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ISOMORPHIC LIMIT ULTRAPOWERS FOR INFINITARY LOGIC

BY

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ABSTRACT

The logic $\mathbb{L}^1_\theta$ was introduced in [She12]; it is the maximal logic below $\mathbb{L}_{\theta,\theta}$ in which a well ordering is not definable. We investigate it for θ a compact cardinal. We prove that it satisfies several parallels of classical theorems on first order logic, strengthening the thesis that it is a natural logic. In particular, two models are $\mathbb{L}^1_\theta$-equivalent iff for some ω-sequence of θ-complete ultrafilters, the iterated ultrapowers by it of those two models are isomorphic.

Also for strong limit $\lambda > \theta$ of cofinality $\aleph_0$, every complete $\mathbb{L}^1_\theta$-theory has a so-called special model of cardinality λ, a parallel of saturated. For first order theory T and singular strong limit cardinal λ, T has a so-called special model of cardinality λ. Using "special" in our context is justified by: it is unique (fixing T and λ), all reducts of a special model are special too, so we have another proof of interpolation in this case.

* The author would like to thank the Israel Science Foundation for partial support of this research (Grant No. 1053/11). References like [She12, 2.11=La18] means we cite from [She12], Claim 2.11 which has label La18; this helps if [She12] will be revised.
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0. Introduction

0(A). Background and results. In the sixties, ultraproducts were very central in model theory. Recall Keisler [Kei61], solving the outstanding problem in model theory of the time, assuming an instance of GCH characterizes elementary equivalence in an algebraic way; that is by proving:

⊞ for any two models M_1, M_2 (of vocabulary τ of cardinality $\leq \lambda$ and) of cardinality[1] $\leq \lambda$, the following are equivalent provided that $2^\lambda = \lambda^+$:
  (a) M_1, M_2 are elementarily equivalent;
  (b) they have isomorphic ultrapowers, that is $M_1^\lambda/D_2 \cong M_2^\lambda/D_1$ for some ultrafilter D_ℓ on a cardinal λ;
  (c) $M^\mu/D \cong M^\mu/D$ for some ultrafilter D on some cardinal μ;
  (d) as in (c) for $\mu = \lambda$.

Kochen [Koc61] uses iteration on taking ultrapowers (on a well ordered index set) to characterize elementary equivalence. Gaifman [Gai74] uses ultrapowers on $\aleph_1$-complete ultrafilters iterated along a linear ordered index set. Keisler [Kei63] uses general $(\aleph_0, \aleph_0)$-l.u.p.; see below, Definition 0.13(4) for $\kappa = \aleph_0$. Shelah [She71] proves ⊞ in ZFC, but with a price: we have to omit clause (d), and the ultrafilter is on $\mu = 2^\lambda$.

Hodges–Shelah [HS81] is closer to the present work (see there for earlier works): it dealt with isomorphic ultrapowers (and isomorphic reduced powers) for the θ-complete ultrafilter (and filter) case, but note that having isomorphic ultrapowers by θ-complete ultrafilters is not an equivalence relation. In particular, assume $\theta > \aleph_0$ is a compact cardinal and little more (we can get it by forcing over a universe with a supercompact cardinal and a class of measurable cardinals). Then two models have isomorphic ultrapowers for some θ-complete ultrafilter iff in all relevant games the isomorphism player does not lose. Those relevant games are of length $\zeta < \theta$ and deal with the reducts to a sub-vocabulary of cardinality $< \theta$ and usually those games are not determined.

The characterization [HS81] of having isomorphic ultrapowers by θ-complete ultra-filters, is necessarily not so "nice" because this relation is not an equivalence relation. Hence having isomorphic ultrapowers is not equivalent to having the same theory in some logic.

[1] In fact "M_ℓ is of cardinality $\leq \lambda^+$" suffices.

Most relevant to the present paper is [She12] which we continue here. For notational simplicity let θ be an inaccessible cardinal. An old problem from the seventies was:

$\boxdot$ is there a logic between $\mathbb{L}_{\theta,\aleph_0}$ and $\mathbb{L}_{\lambda,\theta}$ which satisfies interpolation?

Generally, interpolation had posed a hard problem in soft model theory. Another, not so precise problem was to find generalizations of the Lindstrom theorem; see [Vї1]. Now [She12] solves the first problem and suggests a solution to the second problem, by putting forward the logic $\mathbb{L}^1_\theta$ introduced there. It was proved that it satisfies $\boxdot$ and give a characterization: e.g., it is a maximal logic in the interval mentioned in $\boxdot$ which satisfies non-definability of well order in a suitable sense (see [She12, 3.4=La28]).

Another line of research was investigating infinitary logics for θ a compact cardinal; see [She] and history there. We continue those two lines, investigating $\mathbb{L}^1_\theta$ for θ a compact cardinal. We prove that it is an interesting logic: it shares with first order logic several classical theorems.

We may wonder: do we have a characterization of models being $\mathbb{L}^1_\theta$-equivalent?

In §1 we characterize $\mathbb{L}^1_\theta$-equivalence of models by having isomorphic iterated ultrapowers of length ω. Then in §2 we prove some further generalizations of classical model theoretic theorems, like the existence and uniqueness of special models in λ when $\lambda > \theta + |T|$ is strong limit of cofinality $\aleph_0$. All this seems to strengthen the thesis of [She12] that $\mathbb{L}^1_\theta$ is a natural logic.

Of course, success drives us to consider further problems. For another approach see [She15].

Question 0.1: Assume θ is a strong limit singular cardinal of cofinality $\aleph_0$.

(1) Does the logic $\mathbb{L}_{\theta^+,\theta}$ restricted to models of cardinality θ have interpolation?
(2) Is there a logic $\mathscr{L}$ with interpolation such that: $\mathbb{L}_{\theta^+,\theta} \le \mathscr{L} \le \mathbb{L}_{\theta k,\theta^+}$.

Question 0.2: Let θ be a compact cardinal and $\lambda > \theta$ be a strong limit of cofinality $\aleph_0$.

(1) Does the logic $\mathbb{L}_{\theta,\theta}$ restricted to model of cardinality λ has interpolation?
(2) Can we characterize when a theory $T \subseteq \mathbb{L}^1_\theta$ of cardinality $< \theta$ is categorical in λ?
(2A) Can we then conclude that it is categorical in other such λ-s?
(3) Like parts (2), (2A) for $T \subseteq \mathbb{L}_{\theta,\theta}$?

0(B). Preliminaries.

Hypothesis 0.3: θ is in §1, §2 a compact uncountable cardinal (of course, we use only restricted versions of this).

Notation 0.4: (1) Let $\varphi(\bar{x})$ mean: φ is a formula of $\mathbb{L}_{\theta,\theta}, \bar{x}$ is a sequence of variables with no repetitions including the variables occurring freely in φ and $\ell g(\bar{x}) < \theta$ if not said otherwise. We use φ, ψ, ϑ to denote formulas and for a statement st let $\varphi^{\rm st}$ or $\varphi^{[\rm st]}$ or $\varphi^{\rm if(st)}$ mean φ if st is true or 1 and $\neg\varphi$ if st is false or 0.

(2) For a set u, usually of ordinals, let

$$\bar{x}_{[u]} = \langle x_\varepsilon : \varepsilon \in u \rangle;$$

now u may be an ordinal but, e.g., if $u = [\alpha, \beta)$ we may write $\bar{x}_{[\alpha,\beta)}$; similarly for $\bar{y}_{[u]}, \bar{z}_{[u]}$; let $\ell g(\bar{x}_{[u]}) = u$.

(3) τ denotes a vocabulary, i.e., a set of predicates and function symbols each with a finite number of places, in other words the arity arity$(\tau) = \aleph_0$; see 0.5 on this.

(4) T denotes a theory in $\mathbb{L}_{\theta,\theta}$ or $\mathbb{L}^1_\theta$ (see below), usually complete in the vocabulary τ_T and with a model of cardinality $\geq \theta$ if not said otherwise.

(5) Let Mod_T be the class of models of T.

(6) For a model M let its vocabulary be τ_M.

Remark 0.5: (1) What is the problem with predicates (and function symbols) with infinite arity? If $\langle M_\alpha : \alpha \leq \delta \rangle, \delta$ a limit ordinal is increasing, even if the universe of M_δ is the union of the universes of $M_\alpha, \alpha < \delta$, this does not determine M_δ.

(2) We can still define $\cup\{M_\alpha : \alpha < \delta\}$ by deciding

$$P^{M_\delta} = \cup\{M_\alpha : \alpha < \delta\}$$

for any predicate P and treating function similarly (so the function symbols are interpreted as partial functions) or better, deciding to use predicates only.

Now with care we can use arity$(\tau) \leq \theta$ and we sometimes remark on this.

Notation 0.6: Let ε, ζ, ξ denote ordinals $< \theta$.

Definition 0.7: (1) Let $\mathrm{uf}_\theta(I)$ be the set of θ-complete ultrafilters on I, non-principal if not said otherwise. Let $\mathrm{fil}_\theta(I)$ be the set of θ-complete filters on I; mainly we use (θ,θ)-regular ones (see below).

(2) $D \in \mathrm{fil}_\theta(I)$ is called (λ,θ)-regular when there is a witness

$$\bar{w} = \langle w_t : t \in I \rangle$$

which means: $w_t \in [\lambda]^{<\theta}$ for $t \in I$ and $\alpha < \lambda \Rightarrow \{t : \alpha \in w_t\} \in D$.

(3) Let $\mathrm{ruf}_{\lambda,\theta}(I)$ be the set of (λ,θ)-regular $D \in \mathrm{uf}_\theta(I)$; let $\mathrm{rfil}_{\lambda,\theta}(I)$ be the set of (λ,θ)-regular $D \in \mathrm{fil}_\theta(I)$; when $\lambda = |I|$ we may omit λ.

Definition 0.8: (1) $\mathbb{L}_{\theta,\theta}(\tau)$ is the set of formulas of $\mathbb{L}_{\theta,\theta}$ in the vocabulary τ.

