THE KEISLER-SHELAH ISOMORPHISM THEOREM AND THE CONTINUUM HYPOTHESIS

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ABSTRACT. We show that if for any two elementary equivalent structures $\mathbf{M}, \mathbf{N}$ of size at most continuum in a countable language, $\mathbf{M}^\omega/\mathcal{U} \simeq \mathbf{N}^\omega/\mathcal{U}$ for some ultrafilter $\mathcal{U}$ on ω, then CH holds. We also provide some consistency results about Keisler and Shelah isomorphism theorems in the absence of CH.

§ 1. INTRODUCTION

The Keisler-Shelah isomorphism theorem provides a characterization of elementary equivalence. It says that two models of a theory are elementarily equivalent if and only if they have isomorphic ultrapowers.

Let the *Keisler criterion (for elementary equivalence)* be the statement: for any two structures $\mathbf{M}, \mathbf{N}$ of size $\leq 2^{\aleph_0}$ in a countable language, $\mathbf{M} \equiv \mathbf{N}$ if and only if $\mathbf{M}^\omega/\mathcal{U} \simeq \mathbf{N}^\omega/\mathcal{U}$ for some ultrafilter $\mathcal{U}$ on ω.

In [4] (see also [5]), Keisler showed that the Keisler criterion follows from CH. The result is trivial if at least one of $\mathbf{M}, \mathbf{N}$ is finite, so assume otherwise. He showed that for any non-principal ultrafilter $\mathcal{U}$ on ω the ultrapowers $\mathbf{M}^\omega/\mathcal{U}$ and $\mathbf{N}^\omega/\mathcal{U}$ are $\aleph_1$-saturated of size $2^{\aleph_0}$. Thus, under CH, both are saturated of the same size and the result follows from the uniqueness of saturated models.

Later Shelah [6] removed the CH assumption in Keisler's theorem, by showing that if $\mathcal{L}$ is a countable language and $\mathbf{M}, \mathbf{N}$ are countable $\mathcal{L}$-models, then $\mathbf{M} \equiv \mathbf{N}$ if and only if there exists an ultrafilter $\mathcal{U}$ on 2^ω such that $\mathbf{M}^\omega/\mathcal{U} \simeq \mathbf{N}^\omega/\mathcal{U}$.

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In [8], Shelah has constructed a model of ZFC in which $2^{\aleph_0} = \aleph_2$ and in which there are countable graphs $\mathbf{\Delta} \equiv \mathbf{\Gamma}$ such that for no ultrafilter $\mathcal{U}$ on ω, $\mathbf{\Delta}^\omega/\mathcal{U} \simeq \mathbf{\Gamma}^\omega/\mathcal{U}$. This shows that CH is an essential assumption for Keisler's theorem, even for countable models.

In this paper we discuss some variants of the Keisler's criterion, in particular in the absence of CH, and prove some related results. First we show that Keisler's criterion is indeed equivalent to CH by proving the following theorem.

Theorem 1.1. *Suppose $2^{\aleph_0} \geq \aleph_2$. Then the Keisler criterion fails.*

The counterexample we consider for the above theorem comes from the theory of dense linear orders, see Theorem 2.1.

It is known from the work of Ellentuck and Rucker [2] that if Martin's axiom+$\neg$CH holds, then there exists an ultrafilter $\mathcal{U}$ on ω such that for any countable structure $\mathbf{M}$, the ultrapower $\mathbf{M}^\omega/\mathcal{U}$ is saturated. In particular if $\mathbf{M} \equiv \mathbf{N}$ are countable models of the same vocabulary, then $\mathbf{M}^\omega/\mathcal{U} \simeq \mathbf{N}^\omega/\mathcal{U}$. We consider models of larger size and ask for the same conclusion. In particular, we prove the following:

Theorem 1.2. *Suppose $2^{\aleph_0} > \aleph_1 = \mathrm{cf}(2^{\aleph_0})$ and $\mathrm{Cov}(\text{meagre}) = 2^{\aleph_0}$. If $\mathbf{M}, \mathbf{N}$ are models of size $\aleph_1$ in a countable language, then $\mathbf{M}^\omega/\mathcal{U} \simeq \mathbf{N}^\omega/\mathcal{U}$ for some ultrafilter $\mathcal{U}$ on ω.*

We also prove a related consistency result in the generic extension obtained by adding many Cohen reals, which allows us to remove the cofinality restriction of the above theorem.

