}$. We shall then work over a countable elementary submodel N of $H((2^{\aleph_0})^+)$ and construct two generic sets G_1, G_2 over N that decide $\underset{\sim}{B}$ in two different ways but still give an infinite intersection of the two versions of $\underset{\sim}{B}$. By the absoluteness of $\mathcal{A}$, this will contradict its almost disjointness.

2. The main result

Definition 1. a. Let V be an $\aleph_0$-dimensional vector space over a field F of cardinality $\leq \aleph_0$. We say that the subspaces $S_1, S_2 \subseteq V$ are almost disjoint if $dim(S_1 \cap S_2) < \aleph_0$.

b. We say that $\mathcal{A}$ is a mad family of subspaces of V (or a V-mad family) if $\mathcal{A}$ is infinite, the members of $\mathcal{A}$ are pairwise almost disjoint and $\mathcal{A}$ is not a proper subset of a family $\mathcal{A}'$ with these properties.

Our main result is the following:

Theorem 2. *Let $V = \bigoplus_{n<\omega} \mathbb{F}_2$ be a vector space of $\mathbb{F}_2$, then V has no analytic mad family of subspaces.*

The rest of the paper will be devoted to the proof of Theorem 2.

Notation 2A. a. $(x_n : n < \omega)$ will denote the basis elements of V.
b. Given $u \subseteq \omega$, $\bigoplus_{n \in u} \mathbb{F}_2 x_n$ will denote the subspace generated by $\{x_n : n \in u\}$.
c. For $v \in V$, the minimal $u \subseteq \omega$ such that $v \in \bigoplus_{n \in u} \mathbb{F}_2 x_n$ will be denoted $supp(v)$.
d. For $u \subseteq \omega$, the subspace of V generated by $\{x_n : n \in u\}$ will be denoted $span(u)$.

Definition 3. Given an ultrafilter D on V, we define the forcing $\mathbb{Q} = \mathbb{Q}_D$ as follows:
A. $p \in \mathbb{Q}$ iff $p = (u_p, \mathcal{A}_p) = (u, \mathcal{A})$ where:
a. $u \subseteq V$ is finite and $0 \in u$.
b. $\mathcal{A} \subseteq D$ is finite.
c. If $x \neq y \in u$ then the convex hulls of $supp(x)$ and $supp(y)$ are disjoint.
B. $(u_1, \mathcal{A}_1) \leq (u_2, \mathcal{A}_2)$ iff
a. $u_1 \subseteq u_2$ and for every $x \in u_1$ and $y \in u_2 \setminus u_1$, $max(supp(x)) < min(supp(y))$.
b. $\mathcal{A}_1 \subseteq \mathcal{A}_2$.
c. If $A \in \mathcal{A}_2 \setminus \mathcal{A}_1$ then
α. If $B \in \mathcal{A}_1$ then $A \subseteq B$.
β. If $B \in \mathcal{A}_1$, $x \in span(u_1)$ and $x + B \in D$ then $A \subseteq x + B$.
d. If $x \in span(u_1)$, $y \in span(u_2 \setminus u_1)$, $A \in \mathcal{A}_1$ and $x + A \in D$ then
α. $y \in (x + A) \cup \{0\}$.
β. $x + y + A \in D$.

Definition 4. For D and $\mathbb{Q} = \mathbb{Q}_D$ as in Definition 3, let $\underset{\sim}{A} = \cup\{u_p : p \in \underset{\sim}{G}_{\mathbb{Q}}\}$. If $|\underset{\sim}{A}| = \aleph_0$, we let $(\underset{\sim}{y}_n : n < \omega)$ be an enumeration of $\underset{\sim}{A}$ such that $max(supp(\underset{\sim}{y}_n)) < min(supp(\underset{\sim}{y}_{n+1}))$.

Observation 5. $\mathbb{Q}$ is a partial order.

