Positive logics*

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Abstract

Lindström's Theorem characterizes first order logic as the maximal logic satisfying the Compactness Theorem and the Downward Löwenheim-Skolem Theorem. If we do not assume that logics are closed under negation, there is an obvious extension of first order logic with the two model theoretic properties mentioned, namely existential second order logic. We show that existential second order logic has a whole

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family of proper extensions satisfying the Compactness Theorem and the Downward Löwenheim-Skolem Theorem. Furthermore, we show that in the context of negation-less logics, *positive logics*, as we call them, there is no strongest extension of first order logic with the Compactness Theorem and the Downward Löwenheim-Skolem Theorem.

1 Introduction

Our motivating question in this paper is whether we can generalize Lindström's Theorem from first order logic to Σ^1_1, that is, existential second order logic. In the case of first order logic Lindström's Theorem says that first order logic is maximal with the Compactness Theorem[1] and the Downward Löwenheim-Skolem Theorem[2] among logics satisfying some minimal closure conditions [10]. One of the assumed closure conditions is closure under negation. What happens if we drop this assumption? It seems that this question was first explicitly raised in [7]. The Compactness Theorem and the Downward Löwenheim-Skolem Theorem make perfect sense, whether we have negation or not. These two conditions make no reference to negation.

In earlier related work ([14]) we showed that a strong form of Lindström's Theorem fails for extensions of $L_{\kappa\omega}$ and $L_{\kappa\kappa}$: For weakly compact κ there is no strongest extension of $L_{\kappa\omega}$ with the (κ, κ)-compactness property and the Löwenheim-Skolem Theorem down to κ. With an additional set-theoretic assumption, there is no strongest extension of $L_{\kappa\kappa}$ with the (κ, κ)-Compactness Theorem and the Löwenheim-Skolem theorem down to $< \kappa$.

Obviously first order logic itself is not maximal if negation is dropped because existential second order logic Σ^1_1, and even $\Sigma^1_{1,\delta}$ (also denoted PC_Δ), i.e. existential second order quantifiers followed by a countable conjunction of first order sentences, which clearly satisfy both the Compactness Theorem and the Downward Löwenheim-Skolem Theorem, also properly extend first order logic.

[1]The (κ, λ)-Compactness Theorem says: Every theory of size $\leq \kappa$, every subset of size $< \lambda$ of which has a model, has a model. Compactness Theorem means (ω, ω)-Compactness Theorem.

[2]The Downward Löwenheim-Skolem Theorem down to κ says: Every sentence in a countable vocabulary, which has a model of size $\leq \kappa$. "Down to $< \kappa$" means "has a model of size $< \kappa$". The Downward Löwenheim-Skolem Theorem means the Löwenheim-Skolem Theorem down to $\aleph_0$.

We are led to the following (interrelated) questions, all in the context of logics where closure under negation is *not* assumed:

Question 1: Is Σ^1_1 (or rather $\Sigma^1_{1,\delta}$) maximal among logics satisfying the Compactness Theorem and the Downward Löwenheim-Skolem Theorem?

Question 2: Is there an extension of Σ^1_1 (or $\Sigma^1_{1,\delta}$) which is maximal among logics satisfying the Compactness Theorem and the Downward Löwenheim-Skolem Theorem?

Question 3: Is there a characterization of Σ^1_1 (or $\Sigma^1_{1,\delta}$) as maximal among logics satisfying some model-theoretic conditions?

Question 4: Is there an extension of Σ^1_1 (or $\Sigma^1_{1,\delta}$) which is maximal (or even strongest) among logics satisfying some model-theoretic conditions?

In this paper we formulate Questions 1 and 2 in exact terms. We answer Question 1 negatively. As to Question 2 we show that there is no strongest[3] extension of Σ^1_1 satisfying the Compactness Theorem and the Downward Löwenheim-Skolem Theorem. The existence of a maximal one (which has no proper such extension) remains open. Questions 3 and 4 remain completely unanswered. Admittedly, Question 4 is a little vague as both "extension" and "model-theoretical conditions" are left open.

To answer the above Questions 1 and 2 we introduce a family of new generalized quantifiers associated with the very natural and intuitive concept of the density of a set of reals. These quantifiers are defined for the purpose of solving the said questions and may lack wider relevance, although the general study of logics without negation is so undeveloped that it may be too early to say what is relevant and what is not.

Notation: We use $\mathcal{M}$ and $\mathcal{N}$ to denote structures, and M and N to denote their universes, respectively. For finite sequences s and sets a, we use $s\hat{\ }\langle a\rangle$ to denote the extension of s by the set a. For sequences of length $\leq \omega$, $s \triangleleft s'$ means that s is an initial segment of s'. The empty sequence is denoted $\emptyset$. A subset A of 2^ω is said to be *dense* if for all $s \in 2^{<\omega}$ there is $s' \in A$ such that $s \triangleleft s'$. We use $\mathbb{P}(\omega)$ to denote the power-set of ω.

[3]By strongest extension we mean one which contains every other as a sublogic.

2 Positive logics

We define the concept of a *positive logic,* meaning a logic without negation, except in front of atomic (and first order) formulas. We have to be careful about substitution in this context. If we are too lax about the substitution[4] of formulas into atomic formulas we end up having a logic which is closed under negation, which is not what we want. Substitution is very natural, but it is not needed in Lindström's characterization of first order logic.

One may ask whether a logic deserves to be called a *logic* if it is not closed under negation? We do not try to answer this question, but merely point out that there are several logics that do not have a negation in the sense that we have in mind, i.e. in the sense of classical logic. Take, for example, constructive logic. Although it has a negation, it does not have the Law of Excluded Middle, so its negation does not function in the way we mean when we ask whether a logic is closed under negation. In our sense constructive logic is not closed under negation. Another example is continuous logic [3] and the related positive logic of [4]. We have already mentioned existential second order logic Σ^1_1 and its stronger form, $\Sigma^1_{1,\delta}$. In the same category as Σ^1_1 are Dependence logic [15] and Independence Friendly Logic [11]. Transfinite game quantifiers yield infinitary logics which are not closed under negation, due to non-determinacy [8]. In the finite context there is the complexity class non-deterministic polynomial time NP, which is equivalent to existential second order logic on finite models, of which it is not known whether it is closed under negation. In this paper we introduce new examples of logics without negation.

Definition 1 *A* positive logic *is an abstract logic*[5] *in the sense of [10] (see also [6]) which contains first order logic and is closed under disjunction,*

[4]The Substitution Property for a abstract logic L^* says that if ϕ is in L^*, P is an n-ary predicate symbol in the vocabulary of ϕ and $\psi(x_1,\ldots,x_n)$ is a formula of L^*, then the result of substituting $\psi(t_1,\ldots,t_n)$ to occurrences of $P(t_1,\ldots,t_n)$ in ϕ is again in L^*. For details, see [6, Def. 1.2.3].

[5]An abstract logic (or "a generalized first order logic"), in the sense of [10] is a pair $L=(\Sigma,T)$, where Σ is an arbitrary set and T is a binary relation between members of Σ on the one hand and structures on the other. Members of Σ are called L-sentences. Classes of the form $\{\mathcal{M} : T(\phi,\mathcal{M})\}$, where ϕ is an L-sentence, are called L-characterizable classes. Abstract logics are assumed to satisfy five axioms expressed in terms of L-characterizable classes. The axioms correspond to being closed under isomorphism, conjunction, negation, permutation of symbols, and "free" expansions.

conjunction, and first order quantifiers ∃ and ∀. We do not require closure under negation, nor closure under substitution.

Example 2 *1. First order logic is a positive logic.*

2. Σ^1_1 and $\Sigma^1_{1,\delta}$ are positive logics.

3. If L is a positive logic, then so is $\Sigma^1_1(L)$, the closure of L under existential second order quantification.

3 A class of new quantifiers

In the tradition of [9] we define our new generalized quantifiers by first specifying a class of structures, closed under isomorphisms.

Let τ_d be the vocabulary $\{R_0, R_1, R_2, R_3, R_4\}$ consisting of binary predicates R_0, R_1, R_2 and unary predicates R_3, R_4.