§ 2. Keisler's theorem and the CH

In this section we prove the following theorem which immediately implies Theorem 1.1.

Theorem 2.1. *There are models $\mathbf{M}, \mathbf{N}$ of the theory $\mathrm{Th}(\mathbb{Q}, <)$ of size $\aleph_0, \aleph_2$ respectively such that for no ultrafilter $\mathcal{U}$ on ω, $\mathbf{M}^\omega/\mathcal{U} \simeq \mathbf{N}^\omega/\mathcal{U}$.*

Proof. Let $\mathbf{M} = (\mathbb{Q}, <)$ and let $\mathbf{N}$ be a dense linear order of cardinality $\aleph_2$ such that for some $a, b \in \mathbf{N}$ we have $\mathrm{cf}(\mathbf{N}_a) = \aleph_1$ and $\mathrm{cf}(\mathbf{N}_b) = \aleph_2$, where for each $c \in \mathbf{N}$,

$$\mathbf{N}_c = \{d \in \mathbf{N} : d <_{\mathbf{N}} c\}.$$

We show that $\mathbf{M}$ and $\mathbf{N}$ are as required. Suppose, towards a contradiction, that for some ultrafilter $\mathcal{U}$ on ω, there exists an isomorphism $f : \mathbf{N}^\omega/\mathcal{U} \simeq \mathbf{M}^\omega/\mathcal{U}$. To simplify the notation, let us set $\mathbf{M}_* = \mathbf{M}^\omega/\mathcal{U}$ and $\mathbf{N}_* = \mathbf{N}^\omega/\mathcal{U}$. Let

$$a_* = [\langle a : n < \omega\rangle]_{\mathcal{U}} \in \mathbf{N}_*$$

and

$$b_* = [\langle b : n < \omega\rangle]_{\mathcal{U}} \in \mathbf{N}_*.$$

By the choice of elements a and b we have:

Claim 2.2. $\mathrm{cf}((\mathbf{N}_*)_{a_*}) = \aleph_1$ *and* $\mathrm{cf}((\mathbf{N}_*)_{b_*}) = \aleph_2$.

Proof. Let us show that $\mathrm{cf}((\mathbf{N}_*)_{a_*}) = \aleph_1$. Pick a $<_{\mathbf{N}}$-increasing sequence $\langle a_i : i < \omega_1\rangle$ which is $<_{\mathbf{N}}$-cofinal in a. Then the sequence $\langle (a_i)_* : i < \omega_1\rangle$, where $(a_i)_* = [\langle a_i : n < \omega\rangle]_{\mathcal{U}}$ is an increasing sequence below a_*. Let us show that it is also cofinal in a_*. Thus let $f : \omega \to \mathbf{N}$ and suppose that $[f]_{\mathcal{U}} <_{N_*} a_*$. Without loss of generality $f : \omega \to \mathbf{N}_a$. For each $n < \omega$ pick some $i(n) < \omega_1$ such that $f(n) <_{\mathbf{N}} a_{i(n)}$. Let $j = \sup_{n\to\infty} i(n)$. Then $j < \omega_1$ and for every $n < \omega$

$$f(n) <_{\mathbf{N}} a_{i(n)} <_{\mathbf{N}} a_j,$$

in particular $[f]_{\mathcal{U}} <_{\mathbf{N}_*} (a_j)_*$. Thus the sequence $\langle (a_i)_* : i < \omega_1\rangle$ is increasing and cofinal in a_*. By the regularity of $\aleph_1$, we have $\mathrm{cf}((\mathbf{N}_*)_{a_*}) = \aleph_1$. □

Set $a_\dagger = f(a_*)$ and $b_\dagger = f(b_*)$.