(2) For τ-models M, N let $M \prec_{\mathbb{L}_{\theta,\theta}} N$ mean: if $\varphi(\bar{x}) \in \mathbb{L}_{\theta,\theta}(\tau_M)$ and $\bar{a} \in {}^{\ell g(\bar{x})}M$ then

$$M \models \varphi[\bar{a}] \Leftrightarrow N \models \varphi[\bar{a}].$$

And, of course

Fact 0.9: For a complete $T \subseteq \mathbb{L}_{\theta,\theta}(\tau)$: $(\mathrm{Mod}_T, \prec_{\mathbb{L}_{\theta,\theta}})$ has amalgamation and the joint embedding property (JEP), that is:

(a) amalgamation: if $M_0 \prec_{\mathbb{L}_{\theta,\theta}} M_\ell$ for $\ell = 1, 2$, then there are M_3, f_1, f_2, M_1', M_2' such that
- $M_0 \prec_{\mathbb{L}_{\theta,\theta}} M_3$,
- for $\ell = 1, 2$, f_ℓ is a $\prec_{\mathbb{L}_{\theta,\theta}}$-embedding of M_ℓ into M_3 over M_0, that is, for some τ_T-models M_ℓ' for $\ell = 1, 2$ we have $M_\ell' \prec_{\mathbb{L}_{\theta,\theta}} M_3$ and f_ℓ is an isomorphism from M_ℓ onto M_ℓ' over M_0;

(b) JEP: if M_1, M_2 are $\mathbb{L}_{\theta,\theta}$-equivalent τ-models then there is a τ-model M_3 and $\prec_{\mathbb{L}_{\theta,\theta}}$-embedding f_ℓ of M_ℓ into M_3 for $\ell = 1, 2$.

The well known generalization of the Łos theorem is:

THEOREM 0.10: (1) *If $\varphi(\bar{x}_{[\zeta]}) \in \mathbb{L}_{\theta,\theta}(\tau), D \in \mathrm{uf}_\theta(I)$ and M_s is a τ-model for $s \in I$ and $f_\varepsilon \in \prod_{s\in I} M_s$ for $\varepsilon < \zeta$ then $M \models \varphi[\ldots, f_\varepsilon/D, \ldots]_{\varepsilon<\zeta}$ iff the set*

$$\{s \in I : M_s \models \varphi[\ldots, f_\varepsilon(s), \ldots]_{\varepsilon<\zeta}\}$$

belongs to D.

(2) *Similarly $M \prec_{\mathbb{L}_{\theta,\theta}} M^I/D$.*

Definition 0.11: (0) We say X respects E when for some set I, E is an equivalence relation[2] on I and $X \subseteq I$ and $sEt \Rightarrow (s \in X \Leftrightarrow t \in X)$.

(1) We say $\mathbf{x} = (I, D, \mathscr{E})$ is a (κ, σ)-l.u.f.t.(limit-ultra-filter-iteration triple) when:
   (a) D is a filter on the set I,
   (b) $\mathscr{E}$ is a family of equivalence relations on I,
   (c) $(\mathscr{E}, \supseteq)$ is σ-directed, i.e., if $\alpha(*) < \sigma$ and $E_i \in \mathscr{E}$ for $i < \alpha(*)$, then there is $E \in \mathscr{E}$ refining E_i for every $i < \alpha(*)$
   (d) if $E \in \mathscr{E}$, then D/E is a κ-complete ultrafilter on I/E where $D/E := \{X/E : X \in D \text{ and } X \text{ respects } E\}$.

(1A) Let $\mathbf{x}$ be a (κ, θ)-l.f.t.mean that above we weaken (d) to
   (d)' if $E \in \mathscr{E}$ then D/E is a κ-complete filter.

(2) Omitting "(κ, σ)" means $(\theta, \aleph_0)$, recalling θ is our fixed compact cardinal.

(3) Let $(I_1, D_1, \mathscr{E}_1) \leq^1_h (I_2, D_2, \mathscr{E}_2)$ mean that:
   (a) h is a function from I_2 onto I_1,
   (b) if $E \in \mathscr{E}_1$ then $h^{-1} \circ E \in \mathscr{E}_2$ where

$$h^{-1} \circ E = \{(s, t) : s, t \in I_2 \text{ and } h(s)Eh(t)\},$$

   (c) if $E_1 \in \mathscr{E}_1$ and $E_2 = h^{-1} \circ E_1$ then $D_1/E_1 = h''(D_2/E_2)$.

Remark 0.12: Note that in Definition 0.11(3), if $h = \mathrm{id}_{I_2}$ then $I_1 = I_2$.

Definition 0.13: Assume $\mathbf{x} = (I, D, \mathscr{E})$ is a (κ, σ)-l.u.f.t.

(1) For a function f let $\mathrm{eq}(f) = \{(s_1, s_2) : f(s_1) = f(s_2)\}$. If $\bar{f} = \langle f_i : i < i_* \rangle$ and $i < i_* \Rightarrow \mathrm{dom}(f_i) = I$ then $\mathrm{eq}(\bar{f}) = \cap\{\mathrm{eq}(f_i) : i < i_*\}$.

(2) For a set U let $U^I|\mathscr{E} = \{f \in {}^I U : \mathrm{eq}(f) \text{ is refined by some } E \in \mathscr{E}\}$.

(3) For a model M let

$$\text{l.r.p.}_{\mathbf{x}}(M) = M^I_D|\mathscr{E} = (M^I/D)\restriction\{f/D : f \in {}^I M \text{ and } \mathrm{eq}(f) \text{ is refined by some } E \in \mathscr{E}\},$$

pedantically (as arity(τ_M) may be $> \aleph_0$), $M^I_D|\mathscr{E} = \bigcup\{M^I_D\restriction E : E \in \mathscr{E}\}$; l.r.p. stands for limit reduced power.

(4) If $\mathbf{x}$ is l.u.f.t. we may in part (3) write l.u.p.$_{\mathbf{x}}(M)$.

We now give the generalization of Keisler [Kei63]; Hodges–Shelah [HS81, Lemma 1, p. 80] in the case $\kappa = \sigma$.

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[2] Here, in the interesting cases, the number of equivalence classes of E is infinite, and even $\geq \theta$, pedantically not bounded by any $\theta_* < \theta$.

THEOREM 0.14: (1) *If* $\sigma \leq \kappa$ *and* $(I, D, \mathscr{E})$ *is* (κ, σ)-l.u.f.t.,

$$\varphi = \varphi(\bar{x}_{[\zeta]}) \in \mathbb{L}_{\kappa,\sigma}(\tau)$$

so $\zeta < \sigma, f_\varepsilon \in M^I|\mathscr{E}$ *for* $\varepsilon < \zeta$, *then* $M^I_D|\mathscr{E} \models \varphi[\dots, f_\varepsilon/D, \dots]$ *iff* $\{s \in I : M \models \varphi[\dots, f_\varepsilon(s), \dots]_{\varepsilon<\zeta}\} \in D$.

(2) *Moreover* $M \prec_{\mathbb{L}_{\kappa,\sigma}} M^I_D/\mathscr{E}$, *pedantically* $\mathbf{j} = \mathbf{j}_{M,\mathbf{x}}$ *is a* $\prec_{\mathbb{L}_{\kappa,\sigma}}$*-elementary embedding of* M *into* $M^I_D/\mathscr{E}$ *where* $\mathbf{j}(a) = \langle a : s \in I\rangle/D$.

(3) *We define* $(\prod_{s\in I} M_s)^I_D|\mathscr{E}$ *similarly when* $\mathrm{eq}(\langle M_s : s \in I\rangle)$ *is refined by some* $E \in \mathscr{E}$*; we may use this more at the end of the proof of Claim 1.2.*

CONVENTION 0.15: *Abusing a notation;*

(1) *in* $\prod_{s\in I} M_s/D$ *we allow* f/D *for* $f \in \prod_{s\in S} M_s$ *when* $S \in D$.

(2) *For* $\bar{c} \in {}^{\gamma}(\prod_{s\in I} M_s/D)$ *we can find* $\langle \bar{c}_s : s \in I\rangle$ *such that* $\bar{c}_s \in {}^{\gamma}(M_s)$ *and* $\bar{c} = \langle \bar{c}_s : s \in I\rangle/D$, *which means: if* $i < \ell g(\bar{c})$ *then* $c_{s,i} \in M_s$ *and* $c_i = \langle c_{s,i} : s \in I\rangle/D$.

Remark 0.16: (1) Why the "pedantically" in Definition 0.13(3)? Otherwise if $\mathbf{x}$ is a $(\theta, \sigma) -$ l.u.f.t., $(\mathscr{E}_{\mathbf{x}}, \supseteq)$ is not κ^+-directed, $\kappa <$ arity(τ), then defining l.u.p.$_{\mathbf{x}}(M)$, we have freedom: if $R \in \tau$, arity$_\tau(R) \geq \kappa$, i.e., on

$$R^N {\restriction} \{\bar{a} : \bar{a} \in {}^{\mathrm{arity}(P)}N \text{ and no } E \in \mathscr{E} \text{ refines } \mathrm{eq}(\bar{a})\}$$

so we have no restrictions.

(2) So, e.g., for categoricity we better restrict ourselves to vocabularies τ such that arity$(\tau) = \aleph_0$.

Definition 0.17: We say M is a θ-complete model when for every $\varepsilon < \theta$, $R_* \subseteq {}^{\varepsilon}M$ and $F_* : {}^{\varepsilon}M \to M$ there are $R, F \in \tau_M$ such that $R^M = R_* \wedge F^M = F_*$.

OBSERVATION 0.18: (1) *If* M *is a* τ*-model of cardinality* λ *then there is a* θ*-complete expansion* M^+ *of* M *so* $\tau(M^+) \supseteq \tau(M)$ *and* $\tau(M^+)$ *has cardinality* $|\tau_M| + 2^{(\|M\|^{<\theta})}$.

(2) *For models* $M \prec_{\mathbb{L}_{\theta,\theta}} N$ *and* M^+ *as above the following conditions are equivalent:*

(a) $N =$ l.u.p.$_{\mathbf{x}}(M)$ *identifying* $a \in M$ *with* $\mathbf{j}_{\mathbf{x}}(a) \in N$, *for some* (θ, θ)-l.u.f.t.$\mathbf{x}$

(b) *there is* N^+ *such that* $M^+ \prec_{\mathbb{L}_{\theta,\theta}} N^+$ *and* $N^+{\restriction}\tau_M$ *is isomorphic to* N *over* M, *in fact we can add* $N^+{\restriction}\tau_M = N$.

(3) *[θ is a compact cardinal] For a model M, if $(P^M, <^M)$ is a θ-directed partial order and $\chi = \mathrm{cf}(\chi) \geq \theta$ and $\lambda = \lambda^{\|M\|} + \chi$ then for some (θ,θ)-*l.u.f.t.$\mathbf{x}$, *the model N:=*l.u.p.$_{\mathbf{x}}(M)$ *satisfies $(P^N, <^N)$ has a cofinal increasing sequence of length χ and $|P^N| = \lambda$.*

Proof. Easy, for example:

(3) Let M^+ be as in part (1). Note that M^+ has Skolem functions and let T' be the following set of formulas:

$$\begin{aligned}\mathrm{Th}_{\mathbb{L}_{\theta,\theta}}(M^+) &\cup \{P(x_\varepsilon) : \varepsilon < \lambda \cdot \chi\} \\ &\cup \{P(\sigma(x_{\varepsilon_0}, \ldots, x_{\varepsilon_i}, \ldots)_{i<i(*)}) \to \sigma(x_{\varepsilon_0}, \ldots, x_{\varepsilon_i}, \ldots)_{i<i(*)} < x_\varepsilon : \\ &\qquad \sigma \text{ is a } \tau(M^+)\text{-term so } i(*) < \theta \text{ and } i < i(*) \Rightarrow \varepsilon_i < \varepsilon < \lambda \cdot \chi\}.\end{aligned}$$

Clearly

(*) T' is $(< \theta)$-satisfiable in M^+.

