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Abstract. We show if we use countable support iteration of forcing notions not adding reals that satisfy additional conditions, then the limit forcing does not add reals. As a result we prove that we can amalgamate two earlier methods and prove the consistency with ZFC + GCH of two statements gotten separately earlier: Souslin hypothesis and non-club guessing. We also answer a question of Justin Moore by proving the consistency of one further case of "strong failure of club guessing" with GCH.

Contents

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§ 0. Introduction

One of the major problems in the theory of iterated forcing is to prove some preservation theorems. One example is the preservation of cardinals or cofinalities. By Solovay-Tennenbaum [11], finite support iteration of c.c.c. forcing notions is c.c.c. and hence it preserves all cardinals and cofinalities. In the case of countable support iterations, Shelah [5] proved that the countable support iteration of proper forcing notions is again proper, and hence it preserves $\aleph_1$.

In this paper we are interested in "not adding new reals", abbreviated NNR, in the case of countable support iteration of proper forcing notions. There is a lot of work in this direction, see for example [6, Ch.V,§7, Ch.VIII,§4, Ch.XVIII,§1,§2], [9, §3] and the references there. In this paper we present a more general preservation theorem, and as an application we prove the consistency with ZFC + GCH of Souslin's hypothesis with non-club guessing. We also prove the consistency with GCH of further cases of "strong failure of club guessing", in particular, we answer a question of Moore.

There are several limitations in preserving NNR at limit stages of countable support iterations that we will discuss some of them in Section 2. By results of Shelah, if, e.g. $\mathbf{V} = \mathbf{L}$, then the preservation of NNR fails. Indeed, assuming $\mathbf{V} = \mathbf{L}$, Shelah has built a countable support iteration of length ω^2 of NNR forcing notions of size $\aleph_1$ such that the limit adds a new real (see [6, XVIII. Lemma 1.1]). Justin Moore [2], solving a problem from [9, §3], proved that the continuum hypothesis implies the negation of the forcing axiom for the class of completely proper forcing notions. Indeed he proved in ZFC + CH that some proper forcing notion not adding reals and satisfying a "strong form of the medicine against weak diamond" has no generic, in fact, is a tree with no branches.

The paper is organized as follows. In Section 1, we present some definitions and results which are needed for the rest of the paper. In Section 2, we discuss some obstacles about preservation of NNR in countable support iterations and suggest some ideas about how to overcome them.

In Section 3, we present sufficient conditions for countable support iteration of proper forcing notions, not to add reals. For this we define "reasonable parameters $\mathfrak{p}$" and we have two main demands. One (clause (c) of Definition 3.9) is a weakening of "α-proper for every $\alpha < \omega_1$". This time it has the form (on $\mathbb{Q}_i$), $\mathfrak{p}$-proper which informally says that: if $\mathfrak{p} \in N, Y \subseteq \{M \in N : M$ appropriate$\}$ is α-large, then for some $(N, \mathbb{Q}_i)$-generic condition $q \geq p, q$ forces that $\{M \in Y : M[\underset{\sim}{\mathbf{G}}_{\mathbb{Q}_i}] \cap \mathbf{V} = M\}$ is α-large (the meaning of α-large depends on $\mathfrak{p}$). The other main demand (clause (d) of Definition 3.9) is a "weak diamond preventive".

We then show that α-properness for $\alpha < \omega_1$ is sufficient for the first main demand (in Lemma 3.17(3)). The demand on the games for $\mathfrak{p}$ helps to prove the preservation of $\mathfrak{p}$-properness.

The preservation theorem in Section 3 does not, for standard $\mathfrak{p}$, cover shooting a club $C \subseteq \omega_1$ running away for $C_\delta \subseteq \delta = \sup(C_\delta), C_\delta$ small. For this we will use, in Section 4, $(\mathfrak{p}, \alpha, \beta)$-proper for enough pairs $\alpha \leq \beta < \ell g(\mathfrak{p})$ (so starting from β-large we get α-large; for many α we can choose $\beta = \alpha$, but during the inductive proof we pass through cases of $\alpha < \beta$). Here we introduce various definitions and basic facts needed.

In Section 5, we present the natural forcing showing $\kappa = 2$ is interesting (not only $\kappa = \aleph_0$) (from [5, Ch.VIII,§4]). We show that the natural forcing (see above) for running away from $C_\delta \subseteq \delta$, of small order type (see [6, Ch.XVIII,§2]) falls under our framework for delayed properness. We give examples: running away from $\langle C_{\delta,0}, C_{\delta,1} : \delta < \omega_1$ limit$\rangle, C_{\delta,0}, C_{\delta,1}$ are disjoint closed subsets of δ with no restrictions on their order type so we ask for $C, C \cap C_{\delta,0}$ or $C \cap C_{\delta,1}$ to be bounded in δ and more.

In Section 6, we give a sufficient condition for the limit forcing not to add reals. We here are weakening the demand "$\mathfrak{p}$-proper", using $(\mathfrak{p}, \alpha, f(\alpha))$-proper instead of $(\mathfrak{p}, \alpha, \alpha)$-proper, what we called delayed properness. The price is that here $\mathfrak{p}$ has length of large cofinality, so essentially we catch our tails on a club of it. Also the results here cover the examples.

In Section 7, we derive some forcing axioms from our preservation theorems and give several examples that fit into our axioms.

Finally in Section 8, we answer a question of Justin Moore, which is related to the failure of weak club guessing at ω_1 in the presence of CH.

The results and methods in this paper are all due to the second author. The first author's contribution was to fill in some details and to write the paper.

§ 1. Some preliminaries

In this section we present some preliminaries that are needed for the rest of the paper. We assume familiarity with the theory of iterated forcing and countable support iterations. For a forcing notion $\mathbb{P}$ and conditions $p, q \in \mathbb{P}$, we say q is stronger than p if $q \geq p$.

Definition 1.1. (1) $\mathbb{Q}$ is α-proper if whenever χ is large enough regular, $\bar{N} = \langle N_i : i \leq \alpha \rangle$ is an increasing and continuous chain of countable elementary submodels of $(\mathscr{H}(\chi), \in)$ with $\alpha, \mathbb{Q} \in N_0$ and $\bar{N} \restriction (i+1) \in N_{i+1}$, if $p \in \mathbb{Q} \cap N_0$, then there is q, $p \leq q \in \mathbb{Q}$ such that q is $(N_i, \mathbb{Q})$-generic for each $i \leq \alpha$.

(2) We say $\mathbb{Q}$ is $(< \omega_1)$-proper if $\mathbb{Q}$ is α-proper for any $\alpha < \omega_1$.

(3) We say $\mathbb{Q}$ is $(<^+ \omega_1)$-proper if it satisfies clause (1) for any $\alpha < \omega_1$ even omitting "$\alpha \in N_0$".

Definition 1.2. Suppose $\mathbb{P}$ is a forcing notion, $p \in \mathbb{P}$ and N is a model with $\mathbb{P} \in N$. Then

(1) $\mathrm{Gen}(N, \mathbb{P}) = \{\mathbf{G} \subseteq \mathbb{P} \cap N : \mathbf{G}$ is a $\mathbb{P} \cap N$-generic filter over $N\}$.

(2) $\mathrm{Gen}^+(N, \mathbb{P}) = \{\mathbf{G} \in \mathrm{Gen}(N, \mathbb{P}) : G$ has an upper bound in $\mathbb{P}\}$.

(3) $\mathrm{Gen}(N, \mathbb{P}, p) = \{\mathbf{G} \in \mathrm{Gen}(N, \mathbb{P}) : p \in \mathbf{G}\}$.

One important notion that is useful in proofs for showing that certain countable support iterations do not add reals is Shelah's notion of completeness system.

Definition 1.3. ([6, Ch. V, Definition 5.2]) A *completeness system* for a forcing notion $\mathbb{P}$ is a function $\mathbb{D}$ such that the following statements hold:

(1) For a sufficiently large θ, the domain of $\mathbb{D}$ consists of pairs (N, p), where $N \prec (H(\theta), \in)$ is countable, $\mathbb{P} \in N$ and $p \in \mathbb{P} \cap N$,

(2) For every $(N, p) \in \mathrm{dom}(\mathbb{D})$, $\mathbb{D}(N, p)$ is a collection of subsets of $\mathrm{Gen}(N, \mathbb{P}, p)$.

Definition 1.4. ([6, Ch. V, Definition 5.2]) Suppose κ is a cardinal. We say $\mathbb{D}$ is a κ-completeness system for $\mathbb{P}$, if it is a completeness system for $\mathbb{P}$ and for every $(N, p) \in \mathrm{dom}(\mathbb{D})$, the intersection of fewer than $1 + \kappa$ elements of $\mathbb{D}(N, p)$ is nonempty.

Definition 1.5. ([6, Ch. V, Definition 5.4]) A completeness system $\mathbb{D}$ for $\mathbb{P}$ is *simple* if there is a second order formula Ψ such that $\mathbb{D}(N, p) = \{\mathcal{G}_X : X \subseteq N\}$, where

$$\mathcal{G}_X = \{\mathbf{G} \in \mathrm{Gen}(N, \mathbb{P}, p) : (N, \in, \mathbb{P} \cap N) \models \Psi(\mathbf{G}, X)\}.$$

Definition 1.6. ([6, Ch. V, Definition 5.3]) Suppose $\mathbb{D}$ is a simple completeness system for $\mathbb{P}$. Then $\mathbb{P}$ is said to be $\mathbb{D}$-complete, if for every $(N, p) \in \mathrm{dom}(\mathbb{D})$, $\mathrm{Gen}^+(N, \mathbb{P}, p)$ contains an element of $\mathbb{D}(N, p)$.

The next theorem of Shelah gives a sufficient condition for a countable support iteration of forcing notions to not add new reals.

Theorem 1.7. *([6, Ch. VIII, Theorem 4.5]) A countable support iteration of forcing notions which are $<\omega_1$-proper and $\mathbb{D}$-complete with respect to a simple 2-completeness system does not introduce reals.*

Definition 1.8. ([6, Ch. VII, Definition 1.2]) The forcing notion $\mathbb{P}$ satisfies the κ-e.c.c. (κ-extra chain condition), if there is a binary relation R on $\mathbb{P}$ such that:

- For any sequence $\langle p_i : i < \kappa \rangle$ of elements of $\mathbb{P}$, there are pressing down functions $f_n : \kappa \to \kappa$ (i.e., for all $\alpha < \kappa, f_n(\alpha) < 1 + \alpha$) for $n < \omega$ such that for all $0 < i, j < \kappa$,
$$\bigwedge_{n<\omega} (f_n(i) = f_n(j)) \implies p_i R p_j.$$
- If $\langle p_i : i \leq \omega \rangle$ and $\langle q_i : i \leq \omega \rangle$ are increasing sequences in $\mathbb{P}$ and for all $n < \omega, p_n R q_n$, then there is an r such that for all $n < \omega, r \geq p_n, q_n$.

Definition 1.9. ([6, Ch. VIII, Definition 2.1]) The forcing notion $\mathbb{P}$ satisfies the κ-pic (κ-properness isomorphism condition), if the following holds for any large enough regular cardinal λ: Suppose $i < j < \kappa, N_i, N_j \prec (\mathscr{H}(\lambda), \in, \lhd_\lambda)$ (where $\lhd_\lambda$ is a well-ordering of $\mathscr{H}(\lambda)$) are countable such that $\kappa, \mathbb{P} \in N_i \cap N_j$, $i \in N_i, j \in N_j, N_i \cap \kappa \subseteq j, N_i \cap i = N_j \cap j, p \in N_i \cap \mathbb{P}$ and $h : N_i \cong N_j$ is such that $h \restriction N_i \cap N_j$ is identity and $h(i) = j$. Then there exists $q \in \mathbb{P}$ such that:

- $q \geq p, h(p)$ and for every maximal antichain $\mathscr{I} \in N_i$ of $\mathbb{P}$, we have that $\mathscr{I} \cap N_i$ is predense above q and similarly for $\mathscr{I} \in N_j$,
- for every $r \in N_i \cap \mathbb{P}$ and $q' \geq q$, there is $q'' \geq q'$ such that
$$r \leq q'' \iff h(r) \leq q''.$$

See [6, Ch.VII, §1] and [6, Ch.VIII, §2] for more information about the above two defined notions.

Theorem 1.10. *Assume CH holds.*

(1) *If* $\mathbb{P}$ *is a countable support iteration of length at most* ω_2 *whose iterands are* $< \omega_1$*-proper,* $\mathbb{D}$*-complete for some* $\aleph_1$*-completeness system from* $\mathbf{V}$ *and satisfy the* $\aleph_2$*-e.c.c, then* $\mathbb{P}$ *satisfies the* $\aleph_2$*-c.c. The same result holds if we replace "*$\aleph_1$*-completeness system" by "*$\aleph_0$*-completeness system" or by "2-completeness system".*

(2) *If* $\mathbb{P}$ *is a countable support iteration of length at most* ω_2 *whose iterands satisfy the* $\aleph_2$*-pic, then* $\mathbb{P}$ *satisfies the* $\aleph_2$*-c.c.*

Proof. (1). For the case of $\aleph_1$-completeness system see [6, Ch.VII, Lemmas 1.3]. The case of $\aleph_0$-completeness system follows from [6, Ch.VII, Lemmas 1.6] and the case of 2-completeness system follows from the above resuls combined with [6, Ch.VIII, Theorem 4.5 and Lemma 4.13].

(2). See [6, Ch.VIII, Lemma 2.4]. □

§ 2. Obstacles for NNR preservation

In this section we give lengthy explanation of the problems and proofs for NNR countable support iterations of proper forcing notions, and suggest some ideas about how to overcome them. These ideas will be made precise in the later sections of the paper.

Definition 2.1. (1) Let K_0 be the family of countable support iterations $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \alpha\rangle$. We denote $\mathbb{P}_\alpha = \mathrm{Lim}(\bar{\mathbb{Q}})$.

(2) We say $\bar{\mathbb{Q}} \in K_0$ is proper if for each $i < \alpha$, $\Vdash_{\mathbb{P}_i}$ "$\underset{\sim}{\mathbb{Q}}_i$ is proper". Note that it follows that $\mathbb{P}_j/\underset{\sim}{G}_{\mathbb{P}_i}$ is proper for $i < j \leq \alpha$ (see [5] or [6]).

(3) We say $\bar{\mathbb{Q}} \in K_0$ is ${}^\omega\omega$-bounding if for each $i < \alpha$, $\Vdash_{\mathbb{P}_i}$ "$\underset{\sim}{\mathbb{Q}}_i$ is ${}^\omega\omega$-bounding" [1]. It again follows that $\mathbb{P}_j/\dot{G}_{\mathbb{P}_i}$ is ${}^\omega\omega$-bounding for $i < j \leq \alpha$.

[1]The forcing notion $\mathbb{Q}$ is ${}^\omega\omega$-bounding (in the universe $\mathbf{V}$) if every $f \in ({}^\omega\omega)^{\mathbf{V}^{\mathbb{Q}}}$ is bounded by some $g \in ({}^\omega\omega)^{\mathbf{V}}$.

(4) We say $\bar{\mathbb{Q}}$ is NNR if for each $i < \alpha$, the forcing notion $\mathbb{P}_{i+1}$ adds no reals, or equivalently if for $i < \alpha, \Vdash_{\mathbb{P}_i}$ "$\underset{\sim}{\mathbb{Q}}_i$ adds no reals" and for each $\beta < \alpha, \mathbb{P}_\beta$ adds not reals.

It would be nice if also NNR is preserved in limit stages of the iteration. But this is wrong for at least two known reasons, explained below:

$\bigotimes_1$ weak diamond

$\bigotimes_2$ existence of clubs.

Let us explain these obstacles in more details and the way to avoid them.

Weak diamond:

Let us first explain the obstacle arising from weak diamond. Given a stationary set $S \subseteq \omega_1$, recall that the weak diamond Φ_S says: for each function $F :{}^{<\omega_1}2 \to 2$, there exists $g : \omega_1 \to 2$ such that for each $f : \omega_1 \to 2$, the set

$$\{\delta \in S : g(\delta) = F(f \restriction \delta)\}$$

is stationary. By Devlin-Shelah [1] (see also [5, Ch.XII,§1] or [6, AP,§1]), Φ_{ω_1} is equivalent to $2^{\aleph_0} < 2^{\aleph_1}$.

Now let $\bar{\eta} = \langle \eta_\delta : \delta < \omega_1, \delta \text{ limit}\rangle$ be a ladder system, where $\eta_\delta = \langle \eta_\delta(n) : n < \omega\rangle$ is an increasing ω-sequence of ordinals cofinal in δ. Let D be a non-principal ultrafilter on ω. For $f \in {}^{\omega_1}2$ and a limit ordinal $\delta < \omega_1$ let

$$\mathrm{Av}_D(f, \eta_\delta) = \ell \iff \{n : f(\eta_\delta(n)) = \ell\} \in D.$$

Consider the following natural question:

Question 2.2. (CH) Given $\bar{e} = \langle e_\delta : \delta < \omega_1, \delta \text{ limit}\rangle, e_\delta \in \{0, 1\}$, is there $f \in {}^{\omega_1}2$ such that for a club of $\delta < \omega_1$ we have $e_\delta = \mathrm{Av}_D(f, \eta_\delta)$?

Naturally, trying to prove the consistency of this statement, we should use a countable support iteration $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \omega_2 \rangle$, where for each $i < \omega_2$, for some $\bar{e} \in V^{\mathbb{P}_i}$ as in Question 2.2, $\mathbb{Q}_i$ is defined in $V^{\mathbb{P}_i}$ as $\mathbb{Q}_i = \mathbb{Q}_{\bar{e}}$, where

$$\mathbb{Q}_{\bar{e}} = \{f : \text{ for some } \zeta < \omega_1, f \in {}^{\zeta}2 \text{ and for every limit ordinal } \delta \leq \zeta \text{ we have } \mathrm{Av}_D(f, \eta_\delta) = e_\delta\}.$$

This is a very nice forcing notion, it is proper (even $< \omega_1$-proper, see below) and NNR. For example, let us show that $\mathbb{Q}_{\bar{e}}$ is NNR.

Thus let $p \in \mathbb{Q}_{\bar{e}}$, $\underset{\sim}{\tau} \in V^{\mathbb{Q}_{\bar{e}}}$ and $p \Vdash$ "$\underset{\sim}{\tau} : \omega \to \mathrm{Ord}$ is a function". Let χ be a large enough regular cardinal and let $\bar{N} = \langle N_i : i \leq \omega^2 \rangle$ be an increasing and continuous chain of countable elementary submodels of $(\mathscr{H}(\chi), \in)$ with $\bar{e}, \mathbb{Q}_{\bar{e}}, p, \underset{\sim}{\tau} \in N_0$ and $\bar{N} \restriction (i+1) \in N_{i+1}$. Let $\delta(i) = N_i \cap \omega_1$. So $\langle \delta(i) : i \leq \omega^2 \rangle$ is a strictly increasing and continuous sequence of countable ordinals. Since $\eta_{\delta(\omega^2)}$ has order type ω,

$$W = \{i < \omega^2 : \exists n (\delta_i \leq \eta_{\delta(\omega^2)}(n) < \delta_{i+1})\}$$

has order type ω as well. So, for each $n < \omega$, we can find $\ell_n < \omega$ such that

$$\bigwedge_{n<\omega} \bigwedge_{m<\omega} \omega \cdot n + \ell_n + m \notin W.$$

We choose, by induction on $n < \omega$, $p_n \in \mathbb{Q}_{\bar{e}}$ and $a_n \in \mathrm{Ord}$ such that:

- $p \leq p_n$,
- $p_{n-1} \leq p_n$,
- $p_n \in N_{\omega \cdot n + \ell_n + 1}$,
- p_n forces some value for $\underset{\sim}{\tau}(n)$, say $p_n \Vdash \underset{\sim}{\tau}(n) = \check{a}_n$
- On $[\delta_{\omega \cdot m + \ell_m}, \delta_{\omega \cdot (m+1) + \ell_{m+1}}) \cap \mathrm{Rang}(\eta_\delta) \setminus \mathrm{dom}(p)$, p_n agrees with $e_{\delta(\omega^2)}$.

This is easily seen to be possible by the choice of ℓ_n's. Then $q = \bigcup_{n<\omega} p_n$ is a condition in $\mathbb{Q}_{\bar{e}}$ and it forces $\underset{\sim}{\tau} = \langle \check{a}_n : n < \omega \rangle \in \check{V}$.

Also note that for every $\alpha < \omega_1$, $\mathscr{I}_\alpha = \{f \in \mathbb{Q}_{\bar{e}} : \alpha \subseteq \mathrm{dom}(f)\}$ is a dense open subset of $\mathbb{Q}_{\bar{e}}$, and hence if $\mathbf{G}$ is $\mathbb{Q}_{\bar{e}}$-generic over V, then $f = \bigcup_{f \in \mathbf{G}} f : \omega_1 \to 2$ is as requested in Question 2.2, for $\bar{e}$. But clearly the weak diamond tells us for this case that the answer is no, that is:

$$\exists \bar{e} \forall f \in {}^{\omega_1}2 \exists^{\mathrm{stat}} \delta (e_\delta \neq \mathrm{Av}_D(f, \eta_\delta)).$$

In fact this holds for any function $\mathrm{Av}' : \bigcup_{\delta < \omega_1} {}^{\delta}2 \to \{0, 1\}$. So if $\bar{\mathbb{Q}}$ is going to preserve NNR, the desired demand on $\mathbb{Q}_i$'s should exclude the $\mathbb{Q}_{\bar{e}}$'s. We now explain a way to overcome the above difficulty. Let us first give a definition.

Definition 2.3. (1) Let K_1 be the class of proper ${}^{\omega}\omega$-bounding iterations $\bar{\mathbb{Q}} \in K_0$.

(2) Let K_2 be the class of NNR iterations $\bar{\mathbb{Q}} \in K_1$.

(3) Let K_3 be the class of $\bar{\mathbb{Q}} \in K_2$ such that if

(a) χ is a large enough regular cardinal,

(b) $N \prec (\mathscr{H}(\chi), \in)$ is countable,

(c) $\bar{\mathbb{Q}} \in N$,

(d) $i \in \ell g(\bar{\mathbb{Q}}) \cap N$,

(e) $p \in \mathbb{P}_{i+1} \cap N$,

(f) $q_0, q_1 \in \mathbb{P}_i$ are $(N, \mathbb{P}_i)$-generic (i.e. $q_\ell \Vdash$ "$N[\underset{\sim}{\mathbf{G}}_{\mathbb{P}_i}] \cap \mathbf{V} = N$"),

(g) $q_\ell \Vdash$ "$\underset{\sim}{\mathbf{G}}_{\mathbb{P}_i} \cap N = \mathbf{G}^*$",

(h) $p \restriction i \leq q_\ell$,

then we can find q_0', q_1' and $\mathbf{G}^{**}$ such that for $\ell = 1, 2$ we have

(i) $q_\ell \leq q_\ell' \in \mathbb{P}_{i+1}$,

(j) $p \leq q_\ell'$,

(k) $q_\ell' \Vdash$ "$\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{i+1}} \cap N = \mathbf{G}^{**}$",

(l) q_ℓ' is $(N, \mathbb{P}_{i+1})$-generic, so $\mathbf{G}^{**} \subseteq \mathbb{P}_{i+1} \cap N$ is generic over N.

Clause (3) of the above definition tries to say the following. We know $\mathop{\mathbf{G}}\limits_{\sim}{}_{\mathbb{P}_i} \cap N$ (as being $\mathbf{G}^*$) and we are looking at $N[\mathbf{G}^*]$ (formally, only its isomorphism type). So we know $\mathop{\mathbb{Q}}\limits_{\sim}{}_i^N[\mathbf{G}^*]$. We would like to find $\mathbf{G}' \subseteq \mathop{\mathbb{Q}}\limits_{\sim}{}_i^N[\mathbf{G}^*]$ generic over $N[\mathbf{G}^*]$, so that $\mathbf{G}^*, \mathbf{G}'$ will determine $\mathbf{G}^{**}$. But we need a guarantee that $\mathbf{G}'$ will have an upper bound in $\mathop{\mathbb{Q}}\limits_{\sim}{}_i[\mathbf{G}_{\mathbb{P}_i}]$. If we know $\mathbf{G}_{\mathbb{P}_i}$, this is fine; but in a sense, we are given 2 candidates by q_0, q_1 and can increase them to $q'_0{\restriction}i, q'_1 \restriction i$, and have to find $\mathbf{G}'$ "accepted" by both.

The weak diamond obstacle was overcame in [6, Ch.V,§7] using $\aleph_1$-completeness systems and in [6, Ch.XVIII,§4] using 2-completeness systems. Here we show that being in K_3 is sufficient to overcome this difficulty. Indeed, we will assume something like the following. Many times in some sense $q_0, q_1 \in \mathbb{P}_i$ are $(N, \mathbb{P}_i)$-generic, $p \in \mathop{\mathbb{Q}}\limits_{\sim}{}_i \cap N, q_\ell \Vdash_{\mathbb{P}_i}$ "$\mathop{\mathbf{G}}\limits_{\sim}{}_{\mathbb{P}_i} \cap N = \mathbf{G}^*$" and for some $\mathbf{G}', q'_0 \geq q_0, q'_1 \geq q_1$ in $\mathbb{P}_{i+1}$ we have $\mathbf{G}' \subseteq (\mathop{\mathbb{Q}}\limits_{\sim}{}_i \cap N)[\mathbf{G}^*]$ and $q'_\ell \Vdash_{\mathbb{P}_i}$ "$\mathop{\mathbf{G}}\limits_{\sim}{}_{\mathbb{Q}_i} \cap N[\mathbf{G}^*] = \mathbf{G}'$" and $p \in \mathbf{G}'$.

Unfortunately, this is not sufficient to overcome with the other obstacle $\bigotimes_2$. There is an example where for some incomparable q_0 and q_1 in $\mathbb{Q}_i$, $\mathop{E}\limits_{\sim}{}_i$ a $\mathbb{Q}_i$-name of a club and for some $\alpha(q_0, q_1)$ we have:

$$q_\ell \leq q'_\ell$$

$$q'_\ell \Vdash \text{"}\mathop{E}\limits_{\sim}{}_i \cap \delta = E_i^\delta \Rightarrow E_i^{\delta_0} \cap E_1^{\delta_1} \backslash \alpha(q_0, q_1) \text{ is finite"}.$$

This leads us to suggest another idea to overcome the obstacle which arises from the existence of clubs.

<u>**Existence of clubs**</u>:

Let us now explain the obstacle that arises from working with clubs. This problem was already overcame either by using $(< \omega_1)$-properness or by a kind of "finite powers are proper".

As is shown in [5], [6], $(< \omega_1)$-properness is an antidote to such problems, i.e. against $\bigotimes_2$. Though this is fine for many applications, like specializing an Aronszajn tree and many others, but this requirement is too strong (see [3]). For example consider the following question.

Question 2.4. Let $\bar{C} = \langle C_\delta : \delta < \omega_1, \delta \text{ limit}\rangle$ where $C_\delta \subseteq \delta = \sup(C_\delta), \mathrm{otp}(C_\delta) = \omega$ or at least $< \delta$. Is there a club E of ω_1 such that for each limit ordinal $\delta < \omega_1$, we have $\delta > \sup(C_\delta \cap E)$ (i.e. is this consistent with CH)?

The natural forcing notion for adding such a club is given by

$$\mathbb{Q}^1_{\bar{C}} = \{f : \text{ for some non-limit } \alpha < \omega_1 \text{ we have } f \in {}^\alpha 2, f^{-1}(\{1\}) \\ \text{is closed and } \delta < \alpha \text{ limit } \Rightarrow \sup(f^{-1}(\{1\}) \cap C_\delta) < \delta\}.$$

This forcing notion is not even ω-proper, for if $\langle N_i : i \leq \omega\rangle$ satisfies $C_{N_\omega \cap \omega_1} = \{N_i \cap \omega_1 : i < \omega\}$, then no $f \in \mathbb{Q}^1_{\bar{c}}$ is $(N_i, \mathbb{Q}^1_{\bar{C}})$-generic, for infinitely many i's.

A solution to this problem was suggested in [6, Ch.XVIII,§2]) by demanding that each $\mathbb{P}_i \times \mathbb{P}_i$ is proper for $i < \ell g(\bar{\mathbb{Q}})$. While this is fine for $\mathbb{Q}^1_{\bar{C}}$, this seems to exclude specializing an Aronszajn tree without adding reals.

In the proofs, we usually arrive to a situation as follow:

- $\bar{\mathbb{Q}} \in N_0 \in N$,
- $N_0 \prec (\mathscr{H}(\chi), \in)$ and $N \prec (\mathscr{H}(\chi), \in)$ are countable,
- q_ℓ is $(N, \mathbb{P}_i)$-generic and $(N_0, \mathbb{P}_i)$-generic (for $\ell < 2$),
- q_ℓ forces that $\underset{\sim}{\mathbf{G}}_{\mathbb{P}_i} \cap N = \mathbf{G}_\ell$ (for $\ell < 2$),
- $\mathbf{G}^* = \mathbf{G}_1 \cap N_0 = \mathbf{G}_2 \cap N_0$,
- $i, j, p \in N_0[\mathbf{G}^*], i \leq j \leq \ell g(\bar{\mathbb{Q}})$,
- $p \in P_j, p \restriction i \in \mathbf{G}^*$ (and possibly more).

We would like to find $\mathbf{G}' \subseteq \mathbb{P}^N_j/\mathbf{G}^*$ generic over N_0 such that q_0 and q_1 both force that it has an upper bound in $P_j/\underset{\sim}{\mathbf{G}}_{\mathbb{P}_i}$. If $j = i+1$ this means $\mathbf{G}' \subseteq \underset{\sim}{\mathbb{Q}}_i[\mathbf{G}^*]$ is generic over N_0 such that q_0, q_1 both force that $\mathbf{G}'$ has an upper bound in $\underset{\sim}{\mathbb{Q}}_i[\mathbf{G}_{\mathbb{P}_i}]$.