Claim 2.3. $\mathrm{cf}((\mathbf{M}_*)_{a_\dagger}) = \aleph_1$ *and* $\mathrm{cf}((\mathbf{M}_*)_{b_\dagger}) = \aleph_2$.

Proof. It is trivial by the choice of $a_\dagger$ and $b_\dagger$. □

Claim 2.4. *There is a function $F : \mathbf{M}^3 \to \mathbf{M}$ such that for every $c, d \in \mathbf{M}$, the formula $F(x, c, d)$ defines an automorphism of $\mathbf{M}$ which maps c to d.*

Proof. Define F by $F(x, y, z) = x - y + z$. The function F is easily seen to be as required. □

It follows from Claim 2.4 that for some function F_*, $(\mathbf{M}_*, F_*) = (\mathbf{M}, F)^\omega/\mathcal{U}$. Then by the choice of F, the function F_* has the following property:

(∗): $F_* : \mathbf{M}_*^3 \to \mathbf{M}_*$ is a function such that for all $c, d \in \mathbf{M}_*$, the formula $F_*(x, c, d)$ defines an automorphism of $\mathbf{M}_*$ which maps c to d.

In particular $F_*(x, a_\dagger, b_\dagger)$ defines an automorphism of $\mathbf{M}_*$ which maps $a_\dagger$ to $b_\dagger$. Thus we must have

$$\mathrm{cf}((\mathbf{M}_*)_{a_\dagger}) = \mathrm{cf}((\mathbf{M}_*)_{b_\dagger}),$$

which contradicts Claim 2.3. □

By the above result and Keisler's theorem, we have the following corollary.

Corollary 2.5. *The following are equivalent:*

(a) CH,

(b) *Keisler's criterion: if $\mathcal{L}$ is a countable language and $\mathbf{M}, \mathbf{N}$ are $\mathcal{L}$-models of size $\leq 2^{\aleph_0}$, then $\mathbf{M} \equiv \mathbf{N}$ if and only if there exists an ultrafilter $\mathcal{U}$ on ω such that $\mathbf{M}^\omega/\mathcal{U} \simeq \mathbf{N}^\omega/\mathcal{U}$.*

§ 3. Keisler-Shelah theorem for models of cardinality $\aleph_1$

In this section, we ask to what extent the Keisler and Shelah isomorphism theorems can hold for models of uncountable cardinality. We prove some theorems which by the result of the previous section are, in some sense, optimal.

Definition 3.1. (1) Let Cov(meagre) be the minimal size of a family of meagre subsets of the real line that cover it.

(2) Given an infinite cardinal κ, let MA_κ(countable) be the following statement: if $\mathbb{P}$ is a countable partial order and $\mathcal{A}$ is a family of dense subsets of $\mathbb{P}$ of size κ, then there exists a filter $\mathbf{G} \subseteq \mathbb{P}$ meeting all sets in $\mathcal{A}$.

Our proof relies on the following lemma.

Lemma 3.2. *(see [1, Theorem 7.13]) Suppose* $\mathrm{Cov}(\text{meagre}) = 2^{\aleph_0}$*. Then* $\mathrm{MA}_\kappa(\text{countable})$ *holds for all* $\kappa < 2^{\aleph_0}$*.*

Let us start by proving Theorem 1.2.