[Why? Because if $T'' \subseteq T'$ has cardinality $< \theta$ then the set

$$u = \{\varepsilon < \lambda \cdot \chi : x_\varepsilon \text{ appears in } T''\}$$

has cardinality $< \theta$ and let $i(*) = \mathrm{otp}(u)$; clearly for each $\varepsilon \in u$ the set

$$\Gamma_\varepsilon = T' \cap \{P(\sigma(x_{\varepsilon_0}, \ldots)) \to \sigma(x_{\varepsilon_0}, \ldots, x_{\varepsilon_i}, \ldots)_{i<i(*)} < x_\varepsilon : i(*) < \theta \text{ and } \varepsilon_i < \varepsilon \text{ for } i < i(*)\}$$

has cardinality $< \theta$. Now we choose $c_\varepsilon \in M$ by induction on $\varepsilon \in u$ such that the assignment

$$x_\zeta \mapsto c_\zeta$$

for $\zeta \in \varepsilon \cap u$ in M^+ satisfies Γ_ε, possible because $|\Gamma_\varepsilon| < \theta$, $|u_\varepsilon| < \theta$ and $(P^M, <^M)$ is θ-directed. So the M^+ with the assignment $x_\varepsilon \mapsto c_\varepsilon$ for $\varepsilon \in u$ is a model of T'', so T' is $(< \theta)$-satisfiable indeed.]

Recalling that $|M| = \{c^{M^+} : c \in \tau(M^+) \text{ an individual constant}\}$, T' is realized in some $\prec_{\mathbb{L}_{\theta,\theta}}$-elementary extension N^+ of M^+ by the assignment

$$x_\varepsilon \mapsto a_\varepsilon (\varepsilon < \lambda \cdot \chi).$$

Without loss of generality N^+ is the Skolem hull of $\{a_\varepsilon : \varepsilon < \lambda \cdot \chi\}$, so $N := N^+ {\restriction} \tau(M)$ is as required by the choice of T'. Now $\mathbf{x}$ is as required and exists by part (2) of the claim. $\blacksquare_{0.18}$

Observation 0.19: (1) *If* $\mathbf{x}$ *is a non-trivial* (θ, θ)-l.u.f.t.*and* $\chi = \text{cf}(\text{l.u.p.}(\theta <))$ *then* $\chi = \chi^{<\theta}$.

(2) *Also* $\mu = \mu^{<\theta}$ *when* μ *is the cardinality of* l.u.p.$(\theta, <)$.

Proof. (1) By the choice of $\mathbf{x}$ clearly $\chi \geq \theta$. As χ is regular $\geq \theta$ by a theorem of Solovay [Sol74] we have $\chi^{<\theta} = \chi$.

(2) See the proof of [She, 2.20(3)=La27(3)]. $\blacksquare_{0.19}$

We now quote [She12, Def.2.1+La8]

Definition 0.20: For a vocabulary τ, τ-models M_1, M_2, a set Γ of formulas in the vocabulary τ in any logic (each with finitely many free variables if not said otherwise; see [She, 2.9=La10(4)]), cardinal θ and ordinal α, we define a game $\eth = \eth_{\Gamma,\theta,\alpha}[M_1, M_2]$ as follows, and using $(M_1, \bar{b}_1), (M_2, \bar{b}_2)$ with their natural meaning when $\text{Dom}(\bar{b}_1) = \text{Dom}(\bar{b}_2)$:

- (A) The moves are indexed by $n < \omega$ (but every actual play is finite), just before the n-th move we have a state $\mathbf{s}_n = (A^1_n, A^2_n, h^1_n, h^2_n, g_n, \beta_n, n)$,
- (B) $\mathbf{s} = (A^1, A^2, h^1, h^2, g, \beta, n) = (A^1_{\mathbf{s}}, A^2_{\mathbf{s}}, h^1_{\mathbf{s}}, h^2_{\mathbf{s}}, g_{\mathbf{s}}, \beta_{\mathbf{s}}, n_{\mathbf{s}})$ is a state (or n-state or (θ, n)-state or $(\theta, < \omega)$-state) when:
  - (a) $A^\ell \in [M_\ell]^{\leq \theta}$ for $\ell = 1, 2$,
  - (b) $\beta \leq \alpha$ is an ordinal,
  - (c) h^ℓ is a function from A^ℓ into ω,
  - (d) g is a partial one-to-one function from M_1 to M_2 and let
  $$g^1_{\mathbf{s}} = g^1 = g_{\mathbf{s}} = g \quad \text{and} \quad g^2_{\mathbf{s}} = g^2 = (g^1_{\mathbf{s}})^{-1},$$
  - (e) $\text{Dom}(g^\ell) \subseteq A^\ell$ for $\ell = 1, 2$,
  - (f) g preserves satisfaction of the formulas in Γ and their negations, i.e., for $\varphi(\bar{x}) \in \Gamma$ and $\bar{a} \in {}^{\ell g(\bar{x})}\text{Dom}(g)$ we have
  $$M_1 \models \varphi[\bar{a}] \Leftrightarrow M_2 \models \varphi[g(\bar{a})],$$
  - (g) if $a \in \text{Dom}(g^\ell)$ then $h^\ell(a) < n$,
- (C) we define the state $\mathbf{s} = \mathbf{s}_0 = \mathbf{s}^0_\alpha$ by letting $n_{\mathbf{s}} = 0$, $A^1_{\mathbf{s}} = \emptyset = A^2_{\mathbf{s}}$, $\beta_{\mathbf{s}} = \alpha$, $h^1_{\mathbf{s}} = \emptyset = h^2_{\mathbf{s}}$, $g_{\mathbf{s}} = \emptyset$; so really $\mathbf{s}$ depends only on α (but in general, this may not be a state for our game as possibly for some sentence $\psi \in \Gamma$ we have $M_1 \models \psi \Leftrightarrow M_2 \models \neg\psi$),
- (D) we say that a state $\mathbf{t}$ extends a state $\mathbf{s}$ when $A^\ell_{\mathbf{s}} \subseteq A^\ell_{\mathbf{t}}, h^\ell_{\mathbf{s}} \subseteq h^\ell_{\mathbf{t}}$ for $\ell = 1, 2$ and $g_{\mathbf{s}} \subseteq g_{\mathbf{t}}, \beta_{\mathbf{s}} > \beta_{\mathbf{t}}, n_{\mathbf{s}} < n_{\mathbf{t}}$; we say $\mathbf{t}$ is a successor of $\mathbf{s}$ if, in addition, $n_{\mathbf{t}} = n_{\mathbf{s}} + 1$,

(E) in the n-th move the anti-isomorphism player (AIS) chooses the triple $(\beta_{n+1}, \iota_n, A'_n)$ such that:
- $\iota_n \in \{1,2\}, \beta_{n+1} < \beta_n$ and $A^{\iota_n}_n \subseteq A'_n \in [M_{\iota_n}]^{\leq\theta}$,

the isomorphism player (ISO) chooses a state $\mathbf{s}_{n+1}$ such that:
- $\mathbf{s}_{n+1}$ is a successor of $\mathbf{s}_n$,
- $A^{\iota_n}_{\mathbf{s}_{n+1}} = A'_n$,
- $A^{3-\iota_n}_{\mathbf{s}_{n+1}} = A^{3-\iota_n}_{\mathbf{s}_n} \cup \mathrm{Dom}(g^{3-\iota_n}_{\mathbf{s}_{n+1}})$,
- if $a \in A'_n \backslash A^{\iota_n}_{\mathbf{s}_n}$ then $h^{\iota_n}_{\mathbf{s}_{n+1}}(a) \geq n+1$,
- $\mathrm{Dom}(g^{\iota_n}_{\mathbf{s}_{n+1}}) = \{a \in A^{\iota_n}_{\mathbf{s}_n} : h^{\iota_n}_{\mathbf{s}_n}(a) < n+1\}$ so it includes $\mathrm{Dom}(g^{\iota_n}_{\mathbf{s}_n})$,
- $\beta_{\mathbf{s}_{n+1}} = \beta_{n+1}.$,

(F)
- the play ends when one of the players has no legal moves (always occurs as $\beta_n < \beta_{n-1}$) and then this player loses; this may occur for $n = 0$,
- for $\alpha = 0$ we stipulate that ISO wins iff $\mathbf{s}^0_\alpha$ is a state.

Definition 0.21: (1) Let $\mathscr{E}^{0,\tau}_{\Gamma,\theta,\alpha}$ be the class $\{(M_1, M_2) : M_1, M_2$ are τ-models and in the game $\partial_{\Gamma,\theta,\alpha}[M_1, M_2]$ the ISO player has a winning strategy$\}$ where Γ is a set of formulas in the vocabulary τ, each with finitely many free variables.

(2) $\mathscr{E}^{1,\tau}_{\Gamma,\theta,\alpha}$ is the closure of $\mathscr{E}^{0,\tau}_{\Gamma,\theta,\alpha}$ to an equivalence relation (on the class of τ-models).

(3) Above, we may replace Γ by $\mathrm{qf}(\tau)$, which means Γ = the set $\mathrm{at}(\tau)$ of atomic formulas or $\mathrm{bs}(\tau)$ of basic formulas in the vocabulary τ.

(4) Above, if we omit τ we mean $\tau = \tau_\Gamma$ and if we omit Γ we mean $\mathrm{bs}(\tau)$. Abusing notation we may say M_1, M_2 are $\mathscr{E}^{0,\tau}_{\Gamma,\theta,\alpha}$-equivalent.

The following Definition 0.22 is closely related to the beginning of §1; it quotes [She12, Def. 2.5=La13] .

Definition 0.22: (1) For a vocabulary τ, the τ-models M_1, M_2 are $\mathbb{L}^1_{<\theta}$-equivalent iff for every $\mu < \theta$ and $\alpha < \mu^+$ and $\tau_1 \subseteq \tau$ of cardinality $\leq \mu$, letting Γ = the quantifier free formulas in $\mathbb{L}(\tau)$, the models M_1, M_2 are $\mathscr{E}^{1,\tau_1}_{\Gamma,\mu,\alpha}$.