Example 3 *A canonical example of a τ_d-structure is the model*

$$\mathcal{M}_A = (M, R_0^{\mathcal{M}_A}, R_1^{\mathcal{M}_A}, R_2^{\mathcal{M}_A}, R_3^{\mathcal{M}_A}, R_4^{\mathcal{M}_A}),$$

where $A \subseteq 2^\omega$ and

- $M = 2^{<\omega} \cup A$.
- $R_i^{\mathcal{M}_A} = \{(a, b) \in (2^{<\omega})^2 : b = a\hat{\ }\langle i \rangle\}$ $(i = 0, 1)$.
- $R_2^{\mathcal{M}_A} = \{(a, b) \in M \times M : a \triangleleft b\}$.
- $R_3^{\mathcal{M}_A} = \{\emptyset\}$.
- $R_4^{\mathcal{M}_A} = M$.

Definition 4 *For $n < \omega$ and $\eta \in 2^n$ we define $\psi_\eta(x)$ as:*

$$R_4(x) \wedge \exists y_0 \ldots \exists y_n \, (R_3(y_0) \wedge \textstyle\bigwedge_{i \leq n} R_4(y_i) \wedge \bigwedge_{i<n} y_i R_{\eta(i)} y_{i+1} \wedge \bigwedge_{i \leq n} y_i R_2 x).$$

For a τ_d-model $\mathcal{M}$ and $a \in M$ we define

$$\Omega(\mathcal{M}, a) = \{\eta \in 2^{<\omega} : \mathcal{M} \models \psi_\eta(a)\}$$
$$\Omega(\mathcal{M}) = \{\eta \in 2^\omega : \text{ for some } a \in M,\ \eta \restriction n \in \Omega(\mathcal{M}, a) \text{ for all } n < \omega\}.$$

If $\eta \in \Omega(\mathcal{M}, a)$, we say that a represents *η in $\mathcal{M}$. We also say that $\mathcal{M}$* represents *the set $\Omega(\mathcal{M})$.*

One element a can represent several η, but later in Section 7 we impose a further restriction to the effect that representation is unique.

Note that if $\mathcal{M}$ is a τ_d-model, then the property "$\Omega(\mathcal{M})$ is dense" is a Σ^1_1-property of $\mathcal{M}$. Since we aim at a logic which goes beyond existential second order logic, we have to sharpen the requirement of density. The property of τ_d-models we are interested in is the property that "$\Omega(\mathcal{M}) \setminus A$ is dense" for some preassigned set $A \subseteq 2^\omega$ of reals.

Definition 5 *Let $A \subseteq 2^\omega$. We define the Lindström quantifier Q_A as follows. Suppose $\mathcal{M}$ is a model and $\bar{c} \in M^k$. Then we define*

$$(Q_A x_0 x_1)(\psi_0(x_0,x_1,\bar{c}),\psi_1(x_0,x_1,\bar{c}),\psi_2(x_0,x_1,\bar{c}),\psi_3(x_0,\bar{c}),\psi_4(x_0,\bar{c})) \quad (1)$$

to be true in $\mathcal{M}$ if and only if $\Omega(\mathcal{M}_{\bar{\psi}}) \setminus A$ is dense, where

$$\bar{\psi} = (\psi_0(x_0,x_1,\bar{c}),\psi_1(x_0,x_1,\bar{c}),\psi_2(x_0,x_1,\bar{c}),\psi_3(x_0,\bar{c}),\psi_4(x_0,\bar{c})),$$

$$\mathcal{M}_{\bar{\psi}} = (M, R_0^{\mathcal{N}}, R_1^{\mathcal{N}}, R_2^{\mathcal{N}}, R_3^{\mathcal{N}}, R_4^{\mathcal{N}}),$$

and

- $R_i^{\mathcal{N}} = \{(a,b) \in M^2 : \mathcal{M} \models \psi_i(a,b,\bar{c})\}$ *($i = 0,1,2$).*
- $R_3^{\mathcal{N}} = \{a \in M : \mathcal{M} \models \psi_3(a,\bar{c})\}$.
- $R_4^{\mathcal{N}} = \{a \in M : \mathcal{M} \models \psi_4(a,\bar{c})\}$.

Definition 6 *Suppose $A \subseteq 2^\omega$. We define the positive logic L^d_A as the closure of first order logic under conjunction, disjunction, first order quantifiers $\exists$ and $\forall$, the existential second order quantifier $\exists R$, where R is a relation symbol, and the generalized quantifier Q_A. We denote by $L^{d,\omega}_A$ the extension of L^d_A obtained by allowing countable conjunctions as a logical operation. Finally, the proper class $L^{d,\infty}_A$ denotes the extension of L^d_A obtained by allowing arbitrary set-size conjunctions as a logical operation.*

With the obvious definition of what it means for a positive logic to be a sublogic of another, we can immediately observe that Σ^1_1 is a sublogic of L^d_A, and $\Sigma^1_{1,\delta}$ is a sublogic of $L^{d,\omega}_A$ whatever A is.

Example 7 *Suppose $A \subseteq 2^\omega$. The class K_A of τ_d-models $\mathcal{M}$ satisfying "$\Omega(\mathcal{M}) \setminus A$ is dense" is (trivially) definable in L^d_A, as $\mathcal{M} \in K_A$ if and only if $\mathcal{M} \models \psi_A$, where*

$$\psi_A = (Q_A x_0 x_1)(R_0(x_0, x_1), R_1(x_0, x_1), R_2(x_0, x_1), R_3(x_0), R_4(x_0)).$$

The model $\mathcal{M}_B$ of Example 3 is in K_A, if and only if $B \setminus A$ is dense.

For future reference we make the following observation: If $\eta \in 2^n$, $\bar{y} = (y_0, \dots, y_{n-1})$, $\bar{\psi}$ as in Definition 5 is a 5-tuple of formulas of L^d_A, $\bar{z} = (z_0, \dots, z_{k-1})$, and $\Gamma^{n,k}_{\bar{\psi},\eta}(\bar{y}, x, \bar{z}) \in L^d_A$ is the conjunction of

$$\begin{array}{ll} \psi_4(y_i, \bar{z}) & \text{for } i \leq n \\ \psi_4(x, \bar{z}) \wedge \psi_3(y_0, \bar{z}) & \\ \psi_{\eta(i)}(y_i, y_{i+1}, \bar{z}) & \text{for } i < n \\ \psi_2(y_i, x, \bar{z}) & \text{for } i \leq n, \end{array}$$

then (1) is equivalent to

$$\begin{array}{l} \text{For every } \sigma \in 2^{<\omega} \text{ there are } \eta \in 2^\omega \setminus A \text{ extending } \sigma \\ \text{and } a \in M \text{ such that for some function } n \mapsto \langle b^n_0, \dots, b^n_{n-1} \rangle \\ \text{from } \omega \text{ to } M^n \text{ we have } \mathcal{M} \models \Gamma^{n,k}_{\bar{\psi},\eta\restriction n}(b^n_0, \dots, b^n_{n-1}, a, \bar{c}) \text{ for all } n < \omega. \end{array} \tag{2}$$

We proceed to proving that the logic L^d_A, for suitably chosen $A \subseteq 2^\omega$, satisfies the Compactness Theorem and the Downward Löwenheim-Skolem Theorem, and also properly extends Σ^1_1.

4 The Compactness Theorem

We use the well-established method of ultraproducts to prove the Compactness Theorem of L^d_A.