It is natural to demand $\mathbf{G}' \in N$, as otherwise the two possible generic extensions (for q_0 and q_1) become unrelated. For the case $j = i+1$, the medicine against $\bigotimes_1$ should help us. But we need it for every j. Naturally we prove it by induction on j, and the successor case can be reduced to the case $j = i+1$.

But to continue to a limit case, we need $\mathbf{G}' \in N$ and more: for some intermediate N_1 with $N_0 \in N_1 \in N$, we also need $\bigwedge_\ell [q_\ell \Vdash N_1[\underset{\sim}{\mathbf{G}}_{\mathbb{P}_i}] \cap \mathbf{V} = N_1]$. So the clubs of elementary submodels which q_0, q_1 induce on $\{M \prec N : M \in N\}$ should have non-trivial intersection. This is a major point and it has always appeared in some form. Here the medicine against $\bigotimes_2$ should help, in some way there will be many possible N_1's; but its help has a price, that is we have to carry it during the induction. On the other hand the models playing the role of N_1 may change, we may "consume it and discard it".

Note that the discussion is on two levels. Necessary limitations of universes with CH on the one hand, and how we try to carry the inductive proof on appropriate iterations on the other hand; the connection though is quite tight.

So we shall try for $j \in \ell g(\bar{\mathbb{Q}}) \cap N_0$ to extend the situation with i being replaced by j while $\mathbf{G}^*$ is being increased to $\mathbf{G}^{**}$. We shall prove by induction some suitable facts, with $\mathbf{G}^{**}$ the object we are really interested in. We are given $q_1, q_2 \in \mathbb{P}_i$ and would like to find suitable $q'_1, q'_2 \in \mathbb{P}_j$ such that $q'_\ell \restriction i = q_\ell$. This last requirement helps us in limit steps to find an upper bound.

So the real action occurs for j limit, hence we choose $\zeta_n \in N \cap [i,j)$ such that $\zeta_0 = i, \zeta_n < \zeta_{n+1}$ (sometimes better to have i and each ζ_n non-limit) and $\bigcup_{n<\omega} \zeta_n = \sup(j \cap N)$. You can think of:

> in each case of limit j, proving the inductive statement, we choose a "surrogate" for N called N_1, during the induction it serves like N, in the limit dealing with $\zeta_0, \zeta_1, \ldots$ using the induction hypothesis on N_1 we get $\mathbf{G}^{**}$ which may not be in N_1 but is in N.

So we try to choose by induction on n, the conditions $q_{0,n}, q_{1,n}$ and $\mathbf{G}^*_n$ such that:

- $q_{\ell,n} \in \mathbb{P}_{\zeta_n}$ is $(N, \mathbb{P}_{\zeta_n})$-generic,
- $q_{\ell,0} = q_\ell$,
- $q_{\ell,n+1} \restriction \zeta_n = q_{\ell,n}$,
- $\mathbf{G}^*_n \in N_1$,
- $\mathbf{G}^*_n \subseteq P_{\zeta_n} \cap N$ is generic over N, and
- $q_{\ell,n} \Vdash$ "$\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\zeta_n}} \cap N = \mathbf{G}^*_n$.

The construction of the $\mathbf{G}^*_n$ should use little information on the actual $q_{\ell,n}$ so that the choices of the $\mathbf{G}^*_n$ can be carried say inside N_1 so that $\langle \mathbf{G}^*_n : n < \omega \rangle \in N$. In fact several models will play a role like N_1.

By the proof of the preservation of ${}^\omega\omega$-bounding we can choose some N_1 and demand "$q_{\ell,n}$ gives to each $\mathbb{P}_{\zeta_n}$-name of an ordinal $\underset{\sim}{\tau}_n \in N_1$, only finitely many possibilities".

Let us now explain how $(< \omega_1)$-properness or remaining proper under products can help in such arguments. If the forcings are $(< \omega_1)$-proper , then we can assume in the beginning that $\langle N_{1,\gamma} : \gamma \in A \rangle \in N$ is an increasing and continuous chain of countable elementary submodels of some $(H(\chi), \in)$, $N_0 \prec N_{1,\gamma} \prec N, \langle N_{1,\gamma} : \gamma \leq \beta \rangle \in N_{\beta+1}$ with $A = (j+1) \cap N \backslash i$ and assume q_ℓ is $(N_{1,\gamma}, \mathbb{P}_i)$-generic for $\gamma \in A$ (similarly for q'_0, q'_1, j in the conclusion) and demand $q_{\ell,n}$ is $(N_{1,\gamma}, \mathbb{P}_{\zeta_n})$-generic for $n < \omega$ and $\gamma \in A \backslash \zeta_n$.

If components of the iteration remain proper under products, then we demand things like "(q_0, q_1) is $(N_1, \mathbb{P}_i \times \mathbb{P}_i)$-generic" so this gives many common N_1's, but to preserve this we need more complicated situations. Instead of a "tower" of models of countable length, we have a finite tower of models where on the bottom we are computing $\mathbf{G}^{**} \cap \mathbb{P}_{\zeta_n}$ and as we go up, less and less is demanded.

In this paper, we will deal with a condition which follows from both "$(< \omega_1)$-properness" and (essentially) "the square of the forcing notion is proper". We call this $\mathfrak{p}$-properness where "$\mathbb{Q}$ is $\mathfrak{p}$-proper" says that if Y is a large family of $M \prec N$ and if $p \in \mathbb{Q} \cap N$

and $\mathbb{Q} \in N$, then for some q we have $q \geq p$ is $(N, \mathbb{Q})$-generic and $q \Vdash$ "$\{M \in Y : M[\underset{\sim}{\mathbf{G}}_{\mathbb{Q}}] \cap \mathbf{V} = M\}$ is large".

§ 3. Preservation of not adding reals

In this section we define the notion of $\mathfrak{p}$-properness, for a reasonable parameter $\mathfrak{p}$, and prove some preservation theorems.

Definition 3.1. (1) A pseudo-filter on a set N is a family D of subsets of N which is closed under supersets. If D is a pseudo-filter on N, then we set $D^- = \mathscr{P}(N) \setminus D$.
(2) If D is a filter on N, then set $D^+ = \{X \subseteq N : N \setminus X \notin D\}$.

Definition 3.2. We say $\mathfrak{p} = (\bar{\chi}, \bar{R}, \bar{\mathscr{E}}, \bar{D}) = (\bar{\chi}^{\mathfrak{p}}, \bar{R}^{\mathfrak{p}}, \bar{\mathscr{E}}^{\mathfrak{p}}, \bar{D}^{\mathfrak{p}})$ is a reasonable parameter, when for some ordinal α^*, denoted $\ell g(\mathfrak{p})$, we have:

(a) $\bar{\chi} = \langle \chi_\alpha : \alpha < \alpha^* \rangle$, where χ_α is a regular cardinal and $\mathscr{H}((\bigcup_{\beta<\alpha} \chi_\beta)^+) \in \mathscr{H}(\chi_\alpha)$.
(b) $\bar{R} = \langle R_\alpha : \alpha < \alpha^* \rangle$, where $R_\alpha \in \mathscr{H}(\chi_\alpha)$.
(c) $\bar{\mathscr{E}} = \langle \mathscr{E}_\alpha : \alpha < \alpha^* \rangle$, where $\mathscr{E}_\alpha \subseteq [\mathscr{H}(\chi_\alpha)]^{\leq \aleph_0}$ is stationary.
(d) $\bar{D} = \langle D_\alpha : \alpha < \alpha^* \rangle$, where D_α is a function with domain $\mathscr{E}_\alpha$, and for $a \in \mathscr{E}_\alpha, D_\alpha(a)$ is a pseudo-filter on a.
(e) for $\alpha < \alpha^*$ set $\mathfrak{p}^{[\alpha]} =: \langle \bar{\chi} \restriction \alpha, \bar{R} \restriction (\alpha+1), \bar{\mathscr{E}} \restriction \alpha, \bar{D} \restriction \alpha \rangle$, so it belongs to $\mathscr{H}(\chi_\alpha)$.
(f) if $a \in \mathscr{E}_\alpha$, then for some countable $N \prec (\mathscr{H}(\chi_\alpha), \in)$, a is the universe of N, so we may write $D_\alpha(N)$ instead of $D_\alpha(a)$ and $N \in \mathscr{E}_\alpha$ instead of $|N| \in \mathscr{E}_\alpha$.
(g) if $\alpha < \alpha^*$ and $N \in \mathscr{E}_\alpha$, then $\mathfrak{p}^{[\alpha]} \in N$, so $\alpha \in N$.
(h) for $N \in \mathscr{E}_\alpha$ and $X \subseteq N$ we have:
$$X \in D_\alpha(N) \iff (\bigcup_{\beta<\alpha} \mathscr{E}_\beta) \cap X \in D_\alpha(N).$$
(i) if $N \in \mathscr{E}_\alpha, X \in D_\alpha(N), \beta \in \alpha \cap N$ and $y \in N \cap \mathscr{H}(\chi_\beta)$, then for some $M \in \mathscr{E}_\beta \cap X$ we have $X \cap M \in D_\beta(M)$ and $y \in M$

Let us explain a little about the intended meaning of the above definition. The requirement (a) is just technical. About R_α, we could require R_α is a relation on $\mathscr{H}(\chi_\alpha)$, in a sense it codes a club of $[\mathscr{H}(\chi_\alpha)]^{\leq\aleph_0}$. In clause (e), we considered $\bar{R} \restriction (\alpha+1)$ and not $\bar{R} \restriction \alpha$. This makes it an easy demand on $\mathscr{E}_\alpha$, i.e., if $N \in \mathscr{E}_\alpha$, then $R_\alpha \in N$. Clause (h) says that each $D_\alpha(N)$ has concentrated on $\bigcup_{\beta<\alpha} \mathscr{E}_\beta$, and the last clause (i) is some kind of density, as it implies that $N \cap \mathscr{H}(\chi_\beta) \subseteq \bigcup\{M : M \in \mathscr{E}_\beta \cap X\}$.

Remark 3.3.

(1) Note that $\langle \mathscr{E}_\alpha : \alpha < \ell g(\mathfrak{p})\rangle$ are pairwise disjoint by items (g) and (e), so $D(N)$ can be well defined as $D_\alpha(N)$ for the unique α such that $N \in \mathscr{E}_\alpha$.

(2) Clearly, by clause (h), only $D_\alpha(N) \cap \mathscr{P}(\bigcup_{\beta<\alpha} \mathscr{E}_\beta)$ matters.

Remark 3.4. Some natural choices for $D(N)$ are as follows:

(a) $D(N)$ is a filter on N.

(b) $D(N) = \{X \subseteq N : X \neq \emptyset \mod F\}$ for some filter F on N.

(c) $D(N) = F^+$ for a filter F on N.

Definition 3.5. Suppose $\mathfrak{p} = (\bar{\chi}, \bar{R}, \bar{\mathscr{E}}, \bar{D})$ is a reasonable parameter as in Definition 3.2.

(1) We say $\bar{D}$ is standard, if for every $\alpha < \alpha^*(= \ell g(\mathfrak{p}))$ and $N \in \mathscr{E}_\alpha$ we have

$$\begin{aligned} D_\alpha(N) = \{X \subseteq N : \quad & \text{for every } \gamma \in N \cap \alpha \text{ and} \\ & y \in N \cap \bigcup\{\mathscr{H}(\chi_\beta) : \beta \in N \cap \alpha\}, \\ & \text{for some } \beta \in N \cap (\alpha \setminus \gamma) \text{ and} \\ & M \in X \cap \mathscr{E}_\beta, \text{ we have } y \in M \\ & \text{and } X \cap M \in D_\beta(M)\}. \end{aligned}$$

(2) We say $\mathfrak{p}$ is standard if $\bar{D}$ is standard.

(3) We define the partial order $\leq_{\mathfrak{p}}$ on $\alpha^* = \ell g(\mathfrak{p})$ as follows: $\alpha \leq_{\mathfrak{p}} \beta$ iff

(a) $\alpha \leq \beta$,

(b) $N \in \mathscr{E}_\beta \wedge \alpha \in N \Rightarrow N \cap \mathscr{H}(\chi_\alpha) \in \mathscr{E}_\alpha$,

(c) $N \in \mathscr{E}_\beta \wedge \alpha \in N \wedge Y \in D_\beta(N) \Rightarrow Y \cap \bigcup_{\gamma<\alpha} \mathscr{E}_\gamma \in D_\alpha(M)$, where $M = N \cap \mathscr{H}(\chi_\alpha)$.

(4) We say $\mathfrak{p}$ is simple if $\alpha \leq \beta < \alpha^* \Rightarrow \alpha \leq_{\mathfrak{p}} \beta$.

(5) If $N \prec (\mathscr{H}(\chi), \in)$ and $N \cap \mathscr{H}(\chi_\alpha) \in \mathscr{E}_\alpha$, (hence $\alpha, \mathfrak{p} \restriction \alpha, R_\alpha \in N$), then we let $D_\alpha(N) = D^{\mathfrak{p}}_\alpha(N)$ to be $D_\alpha(N \cap \mathscr{H}(\chi_\alpha))$.

When $\mathfrak{p}$ is standard, we may drop $\bar{D}^{\mathfrak{p}}$ and just write $\mathfrak{p} = (\bar{\chi}^{\mathfrak{p}}, \bar{R}^{\mathfrak{p}}, \bar{\mathscr{E}}^{\mathfrak{p}})$. Also if $\mathfrak{p}$ is clear from the context, we may remove the superscript $\mathfrak{p}$. We now define several games related to a reasonable parameter $\mathfrak{p}$.

Definition 3.6. Suppose $\mathfrak{p}$ is a reasonable parameter.

(1) For $0 < \alpha < \ell g(\mathfrak{p})$ and $N \in \mathscr{E}^{\mathfrak{p}}_\alpha$, the game $\partial_\alpha(N, \mathfrak{p})$ is defined as follows. The play lasts ω moves, in the n-th move:
  (a) the challenger chooses $X_n \in D_\alpha(N)$ such that $m < n \Rightarrow X_n \subseteq X_m$
  (b) the chooser chooses $M_n \in X_n$ and $Y_n \subseteq M_n \cap X_n$ satisfying $Y_n \in D(M_n) \cap N$
  (c) the challenger chooses $Z_n \subseteq Y_n$ such that $Z_n \in D(M_n)$.

  At the end, the chooser wins if $\bigcup\{\{M_n\} \cup Z_n : n < \omega\} \in D_\alpha(N)$.

(2) Assume $N \in N' \prec (\mathscr{H}(\chi), \in)$, $\mathfrak{p} \restriction \alpha \in N'$ and $N \prec N'$ are countable. The game $\partial'_\alpha(N, N', \mathfrak{p})$ is defined similar to $\partial_\alpha(N, \mathfrak{p})$, but during the n-th move, we demand that all the chosen objects belong to N' (this means only then $X_n \in N'$), and at the end of the n-th move, the chooser also chooses $X'_n \subseteq X_n, X'_n \in D_\alpha(N) \cap N'$ and the challenger in the next move has to satisfy $X_{n+1} \subseteq X'_n$.

(3) Omitting N', i.e., writing $\partial'_\alpha(N, \mathfrak{p})$ we mean: for any N' as in (2), the demand $\partial'_\alpha(N, N', \mathfrak{p})$ holds.

(4) We say that $\mathfrak{p}$ is a winner or a ∂-winner (resp. ∂'-winner), if for every $0 < \alpha < \ell g(\mathfrak{p})$ and $N \in \mathscr{E}^{\mathfrak{p}}_\alpha$, the chooser has a winning strategy in the game $\partial_\alpha(N, \mathfrak{p})$ (resp. $\partial'_\alpha(N, \mathfrak{p})$).

(5) We say that $\mathfrak{p}$ is a non-$\Game$-loser (resp. a non-$\Game'$-loser) if for $0 < \alpha < \ell g(\mathfrak{p})$ and $N \in \mathscr{E}_\alpha$ the challenger has no winning strategy in $\Game_\alpha(N, \mathfrak{p})$ (resp. $\Game'_\alpha(N, \mathfrak{p})$).

Lemma 3.7. (1) *If $\mathfrak{p}$ is a reasonable parameter with the standard $\bar{D}^{\mathfrak{p}}$, then $\mathfrak{p}$ is a winner.*

(2) *If $\mathfrak{p}$ is a $\Game_\alpha$-winner, then $\mathfrak{p}$ is a $\Game'_\alpha$-winner. If $\mathfrak{p}$ is a $\Game$-winner, then $\mathfrak{p}$ is a $\Game'$-winner. Similarly for a non-loser.*

Proof. (1). Suppose $\mathfrak{p}$ is a standard reasonable parameter. Let $0 < \alpha < \ell g(\mathfrak{p})$ and $N \in \mathscr{E}^{\mathfrak{p}}_\alpha$. Let $\langle y_n : n < \omega \rangle$ be an enumeration of $N \cap \bigcup \{\mathscr{H}(\chi_\beta) : \beta \in \alpha \cap N\}$ such that each $y \in N \cap \bigcup \{\mathscr{H}(\chi_\beta) : \beta \in \alpha \cap N\}$ appears infinitely often in the enumeration and let $\langle \gamma_n : n < \omega \rangle$ be an increasing sequence of ordinals in $N \cap \alpha$ with $\sup\limits_{n<\omega} \gamma_n = \sup(N \cap \alpha)$. We define the following winning strategy for chooser in the game $\Game_\alpha(N, \mathfrak{p})$: in the n-th move, the challenger chooses some $X_n \in D^{\mathfrak{p}}_\alpha(N)$. In particular, we can find $\beta_n \in N \cap \alpha \setminus \gamma_n$ and $M_n \in X_n \cap \mathscr{E}^{\mathfrak{p}}_{\beta_n}$ such that $y_n \in M_n$ and $M_n \cap X_n \in D^{\mathfrak{p}}_{\beta_n}(M_n)$. Set also $Y_n = M_n \cap X_n$. Then the challenger choose some $Z_n \subseteq Y_n$.

We show that

$$X = \bigcup \{\{M_n\} \cup Z_n : n < \omega\} \in D^{\mathfrak{p}}_\alpha(N).$$

Thus let $\gamma \in N \cap \alpha$ and $y \in N \cap \bigcup \{\mathscr{H}(\chi_\beta) : \beta \in \alpha \cap N\}$. Pick $n < \omega$ such that $\gamma_n > \gamma$ and $y = y_n$. Then β_n and M_n are such that $\beta_n \in N \cap \alpha \setminus \gamma$, $M_n \in X \cap \mathscr{E}^{\mathfrak{p}}_{\beta_n}$ and we have $y_n \in M_n$ and $X \cap M_n \in D^{\mathfrak{p}}_{\beta_n}(M_n)$. Thus $X \in D^{\mathfrak{p}}_\alpha(N)$. Hence the above process defines a winning strategy for chooser, as required.

(2). Suppose $\mathfrak{p}$ is a $\Game_\alpha$-winner. Let $N \in \mathscr{E}_\alpha$ and assume that N' is such that $N \in N' \prec (\mathscr{H}(\chi), \in)$, $\mathfrak{p} \restriction \alpha \in N'$ and N' is countable. We define a winning strategy for chooser in the game $\Game'_\alpha(N, N', \mathfrak{p})$.

Let σ be a winning strategy for chooser in the game $\Game_\alpha(N, \mathfrak{p})$. By elementarity, we may assume that σ is in N'. We define the strategy σ' for chooser in the game $\Game'_\alpha(N, N', \mathfrak{p})$

as follows. At the n-th move, the challenger chooses some $X_n \in N'$. Then chooser picks the sets M_n and Y_n via the strategy σ and he also takes X'_n to be X_n. As σ is in N', all these objects are also in N'. Then challenger chooses some $Z_n \in N'$. It is evident that σ' is a winning strategy for chooser in the game $\Game'_\alpha(N, N', \mathfrak{p})$, as required. The other cases of the lemma can be proved in a similar way. $\square$

Definition 3.8. Assume $\mathfrak{p}$ is a reasonable parameter, $\alpha < \ell g(\mathfrak{p})$, $N \in \mathscr{E}^{\mathfrak{p}}_\alpha$, $y \in N$ and $\mathbb{P} \in N$ is a forcing notion. Set

$$\mathscr{M}_{\mathbb{P}}[\underset{\sim}{\mathbf{G}}_{\mathbb{P}}, N, y] = \{M \in N : \mathbb{P}, y \in M \text{ and } \underset{\sim}{\mathbf{G}}_{\mathbb{P}} \cap M \text{ is } (\mathbb{P} \cap M)\text{-generic over } M\},$$

where $\underset{\sim}{\mathbf{G}}_{\mathbb{P}}$ is the canonical $\mathbb{P}$-name for the generic filter.

We consider $\mathscr{M}_{\mathbb{P}}[\underset{\sim}{\mathbf{G}}_{\mathbb{P}}, N, y]$ as a $\mathbb{P}$-name and then $\mathscr{M}_{\mathbb{P}}[\mathbf{G}, N, y]$ is well defined for any $\mathbb{P}$-generic filter $\mathbf{G}$. If $\mathbb{P}$ is clear from the context, we may omit it. Note that $\mathscr{M}_{\mathbb{P}}[\mathbf{G}, N, y] = \mathscr{M}_{\mathbb{P}}[\mathbf{G} \cap N, N, y]$, so we may write $\mathbf{G} \cap N$ instead of $\mathbf{G}$. If $y = \emptyset$ we may omit it.

Definition 3.9. We say $\bar{\mathbb{Q}} \in K_0$ is a $\mathfrak{p}-\mathrm{NNR}^0_{\aleph_0}$ iteration if the following conditions are satisfied:

(a) $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < j(*) \rangle$ is a countable support iteration of proper forcing notions such that $\bar{\mathbb{Q}}, \mathscr{P}(\mathrm{Lim}(\bar{\mathbb{Q}})) \in \mathscr{H}(\chi^{\mathfrak{p}}_0)$.

(b) forcing with $\mathbb{P}_{j(*)} = \mathrm{Lim}(\bar{\mathbb{Q}})$ does not add reals.

(c) (long properness) suppose that:

$(*)_1$ (α) $i \leq j \leq j(*), \alpha < \ell g(\mathfrak{p})$,

(β) $N \in \mathscr{E}^{\mathfrak{p}}_\alpha, \{i, j, \bar{\mathbb{Q}}\} \in N$,

(γ) the condition $q \in \mathbb{P}_i$ is $(N, \mathbb{P}_i)$-generic,

(δ) $q \Vdash$ "$\underset{\sim}{\mathbf{G}}_{\mathbb{P}_i} \cap N = \mathbf{G}$",

(ε) $p \in \mathbb{P}_j \cap N$ and $p{\restriction}i \in \mathbf{G}$,

(ζ) $Y \subseteq \mathscr{M}_{\mathbb{P}_i}[\mathbf{G}, N, y]$ where $y = \langle \bar{\mathbb{Q}}, i, j \rangle$ and $Y \in D^{\mathfrak{p}}_{\alpha}(N)$ [2],

then there are $\mathbf{G}', q'$ such that:

$(*)_2$ (η) $q' \in \mathbb{P}_j,\ p \leq q'$ and $q \leq q' \restriction i$,

(θ) q' is $(N, \mathbb{P}_j)$-generic,

(ι) $q' \Vdash$ "$\underset{\sim}{\mathbf{G}}_{\mathbb{P}_j} \cap N = \mathbf{G}'$",

(κ) $Y \cap \mathscr{M}_{\mathbb{P}_j}[\mathbf{G}', N, y] \in D^{\mathfrak{p}}_{\alpha}(N)$.

(d) (anti weak diamond or anti-w.d.) suppose that:

$(\bullet)_1$ (α) $i \leq j \leq j(*)$ and $\alpha < \ell g(\mathfrak{p})$,

(β) $N_0 \in N_1 \in \mathscr{E}^{\mathfrak{p}}_{\alpha},\ N_0 \in \bigcup\limits_{\beta<\alpha} \mathscr{E}^{\mathfrak{p}}_{\beta}$,

(γ) $\mathrm{otp}(N_0 \cap [i, j)) < \alpha$ [3],

(δ) $n < \omega$ and for $\ell < n$ we have $q_\ell \in \mathbb{P}_i$ is $(N_1, \mathbb{P}_i)$-generic,

(ϵ) $q_\ell \Vdash$" $\underset{\sim}{\mathbf{G}}_{\mathbb{P}_i} \cap N_1 = \mathbf{G}^\ell$",

(ζ) $\bigwedge\limits_{\ell<n} [\mathbf{G}^\ell \cap N_0 = \mathbf{G}^*]$ where $\mathbf{G}^* \subseteq \mathbb{P}_i \cap N_0$ is generic over N_0,

(η) $Y =: \bigcap\limits_{\ell<n} \mathscr{M}_{\mathbb{P}_i}[\mathbf{G}^\ell, N_1] \in D^{\mathfrak{p}}_{\alpha}(N_1)$,

(θ) $p \in \mathbb{P}_j \cap N_0$ is such that $p \restriction i \in \mathbf{G}^*$.

Then:

$(\bullet)_2$ for some $\mathbf{G}^{**} \subseteq \mathbb{P}_j \cap N_0$ generic over N_0 we have $p \in \mathbf{G}^{**} \in N_1$ and

$$\bigwedge_{\ell<n} \bigvee_{q \in G^\ell} [q \Vdash \text{"}\mathbf{G}^{**} \text{ has an upper bound in } \mathbb{P}_j/\underset{\sim}{\mathbf{G}}_{\mathbb{P}_i}\text{"}].$$

Remark 3.10. We may like to phrase clause (c) as a condition on each $\underset{\sim}{\mathbb{Q}}_i$, for this see Definitions 4.7, 4.9 and 4.12.

We now state and prove the main result of this section.

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[2]Note that the ordinal i is reconstructible from $\bar{\mathbb{Q}}$ and $\mathbf{G}$.

[3]so naturally $\ell g(\mathfrak{p}) = \omega_1$. We use the parallel of "$\aleph_0$-completeness system" rather than "2-completeness system" of [6] as things are complicated enough anyhow; see Definition 3.14 and Theorem 3.15.

Theorem 3.11. *Assume $\mathfrak{p}$ is a reasonable parameter of length ω_1, $\bar{\mathbb{Q}}$ is a countable support iteration such that $\bar{\mathbb{Q}}, \mathscr{P}(\mathrm{Lim}\bar{\mathbb{Q}}) \in \mathscr{H}(\chi_0^{\mathfrak{p}})$, $\delta = \ell g(\bar{\mathbb{Q}})$ is a limit ordinal and for every $\alpha < \delta, \bar{\mathbb{Q}} \restriction \alpha$ is a $\mathfrak{p} - NNR^0_{\aleph_0}$ iteration and $\mathfrak{p}$ is a $\Game$-winner. Then $\bar{\mathbb{Q}}$ is a $\mathfrak{p} - NNR^0_{\aleph_0}$ iteration.*

Proof. We show that items (a)-(d) of Definition 3.9 are satisfied by $\bar{\mathbb{Q}}$.

Proof of clause (a): This holds trivially by our assumptions.

Proof of clause (b): This follows from clause (d) of Definition 3.9 proved below. To see this, let $p \in \mathbb{P}_\delta = \mathrm{Lim}\bar{\mathbb{Q}}$, $\underset{\sim}{r}$ be a $\mathbb{P}_\delta$-name and suppose that $p \Vdash$ "$\underset{\sim}{r}$ is a real". In Definition 3.9(d) set $i = 0, j = \delta, n = 1$ and pick $N_0 \in N_1$ so that the hypotheses in $(\bullet)_1$ are satisfied and $\underset{\sim}{r} \in N_0$. Thus, by $(\bullet)_2$, we can find $\mathbf{G}^{**} \subseteq \mathbb{P}_\delta \cap N_0$ which is generic over N_0, $p \in \mathbf{G}^{**}$ and $\mathbf{G}^{**}$ has an upper bound $q \geq p$ in $\mathbb{P}_\delta$. By genericity of $\mathbf{G}^{**}$, q decides $\underset{\sim}{r}$ to be a real in V, and we are done.

We first prove clause (d) and then return to clause (c).

Proof of clause (d): Let $i, j, \alpha, N_0, N_1, n, q_0, \ldots, q_{n-1}, G^0, \ldots, G^{n-1}, G^*$ and p be as in the assumptions of clause (d). Let $\alpha' = \mathrm{otp}(N_0 \cap [i, j))$. Then $\alpha' < \omega_1$, so $\alpha' \in N_1$. If $j < \delta$, then by our assumption $\bar{\mathbb{Q}} \restriction j$ is a $\mathfrak{p}$$-NNR^0_{\aleph_0}$ iteration and the result follows. Thus assume that $j = \delta$. If $i = j$, the conclusion is trivial, so assume $i < j$.

Let $i_m \in N_0 \cap j$ be such that $i_0 = i$ and $\langle i_m : m < \omega \rangle \in N_1$ is an increasing sequence with $\bigcup_{m<\omega} i_m = \sup(N_0 \cap j)$. Choose $\langle M_k : k < 5 \rangle$ such that:

- $y^* := \{i, j, \alpha, \alpha', \bar{\mathbb{Q}}, N_0, \langle i_m : m < \omega \rangle\} \in M_k$,
- $M_k \in \mathscr{E}^{\mathfrak{p}}_{\alpha'} \cap N_1 \cap \bigcap_{\ell<n} \mathscr{M}_{\mathbb{P}_i}[G^\ell, N_1]$,
- $M_0 \in M_1 \in M_2 \in M_3 \in M_4$,
- $\bigcap_{\ell<n} \mathscr{M}_{\mathbb{P}_i}[\mathbf{G}^\ell, M_0, y^*] \in D^{\mathfrak{p}}_\alpha(M_0)$.