(2) The logic $\mathbb{L}_{\lambda,\kappa}$ is defined like first order logic but we allow conjunctions on sets of $< \lambda$ formulas and we allow quantification of the form $\forall \bar{x}$ for sequences $\bar{x}$ of length $< \kappa$; however each formula has to have $< \kappa$ free

variables, and disjunctions and existential quantifications are defined naturally.

(2A) We define $\mathbb{L}_{<\lambda,<\kappa}$ as $\cup\{\mathbb{L}_{\lambda_1,\kappa_1} : \lambda_1 < \lambda, \kappa_1 < \kappa\}$; we may replace $< \lambda^+$ by λ and $< \kappa^+$ by κ.

(3) The logic $\mathbb{L}^1_{\leq\theta}$ is defined as follows: a sentence $\psi \in \mathbb{L}_{\leq\theta}(\tau)$ iff the sentence is defined using (or by) a triple $(\mathrm{qf}(\tau_1), \theta, \alpha)$ which means: τ_1 is a sub-vocabulary of τ of cardinality $\leq \theta$ and $\alpha < \theta^+$, and for some sequence $\langle M_\beta : \beta < \beta(*)\rangle$ of τ_1-models of length $\beta(*) \leq \beth_{\alpha+1}(\theta)$ we have: $M \models \psi$ iff M is $\mathscr{E}^1_{\mathrm{qf}(\tau_1),\theta,\alpha}$-equivalent to M_α for some $\beta < \beta(*)$.

(4) Let $\mathbb{L}^1_\kappa = \bigcup\{\mathbb{L}^1_{\leq\theta} : \theta < \kappa\}$ so $\mathbb{L}^1_{\theta^+} = \mathbb{L}^1_{\leq\theta}$.

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1. Characterizing equivalence by ω-limit ultrapowers

In [She12], a logic $\mathbb{L}^1_{<\kappa} = \bigcup_{\mu<\kappa} \mathbb{L}^1_{\leq\mu}$ is introduced (here we consider κ is strongly inaccessible for transparency), and is proved to be stronger than $\mathbb{L}_{\kappa,\aleph_0}$ but weaker than $\mathbb{L}_{\kappa,\kappa}$, has interpolation and a characterization, well ordering not definable in it and has an addition theorem. Also it is the maximal logic with some such properties.

For $\kappa = \theta$, we give a characterization of when two models are $\mathbb{L}^1_{<\theta}$-equivalent giving additional evidence for the logic's naturality.

Convention 1.1: *In this section every vocabulary τ has* $\mathrm{arity}(\tau) = \aleph_0$.

Recall [She12, 2.11=La18] which says (we expand it):

Claim 1.2: (1) *We have $M_n \equiv_{\mathbb{L}^1_{\leq\theta}} M_\omega$ for $n < \omega$ when clauses (b), (c) below hold and moreover $M_n \models \psi[\bar{a}] \Leftrightarrow M_\omega \models \psi[\bar{a}]$ when clauses (a)–(e) below hold, where:*

(a) *$\psi(\bar{z}) \in \mathbb{L}^1_{\leq\theta}(\tau)$ a formula,*

(b) *$M_n \prec_{\mathbb{L}_{<\partial,\theta^+}} M_{n+1}$ where $\partial = \beth_{\theta^+}$, recalling Definition 0.22(2A),*

(c) *$M_\omega := \bigcup_{n<\omega} M_n$,*

(d) *$\bar{a} \in {}^{\ell g(\bar{z})}(M_0)$,*

(e) *$\tau = \tau(M_n)$ for $n < \omega$.*

(2) *Assume $|\tau| \leq \mu, M_n$ is a τ-model and $M_n \prec_{\mathbb{L}_{\mu^+,\mu^+}} M_{n+1}$ for $n < \omega$ and $M_\omega = \bigcup\{M_n : n < \omega\}$. Then M_0, M_ω are $\mathbb{L}^1_{\leq\mu}$-equivalent.*

We need two definitions before stating and proving the theorem below. The first definition generalizes common concepts.

Definition 1.3: We say that a pair of models (M_1, M_2) has isomorphic θ-complete ω-iterated ultrapowers iff one can find $D_n \in \mathrm{uf}_\theta(I_n)$ for every $n \in \omega$ such that $M^1_\omega \cong M^2_\omega$, when

$$M^\ell_\omega = \bigcup\{M^\ell_k : k \in \omega\}, \quad M^\ell_0 = M_\ell$$

and

$$M^\ell_n \prec_{\mathbb{L}_{\theta,\theta}} (M^\ell_n)^{I_n}/D_n = M^\ell_{n+1}$$

for $\ell = 1, 2$ and $n < \omega$.

For the second definition, let $\mathbf{x}$ be a l.u.f.t. and in Definition 1.4 below we define "niceness witness". How do we arrive at this definition? If we try to analyze how to prove that two $\mathbb{L}^1_\theta$-equivalent models have isomorphic θ-complete ω-iterated ultrapowers by a sequence of length ω of approximations, it is natural to carry the induction step. The reader may return to this after reading the proof of (a)→(e) of Theorem 1.5.

To understand this (and the proof of Theorem 1.5) the reader may consider the case $\theta = \aleph_0$, which naturally is simpler and tells us that for each coordinate $s \in I$ we play a game of an Ehrenfuecht–Fraïssé game. Note also that Claim 1.2 clarifies why having $\mathrm{arity}(\tau) = \aleph_0$ helps.

Definition 1.4: If $\mathbf{x} = (I, D, \bar{E})$ is an l.u.f.t. and $\bar{E} = \langle E_n : n \in \omega \rangle$ then $\bar{w}$ is a niceness witness for $(I, D, \bar{E})$ when:

(a) $\bar{w} = \langle w_{s,n}, \gamma_{s,n} : s \in I, n < \omega \rangle$,
(b) $w_{s,n} \subseteq \lambda_n$ and $|w_{s,n}| < \theta$ and $|w_{s,n}| \geq |w_{s,n+1}|$,
(c) $\gamma_{s,n} < \theta$ and $(\gamma_{s,n} > \gamma_{s,n+1}) \vee (\gamma_{s,n+1} = 0)$,
(d) $\gamma_{s,n} = 0 \Rightarrow w_{s,n} = \emptyset$ but $w_{s,0} \neq \emptyset$ and for simplicity $w_{s,0}$ is infinite for every $s \in I$,
(e) if $n < \omega, u \in [\lambda_n]^{<\theta}$ then $\{s \in I : u \subseteq w_{s,n}\} \in D$,
(f) $w_{s,n} = w_{t,n}$ and $\gamma_{s,n} = \gamma_{t,n}$ when sE_nt.

Theorem 1.5: *Let θ be a compact cardinal and M_1, M_2 be two τ-models (and* $\mathrm{arity}(\tau) = \aleph_0$*).*

The following conditions are equivalent:

(a) *M_1, M_2 are $\mathbb{L}^1_\theta$-equivalent,*
(b) *there are (θ,θ)-l.u.f.t.$\mathbf{x}_n = (I, D, \mathscr{E}_n)$ and $\mathscr{E}_n \subseteq \mathscr{E}_{n+1}$ for $n < \omega$ and we let $\mathscr{E} = \bigcup\{\mathscr{E}_n : n < \omega\}$ such that $(M_1)^I_D|\mathscr{E}$ is isomorphic to $(M_2)^I_D|\mathscr{E}$,*

(c) (M_1, M_2) *have isomorphic* θ*-complete* ω*-iterated ultrapowers (see Definition 1.3),*

(d) *if* $D_n \in \mathrm{ruf}_{\lambda_n,\theta}(I_n)$ *so* $|I_n| \geq \lambda_n$ *and* $\lambda_{n+1} \geq 2^{|I_n|}$, $\lambda_n > \|M_1\| + \|M_2\| + |\tau|$ *for every* n *then the sequence* $\langle (I_n, D_n) : n < \omega \rangle$ *is as required in clause (c),*

(e) *if* $\mathbf{x} = (I, D, \mathscr{E})$ *is a* l.u.f.t.*(see Definition 0.11(1)),* $\mathscr{E} = \{E_n : n < \omega\}$, *for* $n < \omega$ *we have* E_{n+1} *refines* E_n, $2^{|I/E_n|} \leq \lambda_{n+1}$, D/E_n *is a* (λ_n, θ)*-regular* θ*-complete ultrafilter,* $\lambda_0 \geq \|M_1\| + \|M_2\| + |\tau|$, $\bar{w}$ *is a niceness witness (see Definition 1.4), then* $\mathrm{l.u.p.}_{\mathbf{x}}(M_1) \cong \mathrm{l.u.p.}_{\mathbf{x}}(M_2)$ *(see Definition 0.13(3)).*

Proof. Clause (b)⇒Clause (a):

So let $I, D, \mathscr{E}_n (n < \omega)$ be as in clause (b) and $\mathscr{E} = \bigcup\{\mathscr{E}_n : n < \omega\}$. By the transitivity of being $\mathbb{L}^1_{<\theta}$-equivalent, clearly clause (a) follows from:

$\boxplus_1$ for every model N the models $N, N^I_D|\mathscr{E}$ are $\mathbb{L}^1_\theta$-equivalent.

[Why does $\boxplus_1$ hold? Let $N_n = N^I_D|\mathscr{E}_n$ for $n < \omega$ and $N_\omega = \bigcup\{N_n : n < \omega\}$. So by Theorem 0.14 we have $N \equiv_{\mathbb{L}_{\theta,\theta}} N_0$ and moreover $N_n \prec_{\mathbb{L}_{\theta,\theta}} N_{n+1}$. Hence by Claim 1.2, that is the "Crucial Claim" 1.2 quoting [She12, 2.11=a18], we have $N_n \equiv_{\mathbb{L}^1_{<\theta}} N_\omega$ hence $N \equiv_{\mathbb{L}^1_{<\theta}} N_\omega$.]

Clause (c)⇒Clause (b):

Let

$$I = \prod_{n<\omega} I_n,$$

$$E_n = \{(\eta, \nu) : \eta, \nu \in I \text{ and } \eta{\restriction}n = \nu{\restriction}n\}$$

and

$$D = \Big\{X \subseteq I : \text{for some } n, (\forall^{D_n} i_n \in I_n)(\forall^{D_{n-1}} i_{n-1} \in I_{n-1}) \cdots (\forall^{D_0} i_0 \in I_0)(\forall \eta) \Big[\eta \in I \wedge \bigwedge_{\ell \leq n} \eta(\ell) = i_\ell \rightarrow \eta \in X\Big]\Big\}.$$

Now let $M^\ell_\omega \equiv (M_\ell)^I_D|\{E_n : n < \omega\}$.