Theorem 8 (Łoś Lemma for L^d_A) *Suppose $2^\omega \setminus A$ is dense. Suppose $\mathcal{M}_i$, $i \in I$, are models and D is an ultrafilter on a set I. Let $\mathcal{M} = \prod_{i \in I} \mathcal{M}_i / D$, $f_0, \dots, f_{n-1} \in \prod_{i \in I} M_i$ and $\phi(x_0, \dots, x_{n-1})$ in L^d_A (or even in $L^{d,\infty}_A$). Then*

$$\{i \in I : \mathcal{M}_i \models \phi(f_0(i), \dots, f_{n-1}(i))\} \in D \Rightarrow \mathcal{M} \models \phi(f_0/D, \dots, f_{n-1}/D).$$

Proof: We use induction on ϕ. The cases corresponding to the atomic formulas, the negated atomic fromulas, conjunction (even infinite conjunction), disjunction, $\exists$, $\forall$, and $\exists R$ (see e.g. [5, 4.1.14]) are all standard and well known. In the case of disjunction we use the property of ultrafilters that $I_1 \cup I_2 \in D$ implies $I_1 \in D$ or $I_2 \in D$. We are left with the induction step for Q_A. Let us denote $f_0(i), \ldots, f_{n-1}(i)$ by $\bar{f}(i)$ and $f_0/D, \ldots, f_{n-1}/D$ by $\bar{f}/D$. We assume

$$J = \{u \in I : \mathcal{M}_i \models (Q_A x_0 x_1)(\psi_0(x_0, x_1, \bar{f}(i)), \ldots, \psi_4(x_0, \bar{f}(i)))\} \in D \quad (3)$$

and demonstrate $\mathcal{M} \models (Q_A x_0 x_1)(\psi_0(x_0, x_1, \bar{f}/D), \ldots, \psi_4(x_0, \bar{f}/D))$. For $i \in J$ the set B_i of elements η of $2^\omega \setminus A$ such that there are $a_i \in M_i$ and $b^n_{0,i} \ldots, b^n_{n-1,i}$ in M_i such that $\mathcal{M}_i \models \Gamma^{n,k}_{\bar{\psi},\eta\restriction n}(b^n_{0,i}, \ldots, b^n_{n-1,i}, a_i, \bar{f}(i))$ for all $n < \omega$, is dense.

Case 1: *D* **is $\aleph_1$-incomplete.** Let $J = I_0 \supseteq I_1 \supseteq \ldots$ be a descending chain in D with empty intersection. We show that the set B of $\eta \in 2^\omega$ such that there is $a \in M$ such that for some $b^n_0, \ldots, b^n_{n-1}$ in $\prod_i M_i/D$ we have $\mathcal{M} \models \Gamma^{n,k}_{\bar{\psi},\eta\restriction n}(b^n_0, \ldots, b^n_{n-1}, a, \bar{f}/D)$ for all $n < \omega$, is the full set 2^ω. Since we assume that $2^\omega \setminus A$ is dense, it follows that $B \setminus A$ is dense, as we claim.

Suppose $\eta \in 2^\omega$ is arbitrary. Let $i \in I_{n+1} \setminus I_n$. Because B_i is dense, there are, for all $n < \omega$, extensions $\eta_i \in 2^\omega$ of $\eta \restriction n$ and elements $a_i, b^n_{i,0}, \ldots, b^n_{i,n-1} \in M_i$ such that

$$\mathcal{M}_i \models \Gamma^{n,k}_{\bar{\psi},\eta_i\restriction n}(b^n_{i,0}, \ldots, b^n_{i,n-1}, a_i, \bar{f}(i)) \quad (4)$$

for all $n < \omega$. Let $h(i) = a_i$. For $i \in I_n \setminus I_{n+1}$ and $m < n$, let $g^n_m(i) = b^n_{i,m}$. Now

$$\{i \in I : \mathcal{M}_i \models \Gamma^{n,k}_{\bar{\psi},\eta\restriction n}(g^n_0(i), \ldots, g^n_{n-1}(i), h(i), \bar{f}(i))\} \supseteq I_{n+1} \in D.$$

Hence

$$\mathcal{M} \models \Gamma^{n,k}_{\bar{\psi},\eta\restriction n}(g^n_0/D, \ldots, g^n_{n-1}/D, h/D, \bar{f}/D).$$

Case 2: *D* **is $\aleph_1$-complete.** We show that the set B of $\eta \in 2^\omega \setminus A$ such that for some $a, b^n_0, \ldots, b^n_{n-1} \in M$ we have $\mathcal{M} \models \Gamma^{n,k}_{\bar{\psi},\eta\restriction n}(b^n_0, \ldots, b^n_{n-1}, a, \bar{f}/D)$ for all $n < \omega$, is dense. Suppose $\eta \in 2^n$. By the density of B_i, for each $i \in J$ there is $\nu_i \in B_i$ extending η. There is $J_0 \subseteq J$ in D such that $\nu_i(n)$ is constant for $i \in J_0$. There is $J_1 \subseteq J_0$ in D such that $\nu_i(n+1)$ is constant for $i \in J_1$, etc. By $\aleph_1$-completeness we get $J_\infty \in D$ such that $\nu_i(m)$ is constant, say $\eta^*(m)$ for all $m \geq n$ and all $i \in J_\infty$. Now $\eta^* \in B$ follows easily. $\Box$

Corollary 9 *If $2^\omega \setminus A$ is dense, then:*

1. *L^d_A (even $L^{d,\infty}_A$) satisfies the (full) Compactness Theorem.*
2. *Every sentence of L^d_A with an infinite model has arbitrarily large models.*
3. *The only sentences in L^d_A that have a negation (in the usual sense) are the first order (equivalent) ones.*

Proof: The usual argument gives 1: Suppose T is a finitely consistent theory in L^d_A. Let I be the set of finite subsets of T and for each $i \in I$, let $\mathcal{M}_i \models i$. If $\phi \in T$, let $A_\phi = \{i \in I : \phi \in i\}$. Then the family $\mathcal{J} = \{A_\phi : \phi \in T\}$ has the finite intersection property. Let D be a non-principal ultrafilter on I extending $\mathcal{J}$. Now if $\phi \in T$, then $\prod_D \mathcal{M}_i \models \phi$, as $\{i \in I : \mathcal{M}_i \models \phi\} \supseteq A_\phi \in D$.

Claim 2 follows immediately from Claim from 1. Claim 3 follows from the ultraproduct characterization of first order model classes (see e.g. [5, 4.1.12]) and the characterization of elementary equivalence in terms of ultrapowers [13]. □

Theorem 10 (Robinson's Consistency Lemma for L^d_A) *Suppose $2^\omega \setminus A$ is dense. Suppose T_1 and T_2 are consistent L^d_A-theories with vocabularies τ_1 and τ_2, respectively, such that $T_1 \cap T_2$ is complete with respect to first order logic in the vocabulary $\tau_1 \cap \tau_2$. Then $T_1 \cup T_2$ is consistent.*

Proof: This proof is not specific to L^d_A, but is rather a well-known consequence of Łoś Lemma, Theorem 8. Let $\mathcal{M}_1 \models T_1$ and $\mathcal{M}_2 \models T_2$. Let $\mathcal{M}_l^-$ be the reduct of $\mathcal{M}_l$ to the vocabulary $\tau_d = \tau_1 \cap \tau_2$. Now $\mathcal{M}_1^-$ and $\mathcal{M}_2^-$ are elementarily equivalent in first order logic, for if $\mathcal{M}_1^- \models \phi$ then necessarily $T_1 \cap T_2 \models \phi$, whence $\mathcal{M}_2^- \models \phi$, and vice versa. By [13] there are a set I and an ultrafilter D on I such that if we denote $\mathcal{M}_1^I/D$ by $\mathcal{N}_1$ and $\mathcal{M}_2^I/D$ by $\mathcal{N}_2$, then $\mathcal{N}_1 \restriction \tau_d \cong \mathcal{N}_2 \restriction \tau_d$. W.l.o.g. $\mathcal{N}_1 \restriction \tau_d = \mathcal{N}_2 \restriction \tau_d$. Let $\mathcal{N}$ be a common expansion of $\mathcal{N}_1$ and $\mathcal{N}_2$. By Theorem 8, $\mathcal{N} \restriction \tau_1 \models T_1$ and $\mathcal{N} \restriction \tau_2 \models T_2$. Hence, $\mathcal{N} \models T_1 \cup T_2$. □

5 The Downward Löwenheim-Skolem property

The Downward Löwenheim-Skolem Property, which says that any sentence (of the logic) which has a model has a countable model, is an important ingredient of the Lindström characterization of first order logic. The main examples of logics with this property, apart from first order logic, are $L_{\omega_1\omega}$ and its sublogics $L(Q_0)$ (with the quantifier "there exists infinitely many", see e.g. [1, p. 8]) and the weak second order logic L^2_w (with quantifiers for variables that range over finite sets, see e.g. [1, p. 9]). We now prove this property for L^d_A in a particularly strong form.

