TORSION-FREE ABELIAN GROUPS ARE BOREL COMPLETE

GIANLUCA PAOLINI AND SAHARON SHELAH

ABSTRACT. We prove that the Borel space of torsion-free Abelian groups with domain ω is Borel complete, i.e., the isomorphism relation on this Borel space is as complicated as possible, as an isomorphism relation. This solves a long-standing open problem in descriptive set theory, which dates back to the seminal paper on Borel reducibility of Friedman and Stanley from 1989.

1. INTRODUCTION

Since the seminal paper of Friedman and Stanley on Borel complexity [3], descriptive set theory has proved itself to be a decisive tool in the analysis of complexity problems for classes of countable structures. A canonical example of this phenomenon is the famous result of Thomas from [16] which shows that the complexity of the isomorphism relation for torsion-free abelian groups of rank $1 \leqslant n < \omega$ (denoted as $\cong_n$) is strictly increasing with n, thus, on one hand, finally providing a satisfactory reason for the difficulties found by many eminent mathematicians in finding systems of invariants for torsion-free abelian groups of rank $2 \leqslant n < \omega$ which were as simple as the one provided by Baer for $n = 1$ (see [1]), and, on the other hand, showing that for no $1 \leqslant n < \omega$ the relation $\cong_n$ is universal among countable Borel equivalence relations. As a matter of facts, abelian group theory has been one of the most important fields of mathematics from which taking inspiration for forging the general theory of Borel complexity as well as for finding some of the most striking applications thereof. The present paper continues this tradition solving one of the most important problems in the area, a problem open since the above mentioned paper of Friedman and Stanley from 1989. In technical terms, we prove that the space of countable torsion-free abelian groups with domain ω is *Borel complete*.

As we will see in detail below, saying that a class of countable structures is Borel complete means that the isomorphism relation on this class is as complicated as possible, as an isomorphism relation. The Borel completeness of countable abelian group theory is particularly interesting from the perspective of model theory, as this class is model theoretically "low", i.e., stable (in the terminology of [13]). In fact, as already observed in [3], Borel reducibility can be thought of as a weak version of $\mathfrak{L}_{\omega_1,\omega}$-interpretability, and for other classes of countable structures such as groups or fields much stronger results than Borel completeness exist, as in such cases we can first-order interpret graph theory, but such classes are unstable, while abelian group

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theory is stable. Reference [7] starts a systematic study of the relations between Borel reducibility and classification theory in the context of $\aleph_0$-stable theories.

Coming back to us, we now introduce the notions from descriptive set theory which are necessary to understand our results, and we try to make a complete historical account of the problems which we tackle in this paper. The starting point of the descriptive set theory of countable structures is the following fact:

Fact 1.1. *The set K^L_ω of structures with domain ω in a given countable language L is endowed with a standard Borel space structure $(\mathrm{K}^L_\omega, \mathcal{B})$. Every Borel subset of this space $(\mathrm{K}^L_\omega, \mathcal{B})$ is naturally endowed with the Borel structure induced by $(\mathrm{K}^L_\omega, \mathcal{B})$.*

For example, if take $L = \{e, \cdot, ()^{-1}\}$, and we let K' to be one of the following:
(a) the set of elements of K^L_ω which are groups;
(b) the set of elements of K^L_ω which are abelian groups;
(c) the set of elements of K^L_ω which are torsion-free abelian groups;
(d) the set of elements of K^L_ω which are n-nilpotent groups, for some $n < \omega$;

then we have that K' is a Borel subset of $(\mathrm{K}^L_\omega, \mathcal{B})$, and so Fact 1.1 applies.

Thus, given a class K' as in Fact 1.1, we can consider K' as a standard Borel space, and so we can analyze the complexity of certain subsets of this space or of certain relations on it (i.e., subsets of $K' \times K'$ with the product Borel space structure). Further, this technology allows us to compare pairs of classes of structures or, in another direction, pairs of relations defined on pairs of classes of structures.

Definition 1.2. *Let X_1 and X_2 be two standard Borel spaces, and let also $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$. We say that Y_1 is reducible to Y_2, denoted as $Y_1 \leqslant_R Y_2$, when there is a Borel map $\mathbf{B} : X_1 \to X_2$ such that for every $x \in X_1$ we have:*

$$x \in Y_1 \Leftrightarrow \mathbf{B}(x) \in Y_2.$$

Notice that Definition 1.2 covers in particular the case $X_1 = K' \times K'$ for K' as in Fact 1.1, and so for example Y_1 could be the isomorphism relation on K'. Also, given a Borel space X, we can ask if there are subsets of X which are $\leqslant_R$-maxima with respect to a fixed family of subsets of an arbitrary Borel space (e.g., Borel sets, analytic sets, co-analytic sets, etc). In particular we can define:

Definition 1.3. *Let X_1 be a Borel space and $Y_1 \subseteq X_1$. We say that Y_1 is* complete analytic *(resp. complete co-analytic) if for every Borel space X_2 and analytic subset (resp. co-analytic subset) Y_2 of X_2 we have that $Y_2 \leqslant_R Y_1$.*

We now introduce the notion of Borel reducibility among equivalence relations.

Definition 1.4. *Let X_1 and X_2 be two standard Borel spaces, and let also E_1 be an equivalence relation defined on X_1 and E_2 be an equivalence relation defined on X_2. We say that E_1 is Borel reducible to E_2, denoted as $E_1 \leqslant_B E_2$, when there is a Borel map $\mathbf{B} : X_1 \to X_2$ such that for every $x, y \in X_1$ we have:*

$$xE_1y \ \Leftrightarrow \mathbf{B}(x)E_2\mathbf{B}(y).$$

Remark 1.5. *Notice that in the context of Definitions 1.2 and 1.4, $E_1 \leqslant_R E_2$ and $E_1 \leqslant_B E_2$ have two different meaning, as in the first case the witnessing Borel function has domain $X \times X$, while in the second case it has domain X. Furthermore, notice that $E_1 \leqslant_B E_2$ implies $E_1 \leqslant_R E_2$ (but the converse need not hold, see 1.7).*

We now define *Borel completeness*, the notion at the heart of our paper.

Definition 1.6. *Let* K_1 *be a Borel class of structures with domain* ω *and let* $\cong_1$ *be the isomorphism relation on* K_1. *We say that* K_1 *is* Borel complete *(or, in more modern terminology,* $\cong_1$ *is* S_∞*-complete) if for every Borel class* K_2 *of structures with domain* ω *there is a Borel map* $\mathbf{B} : K_2 \to K_1$ *such that for every* $A, B \in K_2$*:*

$$A \cong B \ \Leftrightarrow \mathbf{B}(A) \cong_1 \mathbf{B}(B),$$

that is, the isomorphism relation on the space K_2 *is Borel reducible (in the sense of Definition 1.4) to the the isomorphism relation on the space* K_1.

The following fact will be relevant for our subsequent historical account.

Fact 1.7 ([3])**.** *Let* K *be a Borel class of structures with domain* ω. *If* K *is Borel complete, then its isomorphism relation is a complete analytic subset of* $K \times K$, *but the converse need not hold, as for example abelian* p*-groups with domain* ω *have complete analytic isomorphism relation but they are* not *a Borel complete space.*

We now have all the ingredients necessary to be able to understand the problems that we solve in this paper and to introduce the state of the art concerning them. But first a useful piece of notation which we will use throughout the paper.

Notation 1.8. *(1) We denote by* Graph *the class of graphs.*
(2) We denote by Gp *the class of groups.*
(3) We denote by AB *the class of abelian groups.*
(4) We denote by TFAB *the class of torsion-free abelian groups.*
(5) Given a class K *we denote by* K_ω *the set of structures in* K *with domain* ω.

Convention 1.9. *To simplify statements, we use the following convention: when we say that a class* K *of countable structures is Borel complete we mean that* K_ω *is Borel complete. Similarly, when we say that a class* K *of countable groups is complete co-analytic we mean that* K_ω *is a complete co-analytic subset of* Gp_ω. *Finally, when we say that the isomorphism relation on a class of countable groups is analytic, we mean that restriction of the isomorphism relation on* K *to* $K_\omega \times K_\omega$ *is an analytic subset of the Borel space* $\mathrm{Gp}_\omega \times \mathrm{Gp}_\omega$ *(as a product space).*

In [3], together with the general notions just defined, the authors studied some Borel complexity problems for specific classes of countable structures of interest. Among other things they showed (we mention only the results relevant to us):

(i) countable graphs, linear orders and trees are Borel complete;
(ii) torsion abelian groups have complete analytic $\cong$ but are *not* Borel complete;
(iii) nilpotent groups of class 2 and exponent p (p a prime) are Borel complete[1];
(iv) the isomorphism relation on finite rank torsion-free abelian groups is Borel.

In [3] Friedman and Stanley state explicitly:

> There is, alas, a missing piece to the puzzle, namely our conjecture that torsion-free abelian groups are complete. [...] We have not even been able to show that the isomorphism relation on torsion-free abelian groups is complete analytic, nor, in another direction, that the class of all abelian groups is Borel complete. We consider these problems to be among the most important in the subject.

[1]As already mentioned in [3], this result is actually a straightforward adaptation of a model theoretic construction due to Mekler [9].

The challenge was taken by several mathematicians. The first to work on this problem was Hjorth, which in [5] proved that any Borel isomorphism relation is Borel reducible (in the sense of Definition 1.4) to the isomorphism relation on countable torsion-free abelian groups, and that in particular the isomorphism relation on TFAB_ω is not Borel (as there is no such Borel equivalence relation), leaving though open the question whether TFAB_ω is a Borel complete class, or even whether the isomorphism relation on TFAB_ω is complete analytic (cf. Def. 1.3 and Fact 1.7).

The problem resisted further attempts of the time and the interest moved to another very interesting problem on torsion-free abelian groups: for $1 \leqslant n < m < \omega$, is the isomorphism relation $\cong_n$ on torsion-free abelian groups of rank n strictly less complex (in the sense of Definition 1.4) than the isomorphism relation on torsion-free abelian groups of rank m? As mentioned above, the isomorphism relation on torsion-free abelian groups of finite rank is Borel while, as just mentioned, the isomorphim relation on countable torsion-free abelian groups is not, and so the two problems are quite different, but obviously related. Also this problem proved to be very challenging, until Thomas finally gave a positive solution to the problem, in a series of two fundamental papers [15, 16], proving in particular that, for every $n < \omega$, $\cong_n$ is not universal among countable Borel equivalence relations.

The fundamental work of Thomas thus resolved completely the case of torsion-free abelian groups of finite rank, leaving open the problem for countable torsion-free abelian groups of arbitrary rank, i.e. the problem referred to as "among the most important in the subject" in [3]. The problem remained "dormant" for various years (at the best of our knowledge), until Downey and Montalbán [2] made some important progress showing that the isomorphism relation on countable torsion-free abelian groups is complete analytic, a necessary but not sufficient condition for Borel completeness, as recalled in Fact 1.7. This was of course possible evidence that the isomorphism relation was indeed Borel complete, as conjectured in [3]. Despite this advancement, the problem of Borel completeness of countable torsion-free abelian groups resisted for other 12 years, until this day, when we prove:

Main Theorem. *The space* TFAB_ω *is Borel complete, in fact there exists a continous map* $\mathbf{B} : \mathrm{Graph}_\omega \to \mathrm{TFAB}_\omega$ *such that for every* $H_1, H_2, \in \mathrm{Graph}_\omega$*:*

$$H_1 \cong H_2 \text{ if and only if } \mathbf{B}(H_1) \cong \mathbf{B}(H_2).$$

The techniques employed in the proof of our Main Theorem lead us to (and at the same time were inspired by) classification questions of "rigid" countable abelian groups. One of the most important notions of rigidity in abelian group theory is the notion of endorigidity, where we say that $G \in \mathrm{AB}$ is endorigid if the only endomorphisms of G are multiplication by an integer. The analysis of endorigid abelian groups is an old topic in abelian group theory, famous in this respect is the result of the second author [14] that for every infinite cardinal λ there is an endorigid torsion-free abelian group of cardinality λ. We prove in Thereom 1.10 below that the classification of the countable endorigid TFAB is an highly untractable problem.

Theorem 1.10. *The set of endorigid torsion-free abelian groups is a complete co-analytic subset of the Borel space space* TFAB_ω*. In fact, more strongly, there is a Borel function* $\mathbf{F}$ *from the set of tree with domain* ω *into* TFAB_ω *such that:*

(i) if T *is well-founded, then* $\mathbf{F}(T)$ *is endorigid;*
(ii) if T *is not well-founded, then* $\mathbf{F}(T)$ *has a* 1*-to-*1 $f \in \mathrm{End}(G)$ *which is not multiplication by an integer and such that* $G/f[G]$ *is not torsion.*

In a work in preparation [11] we extend the ideas behind Theorem 1.10 to a systematic investigation of several classification problems for various rigidity conditions on abelian and nilpotent groups from the perspective of descriptive set theory of countable structures. In another direction, in [12] we study the question of existence (and absolute existence) of uncountable (co-)Hopfian abelian groups.

We conclude with a few words on the history of this article. At the end of the refereeing process, the referee indicated some points which needed correction in the original version of this paper. Around the same time, Laskowski and Ulrich indicated another point which needed correction in our original submission. The referee also asked to change the presentation of our Main Theorem and to simply its proof, in particular separating the algebra from the combinatorics (division which is reflected by the current division in Sections 3 and 4). Here all the points raised there are addressed. We thank the anonymous referee, Laskowski and Ulrich. Meanwhile, Laskowski and Ulrich have found another proof of our Main Theorem, see [8].

2. Notations and Preliminaries

For the readers of various backgrounds we try to make the paper self-contained.

2.1. General notations

Definition 2.1. *(1) Given a set* X *we write* $Y \subseteq_\omega X$ *for* $\emptyset \neq Y \subseteq X$ *and* $|Y| < \aleph_0$.

(2) Given a set X *and* $\bar{x}, \bar{y} \in X^{<\omega}$ *we write* $\bar{y} \triangleleft \bar{x}$ *to mean that* $\lg(\bar{y}) < \lg(\bar{x})$ *and* $\bar{x} \restriction \lg(\bar{y}) = \bar{y}$, *where* $\bar{x}$ *is naturally considered as a function* $\lg(\bar{x}) \to X$.

(3) Given a partial function $f : M \to M$, *we denote by* $\mathrm{dom}(f)$ *and* $\mathrm{ran}(f)$ *the domain and the range of* f, *respectively.*

(4) For $\bar{a} \in B^n$ *we write* $\bar{x} \subseteq B$ *to mean that* $\mathrm{ran}(\bar{x}) \subseteq B$, *where, as usual,* $\bar{a}$ *is considered as a function* $\{0, ..., n-1\} \to B$.

(5) Given a sequence $\bar{f} = (f_i : i \in I)$ *we write* $f \in \bar{f}$ *to mean that there exists* $j \in I$ *such that* $f = f_j$.

2.2. Groups

Notation 2.2. *Let* G *and* H *be groups.*

(1) $H \leqslant G$ *means that* H *is a subgroup of* G.

(2) We let $G^+ = G \setminus \{e_G\}$, *where* e_G *is the neutral element of* G.

(3) If G *is abelian we might denote the neutral element* e_G *simply as* $0_G = 0$.

Definition 2.3. *Let* $H \leqslant G$ *be groups, we say that* H *is pure in* G, *denoted by* $H \leqslant_* G$, *when if* $h \in H$, $0 < n < \omega$, $g \in G$ *and (in additive notation)* $G \models ng = h$, *then there is* $h' \in H$ *s.t.* $H \models nh' = h$. *Given* $S \subseteq G$ *we denote by* $\langle S \rangle^*_S$ *the pure subgroup generated by* S *(the intersection of all the pure subgroups of* G *containing* S*).*

Observation 2.4. $H \leqslant_* G \in \mathrm{TFAB}$, $h \in H$, $0 < n < \omega$, $G \models ng = h \Rightarrow g \in H$.

Observation 2.5. *Let* $G \in \mathrm{TFAB}$, p *a prime and let:*

$$G_p = \{a \in G : a \text{ is divisible by } p^m, \text{ for every } 0 < m < \omega\},$$

then G_p *is a pure subgroup of* G.

Proof. This is well-known, see e.g. the discussion in [4, pg. 386-387]. ∎

Definition 2.6. *Let* p *be a prime.*

(1) We let $\mathbb{Q}_p = \{\frac{m_1}{m_2} : m_1 \in \mathbb{Z}, m_2 \in \mathbb{Z}^+, p$ *and* m_2 *are coprime*$\}$.

(2) We let $\mathbb{Q}_p^{\odot} = \{q \in \mathbb{Q}^+ : q \in \mathbb{Q}_p, \frac{q}{p} \notin \mathbb{Q}_p\}$, *so* $\mathbb{Q}_p^{\odot} \cap \mathbb{Z} = \{q \in \mathbb{Z}^+ : p \not| q\}$.

2.3. Trees

Definition 2.7. *Given an L-structure M by a partial automorphism of M we mean a partial function* $f : M \to M$ *such that* $f : \langle \mathrm{dom}(f)\rangle_M \cong \langle \mathrm{ran}(f)\rangle_M$.

In Section 5 we shall use the following notions.

Definition 2.8. *Let* $(T, <_T)$ *be a strict partial order.*

(1) $(T, <_T)$ *is a* tree *when, for all* $t \in T$, $\{s \in T : s <_T t\}$ *is well-ordered by the relation* $<_T$. *Notice that according to our definition a tree* $(T, <_T)$ *might have more than one root, i.e. more than one* $<_T$*-minimal element. We say that the tree* $(T, <_T)$ *is rooted when it has only one* $<_T$*-minimal element (its root).*
(2) *A* branch *of the tree* $(T, <_T)$ *is a maximal chain of the partial order* $(T, <_T)$.
(3) *A tree* $(T, <_T)$ *is said to be* well-founded *if it has only finite branches.*
(4) *Given a tree* $(T, <_T)$ *and* $t \in T$ *we let the level of t in* $(T, <_T)$, *denoted as* $\mathrm{lev}(t)$, *to be the order type of* $\{s \in T : s <_T t\}$ *(recall item (1)).*

Remark 2.9. *Concerning Def. 2.8(4), we will only consider trees* $(T, <_T)$ *such that, for every* $t \in T$, $\{s \in T : s <_T t\}$ *is finite, so for us* $\mathrm{lev}(t) \in \omega$.

3. The Combinatorial Frame

In Section 4 (where the main construction of this paper occurs) we only use 3.1-3.4, so the reader willing to do so could simply digest 3.1-3.4 and move there.

Notation 3.1. *For Z a set and* $0 < n < \omega$, *we let* $\mathrm{seq}_n(Z) = \{\bar{x} \in Z^n : \bar{x} \text{ injective}\}$.