Note that for $\ell < n$ and $k < 5$, $N_0 \prec M_k \prec N_1$ and $\mathbf{G}^\ell \cap M_k$ is a generic subset of $\mathbb{P}_i \cap M_k$. Now for $\ell < n$ we can choose $q'_\ell \in \mathbf{G}^\ell \cap M_4$ such that:

- q'_ℓ forces (for $\mathbb{P}_{i_0} = \mathbb{P}_i$) a value for $\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{i_0}} \cap M_3$, which necessarily is $\mathbf{G}^\ell \cap M_3$,
- $q_\ell \leq q'_\ell$,
- q'_ℓ is $(M_k, \mathbb{P}_{i_0})$-generic, forcing $\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{i_0}} \cap M_k = \mathbf{G}^\ell \cap M_k$ for $k = 0, 1, 2, 3$,
- q'_ℓ forces $\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{i_0}} \cap N_0 = \mathbf{G}^*$.

Let $\langle \mathscr{I}^*_m : m < \omega\rangle \in M_0$ list the maximal antichains of $\mathbb{P}_j$ that belongs to N_0. We choose, by induction on $m < \omega$, the objects $r_m, p_m, n_m, \mathbf{G}^*_m$, $\langle \mathbf{G}^\ell_m : \ell < n_m\rangle$ and Y_m such that:

$(*)_1$ (a) $r_m \in \mathbb{P}_{i_m} \cap M_4$,

(b) $\mathrm{dom}(r_m) \subseteq [i, i_m)$,

(c) $r_{m+1} \restriction i_m = r_m$,

(d) $q'_\ell \cup r_m \in \mathbb{P}_{i_m}$ and is $(M_k, \mathbb{P}_{i_m})$-generic for $k = 0, 1, 2, 3$ and is $(N_0, \mathbb{P}_{i_m})$-generic,[4]

(e) for every predense subset $\mathscr{I}$ of $\mathbb{P}_{i_m}$ which belongs to M_2, for some finite $\mathscr{J} \subseteq \mathscr{I} \cap M_2$, the set $\mathscr{J}$ is predense above $q'_\ell \cup r_m$, for each $\ell < n$,

(f) $n_m < \omega$ and for $\ell < n_m$ $\mathbf{G}^\ell_m \in M_1$ is a subset of $\mathbb{P}_{i_m} \cap M_0$ generic over M_0,

(g) if $\ell < n_{m+1}$, then $\mathbf{G}^\ell_{m+1} \cap \mathbb{P}_{i_m} \in \{\mathbf{G}^k_m : k < n_m\}$,

(h) $n_0 = n$ and $\mathbf{G}^\ell_0 = \mathbf{G}^\ell \cap M_0$,

(i) $q'_\ell \cup r_m \Vdash_{\mathbb{P}_{i_m}}$ "$\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{i_m}} \cap M_0 \in \{\mathbf{G}^\ell_m : \ell < n_m\}$",

(j) $\mathbf{G}^*_m$ is a subset of $\mathbb{P}_{i_m} \cap N_0$ generic over N_0,

(k) $\mathbf{G}^*_m \subseteq \mathbf{G}^\ell_m$ for $\ell < n_m$,

(l) p_m is such that:

$(l\text{-}1)$ $p_m \in \mathbb{P}_j \cap N_0$,

$(l\text{-}2)$ $p_m \restriction i_m \in \mathbf{G}^*_m$,

$(l\text{-}3)$ $p_{m+1} \in \mathscr{I}^*_m$,

$(l\text{-}4)$ $p_0 = p$,

$(l\text{-}5)$ $p_m \leq p_{m+1}$.

[4]note that q'_ℓ and r_m have disjoint domains.

(m) $Y_m = \bigcap_{\ell<n_m} \mathscr{M}_{\mathbb{P}_{i_m}}[G^\ell_m, M_0, y^*] \in D_{\alpha'}(M_0)$ where $y^* = \{N_0, \langle i_m : m < \omega\rangle, \bar{\mathbb{Q}}, i, j\}$.

The construction is clear for $m = 0$. So suppose that we have it for m and we shall choose for $m+1$. We do this is several steps.

Stage A: Choose $p_{m+1} \in N_0 \cap \mathscr{I}^*_m$ such that $p_m \le p_{m+1}$ and $p_{m+1} \restriction i_m \in \mathbf{G}^*_m$.

Stage B: Choose $\mathbf{G}^*_{m+1} \subseteq \mathbb{P}_{i_{m+1}} \cap N_0$ generic over N_0 such that $\mathbf{G}^*_m \subseteq \mathbf{G}^*_{m+1} \in M_0, p_{m+1} \restriction i_{m+1} \in \mathbf{G}^*_{m+1}$ and

$$\bigwedge_{\ell<n_m} \bigvee_{r\in\mathbf{G}^\ell_m} [r \Vdash_{\mathbb{P}_{i_m}} \text{"}\mathbf{G}^*_{m+1} \text{ has an upper bound in } \mathbb{P}_{i_{m+1}}/\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{i_m}}].$$

This is easy by applying clause (d) of the Definition 3.9 for $i_m, i_{m+1}, \alpha', p_{m+1} \restriction i_m, \mathbf{G}^*_m, \langle \mathbf{G}^\ell_m : \ell < n_m\rangle, N_0, M_0$, for the forcing notion $\bar{\mathbb{Q}} \restriction i_{m+1}$, which is, by induction hypothesis, a $\mathfrak{p}-NNR^0_{\aleph_0}$ iteration, . We also use the fact that $\text{otp}(N_0 \cap [i_m, i_{m+1})) < \text{otp}(N_0 \cap [i_m, j)) = \alpha'$.

Stage C: As $\mathbb{P}_{i_m}$ is proper and adds no new reals, $\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{i_m}} \cap M_1$ is a $\mathbb{P}_{i_m}$-name of an object from $\mathbf{V}$, so

$$\mathscr{I} = \{p \in \mathbb{P}_{i_m} : p \text{ forces a value for } \underset{\sim}{\mathbf{G}}_{\mathbb{P}_{i_m}} \cap M_1 \text{ in } \mathbf{V}\}$$

is a dense open subset of $\mathbb{P}_{i_m}$ and $\mathscr{I} \in M_2$. By clause $(*)_1(e)$ of the induction hypothesis, there is a finite $\mathscr{J} \subseteq \mathscr{I} \cap M_2$ such that: $\ell < n \Rightarrow \mathscr{J}$ is predense above $q'_\ell \cup r_m$. Without loss of generality $\mathscr{J}$ is minimal. Let $n_{m+1} = |\mathscr{J}|$.

Let $\mathscr{J} = \{p^\ell_m : \ell < n_{m+1}\}$ and for each $\ell < n_{m+1}$ choose $H^\ell_m \in M_2$ such that $p^\ell_m \Vdash$ "$\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{i_m}} \cap M_1 = H^\ell_m$". As $\mathscr{J}$ is minimal, $H^\ell_m \cap M_0 \in \{\mathbf{G}^\ell_m : \ell < n_m\}$ so for some $h : n_{m+1} \to n_m$ and every $\ell < n_{m+1}$ we have $H^\ell_m \cap M_0 = \mathbf{G}^{h(\ell)}_m$.

Let $Y = \bigcap_{\ell<m_n} \mathscr{M}_{\mathbb{P}_{i_m}}[\mathbf{G}^\ell_m, M_0, y^*] \in D^{\mathfrak{p}}_{\alpha'}(M_0)$. Now we choose by induction on $\ell \le n_{m+1}$ a condition $r^\ell_m \in M_1$ such that:

$(*)_2$ (α) $r^\ell_m \in \mathbb{P}_{i_{m+1}} \cap M_1$ and $r^\ell_m \restriction i_m \in H^\ell_m$,

(β) r^ℓ_m is $(M_0, \mathbb{P}_{i_m})$-generic and forces a value for $\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{i_m}} \cap M_0$, say $\mathbf{G}^\ell_{m+1}$,

(γ) r^ℓ_m is above $\mathbf{G}^*_{m+1}$, and moreover above $p^{h(\ell)}_m$,

(δ) $Y \cap \bigcap_{k<\ell} \mathscr{M}_{\mathbb{P}_{i_m}}[\mathbf{G}^k_{m+1}, M_0, y^*] \in D^{\mathfrak{p}}_{\alpha'}(M_0)$.

The construction can be easily done by applying clause (c) of Definition 3.9 to $i_m, i_{m+1}, \alpha', M_0$, large enough member of $H^\ell_m, p^{h(\ell)}_m$ and $Y^\ell_m = Y \cap \bigcap_{k<\ell} \mathscr{M}[\mathbf{G}^k_{m+1}, M_0, y^*] \in M_1$, for the $\mathfrak{p} - NNR^0_{\aleph_0}$-iteration $\bar{\mathbb{Q}} \restriction i_{m+1}$.

Stage D: By [6, Ch.XVIII, Claim 2.6], we can choose r_{m+1} as required such that $\{r^\ell_m : \ell < n_{m+1}\}$ is predense over it.

This completes the inductive construction. Let us now show that this is sufficient to get clause (d). Let $\lhd^* \in N_1$ be a well-ordering of M_4. During the construction above we chose inductively members of M_4 and all the parameters used are from M_4, so if we always choose the $\lhd^*$-first object, the construction is determined and is in N_1. By clause $(*)_1(c)$, we have

(α) Let $r = \bigcup_m r_m$ be the unique $r \in \mathbb{P}_j$ satisfying $m < \omega \Rightarrow r \restriction i_m = r_m$. Then $r \in \mathbb{P}_j \cap N_1$.

Also, by the choice of $\langle \mathscr{I}^*_m : m < \omega \rangle$ and clause $(*)_1(k)$, we have

(β) $\mathbf{G}^{**} = \{p' \in \mathbb{P}_j \cap N_0 : \bigvee_{m<\omega} [p' \leq p_m]\}$ belongs to M_4 and is a subset of $\mathbb{P}_j \cap N_0$ generic over N_0.

It is also clear from $(*)_1(\ell)$ that

(γ) $q'_\ell \cup r$ is above $\mathbf{G}^{**}$ (in $\mathbb{P}_j$).

So we have finished proving clause (d).

Proof of clause (c): We prove this by induction on α. Let i, j, α, N, p, q and Y be as there. If $j < \delta$ we can apply "$\bar{\mathbb{Q}} \restriction j$ is a $\mathfrak{p} - NNR^0_{\aleph_0}$ iteration", so without loss of generality $j = \delta$. If $i = j$ the statement is trivial, so assume $i < j$.

Choose $i_n \in N \cap j$, for $n < \omega$, such that $i_0 = i, i_n < i_{n+1}$ and $\bigcup_{n<\omega} i_n = \sup(j \cap N)$. Let $\langle (y_n, \beta_n) : n < \omega \rangle$ list the pairs $(y, \beta) \in N \times (\alpha \cap N)$ such that $y \in \mathscr{H}(\chi^{\mathfrak{p}}_\beta)$. Let σ

be a winning strategy for the chooser in the game $\partial_\alpha(N, \mathfrak{p})$ and let $\langle \mathscr{I}_n : n < \omega \rangle$ list the dense open subsets of $\mathbb{P}_j$ which belong to N.

We choose by induction on $n < \omega$, the objects $q_n, \underset{\sim}{p}_n, \underset{\sim}{M}_n$ and $\underset{\sim}{Y}_n$ such that:

$(*)_3$ (a) $q_n \in \mathbb{P}_{i_n}$ with $q_0 = q$,

(b) q_n is $(N, \mathbb{P}_{i_n})$-generic,

(c) $q_{n+1} \restriction i_n = q_n$,

(d) $\underset{\sim}{p}_n$ is a $\mathbb{P}_{i_n}$-name of a member of $(\mathbb{P}_j / \underset{\sim}{\mathbf{G}}_{\mathbb{P}_{i_n}}) \cap N$

(e) $\underset{\sim}{p}_n$ is forced to belong to $\mathscr{I}_n$,

(f) $\underset{\sim}{M}_n$ is a $\mathbb{P}_{i_n}$-name of a member of $\mathscr{E}^{\mathfrak{p}}_{\beta_n} \cap N$,

(g) if $\mathbf{G}_j \subseteq \mathbb{P}_j$ is generic over $\mathbf{V}$ such that $q_n \in \mathbf{G}_j, \underset{\sim}{p}_n[\mathbf{G}_j \cap \mathbb{P}_{i_n}] \in \mathbf{G}_j$ and if $M = \underset{\sim}{M}_n[\mathbf{G}_j]$, then

(α) $\mathbf{G}_j \cap M$ is a subset of $\mathbb{P}_j \cap M$ generic over M,

(β) $\mathscr{M}_{\mathbb{P}_j}[\mathbf{G}_j \cap M, M, y^*] \cap Y \in D^{\mathfrak{p}}_{\beta_n}[M]$,

(γ) $\underset{\sim}{p}_n[\mathbf{G}_j]$ belongs to M,

(h) $\langle Y_m \cap \mathscr{M}_{\mathbb{P}_{i_m}}[\underset{\sim}{\mathbf{G}}_{i_m}, \underset{\sim}{M}_m, y^*], \underset{\sim}{Y}_m, \mathbb{P}_{i_m}, \underset{\sim}{M}_m : m \leq n \rangle$ is forced by q_n to be an initial segment of a play of the game $\partial_\alpha(N)$ in which the chooser uses the fixed winning strategy σ.

The proof is straight by the induction hypothesis on β, the fact that $\underset{\sim}{M}_n, \underset{\sim}{Y}_n$ are $\mathbb{P}_{i_n}$-names of objects from $\mathbf{V}$ and $\bar{\mathbb{Q}} \restriction i_n$ is a $\mathfrak{p} - NNR^0_{\aleph_0}$-iteration. Now let $q' = \bigcup_{n<\omega} q_n$ and $\mathbf{G}' = \{p' \in \mathbb{P}_j \cap N : \bigvee_{n<\omega} p' \leq q_n\}$. Then $\mathbf{G}'$ and q' are as required; see also the proof of clause $(c)'$ of Theorem 6.3 for more details, where a more general result is proved.

The theorem follows. $\square$

Remark 3.12. (1) It is possible to use "adding no reals+ clause (d)" in the proof of clause (c) in order to weaken "winner" to "not loser". Also we can use $\partial'_\alpha(N, N', \mathbb{P})$, see Section 6. The assumption "$N_0 \in \bigcup_{\beta<\alpha} \mathscr{E}_\beta$" can be replaced by "$N_0 \in \mathscr{E}'_0$ with $\mathscr{E}'_0 \subseteq [\mathscr{H}(\chi_0^{\mathfrak{p}})]^{\aleph_0}$ stationary". We can also put extra restrictions on $\mathbf{G}^*$ (and $\mathbf{G}^{**}$), for example we can require $\mathscr{M}[\mathbf{G}^*, N_0, y^*]$ is large.

(2) The use of $\langle \chi_\alpha : \alpha < \ell g(\mathfrak{p})\rangle$ is not really necessary as all the properties depend just on $N_\ell \cap \mathscr{P}(\mathbb{P}_{\ell g(\bar{\mathbb{Q}})})$.

(3) The proof of clause (c) being preserved can be applied to any $\bar{\mathbb{Q}}$ satisfying (a) + (c) of Definition 3.9 (so possibly adding reals), but then we have to replace $D^{\mathfrak{p}}_\alpha(N)$ by a definition of such pseudo filters with the winning strategy being absolute enough, e.g. for standard $\bar{D}$.

Definition 3.13. Let $\bar{\mathbb{Q}}$ be a countable support iteration of forcing notions. It will be called $\mathfrak{p}$-proper if it satisfies items (a) and (c) of Definition 3.9.

We like to consider the parallel of having 2-completeness systems and also to demand only non-losing rather than winning in the assumption of Theorem 3.11.

Definition 3.14. Let $\kappa \in [2, \omega]$. We say that $\bar{\mathbb{Q}}$ is a $\mathfrak{p} - NNR^0_\kappa$-iteration if items (a)-(c) of Definition 3.9 are satisfied and clause (d) is replaced by κ-anti w.d., which is the same as in (d) there, but with $n < 1 + \kappa$ and $N_0 \in \mathscr{E}^{\mathfrak{p}}_0$.

The next theorem is a natural generalization of Theorem 3.11.

Theorem 3.15. *Assume $\mathfrak{p}$ is a reasonable parameter of length ω_1 which is a non-$\eth'$-loser, $2 \leq \kappa < \aleph_0, \bar{\mathbb{Q}}$ is a countable support iteration with $\mathscr{P}(\mathrm{Lim}(\bar{\mathbb{Q}})) \subseteq \mathscr{H}(\chi_0^{\mathfrak{p}})$, $\delta = \ell g(\bar{\mathbb{Q}})$ is a limit ordinal and for $i < \delta, \bar{\mathbb{Q}} \restriction i$ is a $\mathfrak{p} - NNR^0_\kappa$-iteration. Then $\bar{\mathbb{Q}}$ is a $\mathfrak{p} - NNR^0_\kappa$-iteration.*

Proof. The proof is similar to the proof of Theorem 3.11, with some changes, as in [5, Ch.VIII, Claim 4.10] and [6, Ch.XVIII, Proof 2.10C], so we do not give the details. The only main change is that during the proof of clause (d), we add the following extra conditions to items (a)-(m):

(n) n_m is a power of 2, say $2^{n^*_m}$ and so we can rename $\{\mathbf{G}^\ell_m : \ell < n_m\}$ as $\{\mathbf{G}^\eta_m : \eta \in {}^{n^*_m}2\}$,

(o) The following conditions are satisfied:

(α) for $\eta \in {}^{(n^*_m \geq)}2$, $M_\eta \in M_1 \cap \mathscr{E}^{\mathfrak{p}}_{j_\eta}$, where $j_0 = \text{otp}([i,j) \cap N_0)$ and if $\eta = \nu{}^\frown\langle i\rangle$, then $j_\eta = \text{otp}([i,j) \cap M_\nu)$,

(β) $M_{\langle\rangle} = N_0$,

(γ) $M_\eta \in M_{\eta^\frown\langle 0\rangle} \cap M_{\eta^\frown\langle 1\rangle}$,

(δ) $\eta \triangleleft \nu_1 \in {}^{n^*_m}2 \wedge \eta \triangleleft \nu_2 \in {}^{n^*_m}2 \Rightarrow \mathbf{G}^{\nu_1}_m \cap M_\eta = \mathbf{G}^{\nu_2}_m \cap M_\eta$ so we call it K^η_m

(ε) for $\eta \in {}^{n^*_m >}2$, $M_{\eta^\frown\langle 0\rangle} = M_{\eta^\frown\langle 1\rangle}$, call it N_η,

(ζ) for $\eta \in {}^{(m^*_m >)}2$ and $\ell < 2$, $N_\eta \in \mathscr{E}^{\mathfrak{p}}_{j_{\eta^\frown\langle\ell\rangle}}$,

(η) $Y^\eta_m = \mathscr{M}[K^{\eta^\frown\langle 0\rangle}_m, N_\eta] \cap \mathscr{M}[K^{\eta^\frown\langle 1\rangle}_m, N_\eta] \in D_{j_{\eta^\frown\langle 0\rangle}}(N_\eta)$.

The rest of the argument is essentially the same as before. □

The following is an immediate consequence of Theorems 3.11 and 3.15.

Conclusion 3.16. *Suppose $\mathfrak{p}$ is a non-$\eth'$-loser reasonable parameter with $\ell g(\mathfrak{p}) = \omega_1$, $\bar{\mathbb{Q}}$ is a countable support iteration and $2 \leq n(*) \leq \aleph_0$. Then, $\bar{\mathbb{Q}}$ is a $\mathfrak{p} - NNR^0_{n(*)}$-iteration iff for each $i < \ell g(\bar{\mathbb{Q}})$*

$(*)_i$ *$\underset{\sim}{\mathbb{Q}}_i$ is a proper forcing and $\mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i$ satisfy clauses (d) + (c) of the Definition "$\mathfrak{p} - NNR^0_{n(*)}$-iteration" with $i, i+1$ here standing for i, j there.*

Proof. By induction on $j = \ell g(\bar{\mathbb{Q}})$. For $j = 0$ there is nothing to prove and for j a successor ordinal, this follows easily from the definitions. For j a limit ordinal, the result follows from Theorem 3.11 (for $n(*) = \aleph_0$) or Theorem 3.15 (for $2 \leq n(*) < \aleph_0$). □

We point out here that Clause (c) of Definitions 3.9 and 3.14 really follows from earlier properties which play parallel roles.

Lemma 3.17. (1) *Assume that* $\mathfrak{p}$ *is a standard reasonable parameter,* $\alpha < \ell g(\mathfrak{p}), N \in \mathscr{E}^{\mathfrak{p}}_{\alpha}, Y \in D^{\mathfrak{p}}_{\alpha}(N)$ *and* $\delta \leq \omega_1 \cap N$ *is a limit ordinal. Then we can find sequences* $\bar{N} = \langle N_i : i < \delta \rangle$ *and* $\bar{\gamma} = \langle \gamma_i : i < \delta \rangle$ *such that:*

(a) $N_i \in N$ *is countable,* $N \cap \alpha \subseteq N_i$ *and* $N_i \in Y$ *(for* $i < \delta$*),*

(b) $N_i \subseteq \bigcup_{\beta \in \alpha \cap N} (\mathscr{H}(\chi^{\mathfrak{p}}_{\beta}), \in)$, *and* $\beta \in \alpha \cap N_i \Rightarrow N_i \restriction \mathscr{H}(\chi^{\mathfrak{p}}_{\beta}) \prec (\mathscr{H}(\chi^{\mathfrak{p}}_{\beta}), \in)$,

(c) $i < j \Rightarrow N_i \subseteq N_j$,

(d) *if* i *is a limit ordinal, then* $N_i = \bigcup_{j<i} N_j$ *and* $N \cap \bigcup\{\mathscr{H}(\chi^{\mathfrak{p}}_{\beta}) : \beta \in \alpha \cap N\} = \bigcup_{j<\delta} N_j$, *so we can stipulate* $N_\delta = N$,

(e) $\beta \in \alpha \cap N \Rightarrow \langle \mathscr{H}(\chi^{\mathfrak{p}}_{\beta}) \cap N_j : j \leq i \rangle \in N_{i+1}$,

(f) $\gamma_i \in N_i \cap \alpha, N_i \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\gamma_i}) \in \mathscr{E}^{\mathfrak{p}}_{\gamma_i} \cap Y$ *and* $\bar{\gamma} \restriction (i+1) \in N_{i+1}$,

(g) *if* $i \leq \delta$ *is a limit ordinal and either* ($i = \delta$ *and* $\beta \in \alpha \cap N_i$) *or* ($i < \delta$ *and* $\beta \in \gamma_i \cap N_i$), *then for some* $j < i, \gamma_j = \beta$ *and* $y \in N_j$.

(h) *if* $i \leq \delta$ *is a limit ordinal, then* $\{N_j \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\gamma_j}) : j < i \text{ and } \gamma_j < \gamma_i\} \in D^{\mathfrak{p}}_{\gamma_i}(N) = D^{\mathfrak{p}}_{\gamma_i}(N \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\gamma_i}))$.

(i) *if* $\delta < N \cap \omega_1$ *then* $\delta \in N_0$, *if* $\delta = N \cap \omega_1$ *then* $i < \delta \Rightarrow i \in N_0$.

(2) *If* $\mathfrak{p}$ *is a standard reasonable parameter,* $\bar{\mathbb{Q}}$ *is a countable support iteration,* $\ell g(\bar{\mathbb{Q}}) = \beta + 1, \bar{\mathbb{Q}} \restriction \beta$ *is* $\mathfrak{p} - NNR^0_{k(*)}$*-iteration and* $\Vdash_{\mathbb{P}_\beta}$ "$\underset{\sim}{\mathbb{Q}}_\beta$ *is proper and* $(<^+ \omega_1)$*-proper", then* $\bar{\mathbb{Q}}$ *is* $\mathfrak{p}$*-proper (see Definition 3.13).*

(3) *If* $\ell g(\mathfrak{p}) = \omega_1$, *then in part (2), it suffices to assume* $\Vdash_{\mathbb{P}_i}$ "$\underset{\sim}{\mathbb{Q}}_i$ *is* $(< \omega_1)$*-proper".*

Proof. (1). By induction on $i < \delta$ we prove that there are sequences $\langle N_j : j < i \rangle \in N$ and $\langle \gamma_j : j < i \rangle$ satisfying the relevant requirements, so that for some sequence $\langle N'_j : j < i \rangle \in N$ with $N'_j \prec (\mathscr{H}(\chi^{\mathfrak{p}}_{\alpha}), \in)$, $N_j = N'_j \cap \bigcup_{\beta \in \alpha \cap N} (\mathscr{H}(\chi^{\mathfrak{p}}_{\beta}), \in)$.

For $i = 0$, let $N'_0 \prec (\mathscr{H}(\chi^{\mathfrak{p}}_{\alpha}), \in)$ be a countable model such that $N'_0 \in N \cap Y$ and such that N'_0 contains all relevant information, in particular, $N \cap \alpha \subseteq N'_0, \delta \subseteq N'_0$. Let $N_0 = N'_0 \cap \bigcup_{\beta \in \alpha \cap N} (\mathscr{H}(\chi^{\mathfrak{p}}_{\beta}), \in)$. Pick also γ_0 such that $\gamma_0 \in N_0 \cap \alpha$ and $N_0 \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\gamma_0}) \in \mathscr{E}^{\mathfrak{p}}_{\gamma_0} \cap Y$. Such γ_0 exists as $\mathfrak{p}$ is standard.

If $i = j+1$ is a successor ordinal, let $N'_i \prec (\mathscr{H}(\chi^{\mathfrak{p}}_\alpha), \in)$ be a countable model such that:

- $N'_i \in N \cap Y$,
- $N'_j \subseteq N_i$,
- $\beta \in \alpha \cap N \Rightarrow \langle \mathscr{H}(\chi^{\mathfrak{p}}_\beta) \cap N_k : k \leq j\rangle \in N_i$,
- $\langle \gamma_k : k \leq j\rangle \in N_i$.

Then, using $\mathfrak{p}$ is standard, take $\gamma_i \in N_i \cap \alpha$ such that $N_i \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\gamma_i}) \in \mathscr{E}^{\mathfrak{p}}_{\gamma_i} \cap Y$.

For limit i set $N'_i = \bigcup_{j<i} N_j$ and $N_i = N'_i \cap \bigcup_{\beta \in \alpha \cap N} (\mathscr{H}(\chi^{\mathfrak{p}}_\beta), \in)$. Let also γ_i be such that $\{N_j \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\gamma_j}) : j < i \text{ and } \gamma_j < \gamma_i\} \in D^{\mathfrak{p}}_{\gamma_i}(N) = D^{\mathfrak{p}}_{\gamma_i}(N \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\gamma_i}))$.

As N is countable, we can choose N'_j's such that clause (d) holds as well. This completes our inductive construction.

(2) We show that $\bar{\mathbb{Q}}$ satisfies clause (c) of Definition 3.9. Thus let $i, j, \alpha, N, q, p, \mathbf{G}$ and Y be as there. Without loss of generality, $i = \beta$ and $j = \beta + 1$.

By the definition of $(<^+ \omega_1)$-proper, if $\mathbf{G}_\beta \subseteq \mathbb{P}_\beta$ is generic over N, then $\mathscr{M}_{\mathbb{P}_\beta}[\mathbf{G}_\beta, N, y^*] \in D^{\mathfrak{p}}_\beta(N)$. Let $\delta = N \cap \omega_1$ and let $\langle N_i : i < \delta\rangle$ be as in (1). Without loss of generality $p(\beta) \in N_0 \cap \underset{\sim}{\mathbb{Q}}_\beta[\mathbf{G}_\beta]$. Let $q' \geq p$ be $(N_i, \underset{\sim}{\mathbb{Q}}_\beta[\mathbf{G}_\beta])$-generic for every $i < \delta$ [5] so that $q' \restriction \beta \geq q$. Let also $\mathbf{G}(\beta) \subseteq \underset{\sim}{\mathbb{Q}}_\beta[\mathbf{G}_\beta]$ be generic over N such that $q' \restriction \beta$ forces $\mathbf{G}(\beta)$ has an upper bound in $\underset{\sim}{\mathbb{Q}}_\beta[\mathbf{G}_\beta]$. Set $\mathbf{G}' = \mathbf{G}_\beta * \mathbf{G}(\beta)$. Then $q', \mathbf{G}'$ are as required.

(3) Follows from (2), as $\alpha \in N \cap \omega_1 \Rightarrow \delta = \omega\alpha \in N \cap \omega_1$. □

Remark 3.18. (1) The results of this section include as special cases [5, Ch.V, §5, §7]. There is no direct comparison with [5, Ch.VIII, §4], [6, Ch.VIII, §4], but we can make the notion somewhat more complicated, to include the theorems there in our context, but this is not really needed for the examples discussed there (see Section 5). The condition in [5, Ch.VIII, §4] and [6, Ch.VIII, §4] involves having

[5]formally we only need to look at $\bar{N}' = \langle N'_i : i < \delta\rangle, N'_i = N_i \cap \mathscr{H}(\chi^{\mathfrak{p}}_0)$ and apply the $(<^+ \omega_1)$-properness to it.

many sequences $\langle N_i : i \leq \delta \rangle$ such that if $p_0, p_1 \in \mathbb{P}_\alpha, p_\ell$ is $(N_i, \mathbb{P})$-generic for $i, p_\ell \Vdash$ "$\underset{\sim}{\mathbf{G}}_{\mathbb{P}_\alpha} \cap N_0 = \mathbf{G}^*$", then there is $\mathbf{G}' \subseteq \mathrm{Gen}(N_0[\mathbf{G}^*], \underset{\sim}{\mathbb{Q}}_\alpha[\mathbf{G}^*]), \mathbf{G}' \in N_0$, such that $\nVdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\mathbf{G}}'$ has no bound in $\underset{\sim}{\mathbb{Q}}_\alpha$". This speaks on a family of sequences from $[\mathscr{H}(\chi)]^{\aleph_0}$ rather than members of $\mathscr{H}(\chi)$.