Proof. Suppose that $p_1 \leq p_2$ and $p_2 \leq p_3$, we shall prove that $p_1 \leq p_3$. Denote p_l by $(u_l, \mathcal{A}_l)$ for $l = 1, 2, 3$. Clauses (a) and (b) of Definition 3(B) are immediate. Clause (c)(α) follows from (c)(β), as $0 \in span(u_1)$. For (c)(β), suppose that $A \in \mathcal{A}_3 \setminus \mathcal{A}_1$, $B \in \mathcal{A}_1$, $x \in span(u_1)$ and $x + B \in D$. If $A \in \mathcal{A}_2 \setminus \mathcal{A}_1$, then the desired conclusion follows from the fact that $p_1 \leq p_2$. If $A \in \mathcal{A}_3 \setminus \mathcal{A}_2$, the result follows similarly from the fact that $p_2 \leq p_3$. For clause (d), suppose that $x \in span(u_1)$, $y \in span(u_3 \setminus u_1)$, $A \in \mathcal{A}_1$ and $x + A \in D$. As $y \in span((u_3 \setminus u_2) \cup (u_2 \setminus u_1))$, there are $y_2 \in span(u_2 \setminus u_1)$ and $y_3 \in span(u_3 \setminus u_2)$ such that $y = y_2 + y_3$. WLOG $y_3 \neq 0$, otherwise clause (d) follows from the fact that $p_1 \leq p_2$. As $p_1 \leq p_2$, it follows that $y_2 \in (x+A) \cup \{0\}$ and $x + y_2 + A \in D$. As $p_2 \leq p_3$, replacing (x, y, A) in clause (d) by $(x+y_2, y_3, A)$ here, we get that $y_3 \in x + y_2 + A$ and $x + y_2 + y_3 + A \in D$. It follows that $y = y_2 + y_3 \in y_2 + (x + y_2 + A) = x + A$ (recall that $v + v = 0$ for every $v \in V$) and that $x + y + A \in D$. Therefore, $p_1 \leq p_3$. $\square$

Observation 6. If $p \in \mathbb{Q}$ and $B_1 \in D$, there is $q \in \mathbb{Q}$ and $B_2 \in \mathcal{A}_q$ such that $p \leq q$ and $B_2 \subseteq B_1$.

Proof. Let $p \in \mathbb{Q}$ and $B_1 \in D$. Let $B_2 := \cap\{x + A : x \in span(u_p), A \in \mathcal{A}_p, x + A \in D\} \cap B_1$, then $B_2 \in D$ and $B_2 \subseteq B_1$. Let $q = (u_p, \mathcal{A}_p \cup \{B_2\})$, then $q \in \mathbb{Q}$ and it's easy to verify that $p \leq q$. $\square$

Observation 7. If $B \in D$ and every $p \in \mathbb{Q}$ can be extended to $q \in \mathbb{Q}$ such that $|u_p| < |u_q|$ (so $\Vdash_{\mathbb{Q}}$ "$|\underset{\sim}{A}| = \aleph_0$") then $\Vdash_{\mathbb{Q}}$ "for some k we have $span\{\underset{\sim}{y}_n : k \leq n\} \subseteq B \cup \{0\}$".

Proof. By the previous observation, there is a dense set I of conditions $p \in \mathbb{Q}$ with some $A \in \mathcal{A}_p$ such that $A \subseteq B$. Let $p \in I$, fix $A \in \mathcal{A}_p$ such that $A \subseteq B$ and let $k = |u_p|$; then p forces values $y_0, ..., y_{k-1}$ to $\underset{\sim}{y}_0, ..., \underset{\sim}{y}_{k-1}$. Suppose that q is a condition above p and let $y \in span(u_q \setminus u_p)$, then $y \in A \cup \{0\} \subseteq B \cup \{0\}$ by

Definition 3(B)(d)(α). It follows that $p \Vdash_{\mathbb{Q}}$ "$span\{\underset{\sim}{y}_n : k \leq n\} \subseteq B \cup \{0\}$". As the last claim is true for any p in the dense set I, this completes the proof. $\square$

Towards the proof of Theorem 2, suppose that the theorem fails and fix an analytic $V-$mad family $\mathcal{A}$ (i.e. $\mathcal{A}$ has a Σ^1_1 definition). We shall now derive a contradiction.

Observation 8. For $\mathbb{Q}$ as before, $\Vdash_{\mathbb{Q}}$ "$\mathcal{A}$ is $V-$mad family".

Proof. Observe that as $V = \bigoplus_{n<\omega} \mathbb{F}_2$, given two subspaces $S_1, S_2 \subseteq V$, $dim(S_1 \cap S_2) < \aleph_0$ iff $|S_1 \cap S_2| < \aleph_0$. As $\mathcal{A}$ is Σ^1_1, the statement that $\mathcal{A}$ is maximal is Π^1_2 hence absolute. Similarly, the almost disjointness of $\mathcal{A}$ is Π^1_1 and hence absolute. It follows that $\mathcal{A}$ is mad in $\mathbf{V}^{\mathbb{Q}}$. $\square$

We shall now work with a forcing $\mathbb{Q}_D$ where D is a certain idempotent ultrafilter whose existence will be proved later.