Hypothesis 3.2. *(1)* $\mathbf{K}^{\mathrm{eq}}$ *is the class of models M in a vocabulary* $\{E_i : i < 3\}$ *such that each* E_i^M *is an equivalence relation and* E_2^M *is the equality relation.*
(2) *M is the countable homogeneous universal model in* $\mathbf{K}^{\mathrm{eq}}$.
(3) $\mathcal{G}$ *is essentially the set of finite non-empty partial automorphisms g of M but for technical reasons*[2] *it is the set of objects* $g = (\mathbf{h}_g, \iota_g)$ *where:*
   (A) (a) $\mathbf{h}_g$ *is a finite non-empty partial automorphism of M;*
      (b) $\iota_g \in \{0, 1\}$;
   (B) *for* $g \in \mathcal{G}$ *we let:*
      (a) $g^{-1} = (\mathbf{h}_g^{-1}, 1 - \iota_g)$;
      (b) *for* $a \in M$, $g(a) = \mathbf{h}_g(a)$;
      (c) *for* $\mathcal{U} \subseteq M$, $g[\mathcal{U}] = \{\mathbf{h}_g(a) : a \in \mathcal{U}\}$;
      (d) $g_1 \subseteq g_2$ *means* $\mathbf{h}_{g_1} \subseteq \mathbf{h}_{g_2}$ *and* $\iota_{g_1} = \iota_{g_2}$;
      (e) $g_1 \subsetneq g_2$ *means* $g_1 \subseteq g_2$ *and* $g_1 \neq g_2$;
      (f) $\mathrm{dom}(g) = \mathrm{dom}(\mathbf{h}_g)$ *and* $\mathrm{ran}(g) = \mathrm{ran}(\mathbf{h}_g)$;
      (g) *for* $\mathcal{U} \subseteq M$, $g \restriction \mathcal{U} = (\mathbf{h}_g \restriction \mathcal{U}, \iota_g)$.
(4) *For* $m < \omega$, $\mathcal{G}_*^m = \{(g_0, ..., g_{m-1}) \in \mathcal{G}^m : g_0 \subsetneq \cdots \subsetneq g_{m-1}\}$.
(5) $\mathcal{G}_* = \bigcup\{\mathcal{G}_*^m : m < \omega\}$ *(notice that the empty sequence belongs to* $\mathcal{G}_*$*).*

Notation 3.3. *(1) We use* $s, t, ...$ *to denote finite non-empty subsets of M and* $\mathcal{U}, \mathcal{V}, ...$ *to denote arbitrary subsets of M.*
(2) *For A a set, we let* $s \subseteq_1 A$ *mean* $s \subseteq A$ *and* $|s| = 1$.
(3) *For* $\bar{g} = (g_0, ..., g_{\mathrm{lg}(\bar{g})-1}) \in \mathcal{G}_*^{\mathrm{lg}(\bar{g})}$ *and* $s, t \subseteq_\omega M$, *we let:*
   (a) *for* $a, b \in M$, $\bar{g}(a) = b$ *mean that* $g_{\mathrm{lg}(\bar{g})-1}(a) = b$;

[2] The reason is that we want to force that $g \neq g^{-1}$.

(b) $\bar{g}[s] = t$ *mean that* $g_{\lg(\bar{g})-1}[s] = t$;
(c) $\mathrm{dom}(\bar{g}) = \mathrm{dom}(g_{\lg(\bar{g})-1})$, *and* $\emptyset$ *if* $\lg(\bar{g}) = 0$;
(d) $\mathrm{ran}(\bar{g}) = \mathrm{ran}(g_{\lg(\bar{g})-1})$, *and* $\emptyset$ *if* $\lg(\bar{g}) = 0$;
(e) $\bar{g}^{-1} = (g_i^{-1} : i < \lg(\bar{g}))$;
(f) $\bar{g}((x_\ell : \ell < n)) = (\bar{g}(x_\ell) : \ell < n)$.

Definition 3.4. *In the context of Hyp. 3.2, let* $\mathrm{K}_2^{\mathrm{bo}}(M)$ *be the class of objects (called* systems*)* $\mathfrak{m}(M) = \mathfrak{m} = (X^{\mathfrak{m}}, \bar{X}^{\mathfrak{m}}, \bar{f}^{\mathfrak{m}}, \bar{E}^{\mathfrak{m}}) = (X, \bar{X}, \bar{f}, \bar{E})$ *such that:*

(1) X *is an infinite countable set and* $X \subseteq \omega$;
(2) *(a)* $(X'_s : s \subseteq_1 M)$ *is a partition of* X *into infinite sets;*
(b) *for* $s \subseteq_\omega M$, *let* $X_s = \bigcup_{t \subseteq_1 s} X'_t$;
(c) $\bar{X} = (X_s : s \subseteq_\omega M)$ *and so* $s \subseteq t \subseteq_\omega M$ *implies* $X_s \subseteq X_t$;
(3) *for* $\mathcal{U} \subseteq M$ *let* $X_{\mathcal{U}} = \bigcup\{X_s : s \subseteq_1 \mathcal{U}\}$ *and so* $X = X_M = \bigcup\{X_s : s \subseteq_1 M\}$;
(4) $\bar{f} = (f_{\bar{g}} : \bar{g} \in \mathcal{G}_*)$ *(recall the definition of* $\mathcal{G}_*$ *from 3.2(5)) and:*
(a) $f_{\bar{g}}$ *is a finite partial bijection of* X *and* $f_{\bar{g}}$ *is the empty function iff* $\lg(\bar{g}) = 0$;
(b) $\mathrm{dom}(f_{\bar{g}}) \subseteq X_{\mathrm{dom}(\bar{g})}$ *and* $\mathrm{ran}(f_{\bar{g}}) \subseteq X_{\mathrm{ran}(\bar{g})}$ *(cf. 3.3(3c)(3d)), so* $\mathrm{dom}(f_{()}) = \emptyset$;
(c) *for* $s, t \subseteq_1 M$ *and* $\bar{g}[s] = t$ *we have:*

$$f_{\bar{g}}(x) = y \text{ implies } (x \in X'_s \text{ iff } y \in X'_t).$$

(d) *for* $s, t \subseteq_1 M$, $(f_{\bar{g}}(x) = y, x \in X'_s, y \in X'_t)$ *implies* $(\bar{g}[s] = t)$;
(e) $f_{\bar{g}^{-1}} = f_{\bar{g}}^{-1}$ *(recall that* $\bar{g}^{-1} \neq \bar{g}$, *when* $\mathrm{dom}(\bar{g}) \neq \emptyset$*)*;
(5) $\bar{g}, \bar{g}' \in \mathcal{G}_*$, $\bar{g} \lhd \bar{g}' \Rightarrow f_{\bar{g}} \subsetneq f_{\bar{g}'}$;
(6) *we define the graph* $(\mathrm{seq}_n(X), R_n^{\mathfrak{m}})$ *as* $(\bar{x}, \bar{y}) \in R_n^{\mathfrak{m}} = R_n$ *when* $\bar{x} \neq \bar{y}$ *and:*

$$\text{for some } \bar{g} \in \mathcal{G}_* \text{ we have } f_{\bar{g}}(\bar{x}) = \bar{y},$$

notice that $f_{\bar{g}}^{-1} = f_{\bar{g}^{-1}} \in \bar{f}$, *as* $\bar{g} \in \mathcal{G}_*$ *implies* $\bar{g}^{-1} \in \mathcal{G}_*$;
(7) $\bar{E}^{\mathfrak{m}} = \bar{E} = (E_n : 0 < n < \omega) = (E_n^{\mathfrak{m}} : 0 < n < \omega)$, *and, for* $0 < n < \omega$, E_n *is the equivalence relation corresponding to the partition of* $\mathrm{seq}_n(X)$ *given by the connected components of the graph* $(\mathrm{seq}_n(X), R_n)$;
(8) *if* p *is a prime (recalling* $\mathbb{Q}_p, \mathbb{Q}_p^{\odot}$ *from 2.6),* $k \geqslant 2$, $\bar{x} \in \mathrm{seq}_k(X)$, $\mathbf{y} = (\bar{y}^i : i < i_*) \in (\bar{x}/E_k^{\mathfrak{m}})^{i_*}$, *with the* $\bar{y}^i$*'s pairwise distinct,* $\bar{r} \in \mathbb{Q}^{\mathbf{y}}$, $q_\ell \in \mathbb{Q}_p^{\odot}$, *for* $\ell < k$, *and:*

$$a_{(\mathbf{y},\bar{r})}(y) = a_{(\mathbf{y},\bar{r},y)} = \sum\{r_{\bar{y}} q_\ell : \ell < k, \bar{y} = \bar{y}^i, i < i_*, y = y^i_\ell\},$$

for $y \in \mathrm{set}(\mathbf{y}) = \bigcup\{\mathrm{ran}(\bar{y}^i) : i < i_*\}$, *then* $|\{y \in \mathrm{set}(\mathbf{y}) : a_{(\mathbf{y},\bar{r})}(y) \notin \mathbb{Q}_p\}| \neq 1$;
(9) *if for every* $n < \omega$, $g_n \in \mathcal{G}$ *and* $g_n \subsetneq g_{n+1}$, $\mathcal{U} = \bigcup_{n<\omega} \mathrm{dom}(g_n) \subseteq M$ *and* $\mathcal{V} = \bigcup_{n<\omega} \mathrm{ran}(g_n) \subseteq M$, *then we have the following:*

$$\bigcup_{n<\omega} \mathrm{dom}(f_{(g_\ell : \ell < n)}) = X_{\mathcal{U}} \text{ and } \bigcup_{n<\omega} \mathrm{ran}(f_{(g_\ell : \ell < n)}) = X_{\mathcal{V}}.$$

The rest of this section has as its sole purpose to show that an object as in Definition 3.4 exists. The reader who prefers not to check this might skip the rest of this section and move directly to Section 4, where Definition 3.4 is used to give the group theoretic construction which leads to the Main Theorem.

Definition 3.5. *In the context of Hyp. 3.2, let* $\mathrm{K}_1^{\mathrm{bo}}(M)$ *be the class of objects* $\mathfrak{m}(M) = \mathfrak{m} = (X^{\mathfrak{m}}, \bar{X}^{\mathfrak{m}}, I^{\mathfrak{m}}, \bar{I}^{\mathfrak{m}}, \bar{f}^{\mathfrak{m}}, \bar{E}^{\mathfrak{m}}, Y_{\mathfrak{m}}) = (X, \bar{X}, I, \bar{I}, \bar{f}, \bar{E}, Y)$ *s.t.:*

(1) X *is an infinite countable set and* $X \subseteq \omega$;
(2) *(a)* $(X'_s : s \subseteq_1 M)$ *is a partition of* X *into infinite sets;*
(b) *for* $s \subseteq_\omega M$, *let* $X_s = \bigcup_{t \subseteq_1 s} X'_t$;

(c) $\bar{X} = (X_s : s \subseteq_\omega M)$ *and so* $s \subseteq t \subseteq_\omega M$ *implies* $X_s \subseteq X_t$;

(3) *for* $\mathcal{U} \subseteq M$ *let* $X_{\mathcal{U}} = \bigcup\{X_s : s \subseteq_1 \mathcal{U}\}$ *and so* $X = X_M = \bigcup\{X_s : s \subseteq_1 M\}$;

(4) *(a)* $\bar{I} = (I_n : n < \omega) = (I^{\mathfrak{m}}_n : n < \omega)$ *are pairwise disjoint;*

(b) $\bar{g} \in I_n$ *implies* $\bar{g} \in \mathcal{G}^m_*$ *for some* $m \leqslant n$;

(c) I_n *is finite;*

(5) *if* $\bar{g}' \triangleleft \bar{g} \in I_n$, *then* $\bar{g}' \in I_{<n} := \bigcup_{\ell<n} I_\ell$;

(6) $I = I^{\mathfrak{m}} = \bigcup_{n<\omega} I_n$;

(7) $\bar{f} = (f_{\bar{g}} : \bar{g} \in I)$ *and:*

(a) $f_{\bar{g}}$ *is a finite partial bijection of* X *and* $f_{\bar{g}}$ *is the empty function iff* $\mathrm{lg}(\bar{g}) = 0$;

(b) $\mathrm{dom}(f_{\bar{g}}) \subseteq X_{\mathrm{dom}(\bar{g})}$ *and* $\mathrm{ran}(f_{\bar{g}}) \subseteq X_{\mathrm{ran}(\bar{g})}$ *(cf. Notation 3.3(3c)(3d));*

(c) *for* $s, t \subseteq_1 M$ *and* $\bar{g}[s] = t$ *we have:*

$$f_{\bar{g}}(x) = y \text{ implies } (x \in X'_s \text{ iff } y \in X'_t).$$

(d) *for* $s, t \subseteq_1 M$, $(f_{\bar{g}}(x) = y, x \in X'_s, y \in X'_t)$ *implies* $(\bar{g}[s] = t)$;

(e) *if* $\bar{g} \in I_n$, *then* $\bar{g}^{-1} \in I_n$ *and* $f_{\bar{g}^{-1}} = f^{-1}_{\bar{g}}$;

(8) $\bar{g} \triangleleft \bar{g}' \Rightarrow f_{\bar{g}} \subsetneq f_{\bar{g}'}$;

(9) *we define the graph* $(\mathrm{seq}_n(X), R^{\mathfrak{m}}_n)$ *as* $(\bar{x}, \bar{y}) \in R^{\mathfrak{m}}_n = R_n$ *when* $\bar{x} \neq \bar{y}$ *and:*

$$\text{for some } \bar{g} \in \mathcal{G}_* \text{ we have } f_{\bar{g}}(\bar{x}) = \bar{y},$$

notice that $f^{-1}_{\bar{g}} = f_{\bar{g}^{-1}} \in \bar{f}$, *as* $\bar{g} \in I$ *implies* $\bar{g}^{-1} \in I$;

(10) *(a)* $\bar{E}^{\mathfrak{m}} = \bar{E} = (E_n : n < \omega) = (E^{\mathfrak{m}}_n : n < \omega)$, *and, for* $n < \omega$, E_n *is the equivalence relation corresponding to the partition of* $\mathrm{seq}_n(X)$ *given by the connected components of the graph* $(\mathrm{seq}_n(X), R_n)$;

(b) $Y = Y_{\mathfrak{m}}$ *is a non-empty subset of* X *which include* $\{x \in X :$ *for some* $\bar{g} \in I$, $x \in \mathrm{dom}(f_{\bar{g}})\}$ *and* $\mathrm{seq}_k(\mathfrak{m}) = \{\bar{x} \in \mathrm{seq}_k(X) :$ *for some* $\bar{g} \in I$, $\bar{x} \subseteq \mathrm{dom}(f_{\bar{g}})\}$, *notice* $\mathrm{seq}_k(\mathfrak{m}) \subseteq \mathrm{seq}_k(Y_{\mathfrak{m}})$ *but the converse need not hold;*

(11) *if* p *is a prime,* $k \geqslant 2$, $\bar{x} \in \mathrm{seq}_k(X)$, $\bar{q} \in (\mathbb{Q}^{\odot}_p)^k$, $\mathfrak{s} = (p, k, \bar{x}, \bar{q})$ *and* $\bar{a} \in \mathcal{A}_{\mathfrak{s}}$, *then* $\mathrm{supp}_p(\bar{a})$ *is not a singleton,* <u>*where*</u> *we define* $\mathcal{A}_{\mathfrak{s}}$, $\mathcal{A}_{\mathfrak{m}}$ *and* $\mathrm{supp}_p(\bar{a})$ *as follows:*

(a) $\mathcal{A}_{\mathfrak{s}} \subseteq \mathcal{A}_{\mathfrak{m}} = \{(a_y : y \in Z) : Z \subseteq_\omega Y_{\mathfrak{m}}$ *and* $a_y \in \mathbb{Q}\}$;

(b) *if* $\bar{a} \in \mathcal{A}_{\mathfrak{m}}$, *then we let* $\mathrm{supp}_p(\bar{a}) = \{y \in \mathrm{dom}(\bar{a}) : a_y \notin \mathbb{Q}_p\}$ *and we let:*

$$\bar{a} \approx \bar{b} \;\Leftrightarrow\; \bar{a} \restriction \mathrm{supp}_p(\bar{a}) = \bar{b} \restriction \mathrm{supp}_p(\bar{a});$$

(c) *if* $\mathbf{y} = (\bar{y}^i : i < i_*) \in (\bar{x}/E^{\mathfrak{m}}_k)^{i_*}$ *(but abusing notation we may treat* $\mathbf{y}$ *as a set), with the* $\bar{y}^i$*'s pairwise distinct and* $\bar{r} \in \mathbb{Q}^{\mathbf{y}}$, *then* $\bar{a} \in \mathcal{A}_{\mathfrak{s}}$, *where:*

$$\bar{a} = \bar{a}_{(\mathbf{y}, \bar{r})} = (a_y : y \in \mathrm{set}(\mathbf{y})),$$

$$a_y = a_{(\mathbf{y}, \bar{r})}(y) = a_{(\mathbf{y}, \bar{r}, y)} = \sum\{r_{\bar{y}} q_\ell : \ell < k, \bar{y} = \bar{y}^i, i < i_*, y = y^i_\ell\},$$

$$\mathrm{set}(\mathbf{y}) = \bigcup\{\mathrm{ran}(\bar{y}^i) : i < i_*\};$$

(d) *if* $\mathrm{supp}_p(\bar{a}) \subseteq Z \subseteq \mathrm{dom}(\bar{a})$, *then* $\bar{a} \restriction Z \in \mathcal{A}_{\mathfrak{s}}$;

(e) *if* $\bar{a}, \bar{b} \in \mathcal{A}_{\mathfrak{s}}$, *then* $\bar{c} = \bar{a} + \bar{b} \in \mathcal{A}_{\mathfrak{s}}$, *where* $\mathrm{dom}(\bar{c}) = \mathrm{dom}(\bar{a}) \cup \mathrm{dom}(\bar{b})$ *and:*

(i) $c_y = a_y + b_y$, *if* $y \in \mathrm{dom}(\bar{a}) \cap \mathrm{dom}(\bar{b})$;

(ii) $c_y = a_y$, *if* $y \in \mathrm{dom}(\bar{a}) \setminus \mathrm{dom}(\bar{b})$;

(iii) $c_y = b_y$, *if* $y \in \mathrm{dom}(\bar{b}) \setminus \mathrm{dom}(\bar{a})$;

(f) *if* $\bar{g} \in I^{\mathfrak{m}}$, $Z_1 \subseteq_\omega \mathrm{dom}(f_{\bar{g}})$, $Z_2 = f_{\bar{g}}[Z_1]$ *and* $\bar{a} = (a_y : y \in Z_2) \in \mathcal{A}_{\mathfrak{s}}$, <u>*then*</u>

$$\bar{a}^{[f_{\bar{g}}]} = (a_{f_{\bar{g}}(y)} : y \in Z_1) \in \mathcal{A}_{\mathfrak{s}};$$

(g) $\mathcal{A}_{\mathfrak{s}}$ *is the minimal subset of* $\mathcal{A}_{\mathfrak{m}}$ *satisfying clauses (c)-(f).*

Definition 3.6. *For $\mathfrak{m} \in \mathrm{K}_1^{\mathrm{bo}}(M)$, we say that $\mathfrak{m}$ is full when in addition to (1)-(11) condition 3.4(9) is satisfied and 3.5(4) is strengthen to 3.4(4) (that is we ask $I = \mathcal{G}_*$), explicitly to (1)-(11) from 3.5 we add:*

(12) if for every $n < \omega$, $g_n \in \mathcal{G}$ and $g_n \subsetneq g_{n+1}$, $\mathcal{U} = \bigcup_{n<\omega} \mathrm{dom}(g_n) \subseteq M$ and $\mathcal{V} = \bigcup_{n<\omega} \mathrm{ran}(g_n) \subseteq M$, then we have the following:

$$\bigcup_{n<\omega} \mathrm{dom}(f_{(g_\ell\,:\,\ell<n)}) = X_{\mathcal{U}} \text{ and } \bigcup_{n<\omega} \mathrm{ran}(f_{(g_\ell\,:\,\ell<n)}) = X_{\mathcal{V}};$$

(13) $I = \bigcup_{n<\omega} I_n = \mathcal{G}_*$.