(2) For [6, Ch.XVIII, §2], the comparison is not so easy. Our problem is to "carry" good $(N, \mathbb{P}_i, \langle \mathbf{G}_\ell : \ell < n \rangle), \mathbf{G}_\ell \in \mathrm{Gen}(N, \mathbb{P}_i)$ with a bound, such that we can "increase i" and we can find $N', y \in N' \in N, N' \prec N$ such that $(N', P_i, \langle \mathbf{G}_i \cap N : \ell < n \rangle)$ is good enough. In [6, Ch.XVIII] we are carrying genericity in some $\mathbb{P}_{\bar{\alpha}}$, where $\bar{\alpha} \in$ trind(i)[6], but here we have much less. But what we need is the implication "if $(N, \mathbb{P}_i, \bar{G})$ is good we can extend it to good $(N, \mathbb{P}_{i+1}, \bar{G}')$", so making good weaker generates an incomparable notion and clearly there are other variants.

(3) The iteration theorems proved in this section can be used to give alternative proofs of the consistency results in [6, Ch.XVIII, §1] (see Section 5).

§ 4. Delayed properness

In this section we introduce several notions that will be used in sections 5 and 6. We concentrate on simple reasonable parameters and we present two versions for it. The simpler one is version 2 for which simplicity is a very natural demand. The proof of the next lemma is straightforward in which a general way to create simple reasonable parameters is introduced.

Lemma 4.1. (1) *Assume*

(a) $\bar{\chi} = \langle \chi_\alpha : \alpha < \alpha^* \rangle$ *increases fast enough, so that* $\mathscr{H}((\bigcup_{\beta<\alpha} \chi_\beta)^+) \in \mathscr{H}(\chi_\alpha)$,

(b) $\mathscr{E}_\alpha \subseteq \{N : N$ *is a countable elementary submodel of* $(\mathscr{H}(\chi_\alpha), \in)\}$ *is stationary,*

[6]see [6, Ch.XVIII, Definiton 2.1].

(c) *$R_\alpha \in \mathscr{H}(\chi_\alpha)$ and $N \in \mathscr{E}_\alpha$ implies $\langle \chi_\beta : \beta < \alpha \rangle \in N, \langle R_\beta : \beta \le \alpha \rangle \in N$ and $\langle \mathscr{E}_\beta : \beta < \alpha \rangle \in N$.*

Then there is a standard reasonable parameter $\mathfrak{p}$ with $\ell g(\mathfrak{p}) = \alpha^*, \chi^{\mathfrak{p}}_\alpha = \chi_\alpha$, $\mathscr{E}^{\mathfrak{p}}_\alpha = \mathscr{E}_\alpha$ and $R^{\mathfrak{p}}_\alpha = R_\alpha$.

(2) *If in addition clause (d) below holds, then $\mathfrak{p}$ is a simple standard reasonable parameter (recall Definition 3.5(4)), where*

(d) *$\beta \in N \in \mathscr{E}_\alpha, \beta < \alpha \Rightarrow N \cap \mathscr{H}(\chi_\beta) \in \mathscr{E}_\beta$.*

(3) *If $\chi_\alpha = (\beth_{2\alpha+1})^+$ for $\alpha < \alpha^*, R_\alpha \in \mathscr{H}(\chi_\alpha)$, then χ_α increases fast enough. If $\langle \chi_\alpha : \alpha < \alpha^* \rangle, \langle R_\alpha : \alpha < \alpha^* \rangle$ are as in part (1), $\chi \le \chi_0, \mathscr{E} \subseteq [\mathscr{H}(\chi)]^{\le \aleph_0}$ stationary and we let $\mathscr{E}_\alpha = \{N : N$ is a countable elementary submodel of $(\mathscr{H}(\chi_\alpha), \in)$ and $\langle \chi_\beta : \beta < \alpha \rangle, \langle R_\beta : \beta \le \alpha \rangle, \mathscr{E}$ belong to N and $N \cap \mathscr{H}(\chi) \in \mathscr{E}\}$, then the assumptions of parts (1) and (2) above hold.*

Proof. (1). Let $\mathfrak{p} = \langle \bar{\chi}, \bar{R}, \bar{\mathscr{E}}, \bar{D} \rangle$, where $\bar{\chi}, \bar{R}$ and $\bar{\mathscr{E}}$ are given as above and $\bar{D}$ is defined as in Definition 3.5(1). Then $\mathfrak{p}$ is easily seen to be a standard reasonable parameter as required.

Items (2) and (3) are clear. □

We now define an extension of the games $\Game_\alpha(N, \mathfrak{p})$ and $\Game_\alpha(N, N', \mathfrak{p})$ given in Definition 3.6.

Definition 4.2. Let $\mathfrak{p}$ be a reasonable parameter and $\alpha \le \beta < \ell g(\mathfrak{p})$.

(1) For $N \in \mathscr{E}^{\mathfrak{p}}_\beta$ such that $\alpha \in N$, we define a game $\Game_{\alpha,\beta}(N) = \Game_{\alpha,\beta}(N, \mathfrak{p})$ of length ω as follows. In the n-th move:

(a) the challenger chooses $X_n \in D^{\mathfrak{p}}_\beta(N)$ such that $m < n \Rightarrow X_n \subseteq X_m$,

(b) the chooser chooses $\alpha_n \in \alpha \cap N$,

(c) the challenger chooses $\beta'_n \in \beta \cap N$ and $y'_n \in N \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\alpha_n})$,

(d) the chooser chooses $\beta_n \in \beta \cap N \backslash \beta'_n$ together with $M_n \in X_n \cap \mathscr{E}^{\mathfrak{p}}_{\beta_n}$, $y_n \in M_n \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\alpha_n})$ and $Y_n \in D^{\mathfrak{p}}_{\beta_n}(M_n)$ satisfying:

- $\alpha_n \leq \beta_n$,
- $y'_n \in M_n$,
- $\alpha_n \in M_n$,
- $Y_n \subseteq X_n$ and $Y_n \in N$

(e) the challenger chooses $M'_n \in Y_n \cap \mathscr{E}^{\mathfrak{p}}_{\alpha_n} \cap \big(M_n \cup \{M_n \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\alpha_n})\}\big)$ satisfying $y_n, y'_n \in M'_n$ and chooses $Z_n \in D^{\mathfrak{p}}_{\alpha_n}(M'_n) = D^{\mathfrak{p}}_{\alpha_n}(M'_n \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\alpha_n}))$ such that $Z_n \subseteq Y_n$.

At the end, the chooser wins the play if

$$\bigcup\{Z_n \cup \{M'_n\} : n < \omega\} \in D^+_\alpha(N) = D^+_\alpha(N \cap \mathscr{H}(\chi^{\mathfrak{p}}_\alpha)),$$

where $D_\alpha = D^{\mathfrak{p}}_\alpha$.

(2) We call $\eth_{\alpha,\beta}(N) = \eth_{\alpha,\beta}(N, \mathfrak{p})$, defined in clause (1), version 1 of the game. Version 2 of the game is defined similarly, where

- in clause (d) we require $\alpha_n \leq_{\mathfrak{p}} \beta_n$.
- in clause (e), we add the requirement $M'_n = M_n \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\alpha_n})$,

If we do not mention the version, it means that it holds for both versions.

(3) Assume $N \in N' \prec (\mathscr{H}(\chi), \in)$. We define the game $\eth'_{\alpha,\beta}(N, N', \mathfrak{p})$ similarly, where items (a) - (e) are replaced by:

(a)′ the challenger chooses $X_n \in D^{\mathfrak{p}}_\beta(N) \cap N'$ such that $m < n \Rightarrow X_n \subseteq X'_m$,

(b)′ the chooser chooses $\alpha_n \in \alpha \cap N$ and $X'_n \subseteq X_n$ such that $X'_n \in D^{\mathfrak{p}}_\beta(N) \cap N'$,

(c)′ like (c) above,

(d)′ like (d) above, but we replace "$Y_n \in N$" by "$Y_n \in N'$",

(e)′ like (e) above, but add $Z_n \in N'$.

Note that every proper initial segment of a play belongs to N'.

Definition 4.3. Let $\mathfrak{p}$ be a reasonable parameter.

(1) For $\alpha \leq \beta < \ell g(\mathfrak{p})$, we say $\mathfrak{p}$ is a $\partial_{\alpha,\beta}$-winner (resp. non-$\partial_{\alpha,\beta}$-loser), if for some $x \in \mathscr{H}(\chi^{\mathfrak{p}}_{\beta})$ we have:

if $\{x, \alpha\} \in N \in \mathscr{E}^{\mathfrak{p}}_{\beta}$, then the chooser wins the game $\partial_{\alpha,\beta}(N, \mathfrak{p})$

(resp. the challenger does not win in the game $\partial_{\alpha,\beta}(N, \mathfrak{p})$).

(2) Similarly we can define when $\mathfrak{p}$ is a $\partial'_{\alpha,\beta}$-winner (resp. non-$\partial'_{\alpha,\beta}$-loser).

(3) For any function $f : \ell g(\mathfrak{p}) \to \mathscr{P}(\ell g(\mathfrak{p}))$ we can replace α, β by f, so that $\mathfrak{p}$ is a ∂_f-winner (resp. non-∂_f-loser) if for every $\alpha < \ell g(\mathfrak{p})$ and $\beta \in f(\alpha)$, $\mathfrak{p}$ is a $\partial_{\alpha,\beta}$-winner (resp. non-$\partial_{\alpha,\beta}$-loser).

Given a reasonable parameter $\mathfrak{p}$, we define some families of functions as follows.

Definition 4.4. Let $\mathfrak{p}$ be a reasonable parameter.

(1) Let $\mathscr{F}^{\mathfrak{p}}$ be the family of functions f from $\ell g(\mathfrak{p})$ to $\mathscr{P}(\ell g(\mathfrak{p}))$ such that for each $\alpha < \ell g(\mathfrak{p}), f(\alpha)$ is a nonempty subset of $[\alpha, \ell g(\mathfrak{p}))$.

(2) Let $\mathscr{F}^{\mathfrak{p}}_{\rm club}$ be the set of $f \in \mathscr{F}^{\mathfrak{p}}$ such that for each $\alpha < \ell g(\mathfrak{p}), f(\alpha)$ is a club of $\ell g(\mathfrak{p})$.

(3) Let $\mathscr{F}^{\mathfrak{p}}_{\rm nd}$ be the set of $f \in \mathscr{F}^{\mathfrak{p}}$ such that for each $\alpha < \ell g(\mathfrak{p}), f(\alpha)$ is an end segment of $\ell g(\mathfrak{p})$, we then may identify $f(\alpha)$ with $\min(f(\alpha))$.

(4) Call $f \in \mathscr{F}^{\mathfrak{p}}$ decreasing continuous if

(a) $\alpha < \beta < \ell g(\mathfrak{p}) \Rightarrow f(\alpha) \supseteq f(\beta)$,

(b) for limit $\delta < \ell g(\mathfrak{p})$ we have $f(\delta) = \bigcap\{f(\alpha) : \alpha < \delta\}$.

Let also $f \leq g$ mean that $(\forall \alpha < \ell g(\mathfrak{p}))(g(\alpha) \subseteq f(\alpha))$.

(5) Let $\mathscr{F}^{\mathfrak{p}}_{\rm dc}$ be the set of decreasing continuous functions $f \in \mathscr{F}^{\mathfrak{p}}_{\rm club}$.

We are interested in dealing with winning strategies for the games ∂_f, where f is a function coming from one of the above family of functions. The next lemma gives some obvious monotonicity properties for these classes of functions.

Lemma 4.5. *Assume $\mathfrak{p}$ is a reasonable parameter.*

(1) *If $\alpha \leq_{\mathfrak{p}} \alpha' \leq \beta = \beta' < \ell g(\mathfrak{p})$, and $\mathfrak{p}$ is a $\Game_{\alpha,\beta}$-winner, then it is $\Game_{\alpha',\beta'}$-winner. Similarly for $\Game'$-winner, non-$\Game$-loser and non-$\Game'$-loser.*

(2) *If $\mathfrak{p}$ is a $\Game_{\alpha,\beta}$-winner, then $\mathfrak{p}$ is a $\Game'_{\alpha,\beta}$-winner and a non-$\Game_{\alpha,\beta}$-loser. If $\mathfrak{p}$ is a $\Game'_{\alpha,\beta}$-winner or non-$\Game_{\alpha,\beta}$-loser, then $\mathfrak{p}$ is non-$\Game'_{\alpha,\beta}$-loser.*

(3) *Assume $f, g \in \mathscr{F}^{\mathfrak{p}}$ and $f \leq g$. If $\mathfrak{p}$ is a $\Game_f$-winner (or $\Game'_f$-winner) (or non-$\Game_f$-loser) (or non-$\Game'_f$-loser), then $\mathfrak{p}$ is a $\Game_g$-winner (or $\Game'_g$-winner) (or non-$\Game_g$-loser) (or non-$\Game'_g$-loser).*

Proof. (1) Suppose σ is a winning strategy for chooser in the game $\Game_{\alpha,\beta}$. We show that it is a winning strategy for chooser in the game $\Game_{\alpha',\beta'}$ as well. Suppose not, so, following the notation of Definition 4.2,

$$\bigcup\{Z_n \cup \{M'_n\} : n < \omega\} \notin D^+_{\alpha'}(N) \, (= D^+_{\alpha'}(N \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\alpha'}))).$$

It then follows that

$$N \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\alpha'}) \setminus \bigcup\{Z_n \cup \{M'_n\} : n < \omega\} \in D_{\alpha'}(N).$$

But then, as $\alpha \leq_{\mathfrak{p}} \alpha'$,

$$N \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\alpha}) \setminus \bigcup\{Z_n \cup \{M'_n\} : n < \omega\} \in D_{\alpha}(N).$$

But, as σ is a winning strategy for chooser in the game $\Game_{\alpha,\beta}$, we have

$$\bigcup\{Z_n \cup \{M'_n\} : n < \omega\} \in D^+_{\alpha}(N),$$

a contradiction.

The proof of clause (2) is similar to the proof of Lemma 3.7(2) and the proof of clause (3) is straightforward. $\square$

Lemma 4.6. (1) *Assume $\mathfrak{p}$ is a standard reasonable parameter. Then $\mathfrak{p}$ is a winner.*

(2) *If $\mathfrak{p}$ is a reasonable parameter and it is a winner, then $\mathfrak{p}$ is a $\eth_{\alpha,\alpha}$-winner.*

(3) *If $\mathfrak{p}$ is a reasonable parameter and it is a winner and $\alpha \leq \beta < \ell g(\mathfrak{p})$, then $\mathfrak{p}$ is a $\eth_{\alpha,\beta}$-winner (hence $\eth_f$-winner for any $f : \ell g(\mathfrak{p}) \to \mathscr{P}(\ell g(\mathfrak{p}))$).*

(4) *Similarly with $\eth'$-winner, $\eth'_{\alpha,\beta}$ winner and/or with the "non-loser" cases.*

Proof. Clause (1) follows from Lemma 3.7. The proof of items (2)-(4) is easy and follows from the Definitions 3.6 and 4.2. □

We now define an interpretation of reasonable parameters in the forcing extensions and show that these interpretations are reasonable parameters in the corresponding extension.

Definition 4.7. Let $\mathfrak{p}$ be a reasonable parameter and let $\mathbb{P}$ be a proper forcing notions which adds no new reals. Suppose that $\mathscr{P}(\mathbb{P}) \in \mathscr{H}(\chi_0^{\mathfrak{p}})$ and $\mathbf{G}_{\mathbb{P}} \subseteq \mathbb{P}$ is generic over $\mathbf{V}$. We interpret $\mathfrak{p}$ in $\mathbf{V}^{\mathbb{P}}$ as $\mathfrak{p}' = \mathfrak{p}^{\mathbf{V}[\mathbf{G}_{\mathbb{P}}]}$, defined as follows:

(*a*) $\chi_\alpha^{\mathfrak{p}'} = \chi_\alpha^{\mathfrak{p}}$,

(*b*) $R_\alpha^{\mathfrak{p}'} = \langle R_\alpha^{\mathfrak{p}}, \mathbb{P}, \mathbf{G}_P \rangle$,

(*c*) $\mathscr{E}_\alpha^{\mathfrak{p}'} = \{N[\mathbf{G}_P] : N \in \mathscr{E}_\alpha^{\mathfrak{p}}, \mathbb{P} \in N \text{ and } N[\mathbf{G}_{\mathbb{P}}] \cap V = N\}$,

(*d*) $D_\alpha^{\mathfrak{p}'}(N[\mathbf{G}_{\mathbb{P}}]) = \{\{M[\mathbf{G}_{\mathbb{P}}] \in \mathscr{E}_\alpha^{\mathfrak{p}'} : M \in Y \cap \bigcup_{\beta<\alpha} \mathscr{E}_\beta^{\mathfrak{p}}\} : Y \in D_\alpha^{\mathfrak{p}}(N)\}$.

We also use $\mathfrak{p}^{\mathbb{P}}$ for $\mathfrak{p}' = \mathfrak{p}^{\mathbf{V}[\mathbf{G}_{\mathbb{P}}]}$.

The proof of the next lemma is straightforward.

Lemma 4.8. *Let $\mathfrak{p}, \mathbb{P}$ and $\mathbf{G}_{\mathbb{P}}$ be as in Definition 4.7.*

(1) *$\mathfrak{p}^{\mathbf{V}[\mathbf{G}_{\mathbb{P}}]}$ is a reasonable parameter in $\mathbf{V}[\mathbf{G}_{\mathbb{P}}]$.*

(2) *If $\mathfrak{p}$ is, in $\mathbf{V}$, a $\eth$-winner (or non-$\eth$-loser or $\eth'$-winner or non-$\eth'$-loser), then $\mathfrak{p}^{\mathbf{V}[\mathbf{G}_{\mathbb{P}}]}$ is so in $\mathbf{V}[\mathbf{G}_{\mathbb{P}}]$.*

(3) *If $\mathfrak{p}$ is, in $\mathbf{V}$, a $\eth_{\alpha,\beta}$-winner (or non- $\eth_{\alpha,\beta}$-loser or $\eth'_{\alpha,\beta}$-winner or non-$\eth'_{\alpha,\beta}$-loser), then $\mathfrak{p}^{\mathbf{V}[\mathbf{G}_{\mathbb{P}}]}$ is so in $\mathbf{V}[\mathbf{G}_p]$.*

Proof. (1). It is easily seen, using the fact that $\mathbb{P}$ is an NNR proper forcing notion, that $\mathfrak{p}^{\mathbf{V}[\mathbf{G}_{\mathbb{P}}]}$ satisfies items (a)-(i) of Definition 3.2 in $\mathbf{V}[\mathbf{G}_{\mathbb{P}}]$.

(2). Suppose $\mathfrak{p}$ is a winner in $\mathbf{V}$. We show that $\mathfrak{p}' = \mathfrak{p}^{\mathbf{V}[\mathbf{G}_{\mathbb{P}}]}$ is a winner in $\mathbf{V}[\mathbf{G}_{\mathbb{P}}]$. Suppose $\alpha < \ell g(\mathfrak{p})$ and $N' \in \mathscr{E}^{\mathfrak{p}'}_{\alpha}$. Then for some $N \in \mathscr{E}^{\mathfrak{p}}_{\alpha}, N' = N[\mathbf{G}_{\mathbb{P}}]$. Let σ be a winning strategy for $\mathfrak{p}$ for the game $\partial_{\alpha}(N, \mathfrak{p})$. We define the winning strategy σ' for the game $\partial_{\alpha}(N', \mathfrak{p}')$ in $\mathbf{V}[\mathbf{G}_{\mathbb{P}}]$ as follows. Following the notation of Definition 3.6, in the n-th move, challenger chooses some $X'_n \in D^{\mathfrak{p}'}_{\alpha}(N')$. So for some $X_n \in D^{\mathfrak{p}}_{\alpha}(N)$, we have $X'_n = \{M[\mathbf{G}_{\mathbb{P}}] \in \mathscr{E}^{\mathfrak{p}'}_{\alpha} : M \in X_n \cap \bigcup_{\beta<\alpha} \mathscr{E}^{\mathfrak{p}}_{\beta}\}$. Let $M_n \in X_n$ and $Y_n \subseteq M_n \cap X_n$ with $Y_n \in D^{\mathfrak{p}}(M_n) \cap N$ be the play that chooser does using the strategy σ. Let $M'_n = M_n[\mathbf{G}_{\mathbb{P}}]$ and $Y'_n = \{M[\mathbf{G}_{\mathbb{P}}] : M \in Y_n\}$ be what chooser replies via σ'. Then challenger chooses some $Z'_n \subseteq Y'_n$ with $Z'_n \in D^{\mathfrak{p}'}(M'_n)$. Thus for some $Z_n \subseteq Y_n$ with $Z_n \in D^{\mathfrak{p}}(M_n) \cap \mathbf{V}$, we have $Z'_n = \{M[\mathbf{G}_{\mathbb{P}}] : M \in Z_n\}$. Since σ is a winning strategy, $\bigcup\{\{M_n\} \cup Z_n : n < \omega\} \in D^{\mathfrak{p}}(N)$. It then follows that

$$\bigcup\{\{M'_n\} \cup Z'_n : n < \omega\} \in D^{\mathfrak{p}'}(N').$$

Thus σ' is a winning strategy for chooser for the game $\partial_{\alpha}(N', \mathfrak{p}')$. The proof for non-$\partial$-loser or ∂'-winner or non-∂'-loser is the same.

Clause (3) can be proved similarly. □

Definition 4.9. Let $\mathfrak{p}$ be a reasonable parameter and let $\mathbb{Q}$ be a forcing notion.

(1) For $\alpha \leq \beta < \ell g(\mathfrak{p})$, we say $\mathbb{Q}$ is $(\mathfrak{p}, \alpha, \beta)$-proper if $\mathscr{P}(\mathbb{Q}) \in \mathscr{H}(\chi^{\mathfrak{p}}_0)$ and:

- $(*)$ for some $x \in \mathscr{H}(\chi^{\mathfrak{p}}_{\beta})$, if $N \in \mathscr{E}^{\mathfrak{p}}_{\beta}, \{x, \mathbb{Q}, \alpha\} \in N, p \in N \cap \mathbb{Q}$ and $Y \in D^{\mathfrak{p}}_{\alpha}(N)$, then for some q we have:
  - (a) $p \leq q \in \mathbb{Q}$,
  - (b) q is $(N, \mathbb{Q})$-generic,

(c) for some $N' \in (\mathscr{E}^{\mathfrak{p}}_{\alpha} \cap N \cap Y) \cup \{N \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\alpha})\}$ satisfying $\alpha = \beta \Rightarrow N' = N$ we have $q \Vdash_{\mathbb{Q}}$ "$\mathscr{M}_{\mathbb{Q}}[\underset{\sim}{\mathbf{G}}_{\mathbb{Q}}, N', y^*] \cap Y \in D^{\mathfrak{p}}_{\alpha}(N')$" where $y^* = \langle x, p, \mathbb{Q}\rangle$. Note that this implies q is $(N', \mathbb{Q})$-generic.

We call the above, version 1 of $(\mathfrak{p}, \alpha, \beta)$-properness. Version 2 of $(\mathfrak{p}, \alpha, \beta)$-properness is defined similarly, but we demand $N' = N \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\alpha})$ and $\alpha \leq_{\mathfrak{p}} \beta$.

(2) We say $\mathbb{Q}$ is $(\mathfrak{p}, f)$-proper, where $f \in \mathscr{F}^{\mathfrak{p}}$ (see Definition 4.4), when for every $\alpha < \ell g(\mathfrak{p})$ and $\beta \in f(\alpha)$, the forcing notion $\mathbb{Q}$ is $(\mathfrak{p}, \alpha, \beta)$-proper.

(3) We say $\mathbb{Q}$ is $\mathfrak{p}$-proper if $\mathbb{Q}$ is $(\mathfrak{p}, \alpha, \alpha)$-proper for $\alpha < \ell g(\mathfrak{p})$. We say $\mathbb{Q}$ is almost $\mathfrak{p}$-proper if $\mathbb{Q}$ is $(\mathfrak{p}, f)$-proper for some $f \in \mathscr{F}^{\mathfrak{p}}$.

The next lemma shows some relations between the above defined notions.

Lemma 4.10. *Assume $\mathfrak{p}$ is a simple reasonable parameter.*

(1) *If $\alpha' \leq \alpha \leq \beta \leq \beta' < \ell g(\mathfrak{p})$ (for version 2 we demand $\alpha' \leq_{\mathfrak{p}} \beta'$ and $\alpha \leq_{\mathfrak{p}} \beta$) and $\mathbb{Q}$ is a $(\mathfrak{p}, \alpha, \beta)$-proper forcing notion, then $\mathbb{Q}$ is a $(\mathfrak{p}, \alpha', \beta')$-proper forcing notion.*

(2) *Assume f, f' are in $\mathscr{F}^{\mathfrak{p}}$ and $f \leq f'$. If $\mathbb{Q}$ is a $(\mathfrak{p}, f)$-proper forcing notion, then $\mathbb{Q}$ is a $(\mathfrak{p}, f')$-proper forcing notion.*

Proof. (1) Suppose $\mathbb{Q}$ is $(\mathfrak{p}, \alpha, \beta)$-proper as witnessed by $x \in \mathscr{H}(\chi^{\mathfrak{p}}_{\beta}) \subseteq \mathscr{H}(\chi^{\mathfrak{p}}_{\beta'})$. We show that x witnesses that $\mathbb{Q}$ is $(\mathfrak{p}, \alpha', \beta')$-proper as well. Thus let $N \in \mathscr{E}^{\mathfrak{p}}_{\beta'}$ with $\{x, \mathbb{Q}, \alpha\} \in N$, $p \in N \cap \mathbb{Q}$ and $Y \in D^{\mathfrak{p}}_{\alpha'}(N)$. Then $N \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\beta}) \in \mathscr{E}^{\mathfrak{p}}_{\beta}$. Let $q \geq p$ be as in Definition 4.9 and it witnesses that $\mathbb{Q}$ is $(\mathfrak{p}, \alpha, \beta)$-proper with respect to $N \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\beta}), p$ and Y. Then q witnesses $\mathbb{Q}$ is $(\mathfrak{p}, \alpha', \beta')$-proper with respect to N, p and Y.

(2) is clear, as for every $\alpha < \ell g(\mathfrak{p})$, $f'(\alpha) \subseteq f(\alpha)$. $\square$

It follows from the above lemma that if $\mathfrak{p}$ is a simple reasonable parameter and $f \in \mathscr{F}^{\mathfrak{p}}$, then

$$\mathfrak{p}\text{-proper} \implies (\mathfrak{p}, f)\text{-proper} \implies \text{almost } \mathfrak{p}\text{-proper}.$$

We may like to consider $(\mathfrak{p}, f)$-properness for iterations which may add reals. Then we have to replace $D^{\mathfrak{p}}_\alpha(N)$ by a definition which is absolute enough (and the non-loser versions have to be absolute enough as well). In such situations, it is natural to restrict ourselves to those sets Y which are $\mathfrak{p}$-closed, see below.

Definition 4.11. Let $\mathfrak{p}$ be a simple reasonable parameter, $\alpha < \ell g(\mathfrak{p})$ and $N \in \mathscr{E}^{\mathfrak{p}}_\alpha$. A subset $Y \subseteq N$ is called $\mathfrak{p}$-closed if:

(a) $Y \subseteq N \cap \bigcup\limits_{\beta<\alpha} \mathscr{E}^{\mathfrak{p}}_\beta$,

(b) if $M \in N \cap \mathscr{E}^{\mathfrak{p}}_\beta$, $\beta < \alpha$ (hence $\beta \in \alpha \cap M \subseteq \alpha \cap N$), $\gamma \in M \cap \beta$ and $M \cap \mathscr{H}(\chi^{\mathfrak{p}}_\gamma) \in \mathscr{E}^{\mathfrak{p}}_\gamma$,[7] then

$$M \cap \mathscr{H}(\chi^{\mathfrak{p}}_\gamma) \in Y \iff M \in Y,$$

(c) if $\beta < \alpha, M_\ell \in N \cap \mathscr{E}^{\mathfrak{p}}_\beta$ (hence $\beta \in \alpha \cap N$), $M_\ell \subseteq M_{\ell+1}$ for $\ell < \omega$ and $M = \bigcup\limits_{\ell<\omega} M_\ell \in N \cap \mathscr{E}^{\mathfrak{p}}_\beta$, and even $\langle M_\ell : \ell < \omega\rangle \in N$, then

$$\left(\bigwedge_{\ell<\omega} M_\ell \in Y\right) \Rightarrow M \in Y.$$

Definition 4.12. Assume $\mathfrak{p}$ is a reasonable parameter, $\mathbb{P}$ is a forcing notion and $\underset{\sim}{\mathbb{Q}}$ is a $\mathbb{P}$-name of a forcing notion.

(1) For $\alpha \leq \beta < \ell g(\mathfrak{p})$, we say $\underset{\sim}{\mathbb{Q}}$ has (κ, α, β)-anti w.d. above $\mathbb{P}$ (or $(\mathbb{P}, \underset{\sim}{\mathbb{Q}})$ has (κ, α, β)-anti-w.d.), if clause (A) implies clause (B), where:

(A) (a) $N_0 \in \mathscr{E}^{\mathfrak{p}}_\alpha$ and $N_1 \in \mathscr{E}^{\mathfrak{p}}_\beta$,

(b) $N_0 \in N_1$ and $\{\mathbb{P}, \underset{\sim}{\mathbb{Q}}\} \in N_0$,

(c) $n < 1 + \kappa$,

(d) $p_\ell \in \mathbb{P}$ is $(N_\iota, \mathbb{P})$-generic for $\ell < n$ and $\iota = 0, 1$,

(e) $p_\ell \Vdash$ "$\underset{\sim}{\mathbf{G}}_{\mathbb{P}} \cap N_1 = \mathbf{G}^\ell$" for $\ell < n$,

[7]note that this requirement is redundant if $\mathfrak{p}$ is simple or just $\gamma \leq_{\mathfrak{p}} \beta$.