Fact 9. There exists an ultrafilter D on V such that:
a. $D \cap \mathcal{A} = \emptyset$.
b. For every $A \in D$, for $D-$almost all v, $v + A \in D$.
c. $S \in D$ for every subspace S of finite codimension.

Throughout the rest of the paper, $\mathbb{Q} = \mathbb{Q}_D$ where D is a fixed ultrafilter as in Fact 9 (which will be proved in the end of the paper).

Definition/Observation 10. a. Let $\phi(A) = (\exists x)\psi(x, A)$ be the Σ^1_1 formula that defines $\mathcal{A}$. By the maximality of $\mathcal{A}$ in $\mathbf{V}^{\mathbb{Q}}$, there are $\mathbb{Q}$-names $\underset{\sim}{r}$ and $\underset{\sim}{B}$ such that $\Vdash_{\mathbb{Q}}$ "$\psi(\underset{\sim}{r}, \underset{\sim}{B})$ and $|\underset{\sim}{B} \cap span\{\underset{\sim}{y}_n : n < \omega\}| = \aleph_0$" (by Observation 13 below, $\underset{\sim}{A}$ is infinite, hence $\{\underset{\sim}{y}_n : n < \omega\}$ is an infinite well-defined set).

b. $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{B} \notin \mathbf{V}$".

Proof. Let $A \in \mathcal{A}^{\mathbf{V}}$, then $A \notin D$ hence $V \setminus A \in D$. $\mathbb{Q}$ forces that $span\{\underset{\sim}{y}_n : n < \omega\} \subseteq^* V \setminus A$. As $\mathbb{Q}$ forces that $\underset{\sim}{B}$ contains infinitely many elements from $span\{\underset{\sim}{y}_n : n < \omega\}$, each such element (modulo a finite number) is in $V \setminus A$ and it follows that $\underset{\sim}{B} \neq A$. As $\underset{\sim}{B} \in \mathcal{A}^{\mathbf{V}^{\mathbb{Q}}}$, it follows that $\underset{\sim}{B} \notin V$. $\square$

Let $\kappa = (2^{\aleph_0})^+$ and let N be a countable elementary submodel of $(H(\kappa), \in)$ such that $V, \phi, \underset{\sim}{r}, \underset{\sim}{B} \in N$. Let $(I_n : n < \omega)$ be an enumeration of the dense subsets of $\mathbb{Q}$ that belong to N.

Observation 11. If $p \in \mathbb{Q}$ then $Z^+_p \in D$ where $Z^+_p = \{v \in V :$ some $q \in \mathbb{Q}$ above p forces that $v \in \underset{\sim}{B} \cup \{0\}$, and moreover, $v \in span(u_q \setminus u_p)\}$.