Definition 3.7. *(1) $\mathrm{K}_0^{\mathrm{bo}}(M)$ is the class of $\mathfrak{m} \in \mathrm{K}_1^{\mathrm{bo}}(M)$ such that $Y_{\mathfrak{m}}$ is finite and for some $n < \omega$ we have that for every $m \geqslant n$, $I_m = \emptyset$. In this case we let $n = n(\mathfrak{m})$ to be the minimal such $n < \omega$.*

(2) We say that $\mathfrak{n} \in \mathrm{suc}(\mathfrak{m})$ when:
- *(a) $\mathfrak{n}, \mathfrak{m} \in \mathrm{K}_0^{\mathrm{bo}}(M)$, $X^{\mathfrak{m}} = X^{\mathfrak{n}}$;*
- *(b) for $s \subseteq_1 M$, $(X'_s)^{\mathfrak{m}} = (X'_s)^{\mathfrak{n}}$;*
- *(c) for $t \subseteq_\omega M$, $(X_t)^{\mathfrak{m}} = (X_t)^{\mathfrak{n}}$ (follows);*
- *(d) $n(\mathfrak{n}) = n + 1$, where $n(\mathfrak{m}) = n$;*
- *(e) if $\ell < n(\mathfrak{m})$, then $I_\ell^{\mathfrak{m}} = I_\ell^{\mathfrak{n}}$ and $\bigwedge_{\bar{g} \in I_\ell^{\mathfrak{m}}} f_{\bar{g}}^{\mathfrak{m}} = f_{\bar{g}}^{\mathfrak{n}}$;*
- *(f) for some $\bar{g} \in \mathcal{G}_*$, $I_n^{\mathfrak{n}} = \{\bar{g}, \bar{g}^{-1}\}$, $lg(\bar{g}) \leqslant n$ and $\ell < \lg(\bar{g})$ implies:*

$$\bar{g} \restriction \ell \in \bigcup_{\ell<n} I_\ell^{\mathfrak{m}},$$

  notice that $\bar{g} \notin \bigcup_{\ell<n} I_\ell^{\mathfrak{m}}$ (by Definition 3.5(4a)) and the symmetric condition $\bar{g}^{-1} \restriction \ell \in \bigcup_{\ell<n} I_\ell^{\mathfrak{m}}$ follows from Definition 3.5(7e);
- *(g) (α) if $\bar{x} E_k^{\mathfrak{n}} \bar{y}$ and $\neg(\bar{x} E_k^{\mathfrak{m}} \bar{y})$, then $\bar{x} \notin \mathrm{seq}_k(\mathfrak{m})$ or $\bar{y} \notin \mathrm{seq}_k(\mathfrak{m})$;*
  (β) $E_k^{\mathfrak{n}} \restriction \mathrm{seq}_k(\mathfrak{m}) = E_k^{\mathfrak{m}} \restriction \mathrm{seq}_k(\mathfrak{m})$.

(3) $<_{\mathrm{suc}}$ on $\mathrm{K}_0^{\mathrm{bo}}(M)$ is the transitive closure of the relation $\mathfrak{n} \in \mathrm{suc}(\mathfrak{m})$.

Claim 3.8. *For M as in Hyp. 3.2, there exists $\mathfrak{m} \in \mathrm{K}_1^{\mathrm{bo}}(M) \neq \emptyset$ which is full.*

Proof. $(*)_1$ $\mathrm{K}_0^{\mathrm{bo}}(M) \neq \emptyset$.
[Why? Let $\mathfrak{m}$ be such that:
- (a) $|X| = \aleph_0$, and $X \subseteq \omega$;
- (b) $(X'_s : s \subseteq_1 M)$ is a partition of X into infinite sets;
- (c) for $s \subseteq_\omega M$, $X_s = \bigcup_{t \subseteq_1 s} X'_t$;
- (d) $\bar{X} = (X_s : s \subseteq_\omega M)$;
- (e) $I_0^{\mathfrak{m}} = \{()\}$, $f_{()}$ is the empty function, $\bar{f} = (f_{()})$ and $I_{1+n} = \emptyset$, for every $n < \omega$;
- (f) $Y_{\mathfrak{m}}$ is any finite non-empty subset of X.

Notice that $()$ denotes the empty sequence and under this choice of $\mathfrak{m}$, $n(\mathfrak{m}) = 1$, where we recall that the notation $n(\mathfrak{m})$ was introduced in Definition 3.7(1). Notice also that 3.5(11) is easy to verify for $\mathfrak{m}$ as above, as $\bar{x}/E_k^{\mathfrak{m}}$ is always a singleton.]

$(*)_2$ If $\mathfrak{m} \in \mathrm{K}_0^{\mathrm{bo}}(M)$, $n = n(\mathfrak{m}) > 0$, $\bar{g} = (g_0, ..., g_{m-1}) \in I^{\mathfrak{m}}$ (so $n > m$) and:
- (i) $g \in \mathcal{G}$;
- (ii) for every $\ell < m$, $g_\ell \subsetneq g$;
- (iii) $\bar{g}^\frown(g) \notin I^{\mathfrak{m}}$;

<u>then</u> there is $\mathfrak{n} \in \mathrm{K}_0^{\mathrm{bo}}(M)$ such that:
- (a) $\mathfrak{n} \in \mathrm{suc}(\mathfrak{m})$;
- (b) $\bar{g}^\frown(g) \in I_n^{\mathfrak{n}}$;

(c) if $s \subseteq_1 s^+ = \mathrm{dom}(g) \cup \mathrm{ran}(g)$, then $Y_{\mathfrak{n}}$ contains $\min(X'_s \setminus Y_{\mathfrak{m}})$;
(d) $\mathrm{dom}(f^{\mathfrak{n}}_{\bar{g}^\frown(g)}) = Y_{\mathfrak{m}} \cap X_{\mathrm{dom}(g)}$;
(e) so $n(\mathfrak{n}) = n(\mathfrak{m}) + 1$.

We prove $(*)_2$, where we let $f^{\mathfrak{m}}_{\bar{g}} = f_{\bar{g}}$.

$(*)_{2.1}$ Let $s_* = \mathrm{dom}(g) \subseteq_\omega M$, hence $\mathrm{dom}(\bar{g}) \subsetneq s_*$ and let $u_* = Y_{\mathfrak{m}} \cap X_{s_*}$.

$(*)_{2.2}$ Let f_* be a finite permutation of X satisfying the following:
(a) f_* obeys 3.5(7a)-(7d) and $\mathrm{dom}(f_*) = u_*$;
(b) f_* extends $f_{\bar{g}}$;
(c) $\mathrm{dom}(f_*) \cap \mathrm{ran}(f_*) = \mathrm{ran}(f_{\bar{g}})$;
(d) if $x \in \mathrm{dom}(f_*) \setminus \mathrm{dom}(f_{\bar{g}})$ then $f_*(x) \notin Y_{\mathfrak{m}}$ (so in part. $f_*(x) \notin \mathrm{dom}(f_*)$).

We now define $\mathfrak{n}$, as required in $(*)_2$.

$(*)_{2.3}$ (A) (a) $X^{\mathfrak{n}} = X^{\mathfrak{m}}$ and $\bar{X}^{\mathfrak{n}} = \bar{X}^{\mathfrak{m}}$;
(b) $I^{\mathfrak{n}}_n = \{\bar{g}^\frown(g), (\bar{g}^{-1})^\frown(g^{-1})\}$;
(c) $I^{\mathfrak{n}} = I^{\mathfrak{m}} \cup I^{\mathfrak{n}}_n$;
(d) $I^{\mathfrak{n}}_\ell = I^{\mathfrak{m}}_\ell$, for $\ell \neq n$;
(e) $f^{\mathfrak{n}}_{\bar{h}} = f^{\mathfrak{m}}_{\bar{h}}$, for $\bar{h} \in I^{\mathfrak{m}}$.

(B) (a) $n(\mathfrak{n}) = n + 1$;
(b) $f^{\mathfrak{n}}_{\bar{g}^\frown(g)} = f_*$, $f^{\mathfrak{n}}_{(\bar{g}^{-1})^\frown(g^{-1})} = f_*^{-1}$;
(c) $Y_{\mathfrak{n}} = Z \cup Z^+$, where (noticing $f_*[Y_{\mathfrak{m}}] = \mathrm{ran}(f_*)$):
($\cdot_1$) $Z = Y_{\mathfrak{m}} \cup f_*[Y_{\mathfrak{m}}]$;
($\cdot_2$) $Z^+ = \{\min(X'_s \setminus Y_{\mathfrak{m}}) : s \subseteq_1 s^+\} \setminus Z$, recalling $(*_2)$(c).

$(*)_{2.3.1}$ $R^{\mathfrak{n}}_k$ and $E^{\mathfrak{n}}_k$ are defined from the information in $(*)_{2.3}$.

Comparing $(\mathrm{seq}_k(X), R^{\mathfrak{n}}_k)$ and $(\mathrm{seq}_k(X), R^{\mathfrak{m}}_k)$ the set of new edges are:

$$\{(\bar{x}, \bar{y}) : (\bar{x}, \bar{y}) \in Z^k_1 \cup Z^k_{-1}\},$$

where we let:

$(*)_{2.4}$

$$Z^k_1 = \{(\bar{x}, \bar{y}) : \bar{x} \in \mathrm{seq}_k(\mathrm{dom}(f_*)), f_*(\bar{x}) = \bar{y}, \bar{x} \notin \mathrm{seq}_k(\mathrm{dom}(f_{\bar{g}}))\},$$

$$Z^k_{-1} = \{(\bar{x}, \bar{y}) : (\bar{y}, \bar{x}) \in Z^k_1\},$$

Notice that possibly $\bar{x} \subseteq \mathrm{dom}(f_*) \wedge \bar{x} \notin \mathrm{seq}_k(\mathfrak{m})$, and possibly $\bar{x} \subseteq \mathrm{dom}(f_*) \wedge \bar{x} \not\subseteq \mathrm{dom}(f^{\mathfrak{m}}_{\bar{g}}) \wedge \bar{x} \in \mathrm{seq}_k(\mathfrak{m})$ (as witnessed by some $\bar{g}' \in I^{\mathfrak{m}}_{<n}$), anyhow the union $Z^k_1 \cup Z^k_{-1}$ is disjoint. Notice:

$(*)_{2.4.1}$ if $\bar{x} \in \mathrm{seq}_k(u_*)$ and $\bar{y} = f_*(\bar{x})$, then:

$$\bar{x} \subseteq \mathrm{dom}(f_{\bar{g}}) \Leftrightarrow \bar{y} \subseteq \mathrm{ran}(f_{\bar{g}}) \Rightarrow (\bar{x} \in \mathrm{seq}_k(\mathfrak{m}) \wedge \bar{y} \in \mathrm{seq}_k(\mathfrak{m})).$$

Now, we have:

$(*)_{2.5}$ (a) if $(\bar{x}, \bar{y}) \in Z^k_1$, then:
(α) $\bar{x} \in \mathrm{seq}_k(u_*)$ and $\bar{x} \not\subseteq \mathrm{dom}(f_{\bar{g}})$;
(β) $\bar{y} \subseteq f_*(u_*)$, $\bar{y} \not\subseteq Y_{\mathfrak{m}}$, $\bar{y} \not\subseteq \mathrm{ran}(f_{\bar{g}})$ and $\bar{y} \cap Y_{\mathfrak{m}} \subseteq \mathrm{ran}(f_{\bar{g}})$;
(b) the dual of item (a) for $(\bar{x}, \bar{y}) \in Z^k_{-1}$;
(c) if $\bar{z} \in \mathrm{seq}_k(\mathfrak{n}) \setminus \mathrm{seq}_k(\mathfrak{m})$, then $\bar{z}$ occurs in exactly one edge of $R^{\mathfrak{n}}_k$.

Notice now that:

$(*)_{2.6}$ in the graph $(\mathrm{seq}_k(X), R^{\mathfrak{n}}_k)$ we have:
(i) all the new edges have at least one node in $\mathrm{seq}_k(u_*) \setminus \mathrm{seq}_k(\mathrm{dom}(f_{\bar{g}}))$ and one in $\mathrm{seq}_k(f_*[u_*]) \setminus \mathrm{seq}_k(\mathrm{ran}(f_{\bar{g}}))$;

(ii) every node in $\mathrm{seq}_k(\mathfrak{n}) \setminus \mathrm{seq}_k(Y_\mathfrak{m})$ has valency 1;
(iii) if $\bar{x} \not\subseteq Y_\mathfrak{m}$ and $\bar{x} \not\subseteq \mathrm{ran}(f_*)$, then $\bar{x}/E^\mathfrak{n}_k = \{\bar{x}\}$;
(iv) if $\bar{x} \subseteq Y_\mathfrak{m}$ and $\bar{x} \not\subseteq \mathrm{dom}(f_*)$, then $\bar{x}/E^\mathfrak{n}_k = \bar{x}/E^\mathfrak{m}_k$;
(v) if $\bar{x} \subseteq \mathrm{dom}(f_*)$ (hence $\bar{x} \subseteq Y_\mathfrak{m}$), then:

$$\bar{x}/E^\mathfrak{n}_k = \bar{x}/E^\mathfrak{m}_k \cup \{f_*(\bar{y}) : \bar{y} \in \bar{x}/E^\mathfrak{m}_k, \bar{y} \subseteq u_*\};$$

(vi) if $\bar{x} \subseteq \mathrm{dom}(f_{\bar{g}})$, then $\bar{x}/E^\mathfrak{n}_k = f_*(\bar{x})/E^\mathfrak{m}_k$;
(vii) if $\bar{x} \not\subseteq Y_\mathfrak{m}$ but $\bar{x} \subseteq f_*(u_*)$, then $\bar{x}/E^\mathfrak{n}_k = f_*^{-1}(\bar{x})/E^\mathfrak{n}_k$;

Notice also that:

$(*)_{2.6.1}$ (a) if $\bar{x}_0, ..., \bar{x}_m$ is a path in $(\mathrm{seq}_k(\mathfrak{n}), R^\mathfrak{n}_k)$ with no repetitions and $0 < \ell < m$, then $\bar{x}_\ell \in \mathrm{seq}_k(\mathfrak{m})$;
(b) $E^\mathfrak{n}_k \restriction \mathrm{seq}_k(\mathfrak{m}) = E^\mathfrak{m}_k \restriction \mathrm{seq}_k(\mathfrak{m})$ and $E^\mathfrak{n}_k \restriction \mathrm{seq}_k(Y_\mathfrak{m}) = E^\mathfrak{m}_k \restriction \mathrm{seq}_k(Y_\mathfrak{m})$;

Now, we claim:

$(*)_{2.7}$ $\mathfrak{n} \in \mathrm{K}^{\mathrm{bo}}_0(M)$ and $\mathfrak{n} \in \mathrm{suc}(\mathfrak{m})$.

The only non-trivial thing is to verify that $\mathfrak{n}$ satisfies 3.5(11). To this extent, let $\mathfrak{s} = (p, k, \bar{x}, \bar{q})$ be as there. Now, if $\bar{x} \notin \mathrm{seq}_k(Y_\mathfrak{m})$ and $\bar{x} \notin \mathrm{seq}_k(\mathrm{ran}(f_*))$, then $\bar{x}/E^\mathfrak{n}_k$ is a singleton and so the proof is as in $(*_1)$. Thus, from now on we assume:

$(*)_{2.7.1}$ W.l.o.g. $\bar{x} \in \mathrm{seq}_k(Y_\mathfrak{m})$ or $\bar{x} \in \mathrm{seq}_k(\mathrm{ran}(f_*))$.

$(*)_{2.7.2}$ (a) W.l.o.g. $\bar{x} \in \mathrm{seq}_k(Y_\mathfrak{m})$;
(b) let $\mathfrak{s}(1) = \mathfrak{s}$ and $\bar{x}^1 = \bar{x}$;
(c) let $\mathfrak{s}(2) = (p, k, f_*(\bar{x}), \bar{q})$ and $\bar{x}^2 = f_*(\bar{x})$.

[Why (a)? If $\bar{x} \not\subseteq Y_\mathfrak{m}$, then (by $(*)_{2.7.1}$) necessarily $\bar{x} \subseteq \mathrm{ran}(f_*)$, so $f_*^{-1}(\bar{x}) \in \bar{x}/E^\mathfrak{n}_k$ and $f_*^{-1}(\bar{x}) \subseteq Y_\mathfrak{m}$ and we can replace $\bar{x}$ by $f_*^{-1}(\bar{x})$.]

$(*)_{2.7.3}$ (a) $\mathcal{A}^\mathfrak{m}_{\mathfrak{s}(1)} \subseteq \mathcal{A}^\mathfrak{n}_\mathfrak{s}$, let $\mathcal{A}_{\mathfrak{s}(1)} = \mathcal{A}^\mathfrak{m}_{\mathfrak{s}(1)}$;
(b) let $\mathcal{A}_{\mathfrak{s}(2)} = \{\bar{b}^{[f_*]} : \bar{b} \in \mathcal{A}^\mathfrak{m}_{\mathfrak{s}(1)}$ and $\mathrm{dom}(\bar{b}) \subseteq \mathrm{dom}(f_*)\}$, where for $\bar{b} = (b_y : y \in Z_1)$ with $Z_1 \subseteq \mathrm{dom}(f_*)$ and $Z_2 = f_*[Z_2]$ we have that:

$$\bar{b}^{[f_*^{-1}]} = (b_{f_*^{-1}(y)} : y \in Z_2).$$

$(*)_{2.7.4}$ Let $\mathcal{A}'$ be the set of $\bar{a}$ such that for some $\bar{a}_1 \in \mathcal{A}_{\mathfrak{s}(1)}$ and $\bar{a}_2 \in \mathcal{A}_{\mathfrak{s}(2)}$ and u such that $\mathrm{supp}_p(\bar{a}_1 + \bar{a}_2) \subseteq u \subseteq \mathrm{dom}(\bar{a}_1) \cup \mathrm{dom}(\bar{a}_2)$ we have that $(\bar{a}_1 + \bar{a}_2) \restriction u = \bar{a}$. In this case we call $(\bar{a}_1, \bar{a}_2, u)$ a witness for $\bar{a}$.

Now we crucially claim:

$(*)_{2.7.5}$ $\mathcal{A}^\mathfrak{n}_\mathfrak{s} \subseteq \mathcal{A}'$.

Why $(*)_{2.7.5}$? By 3.5(11)(g) it suffices to prove that $\mathcal{A}'$ satisfies (c)-(f) from 3.5(11).

$(*)_{2.7.5.1}$ $\mathcal{A}'$ satisfies Clause 3.5(11)(c).