(f) $\mathbf{G}^\ell \cap N_0 = \mathbf{G}^*$ for $\ell < n$,

(g) $Y = \bigcap\limits_{\ell<n} \mathscr{M}_{\mathbb{P}}[\mathbf{G}^\ell, N_1, y]$ belongs to $D^{\mathfrak{p}}_\beta(N_1)$, where $y = \langle N_0, \mathbb{P}, \underset{\sim}{\mathbb{Q}}\rangle$,

(h) $\underset{\sim}{q}$ is a $\mathbb{P}$-name of a member of $\underset{\sim}{\mathbb{Q}}$,

(i) $\underset{\sim}{q} \in N_0$,

(B) there is a triple $(\langle p'_\ell : \ell < n\rangle, \underset{\sim}{q}', \mathbf{G}^{**})$ such that:

(a) $\underset{\sim}{q}'$ is a $\mathbb{P}$-name of a member of $\underset{\sim}{\mathbb{Q}}$,

(b) $p'_\ell \Vdash$ "$\underset{\sim}{q} \leq \underset{\sim}{q}'$" for $\ell < n$,

(c) $\mathbf{G}^{**} \in \mathrm{Gen}(N_0, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$,

(d) $(p'_\ell, \underset{\sim}{q}') \Vdash$ "$\mathbf{G}_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}} \cap N_0 = \mathbf{G}^{**}$" for $\ell < n$.

(2) Given a function $f \in \mathscr{F}^{\mathfrak{p}}$, we say $\underset{\sim}{\mathbb{Q}}$ has (κ, f)-anti w.d. above $\mathbb{P}$ (or $(\mathbb{P}, \underset{\sim}{\mathbb{Q}})$ has (κ, f)-anti-w.d.), if for every $\alpha < \ell g(\mathfrak{p})$, $\underset{\sim}{\mathbb{Q}}$ has $(\kappa, \alpha, f(\alpha))$-anti w.d. above $\mathbb{P}$.

Remark 4.13. We may assume that the conditions $p_\ell, \ell < n$, in clause (1)(A)(d) of the above definition are pairwise incompatible. If p_ℓ and p_ι are compatible, then by clause (1)(A)(e), $C^\ell = C^\iota$, so we may replace p_ℓ, p_ι by a common extension $p_{\ell,\iota}$ of them and take $C^{\ell,\iota} = C^\ell$.

As the sets $\mathscr{H}(\chi_\alpha)$ may change with forcing, we may prefer to use $\mathscr{E}_\alpha \subseteq [\chi_\alpha]^{\leq\aleph_0}$. For this reason, we define the notion of ordinal based parameter, and we will show that any ordinal based parameter gives naturally a reasonable parameter.

Definition 4.14. (1) We call $\mathfrak{p}$ an o.b. (ordinal based) parameter if

$$\mathfrak{p} = (\bar{\chi}^{\mathfrak{p}}, \bar{R}^{\mathfrak{p}}, \bar{\mathscr{E}}^{\mathfrak{p}}, \bar{D}^{\mathfrak{p}})(= (\bar{\chi}, \bar{R}, \bar{\mathscr{E}}, \bar{D})),$$

where for some ordinal α^*, called $\ell g(\mathfrak{p})$, we have:

(a) $\bar{\chi} = \langle \chi_\alpha : \alpha < \alpha^*\rangle$, where χ_α is a regular cardinal and $\mathscr{H}((\bigcup\limits_{\beta<\alpha} \chi_\beta)^+) \in \mathscr{H}(\chi_\alpha)$,

(b) $\bar{R} = \langle R_\alpha : \alpha < \alpha^* \rangle$, where R_α is an $n(R_\alpha)$-place relation on some bounded subset of χ_α, [8]

(c) $\bar{\mathscr{E}} = \langle \mathscr{E}_\alpha : \alpha < \alpha^* \rangle$, where $\mathscr{E}_\alpha \subseteq [\chi_\alpha]^{\leq \aleph_0}$ is stationary,

(d) $\bar{D} = \langle D_\alpha : \alpha < \alpha^* \rangle$ and D_α is a function with domain $\mathscr{E}_\alpha$ and for each $a \in \mathscr{E}_\alpha$, $D_\alpha(a)$ is a pseudo-filter on a,

(e) let $\mathfrak{p}^{[\alpha]} = \langle \bar{\chi} \restriction \alpha, \bar{R} \restriction (\alpha+1), \bar{\mathscr{E}} \restriction \alpha, \bar{D} \restriction \alpha \rangle$,

(f) if $a \in \mathscr{E}_\alpha$ and $X \subseteq a$, then

$$X \in D_\alpha(a) \iff X \cap \bigcup_{\beta<\alpha} \mathscr{E}_\beta \in D_\alpha(a).$$

(2) An o.b. parameter $\mathfrak{p}$ is simple if in addition, it satisfies:

(g) if $a \in \mathscr{E}_\alpha$, $X \in D_\alpha(a)$ and $\beta \in \alpha \cap a$, then $a \cap \chi_\beta \in \mathscr{E}_\beta$.

(3) For $\mathfrak{p}$ as above let $\mathfrak{q} = \mathfrak{p}^{\mathbf{V}}$ be defined by

- $\ell g(\mathfrak{q}) = \ell g(\mathfrak{p})$,

and we define by induction on $\alpha < \ell g(\mathfrak{p})$

- $\chi^{\mathfrak{q}}_\alpha = \chi^{\mathfrak{p}}_\alpha$,
- $R^{\mathfrak{q}}_\alpha = R^{\mathfrak{p}}_\alpha$,
- $\mathscr{E}^{\mathfrak{q}}_\alpha = \{N \prec (\mathscr{H}(\chi^{\mathfrak{q}}_\alpha), \in) : N$ is countable, $N \cap \chi^{\mathfrak{p}}_\alpha \in \mathscr{E}^{\mathfrak{p}}_\alpha$ and $\mathfrak{q}^{[\alpha]} \in N\}$. Note that $\mathfrak{q}^{[\alpha]}$ is well defined by the induction hypothesis.
- $D^{\mathfrak{q}}_\alpha(N)$ is defined as

$$D^{\mathfrak{q}}_\alpha(N) = \{Y' : \ Y' \subseteq \bigcup_{\beta<\alpha} \mathscr{E}^{\mathfrak{q}}_\beta \text{ and for some } y \in N \cap \bigcup_{\beta<\alpha} \mathscr{H}(\chi^{\mathfrak{p}}_\beta) \text{ and } Y \in D^{\mathfrak{p}}_\alpha(N \cap \chi^{\mathfrak{p}}_\alpha) \text{ we have } Y' \supseteq \{M : M \in N \cap \bigcup_{\beta<\alpha} \mathscr{E}^{\mathfrak{q}}_\beta \text{ and } y \in M \text{ and } M \cap \chi^{\mathfrak{p}}_\alpha \in Y\}\}.$$

[8] we could have asked "on χ_α", as there is no real difference.

(4) For an o.b. parameter $\mathfrak{p}$, we say $\bar{\mathbb{Q}}$ is an NNR^0_κ-iteration for $\mathfrak{p}$ if it is an NNR^0_κ-iteration for $\mathfrak{p}^{\mathbf{V}}$. We say $\mathfrak{p}$ is simple if $\mathfrak{p}^{\mathbf{V}}$ is. Similarly for Ə-winner, non-Ə-loser, etc.

The next lemma shows that each o.b. parameter leads to a canonical reasonable parameter, namely $\mathfrak{p}^{\mathbf{V}}$, and also shows the advantage of working with o.b. parameters than reasonable parameters.

Lemma 4.15. *Assume $\mathfrak{p}$ is an o.b. (simple) parameter in the universe $\mathbf{V}$.*

(1) $\mathfrak{p}^{\mathbf{V}}$ *is a (simple) reasonable parameter.*

(2) *If $\mathbb{P} \in \mathscr{H}(\chi^{\mathfrak{p}}_0)$ is a proper forcing notion (or at least it preserves the stationarity of $\mathscr{E}^{\mathfrak{p}}_\alpha$, for each $\alpha < \ell g(\mathfrak{p})$), then $\Vdash_{\mathbb{P}}$ "$\mathfrak{p}$ is an o.b. (simple) parameter".*

(3) *If forcing with $\mathbb{P}$ adds no reals, then also Ə-winner, non-Ə-loser, etc., are preserved.*

Proof. Straightforward. $\square$

Definition 4.16. Let $\mathfrak{p}$ be an o.b. parameter.

(1) We say $\mathbb{Q} \in \mathbf{V}$ is an NNR^0_κ-forcing for $\mathfrak{p}$ or is a $\mathfrak{p} - NNR^0_\kappa$-forcing notion, when the following holds:

(∗) if for some transitive class $\mathbf{V}_0$ with $\mathfrak{p} \in \mathbf{V}_0$ and some NNR^0_κ-iteration $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \alpha \rangle \in \mathbf{V}_0$ we have $\mathbf{V}_0^{\mathrm{Lim}(\bar{\mathbb{Q}})} = \mathbf{V}$, then we can let $\mathbb{P}_\alpha = \mathrm{Lim}(\bar{\mathbb{Q}})$, $\underset{\sim}{\mathbb{Q}}_\alpha = \mathbb{Q}$ and get an NNR^0_κ-iteration $\bar{\mathbb{Q}}' = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \alpha + 1 \rangle$. (i.e. $\mathbf{V} = \mathbf{V}_0[\mathbf{G}_\alpha], \mathbf{G}_\alpha \subseteq \mathbb{P}_\alpha$ is generic over $\mathbf{V}_0$ and there is a $\mathbb{P}_\alpha$-name $\underset{\sim}{\mathbb{Q}}_\alpha$ such that $\langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i \leq \alpha \rangle$ is an NNR^0_κ-iteration and $\mathbb{Q} = \underset{\sim}{\mathbb{Q}}_\alpha[\mathbf{G}_\alpha]$). In particular $\mathbb{Q}$ is proper and does not add reals.

(2) If we omit "for $\mathfrak{p}$" we mean for any $\mathfrak{p}$ which makes sense. Alternatively, we can put a family of $\mathfrak{p}$'s.

(3) We add "over x" if this holds whenever $x \in \mathbf{V}_0$. We can use the same definition for other versions of NNR.

In the next section we will present several examples of forcing notions that fit into the above definition, in the sense that they are $\mathfrak{p} - NNR^0_\kappa$ for some $2 \leq \kappa \leq \aleph_0$ and some reasonable parameter $\mathfrak{p}$.

§ 5. Examples: shooting thin clubs

In this section we present some examples that fit into our framework. We already know that $(< \omega_1)$-proper forcing notions are $\mathfrak{p}$-proper for standard reasonable parameter $\mathfrak{p}$ of length ω_1 (by Lemma 3.17). We first deal with a forcing notion which is NNR^0_κ-proper, for some $\kappa < \aleph_0$. Second, we deal with shooting clubs of ω_1 running away from some $C_\delta \subseteq \delta = \sup(C_\delta)$ which are small (see [6, Ch.XVIII, §1]). These are the most natural non-ω-proper forcing notions which do not add reals.

Definition 5.1. (1) Let $\bar{C} = \langle (C_\delta, n_\delta) : \delta < \omega_1,\ \delta \text{ limit}\rangle$, where C_δ is an unbounded subset of δ of order type ω and $1 \leq n_\delta < \omega$. Let $\bar{u} = \langle u_\delta : \delta < \omega_1,\ \delta \text{ limit}\rangle$, where $u_\delta \in [2n_\delta + 1]^{n_\delta}$. Then we define $\mathbb{Q} = \mathbb{Q}_{\bar{C},\bar{u}}$ by

$$\mathbb{Q}_{\bar{C},\bar{u}} = \{f : \text{ for some } \alpha < \omega_1, f \text{ is a function from } \alpha \text{ to } \omega \text{ such that for every limit ordinal } \delta \leq \alpha, \text{ for some } k < 2n_\delta + 1, k \notin u_\delta \text{ and for every } i \in C_\delta \text{ large enough we have } f(i) = k\}.$$

$\mathbb{Q}_{\bar{C},\bar{u}}$ is ordered by inclusion.

(2) Assume $\bar{C} = \langle C_\delta : \delta < \omega_1,\ \delta \text{ limit}\rangle, C_\delta$ a closed subset of δ of order type less than $\omega \cdot \delta$ and for $\delta_1 < \delta_2$ limit ordinals, $\sup(C_{\delta_1} \cap C_{\delta_2}) < \delta_1$, and for limit δ^* we have $\{C_\delta \cap \delta^* : \delta < \omega_1\}$ is countable. Assume further that $\bar{\kappa} = \langle \kappa_\delta : \delta < \omega_1, \delta \text{ limit}\rangle$

with $\kappa_\delta \in \{2, 3, \ldots, \aleph_0\}$ and $\bar{D} = \langle D_\delta : \delta < \omega_1 \rangle, D_\delta$ is a family of subsets of $\mathrm{dom}(D_\delta)$, such that the intersection of any $< \kappa_\delta$ of them is non-empty.

Let $\bar{f} = \langle f_{\delta,x} : \delta < \omega_1, x \in \mathrm{dom}(D_\delta) \rangle$ satisfy $f_{\delta,x} : C_\delta \to \omega$ and $\bar{A} = \langle A_\delta : \delta < \omega_1 \rangle$ satisfy $A_\delta \in D_\delta$. Then we define $\mathbb{Q} = \mathbb{Q}_{\bar{C},\bar{D},\bar{\kappa},\bar{f},\bar{A}}$ as

$$\mathbb{Q}_{\bar{C},\bar{D},\bar{\kappa},\bar{f},\bar{A}} = \{f : \begin{array}{l} \text{for some } \alpha < \omega_1, f \text{ is a function from } \alpha \text{ to } \omega \\ \text{such that for every limit } \delta \le \alpha \text{ and for some} \\ x \in A_\delta \text{ we have } f_{\delta,x} \subseteq^* f \text{ i.e. for every large} \\ \text{enough } i \in C_\delta \text{ we have } f_{\delta,x}(i) = f(i)\}. \end{array}$$

$\mathbb{Q}_{\bar{C},\bar{D},\bar{\kappa},\bar{f},\bar{A}}$ is ordered by inclusion.

Lemma 5.2. (1) *The forcing notion* $\mathbb{Q} = \mathbb{Q}_{\bar{C},\bar{u}}$ *from Definition 5.1(1) is proper, does not add reals, and is* $(< \omega_1)$*-proper. It is also* $\mathbb{D}$*-complete for some simple 2-completeness system, hence*

(∗) *if* $\bar{\mathbb{Q}}$ *is a countable support iteration,* $\ell g(\bar{\mathbb{Q}}) = \alpha + 1, \bar{\mathbb{Q}} \restriction \alpha$ *is* NNR^0_2*-iteration and* $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\mathbb{Q}}_\alpha = \mathbb{Q}_{\bar{C},\bar{u}}$", $(\bar{C}, \bar{u})$ *as above* $\in \mathbf{V}$*, then* $\bar{\mathbb{Q}}$ *is an* NNR^0_2*-iteration.*

(2) *Similarly for the forcing notion* $\mathbb{Q} = \mathbb{Q}_{\bar{C},\bar{D},\bar{\kappa},\bar{f},\bar{A}}$ *from Definition 5.1(2).*

Proof. We only prove (1), as (2) can be proved in a similar way. Set $\mathbb{Q} = \mathbb{Q}_{\bar{C},\bar{u}}$.

Let us start by showing that $\mathbb{Q}$ does not add reals. Thus suppose that $f \in \mathbb{Q}$, $\underset{\sim}{t}$ is a $\mathbb{Q}$-name and $f \Vdash$"$\underset{\sim}{t} : \omega \to 2$ is a real". Let χ be a large enough regular cardinal and let $\langle N_n : n < \omega \rangle$ be a chain of countable elementary submodels of $(H(\chi), \in)$ such that $\mathbb{Q}, f, \underset{\sim}{t} \in N_0$ and for each $n < \omega, N_n \in N_{n+1}$. Set $N = \bigcup_{n<\omega} N_n$ and $\delta = N \cap \omega_1$. Also for each $n < \omega$ set $\delta_n = N_n \cap \omega_1$. Pick some $k < 2n_\delta + 1, k \notin u_\delta$. We define by induction on $n < \omega$ a sequence $\langle f_n : n < \omega \rangle$ of conditions in $\mathbb{Q}$, such that:

(1) $f_n \in N_n$,

(2) $f_0 = f$,

(3) for some $\delta_{n-1} < \alpha_n < \delta_n$, $f_n : \alpha_n \to \omega$, where $\delta_{-1} = 0$,

(4) $f_n \leq f_{n+1}$,

(5) $\alpha_n \notin C_\delta$,

(6) for all $i \in C_\delta \cap \alpha_n \setminus (\alpha_0 + 1)$, $f_n(i) = k$.

(7) f_n decides $\underset{\sim}{t} \restriction n$.

The construction can be done quite easily, noting that for each $n, C_\delta \cap \delta_n$ is bounded in δ_n. Let $f' = \bigcup_{n<\omega} f_n$. Then $f' \in \mathbb{Q}$ extends f and it decides $\underset{\sim}{t}$.

To show that $\mathbb{Q}$ is $< \omega_1$-proper, fix $\alpha < \omega_1$ a limit ordinal and let $\bar{N} = \langle N_\xi : \xi < \alpha \rangle$ be an increasing and continuous chain of countable elementary submodels of some $(H(\chi), \in)$ such that $\alpha, \mathbb{Q} \in N_0$ and $\bar{N} \restriction \xi + 1 \in N_{\xi+1}$. For each $\xi < \alpha$ set $\delta_\xi = N_\xi \cap \omega_1$.

Let $f \in \mathbb{Q} \cap N_0$. By essentially the same argument as above, we can find a sequence $\langle f_\xi : \xi < \alpha \rangle$ of conditions in $\mathbb{Q}$, with $f_\xi : \beta_\xi \to \omega$ such that:

(1) $f_0 = f$,

(2) $f_\xi \leq f_{\xi+1}$,

(3) if ξ is a limit ordinal, then $f_\xi = \bigcup_{\zeta<\xi} f_\zeta$,

(4) $\delta_\xi < \beta_{\xi+1} < \delta_{\xi+1}$,

(5) $f_{\xi+1} \in N_{\xi+1}$ is $(\mathbb{Q}, N_\xi)$-generic,

(6) $f' = \bigcup_{\xi<\alpha} f_\xi$ is a condition.

Then $f' \geq f$ is $(\mathbb{Q}, N_\xi)$-generic for every $\xi < \alpha$.

Let us show that $\mathbb{Q}$ is $\mathbb{D}$-complete for some simple 2-completeness system. Let θ be large enough regular and let $\mathbb{D}$ be a function whose domain consists of those pairs (N, f) where $N \prec (H(\theta), \in)$ is countable, $\mathbb{Q} \in N$ and $f \in \mathbb{Q} \cap N$. Now suppose that $(N, f) \in \text{dom}(\mathbb{D})$. Set

$$\mathbb{D}(N, f) = \{A_x : x \text{ is a finitary relation on } N\},$$

where for any such x,

$$A_x = \{\mathbf{G} \in \mathrm{Gen}(N, \mathbb{Q}, f) : N \cup \mathscr{P}(N) \models \Psi(x, \mathbf{G}, N, \mathbb{Q}, f)\}$$

and the formula Ψ says that "if $x = (y, k)$, where y is an ω-sequence cofinal in $\delta = N \cap \delta$ and $k < 2n_\delta + 1, k \notin u_\delta$, then $\bigcup \mathbf{G} \restriction y$ is eventually equal to k.

Let us show that $\mathbb{D}$ is a simple 2-completeness system. Thus suppose that x_0, x_1 are given, and we have to show that $A_{x_0} \cap A_{x_1}$ is non-empty. The only non-trivial case is when x_0, x_1 satisfy the hypotheses of the formula Ψ. Thus suppose that $x_0 = (y_0, k_0)$ and $x_1 = (y_1, k_1)$, where y_0, y_1 are ω-sequences cofinal in δ and $k_0, k_1 < 2n_\delta + 1, k_0, k_1 \notin u_\delta$. If $y_0 \cap y_1$ is cofinal in δ, then we must have $k_0 = k_1$ and so by the previous arguments we can find an increasing sequence $\langle f_n : n < \omega \rangle$ of extensions of f in N such that $f^* = \bigcup_{n<\omega} f_n$ gives rise to a filter $\mathbf{G} \in \mathrm{Gen}(N, \mathbb{Q}, f)$ such that $f^* \restriction (y_0 \cup y_1)$ is eventually equal to k_0 and we are done. Otherwise, $y_0 \cap y_1$ is bounded in δ, so for some $\eta < \delta, y_0 \cap y_1 \subseteq \eta$. By enlarging η we may also assume that $\mathrm{dom}(f) \subseteq \eta$. Again, by similar arguments as above, it is not difficult to build an increasing sequence $\langle f_n : n < \omega \rangle$ of extensions of f in N such that $f^* = \bigcup_{n<\omega} f_n$ gives rise to a filter $\mathbf{G} \in \mathrm{Gen}(N, \mathbb{Q}, f)$ such that $f^* \restriction (y_0 \setminus \eta)$ is eventually equal to k_0 and $f^* \restriction (y_1 \setminus \eta)$ is eventually equal to k_1. Then $\mathbf{G} \in A_{x_0} \cap A_{x_1}$ and we are done.

It is now clear that $\mathbb{Q}$ is $\mathbb{D}$-complete, which completes the proof. $\square$

Definition 5.3. Assume $\bar{C} = \langle C_\delta : \delta < \omega_1,\ \delta \text{ limit } \rangle$, where C_δ is an unbounded subset of the limit ordinal δ (think of the case C_δ of order type $< \delta$ but not necessarily). Let

$$\begin{aligned} \mathbb{Q}_{\bar{C}} = \{c : \quad & \text{for some } \alpha < \omega_1, c \text{ is a closed subset of } \alpha \\ & \text{and for every limit ordinal } \delta \leq \alpha \text{ we have} \\ & \delta = \sup(c \cap \delta) \Rightarrow c \cap C_\delta \text{ is bounded in } \delta\}. \end{aligned}$$

Order $\mathbb{Q}_{\bar{C}}$ by

$$c_1 < c_2 \iff c_1 \text{ is an initial segment of } c_2.$$

Remark 5.4. (1) For more information about the above forcing notion see [6, Ch.XVIII, §1, 1.9]. Note that $\mathbb{Q}_{\bar{C}}$ may be non-ω-proper.

(2) Note that $c \in \mathbb{Q}_{\bar{C}}$ is a closed subset of α, but not necessarily a closed subset of ω_1.

In general the forcing notion $\mathbb{Q}_{\bar{C}}$ might be trivial, say for example when for every limit ordinal δ, $C_\delta = \delta$. We are interested in the cases that this forcing notion is non-trivial, and we first deal with the simple case of $\text{otp}(C_\delta) = \omega$.

Lemma 5.5. *Assume $\mathfrak{p}$ is a simple reasonable parameter, $\bar{C}$ is as in Definition 5.3 and $\bigwedge_\delta \text{otp}(C_\delta) = \omega$. Let $f \in \mathscr{F}^{\mathfrak{p}}$ be defined as $f(0) = 0$ and $f(\beta) = 1 + \beta$ for $\beta > 0$. Then $\mathbb{Q}_{\bar{C}}$ is $(\mathfrak{p}, f)$-proper.*

In Lemma 5.6 we prove a stronger result, which includes the above lemma as a very special case. Note that if $\bigwedge_\delta \text{otp}(C_\delta) < \delta$, then we can split the analysis by restricting ourselves to $\{N : N \cap \omega_1 \in S_\gamma\}$, where $\gamma < \omega_1$ is such that $S_\gamma = \{\delta : \text{otp}(C_\delta) = \gamma\}$ is stationary.

Lemma 5.6. (1) *Assume*

(a) *$\mathfrak{p}$ is a simple reasonable parameter,*

(b) *$f \in \mathscr{F}^{\mathfrak{p}}$ and $\mathfrak{p}$ is non-$\eth_f$-loser,*

(c) *$\gamma(*) < \omega_1$,*

(d) *$\bar{C} = \langle C_\delta : \delta < \omega_1 \rangle, C_\delta \subseteq \delta = \sup(C_\delta)$.*

(e) *$\text{otp}(C_\delta) \leq \omega^{\gamma(*)}$ for every δ,*

Define $g \in \mathscr{F}^{\mathfrak{p}}$ by recursion as

- *$g(0) = 0$,*

- $g(1) = f(1) + \gamma(*)$,
- $g(\alpha + 1) = f(g(\alpha)) + \gamma(*) + 1$, *for* $\alpha > 0$,
- *for limit ordinals* α, $g(\alpha) = \sup\limits_{\beta<\alpha} g(\beta)$.

Then for every $\alpha < \ell g(\mathfrak{p})$, *the forcing notion* $\mathbb{Q} = \mathbb{Q}_{\bar{C}}$ *is* $(\mathfrak{p}, \alpha, g(\alpha))$*-proper (version 1).*

(2) *In part (1), we can get "version 2" of* $(\mathfrak{p}, \alpha, g(\alpha))$*-properness when the following is satisfied: if* $g(\delta) = \delta, N \in \mathscr{E}^{\mathfrak{p}}_{\alpha}, C \subseteq \omega_1 \cap N = \sup(C), \mathrm{otp}(C) \le \omega^{\gamma(*)}$ *and* $Y \in D^{\mathfrak{p}}_{\alpha}(N)$, *then* $Y' = \{M \in Y : M \cap \omega_1 \notin C\} \in D^{\mathfrak{p}}_{\alpha}(N)$.

(3) *If we weaken clause* (b) *to* $(b)_{f,g}$, *where*

$(b)_{f,g} : f \in \mathscr{F}^{\mathfrak{p}}$, $f(f(\alpha)) = f(\alpha)$ *for every* $\alpha < \ell g(\mathfrak{p})$ *and* $\mathfrak{p}$ *is a non-*$\eth'_f$*-loser,*

then for $\alpha < \ell g(\mathfrak{p})$, *the forcing notion* $\mathbb{Q}_{\bar{C}}$ *is* $(\mathfrak{p}, \alpha, f(\gamma(*) + \alpha))$*-proper.*

Proof. We only prove clause (2). Note that version 2 is harder to prove, and using the extra freedom, we can avoid the need for the extra assumption from (2) to prove (1). First observe that

$(*)$: for each $\alpha < \omega_1$ the set $\mathscr{I}^*_\alpha = \{p \in \mathbb{Q}_{\bar{C}} :$ there is $\beta \in p$ which is $\ge \alpha\}$ is an open dense subset of $\mathbb{Q}_{\bar{C}}$.

We prove (2) by induction on α. Let $\beta = g(\alpha)$. Let $N \in \mathscr{E}^{\mathfrak{p}}_{\beta}$ be countable with $\mathbb{Q}_{\bar{C}}, \alpha, \beta, f, g \in N$, $p \in N \cap \mathbb{Q}_{\bar{C}}$, and suppose $Y \in D^{\mathfrak{p}}_{\beta}(N)$ is given. Let $\delta = \delta_N = N \cap \omega_1$.

<u>Case 1: $\alpha = 0$.</u> In this case, we are reduced to show that $\mathbb{Q}_{\bar{C}}$ is proper. Let $\langle \mathscr{I}_n : n < \omega \rangle$ list the dense open subsets of $\mathbb{Q}_{\bar{C}}$ that belong to N. We shall choose by induction on $n < \omega$, a condition p_n such that:

(i) $p_0 = p$,

(ii) for each n, $p_n \in N$,

(iii) $p_n \le p_{n+1} \in \mathscr{I}_n$,

(iv) the set $p_{n+1} \cup \{\sup p_{n+1}\} \backslash (p_n \cup \{\sup p_n\})$ is disjoint from C_δ.

Set $p_0 = p$. Now assume p_n has been chosen and we shall choose p_{n+1} as requested. Let $F_n \in N$ be a function with domain $\mathbb{Q}_{\bar{C}}$ such that for all $q \in \mathbb{Q}_{\bar{C}}$, $q \leq F_n(q) \in \mathscr{I}_n$.

For $\alpha < \omega_1$ let $q^{[\alpha]} = q \cup \{\sup(q)\} \cup \{\sup(q) + 1 + \alpha\}$, so clearly $q \leq q^{[\alpha]} \in \mathbb{Q}_{\bar{C}}$ and the function $(q, \alpha) \mapsto q^{[\alpha]}$ belongs to N. Define a function $H : \omega_1 \to \omega_1$ by $H(\alpha) = \sup(F_n(p_n^{[\alpha]}))$. Clearly it is well defined and belongs to N. Let

$$C = \{\beta < \omega_1 : \beta \text{ a limit ordinal}, \omega\beta = \beta, (\forall \alpha < \beta)(H(\alpha) < \beta) \text{ and } \sup(p_n) < \beta\}.$$

It is easily seen that C is a club of ω_1 which belongs to N and $\gamma(*) \in N$, hence we can find $\beta^* \in C$ such that $\mathrm{otp}(\beta^* \cap C)$ is divisible by $\omega^{\gamma(*)}$. But $\mathrm{otp}(C_\delta \cap \beta^*) < \omega^{\gamma(*)}$, hence for some $\beta \in C$ we have $\sup(C_\delta \cap \beta) < \beta$. Let $p_{n+1} = F_n(p^{[\sup(C_\delta \cap \beta)+1]})$.

Set $q = \bigcup_{n<\omega} p_n$. Then for each $\alpha < N \cap \omega_1, \mathscr{I}^*_\alpha \in \{\mathscr{I}_n : n < \omega\}$, hence

$$\exists \beta \, (\beta \in q \text{ and } \alpha \leq \beta < \omega_1) \, .$$

It follows that q is $(N, \mathbb{Q}_{\bar{C}})$-generic.