Now it should be clear that $(M_\ell)^I_D|\{E_n : n < \omega\}$ is isomorphic to M^ℓ_ω for $\ell = 1, 2$, so recalling $M^1_\omega \cong M^2_\omega$ by the present assumption, the models $(M_\ell)^I_D|\{E_n : n < \omega\}$ for $\ell = 1, 2$ are isomorphic, so letting $\mathscr{E}_n = \{E_0, \ldots, E_n\}$ we easily see that $(I, D, \mathscr{E}_n)_{n<\omega}$ are as required in clause (b).

Clause (d)⇒Clause (c):

Clause (d) is obviously stronger, but we must point out that there are such I_n, D_n; anyhow we shall elaborate. We can choose

$$\lambda_0 = (\|M_1\| + \|M_2\| + |\tau| + \theta)^{<\theta},$$
$$\lambda_{n+1} = 2^{\lambda_n} \quad \text{for } n < \omega;$$

then letting $I_n = \lambda_n$ there is $D_n \in \mathrm{ruf}_{\lambda_n,\theta}(I_n)$ recalling θ is a compact cardinal, noting $\lambda_n = \lambda_n^{<\theta}$. Now $\langle I_n, D_n : n < \omega\rangle$ is as required in the assumption of clause (d), so as we are now assuming clause (d), also its conclusion holds. Now $\langle (I_n, D_n) : n < \omega\rangle$ are as required in clause (c), in particular the isomorphism holds by the conclusion of clause (d) which, as mentioned in the previous sentence, holds.

Clause (e)⇒Clause (d):

Let $\langle (I_n, D_n, \lambda_n) : n < \omega\rangle$ be as in the assumption of clause (d).

We define $I = \prod_n I_n, E_n = \{(\eta, \nu) : \eta, \nu \in I, \eta{\restriction}(n+1) = \nu{\restriction}(n+1)\}$ and define D as in the proof of (c)⇒(b) above and we choose $\bar{w} = \langle w_{\eta,n} : \eta \in I, n < \omega\rangle$ as follows.

First, choose $\bar{u}_n = \langle u^n_s : s \in I_n\rangle$ which witness D_n is (λ_n, θ)-regular, i.e., $u^n_s \in [\lambda_n]^{<\theta}$ and $(\forall \alpha < \lambda_n)[\{s \in I_n : \alpha \in u^n_s\} \in D_n]$. For $\eta \in I$ and $n < \omega$ let $w_{\eta,n}$ be $u^n_{\eta(n)}$ if $\langle \mathrm{otp}(u_{\eta(\ell)}) : \ell \le n\rangle$ is decreasing and $\emptyset$ otherwise. Let $\gamma_{\eta,n}$ be $\mathrm{otp}(w_{\eta,n})$. Now we can check that the assumptions of clause (e) hold (because of the choice of D); we shall elaborate two points. First the ultrafilter D/E_n is (λ, θ)-regular because $\langle u^n_{\eta(n0)}/E_n : \eta \in I\rangle$ witnesses it.

Second, the main point is to prove that $\bar{w} = \langle (w_{\eta,n}, \gamma_{\eta,n} : \eta \in I, n < \omega\rangle$ is indeed a niceness witness for $(I, D, \bar{E})$. For this, most clauses of Definition 1.4 are easy, but we better elaborate on clause (e) there. For every n:

$(*)_n$ for some $X_n \in D_n$, for every $s_n \in X_n$, for some $X_{n-1} \in D_{n-1}, \ldots$, for some $X_0 \in D_0$ for every $s_0 \in X_0$, if $\langle s_0, \ldots, s_n\rangle \trianglelefteq \eta \in I$, then
- (a) $|w_{\eta,0}| > |w_{\eta,1}| > \ldots > |w_{\eta,0}|$
- (b) $|u^\ell_{s_\ell}| > |u^{\ell+1}_{s_{\ell+1}}|$ for $\ell < n$.

Why does $(*)_n$ hold? Clause (a) holds by clause (b) and the choice of $w_{\eta,n}$ as $u^n_{\eta(n)}$. Clause (b) holds because $u^{\ell+1}_{s_{\ell+1}}$ is of cardinality $< \theta$ and

$$\{s \in I_\ell : |u^{\ell+1}_{s_{\ell+1}}|^+ \subseteq u^\ell_s\} \in D_\ell.$$

Hence the conclusion of clause (e) holds and we are done as in the proof of (c)⇒(b).

Clause (a)⇒Clause (e):

So assume that clause (a) holds, that is M_1, M_2 are $\mathbb{L}^1_\theta$-equivalent and assume $I, D, \mathscr{E}, \langle E_n : n < \omega \rangle$ and $\bar{w}$ are as in the assumption of clause (e); and we should prove that its conclusion holds, that is,

$$\text{l.u.p.}_{\mathbf{x}}(M_1) \cong \text{l.u.p.}_{\mathbf{x}}(M_2).$$

For every $\tau_* \subseteq \tau$ of cardinality $< \theta$ and $\mu < \theta$, by Definition 0.22 we know that $M_1 {\restriction} \tau_*, M_2 {\restriction} \tau_*$ are $\mathbb{L}^1_{\le\mu}$-equivalent, hence for every $\alpha < \mu^+$ there is a finite sequence $\langle N_{\tau_*,\mu,\alpha,k} : k \le \mathbf{k}(\tau_*, \mu, \alpha) \rangle$ such that:

$(*)_1$ (a) $N_{\tau_*,\mu,\alpha,0} = M_1 {\restriction} \tau_*$,
(b) $N_{\tau_*,\mu,\alpha,\mathbf{k}(\tau_*,\mu,\alpha)} = M_2 {\restriction} \tau_*$,
(c) in the game $\partial_{\tau_*,\mu,\alpha}[N_{\tau_*,\mu,\alpha,k}, N_{\tau_*,\mu,\alpha,k+1}]$ the ISO player has a winning strategy for each $k < \mathbf{k}(\tau_*, \mu, \alpha)$, but we stipulate a play to have ω moves, by deciding they continue to choose the moves even when one side already wins using the same state except changing $n_{\mathbf{s}}$.

[Why? By Definition 0.20 which quotes [She12, 2.1=La8]]

$(*)_2$ without loss of generality $\|N_{\tau_*,\mu,\alpha,k}\| \le \lambda_0$ for $k \in \{1, \ldots, \mathbf{k}(\tau_*, \mu, \alpha) - 1\}$ (even $< \theta$).

[Why? By (a degenerated case of) Claim 1.2.]

We can (without loss of generality) assume:

$(*)_3$ (a) above $\mathbf{k}(\tau_*, \mu, \alpha) = \mathbf{k}$,
(b) τ has only predicates.

[Why? Clause (a) by monotonicity in τ^*, μ and in α of $M_1 \mathscr{E}^{1,\tau^*}_{\text{qf}(\tau_*),\mu,\alpha} M_2$. Clause (b) is easy too.]

We denote:

$(*)_4$ (a) $\langle P_\alpha : \alpha < |\tau| \rangle$ list the predicates of τ, recall that $|\tau| \le \lambda_0$,
(b) for $t \in I$ let $\tau_t = \{P_\alpha : \alpha \in w_{t,0} \cap |\tau|\}$.

$(*)_5$ Let $N_{s,k} := N_{\tau_s,|w_{s,0}|,\gamma_{s,0}+1,k}$ for $s \in I$ and $k \le \mathbf{k}$.

For $k \le \mathbf{k}$, let $\bar{f}_{k,n} = \langle f_{k,n,\alpha} : \alpha < 2^{\lambda_n} \rangle$ list the members f of $\prod_{s\in I} N_{s,k}$ such that E_n refines $\text{eq}(f)$, so

$$f_{k,n,\alpha} = \langle f_{k,n,\alpha}(\eta) : \eta \in I \rangle$$

but

$$\eta \in I \wedge \nu \in I \wedge \eta E_n \nu \Rightarrow f_{k,n,\alpha}(\eta) = f_{k,n,\alpha}(\nu).$$

Now

$(*)_6$ (a) for $t \in I$ and $k < \mathbf{k}$ let $\partial_{t,k}$ be the game $\partial_{\tau_t,|w_{t,0}|,\gamma_{t,0}+1}[N_{t,k}, N_{t,k+1}]$,
(b) let $\mathbf{st}_{t,k}$ be a winning strategy for the ISO player in $\partial_{t,k}$,
(c) if $t_1 E_0 t_2$ then $\langle N_{t_\iota,k} : k \le \mathbf{k}\rangle$ are the same for $\iota = 1, 2$, moreover, ($\partial_{t_1,k} = \partial_{t_2,k}$ and) $\mathbf{st}_{t_1,k} = \mathbf{st}_{t_2,k}$ for $k < \mathbf{k}$.

[Why clause (c)? Because by $(*)_5$, $N_{s,k}, N_{\tau_s,|w_{s,0}|,\gamma_{s,0}+1,k}$ are determined by $(w_{s,0}, k)$ and τ_s depends on $w_{s,0}$ only, hence (by clause (e) of Theorem 1.5 and clause (f) from Definition 1.4), $N_{\mathbf{s},k}$ depends just on $(s/E_0, k)$.]

Now for each k by induction on n we choose $\langle \mathbf{s}_{t,k,n} : t \in I\rangle$ such that:

$(*)_7$ (a) $\mathbf{s}_{t,k,n}$ is a state of the game $\partial_{t,k}$,
(b) $\langle \mathbf{s}_{t,k,m} : m \le n\rangle$ is an initial segment of a play of $\partial_{t,k}$ in which the ISO player uses the strategy $\mathbf{st}_{t,k}$,
(c) if $t_1 E_n t_2$ then $\mathbf{s}_{t_1,k,n} = \mathbf{s}_{t_2,k,n}$,
(d) $\beta_{\mathbf{s}_{t,k,n}} = \gamma_{t,n}$, see Definition 0.20,
(e) if $t \in I, n = \iota \mod 2$ and $\iota \in \{0, 1\}$ then

$$A^\iota_{\mathbf{s}_{t,k,n}} \supseteq \{f_{k+\iota,m,\alpha}(t) : m < n \text{ and } \alpha \in w_{t,m}\},$$

see Definition 0.20(E).

$(*)_8$ We can carry the induction on n.

[Why? Straightforward.]

$(*)_9$ For each $k < \mathbf{k}, n < \omega, t \in I$ we define $h_{s,k,n}$, a partial function from $N_{s,k}$ to $N_{s,k+1}$ by $h_{s,k,n}(a_1) = a_2$ iff for some $m \le n, w_{s,m} \ne \emptyset$ and $g_{\mathbf{s}_{t,k,m}}(a_1) = a_2$, see Definition 0.20(E).

Now clearly:

$\boxplus_1$ For each $t \in I, k < \mathbf{k}$ and $n < \omega, h_{s,k,n}$ is a partial one-to-one function and even a partial isomorphism from $N_{s,k}$ to $N_{s,k+1}$, non-empty when $n > 0$ and increasing with n.