Let $\mathbf{y} = (\bar{y}^i : i < i_*) \in (\bar{x}/E^\mathfrak{n}_k)^{i_*}$ and $\bar{r}$ be as there. Recall that abusing notation we treat $\mathbf{y}$ as a set. Let also $\bar{a} = \bar{a}_{(\mathbf{y}, \bar{r})}$. Let:

$$\mathbf{y}_1 = \{\bar{y}^i : i < i_*, \bar{y}^i \subseteq Y_\mathfrak{m}\}$$

$$\mathbf{y}_2 = \{\bar{y}^i : i < i_*, \bar{y}^i \not\subseteq Y_\mathfrak{m} \text{ (so } \bar{y}^i \subseteq \mathrm{ran}(f_*))\}.$$

Easily we have that $\mathbf{y}$ is the disjoint union of $\mathbf{y}_1$ and $\mathbf{y}_2$ and we have:

$$\bar{a}_{(\mathbf{y}, \bar{r})} = \bar{a}_{(\mathbf{y}_1, \bar{r} \restriction \mathrm{set}(\mathbf{y}_1))} + \bar{a}_{(\mathbf{y}_2, \bar{r} \restriction \mathrm{set}(\mathbf{y}_2))} \in \mathcal{A}',$$

provided that we show that $\bar{a}_2 = \bar{a}_{(\mathbf{y}_2, \bar{r} \restriction \mathrm{set}(\mathbf{y}_2))} \in \mathcal{A}_{\mathfrak{s}(2)}$ (as $\bar{a}_1 = \bar{a}_{(\mathbf{y}_1, \bar{r} \restriction \mathrm{set}(\mathbf{y}_1))} \in \mathcal{A}_{\mathfrak{s}(1)}$ is obvious). We do this. Let $\mathbf{y}'_2 = \{f_*^{-1}(\bar{y}) : \bar{y} \in \mathbf{y}_2\}$. Now, if $\bar{y} \in \mathbf{y}_2$, then

$f_*^{-1}(\bar{y}) = f^{-1}_{\bar{g}^\frown(g)}(\bar{y}) \in \bar{x}/E^{\mathfrak{n}}_k$. Let $\bar{r}'_2 = (r'_{(2,y)} : y \in \mathrm{set}(\mathbf{y}'_2))$, where $r'_{(2,y)} = r_{(2,f_*(y))}$. Now, $\mathbf{y}'_2, \bar{r}'_2$ witness that $\bar{a}'_2 \in \mathcal{A}_{\mathfrak{s}(1)}$ and so by the definition of $\mathcal{A}_{\mathfrak{s}(2)}$ we are done.

$(*)_{2.7.5.2}$ $\mathcal{A}'$ satisfies Clause 3.5(11)(d).

This is obvious by the definition of $\mathcal{A}'$.

$(*)_{2.7.5.3}$ $\mathcal{A}'$ satisfies Clause 3.5(11)(e).

Let $\bar{a}, \bar{b} \in \mathcal{A}'$ and let $(\bar{a}_1, \bar{a}_2, u)$ be a witness for $\bar{a}$ and $(\bar{b}_1, \bar{b}_2, v)$ be a witness for $\bar{b}$, now $(\bar{a}_1 + \bar{a}_2, \bar{b}_1 + \bar{b}_2, u \cup v)$ is a witnes for $\bar{a} + \bar{b}$. Hence, $\bar{c} = \bar{a} + \bar{b} \in \mathcal{A}'$.

$(*)_{2.7.5.4}$ $\mathcal{A}'$ satisfies Clause 3.5(11)(f).

Let $\bar{h} \in I^{\mathfrak{m}}$, $Z_1 \subseteq \mathrm{dom}(f_{\bar{h}})$, $Z_2 = f_{\bar{h}}[Z_1]$ and $\mathrm{dom}(\bar{a}) \subseteq Z_2$. We shall prove that $\bar{a}^{[f_{\bar{h}}]} \in \mathcal{A}'$, where $\bar{a} \in \mathcal{A}'$ and $(\bar{a}_1, \bar{a}_2, u)$ is a witness of this.

Case 1. $u \not\subseteq Y_{\mathfrak{m}}$ and $u \not\subseteq \mathrm{ran}(f_*)$.
In this case there is no such $\bar{h}$.

Case 2. $u \not\subseteq Y_{\mathfrak{m}}$ and $u \subseteq \mathrm{ran}(f_*)$.
Necessarily $\bar{h} = (\bar{g}^{-1})^\frown(g^{-1})$, so $f_{\bar{h}} = f_*^{-1}$. Now:

$(\cdot)$ W.l.o.g. $\mathrm{dom}(\bar{a}_1) \subseteq \mathrm{ran}(f_{\bar{g}})$;

[Why? If $y \in \mathrm{dom}(\bar{a}_1) \setminus \mathrm{ran}(f_{\bar{g}})$, then $y \notin u$ so $a_y \in \mathrm{supp}_p(\bar{a})$ and $y \notin \mathrm{dom}(\bar{a}_2)$, hence $a_y = a_{(1,y)}$. Thus, $\bar{a}_1^* = \bar{a}_1 \restriction (\mathrm{dom}(\bar{a}_1) \cap \mathrm{ran}(f_{\bar{g}})) \in \mathcal{A}_{\mathfrak{s}(1)}$ and $(\bar{a}_1 + \bar{a}_2) \restriction u = (\bar{a}_1^* + \bar{a}_2) \restriction u$, so we can replace $\bar{a}_1$ by $\bar{a}_1^*$.]

Let $\bar{a}'_1 = a_1^{[f_{\bar{g}}^{-1}]}$, this is well-defined, it belongs to $\mathcal{A}_{\mathfrak{s}(1)}$ and has domain $\subseteq \mathrm{dom}(f_*)$. Also, $\mathrm{dom}(\bar{a}_2) \subseteq \mathrm{ran}(f_*)$ and $\bar{a}_2 \in \mathcal{A}_{\mathfrak{s}(2)}$, hence $\bar{a}'_2 = a_2^{[f_*^{-1}]} \in \mathcal{A}_{\mathfrak{s}(1)}$ and has domain $\subseteq \mathrm{dom}(f_*)$. Together $\bar{a}' = \bar{a}'_1 + \bar{a}'_2 \in \mathcal{A}_{\mathfrak{s}(1)}$. Also, $\mathrm{supp}_p(\bar{a}') \subseteq f_*^{-1}[u] \subseteq \mathrm{dom}(\bar{a}'_1 + \bar{a}'_2)$, hence $\bar{a}' \restriction f_*^{-1}[u] \in \mathcal{A}_{\mathfrak{s}(1)}$. Hence:

$$\begin{aligned}
\bar{a}^{[f_{\bar{h}}]} &= \bar{a}^{[f_*^{-1}]} \\
&= ((\bar{a}_1 + \bar{a}_2) \restriction u)^{[f_*^{-1}]} \\
&= (\bar{a}_1 + \bar{a}_2)^{[f_*^{-1}]} \restriction f_*^{-1}[u] \\
&= (\bar{a}_1^{[f_*^{-1}]} + \bar{a}_2^{[f_*^{-1}]}) \restriction f_*^{-1}[u] \\
&= (\bar{a}'_1 + \bar{a}'_2) \restriction f_*^{-1}[u] \\
&= \bar{a}' \restriction f_*^{-1}[u] \in \mathcal{A}_{\mathfrak{s}(1)}.
\end{aligned}$$

Case 3. $u \subseteq Y_{\mathfrak{m}}$.
Similar to Case 2.

So indeed $\mathcal{A}^{\mathfrak{n}}_{\mathfrak{s}} \subseteq \mathcal{A}'$, by (g) of the definition of $\mathcal{A}^{\mathfrak{n}}_{\mathfrak{s}}$ in 3.5(11) and $(*)_{2.7.5.1}$-$(*)_{2.7.5.4}$.

$(*)_{2.7.6}$ If $\bar{a} \in \mathcal{A}'$, then $\mathrm{supp}_p(\bar{a})$ is not a singleton.

We prove $(*)_{2.7.6}$. Let $\bar{a} \in \mathcal{A}'$ and let $(\bar{a}_1, \bar{a}_2, u)$ be a witness of this. Now, $\bar{a} \approx \bar{a}_1 + \bar{a}_2$ so $\bar{a} \restriction \mathrm{supp}_p(\bar{a}) = (\bar{a}_1 + \bar{a}_2) \restriction \mathrm{supp}_p((\bar{a}_1 + \bar{a}_2))$. Hence:

$(*)_{2.7.6.1}$ W.l.o.g. $\bar{a} = \bar{a}_1 + \bar{a}_2$ and $\bar{a}_\ell = \bar{a}_\ell \restriction \mathrm{supp}_p(\bar{a}_\ell)$, for $\ell = 1, 2$.

Case A. $\mathrm{supp}_p(\bar{a}_1) \not\subseteq \mathrm{ran}(f_{\bar{g}})$ and $\mathrm{supp}_p(\bar{a}_2) \not\subseteq \mathrm{ran}(f_{\bar{g}})$.
As $\mathrm{supp}_p(\bar{a}_1) \not\subseteq \mathrm{ran}(f_{\bar{g}})$ we can choose $y_1 \in \mathrm{supp}_p(\bar{a}_1) \setminus \mathrm{ran}(f_{\bar{g}})$, and similarly we can choose $y_2 \in \mathrm{supp}_p(\bar{a}_2) \setminus \mathrm{ran}(f_{\bar{g}})$. Now $\mathrm{dom}(\bar{a}_1) \subseteq Y_{\mathfrak{m}}$ and $\mathrm{dom}(\bar{a}_2) \subseteq f_*[Y_{\mathfrak{m}}]$, hence $\mathrm{dom}(\bar{a}_1) \cap \mathrm{dom}(\bar{a}_2) \subseteq Y_{\mathfrak{m}} \cap f_*[Y_{\mathfrak{m}}] = \mathrm{ran}(f_{\bar{g}})$, so necessarily $y_1 \notin \mathrm{dom}(\bar{a}_2)$ and $y_2 \notin \mathrm{dom}(\bar{a}_1)$ (by the choice of y_1 and y_2). Hence, letting $\bar{a} = (a_y : y \in u)$ and recalling the definition of $\bar{a} = \bar{a}_1 + \bar{a}_2$ from 3.5(11e) we have:

$(\cdot)$ $y_1 \in \mathrm{dom}(\bar{a}_1) \setminus \mathrm{dom}(\bar{a}_2)$, so $a_{y_1} = a_{(1,y_1)}$;
$(\cdot)$ $y_2 \in \mathrm{dom}(\bar{a}_2) \setminus \mathrm{dom}(\bar{a}_1)$, so $a_{y_2} = a_{(2,y_2)}$.

But $a_{(1,y_1)}, a_{(2,y_2)} \notin \mathbb{Q}_p$ (as $y_\ell \in \mathrm{supp}_p(\bar{a}_\ell)$, for $\ell = 1,2$) and so $a_{y_1}, a_{y_2} \notin \mathbb{Q}_p$, and, as obviously $y_1 \neq y_2$, we are done. This concludes the proof of Case A.

Case B. $\mathrm{supp}_p(\bar{a}_2) \subseteq \mathrm{ran}(f_{\bar{g}})$, eq. by $(*)_{2.7.6.1}$ we have $\mathrm{dom}(\bar{a}_2) \subseteq \mathrm{ran}(f_{\bar{g}})$.

Define $\mathbf{y}_2' = \{f_*^{-1}(\bar{y}) : \bar{y} \in \mathbf{y}_2\}$, where $\mathbf{y}_2$ is as in the proof of $(*)_{2.7.5.1}$. Let now:

$$\bar{a}_2' = (a_{(2,y)}' : y \in \mathrm{set}(\mathbf{y}_2')),$$

where:

$(\cdot_a)$ if $y \in \mathrm{set}(\mathbf{y}_2')$, then $a_{(2,y)}' = a_{(2,f_*(y))}$.

Clearly we have:

$(\cdot_1)$ $\mathbf{y}_2' \subseteq \bar{x}/E_k^{\mathfrak{m}}$;
$(\cdot_2)$ $\bar{a}_2' \in \mathcal{A}_{\mathfrak{s}(1)}$;
$(\cdot_3)$ $f_*(\bar{a}_2') = \bar{a}_2$;
$(\cdot_4)$ $\mathrm{dom}(\bar{a}_2') \subseteq \mathrm{dom}(f_{\bar{g}})$;
$(\cdot_5)$ $\bar{a}_2' \in \mathcal{A}_{\mathfrak{s}(1)}$;
$(\cdot_6)$ $\bar{a}_2 \in \mathcal{A}_{\mathfrak{s}(1)}$.

[Why? $(\cdot_4)$ holds as $\mathrm{supp}(\bar{a}_2) \subseteq \mathrm{ran}(f_{\bar{g}})$. $(\cdot_5)$ holds by $(\cdot_3)$ and the fact that $\mathfrak{m} \in \mathrm{K}_1^{\mathrm{bo}}(M)$. $(\cdot_6)$ holds by $(\cdot_5)$ and 3.5(11)(f).]

Let now $\bar{a}_* = \bar{a}_1 + \bar{a}_2$, as each summand is in $\mathcal{A}_{\mathfrak{s}(1)}$, then also $\bar{a}_* \in \mathcal{A}_{\mathfrak{s}(1)}$, recalling that $\mathcal{A}_{\mathfrak{s}(1)} = \mathcal{A}_{\mathfrak{s}}^{\mathfrak{m}}$ and $\mathfrak{m}$ satisfies condition 3.5(11e). Also, clearly $\bar{a} = \bar{a}_*$, but the latter belonging to $\mathcal{A}_{\mathfrak{s}(1)}$ we have that $\mathrm{supp}_p(\bar{a})$ is not a singleton, recalling that $\mathcal{A}_{\mathfrak{s}(1)} = \mathcal{A}_{\mathfrak{s}}^{\mathfrak{m}}$ and $\mathfrak{m}$ satisfies 3.5(11).

Case C. $\mathrm{supp}_p(\bar{a}_1) \subseteq \mathrm{ran}(f_{\bar{g}})$, eq. by $(*)_{2.7.6.1}$ we have $\mathrm{dom}(\bar{a}_1) \subseteq \mathrm{ran}(f_{\bar{g}})$.

This case is similar to Case B but we elaborate. Let $\mathbf{y}_2'$, $\bar{a}_2'$ be as in Case B, hence $(\cdot_a)$ from Case B holds (notice that the assumption of Case B was not used there). Let now $Y_1 = \mathrm{supp}_p(\bar{a}_1) \subseteq \mathrm{ran}(f_{\bar{g}})$ and $Y_1' = f_{\bar{g}}^{-1}(Y_1) \subseteq \mathrm{dom}(f_{\bar{g}})$, then we let:

$(\cdot_a)$ $\bar{a}_1' = (a_{(1,y)}' : y \in Y_1')$;
$(\cdot_b)$ $a_{(1,y)}' = a_{(1,f_{\bar{g}}(y))}$;
$(\cdot_c)$ $Y_2 = \mathrm{dom}(\bar{a}_1)$, $Y_2' = f_*^{-1}[Y_2'] = f_{\bar{g}}^{-1}[Y_2']$;
$(\cdot_d)$ $\bar{a}_2' = (a_{(2,y)}' : y \in Y_2')$;
$(\cdot_e)$ $a_{(2,y)}' = a_{(2,f_*(y))}'$.

Then:

$(\cdot_1)$ $\mathbf{y}_2' \subseteq \bar{x}/E_k^{\mathfrak{m}}$;
$(\cdot_2)$ $\bar{a}_2' \in \mathcal{A}_{\mathfrak{s}(1)}$;
$(\cdot_3)$ $f_*(\bar{a}_1') = \bar{a}_1$;
$(\cdot_4)$ $\mathrm{dom}(\bar{a}_1') \subseteq \mathrm{dom}(f_{\bar{g}})$;
$(\cdot_5)$ $\bar{a}_1' \in \mathcal{A}_{\mathfrak{s}(1)}$;
$(\cdot_6)$ $f_*(\bar{a}_2') = \bar{a}_2$.

Let now $\bar{a}_* = \bar{a}_1 + \bar{a}_2$ and let $\bar{a}_*' = \bar{a}_1' + \bar{a}_2'$, so that $f_*(\bar{a}_*') = \bar{a}_*$. As $\bar{a}_1' \in \mathcal{A}_{\mathfrak{s}(1)}$ by $(\cdot_a)$ and $\bar{a}_2' \in \mathcal{A}_{\mathfrak{s}(1)}$ by $(\cdot_2)$, then by 3.5(11e), also $\bar{a}_*' = \bar{a}_1' + \bar{a}_2' \in \mathcal{A}_{\mathfrak{s}(1)}$, hence $\mathrm{supp}_p(\bar{a}_*')$ is not a singleton (as $\mathfrak{m} \in \mathrm{K}_1^{\mathrm{bo}}(M)$) and so $\mathrm{supp}_p(\bar{a}_*)$ is not a singleton.

So we finished proving $(*_{2.7.6})$, i.e., $\bar{a} \in \mathcal{A}'$ implies that $\mathrm{supp}_p(\bar{a})$ is not a singleton. Thus, we also finished proving $(*)_2$, as by $(*_{2.7.5})$ we have $\bar{a} \in \mathcal{A}_{\mathfrak{s}}^{\mathfrak{n}} \Rightarrow \bar{a} \in \mathcal{A}'$, and so by $(*_{2.7.6})$ we are done, i.e., we have verified that $\mathfrak{n}$ satisfies 3.5(11).

$(*_3)$ We can choose an $<_{\mathrm{suc}}$-increasing sequence $(\mathfrak{m}_\ell : \ell < \omega)$ in $\mathrm{K}_0^{\mathrm{bo}}(M)$ whose limit $\mathfrak{m}$ is as wanted, i.e. $\mathfrak{m} \in \mathrm{K}_2^{\mathrm{bo}}(M)$.

We show this. We can find a list $(\bar{g}^\ell : \ell < \omega)$ of $\bigcup_{m<\omega} \mathcal{G}^m_*$ such that:

$(*)_{3.1}$
(i) $\lg(\bar{g}^\ell) \leqslant \ell$;
(ii) if $\bar{g}^\ell \triangleleft \bar{g}^k$, then $\ell < k$;
(iii) $\lg(\bar{g}^\ell) = 0$ iff $\ell = 0$;
(iv) note that for $\ell < \lg(\bar{g})$, $g^k_\ell \neq (g^k_\ell)^{-1}$
(v) $\bar{g}^{2\ell+2} = (\bar{g}^{2\ell+1})^{-1}$;
(vi) if $\lg(\bar{g}^{2\ell+1}) > 1$, then there is a unique $i < \ell$ such that:
($\cdot_1$) $\bar{g}^{2i+1} \triangleleft \bar{g}^{2\ell+2}$;
($\cdot_2$) $\bar{g}^{2i+2} \triangleleft \bar{g}^{2\ell+1}$;
($\cdot_3$) $\lg(\bar{g}^{2\ell+1}) = \lg(\bar{g}^{2\ell+2}) = \lg(\bar{g}^{2i+1}) + 1 = \lg(\bar{g}^{2i+2}) + 1$.