Case 2: $\alpha = 1$. Set $Y' = \{M \in Y : M \cap \omega_1 \notin C_\delta\}$. It is clear that $g(\delta) = \delta$ (as $g \in N$ and $\delta = N \cap \omega_1$), hence by the hypotheses in clause (2) we have $Y' \in D^{\mathfrak{p}}_{g(1)}(N)$. Let $\langle \mathscr{I}_n : n < \omega \rangle$ list the dense open subsets of $\mathbb{Q}_{\bar{C}}$ which belong to N and let $\delta = \lim_{n<\omega} \alpha_n$ where $\langle \alpha_n : n < \omega \rangle$ is increasing. We now simulate a strategy for the challenger in the game $\eth_{\alpha,\beta}(N)$, where in the n-th move, we let the challenger to choose $Z_n = \emptyset$ (so the chooser has to use $Y_n = \emptyset$ as well) and at the end of the n-th move, the challenger also chooses $p_{n+1} \in \mathbb{Q}_{\bar{C}} \cap N$ such that:

- $p_0 = p$,
- $p_n \leq p_{n+1} \in \mathscr{I}_n$,
- p_{n+1} is $(M_n, \mathbb{Q}_{\bar{C}})$-generic and $\sup(p_{n+1}) > \alpha_n$,
- the set $(p_{n+1} \cup \{\sup p_{n+1}\}) \backslash (p_n \cup \{\sup p_n\})$ is disjoint to C_δ.

This is possible by Case 1 and its proof, because $M_n \cap \omega_1 \notin C_\delta$ which holds as $M_n \in Y'$. As this is a legal strategy for the challenger, it cannot be a winning strategy, hence for some such play the chooser wins, hence $\{M_n : n < \omega\} \in D_1^{\mathfrak{p}}(N)$. Now $q = \bigcup_{n<\omega} p_n$ is well defined, and $\sup(q) = \delta$ and $q \cap C_\delta \subseteq p \cup \{\sup(p)\}$ and $q \Vdash_{\mathbb{Q}_{\bar{C}}}$ "$\{M_n : n < \omega\} \subseteq \mathscr{M}[\underset{\sim}{\mathbf{G}}_{\mathbb{Q}_{\bar{C}}}, N]$", so q is as required as the chooser has won the play.

Case 3: $\alpha > 1, \alpha$ successor. The proof is similar to the Case 2, only we use the induction hypothesis instead of using Case 1.

Case 4: α a limit ordinal. The proof is again similar to the Case 2.

This completes the induction hypothesis and hence the proof of clause (2) of the lemma. $\square$

Definition 5.7. Suppose $S \subseteq \omega_1$ is stationary, $\mathscr{D}_{\omega_1}$ is the club filter on ω_1 and $f \in {}^{\omega_1}\omega_1$.

(1) We say f is a $(\mathscr{D}_{\omega_1} + S, \gamma)$ function, when $S \Vdash_{(\mathscr{D}^+_{\omega_1}, \supseteq)}$ "in $\mathbf{V}[\underset{\sim}{\mathbf{G}}], \{x \in \mathbf{V}^{\omega_1}/\underset{\sim}{\mathbf{G}} : \mathbf{V}^{\omega_1}/\underset{\sim}{\mathbf{G}} \models$" x is an ordinal $< f/\underset{\sim}{\mathbf{G}}\}$ has order type γ".

(2) Assume $\bar{C} = \langle C_\delta : \delta < \omega_1 \rangle$, where C_δ is an unbounded subset of δ. We define, by induction on γ, when "$\bar{C}$ obeys f on S" for $f \in {}^{\omega_1}\omega_1$ which is a $(\mathscr{D}_{\omega_1} + S, \gamma)$ function:

- if $\gamma < \omega_1$, this means

$$\{\delta \in S : \mathrm{otp}(C_\delta) \leq \omega^{1+f(\delta)}\} = S \mod \mathscr{D}_{\omega_1}.$$

- if $\gamma \geq \omega_1$, it means that for some $g : \omega_1 \to \omega_1$ and pressing down function h on S, for every $\zeta < \omega_1$ for which $h^{-1}\{\zeta\}$ is stationary, for some $\beta < \gamma$ and f_β, a $(\mathscr{D} + h^{-1}\{\zeta\}, \beta)$ function, we have $\langle C_{g(\delta)} \cap \delta : \delta \in h^{-1}\{\zeta\} \rangle$ obeys f_β.

The next lemma can be proved as in Lemma 5.6.

Lemma 5.8. *Assume*

(a) $\mathfrak{p}$ *is a simple reasonable parameter such that* $\ell g(\mathfrak{p})$ *is of uncountable cofinality,* (*b*) *$S \in \mathscr{D}^+_{\omega_1}$ and $N \in \bigcup_{\alpha} \mathscr{E}^{\mathfrak{p}}_\alpha \Rightarrow N \cap \omega_1 \in S$,*

(*c*) *$\mathfrak{p}$ is a non-$\eth_{\alpha,\alpha}$-loser (or just non $\eth'_{\alpha,\alpha}$-loser) for all $\alpha \in C^*$, where C^* is a club of $\ell g(\mathfrak{p})$ with $0 < \min(C^*)$,*

(*d*) *$\bar{C}$ obeys f on S which is a $(\mathscr{D}_{\omega_1} + S, \gamma)$-function,*

(*e*) *for all $\alpha, g(\alpha) = \min(C^* \backslash \alpha)$.*

Then $\mathbb{Q}_{\bar{C}}$ is $(\mathfrak{p}, g)$-proper.

Let us now give another example which fits into our general framework (see also [6, Ch.XVIII]). Recall that a filter $\mathscr{D}$ on a countable set is called a P-filter if it contains all co-finite sets and if $A_n \in \mathscr{D}$ for $n < \omega$, then for some $A \in \mathscr{D}$ and all $n < \omega$ we have $|A \backslash A_n| < \aleph_0$

Definition 5.9. (1) We say $\bar{\mathscr{D}} = \langle \mathscr{D}_\delta : \delta < \omega_1,\ \delta \text{ limit} \rangle$ is an ω_1-filter-sequence if:

(*a*) $\mathscr{D}_\delta$ is a filter on δ, containing the co-bounded subsets of δ,

(*b*) $\mathscr{D}_\delta$ is a P-filter and some $C_\delta \in \mathscr{D}_\delta$ has order type ω,

(*c*) for every club $C \subseteq \omega_1$ and $\alpha < \omega_1$, the set $A^\alpha_C[\bar{\mathscr{D}}]$ is stationary, where, by induction on α, we define $A^\alpha_C[\bar{\mathscr{D}}]$ by $A^\alpha_C[\bar{\mathscr{D}}] = \{\delta < \omega_1 : \delta$ is a limit ordinal, $\delta \in C$ and for every $\beta < \alpha$ we have $\delta = \sup(\delta \cap A^\beta_C[\bar{\mathscr{D}}])$, moreover $\delta \cap A^\beta_C[\bar{\mathscr{D}}] \in \mathscr{D}_\delta\}$.

(2) A reasonable parameter $\mathfrak{p}$ obeys $\bar{\mathscr{D}}$, if for each $\alpha < \ell g(\mathfrak{p})$ and $N \in \mathscr{E}^{\mathfrak{p}}_\alpha$, $\bar{\mathscr{D}} \in N$ and we have

$$D^{\mathfrak{p}}_\alpha(N) = \{Y: \quad Y \subseteq N \cap \bigcup_{\beta<\alpha} \mathscr{E}^{\mathfrak{p}}_\beta \text{ is } \mathfrak{p}\text{-closed and if } \alpha > 0, \text{ then there are}$$

$$\bar\beta = \langle \beta_n : n < \omega\rangle \text{ and } \bar M = \langle M_n : n < \omega\rangle \text{ satisfying:}$$

(a) $\beta_n \in N \cap \alpha$,

(b) either for all n, $\alpha = \beta_n + 1$

or $\beta_n < \beta_{n+1}, \sup_{n<\omega} \beta_n = \sup(\alpha \cap N)$,

(c) $M_n \in Y \cap \mathscr{E}^{\mathfrak{p}}_{\beta_n}, M_n \in M_{n+1}$,

(d) $\bigcup_{n<\omega} M_n = N \cap \bigcup_{\beta\in\alpha\cap N} \mathscr{H}(\chi^{\mathfrak{p}}_\beta)$,

(e) $\{M_n \cap \omega_1 : n < \omega\} \in \mathscr{D}_{N\cap\omega_1}\}$.

(3) A forcing notion $\mathbb{Q}$ is a $\bar{\mathscr{D}} - NNR^0_\kappa$-forcing if for every reasonable parameter $\mathfrak{p}$ which obeys $\bar{\mathscr{D}}$, $\mathbb{Q}$ is an NNR^0_κ-forcing over $\mathfrak{p}$ (see Definition 4.16).

(4) For a P-filter $\mathscr{D}$ on ω, we say a reasonable parameter $\mathfrak{p}$ obeys $\mathscr{D}$ if for every $N \in \mathscr{E}^{\mathfrak{p}}_\alpha$

$$D^{\mathfrak{p}}_\alpha(N) = \{Y: \quad Y \subseteq N \cap \bigcup_{\beta<\alpha} \mathscr{E}^{\mathfrak{p}}_\beta \text{ is } \mathfrak{p}\text{-closed and if } \alpha > 0, \text{ then there are}$$

$\bar\beta, \bar M$ satisfying items $(a), (b)$ and (d) of clause (2) and

(e) $\{n : M_n \in Y\} \in \mathscr{D}\}$.

(5) In parts (1) - (4) above, we may replace the word "filter" by "ultrafilter" if the $\mathscr{D}_\alpha$'s are ultrafilter.

Lemma 5.10. (1) *If $\diamondsuit_{\aleph_1}$ holds, then there is an ω_1-ultrafilter sequence.*

(2) *If $\bar{\mathscr{D}}$ is an ω_1-filter sequence and $\langle(\chi_\alpha, \mathscr{E}_\alpha) : \alpha < \omega_1\rangle$ is as in Definition 3.2, then there is a reasonable parameter $\mathfrak{p}$ of length ω_1 obeying $\bar{\mathscr{D}}$ which is a non-$\eth$-loser. Furthermore, for all $\alpha < \omega_1$, $\chi^{\mathfrak{p}}_\alpha = \chi_\alpha$ and $\mathscr{E}^{\mathfrak{p}}_\alpha = \mathscr{E}_\alpha$.*

(3) *If $\diamondsuit_{\aleph_1}$ holds, $\langle(\chi_\alpha, \mathscr{E}_\alpha) : \alpha < \omega_1\rangle$ is as above and $\mathscr{D}$ is a P-filter on ω, then some reasonable parameter $\mathfrak{p}$ of length ω_1 is P-filter like, non-$\eth_{\rm id}$-loser with $\chi^{\mathfrak{p}}_\alpha = \chi_\alpha, \mathscr{E}^{\mathfrak{p}}_\alpha = \mathscr{E}_\alpha$. Similarly for ultrafilters.*

(4) *Instead of $\diamondsuit_{\aleph_1}$ it is enough to assume CH and that for some $\langle C_\delta : \delta < \omega_1$ limit$\rangle$ and some normal filter D on ω_1, and for every club C of ω_1, $\{\delta : \delta > \sup(C_\delta \backslash C)\} \in D$.*

Proof. We only prove (1) and (2), the other parts can be proved similarly.

(1). Let $\langle S_\delta : \delta < \omega_1\rangle$ be a $\diamondsuit_{\aleph_1}$-sequence, where each $S_\delta \subseteq \delta$ and let $\langle E_\delta : \delta < \omega_1 \text{ limit}\rangle$ be a ladder system, where each $E_\delta \subseteq \delta$ has order type ω and $E_\delta = S_\delta$ if S_δ is an ω-sequence cofinal in δ. Let $\bar{\mathscr{D}} = \langle \mathscr{D}_\delta : \delta < \omega_1 \text{ limit}\rangle$, where $\mathscr{D}_\delta$ is any P-ultrafiler on δ extending $\mathscr{D}^{\mathfrak{c}}_\delta \cup \{E_\delta\}$, where $\mathscr{D}^{\mathfrak{c}}_\delta$ is the filter of co-bounded subsets of δ. We show that $\bar{\mathscr{D}}$ is as required. Items (a) and (b) of Definition 5.9(1) are clearly satisfied. Let us prove clause (3). Thus suppose that $C \subseteq \omega_1$ is a club, $\alpha < \omega_1$, and suppose by induction that for all $\beta < \alpha$, the set $A^\beta_C(\bar{\mathscr{D}})$ is stationary. We show that $A^\alpha_C(\bar{\mathscr{D}})$ is stationary as well. We may assume that C only contains limit ordinals. Then

$$A^\alpha_C(\bar{\mathscr{D}}) = C \cap \bigcap_{\beta<\alpha} \{\delta : \delta = \sup(\delta \cap A^\beta_C(\bar{\mathscr{D}}))\} \cap \{\delta : \forall \beta < \alpha,\ \delta \cap A^\beta_C(\bar{\mathscr{D}}) \in \mathscr{D}_\delta\}.$$

It follows from the induction hypothesis that the set $C \cap \bigcap_{\beta<\alpha} \{\delta : \delta = \sup(\delta \cap A^\beta_C(\bar{\mathscr{D}}))\}$ is a club. Suppose by contradiction that the set $A = \{\delta : \forall \beta < \alpha,\ \delta \cap A^\beta_C(\bar{\mathscr{D}}) \in \mathscr{D}_\delta\}$ is non-stationary. Thus for some club $D \subseteq C$ and for all $\delta \in D$, there exists some $\beta_\delta < \alpha$ such that $\delta \cap A^{\beta_\delta}_C(\bar{\mathscr{D}}) \notin \mathscr{D}_\delta$. As $\alpha < \omega_1$, it follows from Födor's lemma that there are a stationary set $S \subseteq D$ and some fixed $\beta_* < \alpha$ such that

$$\delta \in S \implies \delta \cap A^{\beta_*}_C(\bar{\mathscr{D}}) \notin \mathscr{D}_\delta$$

On the other hand, by the $\diamondsuit_{\aleph_1}$-assumption, the set

$$T = \{\delta \in S : \delta \cap A^{\beta_*}_C(\bar{\mathscr{D}}) = S_\delta\}$$

is stationary. We may assume without loss of generality that for all $\delta \in T, \delta \cap A_C^{\beta_*}(\bar{\mathscr{D}})$ has order type ω. But then

$$\delta \in T \implies \delta \cap A_C^{\beta_*}(\bar{\mathscr{D}}) = S_\delta = E_\delta \in \mathscr{D}_\delta,$$

which is a contradiction.

(2) Define $\mathfrak{p}$ of length ω_1 such that for all $\alpha < \omega_1$,

- $\chi_\alpha^{\mathfrak{p}} = \chi_\alpha$,
- $R_\alpha^{\mathfrak{p}} = [\mathscr{H}((\bigcup_{\beta<\alpha} \chi_\beta)^+)]^{\leq\aleph_0}$,
- $\mathscr{E}_\alpha^{\mathfrak{p}} = \mathscr{E}_\alpha$,
- for $N \in \mathscr{E}_\alpha$, $D_\alpha^{\mathfrak{p}}(N)$ is defined as in Definition 5.9(2).

Then $\mathfrak{p}$ is as required. □

Lemma 5.11. (1) *If $\mathscr{D}$ is a P-filter on ω (or P-ultrafilter on ω) and $\mathfrak{p}$ is a reasonable parameter obeying $\mathscr{D}$, then for some $\bar{\mathscr{D}}, \bar{\mathscr{D}}$ is an ω_1-filter-sequence (or ω_1-ultrafilter-sequence) and $\mathfrak{p}$ obeys $\bar{\mathscr{D}}$.*

(2) *If $\mathfrak{p}$ is a P-point filter (or ultrafilter), then $\mathfrak{p}$ is a non-$\eth$-loser.*

Proof. (1) For each limit ordinal $\delta < \omega_1$ fix a bijection $f_\delta : \omega \leftrightarrow \delta$ and set $\mathscr{D}_\delta = \{f''[X] : X \in \mathscr{D}\}$. Then $\bar{\mathscr{D}}$ is as required.

Proof of (2) is essentially similar to the proof of Lemma 3.7. □

We now consider the case where the order type of the club sets $C_\delta$is higher than ω.

Lemma 5.12. (1) *Assume*

(a) $\kappa \leq \omega$ *and* $\bar{C} = \langle C_{\delta,\ell} : \ell < k_\delta, \delta < \omega_1$ *limit*$\rangle$, *where* $1 + \kappa \leq k_\delta \leq \omega, C_{\delta,\ell}$ *is a closed unbounded subset of* δ *and* $\ell < m < k_\delta \Rightarrow C_{\delta,\ell} \cap C_{\delta,m} = \emptyset$,

(b) $\mathbb{Q} = \mathbb{Q}_{\bar{C}} = \{C : C$ *is a closed bounded subset of* ω_1 *such that for every limit* $\delta < \sup(C)$, *and for every* $\ell < k_\delta$ *except* $< 1 + \kappa$ *many,* $\delta \leq \sup(C \cap C_{\delta,\ell})\}$,[9]

[9]i.e., $\{\ell < k_\delta : \delta > \sup(C \cap C_{\delta,\ell})\}$ has size $< 1 + \kappa$.

(c) $\mathfrak{p}$ *is a reasonable parameter, obeying the* P*-ultrafilter* $\mathscr{D}$.

Then $\mathbb{Q}$ *is a* $\mathfrak{p}-NNR^0_\kappa$ *forcing notion.*

(2) *In part (1), if we add:*

$N \in \mathscr{E}^{\mathfrak{p}}_\alpha$ *and* $D^{\mathfrak{p}}_\alpha(N) = \{Y : \{n : M_n \in Y\} \in \mathscr{D}_N\}, \mathscr{D}_N$ *a* P*-ultrafilter and*

$\ell < k_\delta \Rightarrow \{n < \omega : M_n \cap \omega_1 \in C_{\delta,\ell}\} = \emptyset \mod \mathscr{D}_N$,

then we can allow $C_{\delta,0} = C_{\delta,1}$.

(3) *Assume*

(a) D_δ *is a family of subsets of* $\mathrm{dom}(D_\delta)$*, the intersection* Y *of any* $< 1 + \kappa$ *of them satisfies,*

(∗) $$\exists n(\exists y_1, \ldots, y_n \in Y)[\delta > \sup(\bigcap_{\ell=1}^{n} C_{\delta,y_\ell})],$$

(b) $\bar{C} = \langle C_{\delta,x} : x \in \mathrm{dom}(D_\delta)$ *and* δ *is a limit ordinal* $< \omega_1\rangle$,

(c) $\langle C_{\delta,x} : x \in \mathrm{dom}(D_\delta)\rangle$ *is a sequence of pairwise disjoint subsets of* δ,

(d) $\bar{X} \in \prod_{\delta<\omega_1} \mathrm{dom}(D_\delta)$,

(e) $\mathbb{Q}_{\bar{C},\bar{X},\bar{D}} = \{C : C$ *is a closed bounded subset of* ω_1 *such that for every limit* $\delta \le \sup(C)$ *we have* $(\exists x \in X_\delta)(\delta > \sup(C \cap C_{\delta,x}))\}$ *ordered by end extension,*

(f) $\bar{\mathscr{D}}$ *is a* P*-ultrafilter sequence,*

(g) $\mathfrak{p}$ *is a reasonable parameter which obeys* $\bar{\mathscr{D}}$.

Then $\mathbb{Q}$ *is a* $\mathfrak{p} - NNR^0_\kappa$ *forcing notion.*

Proof. We prove clause (1), as other items can be proved in a similar way. So let $\mathbf{V}_0$ be some transitive class with $\mathfrak{p} \in \mathbf{V}_0$ and suppose that $\bar{\mathbb{Q}} \in \mathbf{V}_0$ is an NNR^0_κ-iteration with $\mathbb{P}_\alpha = \lim(\bar{\mathbb{Q}})$ such that $\mathbf{V} = \mathbf{V}_0^{\mathbb{P}_\alpha}$ and $\Vdash^{\mathbf{V}_0}_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\mathbb{Q}}_\alpha = \mathbb{Q}_{\underset{\sim}{\bar{C}}}$ is as above". Set $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \underset{\sim}{\mathbb{Q}}_\alpha$. We have to show that items (a)-(d) of Definition 3.9 (for $\kappa = \aleph_0$) or 3.14 (for $\kappa < \aleph_0$) are satisfied. We only check clause (d), as other items are easier to prove. Suppose that in $\mathbf{V}_0$,

- $N_0 \in \bigcup_{\beta<\alpha} \mathscr{E}^{\mathfrak{p}}_\beta$,

- $N_0 \in N_1 \in \mathscr{E}^{\mathfrak{p}}_\alpha$,
- $N_1 = \{M_n : n < \omega\}$,
- $\mathscr{D}_{N_0}$ is a P-ultrfiler as in Definition 5.12(2),
- $\mathbf{G}_m \subseteq N_1 \cap \mathbb{P}_\alpha$ is generic over N_1 for $m < k < 1 + \kappa$,
- $\bigwedge\limits_{m<k} [\mathbf{G}_m \cap N_0 = \mathbf{G}^*]$,
- $p \in \mathbb{P}_{\alpha+1} \cap N_0$ is such that $p \restriction \alpha \in \mathbf{G}^*$,

Clearly, without loss of generality $\langle M_n : n < \omega\rangle$, $\mathscr{D}_{N_0} \in N_1$. For $\iota = 0, 1$ set $\delta_{N_\iota} = N_\iota \cap \omega_1$. So for each $\ell < k_{\delta_{N_0}}$, we have $\underset{\sim}{C}_{\delta_{N_0},\ell}[\mathbf{G}_m]$ is a closed subset of δ and for $\ell_1 < \ell_2 < k_{\delta_{N_0}}$ we have $\underset{\sim}{C}_{N_0\cap\omega_1,\ell_1}[\mathbf{G}_m] \cap \underset{\sim}{C}_{N_0\cap\omega_1,\ell_2}[\mathbf{G}_m] = \emptyset$. So for some $\ell(m) \in \{0, 1, \ldots, k_{\delta_{N_0}} - 1\}$ we have

$$\ell \neq \ell(m) \Rightarrow \underset{\sim}{C}_{N_0\cap\omega_1,\ell}[\mathbf{G}_m] = \emptyset \mod \mathscr{D}_{N_0}.$$

Now let

$$B = \{n : \text{if } \ell < k_{\delta_{N_0}}, \ell \notin \{\ell(m) : m < k\} \text{ and } \ell < n, \text{ then } M_n \cap \omega_1 \notin \underset{\sim}{C}_{\delta_{N_0},\ell}[\mathbf{G}_m] \text{ for } m < k \text{ and } p \in M_n\}.$$

Then B belongs to $N_1 \cap \mathscr{D}_{N_0}$. Let $B = \{n_i : i < \omega\}$ be an increasing enumeration of B and let $\langle \mathscr{I}_n : n < \omega\rangle$ list the dense open subsets of $\mathbb{P}_{\alpha+1}$ which belong to N_0. We choose p_i, by induction on $i < \omega$, such that:

(a) $p_i \in N_1 \cap P_{\alpha+1}$,

(b) $p_i \restriction \alpha \in \bigcap\limits_{m<k} \mathbf{G}_m$,

(c) $p_i \in \bigcap\{\mathscr{I}_n : n < n_i, \mathscr{I}_n \in N_1 \text{ and } i > 0\}$,

(d) $p \le p_i$,

(e) $p_i \le p_{i+1}$,

(f) $p_{i+1} \backslash p_i$ is disjoint to $\bigcup\{\underset{\sim}{C}_{\delta_{N_0},\ell}[\mathbf{G}_m] : \ell < k_{\delta_{N_0}}, \ell < n_i \text{ and } m < k \Rightarrow \ell \neq \ell(m)\}$.

This is possible as, for each $i < \omega$,

$$\bigcup\{\underset{\sim}{C}_{\delta_{N_0},\ell}[\mathbf{G}_m] \cap M_{n_i} \cap \omega_1 : \ell < k_{\delta_{N_0}}, \ell < n_i \text{ and } m < k \Rightarrow \ell \neq \ell(m)\}$$

is a bounded subset of $M_{n_i} \cap \omega_1$.

Set

$$\mathbf{G}^{**} = \{q \in \mathbb{P}_{\alpha+1} \cap N_0 : \exists i < \omega (p_i \leq q)\}.$$

Then $\mathbf{G}^{**} \subseteq \mathbb{P}_{\alpha+1} \cap N_0$ is generic over $N_0, p \in \mathbf{G}^{**}$ and if we set $q = \bigcup_{i<\omega} p_i$, then q witnesses that $\mathbf{G}^{**}$ has an upper bound in $\mathbb{P}_{\alpha+1}/\underset{\sim}{G}_{\mathbb{P}_\alpha}$. □

The proof of the next lemma is similar to the above proofs.

Lemma 5.13. (1) *The forcing notion $\mathbb{Q}_{\bar{C}}$ from Lemma 5.12(1) is $NNR^0_{\aleph_0}$-forcing notion for every $\mathfrak{p}$, non-$\eth_{\rm id}$-loser.*

(2) *If $\bar{\mathscr{D}}$ is an ω_1-filter sequence and $\mathfrak{p}$ is a reasonable parameter obeying $\bar{\mathscr{D}}$, then any $(< \omega_1)$-proper forcing notion is $\mathfrak{p}$-proper.*

§ 6. Second preservation of not adding reals

In this section we present our second preservation theorem. We shall concentrate on the simple case.

Definition 6.1. Let $\mathfrak{p}$ be a reasonable parameter and let $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \ell g(\bar{\mathbb{Q}}) \rangle$ be an iteration of forcing notions. We say that $\bar{\mathbb{Q}}$ is a $\mathfrak{p}$-NNR^1_κ iteration, where $2 \leq \kappa \leq \aleph_0$,[10] when for some $f_i, g_i \in \mathscr{F}^{\mathfrak{p}}_{\rm dc}$, for $i < \ell g(\bar{\mathbb{Q}})$ (see 4.4), we have:

(a) $\mathrm{cf}(\ell g(\mathfrak{p})) > \ell g(\bar{\mathbb{Q}})$,

(b) $\bar{\mathbb{Q}}$ is a countable support iteration of proper forcing notions such that for each $i < \ell g(\bar{\mathbb{Q}}), \mathbb{P}_i$ adds no reals, [11]

(c) (long properness) for each $i < \ell g(\bar{\mathbb{Q}})$, we have $\Vdash_{\mathbb{P}_i}$ "$\underset{\sim}{\mathbb{Q}}_i$ is $(\mathfrak{p}^{\mathbb{P}_i}, f_i)$-proper",

(d) (κ-anti w.d.) if $i < \ell g(\bar{\mathbb{Q}})$ and $\beta \in g_i(\alpha)$, then $\underset{\sim}{\mathbb{Q}}_i$ has (κ, α, β)-anti w.d. above $\mathbb{P}_i$.

[10] we omit $\kappa = \aleph_1$ for convenience.

[11] this follows from other parts (close (d)), even for $\mathbb{P}_{i+1}, i < \ell g(\bar{\mathbb{Q}})$.

Remark 6.2. (1) κ is the amount of "$\mathbb{D}$-completeness", in other words what versions of weak diamond we kill by our iteration. So the case $\kappa = \aleph_0$ is easier, and we first deal with it in Theorem 6.3.

(2) Note that we ask for $f_i \in \mathbf{V}$ and not a $\mathbb{P}_i$-name $\underset{\sim}{f}_i$ of such a function. The reason is that if for $i < \ell g(\bar{\mathbb{Q}}), \mathbb{P}_i$ satisfies the $\mathrm{cf}(\ell g(\mathfrak{p}))$-c.c., then we can find $f'_i \in \mathscr{F}^{\mathfrak{p}}, f'_i \geq \underset{\sim}{f}_i$. As in practice we usually have $\mathrm{cf}(\ell g(\mathfrak{p})) > |\mathbb{P}_\alpha|$, there is no point at present for f_i to be a $\mathbb{P}_i$-name.

(3) In clause (d) we have implicitly used:

(∗) if $\alpha \leq \beta' < \beta$, then clause (d) for (α, β) and κ implies clause (d) for (α, β') and κ.

This holds by clause (i) of Definition 3.2.

(4) We could replace f_i by a club E_i of $\ell g(\mathfrak{p})$, letting $f_i(\alpha) = E_i \backslash \alpha$.

(5) In clause (c), for a club C of $\ell g(\mathfrak{p})$ we catch our tail, that is $f_i(\alpha) \cap C = C \backslash \alpha$ for a club of $\alpha < \ell g(\mathfrak{p})$.

(6) In clause (d), much of the freedom/variation will be due to the decision how "similar" are $\langle G^\ell : \ell < k \rangle$ such that $\mathbf{G}^{**}$ exists. Here we demand

(α) $Y \in D^{\mathfrak{p}}_\beta(N_1)$.

In [6, Ch.VIII], it is essentially required that

(β) $\mathbf{G}^0 \times \mathbf{G}^1 \times \cdots \times \mathbf{G}^{k-1} \subseteq (\mathbb{P}_i \times \cdots \times \mathbb{P}_i) \cap N_1$ (k times) is generic over N_1.

In [6, Ch.V], it is required that

(γ) the common Y is a pre-determined increasing sequence of models.

Clause (β) makes demand (d) in Definition 6.1 easier, but the parallel of (c) is harder compared to clause (α).

We now state and prove the main results of this section. We first deal with the $\mathfrak{p}$-$\mathrm{NNR}^1_{\aleph_0}$ iterations.