[Why? By the choice of $\mathbf{st}_{t,k}$ and $(*)_7$(a).]

$\boxplus_2$ Let

$$Y_{k,n} = \Big\{(f_1, f_2) : f_\ell \in \prod_{s\in I} \mathrm{Dom}(h_{s,k,n}) \text{ for } \ell = 1, 2 \text{ and } s \in I \Rightarrow f_2(s) = h_{s,k,n}(f_1(s))\Big\}.$$

$\boxplus_3$ $\mathbf{f}_{k,n} = \{(f_1/D, f_2/D) : (f_1, f_2) \in Y_{k,n}\}$ is a partial isomorphism from

$$M_1^I \restriction \Big\{ f/D : f \in \prod_s N_{s,k} \text{ and } f \text{ respects } E_n \Big\}$$

to

$$M_2^I \restriction \Big\{ f/D : f \in \prod_s N_{s,k+1} \text{ and } f \text{ respects } E_n \Big\}.$$

$\boxplus_4$ $\mathbf{f}_{k,n} \subseteq \mathbf{f}_{k,n+1}$.

$\boxplus_5$ (a) If $f_1 \in \prod_s N_{s,k}$ and $\mathrm{eq}(f_1)$ is refined by E_n then for some $n_1 > n$ and $f_2 \in \prod_s N_{s,k+1}$ the pair $(f_1/D, f_2/D)$ belongs to $\mathbf{f}_{k,n_1}$.

(b) If $f_2 \in \prod_s N_{s,k+1}$ and $\mathrm{eq}(f_2)$ is refined by E_n then for some $n_1 > n$ and $f_1 \in \prod_s N_{s,k}$ the pair $(f_1/D, f_2/D)$ belongs to $\mathbf{f}_{k,n_1}$.

[Why? By symmetry it suffices to deal with clause (a). For some α, $f_1 = f_{k,n,\alpha}$, hence for every $t \in \mathrm{Dom}(f_1)$, $f_1(t) \in A^1_{\mathbf{s}_{t,k,n}}$. We use the "delaying function", $h_{\mathbf{s}_{t,k,n}}(f_1(t)) < \omega$, so for some m the set $\{t \in I : h_{\mathbf{s}_{t,k,n}}(f_1(t)) \leq m\}$ which respects E_n belongs to D. In particular $\{s : \gamma_{s,k,n} > m\} \in D$; the rest should be clear recalling the regularity of each D/E_m.]

Letting $\mathscr{E} = \{E_n : n < \omega\}$, putting together

$(*)_{10}$ $\mathbf{f}_k = \bigcup_n \mathbf{f}_{k,n}$ is an isomorphism from $(\prod_s N_{k,s})_D|\mathscr{E}$ onto $(\prod_s N_{k+1,s})_D|\mathscr{E}$.

Hence

$(*)_{11}$ $\mathbf{f}_{\mathbf{k}-1} \circ \cdots \circ \mathbf{f}_0$ is an isomorphism from $(M_1)^I_D|\mathscr{E}$ onto $(M_2)^I_D|\mathscr{E}$.

So we are done. $\blacksquare_{1.5}$

Discussion 1.6: (1) So for our θ, we get another characterization of $\mathbb{L}^1_\theta$.

(2) We may deal with universal homogeneous (θ, σ)-l.u.p.$\mathbf{x}$, at least for $\sigma = \aleph_0$, using Definition 0.11.

Claim 1.7: *In Theorem 1.5, if $\kappa = \kappa^{<\theta} \geq \|M_1\| + \|M_2\|$ we can add:*

(b)$^+$ *like clause* (b) *of 1.5 but* $|I| \leq 2^\kappa$.

Remark 1.8: Note that we do not restrict $\tau = \tau(M_\ell)$. See proof of $(*)_9$ below.

Proof. Clearly (b)$^+$ $\Rightarrow$(b), so it is enough to prove (b)$\Rightarrow$(b)$^+$; we shall assume $M_1, M_2, \kappa, \mathbf{x}_n, D, \mathscr{E}_n, \mathscr{E}$ are as in (b) and let g be an isomorphism from $(M_1)^I_D/\mathscr{E}$ onto $(M_2)^I_D/\mathscr{E}$.

Let

$(*)_1$ (a) $\mathscr{E}'_n = \{E{:}E$ is an equivalence relation on I with $\le \kappa$ equivalence classes such that some $E' \in \mathscr{E}_n$ refines $E\}$,

(b) let $\mathscr{E}' = \bigcup\{\mathscr{E}'_n : n \in \mathbb{N}\}$.

Clearly

$(*)_2$ $(M_\ell)^I_D|\mathscr{E} = (M_\ell)^I_D|\mathscr{E}'$ for $\ell = 1, 2$.

Let χ be large enough such that $M_1, M_2, \kappa, D, I, \mathscr{E}, \bar{\mathscr{E}}' = \langle \mathscr{E}'_n : n \in \mathbb{N}\rangle, g$ and $(M_\ell)^I_D|\mathscr{E}$ for $\ell = 1, 2$ belong to $\mathscr{H}(\chi)$. We can choose $\mathfrak{B} \prec_{\mathbb{L}_{\kappa^+,\kappa^+}} (\mathscr{H}(\chi), \in)$ of cardinality 2^κ to which all the members of $\mathscr{H}(\chi)$ mentioned above belong and such that $2^\kappa + 1 \subseteq \mathfrak{B}$. So as $\tau = \tau(M_1) \in \mathfrak{B}$ and without loss of generality $|\tau| \le 2^{\|M_1\|+\|M_2\|} \le 2^\kappa$; necessarily $\tau \subseteq \mathfrak{B}$ (alternatively see the end of the proof).

$(*)_3$ Let

(a) $I^* = I \cap \mathfrak{B}$,

(b) $\mathscr{E}^*_n = \{E{\restriction}I^* : E \in \mathscr{E}'_n \cap \mathfrak{B}\}$,

(c) $\mathscr{E}^* = \bigcup\{\mathscr{E}^*_n : n \in \mathbb{N}\}$,

(d) let D^* be any ultrafilter on I^* which includes $\{I \cap I^* : I \in D \cap \mathfrak{B}\}$.

It is enough to check the following points:

$(*)_4$ $\mathbf{x}^*_n := (I^*, D^*, \mathscr{E}^*_n)$ is a (θ, θ)-l.u.f.t.for every $n \in \omega$.

Why? For example, note that if $E \in \mathscr{E}^*_n$, then for some $E' \in \mathscr{E}'_n \cap \mathfrak{B}$ we have $E'{\restriction}I^* = E$, hence E has $\le \kappa$ equivalence classes. Now for any such E', as E' has $\le \kappa$-equivalence classes and belongs to $\mathfrak{B}$, clearly every E'-equivalence class is not disjoint to I^* and every $A \subseteq I^*$ respecting E is $A' \cap I^*$ for some $A' \in \mathfrak{B}$ respecting E'. So $D/E'_n, D^*/E$ are essentially equal, etc., that is, let $\pi_n : \mathscr{E}^*_n \to \mathscr{E}'_n$ be such that $E \in \mathscr{E}^*_n \Rightarrow \pi_n(E){\restriction}I^* = E$ and let $\pi_{n,E} : \{A : A \subseteq I^*$ respects $E\} \to \{A \subseteq I : A$ respects $\pi_n(E)\}$ be such that $\pi_{n,E}(A) = B \Rightarrow B \cap I^* = A$; in fact, those functions are uniquely determined.

So clearly $(*)_4$ follows by

$(*)_5$ (a) π_n is a one-to-one function from $\mathscr{E}^*_n$ onto $\mathscr{E}'_n \cap \mathfrak{B}$,

(b) π_n preserves "E^1 refines E^2" and its negation,

(c) $\mathscr{E}^*_n$ is $(< \theta)$-directed,

(d) if $n = m + 1$ then $\mathscr{E}^*_m \subseteq \mathscr{E}^*_n$ and $\pi_m \subseteq \pi_n$.

Moreover

$(*)_6$ (a) if $E \in \mathscr{E}^*_n$, then $\mathrm{Dom}(\pi_{n,E}) \subseteq \mathfrak{B}$ (because $2^\kappa \subseteq \mathfrak{B}$ is assumed),
(b) $\pi_{n,E}$ is an isomorphism from the Boolean Algebra $\mathrm{Dom}(\pi_{n,E})$ onto $\{A \subseteq I : A$ respects $\pi_n(E)\}$ which is canonically isomorphic to the Boolean Algebra $\mathscr{P}(I/\pi_n(E))$ and also to $\mathscr{P}(I^*/E)$,
(c) $D^* \cap \mathrm{Dom}(\pi_{n,E})$ is an ultrafilter which $\pi_{n,E}$ maps onto $D \cap \mathrm{Rang}(\pi_{n,E})$ which is an ultrafilter; those ultrafilters are θ-complete,

$(*)_7$ I^* has cardinality $\leq 2^\kappa$.

[Why? Because $\mathfrak{B}$ has cardinality $\leq 2^\kappa$.]

$(*)_8$ $(M_\ell)^{I^*}_{D^*}|\mathscr{E}^*$ is isomorphic to $((M_\ell)^I_D|\mathscr{E}')\restriction\mathfrak{B}$ for $\ell = 1, 2$.

[Why? Let $\varkappa$ be the following function:

$(*)_{8.1}$ (a) $\mathrm{Dom}(\varkappa) = (M_1)^{I_*}|\mathscr{E}^*$,
(b) if $f_1 \in (M_1)^{I_*}$ and $E \in \mathscr{E}^*$ refines $\mathrm{eq}(f_1)$, then $f_2 := \varkappa(f_1)$ is the unique function with domain I such that $(\bigcup_n \pi_n)(E) \in \mathscr{E}'$ refines $\mathrm{eq}(f_2)$ and $f_2\restriction I^* = f_1$.

Now easily $\varkappa$ induces an isomorphism as promised in $(*)_8$.]

$(*)_9$ $((M_1)^I_D|\mathscr{E}')\restriction\mathfrak{B}$ is isomorphic to $(M_2)^I_D|\mathscr{E}')\restriction\mathfrak{B}$.

[Why? By $(*)_2$ and the choices of g (in the beginning) and of $\mathfrak{B}$ after $(*)_2$, this is obvious when $\tau = \tau(M_1)$ is included in $\mathfrak{B}$, which is equivalent to $|\tau| \leq 2^\kappa$. By recalling that $\mathrm{arity}(\tau) \leq \aleph_0$, i.e., every predicate and function symbol of τ has finitely many places (see Theorem 1.5), without loss of generality this holds. That is, let $\tau' \subseteq \tau$ be such that for every predicate $P \in \tau$ there is one and only one $P' \in \tau'$ such that

$$\ell \in \{1, 2\} \Rightarrow P^{M_\ell} = (P')^{M_\ell}$$

and similarly for every function symbol; clearly it suffices to deal with $M_1\restriction\tau', M_2\restriction\tau'$ and $|\tau'| \leq 2^{\|M_1\|} \leq 2^\kappa$.]