Now, by induction on $\ell < \omega$, we choose $\mathfrak{m}_\ell \in \mathrm{K}^{\mathrm{bo}}_0$ such that $n(\mathfrak{m}_\ell) \leqslant \ell + 1$ and $\mathfrak{m}_{\ell+1} \in \mathrm{suc}(\mathfrak{m}_\ell)$ or $\mathfrak{m}_{\ell+1} = \mathfrak{m}_\ell$. We proceed as follows:

$(*_{3.2})$ ($\ell = 0$) use $(*)_1$;
($\ell = k+1$) ($\cdot_1$) if $\bar{g}^{k+1} \in I^{\mathfrak{m}_k}$, then $\mathfrak{m}_\ell = \mathfrak{m}_k$ (if this occurs, then k is odd);
($\cdot_2$) if $\bar{g}^{k+1} \notin I^{\mathfrak{m}_k}$, let $m_k = \lg(\bar{g}^{k+1}) - 1$, so $\bar{g}^{k+1} \restriction m_k \in I^{\mathfrak{m}_k}$, and use $(*)_2$ with the pair $n(\mathfrak{m}_k)$, $\bar{g}^{2k+1}$ here standing for $n, \bar{g}^\frown(g)$ there.

Clearly $\mathfrak{m} = \lim_{\ell<\omega}(\mathfrak{m}_\ell) \in \mathrm{K}^{\mathrm{bo}}_1(M)$. Notice that by $(*)_{3.1}$ we have:

$(*)_{3.3}$ if $\bar{g}^k \triangleleft \bar{g}^\ell \triangleleft \bar{g}^m$, then:
(i) $f_{\bar{g}^k} \subseteq f_{\bar{g}^\ell} \subseteq f_{\bar{g}^m}$;
(ii) $Y_{\mathfrak{m}_k} \cap X_{\mathrm{dom}(f_{\bar{g}^k})} \subseteq \mathrm{dom}(f_{\bar{g}^m})$;
(iii) $Y_{\mathfrak{m}_k} \cap X_{\mathrm{ran}(f_{\bar{g}^k})} \subseteq \mathrm{ran}(f_{\bar{g}^m})$;
(iv) if $s \subseteq_1 \mathrm{dom}(f_{\bar{g}^k})$, then $\min(X'_s \setminus Y_{\mathfrak{m}_k}) \in \mathrm{dom}(f_{\bar{g}^m})$ (see $(*)_{2.3}$(B)(c)($\cdot_1$));
(v) if $s \subseteq_1 \mathrm{ran}(f_{\bar{g}^k})$, then $\min(X'_s \setminus Y_{\mathfrak{m}_k}) \in \mathrm{ran}(f_{\bar{g}^m})$ (see $(*)_{2.3}$(B)(c)($\cdot_2$)).

Thus we are only left to show that $\mathfrak{m} \in \mathrm{K}^{\mathrm{bo}}_1(M)$ is full, that this, that $\mathfrak{m}$ satisfies conditions (12)(13) from 3.6. For this notice:

(i) Def. 3.6(12) holds by the definition of $\mathfrak{m}_{k+1} \in \mathrm{suc}_{\mathfrak{m}_k}$, recalling $(*)_{3.3}$(iv)(v);
(ii) Def. 3.6(13) holds as the $\bar{g}^\ell$'s list $\mathcal{G}_*$.

■

Corollary 3.9. $\mathrm{K}^{\mathrm{bo}}_2(M) \neq \emptyset$.

Proof. This is obvious by 3.8 simply comparing Def. 3.4 and Def. 3.5+3.6. ■

4. Borel Completeness of Torsion-Free Abelian Groups

4.1. The Definition of the Groups $G_{(1,\mathcal{U})}$

Definition 4.1. *Let $\mathrm{K}^{\mathrm{bo}}_3(M)$ be the class of $\mathfrak{m} \in \mathrm{K}^{\mathrm{bo}}_2(M)$ expanded with a sequence $\bar{p} = \bar{p}^{\mathfrak{m}}$ of prime numbers without repetitions such that we have the following:*

(1) $\bar{p} = (p_{(e,\bar{q})} : e \in \mathrm{seq}_n(X)/E^{\mathfrak{m}}_n$ for some $0 < n < \omega$ and $\bar{q} \in (\mathbb{Z}^+)^n)$;

(2) if $e \in \mathrm{seq}_n(X)/E^{\mathfrak{m}}_n$, $\bar{q} = (q_0, ..., q_{n-1}) \in (\mathbb{Z}^+)^n$ and $\ell < n$, then $p_{(e,\bar{q})} \nmid q_\ell$, so, recalling 2.6(2), we have that $q_\ell \in \mathbb{Q}^{\odot}_p$ (this is relevant for applying 3.4(8)).

Fact 4.2. *Clearly every element of $\mathfrak{m} \in \mathrm{K}^{\mathrm{bo}}_2(M)$ can be expanded to an element of $\mathfrak{m} \in \mathrm{K}^{\mathrm{bo}}_3(M)$, and, as we showed in 3.9, that $\mathrm{K}^{\mathrm{bo}}_2(M) \neq \emptyset$ we have $\mathrm{K}^{\mathrm{bo}}_3(M) \neq \emptyset$.*

Definition 4.3. *Let $\mathfrak{m} \in \mathrm{K}^{\mathrm{bo}}_3(M)$.*

(1) Let $G_2 = G_2[\mathfrak{m}]$ be $\bigoplus\{\mathbb{Q}x : x \in X\}$.

(2) Let $G_0 = G_0[\mathfrak{m}]$ be the subgroup of G_2 generated by X, i.e. $\bigoplus\{\mathbb{Z}x : x \in X\}$.

(3) Let $G_1 = G_1[\mathfrak{m}]$ be the subgroup of G_2 generated by:

(a) G_0;

(b) $p^{-m}(\sum_{\ell<n} q_\ell x_\ell)$, where:

(i) $0 < m < \omega$;

(ii) $\bar{x} = (x_\ell : \ell < n) \in \mathrm{seq}_n(X)$, $e = \bar{x}/E^{\mathfrak{m}}_n$, $n > 0$;

(iii) $\bar{q}$ is as in 4.1;

(iv) $p = p_{(e,\bar{q})}$ (so a prime, recalling Definition 4.1);

(c) [follows] for every $a \in G_1$ there are $i_* < \omega$ and, for $i < i_*$, k_i, $\bar{x}_i \in \mathrm{seq}_{k_i}(X)$, $\bar{q}_i \in (\mathbb{Z}^+)^{k(i)}$, $e_i = \bar{x}_i/E^{\mathfrak{m}}_{k_i}$, $p_i = p_{(e_i,\bar{q}_i)}$ (hence $\bar{q}_i$ is as in 4.1), $m(i) \geqslant 0$ and $r^i \in \mathbb{Z}^+$ such that the following condition holds:

$$a = \sum\{p_i^{-m(i)} r^i q_{(i,\ell)} x_{(i,\ell)} : i < i_*, \ell < k_i\}.$$

(4) For a prime p, let $G_{(1,p)} = \{a \in G_1 : a$ is divisible by p^m, for every $0 < m < \omega\}$ (notice that, by Observation 2.5, $G_{(1,p)}$ is always a pure subgroup of G_1).

(5) For $\mathcal{U} \subseteq M$, we let:

$$G_{(1,\mathcal{U})}[\mathfrak{m}] = G_{(1,\mathcal{U})}[\mathfrak{m}(M)] = G_{(1,\mathcal{U})} = \langle y : y \in X_u, u \subseteq_1 \mathcal{U}\rangle^*_{G_1} = \langle X_{\mathcal{U}}\rangle^*_{G_1}.$$

The notation $\mathfrak{m}(M)$ is from the second line of Def. 3.4 and $X_{\mathcal{U}}$ is from 3.4(3).

(6) For $f_{\bar{g}} \in \bar{f}^{\mathfrak{m}}$ (cf. Definition 3.4(4)), let $\hat{f}^2_{\bar{g}}$ be the unique partial automorphism of G_2 which is induced by $f_{\bar{g}}$ (see 4.4(2)), explicitly: if $k < \omega$ and for every $\ell < k$ we have that $y^1_\ell \in \mathrm{dom}(f_{\bar{g}})$, $y^2_\ell = f_{\bar{g}}(y^1_\ell)$, $q_\ell \in \mathbb{Q}^+$, then:

$$a = \sum_{\ell<k} q_\ell y^1_\ell \in G_2 \ \Rightarrow\ \hat{f}^2_{\bar{g}}(a) = \sum_{\ell<k} q_\ell y^2_\ell.$$

(7) For $\ell \in \{0,1\}$ we let $\hat{f}^2_{\bar{g}} \restriction G_\ell = \hat{f}^\ell_{\bar{g}}$ and $\hat{f}_{\bar{g}} = \hat{f}^1_{\bar{g}}$ (see 4.4(2)).

(8) For $i \in \{0,1,2\}$, $a = \sum_{\ell<m} q_\ell x_\ell \in G_i$, with $(x_\ell : \ell < k) \in \mathrm{seq}_k(X)$ and $q_\ell \in \mathbb{Q}^+$, let $\mathrm{supp}(a) = \{x_\ell : \ell < m\}$, i.e., when $a \in G^+_i$, $\mathrm{supp}(a) \subseteq_\omega X$ is the smallest subset of X such that $a \in \langle\mathrm{supp}(a)\rangle^*_{G_i}$.

(9) For p a prime and $a \in G^+_2$ we define the p-support of a, denoted as $\mathrm{supp}_p(a)$, as: if $a = \sum\{q_\ell x_\ell : \ell < k\}$ with $(x_\ell : \ell < k) \in \mathrm{seq}_k(X)$ and $q_\ell \in \mathbb{Q}^+$, then:

$$\mathrm{supp}_p(a) = \{x_\ell : \ell < k \text{ and } q_\ell \notin \mathbb{Q}_p\},$$

where we recall that $\mathbb{Q}_p$ was defined in 2.6.

Lemma 4.4. *Let $\mathfrak{m} \in \mathrm{K}^{\mathrm{bo}}_3$ and $\ell \in \{0,1,2\}$.*

(1) $G_\ell[\mathfrak{m}] \in \mathrm{TFAB}$ and $|G_\ell[\mathfrak{m}]| = \aleph_0$.

(2) (a) $\hat{f}^2_{\bar{g}}$ is a partial automorphisms of $G_2[\mathfrak{m}]$ mapping $G_0[\mathfrak{m}]$ into itself;

(b) $\hat{f}_{\bar{g}} = \hat{f}^1_{\bar{g}} = \hat{f}^2_{\bar{g}} \restriction G_{(1,\mathrm{dom}(\bar{g}))}$ (cf. Def. 4.3(5)(7)), we have that the map $\hat{f}_{\bar{g}}$ is a well-defined partial automorphism of G_1, and $\mathrm{dom}(\hat{f}_{\bar{g}})$ is a pure subgroup of $G_1[\mathfrak{m}]$, in fact $\mathrm{dom}(\hat{f}_{\bar{g}})$ is the pure closure in G_1 of $\mathrm{dom}(\hat{f}^0_{\bar{g}})$;

(c) $\hat{f}_{\bar{g}^{-1}} = \hat{f}^{-1}_{\bar{g}}$;

(d) $\bar{g}_1 \subseteq \bar{g}_2 \Rightarrow \hat{f}_{\bar{g}_1} \subseteq \hat{f}_{\bar{g}_2}$;

(e) $f_{\bar{g}} \subseteq \hat{f}^0_{\bar{g}} \subseteq \hat{f}^1_{\bar{g}} \subseteq \hat{f}^2_{\bar{g}}$.

(3) If $p = p_{(e,\bar{q})}$, $e \in \mathrm{seq}_n(X)/E^{\mathfrak{m}}_n$, $\bar{q} = (q_\ell : \ell < n)$ is as in 4.1 (so $\bar{q} \in (\mathbb{Q}^{\odot}_p)^n$), and $n \geqslant 1$, then:

(a) $\langle\sum_{\ell<n} p^{-m} q_\ell y_\ell : m < \omega, \bar{y} \in e\rangle^*_{G_1} \leqslant G_{(1,p)}$;

(b) $G_1 \leqslant \langle\{p^{-m}\sum_{\ell<n} q_\ell y_\ell : \bar{y} \in e\} \cup \mathbb{Q}_p G_0\rangle_{G_2}$;

(c) if $a \in G_1$, then there are $k < \omega$, and, for $i < k$, $\bar{y}^i \in e$, $r_i \in \mathbb{Q}^+$ s.t.:
(i) $a = \sum_{i<k} r_i(\sum_{\ell<n} q_\ell y^i_\ell) \mod(\mathbb{Q}_p G_0)$ (and $\mod(\mathbb{Q}_p G_0 \cap G_1)$);
(ii) for all $i < k$, $r_i \sum_{\ell<n} q_\ell y^i_\ell \notin \mathbb{Q}_p G_0$, and $\ell < n$ implies $r_i q_\ell y^i_\ell \notin \mathbb{Q}_p G_0$;
(iii) $\sum\{q^i_\ell y^i_\ell : \ell < n\} \in G_0$;
(iv) $r_i \sum\{q^i_\ell y^i_\ell : \ell < n\} \in G_1$.

(4) In 4.4(3) we may add: $((\bar{y}^i, \bar{q}_i) : i < i_*)$ and $(p_i : i < i_*)$ are with no repetitions.

Proof. Item (1) is clear. Concerning item (2), clause (a) holds as $f_{\bar{g}}$ is a partial 1-to-1 function from X to X; while for clause (b) it suffices to prove that given $\sum_{\ell<k} q_\ell y^1_\ell$ and $\sum_{\ell<k} q_\ell y^2_\ell$ as in Definition 4.3(6) we have that:

$$\sum_{\ell<k} q_\ell y^1_\ell \in G_1 \Rightarrow \sum_{\ell<k} q_\ell y^2_\ell \in G_1.$$

In order to verify this it suffices to consider the case in which $a := \sum_{\ell<k} q_\ell y^1_\ell$ is one of the generators of G_1 from 4.3(3). Thus, to conclude it suffices to notice that $f_{\bar{g}}$ maps $\bar{y}^1 = (y^1_\ell : \ell < k)$ to $\bar{y}^2 = (y^2_\ell : \ell < k)$, hence $\bar{y}^2 \in \bar{y}^1/E_k$ and recall 4.3(3b). This shows (2)(b). Finally, items (2)(c)-(e) are easy and so we omit details.

Concerning item (3), if $\bar{y} \in e$ and $0 < m < \omega$, then $p^{-m} \sum_{\ell<k} q_\ell y_\ell$ is one of the generators of G_1, as this holds for every $0 < m < \omega$ it follows that $\sum_{\ell<k} p^{-m} q_\ell y_\ell \in G_{(1,p)}$, by the definition of $G_{(1,p)}$. As $G_{(1,p)}$ is a subgroup of G_1, for every $\bar{y} \in e$ we have that $\sum_{\ell<n} q_\ell y_\ell \in G_{(1,p)} \leqslant G_1$. Let $Z_{(e,\bar{q})} = \{\sum_{\ell<n} q_\ell y_\ell : \bar{y} \in e\} \subseteq G_{(1,p)}$, then $\langle Z_{(e,\bar{q})}\rangle^*_{G_1} \leqslant G_{(1,p)}$, because by Definition 4.3(4) we have that $G_{(1,p)}$ is a pure subgroup of G_1 (cf. Obs. 2.5). This proves (3)(a). Concerning (3)(b)(c), assume:

$(*_1)$ $a \in G^+_1$.

By 4.3(3c), we have:

$(*_2)$ As $a \in G_1$ we can find:
(a) $i_* < \omega$;
(b) for $i < i_*$, $e_i = \bar{x}_i/E_{k_i}$, $\bar{x}_i \in \mathrm{seq}_{k_i}(X)$, $\bar{q}^i = (q^i_\ell : \ell < k_i) \in (\mathbb{Z}^+)^{k_i}$;
(c) $r^i \in \mathbb{Z}^+$, $\bar{y}^i \in e_i$, $b_i = \sum_{\ell<k_i} q^i_\ell y^i_\ell \in G_0$;
(d) $p_i = p_{(e_i,\bar{q}^i)}$;
(e) $a = \sum_{i<i(*)} p_i^{-m(i)} r^i b_i$, where $m(i) < \omega$;
(f) $(b_i : i < i_*)$ is with no repetitions (hence also $(p_i : i < i_*)$);
(g) $p_i^{-m(i)} r^i b_i \in G_1$.

Let now:

$(*_3)$ $V = \{i < i_* : p_i = p = p_{(e,\bar{q})}$ and $p_i^{-m(i)} r^i b_i \notin \mathbb{Q}_p G_0\}$.

Hence, we have:

$(*_4)$ (a) if $i \in i_* \setminus V$, then $p_i^{-m(i)} r^i b_i \in \mathbb{Q}_p G_0$;
(b) $i \in V$ implies $\bar{y}^i \in e$ and $\bar{q}^i = \bar{q}$;
(c) if $i \in V$ and $\ell(1), \ell(2) < k$, then:

$$p_i^{-m(i)} r^i q^i_{\ell(1)} \in \mathbb{Q}_p \Leftrightarrow p_i^{-m(i)} r^i q^i_{\ell(2)} \in \mathbb{Q}_p \Leftrightarrow p_i^{-m(i)} r^i b_i \in \mathbb{Q}_p G_0;$$

(d) if $i \in V$, then $p_i^{-m(i)} r^i b_i \notin \mathbb{Q}_p G_0$;
(e) if $i \in V$ and $\ell < k$, then $p_i^{-m(i)} r^i q^i_\ell y^i_\ell \notin \mathbb{Q}_p G_0$.

[Notice that in the first equivalence of $(*_4)$(c) we use: $\ell < k \Rightarrow q_\ell \in \mathbb{Z}^+, p \nmid q_\ell$.]

By $(*_4)$ we have:

$(*_5)$ (a) $a = \sum\{p_i^{-m(i)} r^i b_i : i \in V\} \mod(\mathbb{Q}_p G_0)$;

(b) $i \in V$ implies $p_i^{-m(i)} r^i b_i \notin \mathbb{Q}_p G_0$.

So, defining r_i as $p^{-m(i)} r^i$, we are done proving (3)(b)(c). Finally, (4) is easy. ■

Fact 4.5. *Assume that $\mathfrak{m} \in \mathrm{K}_3^{\mathrm{bo}}(M)$, $\mathcal{U}, \mathcal{V} \subseteq M$ and $|\mathcal{U}| = |\mathcal{V}| = \aleph_0$. Suppose further that there is $h : M \restriction \mathcal{U} \cong M \restriction \mathcal{V}$. Then there is $\bar{g} = (g_k : k < \omega)$ such that:*
(a) for every $k < \omega$, $g_k \in \mathcal{G}$ (cf. 3.2(3));
(b) for every $k < \omega$, $g_k \subsetneq g_{k+1}$;
(c) $\bigcup_{k<\omega} g_k : M \restriction \mathcal{U} \cong M \restriction \mathcal{V}$.

Proof. Let $h : M \restriction \mathcal{U} \cong M \restriction \mathcal{V}$. We can choose an increasing sequence $(n_k : k < \omega)$ such that $g_k = h \cap (n_k \times n_k)$ (pedantically $g = (h \cap (n_k \times n_k), 1)$ recalling 3.2(3)) is strictly increasing and $\bigcup_{k<\omega} g_k = h$. ■

Claim 4.6. *Assume that $\mathfrak{m} \in \mathrm{K}_3^{\mathrm{bo}}(M)$, $\mathcal{U}, \mathcal{V} \subseteq M$ and $|\mathcal{U}| = |\mathcal{V}| = \aleph_0$. Then:*

$$M \restriction \mathcal{U} \cong M \restriction \mathcal{V} \;\Rightarrow\; G_{(1,\mathcal{U})}[\mathfrak{m}] \cong G_{(1,\mathcal{V})}[\mathfrak{m}].$$

Proof. Let $(g_k : k < \omega)$ be as in Fact 4.5, $s_k = \mathrm{dom}(g_k)$ and $t_k = \mathrm{ran}(g_k)$. Then:
(i) for $k < \omega$, $\bar{g}_k = (g_\ell : \ell \leqslant k)$, so $\bar{g}_k \in \mathcal{G}_*^{k+1}$ (cf. 3.2(4) and 4.5(a)(b));
(ii) $\bigcup_{k<\omega} g_k : M \restriction \mathcal{U} \cong M \restriction \mathcal{V}$ (cf. 4.5(c));
(iii) for every $k < \omega$ we have that $\bar{g}_k \in \mathcal{G}_*$ and so, by Definition 3.4(4), $f_{\bar{g}_k} \in \bar{f}^{\mathfrak{m}}$.