Theorem 6.3. *Assume $\bar{\mathbb{Q}}$ is a $\mathfrak{p}$-$NNR^1_{\aleph_0}$ iteration, $\mathfrak{p}$ is a reasonable parameter and $\mathfrak{p}$ is a $\eth_f$-winner for some $f \in \mathscr{F}^{\mathfrak{p}}_{\text{club}}$ (or at least is $\eth'_f$-non loser).*

(1) *Forcing with $\mathbb{P}_{\ell g(\bar{\mathbb{Q}})} = \text{Lim}(\bar{\mathbb{Q}})$ does not add reals (so consequently adds no ω-sequences, as we are assuming properness).*

(2) *If $i \leq j \leq \ell g(\bar{\mathbb{Q}})$, then*

$(b)'$ *$\mathbb{P}_j/\mathbb{P}_i$ is proper,*

$(c)'$ *$\mathbb{P}_j/\mathbb{P}_i$ is $(\mathfrak{p}, f_{i,j})$-proper, where $f_{i,j} \in \mathscr{F}^{\mathfrak{p}}_{\text{club}}$ is increasing continuous and is computable from the $f_\varepsilon \in \mathscr{F}^{\mathfrak{p}}$ for $\varepsilon \in [i, j)$,*

$(d)'$ *we have the parallel of clause (d) in the following sense: if $i < j < \ell g(\bar{\mathbb{Q}})$, then for some function $g \in \mathscr{F}^{\mathfrak{p}}_{cd}$ in $\mathbf{V}$ and for all $\alpha < \ell g(\mathfrak{p})$ and $\beta \in g(\alpha)$ we have $\mathbb{P}_j/\mathbb{P}_i$ is $(\aleph_0, \alpha, \beta)$-anti-w.d above $\mathbb{P}_i$.*

Proof. The proof is by induction on $\ell g(\bar{\mathbb{Q}})$. For notational simplicity we assume that:

$\boxtimes$: all f_i's are also in $\mathscr{F}^{\mathfrak{p}}_{\text{nd}}$, so we can consider them as increasing and continuous functions from $\ell g(\mathfrak{p})$ to $\ell g(\mathfrak{p})$. We also demand that the $f_{i,j}$'s are also like that, are increasing continuous and moreover $f_{i,j}(f_{i,j}(\alpha)) = f_{i,j}(\alpha)$, and they are $\geq f^*$ where $f^* \in \mathscr{F}^{\mathfrak{p}}_{\text{nd}}$ is increasing continuous and $\mathfrak{p}$ is $\eth_{f^*}$-winner (or at least $\eth'_{f^*}$-non-loser).

Case 1: $\ell g(\bar{\mathbb{Q}}) = 0$. This is trivial.

Case 2: $\ell g(\bar{\mathbb{Q}}) = i(*) + 1$ is a successor ordinal. We show that items (1) and (2) are satisfied.

Clause (1): $\mathbb{P}_{i(*)}$ adds no reals by the induction hypothesis and $\Vdash_{\mathbb{P}_{i(*)}}$ "$\underset{\sim}{\mathbb{Q}}_{i(*)}$ adds no reals", by clause (d) in Definition 6.1, hence $\mathbb{P}_{i(*)+1} = \mathbb{P}_{i(*)} * \underset{\sim}{\mathbb{Q}}_{i(*)}$ adds no reals.

Clause (2): We have to show that items $(b)', (c)'$ and $(d)'$ are satisfied.

Clause $(b)'$: By [6, Ch. III], $\mathbb{P}_j/\mathbb{P}_i$ is proper.

Clause $(c)'$: Given $i \leq j \leq \ell g(\bar{Q})$, if $j < i(*) + 1$ the conclusion follows by the induction hypothesis. So assume $j = i(*) + 1$. If $i = j$, the required demand is trivial, so assume $i < j$. If $i = i(*)$, use clause (c) of Definition 6.1 for i to get the conclusion. So assume that $i < i(*)$. Let

- $f_{i,j,0} = f_{i(*)}$,
- $f_{i,j,m+1} = f_{i(*)} \circ f_{i,i(*)} \circ f_{i,j,m}$,
- $f_{i,j}(\alpha) = \sup\limits_{m<\omega} f_{i,j,m}(\alpha)$.

Then the $f_{i,j}$'s are as required in $\boxtimes$. To prove "$\mathbb{P}_j/\mathbb{P}_i$ is $(\mathfrak{p}, f_{i,j})$-proper", assume that

$(*)_1$ (a) $N \prec (\mathscr{H}(\chi), \in)$ is countable,

(b) $\{\bar{\mathbb{Q}}, i, j, \alpha, \beta, f_{i,i(*)}, f_{i(*)}, f_{i,j}\} \in N$,

(c) $\alpha \leq f_{i,j}(\alpha) \leq \beta < \ell g(\mathfrak{p})$,

(d) $q \in \mathbb{P}_i$ is $(N, \mathbb{P}_i)$-generic,

(e) $p \in N \cap \mathbb{P}_j, p \restriction i \leq q$,

(f) $Y \in D^{\mathfrak{p}}_{\beta}(N)$,

(g) $q \Vdash$ "$Y \subseteq \mathscr{M}_{\mathbb{P}_i}[\underset{\sim}{\mathbf{G}}_{\mathbb{P}_i}, N]$".

First we deal with version 2, and assume that $\mathfrak{p}$ is simple. Choose $y^* \in N$ which codes enough information. Clearly $\beta' = f_{i(*)}(\alpha)$ belongs to N. So $f_{i,i(*)}(\beta') \leq \beta$, hence by the induction hypothesis there are q', Y' such that:

- $q \leq q' \in \mathbb{P}_{i(*)}$,
- $p \restriction i(*) \leq q'$,
- q' is $(N, P_{i(*)})$-generic,
- $Y' \subseteq Y, Y' \in D^{\mathfrak{p}}_{\beta'}(N)$,
- $q' \Vdash$ "$Y' \subseteq \mathscr{M}_{\mathbb{P}_{i(*)}}[\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{i(*)}}, N, y^*]$".

Next, we apply clause (c) in the Definition 6.1 for $i(*)$, so there are q'', Y'' such that

- $q' \leq q'' \in \mathbb{P}_{i(*)+1} = \mathbb{P}_j$,
- $p \leq q''$,
- q'' is $(N, \mathbb{P}_j)$-generic,
- $Y'' \subseteq Y', Y'' \in D^{\mathfrak{p}}_\alpha(N)$,
- $q'' \Vdash$ "$Y'' \subseteq \mathscr{M}[\underset{\sim}{\mathbf{G}}_{\mathbb{P}_j}, N, y^*]$".

The result follows immediately. The proof for version 1 is similar.

Clause $(d)'$: Recall that we have demanded $f_{i,j}(f_{i,j}(\alpha)) = f_{i,j}(\alpha)$ (see ⊠ at the beginning of the proof).

Let $N_0, N_1, \alpha, \beta, i, j, p, k, q_\ell$ (for $\ell < k$), $\mathbf{G}^\ell$ (for $\ell < k$) and $\mathbf{G}^*$ be as in the assumptions of Definition 6.1(d) (see Definition 4.12).

Without loss of generality $i < i(*) < j = i(*) + 1$, since the other cases are trivial as in the proof of clause $(c)'$. First choose $\mathbf{G}^{**} \in N_1$ for $\mathbf{G}^*, \alpha, \beta, i, i(*)$. For each $\ell < k$, if for some $s_\ell \in \mathbf{G}^\ell$ we have

$(*)_2$ $\qquad s_\ell \Vdash_{\mathbb{P}_i}$ " there is an upper bound for $\mathbf{G}^{**}$ in $\mathbb{P}_{i(*)}/\underset{\sim}{\mathbf{G}}_{\mathbb{P}_i}$",

then, as $\mathbf{G}^\ell$ is generic over N_0, by increasing s_ℓ if necessary, there are $s_\ell \in \mathbf{G}^\ell$ and $r_\ell \in P_{i(*)} \cap N_1$ such that s_ℓ forces that r_ℓ is an upper bound for $\mathbf{G}^{**}$, and without loss of generality $r_\ell \restriction i \leq s_\ell$. Now without loss of generality

$$\mathbf{G}^{\ell_1} = \mathbf{G}^{\ell_2} \Rightarrow s_{\ell_1} = s_{\ell_2}$$

and

$$\mathbf{G}_{\ell_1} \neq \mathbf{G}_{\ell_2} \Rightarrow s_{\ell_1}, s_{\ell_2} \text{ are incompatible.}[12]$$

Now choose $r \in \mathbb{P}_{i(*)} \cap N_1$ with domain $\subseteq i(*) \backslash i$ as follows:

- $\mathrm{dom}(r) = \bigcup\limits_{\ell<k} \mathrm{dom}(r_\ell) \backslash i$,
- $r(\alpha) = r_\ell(\alpha)$ if $s_\ell \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}_i}, \ell < k$,
- $r(\alpha) = \emptyset_{\mathbb{P}_\alpha}$ if this occurs for no ℓ.

[12]see Remark 4.13.

Renaming $r \in N_i \cap \mathbb{P}_{i(*)}, \mathrm{dom}(r) \subseteq i(*)\backslash i$ and $s_\ell \in \mathbf{G}^\ell, r_\ell = s_\ell \cup r$ is above $\mathbf{G}^{**}$ in $\mathbb{P}_{i(*)}$. Let $\beta_\ell = f_{i,j,1+\ell}(\alpha)$ for $\ell \leq k$.

We choose, by induction on $\ell \leq k$, the objects Y_ℓ, q'_ℓ, M_ℓ such that:

$(*)_3$ (a) $Y_0 = Y$,

(b) $M_0 = N_1$,

(c) $N_0 \in M_{\ell+1}$,

(d) $M_{\ell+1} \in M_\ell \cap \mathscr{E}^{\mathfrak{p}}_{\beta_{k-\ell}}$,

(e) $Y_{\ell+1} \subseteq Y_\ell$,

(f) $Y_\ell \in D^{\mathfrak{p}}_{\beta_\ell}(M_\ell)$,

(g) $M_{\ell+1} \in Y_\ell$,

(h) $q_\ell \leq q'_\ell \in P_{i(*)}$,

(i) q'_ℓ is $(M_{\ell+1}, \mathbb{P}_{i(*)})$-generic,

(j) q'_ℓ forces a value for $\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{i(*)}} \cap M_{\ell+1}$,

(k) q'_ℓ is $(N_0, \mathbb{P}_{i(*)})$-generic,

(l) $q'_\ell \Vdash$ "$Y_{\ell+1} \subseteq \mathscr{M}_{\mathbb{P}_{i(*)}}[\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{i(*)}}, M_\ell]$",

(m) $q'_\ell \restriction i \in G^\ell$.

Now apply clause (d) of the definition for $i(*), N_0, M_k, \langle q'_\ell : \ell < k\rangle, Y_k$ and $\mathbf{G}^{**}$ and get $\mathbf{G}^{***}$ as required.

Case 3: $\delta = \ell g(\bar{\mathbb{Q}})$ is a limit ordinal. We show that items (1) and (2) are satisfied in this case as well.

Clause (1): This follows from clause $(2)(d)'$ proved below.

Clause (2): Again, we have to check items $(b)'$, $(c)'$ and $(d)'$. Let $f_{i,j}$ be fast enough functions.

Clause $(b)'$: This is obvious.

Clause $(d)'$: We first prove clause $(d)'$ and later prove clause $(c)'$. As before, without loss of generality $i < j = \delta$. Let $N_0, N_1, p, \mathbf{G}^*, \alpha, \beta, k < \aleph_0$ and $\mathbf{G}^\ell, q_\ell$ for $\ell < k$ be as in the assumptions of clause (d) of Definition 6.1.

Choose $\gamma \in N_0, \alpha < \gamma < \beta$ such that γ is large enough, in particular,

$$i \leq i' < j' < j \Rightarrow f_{i',j'}(\gamma) = \gamma.$$

Let $\langle i_m : m < \omega \rangle \in N_1$ be such that

$(*)_4$ (a) $i_0 = i$,

(b) $i_m < i_{m+1}$,

(c) $\sup\limits_{m<\omega} i_m = \sup(j \cap N_0)$.

Choose $y^* \in N_1 \cap \mathscr{H}(\chi_\gamma)$ coding enough information. We choose

$$M_0, M_1, M_2, M_3, M_4 \in N_1 \cap \mathscr{E}^{\mathfrak{p}}_\gamma \cap Y$$

such that

$(*)_5$ (a) $N_0 \in M_0 \in M_1 \in M_2 \in M_3 \in M_4$,

(b) $Y \cap M_m \in D^{\mathfrak{p}}_\gamma(M_m)$ for $m < 5$.

Choose $q'_\ell \in \mathbf{G}^\ell \cap M_4$ above $\mathbf{G}^\ell \cap M_3$ so that q'_ℓ is $(M_t, \mathbb{P}_i)$-generic for $t < 4$. Let $\langle \mathscr{I}_m : m < \omega \rangle \in M_0$ list the dense open subsets of $\mathbb{P}_j$ from N_0. Now we shall use the diagonal argument and choose $\underset{\sim}{\mathbf{G}}_{\mathbb{P}} \cap N_0, p_m \in \mathbb{P}_{i_m} \cap N_0, r_m \in \mathbb{P}_{i_m}$. We fulfill the above in M_4, so that at the end can find a solution in N_1, by using a canonical construction.

But to carry this, we need to have finitely many candidates for $\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{i_m}} \cap M_0$ with a common Y_m. To get this in the inductive step, we need in step $m-1$ that for M_1 we just have finitely many candidates for $\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{i_m}} \cap M_1$, and in turn to get this in the step $m-1$, we use that in step $m-2$ for M_2 and from every maximal antichain we choose a finite subset. To get this we use that for M_3

we just ask $M_3[\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{i_{m-3}}}] \cap \mathbf{V} = M_3$. So along the way N_0, M_0, M_1, M_2, M_3 our induction demands go down, but slowly, so that in each step m, advancing for say M_0, we have to preserve less than really knowing $\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{i_m}} \cap M_0$, and are helped by our demand on M_1, just like in [6, Ch. XVIII]. So compared to [6, Ch. V], we have a finite tower.

Thus we choose by induction on $m < \omega$ the objects $r_m, \mathbf{G}^*_m, p_m, n_m$, $\langle \mathbf{G}^\ell_m : \ell < n_m \rangle$ and Y_m such that:

$(*)_6$ (a) $r_m \in \mathbb{P}_{i_m} \cap M_4$,

(b) $\mathrm{dom}(r_m) \subseteq [i, i_m)$,

(c) $r_{m+1} \restriction i_m = r_m$,

(d) $q'_\ell \cup r_m \in \mathbb{P}_{i_m}$ is $(M_t, \mathbb{P}_{i_m})$-generic for $t < 4$,

(e) if $\ell < k$, $\mathscr{J} \subseteq \mathbb{P}_{i_m}$ is dense open and $\mathscr{J} \in M_2$, then for some finite $\mathscr{J} \subseteq \mathscr{I} \cap M_2$, $\mathscr{J}$ is predense above $q'_\ell \cup r_m$,

(f) $n_m < \omega$, and for $\ell < n_m$, $\mathbf{G}^\ell_m$ is a subset of $\mathbb{P}_{i_m} \cap M_0$ generic over M_0 and $\mathbf{G}^\ell_m \in M_1$,

(g) $\mathbf{G}^\ell_{m+1} \cap \mathbb{P}_{i_m} \in \{\mathbf{G}^\ell_m : \ell < n_m\}$,

(h) $n_0 = k$ and $\mathbf{G}^\ell_0 = \mathbf{G}^\ell \cap M_1$,

(i) $q_\ell \cup r_m \Vdash$ "$\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{i,m}} \cap M_1 \in \{\mathbf{G}^\ell_m : \ell < n_m\}$",

(j) $\mathbf{G}^*_m$ is a subset of $\mathbb{P}_{i_m} \cap N_0$ generic over N_0,

(k) $\mathbf{G}^*_m \subseteq \mathbf{G}^\ell_m$, so that $\mathbf{G}^*_m \subseteq \mathbf{G}^*_{m+1}$ and $\mathbf{G}^*_0 = \mathbf{G}^*$,

(l) $p_m \in \mathbb{P}_j \cap N_1$,

(m) $p_0 = p$,

(n) $p_m \restriction i_m \in \mathbf{G}^*_m$,

(o) $p_m \leq p_{m+1} \in \mathscr{I}_m$,

(p) $Y_m \subseteq \mathscr{M}_{\mathbb{P}_{i_m}}[\mathbf{G}^\ell_m, M_0, y^*]$,

(q) $Y_m \in D^{\mathfrak{p}}_\gamma(M_0)$.

Let us now explain the induction construction. If $m = 0$, this is trivial, so suppose that it holds for m and we do it for $m + 1$. This is done in several stages.

Stage A: Choosing p_{m+1} is trivial, the demands are: $p_{m+1} \geq p_m, p_{m+1} \restriction i_m \in \mathbf{G}^*_m$ and $p_{m+1} \in \mathscr{I}_m$.

Stage B: To choose $\mathbf{G}^*_{m+1}$, apply the induction hypothesis using clause $(d)'$ of what we are proving with $i_m, i_{m+1}, \gamma, f_{i_m,i_{m+1}}(\gamma), N_0, M_1$ here standing for $i, j, \alpha, \beta, N_0, N_1$ there.

Stage C: Let $\{H^\ell_m : \ell < n_{m+1}\}$ list the possibilities of $\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{i_m}} \cap M_1$ (by clause (e) this exists). Without loss of generality $H^\ell_m \cap M_0 = G^{h(\ell)}_m$, for some function $h = h_m : n_{m+1} \to n_m$. We choose $s^\ell_m \in \mathbb{P}_{i_{m+1}} \cap M_1$ above $\mathbf{G}^*_{m+1}$, such that $s^\ell_m \restriction i_m \in \mathbf{G}^{h(\ell)}_m$. Now we repeat the argument of the successor stage of shrinking Y, so we can find t^ℓ_m such that

- $t^\ell_m \in \mathbb{P}_{i_{m+1}} \cap M_1$, above s^ℓ_m,
- $t^\ell_m \restriction i_m \in H^\ell_m$,
- $t^\ell_n \Vdash$ "$\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{i_{m+1}}} \cap M_0 =: \mathbf{G}^\ell_{m+1}$",

and such that

$(*)_7$ $Y_{m+1} =: \bigcap_{i<n_{m+1}} \mathscr{M}[\mathbf{G}^\ell_{m+1}, M_0, y^*] \in D^{\mathfrak{p}}_\gamma(M_0)$.

The rest is as in the proof of Theorem 3.11.

Now, without loss of generality the construction belongs to N_1. So

$$\mathbf{G}^{**} = \{s \in \mathbb{P}_j \cap N_0 : \bigvee_{n<\omega} s \leq p_m\}$$

is as required, as $q'_\ell =: q_\ell \cup \bigcup_m r_m \in \mathbb{P}_j \cap N_1$, and is above $\mathbf{G}^{**}$ and $p \leq q'_\ell$. This finishes proving clause $(d)'$ in the case $\ell g(\bar{\mathbb{Q}})$ is a limit ordinal.

Clause $(c)'$: Again, without loss of generality $i < j = \delta$. So assume $f_{i,j}(\alpha) \leq \beta, \{i, j, \alpha, \beta\} \in N^* \in \mathscr{E}^{\mathfrak{p}}_\beta, q$ is $(N^*, \mathbb{P}_i)$-generic, $Y^* \in D^{\mathfrak{p}}_\beta(N^*), q \in \mathbb{P}_\alpha$ and $q \Vdash$ "$Y^* \subseteq \mathscr{M}_{\mathbb{P}_i}[\underset{\sim}{\mathbf{G}}_{\mathbb{P}_i}, N^*, y^*]$" are given. We prove the desired conclusion by induction on α. For each α, we would like to simulate a play of $\eth_{\alpha,\beta}(N^*)$, supplying the challenger with a strategy. For this we apply the proof of clause $(d)'$. Choose $N_0, N_1, M_0, \ldots, M_4, q_0, \mathbf{G}_0, \mathbf{G}^*$ (and $k = 1$) as there so that for some $\alpha' < \gamma' < \beta'$ as there, $\beta < \alpha'$ and $N^*, q, Y^* \in N_0$.

During the construction, we demand $p_m \in N^* \cap \mathbb{P}_j$, so a generic for N_0 is not necessarily created. But still $p_m \le p_{m+1}, p_m \restriction i_m \in \mathbf{G}^*_m$. Now p_m will be played by the chooser. Now $g(1+\alpha)$ will be a fixed point of f_{i_m,i_m+1}. So we can add the demand $N^*[\mathbf{G}^*_m \cap N^*] \cap \mathbf{V} = N^*$, i.e. $\mathbf{G}^*_m$ is generic over N^* and

$(*)_8$ $\mathscr{M}[\mathbf{G}^*_m \cap N^*, N^*, y^*] \in D^{\mathfrak{p}}_{g(1+\alpha)}(N^*)$.

We will define the game so that the following are satisfied:

- The challenger chooses

  $(*)_9$ $X_{m+1} = \mathscr{M}[\mathbf{G}^*_m \cap N^*, N^*, y^*] \cap \bigcup\limits_{\xi < g(1+\alpha)} \mathscr{E}^{\mathfrak{p}}_\xi \cap \{M : p_m \in M\} \in N_0$.
- Now the chooser chooses α_m, β'_m and then the challenger chooses $\beta_m \ge \beta'_m, f_{i_m,i_{m+1}}(\alpha)$ in $N^*_0 \cap j$ and the chooser chooses M^*_{m+1} such that $p_m \in M^*_{m+1}$.
- Now the chooser chooses $Y_m \in D^{\mathfrak{p}}_{\beta_m}(M_0), Y_m \subseteq X_m \cap M_0, Y_m \in N_0$.
- Now we play Z_m for the challenger as follows: there is $p_{m+1} \ge p_m$ which is $(M^*_{m+1}, \mathbb{P}_{i_{m+1}})$-generic such that $p_{m+1} \restriction i_m \in \mathbf{G}^*_m$, p'_{m+1} decides $\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{i_{m+1}}} \cap N^*$ and forces $Z_m = Y_m \cap \mathscr{M}_{\mathbb{P}_{i_{m+1}}}[\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{i_{m+1}}}, M^*_m, y^*] \in D^{\mathfrak{p}}_{\alpha_m}(M^*_m)$.

Let us now give the details. Choose $\langle i'_m : m < \omega\rangle \in N_0$ such that $i_m \in N^*, i_0 = i, i_m < i_{m+1}$ and $\sup\{i_m : m < \omega\} = \sup(N^* \cap j)$ and let $\langle \mathscr{I}'_m : m < \omega\rangle$ list the dense open subsets of $\mathbb{P}_j$ from N^*. For $\mathbf{m} < \omega$ let $\mathscr{T}_{\mathbf{m}}$ be the set of finite sequences $\mathfrak{x}$ from M_4 coding

- $\langle r_{\mathfrak{x},m} : m \le \mathbf{m}\rangle$,
- $\langle \mathbf{G}_{\mathfrak{x},m} : m \le \mathbf{m}\rangle$,
- $\langle p_{\mathfrak{x},m} : m \le \mathbf{m}\rangle$,
- $\langle n_{\mathfrak{x},m} : m \le \mathbf{m}\rangle$,

- $\langle \mathbf{G}^{\ell}_{\mathfrak{x},m} : \ell \le n_{\mathfrak{x},m}, m \le \mathbf{m}\rangle$,
- $\langle Y_{\mathfrak{x},m} : m \le \mathbf{m}\rangle$,
- $(X_{\mathfrak{x},m}, \alpha_{\mathfrak{x},m}, \beta'_{\mathfrak{x},m}, \beta_{\mathfrak{x},m}, M_{\mathfrak{x},n}, y'_{\mathfrak{x},m}, M'_{\mathfrak{x},m}, y_{\mathfrak{x},m})$ for $m \le \mathbf{m}$,
- $Z_{\mathfrak{x},m}$ for $m < \mathbf{m}$,

satisfying items (a)-(k), (m), (n), (p) and (q) from the proof of $(d)'$ above and

$(*)_{10}$ $(l)'$ $p_m \in \mathbb{P}_j \cap N^*$,

$(o)'$ $p_m \le p_{m+1} \in \mathscr{I}'_m$,

$(r)'$ $r_{\mathfrak{x},m}$ is $(N^*, \mathbf{G}_{i_m})$-generic for $m \le \mathbf{m}$,

$(s)'$ $\langle (X_{\mathfrak{x},m}, \alpha_{\mathfrak{x},m}, \beta'_{\mathfrak{x},m}, \beta_{\mathfrak{x},m}, M_{\mathfrak{x},m}, Y_{\mathfrak{x},m}, M'_{\mathfrak{x},m}, Z_{\mathfrak{x},m'}) : m \le \mathbf{m}\rangle$ belongs to N and is an initial segment of a play of the game $\partial'_{\alpha,\beta}(N^*, \mathfrak{p})$ or just $\partial'_{\alpha,\beta}(N^*, N, \mathfrak{p})$,[13]

$(t)'$ $Z_{\mathfrak{x},m} \subseteq Y_{m+1}$,

$(u)'$ $y_{\mathfrak{x},m}$ codes $p_m, \langle i_m : m < \omega\rangle$,

$(v)'$ $f_{i_m, i_{m+1}}(\alpha_m) \le \beta'_m$ for $m \le \mathbf{m}$.

We let $\mathfrak{x} \triangleleft \mathfrak{y}$ to have the natural meaning for $\mathfrak{x} \in \mathscr{T}_{\mathbf{m}_1}, \mathfrak{y} \in \mathscr{T}_{\mathbf{m}_2}, \mathbf{m}_1 < \mathbf{m}_2$. Note that

$\boxtimes_1$ $\mathscr{T}_{\mathbf{m}} \subseteq N$ for $\mathbf{m} < \omega$,

$\boxtimes_2$ $\mathscr{T}_0 \neq \emptyset$,

$\boxtimes_3$ if $\mathbf{x} \in \mathscr{T}_{\mathbf{m}}$, then $\mathfrak{x}$ is an initial segment of a play of the game $\partial_{\alpha,\beta}(N^*, \mathfrak{p})$ (see clause $(s)'$ above).

Now we show that $(*)_{11} \Rightarrow (*)_{12}$, where

$(*)_{11}$ (a) $M'_{\mathbf{m}} \in Y_{\mathfrak{x},\mathbf{m}} \cap \mathscr{E}^{\mathfrak{p}}_{\alpha_{\mathbf{m}}} \cap (M_{\mathfrak{x},\mathbf{m}} \cup \{M_{\mathfrak{x},\mathbf{m}} \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\mathbf{m}})\}$ satisfies $y_{\mathbf{m}}, y'_{\mathbf{m}} \in M'_{\mathbf{m}}$,

(b) $Z_{\mathbf{m}} \subseteq \mathscr{D}_{\alpha_{\mathfrak{x},m}}(M'_m)$,

[13] note that in the $\mathbf{m}$-th move the challenger has not yet chose $Z_{\mathfrak{x},\mathbf{m}}$, (see clause (e) of Definition 4.2(1).

(c) $Z_{\mathbf{m}} \subseteq Y_{\mathbf{m}}$ (hence $Z_{\mathbf{m}} \subseteq X_{\mathfrak{x},\mathbf{m}}$),

(d) $X_{\mathbf{m}+1} \in D^{\mathfrak{p}}_{\beta}(N^*) \cap X_{\mathfrak{x},\mathbf{m}}$ is such that $Z_{\mathbf{m}} \subseteq X_{\mathbf{m}+1}$,

(e) $\alpha_{\mathbf{m}+1} \in \alpha \cap N^*$,

(f) $\beta'_{\mathbf{m}+1} \in \beta \cap N^* \backslash p_{i_{\mathbf{m}+1}, i_{\mathbf{m}_r}}(\alpha_{\mathbf{m}+1})$,

(g) $y'_{\mathbf{m}+1} \in N \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\alpha_{\mathbf{m}+1}})$ and $y'_{\mathbf{m}+1} \in M_{\mathbf{m}+1}$,

(h) $\beta_{\mathbf{m}} \in \beta \cap N \backslash \beta'_n \backslash \alpha_n$ and $M_{\mathbf{m}+1} \in X_{\mathbf{m}+1} \cap \mathscr{E}^{\mathfrak{p}}_{\beta_{\mathbf{m}+1}}$,

(i) $y_{\mathbf{m}+1} \in M_{\mathbf{m}+1} \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\alpha_{\mathbf{m}}})$,

(j) $Y_{\mathbf{m}+1} \in N \cap D^{\mathfrak{p}}_{\beta_{\mathbf{m}+1}}(M_{\mathbf{m}+1})$,

(k) any $M'_{\mathbf{m}+1} \in Y_{\mathbf{m}+1} \cap \mathscr{E}^{\mathfrak{p}}_{\alpha_m} \cap (M_{\mathbf{m}+1} \cup \{M_{\mathbf{m}+1} \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\alpha_m})\}$ satisfies $y_{\mathbf{m}+1}, y'_{\mathbf{m}+1} \in M'_{\mathbf{m}+1}$.

and

$(*)_{12}$ there is $\mathfrak{y} \in \mathscr{T}_{\mathbf{m}+1}$ such that $\mathfrak{x} \triangleleft \mathfrak{y}$ and $(Z_{\mathfrak{y},\mathbf{m}}, X_{\mathfrak{y},\mathbf{m}+1},\ \alpha_{\mathfrak{y},\mathbf{m}+1}, \beta'_{\mathfrak{y},m+1}, y'_{\mathfrak{y},\mathbf{m}+1}, \beta_{\mathfrak{y},\mathbf{m}+1}, y_{\mathfrak{y},\mathbf{m}+1}, M_{\mathfrak{y},\mathbf{m}+1}, Y_{\mathfrak{y},\mathbf{m}+1}, M'_{\mathfrak{y},\mathbf{m}+1})$ is equal to $(Z_{\mathbf{m}}, X_{\mathbf{m}+1}, \alpha_{\mathbf{m}+1}, \beta'_{\mathbf{m}+1}, y'_{\mathbf{m}+1}, \beta_{\mathbf{m}+1}, y_{\mathbf{m}+1},\ M_{\mathbf{m}+1}, Y_{\mathbf{m}+1}, M'_{\mathbf{m}+1})$.