Because of lack of negation the elementary submodel relation $\mathcal{M} \preccurlyeq \mathcal{N}$ splits into two different concepts $\mathcal{M} \preccurlyeq^+ \mathcal{N}$ and $\mathcal{M} \preccurlyeq^- \mathcal{N}$:

Definition 11 *Let us write $\mathcal{N} \preccurlyeq^-_{L^d_A} \mathcal{M}$ if $\mathcal{N} \subseteq \mathcal{M}$ and for all $a_1, \ldots, a_n$ in N and all formulas $\phi(x_1, \ldots, x_n)$ of L^d_A we have*

$$\mathcal{M} \models \phi(a_1, \ldots, a_n) \Rightarrow \mathcal{N} \models \phi(a_1, \ldots, a_n).$$

Respectively, we write $\mathcal{N} \preccurlyeq^+_{L^d_A} \mathcal{M}$ if $\mathcal{N} \subseteq \mathcal{M}$ and for all $a_1, \ldots, a_n$ in N and all formulas $\phi(x_1, \ldots, x_n)$ of L^d_A we have

$$\mathcal{N} \models \phi(a_1, \ldots, a_n) \Rightarrow \mathcal{M} \models \phi(a_1, \ldots, a_n).$$

Similar definitions can be given for $L^{d,\omega}_A$ and $L^{d,\infty}_A$, and for fragments (i.e. subsets closed under subformulas) Γ thereof.

The Compactness Theorem implies that every infinite model $\mathcal{N}$ has arbitrarily large $\mathcal{M}$ such that $\mathcal{N} \preccurlyeq^+_{L^d_A} \mathcal{M}$ (Corollary 9).

Theorem 12 (Downward Löwenheim-Skolem-Tarski Theorem) *Suppose $\kappa \geq \aleph_0$, $A \subseteq 2^\omega$, $\mathcal{M}$ is a model for a vocabulary of cardinality $\leq \kappa$, and $X \subseteq M$ such that $|X| \leq \kappa$. Then there is $\mathcal{N} \preccurlyeq^-_{L^d_A} \mathcal{M}$ (even $\mathcal{N} \preccurlyeq^-_\Gamma \mathcal{M}$ for any fixed fragment Γ of $L^{d,\infty}_A$ of size $\leq \kappa$) such that $X \subseteq N$ and $|N| = \kappa$.*

Proof: We first expand $\mathcal{M}$ as follows: For every L^d_A-formula $\phi(R, \bar{z})$, where R is n-ary and $\bar{z} = z_0, \ldots, z_{k-1}$, we make sure there is a predicate symbol R^* of arity $k + n$ such that if $\mathcal{M} \models \exists R\phi(R, \bar{c})$, then $\mathcal{M} \models \phi(R^*(\bar{c}, \cdot), \bar{c})$. Likewise, we may assume the vocabulary of $\mathcal{M}$ has a Skolem function f_ϕ for each formula $\phi(x, \bar{z})$ such that if $\mathcal{M} \models \exists x\phi(x, \bar{c})$, then $\mathcal{M} \models \phi(f_\phi(\bar{c}), \bar{c})$. Let τ be the original vocabulary of $\mathcal{M}$ and τ^* the vocabulary of the expansion, which we denote $\mathcal{M}^*$. For any formulas $\bar{\psi}$ in L^d_A of the vocabulary τ^* and $\bar{c} \in M^k$ such that (1) in Definition 5 holds, let $g(n, k, \bar{\psi}, \eta, \bar{c})$ be the function which maps $n, k, \bar{\psi}, \bar{c}$ and $\eta \in 2^n$ to $\langle b_0^n, \ldots, b_{n-1}^n \rangle \in M^n$ such that $\mathcal{M}^* \models \Gamma^{n,k}_{\bar{\psi},\eta\restriction n}(b_0^n, \ldots, b_{n-1}^n, a, \bar{c})$ for all $n < \omega$. Denoting for any cardinal θ the set of sets of hereditary cardinality $< \theta$ by H_θ, let $\theta \geq (\kappa + 2^\omega)^+$such that $M \subseteq H_\theta$, and $K \prec H_\theta$, such that $|K| = \kappa$ and $\{A, \kappa, \tau, \mathcal{M}^*, X, g\} \cup \kappa \cup \tau^* \cup X \subseteq K$. Let $\mathcal{N}$ be the restriction of $\mathcal{M}^*$ to K, i.e. the universe N of $\mathcal{N}$ is $M \cap K$ and the constants, relations and functions of $\mathcal{M}^*$ are relativized to N.

We need to check that N is closed under the interpretations of function symbols of the vocabulary of $\mathcal{M}^*$. Let f be such a function symbol. Suppose f is s-ary and $\bar{c} \in N^s$. The sentence $\exists x(x \in M \wedge x = f^{\mathcal{M}^*}(\bar{c}))$ is true in H_θ, hence true in K. Thus there is $b \in N (= M \cap K)$ such that $b = f^{\mathcal{M}^*}(\bar{c})$ is true in K. Therefore $f^{\mathcal{M}^*}(\bar{c}) = b \in N$. We can conclude that $\mathcal{N}$ is a substructure of $\mathcal{M}^*$.

Claim: If $\phi(\vec{x})$ is a τ-formula in L^d_A and $\vec{a} \in N$, then $\mathcal{M}^* \models \phi(\vec{a}) \Rightarrow \mathcal{N} \models \phi(\vec{a})$.

We use induction on ϕ. The claim follows from $\mathcal{N} \subseteq \mathcal{M}^*$ for atomic and negated atomic ϕ. The claim is clearly preserved under conjunction and disjunction. It is also trivially preserved under universal quantifier, since $\mathcal{N} \subseteq \mathcal{M}^*$. The induction steps for both first and second order existential quantifiers are trivial because of the expansion we have performed on $\mathcal{M}^*$. We are left with the quantifier Q_A.

Suppose $\mathcal{M}^*$ satisfies (1) of Definition 5 with $\bar{c} \in N^k$. Thus (2) holds and we want to prove (2) with $\mathcal{M}^*$ replaced by $\mathcal{N}$. Note that (2) also holds in K. Suppose $\sigma \in 2^n$ is given. There is $\eta \in K \cap (2^\omega \setminus A)$ extending σ such that K satisfies

> There is $a \in M$ such that for some function $n \mapsto \langle b_0^n, \ldots, b_{n-1}^n \rangle$ from ω to M^n we have $\mathcal{M}^* \models \Gamma^{n,k}_{\bar{\psi},\eta\restriction n}(b_0^n, \ldots, b_{n-1}^n, a, \bar{c})$ for all $n < \omega$.

Thus there are $a \in N$ and a function $n \mapsto \langle b_0^n, \ldots, b_{n-1}^n \rangle$ from ω to N^n such that $\mathcal{M}^* \models \Gamma^{n,k}_{\bar{\psi},\eta\restriction n}(b_0^n, \ldots, b_{n-1}^n, a, \bar{c})$ for all $n < \omega$. By the Induction Hypothesis, $\mathcal{N} \models \Gamma^{n,k}_{\bar{\psi},\eta\restriction n}(b_0^n, \ldots, b_{n-1}^n, a, \bar{c})$ for all $n < \omega$ follows. □

We conclude that every sentence of L^d_A which has an infinite model has a countable model and an uncountable model.

The following examples show that Theorem 12 is in a sense optimal:

Example 13 *There is an uncountable model $\mathcal{M}$, namely $(\mathbb{P}(\omega), a, \in)$, where a is the element ω of $\mathbb{P}(\omega)$, such that there is no countable model $\mathcal{N}$ with*

$\mathcal{N} \preccurlyeq^{+}_{\Sigma^1_1} \mathcal{M}$. There is a countable model $\mathcal{N}$, namely $(\omega, <)$, such that there is no uncountable model $\mathcal{M}$ with $\mathcal{N} \preccurlyeq^{-}_{\Sigma^1_1} \mathcal{M}$.