Proof. Suppose towards contradiction that $Z^+_p \notin D$, then $V \setminus Z^+_p \in D$. By the proof of Observation 6, there is a condition p_1 above p of the form $p_1 = (u_p, \mathcal{A}_p \cup \{Z\})$ where $Z \subseteq V \setminus Z^+_p$. If p_2 is a condition above p_1 such that $u_{p_2} \neq u_{p_1}$ (such a condition exists by Observation 13), then $span(u_{p_2} \setminus u_p) \setminus \{0\} \subseteq Z \subseteq V \setminus Z^+_p$: Let $y \in span(u_{p_2} \setminus u_p) \setminus \{0\}$, hence $y \in span(u_{p_2} \setminus u_{p_1})$. As $Z \in \mathcal{A}_{p_1}$ and $p_1 \leq p_2$, it follows that $y \in Z$, by the definition of the partial order. As $\Vdash_{\mathbb{Q}}$ "$|\underset{\sim}{B} \cap span\{\underset{\sim}{y}_k : k < \omega\}| = \aleph_0$", given $\{z_n : n < \omega\} \subseteq \underset{\sim}{B} \cap span\{\underset{\sim}{y}_k : k < \omega\}$, letting $z_n = z^1_n + z^2_n$ where $z^1_n \in span\{\underset{\sim}{y}_k : k \leq |u_{p_1}|\}$ and $z^2_n \in span\{\underset{\sim}{y}_k : k > |u_{p_1}|\}$, there is an infinite set $\{n_k : k < \omega\}$ such that $z^1_{n_k} = z^1_*$ for all $k < \omega$. Therefore, for $k \neq k'$, as $\underset{\sim}{B}$ is a subspace, $z_{n_k} + z_{n'_k} \in \underset{\sim}{B}$ and moreover $z_{n_k} + z_{n'_k} \in span\{\underset{\sim}{y_k} : |u_{p_1}| < k\}$. Therefore, there is p_2 above p_1 and y such that $p_2 \Vdash_{\mathbb{Q}}$ "$y \in span\{\underset{\sim}{y_k} : k > |u_{p_1}|\} \cap \underset{\sim}{B} \setminus \{0\}$". By the definition of $\{\underset{\sim}{y_k} : k < \omega\}$ and $\leq_{\mathbb{Q}}$, it follows that $y \in span(u_{p_2} \setminus u_{p_1}) = span(u_{p_2} \setminus u_p)$. By the definition of Z_p^+, it follows that $y \in Z_p^+$. As $y \in span(u_{p_2} \setminus u_p) \setminus \{0\} \subseteq V \setminus Z_p^+$, we obtain a contradiction. Therefore, $Z_p^+ \in D$. □

Observation 12. If $p \in \mathbb{Q}$ then $Z_p^- \in D$ where $Z_p^- = \{v \in V : \text{some } q \text{ above } p \text{ forces } v \notin \underset{\sim}{B}\}$.

Proof. Suppose towards contradiction that $Z_p^- \notin D$, then $\{v \in V : p \Vdash_{\mathbb{Q}}$ "$v \in \underset{\sim}{B}$"$\} = V \setminus Z_p^- \in D$. By the madness of $\mathcal{A}^{\mathbf{V}}$ in $\mathbf{V}$, there is $B_1 \in \mathcal{A}^{\mathbf{V}}$ such that $B_2 := B_1 \cap (V \setminus Z_p^-)$ is infinite. Note that $p \Vdash$ "$B_2 \subseteq V \setminus Z_p^- \subseteq \underset{\sim}{B}$", hence $p \Vdash_{\mathbb{Q}}$ "$|\underset{\sim}{B} \cap B_1| = \aleph_0$". By absoluteness, $p \Vdash_{\mathbb{Q}}$ "$B_1 \in \mathcal{A}$", and by the choice of $\underset{\sim}{B}$, $p \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{B} \in \mathcal{A}$". As $B_1 \in \mathbf{V}$ and $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{B} \notin \mathbf{V}$" it follows that $p \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{B} \neq B_1$". This contradicts the fact that $p \Vdash_{\mathbb{Q}}$ "$\mathcal{A}$ is almost disjoint". It follows that $Z_p^- \in D$, as required. □

Observation 13. $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{A}$ is infinite". Moreover, for every $p \in \mathbb{Q}$ there exists $q \in \mathbb{Q}$ such that $p \leq q$ and $|u_p| < |u_q|$.

Proof. Let $p \in \mathbb{Q}$, we shall prove that there exists $q \in \mathbb{Q}$ above p such that $u_p \neq u_q$. Let $B_1 = \cap\{x + A : x \in u_p,\ A \in \mathcal{A}_p \text{ and } x + A \in D\}$. As D is a filter and $u_p, \mathcal{A}_p$ are finite, $B_1 \in D$. By Fact 9, the set $B_2 = \{v \in V : v + B_1 \in D\}$ is in D. Let n_* be large enough such that $\bigoplus_{l<n_*} \mathbb{F}_2 x_l$ includes u_p, then by Fact 9, $B_3 := \bigoplus_{n_*<l} \mathbb{F}_2 x_l \in D$. Therefore, $B_1 \cap B_2 \cap B_3 \in D$ and hence is non-empty. Let $y \in B_1 \cap B_2 \cap B_3$ and let $q = (u_p \cup \{y\}, \mathcal{A}_p)$. Obviously, $q \in \mathbb{Q}$. It's easy to verify that $p \leq q$, for example, we shall verify clause (B)(d)(β) in Definition 3: Suppose that $x \in span(u_p)$, $A \in \mathcal{A}_p$ and $x + A \in D$. $y \in B_2$, hence $y + B_1 \in D$. $B_1 \subseteq x + A$, hence $y + B_1 \subseteq y + x + A$. As $y + B_1 \in D$, it follows that $y + x + A \in D$, as required. □