Theorem 3.3. *Suppose* $2^{\aleph_0} > \aleph_1 = \mathrm{cf}(2^{\aleph_0})$ *and* $\mathrm{Cov}(\text{meagre}) = 2^{\aleph_0}$*. Suppose* $\mathbf{M}_0 \equiv \mathbf{M}_1$ *are models of size* $\leq \aleph_1$ *in the same countable vocabulary* $\mathcal{L}$*. Then for some ultrafilter* $\mathcal{U}$ *on* ω*,* $\mathbf{M}_0^\omega/\mathcal{U} \simeq \mathbf{M}_1^\omega/\mathcal{U}$*.*

Proof. Before giving the details of the proof, let us sketch the main idea. We would like to find an ultrafilter $\mathcal{U}$ on ω and enumerations $\langle g^0_\alpha : \alpha < 2^{\aleph_0}\rangle$ and $\langle g^1_\alpha : \alpha < 2^{\aleph_0}\rangle$ of $\mathbf{M}_0^\omega$ and $\mathbf{M}_1^\omega$ respectively, such that

$$((\mathbf{M}_0)^\omega/\mathcal{U}, [g^0_0]_\mathcal{U}, \cdots, [g^0_\alpha]_\mathcal{U}, \cdots) \equiv ((\mathbf{M}_1)^\omega/\mathcal{U}, [g^1_0]_\mathcal{U}, \cdots, [g^1_\alpha]_\mathcal{U}, \cdots). \tag{3.1}$$

This will show that the function $\langle([g^0_\alpha]_\mathcal{U}, [g^1_\alpha]_\mathcal{U}) : \alpha < 2^{\aleph_0}\rangle$ is an isomorphism between $\mathbf{M}_0^\omega/\mathcal{U}$ and $\mathbf{M}_1^\omega/\mathcal{U}$. On the other hand, by Loś theorem, 3.1 is equivalent to saying that for all $\mathcal{L}$-formula $\phi(x_0, \cdots, x_{n-1})$ and all $\beta_0, \cdots, \beta_{n-1} < 2^{\aleph_0}$,

$$\left\{k < \omega : \mathbf{M}_0 \models \phi(g^0_{\beta_0}(k), \cdots, g^0_{\beta_{n-1}}(k)) \Leftrightarrow \mathbf{M}_1 \models \phi(g^1_{\beta_0}(k), \cdots, g^1_{\beta_{n-1}}(k))\right\} \in \mathcal{U}.$$

We define by induction on $\alpha < 2^{\aleph_0}$, a sequence $\langle(\mathcal{U}_\alpha, g^0_\alpha, g^1_\alpha) : \alpha < 2^{\aleph_0}\rangle$, where $\langle\mathcal{U}_\alpha : \alpha < 2^{\aleph_0}\rangle$ is an increasing and continuous chain of filters on ω such that 3.1 holds whenever $\mathcal{U}$ replaced by $\mathcal{U}_{\alpha+1}$. To make sure that g^0_α's and g^1_α's enumerate all elements of $\mathbf{M}_0^\omega$ and $\mathbf{M}_1^\omega$ respectively, we use a back and forth construction. To make sure that the construction continues to work at all levels below $2^{\aleph_0}$, we use the assumption $\mathrm{Cov}(\text{meagre}) = 2^{\aleph_0}$ and proceed in such a way that $\mathcal{U}_\alpha$ is generated by $\leq \aleph_0 + |\alpha|$ many elements.

Let us now go into the details of the proof. Let $\langle\lambda_i : i < \omega_1\rangle$ be an increasing and continuous sequence of cardinals $\geq \aleph_1$ cofinal in $2^{\aleph_0}$ and for $\ell < 2$ let $\langle\mathbf{M}^\ell_i : i < \omega_1\rangle$ be an increasing and continuous chain of elementary submodels of $\mathbf{M}_\ell$ such that for all $i < \omega_1, ||\mathbf{M}^\ell_i|| = \aleph_0$ and $\mathbf{M}_\ell = \bigcup\limits_{i<\omega_1} \mathbf{M}^\ell_i$. Let $\langle f^\ell_\alpha : \alpha < 2^{\aleph_0}\rangle$ be an enumeration of $\mathbf{M}_\ell^\omega$ such that

$$\alpha < \lambda_i \implies f^\ell_\alpha \in (\mathbf{M}^\ell_i)^\omega.$$

Let also $\langle X_\alpha : \alpha < 2^{\aleph_0}\rangle$ enumerate $\mathcal{P}(\omega)$. By induction on $\alpha < 2^{\aleph_0}$ and using a back and forth construction, we build the triple $(\mathcal{U}_\alpha, g^0_\alpha, g^1_\alpha)$ such that:

(a) $g^0_\alpha \in \mathbf{M}_0^\omega$, furthermore if $\alpha < \lambda_i$, then $g^0_\alpha \in (\mathbf{M}^0_i)^\omega$,

(b) $g^1_\alpha \in \mathbf{M}_1^\omega$, furthermore if $\alpha < \lambda_i$, then $g^1_\alpha \in (\mathbf{M}^1_i)^\omega$,

(c) for $i < \omega_1$ and $\ell < 2$, $\{g^\ell_\alpha : \alpha < \lambda_i\} = \{f^\ell_\alpha : \alpha < \lambda_i\}$,

(d) $\mathcal{U}_\alpha$ is a filter on ω generated by $\leq \aleph_0 + |\alpha|$ sets containing all co-finite subsets of ω,

(e) if $\phi(x_0, \cdots, x_{n-1})$ is a formula of $\mathcal{L}$ and $\beta_0, \cdots, \beta_{n-1} \leq \alpha$, then the set $Y_{\phi,\langle\beta_0,\cdots,\beta_{n-1}\rangle}$ defined as

$$\Big\{k < \omega : \mathbf{M}_0 \models \phi(g^0_{\beta_0}(k), \cdots, g^0_{\beta_{n-1}}(k)) \Leftrightarrow \mathbf{M}_1 \models \phi(g^1_{\beta_0}(k), \cdots, g^1_{\beta_{n-1}}(k))\Big\},$$

belongs to $\mathcal{U}_{\alpha+1}$,

(f) if $\alpha < \beta < 2^{\aleph_0}$, then $\mathcal{U}_\alpha \subseteq \mathcal{U}_\beta$.

(g) if α is a limit ordinal, then $\mathcal{U}_\alpha = \bigcup\limits_{\beta<\alpha} \mathcal{U}_\beta$,

(h) for all $\alpha < 2^{\aleph_0}$, either $X_\alpha \in \mathcal{U}_{\alpha+1}$ or $\omega \setminus X_\alpha \in \mathcal{U}_{\alpha+1}$.

As in Shelah [7, Ch VI, §3], there is no problem in carrying the induction, however let us elaborate the main point of the proof. The only difficulty in carrying the induction is clause (e). Thus suppose that $\alpha < 2^{\aleph_0}$ and the construction is done up to α. Let also $i < \omega_1$ be such that $\alpha < \lambda_i$. First suppose that α is an even ordinal. Let $g^0_\alpha = f^0_{\gamma_\alpha}$, where γ_α is the least ordinal such that $f^0_{\gamma_\alpha} \notin \{g^0_\beta : \beta < \alpha\}$. Note that $\gamma_\alpha < \lambda_i$. Let also $\mathscr{G}(\mathcal{U}_\alpha)$ be a set of generators of $\mathcal{U}_\alpha$ of size $\leq \aleph_0 + |\alpha|$.

Let $\mathbb{P}$ be the forcing notion consisting of all maps $p : \mathrm{dom}(p) \to M^1_i$, where $\mathrm{dom}(p)$ is a finite subset of ω, ordered by inclusion. $\mathbb{P}$ is countable. Define the following sets:

- $D_n = \{p \in \mathbb{P} : n \in \mathrm{dom}(p)\}$, where $n < \omega$.
- For any set $A \in \mathscr{G}(\mathcal{U}_\alpha)$, any finite sequence $\vec{\phi} = \langle \phi_\iota(x_0, \cdots, x_{n_\iota-1}, y) : \iota \in I\rangle$, any finite sequence $\langle \vec{\beta}^\ell_\iota : \iota \in I, \ell \in J_\iota\rangle$, where $\vec{\beta}^\ell_\iota = \langle \beta^\ell_{\iota,0}, \cdots, \beta^\ell_{\iota,n_\iota-1}\rangle$ consists of ordinals less than α and $m < \omega$ let $\Sigma_{A,\vec{\phi},\langle\vec{\beta}^\ell_\iota:\iota\in I,\ell\in J_\iota\rangle,m}$ be the set of all conditions $p \in \mathbb{P}$ such that for some $k \in \mathrm{dom}(p) \cap A$ with $k > m$