6 Proper extensions of Σ^1_1 and $\Sigma^1_{1,\delta}$

Our goal in this section is to show that for many $A \subseteq 2^\omega$ the logic L^d_A properly extends Σ^1_1 and $L^{d,\omega}_A$ properly extends $\Sigma^1_{1,\delta}$. We have a spectrum of results to this effect but nothing as conclusive as being able to explicitly point out such a set A. There are obvious reasons for this. The logic Σ^1_1 is very powerful and any "simple" $A \subseteq 2^\omega$ is likely to yield L^d_A which is equivalent to Σ^1_1 rather than properly extending it. This is even more true with $L^{d,\omega}_A$ and $\Sigma^1_{1,\delta}$.

We first establish the basic existence of sets $A \subset 2^\omega$ with the desired properties. We shall then refine the result with further arguments.

Theorem 14 *There are sets $A \subseteq 2^\omega$ such that $2^\omega \setminus A$ is dense and Q_A is not definable in Σ^1_1, nor in $\Sigma^1_{1,\delta}$, nor in $L_{\omega_1\omega_1}$.*

Proof: Let A_α, $\alpha < 2^\omega$, be disjoint dense subsets of 2^ω. For any $X \subseteq 2^\omega$, let

$$A_X = \bigcup_{\alpha \in X} A_\alpha.$$

Note that if $X \neq Y$, then $A_X \setminus A_Y$ or $A_Y \setminus A_X$ is dense. Let K_A and K_B be as in Example 7 and $\mathcal{M}_A$ is as in Example 3. If $A \setminus B$ is dense, then $K_A \neq K_B$, as $\mathcal{M}_A \in K_B$ but $\mathcal{M}_A \notin K_A$. Thus the classes K_{A_X}, $X \subseteq 2^\omega$, are all different. For cardinality reasons there is $X \subseteq 2^\omega$ so that K_{A_X} is not definable in $\Sigma^1_{1,\delta}$, nor in $L_{\omega_1\omega_1}$. But K_{A_X} is always definable in $L^d_{A_X}$. $\square$

The following result merely improves the previous result:

Theorem 15 *There is a countable $A_0 \subseteq 2^\omega$ such that if $A_0 \subseteq A \subseteq 2^\omega$ with $2^\omega \setminus A$ dense, then Q_A is not Σ^1_1-definable.*

Proof: Let us consider $\mathcal{M}_B$, where $B = 2^\omega$ (defined in Example 3). Let $\mathcal{M}$ be resplendent (see [2]) such that $\mathcal{M}_B \prec \mathcal{M}$. Let $\mathcal{M}^+$ be an expansion of $\mathcal{M}$ such that every Σ^1_1-sentence true in $\mathcal{M}^+$ has a witness in the vocabulary (countable). Let $\mathcal{N}^+ \prec \mathcal{M}^+$ be countable. Let A_0 be the countable set $\Omega(\mathcal{N}^+)$. Suppose now $A_0 \subseteq A \subseteq 2^\omega$, but ψ_A, as defined in Example 7, is definable by a Σ^1_1-sentence ϕ. Since $2^\omega \setminus A$ dense, $\mathcal{M}_B \models \psi_A$, whence

$\mathcal{M}_B \models \phi$. Hence all the first order consequences of ϕ are true in $\mathcal{M}_B$. Since $\mathcal{M}$ is resplendent, $\mathcal{M} \models \phi$. Since ϕ has a witness in $\mathcal{M}^+$, $\mathcal{N}^+ \models \phi$. Hence $\mathcal{N}^+ \models \psi_A$ whence $\Omega(\mathcal{N}^+) \setminus A$ is dense. This is a contradiction, as $\Omega(\mathcal{N}^+) \setminus A = \emptyset$. □

Theorem 16 *Let* $\mathbf{P}$ *be the poset of finite partial functions* $(\omega_1+\omega_1)\times\omega \to 2$ *i.e. the forcing for adding* $\omega_1+\omega_1$ *Cohen reals. Let* G *be* $\mathbf{P}$*-generic and* $\eta_\alpha \in 2^\omega$, $\alpha < \omega_1+\omega_1$, *the Cohen reals added by* G. *Let* A *be the set of* η *such that* $\eta = \eta_\alpha$ *(mod finite) for some* $\alpha < \omega_1$. *Then in* $V[G]$, Q_A *is not* Σ^1_1*-definable.*

Proof: Let B be the set of η such that $\eta = \eta_\alpha$(mod finite) for some $\alpha < \omega_1+\omega_1$. Then $\mathcal{M}_B \models \psi_A$. Suppose ϕ is a Σ^1_1-sentence logically equivalent to ψ_A. Thus $\mathcal{M}_B \models \phi$. Let f be a bijection (in V) of $\omega_1+\omega_1$ onto ω_1. The function f induces an complete embedding $\bar{f}$ of $\mathbf{P}$ into $\mathbf{P}$. The mapping $\bar{f}$ induces a mapping $\tau \mapsto \tau_{\bar{f}}$ between $\mathbf{P}$-terms. Let $\mathcal{N}$ be the image of $\mathcal{M}_B$ under this mapping. Now $\mathcal{N} \not\models \psi_A$. However, $\mathcal{N} \models \phi$, whence $\mathcal{N} \models \psi_A$, a contradiction. □

Theorem 17 *Assume* $A = 2^\omega \setminus D$, *where* $D \subseteq 2^\omega$ *is dense,* $\omega < |D| < 2^{\aleph_0}$ *and there is an open set* U *such that* $D \cap V$ *is uncountable for every non-empty open* $V \subseteq U$. *Then the quantifier* Q_A *is not* Σ^1_1*-definable.*

Proof: Recall that ψ_A is a sentence of L^d_A in the vocabulary τ_d saying that $\Omega(M) \setminus A$ is dense. Thus ψ_A says $\Omega(M) \cap D$ is dense. Let ϕ be a Σ^1_1 sentence $\exists R\phi_0$ such that ψ_A and ϕ are logically equivalent, contradicting our desired conclusion. Let $\langle D_\alpha : \alpha < \omega_1\rangle$ be a sequence of disjoint countable dense subsets of $D \cap U$. Let $\mathcal{N}_\alpha$ be a countable model representing the set D_α, whence it satisfies ψ_A, hence ϕ, and there is an expansion $\mathcal{N}^*_\alpha$ of $\mathcal{N}_\alpha$ to a model of ϕ_0. Let $\mathcal{N} = \langle N^*_\alpha : \alpha < \omega_1\rangle$. Let $\mathcal{B} = (H_\theta, \in, <)$, for a large enough cardinal θ and for a well-ordering $<$ of H_θ. We choose a countable elementary submodel $\mathcal{B}^*$ of $\mathcal{B}$ such that $\{\mathcal{N}, A, \omega_1\} \subset \mathcal{B}^*$.

By Theorem IV.5.19 of [12] there is a sequence $\langle \mathcal{B}_\alpha : \alpha < 2^\omega\rangle$ of countable elementary extensions of $\mathcal{B}^*$ such that for every $\alpha < \beta < 2^\omega$:

(a) $\mathcal{B}_\alpha$ has standard ω.

(b) $\mathcal{B}_\alpha$ has a (possibly non-standard) member c_α of $(\omega_1)^{\mathcal{B}^*}$.

(c) If an element of ${}^{\omega}2$ is definable in both $\mathcal{B}_\alpha$ and $\mathcal{B}_\beta$, then it is in $\mathcal{B}^*$.

Let $\mathcal{N}^+_\eta$ be the c_η'th member of the sequence $\langle \mathcal{N}^*_\alpha : \alpha < \omega_1 \rangle$ as interpreted in $\mathcal{B}_\eta$. So necessarily $\mathcal{N}^+_\eta$ is a model of ϕ_0 and hence its reduct $\mathcal{N}^+_\eta \restriction \tau_d$ is a model of ϕ, and further of ψ_A. We have continuum many models $\mathcal{N}^+_\eta \restriction \tau_d$ of ψ_A. However, we will now show that the number of η for which the model $\mathcal{N}^+_\eta \restriction \tau_d$ satisfies ψ_A is at most $|D| < 2^\omega$, a contradiction. Suppose $\mathcal{N}^+_\eta \restriction \tau_d \models \psi_A$. Then the subset of 2^ω represented by $\mathcal{N}^+_\eta = (\mathcal{N}^*_{c_\eta})^{\mathcal{B}_\eta}$, i.e. $(D_{c_\eta})^{\mathcal{B}_\eta}$, meets D in a dense set. Every element of $(D_{c_\eta})^{\mathcal{B}_\eta}$ is definable in $\mathcal{B}_\eta$. By the disjointness clause (c) above we get the claimed contradiction.