Notice also that by 3.4(9) we have:

($\star_1$) (d) $\bigcup_{k<\omega} \mathrm{dom}(f_{\bar{g}_k}) = \bigcup_{k<\omega} X_{s_k} = X_{\mathcal{U}}$;
(e) $\bigcup_{k<\omega} \mathrm{ran}(f_{\bar{g}_k}) = \bigcup_{k<\omega} X_{h[s_k]} = X_{\mathcal{V}}$.

Hence, we have:

($\star_2$) $\bigcup_{k<\omega} \hat{f}_{\bar{g}_k}$ is an isomorphism from $G_{(1,\mathcal{U})}$ onto $G_{(1,\mathcal{V})}$ (cf. Def. 4.3(5)(7)).
[Why? By 4.3(5)(6)(7), 4.4(2b) and 3.4(9).] ■

4.2. Analyzing Isomorphism

Our aim in this subsection is to prove the converse of Claim 4.6.

Hypothesis 4.7. *Throughout this subsection the following hypothesis holds:*
(1) $\mathfrak{m} \in \mathrm{K}_3^{\mathrm{bo}}(M)$;
(2) $\mathcal{U}, \mathcal{V} \subseteq M$;
(3) $|\mathcal{U}| = \aleph_0 = |\mathcal{V}|$;
(4) π is an isomorphism from $G_{(1,\mathcal{U})}[\mathfrak{m}]$ onto $G_{(1,\mathcal{V})}[\mathfrak{m}]$.

Lemma 4.8. *Let $a \in G_{(1,\mathcal{U})}[\mathfrak{m}]$ and let $b = \pi(a)$.*
(1) For a prime p, $a \in G_{(1,p)} \Leftrightarrow b \in G_{(1,p)}$;
(2) if $a = qx$, for some $q \in \mathbb{Q}^+$ and $x \in X_{\mathcal{U}}$, then for some $y \in X_{\mathcal{V}}$:
(a) $(x)E_1^{\mathfrak{m}}(y)$;
(b) $b \in \mathbb{Q}y$, i.e. there exist $m_1, m_2 \in \mathbb{Z}^+$ such that $m_1 b = m_2 y$.

Proof. Item (1) is obvious by Hypothesis 4.7(4). Notice now that:

($*_0$) It suffices to prove (2)(b).

Why ($*_0$)? Suppose that $b = \frac{m_2}{m_1} y$ and let $e' = (x)/E_1^{\mathfrak{m}}$ and $p' = p_{(e',(1))}$, then $x \in G_{(1,p')}$, but $a = qx$ and $a \in G_1$, hence $a \in G_{(1,p)}$. Now, applying (1) with (a, b, p') here standing for (a, b, p) there, we get that $b \in G_{(1,p')}$. As $b = \frac{m_2}{m_1} y \in G_1$, we have that $y \in G_{(1,p')}$ and thus:

($\cdot$) $G_1 \models (p')^\infty | \, x$ and $G_1 \models (p')^\infty | \, y$.

Now, letting $H_{(p',0)} = \langle x/E_1^{\mathfrak{m}} \rangle_{G_0}$ and $H_{(p',1)} = \langle x/E_1^{\mathfrak{m}} \rangle^*_{G_1}$ we have that:

$(*_{0.1})$ (i) $G_0/H_{(p',0)}$ is canonically $\cong$ to the direct sum of $\langle \mathbb{Z}y : y \in X \setminus x/E_1^{\mathfrak{m}} \rangle$;
(ii) $H_{(p',1)} \cap G_0 = H_{(p',0)}$;
(iii) $G_1/H_{(p',1)}$ naturally extends $G_0/H_{(p',0)}$;
(iv) no non-zero element of $G_1/H_{(p',1)}$ is divisible by $(p')^\infty$.

Why $(*_{0.1})$? Straightforward or see a detailed proof of a more complicated case in 5.17(1). This concludes the proof of $(*_0)$.

Coming back to the proof:

$(*_1)$ Let $n < \omega$, $\bar{y} \in \mathrm{seq}_n(X_{\mathcal{V}})$ and $\bar{q} \in (\mathbb{Q}^+)^n$ be such that $b = \sum\{q_\ell y_\ell : \ell < n\}$.

Trivially, $n > 0$, we shall show that $n = 1$. Let $q_* \in \omega \setminus \{0\}$ be such that $b_1 := q_* b \in G_0[\mathfrak{m}]$. Let $e = \bar{y}/E_n$, $q'_\ell = q_* q_\ell$ and $\bar{q}' = (q'_\ell : \ell < n)$, so that $q_* q_\ell y_\ell = q'_\ell y_\ell$ and $q'_\ell \in \mathbb{Z}^+$. Let $p = p_{(e,\bar{q}')}$ and let $b_1 = q_* b$. Notice:

$(*_2)$ For $\ell < k$, $p \not| q'_\ell$, $q_\ell \in \mathbb{Q}^+$ and $q'_\ell \in \mathbb{Z}^+$.

[Why? Because $p = p_{(e,\bar{q}')}$ has been chosen in 4.1 so that this happens.]
Then we have:

$(*_3)$ (i) $b \in G_{(1,p)}$;
(ii) $a \in G_{(1,p)}$;
(iii) if $m < \omega$, then $p^{-m} a \in G_{(1,p)} \leqslant G_1$.

[Why (i)? By the choice of p we have that $b_1 \in G_{(1,p)}$ (cf. Def. 4.3(3)(4)) and so, as $G_{(1,p)}$ is pure in G_1 (cf. Observation 2.5), $b_1 = q_* b$ and $q_* \in \mathbb{Z}$, we have $b \in G_{(1,p)}$ (cf. Observation 2.4). Why (ii)? By (1) and (i), recalling Hyp. 4.7(4). Lastly, (iii) is immediate by the definition of G_1 and of $G_{(1,p)}$ (Definition 4.3(3)(4)).]

$(*_4)$ W.l.o.g. $a \notin G_0$, $pa \in G_0$ and $a \notin \mathbb{Q}_p G_0$.

We prove $(*_4)$. First of all:

$(*_{4.1})$ There is $q' \in \{p^m r : m \in \mathbb{Z}, r \in \mathbb{Z}^+, (r,p) = 1\}$ such that:
(a) $q'qx \notin G_0$;
(b) $pq'qx \in G_0$.

Secondly, as $a = qx \in G_{(1,p)}$ and $q' \in \{p^m r : m \in \mathbb{Z}, r \in \mathbb{Z}^+, (r,p) = 1\}$, clearly $q'qx \in G_{(1,p)} \leqslant G_1$. Now, for the sake of contradiction, suppose that $q'qx \in \mathbb{Q}_p G_0$, then for some $m \in \mathbb{Z}^+$ we have:

$(*_{4.2})$ $(m,p) = 1$ and $m(q'qx) \in G_0$.

As $(m,p) = 1$ and $m, p \in \mathbb{Z}^+$ there are $r_1, r_2 \in \mathbb{Z}^+$ such that $r_1 m + r_2 p = 1$. Hence:

$$\begin{aligned} q'qx &= 1q'qx \\ &= (r_1 m + r_2 p) q'qx \\ &= r_1(mq'qx) + r_2(pq'qx) \in G_0, \end{aligned}$$

where at the end we used that $mq'qx \in G_0$ (by $(*_{4.2})$) and that $pq'qx \in G_0$ by $(*_{4.1})$(b)). Hence, we have that $q'qx \in G_0$, which contradicts $(*_{4.1})$(a). Thus, $q'qx \notin \mathbb{Q}_p G_0$. Also as $\pi(q'qx) = q'b$, necessarily $q'b \in G_1$. Thus, $(q'a, q'b, q'q, \frac{q_*}{q'}, b, \bar{q}, \bar{q}')$ satisfies all the demands on $(a, b, q, q_*, b, \bar{q}, \bar{q}')$ and so replacing the latter with the former we loose nothing and gain $(*_4)$. This concludes the proof of $(*_4)$.

Now, by 4.4(3), there are $k < \omega$, and, for $i < k$, $\bar{y}^i \in \bar{y}/E_n$ and $r_i \in \mathbb{Q}^+$ such that:

$(*_5)$ (a) $a = \sum_{i<k} r_i (\sum_{\ell<n} q'_\ell y^i_\ell) = \sum_{i<k} (\sum_{\ell<n} r_i q'_\ell y^i_\ell) \mod(\mathbb{Q}_p G_0 \cap G_1)$;
(b) $r_i \sum_{\ell<n} q'_\ell y^i_\ell \in G_1$ and $r_i q'_\ell \notin \mathbb{Q}_p$.

By $(*_4)$, clearly $k > 0$, and it suffices to prove that $k = 1$, which by $(*_5)$ implies that $n = 1$, i.e., there is $y \in X_{\mathcal{V}}$ such that $b \in \mathbb{Q}y$. Why it follows that $n = 1$? As otherwise in $(*_2)$ the LHS has as p-support a singleton but the RHS has p-support of size at least two. So toward contradiction assume that $k \geqslant 2$. Notice that by the assumption in item (2) of the present lemma we have that $a = qx \in G_1$, for some $x \in X$ and $q \in \mathbb{Q}^+$, and so we have:

$(*_6)$ $qx = a = \sum_{\ell<n}(\sum_{i<k} r_i q'_\ell y^i_\ell) \mod(\mathbb{Q}_p G_0 \cap G_1)$;

Now, let $Z = \{y^i_\ell : i < i_*, \ell < k\}$ and, for $y \in Z$, let:

$$a_y = \sum\{r_i q'_\ell : i < i_*, \ell < k, y^i_\ell = y\}.$$

So, by $(*_6)$ we have:

$(*_7)$ $qx = \sum\{a_y y : y \in Z\} \mod(\mathbb{Q}_p G_0 \cap G_1)$.

Now, as we are assuming that $k \geqslant 2$, and recalling that by $(*_1)$ we have that $q'_\ell \in \mathbb{Q}^{\odot}_p$, by 3.4(8), we have:

$(*_8)$ $\mathrm{supp}_p(\sum_{y\in Y} a_y y) = \{y \in Y : a_y \notin \mathbb{Q}_p\}$ is not a singleton.

Now recall that, by $(*_4)$, $qx = a \notin \mathbb{Q}_p G_0 \cap G_1$, so by $(*_7)$, for some $y_1 \in Y$ we have $a_{y_1} \notin \mathbb{Q}_p G_0$. By $(*_8)$, necessarily there is $y_2 \in Y \setminus \{y_1\}$ such that $a_{y_2} \notin \mathbb{Q}_p$. It follows that in the RHS of $(*_7)$, y_1 and y_2 belong to the p-support of a, but the LHS of $(*_7)$ has as p-support a singleton (such a support is non-empty by $(*_4)$), a contradiction. Hence, we are done proving (2) and the lemma. ■

Conclusion 4.9. *(1) There is a sequence $(q^1_x : x \in X_{\mathcal{U}})$ of non-zero rationals and a function $\pi_1 : X_{\mathcal{U}} \to X_{\mathcal{V}}$ such that for every $x \in X_{\mathcal{U}}$ we have that:*

$$\pi(x) = q^1_x(\pi_1(x)) \quad \text{and} \quad \pi_1(x) \in x/E^{\mathfrak{m}}_1.$$

(2) There is a sequence $(q^2_x : x \in X_{\mathcal{V}})$ of non-zero rationals and a function $\pi_2 : X_{\mathcal{V}} \to X_{\mathcal{U}}$ such that:

$$\pi^{-1}(x) = q^2_x(\pi_2(x)).$$

(3) (i) $\pi_2 \circ \pi_1 : X_{\mathcal{U}} \to X_{\mathcal{U}} = id_{\mathcal{U}}$;
(ii) $\pi_1 \circ \pi_2 : X_{\mathcal{V}} \to X_{\mathcal{V}} = id_{\mathcal{V}}$;
(iii) $\pi_1 : X_{\mathcal{U}} \to X_{\mathcal{V}}$ is a bijection.

Proof. (1) is by 4.8, we elaborate. To this extent, let $R = \{(x, y) : x, y \in X \text{ and } \pi(x) \in \mathbb{Q}^+ y\}$. Now, we have:

$(*_1)$ For all $x \in X$ there is $y \in X$ such that $R(x, y)$.

[Why? By 4.8(2b) there is $y \in X$ such that $\pi(x) \in \mathbb{Q}y$, as π is an automorphism, necessarily $\pi(x) \neq 0$ and so $\pi(x) \in \mathbb{Q}^+ y$.]

$(*_2)$ If $x \in X$ and $(x, y_1), (x, y_2) \in R$, then $y_1 = y_2$.

[Why? By the definition of R, there are $q_1, q_2 \in \mathbb{Q}^+$ such that $q_1 y_1 = \pi(x) = q_2 y_2$. As $q_1, q_2 \neq 0$, necessarily $q_1 = q_2$ and $y_1 = y_2$.]

Together, R is the graph of a function which we call π_1. Lastly, $\pi_1(x) \in x/E^{\mathfrak{m}}_1$ by 4.8(2a). Thus we proved (1).

(2) is by part (1) applied to π^{-1} (and $\mathcal{V}, \mathcal{U}$).

(3) is by (1) and (2). Why? E.g., for (i) we have that:

$$\pi^{-1} \circ \pi(x) = \pi^{-1}(q^1_x(\pi_1(x))) = q^2_{\pi_1(x)} q^1_x(\pi_2 \circ \pi_1(x)) = x,$$

which implies that $\pi_2 \circ \pi_1(x) = x$. (ii) is similar and (iii) follows from (i)+(ii). ■

Definition 4.10. *For $i < 3$, let:*

$$\mathcal{E}_i = \{(x, y) : \text{ for some } (a, b) \in E^M_i, x \in X'_{\{a\}} \text{ and } y \in X'_{\{b\}}\}.$$

Claim 4.11. *(1) If $(y_0, y_1) \in (x_0, x_1)/E^M_2$, $x_0, x_1, y_0, y_1 \in X$ and $i < 3$, then:*

$$x_0\mathcal{E}_i x_1 \Leftrightarrow y_0\mathcal{E}_i y_1.$$

(2) The mapping π_1 from 4.9 preserves $\mathcal{E}_i$ and its negation, for all $i < 3$.

Proof. (1) Suppose that $(y_0, y_1) \in (x_0, x_1)/E^M_2$. Then it is enough to prove:

$(\star_1)$ If $\bar{g} \in \mathcal{G}_*$, $f_{\bar{g}}(x_\ell) = y_\ell$, for $\ell = 0, 1$, then $x_0\mathcal{E}_i x_1 \Leftrightarrow y_0\mathcal{E}_i y_1$.

For $\ell = 0, 1$, let $x_\ell \in X'_{s_\ell}$, for $s_\ell \subseteq_1 M$ and $y_\ell \in X'_{t_\ell}$, for $t_\ell \subseteq_1 M$. Now, as $f_{\bar{g}}(x_\ell) = y_\ell$, clearly $\bar{g}[s_\ell] = t_\ell$. So $\bar{g}(s_0, s_1) = (t_0, t_1)$, and so, as $\bar{g} \in \mathcal{G}_*$ (cf. 3.4(4d)), $s_0 E^M_i s_1 \Leftrightarrow t_0 E^M_i t_1$. This implies $x_0\mathcal{E}_i x_1 \Leftrightarrow y_0\mathcal{E}_i y_1$.

Concerning (2). Using also $\pi_2, \mathcal{V}, \mathcal{U}$ it suffices to prove that for $x, y \in X_{\mathcal{U}}$:

$$x\mathcal{E}_i y \Rightarrow \pi_1(x)\mathcal{E}_i\pi_1(y).$$

To this extent, suppose that $x\mathcal{E}_i y$ and let $s \subseteq_1 \mathcal{U}$ be such that $x, y \in X_{s/E^M_i}$. If $x = y$, then the conclusion is trivial, so we assume that $x \neq y$.

$(\star_{1.1})$ Let $e = (x, y)/E^{\mathfrak{m}}_2$, $\bar{q} = (1, 1)$ and $p = p_{(e,\bar{q})}$.

Now, we can assume:

$(\star_{1.2})$ Let $m < \omega$ be such that $p^{-m}q_x\pi_1(x) + p^{-m}q_y\pi_1(y) \notin \mathbb{Q}_pG_0 \cap G_1$.

[Why? First of all, as $q_x, q_y \in \mathbb{Q}^+$ and $x \neq y \Rightarrow \pi_1(x) \neq \pi_1(y)$ and $\pi(x+y) \in G_1$, we have that $0 \neq a = q_x\pi_1(x) + q_y\pi_1(y) \in G_1$ and so we are done, recalling that by the definition of $\mathbb{Q}_p$ we have that for every $b \in G^+_1$ there is $m < \omega$ s.t. $p^{-m}b \notin \mathbb{Q}_pG_0$.]

Now, by the choice of p, we have that $p^{-m}(x + y) \in G_{(1,p)} \leqslant G_1$ and so:

$$p^{-m}q_x\pi_1(x) + p^{-m}q_y\pi_1(y) = p^{-m}\pi(x) + p^{-m}\pi(y) = \pi(p^{-m}(x + y)) \in G_{(1,p)}.$$

So, by 4.4(3) applied with $((x, y)/E^{\mathfrak{m}}_2, (1, 1), p, p^{-m}q_x\pi_1(x) + p^{-m}q_y\pi_1(y))$ standing for $(e, \bar{q}, p_{(e,\bar{q})}, a)$ there, there are $(x_j, y_j) \in (x, y)/E^{\mathfrak{m}}_2$ and $r_j \in \mathbb{Q}^+$, for $j < j_*$, s.t.:

$(\star_2)$ (a) $((x_j, y_j) : j < j_*)$ is with no repetitions;
(b) $p^{-m}q_x\pi_1(x) + p^{-m}q_y\pi_1(y) = \sum_{j<j_*} r_j(x_j + y_j) \mod(\mathbb{Q}_pG_0 \cap G_1)$.

Now, by (1), recalling $x\mathcal{E}_i y$, for $j < j_*$, there are $s_j \subseteq_1 M$ such that $x_j, y_j \in X_{s_j/E^M_i}$. Now, by $(\star_{1.2})$, the LHS of $(\star_2)$(b) is not in $\mathbb{Q}_pG_0 \cap G_1$, so the same happens for the RHS of $(\star_2)$(b), hence, necessarily, $\{s_j/E^M_i : j < j_*\} \neq \emptyset$ (i.e., $j_* \geqslant 1$), let $(t_\ell/E^M_i : \ell < \ell_*)$ list it without repetitions, with $t_\ell \in \{s_j : j < j_*\}$ for each $\ell < \ell_*$. Let then:

$$u_\ell = \{j < j_* : s_j/E^M_i = t_\ell/E^M_i\}.$$

So we have:

$(\star_3)$ $p^{-m}q_x\pi_1(x) + p^{-m}q_y\pi_1(y) = \sum_{\ell<\ell_*}\sum_{j\in u_\ell} r_j(x_j + y_j) \mod(\mathbb{Q}_pG_0 \cap G_1)$.