Finishing the proof of Theorem 2. We shall now choose $(v_n, B_n^1, B_n^2, p_n^1, p_n^2)$ by induction on n such that:

a. $v_n \in V$.
b. $B_n^l \subseteq V$ $(l = 1, 2)$.
c. $p_n^l \in N \cap \mathbb{Q}$ $(l = 1, 2)$.
d. For every $n < \omega$, $p_n^l \leq p_{n+1}^l$ $(l = 1, 2)$.

Case I ($n = 4i$): We choose $p_{n+1}^l \in I_i$ above p_n^l (recall that I_i is dense).

Case II ($n = 4i + 1$): Suppose that $v_0, ..., v_{n-1}$ and $p_0^l, ..., p_n^l$ have already been chosen. By Observation 11, $Z_{p_n^l}^+ \setminus \{v_0, ..., v_{n-1}\} \in D$, hence $(Z_{p_n^1}^+ \cap Z_{p_n^2}^+) \setminus \{v_0, ..., v_{n-1}\} \in D$ and hence $(Z_{p_n^1}^+ \cap Z_{p_n^2}^+) \setminus \{v_0, ..., v_{n-1}\} \neq \emptyset$. Choose $v_n \in Z_{p_n^1}^+ \cap Z_{p_n^2}^+ \setminus \{v_0, ..., v_{n-1}\}$, then there are conditions q_{n+1}^1 and q_{n+1}^2 above p_n^1 and p_n^2, respectively, such that $q_{n+1}^l \Vdash_{\mathbb{Q}}$ "$v_n \in \underset{\sim}{B}$" $(l = 1, 2)$. Let p_{n+1}^l be an extension of q_{n+1}^l that decides $\underset{\sim}{B} \cap \bigoplus_{k<n} \mathbb{F}_2 x_k$ and let B_n^l be a subset of V such that $p_{n+1}^l \Vdash_{\mathbb{Q}}$ "$B_n^l = \underset{\sim}{B} \cap \bigoplus_{k<n} \mathbb{F}_2 x_k$".

Case III ($n = 4i + 2$): As in the previous case (using $Z_{p_n^1}^+$ and $Z_{p_n^2}^-$), we choose $v_n \notin \{v_0, ..., v_{n-1}\}$ and conditions p_{n+1}^1, p_{n+1}^2 such that $p_{n+1}^1 \Vdash_{\mathbb{Q}}$ "$v_n \in \underset{\sim}{B}$" and $p_{n+1}^2 \Vdash_{\mathbb{Q}}$ "$v_n \notin \underset{\sim}{B}$".

Case IV ($n = 4i + 3$): As in Case III (this time using $Z_{p_n^1}^-$ and $Z_{p_n^2}^+$), we choose $v_n \notin \{v_0, ..., v_{n-1}\}$ and conditions p_{n+1}^1, p_{n+1}^2 such that $p_{n+1}^1 \Vdash_{\mathbb{Q}}$ "$v_n \notin \underset{\sim}{B}$" and $p_{n+1}^2 \Vdash_{\mathbb{Q}}$ "$v_n \in \underset{\sim}{B}$".

Finally, having carried the induction, let $G_l = \{p \in N \cap \mathbb{Q} : p \text{ be below some } p_n^l\}$ $(l = 1, 2)$, then by Case I of the induction, G_l is $(N, \mathbb{Q})$-generic. For $l = 1, 2$, let $S^l = \bigcup_{n<\omega} B_n^l$, then by the genericity of G_l, the choice of the B_n^ls, and $\mathcal{A}$ being analytic, it follows that $S_l \in \mathcal{A}$. By Cases II-IV of the induction, $S_1 \neq S_2$

and $|S_1 \cap S_2| = \aleph_0$, contradicting the almost disjointness of $\mathcal{A}$. This proves Theorem 2 modulo Fact 9 that will be proved below. □

Proof of Fact 9. For $S \in \mathcal{A}$ and $n < \omega$, let $S[n] = \bigoplus_{l<n} \mathbb{F}_2 x_l + S$. Let $\mathcal{D}$ be the set of all nonprincipal ultrafilters on V that contain all subspaces of V of finite codimension and all sets of the form $V \setminus S[n]$ for $S \in \mathcal{A}$ and $n < \omega$. Let $X = \{V \setminus S[n] : S \in \mathcal{A}, n < \omega\} \cup \{W : W \subseteq V, \text{ is a subspace of finite codimension}\}$, by the definition of the topology on $\beta(V)$ (the space of ultrafilters on V), $\mathcal{D}$ is closed in $\beta(V)$.