We now finish the proof of Theorem 17: Suppose η is such that $\mathcal{N}^+_\eta \not\models \psi_A$. This is a contradiction because $\mathcal{N}^+_\eta \models \phi$. $\square$

7 No strongest extension

We show that there is no strongest extension among positive logics of first order logic, or Σ^1_1, or $\Sigma^1_{1,\delta}$, with the Compactness Theorem and the Downward Löwenheim-Skolem Theorem.

We consider sequences $\mathcal{A} = \langle A_\alpha : \alpha \leq \omega_1 \rangle$ such that each A_α, $\alpha < \omega_1$, is a countable dense subset of 2^ω, $\alpha < \beta$ implies $A_\alpha \subset A_\beta$, $A_{\omega_1} = \bigcup_{\alpha<\omega_1} A_\alpha$, and the set $S = \{\alpha < \omega_1 : A_\alpha = \bigcup_{\beta<\alpha} A_\beta\}$ is stationary.

Let $\Theta_{\rm TL}$ be the first order sentence

$$\begin{array}{l}
\exists x(R_3(x) \wedge \forall y(\neg R_4(y) \vee R_2(x,y)))\wedge \\
\forall x \forall y(\neg R_0(x,y) \vee R_2(x,y))\wedge \\
\forall x \forall y(\neg R_1(x,y) \vee R_2(x,y))\wedge \\
\forall x \forall y(\neg R_2(x,y) \vee (R_4(x) \wedge R_4(y)))\wedge \\
\forall x(\neg R_4(x) \vee R_2(x,x))\wedge \\
\forall x \forall y \forall z(\neg R_2(x,y) \vee \neg R_2(y,z) \vee R_2(x,z))\wedge \\
\forall x \forall y \forall z(\neg R_2(y,x) \vee \neg R_2(z,x) \vee R_2(y,z) \vee R_2(z,y)).
\end{array}$$

Intuitively, $\Theta_{\rm TL}$ says that R_2 is a tree-like partial order extending R_0 and R_1. For example, the model $\mathcal{M}_A$ of Example 3 always satisfies $\Theta_{\rm TL}$. If $\mathcal{M} \models \Theta_{\rm TL}$, then one element a of M can represent only one η, i.e.

$$\eta, \eta' \in \Omega(\mathcal{M}, a) \text{ implies } \eta = \eta'. \tag{5}$$

Definition 18 *We define the Lindström quantifier Q_A as follows. Suppose $\mathcal{M}$ is a model and $\bar{c} \in M^k$. Then we define that $\mathcal{M}$ satisfies*

$$(Q_{\mathcal{A}} x_0 x_1)(\psi_0(x_0,x_1,\bar{c}), \psi_1(x_0,x_1,\bar{c}), \psi_2(x_0,x_1,\bar{c}), \psi_3(x_0,\bar{c}), \psi_4(x_0,\bar{c})) \tag{6}$$

if and only if $\mathcal{M}_{\bar\psi} \models \Theta_{\mathrm{TL}}$ *and* $\Omega(\mathcal{M}_{\bar\psi}) \cap A_{\omega_1} \in \mathcal{A}$, *where* $\mathcal{M}_{\bar\psi}$ *is as in Definition 5 and* $\Omega(\mathcal{M}_{\bar\psi})$ *is as in Definition 4.*

Definition 19 *We define* $L^d_{\mathcal{A}}$ *as the closure of first order logic under* $\wedge, \vee, \exists$, $\forall, \exists R$ *and* $Q_{\mathcal{A}}$. *The fragment, where* $Q_{\mathcal{A}}$ *is applied to first order formulas* $\bar\psi$ *only is denoted* $L^{d^-}_{\mathcal{A}}$. *Similarly,* $L^{d,\omega}_{\mathcal{A}}$, $L^{d,\infty}_{\mathcal{A}}$, $L^{d^-,\omega}_{\mathcal{A}}$, *and* $L^{d^-,\infty}_{\mathcal{A}}$.

Theorem 20 (Łoś Lemma for $L^d_{\mathcal{A}}$) *Suppose* $\mathcal{M}_i$, $i \in I$, *are models and* D *is an* ω_1*-incomplete ultrafilter on a set* I. *Let* $\mathcal{M} = \prod_{i\in I} \mathcal{M}_i/D$, $f_0, \ldots, f_{n-1} \in \prod_{i\in I} M_i$ *and* $\phi(x_0, \ldots, x_{n-1})$ *in* $L^d_{\mathcal{A}}$ *(even in* $L^{d,\infty}_{\mathcal{A}}$*). Then*

$$\{i \in I : \mathcal{M}_i \models \phi(f_0(i), \ldots, f_{n-1}(i))\} \in D \Rightarrow \mathcal{M} \models \phi(f_0/D, \ldots, f_{n-1}/D).$$

Proof: We follow the proof of Theorem 8. The only point that requires attention is the induction step for $Q_{\mathcal{A}}$. We assume

$$J = \{u \in I : \mathcal{M}_i \models Q_{\mathcal{A}} x_0 x_1 \psi_0(x_0, x_1, \bar f(i)) \ldots \psi_4(x_0, \bar f(i))\} \in D \qquad (7)$$

and demonstrate $M \models Q_{\mathcal{A}} x_0 x_1 \psi_0(x_0, x_1, \bar f/D) \ldots \psi_4(x_0, \bar f/D)$. As in the proof of Theorem 8, it can be shown that the set B of $\eta \in 2^\omega$ such that there is $a \in M$ such that for some $b^n_0, \ldots, b^n_{n-1}$ in $\prod_i M_i/D$ we have $\mathcal{M} \models \Gamma^{n,k}_{\bar\psi,\eta\restriction n}(b^n_0, \ldots, b^n_{n-1}, a, \bar f/D)$ for all $n < \omega$, is the full set 2^ω. It follows that $2^\omega \cap A_{\omega_1} = A_{\omega_1}$ and hence that $2^\omega \cap A_{\omega_1} \in \mathcal{A}$, as claimed. $\square$

Corollary 21 *If* $2^\omega \setminus A$ *is dense, then* $L^d_{\mathcal{A}}$ *(even* $L^{d,\infty}_{\mathcal{A}}$*) satisfies the (full) Compactness Theorem.*

Proof: The ultrafilter we used in the proof of Corollary 9 was regular, hence ω_1-incomplete. $\square$

We can prove the Downward Löwenheim-Skolem-Tarski Theorem for $L^{d^-}_{\mathcal{A}}$ only (see Proposition 25 and Theorem 27).

Theorem 22 (Downward Löwenheim-Skolem-Tarski Theorem) *Suppose* $\mathcal{M}$ *is a model for a countable vocabulary and* $X \subseteq M$ *is countable. Then there is* $\mathcal{N} \preccurlyeq^-_{L^{d^-}_{\mathcal{A}}} \mathcal{M}$ *(even* $\mathcal{N} \preccurlyeq^-_{\Gamma} \mathcal{M}$ *for any fixed countable fragment of* $L^{d,\omega}_{\mathcal{A}}$*) such that* $X \subseteq N$ *and* $|N| \leq \aleph_0$.