Together we are done. $\blacksquare_{1.7}$

Note that the proof of Claim 1.7 really uses $\kappa = \kappa^{<\theta}$, as otherwise $\mathscr{E}'_n$ is not $(< \theta)$-directed. How much is the assumption $\kappa = \kappa^{<\theta}$ needed in Claim 1.7? We can say something in Claim 1.9.

Claim 1.9: *Assume that* $\kappa \geq 2^\theta$ *but* $\kappa^{<\theta} > \kappa$, *hence for some regular* $\sigma < \theta$ *we have* $\kappa^{<\sigma} = \kappa < \kappa^\sigma$ *and* $\mathrm{cf}(\kappa) = \sigma$ *and, by* [Sol74], *we have* $(\forall \mu < \kappa)(\mu^\theta < \kappa)$*; recall* $\mathrm{arity}(\tau) = \aleph_0$.

(1) *If* $\langle \mathfrak{B}_i : i \leq \sigma \rangle$ *is a* $\subseteq$*-increasing continuous sequence of* τ*-models and* $\mathbf{x}$ *is a* (θ, θ)-l.u.f.t.*then* $\mathrm{l.u.p.}_{\mathbf{x}}(\mathfrak{B}_\sigma) = \bigcup\{\mathrm{l.u.p.}_{\mathbf{x}}(\mathfrak{B}_i) : i < \sigma\}$ *and*

$$i < j \Rightarrow \mathrm{l.u.p.}_{\mathbf{x}}(\mathfrak{B}_i) \subseteq \mathrm{l.u.p.}_{\mathbf{x}}(\mathfrak{B}_j).$$

(2) *If* J *is a directed partial order of cardinality* $\leq \sigma$ $(<\theta)$ *and* $\mathbf{x}_s = (I, D, \mathscr{E}_s)$ *is a* (θ, θ)-l.u.f.t.*for* $s \in J$ *such that* $s <_J t \Rightarrow \mathscr{E}_s \subseteq \mathscr{E}_t$ *and* M *is a* τ*-model then* $\mathrm{l.u.p.}_{\mathbf{x}}(\mathfrak{B}) = \bigcup\{\mathrm{l.u.p.}_{\mathbf{x}_s}(\mathfrak{B}) : s \in J\}$ *and*

$$s <_J t \Rightarrow \mathrm{l.u.f.t.}_{\mathbf{x}_s}(\mathfrak{B}) \subseteq \mathrm{l.u.p.}_{\mathbf{x}_t}(\mathfrak{B})$$

under the natural identification.

(3) *In Claim 1.7,* $|I^*| \leq \Sigma\{2^\partial : \partial < \kappa\}$ *is enough.*

Proof. Straightforward. $\blacksquare_{1.9}$

2. Special models

Note that in Definition 2.1 below, $M_n \prec_{\mathbb{L}_{\theta,\theta}} M$ is not required. The reader may in a first reading ignore the special$^\bullet$ case.

Definition 2.1: (1) Assume $\lambda > \theta$ is strong limit of cofinality $\aleph_0$.

We say a model M is λ-special when there are $\bar{\lambda}, \bar{M}$ such that (we also may say $\bar{M}$ is a λ-special sequence):

(a) M is a model of cardinality λ with $|\tau(M)| < \lambda$,

(b) (α) $\bar{\lambda} = \langle \lambda_n : n \in \mathbb{N} \rangle$,

(β) $\lambda_n \leq \lambda_{n+1}$,

(γ) $\theta \leq \lambda_n < \lambda_{n+1} < \lambda = \sum_k \lambda_k$ and stipulate $\lambda_{-1} = \theta$,

(c) (α) $\bar{M} = \langle M_n : n < \omega \rangle$,

(β) $M_n \prec_{\mathbb{L}_{\theta,\theta}} M_{n+1}$,

(γ) $M = \bigcup_n M_n$,

(δ) $\lambda_n \geq \|M_n\| \geq \lambda_{n-1}$ recalling $\lambda_{-1} = \theta$,

(d) (α) $\bar{D} = \langle D_n : n \in \mathbb{N} \rangle$ and $\|M_n\| \leq \lambda_n$,

(β) $D_n \in \mathrm{ruf}_{\lambda_{n+1},\theta}(\lambda_{n+1})$,

(γ) $M_n^{\lambda_n}/D_n \prec_{\mathbb{L}_{\theta,\theta}} M_{n+1}$ under the canonical identification (so hence $2^{\lambda_n} \leq \lambda_{n+1}$).

(2) We say that the model M is λ-special$^\bullet$ when clauses (a),(b),(c) above hold but instead of clause (d) we have

(d)′ if Γ is an $\mathbb{L}_{\theta,\theta}$-type on M_n of cardinality $\leq \lambda_n$ with $\leq \lambda_n$ free variables, then Γ is realized in M_{n+1}.

Claim 2.2: (1) *If for every $n < \omega$ we have D_n is a (λ_n, θ)-regular θ-complete ultra-filter on I_n, $|I_n| \leq \lambda_{n+1}$, $M_{n+1} = (M_n)^{I_n}/D_n$ identifying M_n with its image under the canonical embedding into M_{n+1} so $M_n \prec_{\mathbb{L}_{\theta,\theta}} M_{n+1}$ and $\lambda_n \geq \|M_n\|$, $\lambda = \sum_n \lambda_n \geq \theta$ (equivalently $> \theta$) then $\langle M_n : n \in \mathbb{N}\rangle$ is a λ-special sequence, so $M = \bigcup_n M_n$ is a λ-special model and M is a model of* $\mathrm{Th}_{\mathbb{L}^1_\theta}(M_1)$.

(2) *Assume $\lambda > \theta, \mathrm{cf}(\lambda) = \aleph_0$. In Definition 2.1, clause (d) indeed implies clause (d)′; so every λ-special model/sequence is a λ-special$^\bullet$ model/sequence.*

(3) *In Definition 2.1, M is a model of* $\mathrm{Th}_{\mathbb{L}^1_\theta}(M)$, *in fact this follows by Definition 2.1(1)(d)$(\alpha), (\beta), (\gamma)$.*

(4) *Assume $\lambda > \theta$ is a strong limit cardinal of cofinality $\aleph_0$. If M is a model of cardinality $\geq \theta$ but $< \lambda$ then:*

- (A) (a) *There is a λ-special sequence $\bar{M}$ with $M_0 = M$,*
  - (b) *there is a λ-special model N which is a $\prec_{\mathbb{L}^1_\theta}$-extension of M,*
  - (c) $\mathrm{Th}_{\mathbb{L}^1_\theta(M)}$ *has a λ-special model.*
- (B) *If M is a model of cardinality λ then for some $N, \bar{M}, \bar{N}$ we have:*
  - (a) *$\bar{M} = \langle M_n : n < \omega\rangle$ satisfies clauses (a), (b), (c) of 2.1. with union M,*
  - (b) *$\bar{N} = \langle N_n : n < \omega\rangle$ is a λ-special$^\bullet$ sequence with union N,*
  - (c) $M_n \prec_{\mathbb{L}_{\theta,\theta}} N_n$.
- (C) *If M is a λ-special model and $\tau \subseteq \tau_M$ then $M{\restriction}\tau$ is also a λ-special model.*

(5) *Assume $\lambda > \theta > \aleph_0 = \mathrm{cf}(\lambda)$. If M is a λ-special$^\bullet$ model and $\tau \subseteq \tau_M$ then $M{\restriction}\tau$ is also a λ-special$^\bullet$ model*

(6) *If λ is strong limit $> \theta$ of cofinality $\aleph_0$, a model M is λ-special iff it is λ-special$^\bullet$.*

Proof. (1) If we assume clause (d) in Definition 2.1, then just by the definition. If we assume clause (d)′ in Definition 2.1, then use part (2).

(2) It follows by the (λ_n, θ)-regularity of D_n.

(3) Check the definition.

(4) Clause (A):

We can choose an increasing sequence $\langle \lambda_n : n < \omega \rangle$ with limit λ such that $\lambda_0 = \|M\|^\theta$ and $2^{\lambda_n} < \lambda_{n+1} = \lambda^\theta_{n+1}$. For each n we can choose a (λ, θ)-regular θ-complete ultrafilter D_n on λ_n, and define M_n as in part (1). Now use the conclusion of part (1).

Clause (B):

Without loss of generality the universe of M is λ. Choose $\langle \lambda_n : n < \omega \rangle$ as above (except $\lambda_0 \geq \|M\|$ of course), and by induction on n choose $M_n \prec_{\mathbb{L}^1_\theta} M$ of cardinality λ_n which includes $\cup\{M_k : k < n\} \cup \lambda_n$. We now choose

$$\langle M^*_k, M^*_{k,n} : n < \omega \rangle$$

by induction on k such that:

(a) for $k = 0$ we let $M^*_k = M$ and $M^*_{k,n} = M_n$,
(b) for $k = \ell + 1$ let $M^*_k = (M^*_\ell)^{\lambda_k}/D_k$ and $M^*_{k,n} = (M^*_{\ell,n})^{\lambda_k}/D_k$.

There is no problem to carry the induction and we let $N = \cup\{M^*_{k,k} : k < \omega\}$ and $N_k = M^*_{k,k}$; now check.

Clause (C):

Just read the definition.

(5) Again just read the definition.

(6) Easy too. $\blacksquare_{2.2}$

Remark 2.3: (1) In Claim 2.4 below we do not require that the λ_n-s are the same and, of course, we do not require that the D_n are the same. Part (3) clarifies this.

(2) In Definition 2.1 clause (c)(δ), it is enough to demand $\lambda_n \geq \|M_n\| \geq \theta$.

CLAIM 2.4: (1) *If $\langle M^\ell_n : n \in \mathbb{N} \rangle$ is a λ-special sequence (or just a λ-special$^\bullet$ sequence) with union M_ℓ for $\ell = 1, 2$ and $\mathrm{Th}_{\mathbb{L}_{\theta,\theta}}(M^1_0) = \mathrm{Th}_{\mathbb{L}_{\theta,\theta}}(M^2_0)$ then M_1, M_2 are isomorphic.*

(2) *Moreover, if $n < \omega$ and f is a partial function from M^1_n into M^2_n which is $(M^1_n, M^2_n, \mathbb{L}_{\theta,\theta})$-elementary, that is,*

$$\bar{a} \in {}^{\theta>}(\mathrm{Dom}(f)) \Rightarrow f(\mathrm{tp}_{\mathbb{L}_{\theta,\theta}}(\bar{a}, \emptyset, M^1_n)) = \mathrm{tp}_{\mathbb{L}_{\theta,\theta}}(f(\bar{a}), \emptyset, M^2_n),$$

then f can be extended to an isomorphism from M_1 onto M_2.