Subclaim 1. $\mathcal{D} \neq \emptyset$.

Proof. In order to show that $\mathcal{D} \neq \emptyset$, we shall prove that X has the FIP (Finite Intersection Property). Let $W \subseteq V$ be a subspace of finite codimension, let $S_1, ..., S_k \in \mathcal{A}$ and let $n_1, ..., n_k < \omega$.

Subclaim 1(a). Given $S^1 \neq S^2 \in \mathcal{A}$ and $n < \omega$, $S^1[n] \cap S^2[n]$ is finite.

Proof. Let $k = |S^1 \cap S^2|$, we shall prove that $|S^1[n] \cap S^2[n]| \leq k2^{2n}$. Suppose towards a contradiction that $|S^1[n] \cap S^2[n]| > k2^{2n}$ and let $m := k2^{2n}$. Let $\{r_j : j \leq m\}$ be pairwise distinct elements of $S^1[n] \cap S^2[n]$. For each $j \leq m$ and $l \in \{1, 2\}$, there are t^l_j and $a^l_{j,i}$ $(i < n)$ such that:
a. $t^l_j \in S^l$, $a^l_{j,i} \in \mathbb{F}_2$.
b. $r_j = \sum_{i<n} a^l_{j,i} x_i + t^l_j$.

Let E be the equivalence relation on $\{j : j \leq m\}$ defined by $j_1 E j_2$ iff $\bigwedge_{l=1,2} \bigwedge_{i<n} a^l_{j_1,i} = a^l_{j_2,i}$. E has $\leq 2^{2n}$ equivalence classes, hence there is $j_* \leq m$ such that $\frac{m+1}{2^{2n}} \leq |j_*/E|$, hence $k < |j_*/E|$. By renaming, we may assume wlog that $\{0, 1, ..., k\} \subseteq j_*/E$. For $l \in \{1, 2\}$ and $j < k + 1$, $r_j - r_0 = t^l_j - t^l_0$, and as $t^l_j, t^l_0 \in S^l$, it follows that $r_j - r_0 \in S^l$. Therefore, $r_j - r_0 \in S^1 \cap S^2$ for every $j < k + 1$. As $\{r_j : j < k + 1\}$ is without repetition, so is $\{r_j - r_0 : j < k + 1\}$, contradicting the fact that $|S^1 \cap S^2| = k$. This proves Subclaim 1(a).

Back to the proof of Subclaim 1, choosing $S' \in \mathcal{A} \setminus \{S_1, ..., S_k\}$, $S' \cap S_l[n_l] \subseteq S'[n_l] \cap S_l[n_l]$ is finite. WLOG suppose that $W = \bigoplus_{m \leq n} \mathbb{F}_2 x_n$ and let $\{z_n : n < \omega\}$ be an infinite subset of S' such that $max(supp(z_n)) < max(supp(z_{n+1}))$, wlog $\{z_n : n < \omega\}$ is disjoint to $S_l[n_l]$ for every $l \leq k$. By the same argument as in the proof of Observation 11, we may assume wlog that for each n, $z_n = z \oplus z'_n$ for a fixed $z \in \bigoplus_{l<m} \mathbb{F}_2 x_l$. Now consider the set $\{z_0 + z_n : n < \omega\} \subseteq S'$. As $max(supp(z_n)) < max(supp(z_{n+1}))$, this set is infinite, and therefore contains an element $z_0 + z_n$ such that $z_0 + z_n \notin \bigcup_{l \leq k} S_l[n_l]$. Obviously, $z_0 + z_n \in W$. Therefore, $W \cap (\bigcap_{l \leq k} (V \setminus S_l[n_l])) \neq \emptyset$ and it follows that X has the FIP. This completes the proof of Subclaim 1.