Proof: We first expand $\mathcal{M}$ as follows: For every $L^{d-}_{\mathcal{A}}$-formula $\phi(R,\bar{z})$, where R is n-ary and $\bar{z}=z_0,\ldots,z_{k-1}$, there is a predicate symbol R^* of arity $k+n$ such that if $\mathcal{M}\models\exists R\phi(R,\bar{c})$, then $\mathcal{M}\models\phi(R^*(\bar{c},\cdot),\bar{c})$. Likewise, we may assume the vocabulary of $\mathcal{M}$ has a Skolem function f_ϕ for each formula $\phi(x,\bar{z})$ such that if $\mathcal{M}\models\exists\phi(x,\bar{c})$, then $\mathcal{M}\models\phi(f_\phi(\bar{c}),\bar{c})$. Let τ be the original vocabulary of $\mathcal{M}$ and τ^* the vocabulary of the expansion, which we also denote $\mathcal{M}$. For any formulas $\bar{\psi}$ in $L^{d-}_{\mathcal{A}}$ of the vocabulary τ^* let $g(n,k,\bar{\psi},\eta)$ be the function which maps $n,k,\bar{\psi}$ and $\eta\in 2^n$ to $\Gamma^{n,k}_{\bar{\psi},\eta}(y_0,\ldots,y_n,x,\bar{z})$. Recall that $S=\{\alpha<\omega_1 : A_\alpha=\bigcup_{\beta<\alpha}A_\beta\}$ is stationary. Let $K\prec H_\theta$, where $\theta\geq(2^\omega)^+$ such that $M\subseteq H_\theta$, $|K|=\aleph_0$, $\{\mathcal{A},\omega_1,\tau^*,\mathcal{M},X,g\}\cup\omega_1\cup\tau\cup X\subseteq K$, and $\delta=K\cap\omega_1\in S$. Let $\mathcal{N}$ be the restriction of $\mathcal{M}$ to K, i.e. the universe N of $\mathcal{N}$ is $M\cap K$ and the constants, relations and functions of $\mathcal{M}$ are relativized to N.

As in the proof of Theorem 12, N is closed under the interpretations of function symbols of the vocabulary of $\mathcal{M}$.

Claim: If $\phi(\vec{x})$ is a τ^*-formula in $L^{d-}_{\mathcal{A}}$, then $\mathcal{M}\models\phi(\vec{c})\Rightarrow\mathcal{N}\models\phi(\vec{c})$.

We use induction on ϕ. In light of the proof of Theorem 12, we only need to consider the quantifier $Q_{\mathcal{A}}$. Suppose $\mathcal{M}$ satisfies (6) with $\bar{c}\in N^k$. Thus $\Omega(\mathcal{M}_{\bar{\psi}})\cap A_{\omega_1}\in\mathcal{A}$. Hence

$$K\models\text{“}\Omega(\mathcal{M}_{\bar{\psi}})\cap A_{\omega_1}\in\mathcal{A}\text{”}.$$

Note that since $\delta\in S$, $K\cap A_{\omega_1}=A_\delta$.
Case 1: $\Omega(\mathcal{M}_{\bar{\psi}})\cap A_{\omega_1}=A_\alpha$ for some $\alpha<\omega_1$. Then $\alpha<\delta$ and

$$K\models\text{“}\Omega(\mathcal{M}_{\bar{\psi}})\cap A_{\omega_1}=A_\alpha\text{”}.$$

We prove $\Omega(\mathcal{N}_{\bar{\psi}})\cap A_{\omega_1}=A_\alpha$, from which $\Omega(\mathcal{N}_{\bar{\psi}})\cap A_{\omega_1}\in\mathcal{A}$ follows.

Let first $\eta\in\Omega(\mathcal{N}_{\bar{\psi}})\cap A_{\omega_1}$. There are $a\in N$ and $b^n_0,\ldots,b^n_{n-1}$ in N such that $\mathcal{N}\models\Gamma^{n,k}_{\bar{\psi},\eta\restriction n}(b^n_0,\ldots,b^n_{n-1},a,\bar{c})$ for all $n<\omega$. Since the formulas of $\bar{\psi}$ are first order, $\mathcal{M}\models\Gamma^{n,k}_{\bar{\psi},\eta\restriction n}(b^n_0,\ldots,b^n_{n-1},a,\bar{c})$ for all $n<\omega$. Hence $\eta\in\Omega(\mathcal{M}_{\bar{\psi}})\cap A_{\omega_1}=A_\alpha$.

For the converse, let $\eta\in A_\alpha$. Note that now $\eta\in K$. By the choice of α, $\eta\in\Omega(\mathcal{M}_{\bar{\psi}})$. Hence there is $a\in M$ such that for some $b^n_0,\ldots,b^n_{n-1}$ in M we have $\mathcal{M}\models\Gamma^{n,k}_{\bar{\psi},\eta\restriction n}(b^n_0,\ldots,b^n_{n-1},a,\bar{c})$ for all $n<\omega$. Such an a and such $b^n_0,\ldots,b^n_{n-1}$ exist also in K, by elementarity, as $\eta\in K$. By Induction Hypothesis, $\mathcal{N}\models\Gamma^{n,k}_{\bar{\psi},\eta\restriction n}(b^n_0,\ldots,b^n_{n-1},a,\bar{c})$ for all $n<\omega$. Thus $\eta\in\Omega(\mathcal{N}_{\bar{\psi}})$.

Case 2: $\Omega(\mathcal{M}_{\bar{\psi}}) \cap A_{\omega_1} = A_{\omega_1}$. Then $K \models$ "$\Omega(\mathcal{M}_{\bar{\psi}}) \cap A_{\omega_1} = A_\delta$". We prove $\Omega(\mathcal{N}_{\bar{\psi}}) \cap A_{\omega_1} = A_\delta$, from which $\Omega(\mathcal{N}_{\bar{\psi}}) \cap A_{\omega_1} \in \mathcal{A}$ follows.

Let first $\eta \in \Omega(\mathcal{N}_{\bar{\psi}}) \cap A_{\omega_1}$. As in Case 1, $\eta \in \Omega(\mathcal{M}_{\bar{\psi}})$. Because we have (5), that is, η is determined by an element of N, we may conclude $\eta \in K$. By $K \prec H_\theta$, $\eta \in (A_{\omega_1})^K$. Hence $\eta \in A_\delta$.

For the converse, let $\eta \in A_\delta$. Since $\delta \in S$, $A_\delta \subset K$, and hence $\eta \in K$. On the other hand, $\eta \in \Omega(\mathcal{M}_{\bar{\psi}})$, since $A_\delta \subset A_{\omega_1} = \Omega(\mathcal{M}_{\bar{\psi}}) \cap A_{\omega_1}$. Now we can argue as in Case 1 to conclude $\eta \in \Omega(\mathcal{N}_{\bar{\psi}})$. □

A consequence of Corollary 20 and Theorem 22 is that the positive logic $L^{d^-}_{\mathcal{A}}$ is an extension of Σ^1_1 with both the Compactness Theorem and the Downward Löwenheim-Skolem Theorem. Similarly, $L^{d^-,\omega}_{\mathcal{A}}$ is such an extension of $\Sigma^1_{1,\delta}$.

Theorem 23 *There are positive logics L_1 and L_2 such that*

1. *L_1, L_2 both (properly) extend Σ^1_1.*
2. *L_1, L_2 both satisfy the Compactness Theorem and the Downward Löwenheim-Skolem Theorem.*
3. *There is no logic L_3 such that $L_1 \leq L_3$, $L_2 \leq L_3$, and L_3 satisfies the Downward Löwenheim-Skolem Theorem.*

We can replace Σ^1_1 by $\Sigma^1_{1,\delta}$.