To see this, note that $f_{i_{\mathbf{m}}, i_{\mathbf{m}+1}}(\alpha_{\mathbf{m}}) \le \beta_{\mathbf{m}}$, hence $\mathbb{P}_{i_{\mathbf{m}+1}}/\mathbb{P}_{i_{\mathbf{m}}}$ is $(\mathfrak{p}, \alpha_{\mathbf{m}}, \beta_{\mathbf{m}})$-proper; thus let $\mathbf{G}_{i_{\mathbf{m}}} \subseteq \mathbb{P}_{i_{\mathbf{m}}}$ be generic over $\mathbf{V}, r_{\mathbf{m}} \in \mathbf{G}_{i_{\mathbf{m}}}$ to the model $M'_{\mathfrak{x},\mathbf{m}}$ and the set $Y_{\mathfrak{x},\mathbf{m}}$. Thus we can describe a strategy for the challenger in the game $\eth_{\alpha,\beta}(N^*, \mathfrak{p})$ (or $\eth'_{\alpha,\beta}(N^*, N, \mathfrak{p})$) delaying his choice of $M'_{\mathbf{m}}, Z_{\mathbf{m}}$ to the $(\mathbf{m}+1)$-th move, he just chose on the side $\mathfrak{x}_{\mathbf{m}} \in \mathscr{T}_{\mathbf{m}}$ which "codes" what they played so far and preserve $\mathfrak{x}_{\mathbf{m}} \triangleleft \mathfrak{x}_{\mathbf{m}+1}$.

By $\boxtimes_3$ this is possible, all possible choices of the chooser are allowed, that is this gives a well defined strategy for the challenger. Now take $\mathfrak{y} \in \mathscr{T}_{\mathbf{m}+1}$ be such that for all $\mathbf{m}$, $\mathfrak{x}_{\mathbf{m}} \triangleleft \mathfrak{y}$

As the challenger does not have a winning strategy, there is a play where the chooser wins. This gives us

$$\mathbf{G}'' = \bigcup_{n<\omega} \mathbf{G}_n^* \cap N^*$$

with a bound. Also, there is such a choice $\langle \mathfrak{x}_{\mathbf{m}} : \mathbf{m} < \omega \rangle$ with $\bigcup \{(M'_{\mathfrak{x}_{\mathbf{m}+1},\mathbf{m}}\} \cup Y_{\mathfrak{x}_{\mathbf{m}+1},\mathbf{m}} : \mathbf{m} < \omega\} \in D^{\mathfrak{p}}_{\alpha}(N)$ and $q' = \bigcup_{\mathbf{m}<\omega} r_{\mathbf{m}}$ is as required.

The proof is complete. $\square$

Now we deal with the case of $\mathfrak{p}$-NNR$^1_\kappa$ iteration, where $2 \leq \kappa < \aleph_2$. The adaptation for the proof of Theorem 6.3 when $2 \leq \kappa < \aleph_2$ should be clear.

Theorem 6.4. *Assume $\bar{\mathbb{Q}}$ is a $\mathfrak{p}$-NNR$^1_\kappa$ iteration where $2 \leq \kappa < \aleph_2$, $\mathfrak{p}$ is a reasonable parameter and $\mathfrak{p}$ is a $\eth_f$-winner for some $f \in \mathscr{F}^{\mathfrak{p}}_{\text{club}}$ (or at least is $\eth'_f$-non loser).*

(1) *Forcing with $\mathbb{P}_{\ell g(\bar{\mathbb{Q}})} = \text{Lim}(\bar{\mathbb{Q}})$ does not add reals.*
(2) *If $i \leq j \leq \ell g(\bar{\mathbb{Q}})$, then*
  - *(b)′ $\mathbb{P}_j/\mathbb{P}_i$ is proper,*
  - *(c)′ $\mathbb{P}_j/\mathbb{P}_i$ is $(\mathfrak{p}, f_{i,j})$-proper, where $f_{i,j} \in \mathscr{F}^{\mathfrak{p}}_{\text{club}}$ is increasing continuous and is computable from the $f_\varepsilon \in \mathscr{F}^{\mathfrak{p}}$ for $\varepsilon \in [i, j)$,*
  - *(d)′ if $i < j < \ell g(\bar{\mathbb{Q}})$, then for some function $g \in \mathscr{F}^{\mathfrak{p}}_{cd}$ in $\mathbf{V}$ and for all $\alpha < \ell g(\mathfrak{p})$ and $\beta \in g(\alpha)$ we have $\mathbb{P}_j/\mathbb{P}_i$ is $(\aleph_0, \alpha, \beta)$-anti-w.d above $\mathbb{P}_i$.*

Proof. Similar to the proof of Theorem 6.3, with some changes as in the proof of Theorem 3.15. $\square_{5.5}$

Remark 6.5. We may be interested in non-proper forcing notions, say semi-proper and UP ones (see [6, Ch. X, XI, XV]). Here the change from reasonable parameter $\mathfrak{p} = \mathfrak{p}^V$ to $\mathfrak{p}^{\mathbf{V}[\mathbf{G}]}$ is more serious as $\{N \cap \chi_\alpha : N \in \mathscr{E}^{\mathfrak{p}^{\mathbf{V}[\mathbf{G}]}}\}$ is in general not equal to $\{N \cap \chi_\alpha : N \in \mathscr{E}^{\mathfrak{p}}_\alpha\}$. This is treated in [7].

§ 7. Forcing axioms compatible with CH

As is well known, iteration theorems give us consistency of axioms and in this section we present a few of such examples. We consider $\kappa \in \{2, \aleph_0\}$, but could also have $\kappa = \aleph_1$ at some points.

Definition 7.1. Suppose $\mathfrak{p}$ is an o.b. parameter. Then $\mathrm{Ax}^\alpha_\lambda(\mathfrak{p}, \kappa, 0)$ means: if $\mathbb{Q}$ is $\aleph_2$-e.c.c. ($\aleph_2$-pic if $\lambda = \aleph_2$) and an NNR^0_κ-forcing notion for $\mathfrak{p}$, $\mathscr{I}_\beta$ is a dense open subset of $\mathbb{Q}$, for $\beta < \beta^* < \lambda$, and $\underset{\sim}{S}_i$ is a $\mathbb{Q}$-name of a stationary subset of ω_1, for $i < i^* < \alpha$, then for some directed $\mathbf{G} \subseteq \mathbb{Q}$ we have:

- $\beta < \beta^* \Rightarrow \mathbf{G} \cap \mathscr{I}_\beta \neq \emptyset$,
- $i < i^* \Rightarrow \underset{\sim}{S}_i[\mathbf{G}] = \{\gamma < \omega_1 : (\exists r \in \mathbf{G})(r \Vdash_{\mathbb{Q}} \text{“}\gamma \in \underset{\sim}{S}_i\text{”})\}$ is a stationary subset of ω_1.

We remove α when $\alpha = 0$.

Now we use our results on preservation of being an NNR^0_κ-forcing notion for $\mathfrak{p}$ to get the consistency of $\mathrm{Ax}^\alpha_\lambda(\mathfrak{p}, \kappa, 0)$.

Lemma 7.2. (1) *If $\mathfrak{p}$ is an o.b. parameter of length $\ell g(\mathfrak{p}) = \omega_1$ which is non-$\eth'_{\mathrm{id}}$-loser, and if $\bar{\mathbb{Q}}$ is a countable support iteration such that for $\alpha < \ell g(\bar{\mathbb{Q}})$, $\Vdash_{\mathbb{P}_\alpha}$ “$\underset{\sim}{\mathbb{Q}}_\alpha$ is an NNR^0_κ-forcing notion for $\mathfrak{p}$”, then $\bar{\mathbb{Q}}$ is an NNR^0_κ-iteration for $\mathfrak{p}$*

(2) *Assume $\mathrm{CH} + \mu = \mu^{<\mu} \geq \lambda$. If $\mathfrak{p}$ is a non$-\eth'_{\mathrm{id}}$-loser o.b. parameter, $\chi^{\mathfrak{p}}_0 > 2^\lambda$, then for some $\aleph_2$-e.c.c. ($\aleph_2$-pic, if $\lambda = \aleph_2$) NNR^0_κ-forcing notion $\mathbb{P}$ of size μ we have $\Vdash_{\mathbb{P}}$ “$\mathrm{Ax}_\lambda(\mathfrak{p}, \kappa, 0)$”.*

Proof. (1). Follows from Theorem 3.11,

(2). It follows using a suitable countable support iteration of length μ, forcing all instances of the axiom $\mathrm{Ax}_\lambda(\mathfrak{p}, \kappa, 0)$ at some stage of the iteration. Clause (1) and Theorem 1.10 guarantee that the iteration is as required.. □

Definition 7.3. Assume $\lambda = \lambda^{<\lambda} \gg \aleph_1 \geq \kappa \geq 2$ and $\mathfrak{p}$ is a reasonable parameter such that $\lambda < \chi_0^{\mathfrak{p}}$ and $\lambda \leq \mathrm{cf}(\ell g(\mathfrak{p}))$. Let also

$$\mathbb{R} = \mathbb{R}_{\lambda,\mathfrak{p}} = (\{\bar{\mathbb{Q}} : \bar{\mathbb{Q}} \in \mathscr{H}(\lambda) \text{ is a } \mathfrak{p}\text{-NNR}^0_\kappa \text{ iteration}\}, \leq_{\mathbb{R}})$$

where

$$\bar{\mathbb{Q}}^1 \leq_{\mathbb{R}} \bar{\mathbb{Q}}^2 \Leftrightarrow \bar{\mathbb{Q}}^1 = \bar{\mathbb{Q}}^2 \restriction \ell g(\bar{\mathbb{Q}}^1).$$

(1) $\underset{\sim}{\mathbb{Q}}$ is absolutely $(\lambda, \mathfrak{p}, \mathbb{R})$-NNR$^0_\kappa$ forcing above $\bar{\mathbb{Q}}$, when

 (a) $\bar{\mathbb{Q}} \in \mathbb{R}$,

 (b) $\underset{\sim}{\mathbb{Q}}$ is a $\mathrm{Lim}(\bar{\mathbb{Q}})$-name of a forcing notion from $\mathscr{H}(\lambda)^{\mathbf{V}[\mathbf{G}_{\mathrm{Lim}(\bar{\mathbb{Q}})}]}$,

 (c) if $\bar{\mathbb{Q}} \leq_{\mathbb{R}} \bar{\mathbb{Q}}^1$, then $\underset{\sim}{\mathbb{Q}}$ is $(\bar{\mathbb{Q}}^1, \bar{\mathbb{Q}}, \mathfrak{p})$-NNR$^0_\kappa$, which means that there is $\bar{\mathbb{Q}}^2 \in \mathbb{R}, \bar{\mathbb{Q}}^1 \leq_{\mathbb{R}} \bar{\mathbb{Q}}^2$ and $\bar{\mathbb{Q}}^2_{\ell g(\bar{\mathbb{Q}}^1)} = \underset{\sim}{\mathbb{Q}}$ so $\underset{\sim}{\mathbb{Q}}$ is a $\mathrm{Lim}(\bar{\mathbb{Q}}^1)$-name $\in \mathscr{H}(\lambda)$.

(2) $(\underset{\sim}{\mathbb{Q}}, \underset{\sim}{\bar{\mathscr{I}}})$ is an absolute $(\lambda, \mathfrak{p}, \mathbb{R})$-NNR$^0_\kappa$-problem above $\bar{\mathbb{Q}}$, when

 (a) $\underset{\sim}{\mathbb{Q}}$ is a $\mathrm{Lim}(\bar{\mathbb{Q}})$-name of a forcing notion from $\mathscr{H}(\lambda)^{\mathbf{V}[\mathbf{G}_{\mathrm{Lim}(\bar{\mathbb{Q}})}]}$,

 (b) $\underset{\sim}{\bar{\mathscr{I}}}$ is a $\mathrm{Lim}(\bar{\mathbb{Q}})$-name for a sequence of $< \lambda$ subsets of $\underset{\sim}{\mathbb{Q}}$,

 (c) if $\bar{\mathbb{Q}} \leq_{\mathbb{R}} \bar{\mathbb{Q}}^1$, then there is $\bar{\mathbb{Q}}^2$ such that $\bar{\mathbb{Q}}^1 \leq_{\mathbb{R}} \bar{\mathbb{Q}}^2$ and $\Vdash_{\mathrm{Lim}(\bar{\mathbb{Q}}^2)}$ "$(\underset{\sim}{\mathbb{Q}}, \underset{\sim}{\bar{\mathscr{I}}})$ is solved", which means there is a directed $\mathbf{G} \subseteq \underset{\sim}{\mathbb{Q}}$ meeting $\underset{\sim}{\mathscr{I}}_\varepsilon$ for every $\varepsilon < \ell g(\underset{\sim}{\bar{\mathscr{I}}})$".

Lemma 7.4. *Suppose that* CH *holds,* $\lambda = \lambda^{<\lambda} \gg \aleph_1 \geq \kappa \geq 2$ *and* $\mathfrak{p}$ *is a reasonable parameter such that* $\lambda < \chi_0^{\mathfrak{p}}$ *and* $\lambda \leq \mathrm{cf}(\ell g(\mathfrak{p}))$*. Then there is a proper* λ*-c.c. forcing notion* $\mathbb{P}_*$ *of cardinality* λ*, such that*

(a) *Forcing with* $\mathbb{P}_*$ *adds no reals,*

(b) $\mathbb{P}_* = \mathrm{Lim}(\bar{\mathbb{Q}}_*)$*, where* $\bar{\mathbb{Q}}_*$ *is a countable support iteration* $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \lambda \rangle$ *with* $\Vdash_{\mathbb{P}_\alpha}$ "$|\mathbb{Q}_\alpha| < \lambda$"*, such that* $\bar{\mathbb{Q}}_* \restriction \alpha \in \mathscr{H}(\lambda)$ *for* $\alpha < \lambda$*. In particular,* $\mathbb{P}_* = \bigcup_{\alpha<\lambda} \mathbb{P}_\alpha$*,*

(c) $\bar{\mathbb{Q}}_*$ *is* $\mathfrak{p}$*-NNR*$^1_\kappa$*-iteration,*

(d) *if* $\mathbf{I}$ *is a dense open subset of*

$$\mathbb{R} = \mathbb{R}_{\lambda,\mathfrak{p}} = (\{\bar{\mathbb{Q}} : \bar{\mathbb{Q}} \in \mathscr{H}(\lambda) \textit{ is a } \mathfrak{p}\textit{-NNR}^0_\kappa \textit{ iteration}\}, \leq_{\mathbb{R}}),$$

where $\bar{\mathbb{Q}}^1 \leq_{\mathbb{R}} \bar{\mathbb{Q}}^2 \Leftrightarrow \bar{\mathbb{Q}}^1 = \bar{\mathbb{Q}}^2 \restriction \ell g(\bar{\mathbb{Q}}^1)$, *and if* $\mathbf{I}$ *is definable in* $(\mathscr{H}(\lambda), \in)$ *from a parameter, then* $\lambda = \sup\{\alpha < \lambda : \bar{\mathbb{Q}}_* \restriction \alpha \in \mathbf{I}\}$,

(e) *if* $\mathbb{Q} \in \mathbf{V}^{\mathbb{P}_*}$ *is a forcing notion of cardinality* $\aleph_1$, *so without loss of generality whose set of elements is a subset of* ω_1, *and if* $\alpha < \lambda$ *is such that* $\mathbb{Q} \in \mathbf{V}^{\mathbb{P}_\alpha}$ *and* $\mathbb{Q}$ *is absolute* $(\mathfrak{p} \restriction \alpha, \lambda, \mathbb{R})$*-proper, then for some* $\beta \in (\alpha, \lambda), \mathbb{Q}_\beta = \mathbb{Q}$, *hence* $\mathbf{V}^{\mathbb{P}_*} \models \mathrm{Ax}_\lambda(\mathbb{Q})$.

(f) *similarly for* $(\mathbb{Q}, \bar{\mathscr{I}})$.

Proof. First note that $\mathbb{R}$ is non-empty and with no maximal member. Indeed, the iteration of length zero belongs to $\mathbb{R}$ and if $\bar{\mathbb{Q}} = \langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \alpha \rangle \in \mathbb{R}$, we can define $\bar{\mathbb{Q}}' \in \mathbb{R}$ above it by letting $\bar{\mathbb{Q}}' = \langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta \leq \alpha \rangle$, where $\mathbb{P}_\alpha = \lim(\bar{\mathbb{Q}})$ and $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\mathbb{Q}}_\alpha = ({}^{\omega_1 >}2, \triangleleft)$".

Now fix $\Phi : \lambda \to \mathscr{H}(\lambda)$ such that for each $x \in \mathscr{H}(\lambda)$, the set $\Phi^{-1}\{x\}$ is stationary in λ. Let $\bar{\mathbb{Q}}_* = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \lambda \rangle$ be a countable support iteration of forcing notions, where at stage α, if $\Phi(\alpha)$ is a $\mathbb{P}_\alpha$-name for a forcing notion as in (e), then we let $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\mathbb{Q}}_\alpha = \Phi(\alpha)$". Otherwise, $\underset{\sim}{\mathbb{Q}}_\alpha$ is forced to be the trivial forcing notion.

Using the preservation theorems we have proved earlier, we can easily show that $\mathbb{P}_* = \lim(\bar{\mathbb{Q}}_*)$ is as required. □

It is natural to restrict ourselves to the linear case, but this is not a real difference when we allow to change the f a little.

Definition 7.5. Let $\mathfrak{p}$ be a reasonable parameter.

(1) $\mathfrak{p}$ is linear, if whenever $\alpha < \ell g(\mathfrak{p}), N \in \mathscr{E}^{\mathfrak{p}}_\alpha$ and $Y \in D^{\mathfrak{p}}_\alpha(N)$, then there is Z such that:

   (a) $Z \in D^{\mathfrak{p}}_\alpha(N)$,

   (b) $Z \subseteq Y$,

   (c) if $a \in Z \cap \mathscr{E}^{\mathfrak{p}}_\gamma$ then $N \restriction a \prec N \restriction \mathscr{H}(\chi_\gamma)$,

   (d) Z is linear, which means

(α) Z is well ordered by $\in$ (and by $\subseteq$),

(β) if $a \in Z$ then $Z \cap a \in N$ and $\langle N{\restriction}a : a \in Z\rangle$ is $\subseteq$-increasing continuous.

(2) $\mathfrak{p}$ is linearly standard when it is standard and linear.

(3) Assume $f \in \mathscr{F}^{\mathfrak{p}}$ and $g \in \mathscr{F}^{\mathfrak{p}}$ is defined as

$$g(\alpha) = \bigcup\{f(\beta) : \beta \in f(\alpha)\}.$$

Let $\mathfrak{q} = \mathfrak{p}^{[f]}$ be defined as in $\mathfrak{p}$, except that for each $\alpha < \ell g(\mathfrak{p})$ and N, $D^{\mathfrak{q}}_{\alpha}(N) = \{Y \in D^{\mathfrak{p}}_{\alpha}(N)$: for some $\beta \in N \cap f(\alpha)$ there is $Z \subseteq Y, Z$ linear and $Z \in N \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\beta})\}$.

Lemma 7.6. *If $\mathfrak{p}$ is a reasonable parameter and f, g and $\mathfrak{q}$ are as in Definition 7.5(3), then*

(a) *$\mathfrak{q}$ is a reasonable parameter,*

(b) *if f is increasing continuous then so is g,*

(c) *if $f(\alpha) = \alpha$ then $g(\alpha) = \alpha$ and $D^{\mathfrak{q}}_{\alpha}(N) = D^{\mathfrak{p}}_{\alpha}(N)$.*

Proof. Straightforward. □

Lemma 7.7. *Assume $\mathfrak{p}$ is a reasonable parameter, $f \in \mathscr{F}^{\mathfrak{p}}, \lambda = \lambda^{<\lambda}$ is large enough regular and $\mathbb{R} = \mathbb{R}_{\lambda,\mathfrak{p}}$. Let $\bar{\mathbb{Q}} \in \mathbb{R}$ and $\mathbb{P} = \mathrm{Lim}(\bar{\mathbb{Q}})$.*

(1) *If $\mathfrak{p}$ is linear and $\underset{\sim}{\mathbb{Q}}$ is a $\mathbb{P}$-name of a $(<^{+} \omega_1)$-proper forcing notion from $\mathscr{H}(\lambda)$ and $f \in \mathscr{F}^{\mathfrak{p}}_{\mathrm{dc}}$, then, $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}}$ is $(\mathfrak{p}^{\mathbb{P}}, f)$-proper".*

(2) *if $\underset{\sim}{\mathbb{Q}}$ satisfies the κ-completeness system $\mathbb{D} \in \mathbf{V}$ over $\mathbb{P}$ and $\mathbb{P}$ forces it is an NNR proper forcing notion, then for some $f \in \mathscr{F}^{\mathfrak{p}}_{\mathrm{dc}}$, we have $(\mathbb{P}, \underset{\sim}{\mathbb{Q}})$ is (κ, f)-anti w.d. (see Definition 4.12),*

Proof. (1). The proof is essentially the same as the proof of Lemma 3.17.

(2). Define the function f such that for each $\alpha < \ell g(\mathfrak{p})$, $f(\alpha) = [\theta_\alpha, \ell g(\mathfrak{p}))$, where $\theta_\alpha \geq \alpha$ is large enough so that $\mathscr{H}(\chi_{\theta_\alpha})$ contains all the relevant information and the cardinal θ from Definition 1.3 witnessing $\mathbb{D}$ is a completeness system is below χ_{θ_α}.

Suppose $\alpha < \ell g(\mathfrak{p})$, $\beta \in f(\alpha)$ and suppose that $N_0, N_1, n, \langle p_\ell : \ell < n\rangle, \langle \mathbf{G}^\ell : \ell < n\rangle, \mathbf{G}^*, Y$ and $\underset{\sim}{q}$ are as in Definition 4.12(1)(A). Without loss of generality, we can assume that the p_ℓ's are pairwise incompatible.

Pick some $M \in Y$, so that for each $\ell < n, (\mathbf{G}^\ell \cap M)$ is $(\mathbb{P} \cap M)$-generic over M. Fix some $\ell < n$. Consider the pair $(M[\mathbf{G}^\ell], \underset{\sim}{q}[\mathbf{G}^*]) \in \mathrm{dom}(\mathbb{D})$. By the assumption, there are $p'_\ell \geq p^\ell$ and $\underset{\sim}{\mathbf{H}}^\ell$ such that

$$p'_\ell \Vdash \text{“}\underset{\sim}{\mathbf{H}}^\ell \text{ is in } \mathrm{Gen}^+(M[\mathbf{G}^\ell], \underset{\sim}{\mathbb{Q}}, \underset{\sim}{q})\text{”}.$$

By extending p'_ℓ if necessary, we can assume that for some $\underset{\sim}{q}_\ell$ we also have

$$p'_\ell \Vdash \text{“}\underset{\sim}{q}_\ell \geq \underset{\sim}{q} \text{ is an upper bound for } \underset{\sim}{\mathbf{H}}^\ell\text{”},$$

Set $\mathbf{K}^\ell = \mathbf{G}^\ell * \underset{\sim}{\mathbf{H}}^\ell$. As the iteration $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ does not add any new ω-sequences of elements of $\mathbf{V}$, and by our choice of $\mathbf{G}^*$, by extending $(p'_\ell, \underset{\sim}{q}_\ell)$, we may assume that for some fixed $\mathbf{G}^{**}$ and for all $\ell < n$, we have

$$(p'_\ell, \underset{\sim}{q}_\ell) \Vdash \text{“}\mathbf{K}^\ell \cap N_0 = \mathbf{G}^{**}\text{”}.$$

It follows that $\mathbf{G}^{**} \in \mathrm{Gen}(N_0, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$. Let $\underset{\sim}{q}'$ be such that for all $\ell < n, p'_\ell \Vdash$“$\underset{\sim}{q}' = \underset{\sim}{q}_\ell$”. It is clear that $\langle p'_\ell : \ell < n\rangle, \underset{\sim}{q}'$ and $\mathbf{G}^{**}$ are as required by Definition 4.12(1)(B). □

For the rest of this section, assume that CH holds, $\lambda = \lambda^{<\lambda} \gg \aleph_1 \geq \kappa \geq 2$ and $\mathfrak{p}$ is a reasonable parameter such that $\lambda < \chi_0^{\mathfrak{p}}$ and $\lambda \leq \mathrm{cf}(\ell g(\mathfrak{p}))$. Let $\mathbb{P}_*$ be as in Lemma 7.4.

Lemma 7.8. *Under the above assumptions, if $\mathbb{Q} \in \mathbf{V}^{\mathbb{P}_*}$ is one of the following forcing notions, then it satisfies clause (e) of Lemma 7.4, in particular $\mathbf{V}^{\mathbb{P}_*} \models \mathrm{Ax}_\lambda(\mathbb{Q})$.*

(a) $\mathbb{Q} = \mathbb{Q}_{\bar{C},\bar{u}}$ is as in Definition 5.1(1),

(b) $\underset{\sim}{\mathbb{Q}} = \mathbb{Q}_{\bar{C}}$ *is as in Definition 5.3, for* $\bar{C} = \langle C_\delta : \delta < \omega_1 \text{ limit}\rangle$, *where for each limit ordinal* $\delta, \operatorname{otp}(C_\delta) = \omega$.

(c) $\underset{\sim}{\mathbb{Q}} = \mathbb{Q}_{\bar{C}}$ *is as in Definition 5.3, for* $\bar{C} = \langle C_\delta : \delta < \omega_1 \text{ limit}\rangle$, *where for some countable ordinal* $\gamma(*), \delta < \omega_1 \Rightarrow \operatorname{otp}(C_\delta) \leq \omega^{\gamma(*)}$.

Proof. The $(\mathfrak{p}, f)$-properness of $\mathbb{Q}_{\bar{C},\bar{u}}$ follows from Lemmas 5.2 and 7.7. The $(\mathfrak{p}, f)$-properness of $\mathbb{Q}_{\bar{C}}$ follows from Lemma 5.5 for $\bar{C}$ as in (b) and from Lemma 5.6 for $\bar{C}$ as in (c). The $(\mathfrak{p}, g)$-anti w.d. is straightforward. □

Given an Aronszajn tree T, let $\mathbb{Q}_T$ be the forcing notion of [6, Ch. V, Definition 6.5]. Let also $\bar{\mathscr{I}}_T = \langle \mathscr{I}_{T,\alpha} : \alpha < \omega_1\rangle$ where

$$\mathscr{I}_{T,\alpha} = \{(f, C, \Psi) \in \mathbb{Q}_T : T_{\leq\alpha} \subseteq \operatorname{dom}(f)\}.$$

Lemma 7.9. *Under the above assumption, we have the following:*

(1) *If* $\underset{\sim}{T}$ *is a* $\mathbb{P}_*$*-name of an Aronszajn tree, then the pair* $(\underset{\sim}{\mathbb{Q}}_{\underset{\sim}{T}}, \underset{\sim}{\bar{\mathscr{I}}_T})$ *is an absolute* $(\lambda, \mathfrak{p}, \mathbb{R})$*-NNR*$^0_{\aleph_1}$ *problem over* $\mathbb{P}_*$.

(2) *If* λ *is strongly inaccessible*[14], *then every Aronszajn tree is special.*

Proof. (1) follows from [6, Ch. V, Theorem 6.1] and Lemma 7.7.

(2) is clear, as for any Aronszajn tree T, by (1), the forcing notion $\mathbb{Q}_T$ satisfies clause (e) of Lemma 7.4. □

§ 8. On Moore's question

In this section we answer a question of Justin Moore about the consistency of strong failure of club guessing sequences with CH.

[14] We may avoid this, if we use iterations as in [6, Ch. VIII], i.e. $\bar{\mathbb{Q}} \in \mathbb{R}$ is only a class of $(\mathscr{H}(\lambda), \in)$, satisfying a strong version of λ-c.c., so $\lambda = \aleph_2$ is sufficient

Definition 8.1. Let $\mathbf{cd}:\mathscr{H}(\aleph_1) \to \omega_1$ be one-to-one. We say E solves $\mathbf{cd}$, when E is a club of ω_1 and for every $\alpha \in E$ we have $\mathbf{cd}(E \cap (\alpha+1)) < \min(E \backslash (\alpha+1))$.

Justin Moore asked the following question.

Question 8.2. Is the following consistent:

(a) CH,

(b) for every one-to-one function $\mathbf{cd}$ from $\mathscr{H}(\aleph_1)$ to ω_1, there is some E which solves it.

(c) if $\bar{C} = \langle C_\delta : \delta < \omega_1 \text{ limit}\rangle$, where $C_\delta \subseteq \delta = \sup(C_\delta)$ and $\mathrm{otp}(C_\delta) = \omega$, then for some club E of $\omega_1, (\forall \delta)(\delta > \sup(C_\delta \cap E))$.

We give a positive answer to the above question by proving the following theorem.

Theorem 8.3. *Suppose* CH *holds,* $\lambda = \lambda^{<\lambda} \gg \aleph_1 \geq \kappa \geq 2$ *and* $\mathfrak{p}$ *is a reasonable parameter such that* $\lambda < \chi_0^{\mathfrak{p}}$ *and* $\lambda \leq \mathrm{cf}(\ell g(\mathfrak{p}))$. *Let* $\mathbb{P}_*$ *be as in Lemma 7.4 and set* $\mathbf{V}_1 = \mathbf{V}^{\mathbb{P}_*}$. *Then* $\mathbf{V}_1$ *satisfies the requirements of Question 8.2.*

Proof. In $\mathbf{V}_1$, CH holds by Lemma 7.4. Clause (c) of 8.1 holds by Lemma 7.8(b). To show that clause (b) of 8.2 is satisfied, let $\mathbf{cd}:\mathscr{H}(\aleph_1) \to \omega_1$ be a one-to-one function. Define $\mathbb{Q} = \mathbb{Q}_{\mathbf{cd}}$ as follows:

(a) $p \in \mathbb{Q}$ iff p is a closed bounded subset of ω_1 satisfying

$$(\forall \alpha \in p)[\alpha \neq \max(p) \Rightarrow \mathbf{cd}(p \cap (\alpha+1)) < \min(p \backslash (\alpha+1))].$$

(b) $p \leq_{\mathbb{Q}} q \iff p, q \in \mathbb{Q}$ and p is an initial segment of q.

It is easily seen that $\mathbb{Q}_{\mathbf{cd}}$ is $(<^+ \omega_1)$-proper and that it satisfies clause (e) of Lemma 7.4. The result follows immediately. $\square$

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