and all $\iota \in I$ and $\ell \in J_\iota$:

$$\mathbf{M}^0_i \models \phi_\iota(g^0_{\beta^\ell_{\iota,0}}(k), \cdots, g^0_{\beta^\ell_{\iota,n_\iota-1}}(k), g^0_\alpha(k)) \Leftrightarrow \mathbf{M}^1_i \models \phi_\iota(g^1_{\beta^\ell_{\iota,0}}(k), \cdots, g^1_{\beta^\ell_{\iota,n_\iota-1}}(k), p(k)).$$

Let us show that each of the sets defined above are dense in $\mathbb{P}$. This is clear for the sets D_n. Now suppose that $A, \vec{\phi}$, $\langle \vec{\beta}^\ell_\iota : \iota \in I, \ell \in J_\iota \rangle$ and m are given as above and suppose that $p \in \mathbb{P}$. We find some $q \supseteq p$ in $\Sigma_{A,\vec{\phi},\langle \vec{\beta}^\ell_\iota : \iota \in I, \ell \in J_\iota \rangle, m}$. Let $k > m, \max(\mathrm{dom}(p))$ be such that $k \in A$. Such a k exists as A is unbounded in ω. Now $x = g^0_\alpha(k)$ witnesses

$$\mathbf{M}^0_i \models \exists x \bigwedge_{\iota \in I, \ell \in J_\iota} \phi_\iota(g^0_{\beta^\ell_{\iota,0}}(k), \cdots, g^0_{\beta^\ell_{\iota,n_\iota-1}}(k), x),$$

and hence by our induction hypothesis we can find some $y \in \mathbf{M}^1_i$ such that

$$M^1_i \models \bigwedge_{\iota \in I, \ell \in J_\iota} \phi_\iota(g^1_{\beta^\ell_{\iota,0}}(k), \cdots, g^1_{\beta^\ell_{\iota,n_\iota-1}}(k), y).$$

Set $q = p \cup \{(k, y)\}$. Then $q \in \mathbb{P}$ is as required.

The number of the sets we defined above is at most

$$\aleph_0 + |\mathscr{G}(\mathcal{U}_\alpha)| \cdot \aleph_0 \cdot |\alpha|^{<\aleph_0} \cdot \aleph_0 = \aleph_0 + |\alpha|$$

which is less than λ_i, and hence as MA_{λ_i}(countable) holds, there exists a filter $\mathbf{G} \subseteq \mathbb{P}$ meeting all the above dense sets. Set $g^1_\alpha = \bigcup_{p \in \mathbf{G}} p$. Let $\mathcal{U}'_{\alpha+1}$ be the filter generated by

$$\mathcal{U}_\alpha \cup \{Y_{\phi,\langle \beta_0, \cdots, \beta_{n-1} \rangle} : \phi, \beta_0, \cdots, \beta_{n-1} \text{ as in clause (e)}\}.$$

By the choice of the sets $\Sigma_{A,\vec{\phi},\langle \vec{\beta}^\ell_\iota : \iota \in I, \ell \in J_\iota \rangle, m}$, the above set has the finite intersection property and hence $\mathcal{U}'_{\alpha+1}$ is a proper filter. Now let $\mathcal{U}_{\alpha+1}$ be the filter generated by $\mathcal{U}'_{\alpha+1} \cup \{X_\alpha\}$ if this is a proper filter and let $\mathcal{U}_{\alpha+1}$ be the filter generated by $\mathcal{U}'_{\alpha+1} \cup \{\omega \setminus X_\alpha\}$ otherwise. If α is an odd ordinal, proceed in the same way, changing the role of the indices 0 and 1.