Proof: Let $\mathcal{A}$ be as above but $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ for all limit α. Let $S, S' \subseteq \omega_1$ be disjoint stationary sets. Note that the set of elements of S that are limits of elements of S is stationary, because it contains the intersection of S with the closed unbounded set of limits of elements of S. Similarly, the set of elements of S' that are limits of elements of S' is stationary. Let $\mathcal{A} = \langle A_\alpha : \alpha \in S\rangle\hat{}\langle A_{\omega_1}\rangle$ and $\mathcal{A}' = \langle A_\alpha : \alpha \in S'\rangle\hat{}\langle A_{\omega_1}\rangle$. Now both $\{\alpha \in S : A_\alpha = \bigcup_{\beta\in\alpha\cap S} A_\beta\}$ and $\{\alpha \in S' : A_\alpha = \bigcup_{\beta\in\alpha\cap S'} A_\beta\}$ are stationary. Let

$$\psi_{\mathcal{A}} = (Q_{\mathcal{A}} x_0 x_1)(R_0(x_0, x_1), R_1(x_0, x_1), R_2(x_0, x_1), R_3(x_0), R_4(x_0))$$

and similarly $\psi_{\mathcal{A}'}$. Let ϕ be the sentence $\psi_{\mathcal{A}} \wedge \psi_{\mathcal{A}'} \wedge \Theta_{TL}$. This sentence has a model, namely $\mathcal{M}_{A_{\omega_1}}$. Suppose it has a countable model $\mathcal{N}$. Then $\Omega(\mathcal{N}) \cap A_{\omega_1} \in \mathcal{A} \cap \mathcal{A}'$. Hence $\Omega(\mathcal{N}) \cap A_{\omega_1} = A_{\omega_1}$. Since $\mathcal{N} \models \Theta_{TL}$, N must be uncountable, a contradiction. □

Corollary 24 *No extension of Σ^1_1 is strongest with respect to the Compactness Theorem and the Downward Löwenheim-Skolem Theorem, among positive logics.*

We shall now prove that Theorem 22 does not hold with $L^{d-}_{\mathcal{A}}$ replaced by $L^{d}_{\mathcal{A}}$.

Proposition 25 *Suppose $\mathcal{A}$ is as above. There is an uncountable model $\mathcal{M}$ for a countable vocabulary such that there is no countable $\mathcal{N} \preccurlyeq^{-}_{L^d_{\mathcal{A}}} \mathcal{M}$.*

Proof: Let M be the union of 2^ω, $2^{<\omega}$ and $2^\omega \times \omega_1$. The relations of the structure $\mathcal{M}$ are

1. $R_i^{\mathcal{M}} = \{a\hat{\ }\langle i\rangle : a \in 2^{<\omega}\}$ $(i = 0, 1)$.
2. $R_2^{\mathcal{M}} = \{(a, b) \in (2^{<\omega}) \times 2^\omega : a \lhd b\}$.
3. $R_3^{\mathcal{M}} = \{\emptyset\}$.
4. $R_4^{\mathcal{M}} = 2^\omega \cup 2^{<\omega}$.
5. $R_5^{\mathcal{M}} = \{(a, (a, \alpha)) : a \in 2^\omega, \alpha < \omega_1\}$.
6. $R_6^{\mathcal{M}} = 2^\omega$.
7. $R_7^{\mathcal{M}} = 2^{<\omega}$.
8. $Q_1^{\mathcal{M}} = 2^\omega \times \omega_1$.
9. $Q_2^{\mathcal{M}} = \{(a, \alpha) \in Q_1 : (a \in A_0 \wedge \alpha < \omega) \vee (a \in A_2 \setminus A_1 \wedge \alpha < \omega_1)\}$.
10. $Q_3^{\mathcal{M}} = \{(a, b) : \exists \alpha (a \in A_\alpha \wedge b \in A_{\omega_1} \setminus A_\alpha)\}$.

Suppose $\mathcal{N} \preccurlyeq^{-}_{L^d_{\mathcal{A}}} \mathcal{M}$ is countable. Let $\phi(x)$ be the existential second order formula

$$R_6(x) \wedge \exists F (F \text{ is a one-one function from } R_7 \text{ onto } \{y : Q_2(x, y)\}).$$

- $\mathcal{M} \models \phi(a)$ if and only if $a \in A_0^{\mathcal{M}}$.
- $\mathcal{N} \models \phi(a)$ if and only if $a \in A_0^{\mathcal{N}} \cup (A_2^{\mathcal{N}} \setminus A_1^{\mathcal{N}})$.

Thus

$$\mathcal{M} \models (Q_{\mathcal{A}} x_0 x_1)(\psi_0(x_0, x_1, \bar{c}), \psi_1(x_0, x_1, \bar{c}), \psi_2(x_0, x_1, \bar{c}), \psi_3(x_0, \bar{c}), \psi_4(x_0, \bar{c}))$$

but

$$\mathcal{N} \not\models (Q_{\mathcal{A}} x_0 x_1)(\psi_0(x_0, x_1, \bar{c}), \psi_1(x_0, x_1, \bar{c}), \psi_2(x_0, x_1, \bar{c}), \psi_3(x_0, \bar{c}), \psi_4(x_0, \bar{c}))$$

□

Despite the negative result of Theorem 25, Theorem 22 still holds for the fragment of $L^d_{\mathcal{A}}$ obtained by dropping existential second order quantifiers.

Definition 26 *Let $L^{d0}_{\mathcal{A}}$ be defined as $L^d_{\mathcal{A}}$ (Definition 19) except that existential second order quantification is not allowed. Let $L^{d1}_{\mathcal{A}}$ be defined as the extension of $L^d_{\mathcal{A}}$ by adding negation to the logical operations.*

Clearly, $L^{d0}_{\mathcal{A}}$ is a positive logic and it satisfies the Compactness Theorem because even $L^d_{\mathcal{A}}$ does. The logic $L^{d1}_{\mathcal{A}}$ is an abstract logic in the sense of [10]. Unlike our positive logics, it is closed under negation and also closed under substitution. Note that $L^{d0}_{\mathcal{A}} \leq L^{d1}_{\mathcal{A}}$.

Theorem 27 (Downward Löwenheim-Skolem-Tarski Theorem) *Suppose $\mathcal{M}$ is a model for a countable vocabulary and $X \subseteq M$ is countable. Then there is $\mathcal{N} \preccurlyeq_{L^{d1}_{\mathcal{A}}} \mathcal{M}$ such that $X \subseteq N$ and $|N| \leq \aleph_0$. In particular, $\mathcal{N} \preccurlyeq_{L^{d0}_{\mathcal{A}}} \mathcal{M}$.*

Proof: This is as in the proof of Theorem 22. We first expand $\mathcal{M}$ as follows: For every $L^{d1}_{\mathcal{A}}$-formula $\phi(\bar{z})$, where $\bar{z} = z_0, \ldots, z_{k-1}$, there is a predicate symbol R_ϕ of arity k such that $\mathcal{M} \models \forall \bar{z}(\phi(\bar{z}) \leftrightarrow R_\phi(\bar{z}))$. Let τ be the original vocabulary of $\mathcal{M}$ and τ^* the vocabulary of the expansion. For any atomic formulas $\bar{\psi}$ of the vocabulary τ^* let $g(n, k, \bar{\psi}, \eta)$ be the function which maps $n, k, \bar{\psi}$ and $\eta \in 2^n$ to $\Gamma^{n,k}_{\bar{\psi},\eta}(y_0, \ldots, y_n, x, \bar{z})$. Let $K \prec H_\theta$, where $\theta \geq (2^\omega)^+$ such that $M \subseteq H_\theta$, $|K| = \aleph_0$, $\{\mathcal{A}, \omega_1, \tau^*, \mathcal{M}, X, g\} \cup \omega_1 \cup \tau \cup X \subseteq K$, and $\delta = K \cap \omega_1 \in S$. Let $\mathcal{N}$ be the restriction of $\mathcal{M}$ to K, i.e. the universe N of $\mathcal{N}$ is $M \cap K$ and the constants, relations and functions of $\mathcal{M}$ are relativized to N.

As in the proof of Theorem 12, N is closed under the interpretations of function symbols of the vocabulary of $\mathcal{M}$.

Claim: If $\phi(\vec{x})$ is a τ^*-formula in $L^{d1}_{\mathcal{A}}$ and $\vec{c} \in N$, then $\mathcal{N} \models \phi(\vec{c}) \leftrightarrow R_\phi(\vec{c})$.

The proof of this claim is as in the proof of Theorem 22. Since $\mathcal{N} \subseteq \mathcal{M}$ in the vocabulary τ^*, the claim implies $\mathcal{N} \preccurlyeq_{L^{d1}_{\mathcal{A}}} \mathcal{M}$. $\square$

The logic $L^{d1}_{\mathcal{A}}$ is closed under negation and satisfies the Downward Löwenheim-Skolem Theorem. Thus it cannot satisfy the Compactness Theorem, although its sublogic $L^{d0}_{\mathcal{A}}$ does.

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