Subclaim 2. If $D_1, D_2 \in \mathcal{D}$ then $D_1 \oplus D_2 \in \mathcal{D}$ where $D_1 \oplus D_2 = \{A : (\forall^{D_1} s_1)(\forall^{D_2} s_2)(s_1 + s_2 \in A)\}$.

Proof. Obviously, $D_1 \oplus D_2 \in \beta(V)$. For every space of the form $W = \bigoplus_{m \leq n} \mathbb{F}_2 x_n$ we have $W \in D_1$, $W \in D_2$ and $s_1 + s_2 \in W$ for every $s_1, s_2 \in W$. Therefore, $W \in D_1 \oplus D_2$ and the ultrafilter contains all subspaces of finite codimension. Now let $S \in \mathcal{A}$ and $n < \omega$, we shall prove that $S[n] \notin D_1 \oplus D_2$. Fix $s_1 \in V$, letting $m > n$ such that $s_1 \in \bigoplus_{i<m} \mathbb{F}_2 x_i$, we have $\{s_2 : s_1 + s_2 \in S[n]\} \subseteq S[m]$ (if $s_1 + s_2 \in S[n]$, then $s_1 + s_2 = x + y$ for some $x \in \bigoplus_{i<n} \mathbb{F}_2 x_i$ and $y \in S$. Hence, $s_2 = s_1 + s_1 + s_2 = (s_1 + x) + y \in \bigoplus_{i<m} \mathbb{F}_2 x_i + S = S[m]$). As $S[m] \notin D_2$, it follows that $S[n] \notin D_1 \oplus D_2$. This completes the proof of Subclaim 2.

By Glazer's argument in the proof of Hindman's theorem (see Lemma 10.1, page 449 in [1] or Lemma 2.7 in [6]), the fact that $\mathcal{D}$ is a nonempty closed subset of $\beta(V)$ that is closed under the $\oplus$-operation implies that there exists $D \in \mathcal{D}$ such that $D \oplus D = D$. We shall now check that D is as required in Fact 9. Clauses (a) and (c) are immediate from the fact that $D \in \mathcal{D}$, so it remains to show that D satisfies clause (b). Let $A \in D$, we shall prove that $v + A \in D$ for D-almost all v. Given $v \in V$, let $v \oplus A = \{z \in V : v + z \in A\}$. As $D \oplus D = D$, $(\forall^D s_1)(\forall^D s_2)(s_1 + s_2 \in A)$. Therefore, $(\forall^D s_1)(s_1 \oplus A \in D)$. Note that $s_1 \oplus A = \{s_2 : s_2 \in s_1 + A\} = s_1 + A$, therefore, for D-almost all s_1 we have $s_1 + A \in D$, as required. This completes the proof of Fact 9. $\square$

The following question remains open:

Question. Let $F \neq \mathbb{F}_2$ be a field of cardinality $\leq \aleph_0$, is there an analytic mad family of subspaces of $\bigoplus\limits_{n<\omega} F$?

It is conceivable that a method similar to that of [2] might allow us to construct a Borel mad family for fields other than $\mathbb{F}_2$.

Finally, we observe that combining the proof of [3] with the results from this paper we obtain the following:

Theorem. *Let $V = \bigoplus\limits_{n<\omega} \mathbb{F}_2$, then $ZF + DC +$ "there are no mad families of subspaces of V" is equiconsistent with ZFC.*

The proof is almost identical to [3], where instead of using Mathias forcing, we now use the forcing $\mathbb{Q}_D$ from this paper where D is as in Fact 9. We shall elaborate on the proof in a subsequent paper.

References

[1] W. Wistar Comfort, Ultrafilters: some old and some new results, Bull. Am. Math. Soc. 83 (1977) 417–455.
[2] Haim Horowitz, Saharon Shelah, A Borel maximal eventually different family, arXiv:1605.07123.
[3] Haim Horowitz, Saharon Shelah, Can you take Toernquist's inaccessible away?, arXiv:1605.02419.
[4] Haim Horowitz, Saharon Shelah, A Borel maximal cofinitary group, arXiv:1610.01344.
[5] A.R.D. Mathias, Happy families, Ann. Math. Log. 12 (1) (1977) 59–111.
[6] Andrzej Roslanowski, Saharon Shelah, Partition theorems from creatures and idempotent ultrafilters, Ann. Comb. 17 (2013) 353–378.
[7] Iian Smythe, Madness in vector spaces, arXiv:1712.00057.