This completes the induction construction. Set

$$\mathcal{U} = \bigcup \{\mathcal{U}_\alpha : \alpha < 2^{\aleph_0}\}.$$

Then $\mathcal{U}$ is a non-principal ultrafilter on ω and $\mathbf{M}_0^\omega/\mathcal{U} \simeq \mathbf{M}_1^\omega/\mathcal{U}$ as witnessed by the function

$$\langle([g^0_\alpha]_\mathcal{U}, [g^1_\alpha]_\mathcal{U}) : \alpha < 2^{\aleph_0}\rangle.$$

This completes the proof of the theorem. □

We close the paper by proving the following consistency result, which is an analogue of Theorem 1.2, but the cofinality restriction on $2^{\aleph_0}$ is removed.

Let us recall that the Cohen forcing $\mathrm{Add}(\omega, \lambda)$ for adding λ many new Cohen reals is defined as $\mathrm{Add}(\omega, \lambda) = \{p : p \text{ is a finite partial function from } \omega \times \lambda \text{ into } 2\}$, ordered by inclusion.

Remark 3.4. (see [3, Proposition 22.10]) $\mathrm{Add}(\omega, \lambda)$ forces $\mathrm{Cov}(\text{meagre}) = 2^{\aleph_0}$.

Theorem 3.5. *Suppose $\lambda > \aleph_1$ and $\lambda^{\aleph_0} = \lambda$. Let $\mathbb{P} = \mathrm{Add}(\omega, \lambda)$. Then in $V[\mathbf{G}_\mathbb{P}]$, the following holds: if $\mathbf{M}_0 \equiv \mathbf{M}_1$ are models of size $\leq \aleph_1$ of the same countable vocabulary $\mathcal{L}$, then for some ultrafilter $\mathcal{U}$ on ω, $\mathbf{M}_0^\omega/\mathcal{U} \simeq \mathbf{M}_1^\omega/\mathcal{U}$.*

Proof. We may assume that $\mathrm{cf}(\lambda) > \aleph_1$, as otherwise the result follows from Theorem 1.2. Now suppose that $\mathbf{M}_0 \equiv \mathbf{M}_1$ are models of size $\leq \aleph_1$ of a countable vocabulary in $V[\mathbf{G}_\mathbb{P}]$. Then for some $\bar{\lambda} < \lambda$, $\mathbf{M}_0, \mathbf{M}_1 \in V[\mathbf{G}_{\mathbb{P}\restriction\bar{\lambda}}]$. By replacing V by $V[\mathbf{G}_{\mathbb{P}\restriction\bar{\lambda}}]$, we may assume that $\mathbf{M}_0, \mathbf{M}_1 \in V$.

As $|\lambda \cdot \omega_1| = \lambda$, we may assume that $\mathbb{P}$ is $\mathrm{Add}(\omega, \lambda \cdot \omega_1)$ so that forcing with $\mathbb{P}$ adds a sequence $\langle r_{\alpha,i} : \alpha < \lambda, i < \omega_1\rangle$ of reals of order type $\lambda \cdot \omega_1$.

For $i < \omega_1$, set $\mathbb{P}_i = \mathrm{Add}(\omega, \lambda \cdot i)$. As $\mathbb{P}$ is c.c.c., for every $X \subseteq \omega, X \in V[\mathbf{G}_\mathbb{P}]$, there exists some $i < \omega_1$ such that $X \in V[\mathbf{G}_{\mathbb{P}_i}]$. Proceed as in the proof of Theorem 1.2 with:

- $\lambda_i = \lambda \cdot (1 + i)$,
- $\langle \mathbf{M}^\ell_i : i < \omega_1\rangle$ as there,
- $\langle f^\ell_\alpha : \alpha < 2^{\aleph_0}\rangle$ is an enumeration of $\mathbf{M}^\omega_\ell$ in such a way that for $\alpha < \lambda\cdot(1+i)$, $f^\ell_\alpha \in \mathbf{M}^\ell_i$.

The rest of the argument is essentially as before. □

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