(3) *If we weaken clause (d)$'$ of Definition 2.1 by weakening the conclusion to: for some $k > n, \Gamma$ is realized in M_k, then we get an equivalent definition.*

Proof. (1) By the hence and forth argument; but we elaborate. Let $\mathscr{F}_n$ be the set of f such that:

(a) f is a one-to-one function,
(b) the domain of f is included in M^1_n,
(c) the range of f is included in M^2_n,
(d) if $\zeta < \theta$ and $\bar{a} \in {}^\zeta(M^1_n)$ and $\bar{b} = f(\bar{a}) \in {}^\zeta(M^2_n)$ and $\varphi(\bar{x}_{[\zeta]} \in \mathbb{L}_{\theta,\theta}(\tau(M_\ell))$ then $M^1_n \models \varphi[\bar{a}]$ iff $M^2_n \models \varphi[\bar{b}]$.

Easily

$(*)_1$ the set $\mathscr{F}_n$ is not empty.

[Why? Because the empty function belongs to $\mathscr{F}_n$.]

$(*)_2$ If $f \in \mathscr{F}_n$, then some $g \in \mathscr{F}_{n+1}$ extends f and $M^1_n \subseteq \mathrm{Dom}(g)$.

[Why? By clause (d)' of Definition 2.1(2)]

$(*)_3$ If $f \in \mathscr{F}_n$, then some $g \in \mathscr{F}_{n+1}$ extends f and $M^1_n \subseteq \mathrm{Rang}(g)$.

[Why? Similarly.]

Together clearly we are done.

(2) Same proof.

(3) Use suitable sub-sequences (using monotonicity). $\blacksquare_{2.4}$

Note that comparing Definition 2.1 with the first order parallel, in Claim 2.4(1), a priori it is not given that $\mathrm{Th}_{\mathbb{L}_{\theta,\theta}}(M_1) = \mathrm{Th}_{\mathbb{L}_{\theta,\theta}}(M_2)$ suffices. Also Claim 2.4 does not say that $\mathrm{Th}_{\mathbb{L}^1_\theta}(M)$ and λ determines M up to isomorphism because we demand that M^1_0, M^2_0 are $\mathbb{L}^1_\theta$-equivalent. However:

Claim 2.5: *Assume $\lambda > \theta$ is of cofinality $\aleph_0$ and T is a complete theory in $\mathbb{L}^1_\theta(\tau_T), |T| < \lambda$, equivalently $|\tau_T| < \lambda$.*

(1) *If λ is strong limit then T has exactly one λ-special model (up to isomorphism).*
(2) *T has at most one λ-special$^\bullet$ model of cardinality λ up to isomorphism.*

Proof. (1) Assume N_1, N_2 are special models of T of cardinality λ. By Definition 2.1 for $\ell = 1, 2$ there is a triple $(\bar{\lambda}_\ell, \bar{M}_\ell, \bar{D}_\ell)$ witnessing N_ℓ is λ-special as there.

As $M_{\ell,0} \prec_{\mathbb{L}_{\theta,\theta}} M_{\ell,n} \prec_{\mathbb{L}_{\theta,\theta}} M_{\ell,n+1} \prec_{\mathbb{L}^1_\theta} \bigcup_m M_{\ell,m} = N_\ell$ for $n \in \mathbb{N}$, by Theorem 0.10 and Claim 1.2, we know that $M_{\ell,0} \equiv_{\mathbb{L}^1_\kappa} N_\ell$, so we can conclude that $M_{1,0} \equiv_{\mathbb{L}^1_\kappa} M_{2,0}$ and both are models of T.

By Theorem 1.5 there is a sequence $\langle(\lambda_n, D_n) : n \in \mathbb{N}\rangle$ with $\Sigma_{n<\omega}\lambda_n > \lambda$, $2^{\lambda_n} \leq \lambda_{n+1}$ and D_n a (λ_n, θ)-regular ultrafilter on λ_n such that $M'_1 \cong M'_2$ when:

(∗) $M'_{\ell,0} = M_{\ell,0}, M'_{\ell,n+1} = (M'_{\ell,n})^{\lambda_n}/D_n$ and $M'_\ell = \bigcup_n M'_{\ell,n}$.

Let $\langle\mu_n : n < \omega\rangle$ be such that: $2^{\mu_n} < \mu_{n+1} < \lambda = \Sigma\{\mu_k : k < \omega\}$ for $n < \omega$.

Next let $M''_{\ell,n}$ for $\ell = 1, 2$ and $n < \omega$ be such that $M''_{\ell,n} \prec_{\mathbb{L}_{\mu_n^+,\mu_n^+}} M'_{\ell,n}$ and $M''_{\ell,n}$ has cardinality 2^{μ_n} and $M''_{\ell,n} \prec_{\mathbb{L}_{\mu_n^+,\mu_n^+}} M'_{\ell,n+1}$ and f maps $M''_{1,n}$ onto $M''_{2,n}$.

Now let $M''_\ell = \bigcup\{M''_{\ell,n} : n < \omega\}$ for $\ell = 1, 2$.

Easily $\langle M''_{\ell,n} : n < \omega\rangle$ witness that M''_ℓ is λ-special$^\bullet$ and f witness that $M''_1 \cong M''_2$.

Also, $M''_{\ell,n}, M'_{\ell,n}, M_{\ell,0}$ are $\mathbb{L}_{\theta,\theta}$-equivalent, hence $N_1 \cong M''_1$ by 2.4(1) and $N_2 \cong M''_2$ similarly. Together $N_1 \cong N_2$ is promised.

(2) The proof is similar to part of the proof of Theorem 1.5 clause (a) implies clause (e), i.e., by the hence and forth argument. $\blacksquare_{2.5}$

Now we can generalize the Robinson lemma, hence (see, e.g., [Mak85]) giving an alternative proof of the interpolation theorem (recall though that in [She12] we do not assume the cardinal θ is compact).

Claim 2.6: (1) *Assume $\tau_1 \cap \tau_2 = \tau_0, T_\ell$ is a complete theory in $\mathbb{L}^1_\theta(\tau_\ell)$ for $\ell = 1, 2$ and $T_0 = T_1 \cap T_2$. Then $T_1 \cup T_1$ has a model.*

(2) *We can allow in (1) the vocabularies to have more than one sort.*

(3) *The logic $\mathbb{L}^1_\theta$ satisfies the interpolation theorem.*

(4) *$\mathbb{L}^1_\theta$ has disjoint amalgamation, i.e., if $M_0 \prec_{\mathbb{L}^1_\theta} M_\ell$ for $\ell = 1, 2$, that is, $(M_0, c)_{c\in M_0}, (M_\ell, c)_{c\in M_0}$ has the same $\mathbb{L}^1_\theta$-theory and $|M_1|\cap|M_2| = |M_0|$, then there is M_3 such that $M_\ell \prec_{\mathbb{L}^1_\theta} M_3$ for $\ell = 0, 1, 2$ (hence orbital types are well defined).*

(5) *$\mathbb{L}^1_\theta$ has the JEP.*[3]

Proof. (1) Let $\lambda > |\tau_1| + |\tau_2| + \theta$ be a strong limit cardinal of cofinality $\aleph_0$. For $\ell = 1, 2$ there is a λ-special model M_ℓ of T_ℓ by Claim 2.2(3). Now $N_\ell = M_\ell{\restriction}\tau_0$ is a λ-special model of T.

By Claim 2.5(1), $N_1 \cong N_2$ so without loss of generality $N_1 = N_2$, and let M be the expansion of $N_1 = N_2$ by the predicates and functions of M_1 and of M_2. Clearly M is a model of $T_1 \cup T_2$.

(2) Similarly.

[3] But the disjoint version may fail, e.g., if we have individual constants.

(3) Follows, as $\mathbb{L}^1_\theta$ being $\subseteq \mathbb{L}_{\theta,\theta}$ satisfies θ-compactness and part (1).

(4) Follows by (1), that is, let $\mathbf{x}$ be as in Theorem 1.5(c) for M_1, M_2. So for every $C \subseteq M_0$ of cardinality $< \theta$, letting $M_{C,\ell} = (M_\ell, c)_{c \in C}$ we have $N_{C,1} \cong N_{C,2} \cong N_{C,0}$ where $N_{C,\ell} = \text{l.u.p.}_{\mathbf{x}}(M_{C,\ell})$. Hence $N_{C,0} \prec_{\mathbb{L}_{\theta,\theta}} N_{C,\ell}$ for $\ell = 1, 2$ and we use "$\mathbb{L}_{\theta,\theta}$ has disjoint amalgamation".

(5) Follows by Theorem 1.5. $\blacksquare_{2.6}$

Remark 2.7: This proof implies the generalization of preservation theorems; see [CK73].

Recall that the aim of Ehrenefucht–Mostowski [EM56] was: every first order theory T with infinite models has models with many automorphisms. This fails for $\mathbb{L}_{\theta,\theta}$ and even $\mathbb{L}_{\aleph_1,\aleph_1}$ as we can express "$<$ is a well ordering". What about $\mathbb{L}^1_\theta$?

Claim 2.8: *Assume (λ, T are as above in Claim 2.5 and) M is a special model of T of cardinality λ. Then M has 2^λ automorphisms.*

Proof. Let $\langle M_n : n < \omega \rangle$ witness M is special. The result follows by the proof of 2.4(2) noting that

- $(*)$ if f_n is an $(M_n, M_n, \mathbb{L}_{\theta,\theta}(\tau_M))$-elementary mapping then there are $a \in M_{n+1}, a_2 \in {}^\lambda(M_{n+1})$ and $f_\alpha, a_{2,\alpha} \in (M_{n+1})$ for $\alpha < \lambda_n$ such that
  - (a) $a_{2,\alpha} \neq a_{2,\beta}$ for $\alpha < \beta < \lambda_n$,
  - (a) f_α is an $(M^1_{n+1}, M^2_{n+1}, \mathbb{L}_{\theta,\theta}(\tau_M))$-elementary mapping,
  - (b) $f_\alpha \supseteq f$ and maps a to a_α.

Why is this possible? Choose $a' \in M_{n+2} \setminus M_{n+1}$ and choose $a_\alpha \in M_{n+1} \setminus \{a_\beta : \beta < \alpha\}$ by induction on $\alpha < \lambda_n$ realizing $\text{tp}_{\mathbb{L}_{\theta,\theta}(\tau_T)}(a', M_n, M_{n+2})$.

Lastly, let $f_\alpha = f \cup \{(a_0, g(a_\alpha))\}$.

Why is this enough? It should be clear. $\blacksquare_{2.8}$

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