Now, for $\ell < \ell_*$, let $c_\ell = \sum_{j\in u_\ell} r_j(x_j + y_j)$, then:

$(\star_4)$ $p^{-m}q_x\pi_1(x) + p^{-m}q_y\pi_1(y) = \sum_{\ell<\ell_*} c_\ell \mod(\mathbb{Q}_pG_0 \cap G_1)$.

$(\star_5)$ $(\mathrm{supp}_p(c_\ell) : \ell < \ell_*)$ is a sequence of pairwise disjoint sets.

[Why? As $\mathrm{supp}(c_\ell) \subseteq X_{t_\ell/E^M_i}$, recalling the t_ℓ/E^M_i's are with no repetitions.]

$(\star_6)$ If $c_\ell \notin \mathbb{Q}_pG_0$, then $|\mathrm{supp}_p(c_\ell)| \geqslant 2$.

[Why? Recall that $c_\ell = \sum_{j\in u_\ell} r_j(x_j+y_j)$ and let $Y_\ell = \bigcup\{\{x_j,y_j\} : j\in u_\ell\}$ and, for $z\in Y_\ell$, let $a_z = \sum\{r_j : j\in u_\ell,\ x_j = z\} + \sum\{r_j : j\in u_\ell,\ y_j = z\}$. Now we can apply 3.4(8) with $(p,2,(x,y),((x_j,y_j) : j\in u_\ell),(1,1),(r_j : j\in u_\ell),(a_z : z\in Y_\ell))$ here standing for $(p,k,\bar{x},\mathbf{y},\bar{r},\bar{a}_{(\mathbf{y},r)})$ there and get $|\{z\in Y_\ell : a_z\notin \mathbb{Q}_p\}|\neq 1$. But this means that $|\mathrm{supp}_p(c_\ell)|\neq 1$, but $|\mathrm{supp}_p(c_\ell)|\neq 0$ by an assumption of $(\star_6)$, hence $|\mathrm{supp}_p(c_\ell)|\geqslant 2$, as promised. This concludes the proof of $(\star_6)$.]

$(\star_7)$ $V=\{\ell<\ell_* : c_\ell\notin \mathbb{Q}_pG_0\}$ has exactly one member.

[Why? If $V=\emptyset$, then the RHS of $(\star_4)$ is in $\mathbb{Q}_pG_0$ but not the LHS, recalling $(\star_5)$ and the choice of $m<\omega$ in $(\star_{1.2})$, a contradiction. On the other hand, if $|V|\geqslant 2$, then the RHS of $(\star_3)$ has p-support of size $\sum_{\ell<\ell_*}|\mathrm{supp}_p(c_\ell)|\geqslant 2|V|>2$, but the p-support of the RHS of $(\star_4)$ has cardinality 2, a contradiction.]

Let k be the unique member of V, then we have the following:

$$\begin{aligned}\{\pi_1(x),\pi_1(y)\} &= \mathrm{supp}_p(p^{-m}q_x\pi_1(x)+p^{-m}q_y\pi_1(y))\\ &= \mathrm{supp}_p(\textstyle\sum_{\ell<\ell_*}c_\ell)\\ &= \mathrm{supp}_p(c_k)\subseteq X_{t_k/E_i^M}.\end{aligned}$$

So $\pi_1(x),\pi_1(y)\in X_{t_k/E_i^M}$ and as X_{t_k/E_i^M} is an $\mathcal{E}_i$-equivalence class (by the definition of $\mathcal{E}_i$), then $\pi_1(x)\mathcal{E}_i\pi_1(y)$, as desired. This concludes the proof of the claim. ∎

Claim 4.12. *There is a bijection $h:\mathcal{U}\to\mathcal{V}$ preserving E_i^M and $\neg E_i^M$, for all $i<3$.*

Proof. By 4.11(2), we have:

$(*_1)$ If $x,y\in X_\mathcal{U}$ and $i<3$, then $x\mathcal{E}_iy \Leftrightarrow \pi_1(x)\mathcal{E}_i\pi_1(y)$.

Now apply $(*_1)$ for $i=2$ and recall that by 3.2(1) E_2^M is equality on M. Then:

$(*_2)$ $\exists s\subseteq_1\mathcal{U}(x,y\in X'_s)\Leftrightarrow \exists t\subseteq_1\mathcal{V}(\pi_1(x),\pi_1(y)\in X'_t)$.

Now, as $X_\mathcal{U}=\bigcup_{s\subseteq_1\mathcal{U}}X'_s$ and $X_\mathcal{V}=\bigcup_{s\subseteq_1\mathcal{V}}X'_s$, there is a function $\mathbf{h}_1$ from $\mathcal{U}$ into $\mathcal{V}$ such that (not distinguishing $a\in\mathcal{U}$ with $\{a\}\subseteq_1\mathcal{U}$):

$(*_3)$ If $x\in X'_s$, $s\subseteq_1\mathcal{U}$, then $\pi_1(x)\in X'_{\mathbf{h}_1(s)}$.

As $\pi_2=\pi_1^{-1}$ and π_2 is a function from $X_\mathcal{V}$ onto $X_\mathcal{U}$ (cf. 4.9) we have that:

$(*_4)$ $\mathbf{h}_1:\mathcal{U}\to\mathcal{V}$ is 1-to-1 and onto.

Finally, applying 4.11(2) to i and recalling the definition of $\mathcal{E}_i$ we get:

$(*_5)$ For $i=0,1$, $a\neq b\in\mathcal{U}$ implies $aE_i^Mb\Leftrightarrow \pi_1(a)E_i^M\pi_1(b)$.

∎

Conclusion 4.13. *$M\restriction\mathcal{U}$ and $M\restriction\mathcal{V}$ are isomorphic members of $\mathbf{K}^{\mathrm{eq}}$.*

The following three items are not essential for the proof of the Main Theorem but we think that they are nice to know, and so we chose to retain them. Also, the argument proposed in 4.15 is useful for Section 5. (Notice that the assumption "$X_\mathcal{U}/E_1$ is not a singleton" in 4.14 is not serious, as the proof in Section 3 ensures this.)

Claim 4.14. *In the context of Conclusion 4.9. Assuming that $X_\mathcal{U}/E_1$ is not a singleton, we have that for some $q_*\in\mathbb{Q}^+$ the following holds:*

$$\forall x\in X_\mathcal{U},\ q_x^1=q_*\ (\text{so } q_x^2=q_*^{-1}).$$

Proof. Let $x\neq y\in X_\mathcal{U}$ be such that $(x)/E_1^{\mathfrak{m}}\neq(y)/E_1^{\mathfrak{m}}$. Let then $e=(x,y)/E_2^{\mathfrak{m}}$, $\bar{q}=(1,1)$ and $p=p_{(e,\bar{q})}$. Now, by the choice of p, we have that $x+y\in G_{(1,p)}$ and so $q_x\pi_1(x)+q_y\pi_1(y)=\pi(x)+\pi(y)=\pi(x+y)\in G_{(1,p)}$. So, by 4.4(3), there are $(x_i,y_i)\in(x,y)/E_2^{\mathfrak{m}}$ and $r_i\in\mathbb{Q}^+$, for $i<k$, s.t.:

$(\star_1)$ (a) $((x_i, y_i) : i < k)$ is with no repetitions;
(b) $q_x\pi_1(x) + q_y\pi_1(y) = (\sum_{i<k} r_i x_i) + (\sum_{i<k} r_i y_i) \mod(\mathbb{Q}_p G_0 \cap G_1)$.

Let $(x_k, y_k) = (\pi_1(x), \pi_1(y))$. Now, $(\pi_1(x), \pi_1(y)) \in \mathrm{seq}_2(X)$ by Conclusion 4.9. Let $e_1 = (x)/E_1^{\mathfrak{m}}$, $e_2 = (y)/E_1^{\mathfrak{m}}$ and, for $\ell = 1, 2$, $p_\ell = p_{(e_\ell, 1)}$. So:

$(\star_{1.1})$ for $z \in X$, $z \in G_{(1,p_\ell)} \Leftrightarrow z \in e_\ell$.

Hence, $x_k = \pi_1(x) \in e_1$ and $y_k = \pi_1(y) \in e_2$, since π is an isomorphism and (by Lemma 4.9(1)) π_1 is such that for every $x \in X_{\mathcal{U}}$ we have $\pi(x) = q_x^1(\pi_1(x))$, hence $x_k = \pi_1(x) \in x/E_1 = e_1$ and $y_k = \pi_1(x) \in y/E_1 = e_2$. Also, $x_i \in e_1$ and $y_i \in e_2$, for every $i < k$, as $(x_i, y_i) \in (x, y)/E_2^{\mathfrak{m}}$ and so $x_i \in x/E_2^{\mathfrak{m}}$ and $y_i \in y/E_2^{\mathfrak{m}}$, for $i \leqslant k$. Lastly, clearly $e_1 \cap e_2 = \emptyset$, by the choice of (x, y). Hence, the sets $\{x_i : i \leqslant k\}$ and $\{y_i : i \leqslant k\}$ are disjoint. But this means that:

$(\star_{1.2})$ the sets $\{x_i : i < k\} \cup \{\pi_1(x)\}$ and $\{y_i : i < k\} \cup \{\pi_1(y)\}$ are disjoint.

Thus, by $(\star_1)$ and $(\star_{1.2})$ we have:

$(\star_2)$ (a) $q_x\pi_1(x) = \sum\{r_i x_i : i < k\} \mod(\mathbb{Q}_p G_0 \cap G_1)$;
(b) $q_y\pi_1(y) = \sum\{r_i y_i : i < k\} \mod(\mathbb{Q}_p G_0 \cap G_1)$.

Why $(\star_2)$? By $(\star_1)$ we have that:

$$a := q_x\pi_1(x) - \sum\{q^i x_i : i < k\} = -q_y\pi_1(y) + \sum\{q^i y_i : i < k\}.$$

But then (recalling that $x_k = \pi_1(x)$ and $y_k = \pi_1(y)$) by $(\star_{1.2})$ we have that:

$$a \in \langle x_i : i \leqslant k\rangle^*_{G_2} \cap \langle y_i : i \leqslant k\rangle^*_{G_2} = \{0\} \text{ in } G_2/\mathbb{Q}_p G_0,$$

and so we have that:

$$q_x\pi_1(x) - \sum\{q^i x_i : i < k\} = 0 = -q_y\pi_1(y) + \sum\{q^i y_i : i < k\} \mod(\mathbb{Q}_p G_0 \cap G_1).$$

Let $E_x = \{(i, j) : i, j < k, x_i = x_j\}$, so E_x is an equivalence relation on k. By $(\star_2)$(a), $\pi_1(x) \in \{x_i : i < k\}$, and so for some $i_* < k$ we have that $x_{i_*} = \pi_1(x)$. Hence, again by $(\star_2)$(a), we have:

$(\star_{2.5})$ (a) $q_x = \sum\{q^i : i \in i_*/E_x\} \mod(\mathbb{Q}_p G_0 \cap G_1)$;
(b) if $i < k$ and $i \notin i_*/E_x$, then $\sum\{q^j : j \in i/E_x\} = 0 \mod(\mathbb{Q}_p G_0 \cap G_1)$.

Together we have:

$(\star_3)$ $q_x = \sum\{q^i : i < k\}$.

Similarly using $(\star)_2$(b) we get that:

$(\star_4)$ $q_y = \sum\{q^i : i < k\}$.

Thus, $q_x = q_y$. So $x, y \in X_{\mathcal{U}}$ and $x/E_1^{\mathfrak{m}} \neq y/E_1^{\mathfrak{m}}$ implies $q_x = q_y$. Hence, for every $x, y \in X_{\mathcal{U}}$ we have $q_x = q_y$. Why? If $x/E_1^{\mathfrak{m}} \neq y/E_1^{\mathfrak{m}}$ see above. Otherwise, by our assumption, there is $z \in X \setminus (x/E_1^{\mathfrak{m}}) = X \setminus (y/E_1^{\mathfrak{m}})$, so $q_x = q_z = q_y$. ■

Claim 4.15. *In the context of Claim 4.14, q_* is an integer.*

Proof. If not, then $q_* = \frac{m}{k}$, for $m \in \mathbb{Z}^+$ and $k \in \omega \setminus \{0, 1\}$. W.l.o.g. no prime divides both k and m. Toward contradiction, let p be a prime dividing k. Let $x_1 \in X_{\mathcal{U}}$. If in G_1 we have that x_1 is not divisible by p, then we are done (since then $\pi(x_1)$ cannot be $q_* x_1$). Similarly, we are done if $x_1 \notin \bigcap p^{-n} G_1$. Hence we can assume that $x \in G_{(1,p)}$ and so for some $1 \leqslant k < \omega$, $\bar{y} \in \mathrm{seq}_k(X_{\mathcal{U}})$ and $\bar{q} \in (\mathbb{Z}^+)^k$ we have that $p = p_{(\bar{y}/E_k, \bar{q})}$. If $k \geqslant 2$, then as in the proof of 4.8 we get a contradiction. Furthermore, as π is an isomorphism $k > 0$, so necessarily $k = 1$. Now, by an assumption in 4.14 we can find $z \in X$ such that $z \notin \bigcup\{y_\ell/E_1 : \ell < k\} = y_0/E_1$

and $z \in z/E_1 \cap X_{\mathcal{U}}$. Clearly, $G_{(1,p)} \subseteq \langle x' : x' \in y_0/E_1 \rangle^*_{G_1}$ by Lemma 4.4(3), and so also in this case we reach a contradiction. Thus, q_* is an integer, as promised. ∎

Claim 4.16. *In the context of Claim 4.14, $q_* \in \{1, -1\}$.*

Proof. If not, then we contradict Claim 4.15 when applied to π^{-1}. ∎

4.3. **The Proof of the Main Theorem**

Notice that in this subsection Hypothesis 4.7 is no longer assumed.

Conclusion 4.17. *Let $\mathfrak{m}[M] \in \mathrm{K}_3^{\mathrm{bo}}$, $\mathcal{U}, \mathcal{V} \subseteq M$ and $|\mathcal{U}| = |\mathcal{V}| = \aleph_0$. Then:*

$$(\star) \qquad M \restriction \mathcal{U} \cong M \restriction \mathcal{V} \Leftrightarrow G_{(1,\mathcal{U})}[\mathfrak{m}] \cong G_{(1,\mathcal{V})}[\mathfrak{m}].$$

Proof. If the LHS of $(\star)$ holds, then by 4.6 also the RHS of $(\star)$ holds. If the RHS of $(\star)$ holds, then the assumptions in 4.7 are fulfilled and thus 4.13 holds, and so the LHS of $(\star)$ holds. ∎

Convention 4.18. *In Fact 1.1 and Notation 1.8(5) instead of considering structures with domain ω we could have considered structures with domain an infinite subset of ω. We take the liberty of not distinguishing between these two variants. This happens most notably in the Proof of Main Theorem right below.*

Recall:

Fact 4.19. *The class $\mathbf{K}^{\mathrm{eq}}_\omega$ is Borel complete. In fact, there is a continuous map from Graph_ω into $\mathbf{K}^{\mathrm{eq}}_\omega$ which preserves isomorphism and its negation.*

Proof. See e.g. [10, pg. 295]. ∎

Proof of Main Theorem. Let M be as in Hypothesis 3.2. Fix $\mathfrak{m} \in \mathrm{K}_3^{\mathrm{bo}}(M)$ (cf. Fact 4.2) and assume without loss of generality that $G_1[\mathfrak{m}]$ has set of elements ω. For every $H \in \mathbf{K}^{\mathrm{eq}}_\omega$ we define $F[H] : H \to M$ by defining $F[H](n)$ by induction on $n < \omega$ as the minimal $k < \omega$ such that $\{(\ell, F[H](\ell)) : \ell < n\} \cup \{(n, k)\}$ is an isomorphism from $H \restriction (n+1)$ onto $M \restriction (\{F[H](\ell) : \ell < n\} \cup \{k\})$. The map $H \mapsto M \restriction \{F[H](n) : n < \omega\}$ is clearly continuous. We will show that the map $F' : M \restriction \mathcal{U} \mapsto G_{(1,\mathcal{U})}[\mathfrak{m}]$, for $\mathcal{U} \subseteq M$ infinite, is also continuous (cf. 4.18), thus concluding that the map $\mathbf{B} := F' \circ F : H \mapsto G_{(1,\{F[H](n) : n < \omega\})}[\mathfrak{m}]$ is a continuous map from $\mathbf{K}^{\mathrm{eq}}_\omega$ into TFAB_ω (cf. 4.18), so, by 4.17 and 4.19, we are done.

In order to show that F' is continuous, first recall that $\mathfrak{m} \in \mathrm{K}_3^{\mathrm{bo}}$ is fixed (cf. 4.1), and so in particular $\bar{p}$ is fixed. Now, given $a \in G_1[\mathfrak{m}]$, we have to compute from $\mathcal{U}$ whether $a \in G_{(1,\mathcal{U})}[\mathfrak{m}]$ or not. To this extent, let $a = \sum\{q^a_\ell x^a_\ell : \ell < n\}$ with the x_ℓ's pairwise distinct and $q_\ell \in \mathbb{Q}^+$. Now, as by 3.4(3), $X_{\mathcal{U}} = \bigcup\{X_s : s \subseteq_\omega \mathcal{U}\} = \bigcup\{X'_s : s \subseteq_1 \mathcal{U}\}$ and the latter is a partition of X, for every $\ell < n$, there is a unique finite $s^a_\ell \subseteq M$ such that the following conditions holds:

$$a \in G_{(1,\mathcal{U})}[\mathfrak{m}] \Leftrightarrow \bigwedge_{\ell < n} s^a_\ell \subseteq \mathcal{U}.$$

This suffices to show continuity of F', thus concluding the proof of the theorem. ∎

Remark 4.20. *We observe that in the context of the Proof of Main Theorem we can choose both M and $\mathfrak{m}$ to be computable stuctures, in the sense of computable model theory, i.e., all the relations and functions of the structure are computable.*

5. Completeness of Endorigid Torsion-Free Abelian Groups

Hypothesis 5.1. *Throughout this section the following hypothesis stands:*

(1) $T = (T, <_T)$ *is a rooted tree with* ω *levels and we denote by* $\mathrm{lev}(t)$ *the level of* t;
(2) $T = \bigcup_{n<\omega} T_n$, $T_n \subseteq T_{n+1}$, *and* $t \in T_n$ *implies that* $\mathrm{lev}(t) \leqslant n$;
(3) $T_0 = \emptyset$, T_n *is finite, and we let* $T_{<n} = \bigcup_{\ell<n} T_\ell$ *(so* $T_{<(n+1)} = T_n$*)*;
(4) *if* $s <_T t \in T_{n+1}$, *then* $s \in T_n$;
(5) T *is countable.*

Definition 5.2. *Let* $\mathrm{K}_1^{\mathrm{ri}}(T)$ *be the class of objects:*

$$\mathfrak{m}(T) = \mathfrak{m} = (X, X_n^T, \bar{f}^T : n < \omega) = (X, X_n, \bar{f} : n < \omega)$$

satisfying the following requirements:

(a) $X_0 \neq \emptyset$ *and, for* $n < \omega$, X_n *is finite and* $X_n \subsetneq X_{n+1}$, *and* $X_{<n} = \bigcup_{\ell<n} X_\ell$;
(b) $\bar{f} = (f_t : t \in T)$;
(c) *if* $n > 0$ *and* $t \in T_n \setminus T_{<n}$, *then* f_t *is a one-to-one function from* X_{n-1} *into* X_n;
(d) *for every* $t \in T$, $X_0 \cap \mathrm{ran}(f_t) = \emptyset$;
(e) *if* $s \leqslant_T t \in T_n$, *then* $f_s \subseteq f_t$;
(f) *if* $t \in T_{n+1} \setminus T_n$, $f_t(x) = y$ *and* $y \in X_n$, *then for some* $s <_T t$, $x \in \mathrm{dom}(f_s)$;
(g) *if* $s, t \in T_n$ *and* $y \in \mathrm{ran}(f_s) \cap \mathrm{ran}(f_t)$, *then for some* $r \in T_n$ *such that* $r \leqslant_T s, t$ *we have that* $y \in \mathrm{ran}(f_r)$, *equivalently,* $\mathrm{ran}(f_s) \cap \mathrm{ran}(f_t) = \mathrm{ran}(f_r)$, *for* $r = s \wedge t$, *where* $\wedge$ *is the natural semi-lattice operation taken in the tree* $(T, <_T)$;
(h) $X_{n+1} \supsetneq \bigcup \{\mathrm{ran}(f_t) : t \in T_{n+1} \setminus T_n\} \cup X_n$;
(i) *we let* $X = X^{\mathfrak{m}} = \bigcup_{n<\omega} X_n$;
(j) *if* $f_s(x) = f_t(x)$ *and* $x \in X_n \setminus X_{<n}$, *then* $f_s \restriction X_n = f_t \restriction X_n$ *and* $X_n \subseteq \mathrm{dom}(f_s) \cap \mathrm{dom}(f_t)$.

Notation 5.3. *For* $x \in X$, *we let* $\mathbf{n}(x)$ *be the unique* $n < \omega$ *such that* $x \in X_n \setminus X_{<n}$ *(so* $x \in X_0$ *implies* $\mathbf{n}(x) = 0$*)*.

Convention 5.4. $\mathfrak{m} = (X, X_n, \bar{f} : n < \omega) \in \mathrm{K}_1^{\mathrm{ri}}(T)$ *(cf. 5.2 and 5.6 below)*.

Observation 5.5. *In the context of Definition 5.2, we have:*

(1) *If* $m < n < \omega$, $t \in T_n \setminus T_{<n}$ *and for every* $s <_T t$ *we have* $s \in T_m$, *then:*

$$(X_{n-1} \setminus X_m) \cap \mathrm{ran}(f_t) = \emptyset.$$

(2) *If* $t \in T$, *then for every* $x \in \mathrm{dom}(f_t)$ *we have that* $x \neq f_t(x)$, *moreover there is a unique* $0 < n < \omega$ *such that* $x \in X_{n-1}$ *and* $f_t(x) \in X_n \setminus X_{n-1}$, *and for some* $s \in T_n \setminus T_{<n}$ *we have* $s \leqslant_T t$ *and* $f_s(x) = f_t(x)$.

Proof. We prove (1), by Definition 5.2(c) we know that f_t is one-to-one from X_{n-1} into X_n. If $n = 1$, then $m = 0$ and so $X_{n-1} = X_0 = X_m$, thus the conclusion is trivial. Suppose then that $n > 1$ and let $y \in (X_{n-1} \setminus X_m) \cap \mathrm{ran}(f_t)$, and let $x \in \mathrm{dom}(f_t)$ be such that $f_t(x) = y$. Then, by Definition 5.2(f) there exists $s <_T t$ such that $x \in \mathrm{dom}(f_s)$. But then, using the assumption in (1), we have that $s \in T_m$ (so $m = 0$ is impossible by Definition 5.1(3)). Hence, by Definition 5.2(c), $\mathrm{ran}(f_s) \subseteq X_m$, so $y = f(x) \in X_m$, contradicting the fact that $y \in (X_{n-1} \setminus X_m)$.

We prove (2). Assume that x, t, and thus also f_t, are fixed and $x \in \mathrm{dom}(f_t)$. Let $s \leqslant_T t$ be $\leqslant_T$-minimal such that $f_s(x)$ is well-defined, and let $n < \omega$ be such that $s \in T_n \setminus T_{<n}$ (notice that $n \geqslant 1$ since $T_0 = \emptyset$). Clearly, there is unique $m < \omega$ such that $x \in X_m \setminus X_{<m}$. As $x \in \mathrm{dom}(f_s)$ and $s \in T_n \setminus T_{<n}$ necessarily $m < n$, so

$x \in X_{<n}$. But by the choice of s we have that $r <_T s$ implies $x \notin \mathrm{dom}(f_r)$. By the last two sentences and Def. 5.2(f) we have $f_s(x) \in X_n \setminus X_{<n}$, but $f_t(x) = f_s(x)$. ■

Claim 5.6. *For T as in Hypothesis 5.1, $\mathrm{K}_1^{\mathrm{ri}}(T) \neq \emptyset$ (cf. Definition 5.2).*

Proof. Straightforward. ■

Definition 5.7. *On X (cf. Convention 5.4) we define:*

(1) for $x \in X$, $\mathrm{suc}(x) = \{f_t(x) : t \in T, x \in \mathrm{dom}(f_t)\}$;
(2) for $x, y \in X$, we let $x <_X y$ if and only if for some $0 < n < \omega$ and $x_0, ..., x_n \in X$ we have that $\bigwedge_{\ell<n} x_{\ell+1} \in \mathrm{suc}(x_\ell)$, $x = x_0$ and $y = x_n$;
(3) $\mathrm{seq}_k(X) = \{\bar{x} \in \mathrm{seq}_k(X) : \bar{x}$ is injective$\}$;
(4) we say that $\bar{x} \in \mathrm{seq}_k(X)$ is reasonable when the following happens:

$$n(1) < n(2),\ x_{i(1)} \in X_{n(1)} \setminus X_{<n(1)},\ x_{i(2)} \in X_{n(2)} \setminus X_{<n(2)} \Rightarrow i(1) < i(2);$$

(5) $<^k_X$ is the partial order on $\mathrm{seq}_k(X)$ defined as $\bar{x}^1 <^k_X \bar{x}^2$ if and only if $\bar{x}^1, \bar{x}^2 \in \mathrm{seq}_k(X)$ and there are $0 < n < \omega$, $\bar{y}^0, ..., \bar{y}^n \in \mathrm{seq}_k(X)$ and $t_0, ..., t_{n-1} \in T$ such that for every $\ell < n$ we have that $f_{t_\ell}(\bar{y}^\ell) = \bar{y}^{\ell+1}$, and $(\bar{x}^1, \bar{x}^2) = (\bar{y}^0, \bar{y}^n)$;
(6) notice that for $k = 1$ we have that $<^k_X = <_X$, where $<_X$ is as in (2) (ignoring the difference between x and (x), for $x \in X$);
(7) for $k \geqslant 1$, let E_k be the closure of $\{(\bar{x}, \bar{y}) : \bar{x} <^k_X \bar{y}\}$ to an equivalence relation.

Observation 5.8. *If $\bar{x}^1 <^k_X \bar{x}^2$ (cf. 5.7(5)), then there is a unique $\bar{t}$ such that:*

(a) $\bar{t} \in T^n$ for some $0 < n < \omega$;
(b) $f_{\bar{t}}(\bar{x}^1) = \bar{x}^2$, where $f_{\bar{t}} = (f_{t_{n-1}} \circ \cdots \circ f_{t_0})$ and $t_0, ..., t_{n-1}$ are as as in 5.7(5);
(c) for every $\ell < n$, there is no $s <_T t_\ell$ such that $f_s(\bar{y}_\ell)$ is well-defined, where $(\bar{y}_0, ..., \bar{y}_{n-1})$ are as in 5.7(5);
(d) if $\bar{t}' \in T^n$ is as in clauses (a)-(b) above and $\ell < n$, then $t_\ell \leqslant_T t'_\ell$.

Observation 5.9. *(1) $(X, <_X)$ is a tree with ω levels;*

(2) every $z \in X_0$ is a root of the tree $(X, <_X)$, further, for every $n < \omega$, some $z \in X_{n+1} \setminus X_n$ is a root of the tree $(X, <_X)$, and so $z/E_1 \cap X_n = \emptyset$;
(3) if $y \in X_{n+1} \setminus X_n$, then for at most one $x \in X_n$ we have $y \in \mathrm{suc}(x)$;
(4) if $y \in \mathrm{suc}(x)$, then $\{t \in T : f_t(x) = y\}$ is a cone of T;
(5) if $\bar{x} \in \mathrm{seq}_k(X)$, then some permutation of $\bar{x}$ is reasonable (cf. Definition 5.7(4));
(6) if $f_t(\bar{x}) = \bar{y}$ and $\bar{x}$ is reasonable, then so is $\bar{y}$;
(7) for every $k \geqslant 1$, $(\mathrm{seq}_k(X), <^k_X)$ is a tree;
(8) if $\bar{x} <^k_X \bar{y}$ and $\bar{x}$ is reasonable, then $\bar{y}$ is also reasonable;
(9) if $\bar{x} \in \mathrm{seq}_k(X)$ is reasonable, $\bar{x} \leqslant^k_X \bar{y}^1 = (y^1_0, ..., y^1_{k-1})$, $\bar{x} \leqslant^k_X \bar{y}^2 = (y^2_0, ..., y^2_{k-1})$ and $y^1_{k-1} = y^2_{k-1}$, then $\bar{y}^1 = \bar{y}^2$.

Proof. Items (1)-(2) are clear, where (2) is by 5.2(h). Item (3) is by Definition 5.2(f)-(g). Items (4) and (5) are also easy (and (4) is not used (except in 5.11(1)) but we retain it to give the picture). Item (6) can be proved for $t \in T_n \setminus T_{<n}$ by induction on $n < \omega$. Finally, (7) and (8) are easy, and (9) is easy to see using 5.8 and 5.2(j). ■

Definition 5.10. *Let $\mathfrak{m} \in \mathrm{K}_1^{\mathrm{ri}}(T)$ (i.e. as in Convention 5.4).*

(1) Let $G_2 = G_2[\mathfrak{m}]$ be $\bigoplus\{\mathbb{Q}x : x \in X\}$.
(2) Let $G_0 = G_0[\mathfrak{m}]$ be the subgroup of G_2 generated by X, i.e., $\bigoplus\{\mathbb{Z}x : x \in X\}$.
(3) For $t \in T$, let:
   (a) $H_{(2,t)} = \bigoplus\{\mathbb{Q}x : x \in \mathrm{dom}(f_t)\}$;
   (b) $I_{(2,t)} = \bigoplus\{\mathbb{Q}x : x \in \mathrm{ran}(f_t)\}$;

(c) $\hat{f}^2_t$ is the (unique) isomorphism from $H_{(2,t)}$ onto $I_{(2,t)}$ such that $x \in \mathrm{dom}(f_t)$ implies that $\hat{f}^2_t(x) = f_t(x)$ (cf. Definition 5.2(c)).

(4) For $t \in T$, we define $H_{(0,t)} := H_{(2,t)} \cap G_0$ and $I_{(0,t)} := I_{(2,t)} \cap G_0$;

(5) For $\hat{f}^2_t$ as above, we have that $\hat{f}^2_t[H_{(0,t)}] = I_{(0,t)}$. We define $\hat{f}^0_t$ as $\hat{f}^2_t \restriction H_{(0,t)}$.

(6) We define the partial order $<_*$ on G^+_0 by letting $a <_* b$ if and only if $a \neq b \in G^+_0$ and, for some $0 < n < \omega$, $a_0, ..., a_n \in G_0, a_0 = a, a_n = b$ and:

$$\ell < n \Rightarrow \exists t \in T(\hat{f}^0_t(a_\ell) = a_{\ell+1}).$$

(7) For $a = \sum_{\ell<m} q_\ell x_\ell$, with $x_\ell \in X$ and $q_\ell \in \mathbb{Q}^+$, let $\mathrm{supp}(a) = \{x_\ell : \ell < m\}$.

(8) For $a \in G^+_2$, let $\mathbf{n}(a)$ be the minimal $n < \omega$ such that $a \in \langle X_n \rangle^*_{G_2}$.

Lemma 5.11. *(1) If $\{t \in T : \hat{f}^2_t(a) = b\} \neq \emptyset$, then it is a cone of T.*

(2) $<_* \restriction X = <_X$ (where $<_X$ is as in Definition 5.7(2)).

(3) $(G^+_0, <_*)$ is a countable tree with ω levels (recall Hypothesis 5.1(1)).

(4) If $s \leqslant_T t$, then $\hat{f}^\ell_s \subseteq \hat{f}^\ell_t$, for $\ell \in \{0, 2\}$.

(5) If $t \in T$, $\hat{f}^2_t(a) = b$ and $a \in G^+_0$, then $\mathbf{n}(a) < \mathbf{n}(b)$ (cf. Definition 5.10(8)).

(6) If $a <_* b$ (so $a, b \in G^+_0$), then the sequence $(a_\ell : \ell \leqslant n)$ from 5.10(6) is unique.

(7) If $a_1 <_* a_2$, and, for $\ell \in \{1, 2\}$, $a_\ell = \sum_{i<k} q^\ell_i x^\ell_i$, $q^\ell_i \in \mathbb{Q}^+$, $\bar{x}^\ell = (x^\ell_i : i < k) \in \mathrm{seq}_k(X)$, then maybe after replacing $\bar{x}^1$ with a permutation of it we have:

$$\bar{x}^0 \leqslant^k_X \bar{x}^1 \text{ and } q^1_i = q^2_i \text{ (for } i < k).$$

Proof. Unraveling definitions, we elaborate only on item (5). Concerning item (5), as $a \neq 0$, let $a = \sum_{i\leqslant n} q_i x_i$, $x_i \in X$ with no repetitions, $q_i \in \mathbb{Q}^+$. Let $x_i \in X_{k(i)} \setminus X_{<k(i)}$ and w.l.o.g. $k(i) \leqslant k(i+1)$, for $i < n$ (cf. Observation 5.9(5)). Clearly $a \in \langle X_{k(n)} \rangle^*_{G_2}$ but $a \notin \langle X_{<k(n)} \rangle^*_{G_2}$. As $\hat{f}^2_t(a)$ is well-defined, clearly $\{x_i : i \leqslant n\} \subseteq \mathrm{dom}(f_t)$ and $b = \hat{f}^2_t(a) = \sum_{i\leqslant n} q_i f_t(x_i)$ and, as f_t is 1-to-1, the sequence $(f_t(x_i) : i \leqslant n)$ is with no repetitions. By Observation 5.5(2) applied with n there as $k(n)$ here, $f_t(x_n) \notin \langle X_{k(n)} \rangle^*_{G_2}$, hence we have that $\mathbf{n}(b) \geqslant n(f_t(x_n)) > k(n) = \mathbf{n}(a)$. ■

Claim 5.12. *If (A), then (B), where:*

(A) (a) $a, b_\ell \in G^+_2$, for $\ell < \ell_*$;
(b) $a \leqslant_* b_\ell$ and the b_ℓ's are with no repetitions;
(c) $a = \sum\{q_i x_i : i < j\}$, $q_i \in \mathbb{Q}^+$;
(d) $\bar{x} = (x_i : i < j) \in \mathrm{seq}_j(X)$ and it is reasonable;

(B) there are, for $\ell < \ell_*$, $\bar{y}^\ell = (y_{(\ell,i)} : i < j)$ such that:
(a) $y_{(\ell,i)} =: y^\ell_i \in X$ and $\bar{x} \leqslant^j_X \bar{y}^\ell$ (cf. Definition 5.7(5));
(b) $b_\ell = \sum\{q_i y_{(\ell,i)} : i < j\}$, (so the $\bar{y}^\ell$ are pairwise distinct, as the b_ℓ's are);
(c) $(y_{(\ell,i)} : i < j) \in \mathrm{seq}_j(X)$ and it is reasonable;
(d) if $j > 1$ and $\ell_* > 1$, then there are $\ell_1 \neq \ell_1 < \ell_*$ and $i_1, i_2 < j$ such that:
(i) if $\ell < \ell_*$, $i < j$ and $y_{(\ell,i)} = y_{(\ell_1,i_1)}$, then $(\ell, i) = (\ell_1, i_1)$;
(ii) if $\ell < \ell_*$, $i < j$ and $y_{(\ell,i)} = y_{(\ell_2,i_2)}$, then $(\ell, i) = (\ell_2, i_2)$.

Proof. By the definition of $\leqslant_*$ there are $(y_{(\ell,i)} : i < j,\ \ell < \ell_*)$ and by by 5.9(6) and 5.11(7) they satisfying clauses (a)-(c) of (B). Recall that $(\{\bar{y} : \bar{x} \leqslant^j_X \bar{y}\}, \leqslant^j_X)$ is a tree (as $(\mathrm{seq}_j(X), \leqslant^j_X)$ is a tree). We now show (B)(d). There are two cases.

<u>Case 1</u>. $\{\bar{y}^\ell : \ell < \ell_*\}$ is not linearly ordered by $\leqslant^j_X$.

Then there are $\ell(1) \neq \ell(2) < \ell_*$ such that $\bar{y}^{\ell(1)}, \bar{y}^{\ell(2)}$ are locally $\leqslant^j_X$-maximal. So we can choose $i_1, i_2 < j$ such that we have the following:

$$x^{\ell_1}_{i_1} \in X_{\mathbf{n}(b_{\ell_1})} \setminus X_{<\mathbf{n}(b_{\ell_1})} \text{ and } x^{\ell_2}_{i_2} \in X_{\mathbf{n}(b_{\ell_2})} \setminus X_{<\mathbf{n}(b_{\ell_2})},$$

notice that using the assumption that the sequences are reasonable we can choose $i_1 = j-1 = i_2$, see 5.11(5) and 5.9(9). Hence, ℓ_1, ℓ_2, i_1, i_2 are as required for (d).

Case 2. Not Case 1.

So w.l.o.g. we have that, for every $\ell < \ell_* - 1$, $\bar{y}^\ell <^j_X \bar{y}^{\ell+1}$. Now, for $\ell < \ell_*$ and $i < j$, let $n(\ell, i) = \mathbf{n}(y^\ell_i)$. Let then:

$(\cdot_1)$ $i(1) < j$ be such that $i < j$ implies $n(0,i) \geqslant n(0, i(1))$;

$(\cdot_2)$ $i(2) < j$ be such that $i < j$ implies $n(\ell_* - 1, i) \leqslant n(\ell_* - 1, i(2))$.

Then $(0, i(1))$, $(\ell_* - 1, i(2))$ are as required. As, for $\ell < \ell_*$, $\bar{y}^\ell$ is reasonable we can actually choose $i(1), i(2)$ such that $i(1) = 0$ and $i(2) = j_* - 1$. ■

Definition 5.13. *Let $(p_a : a \in G^+_0)$ be a sequence of pairwise distinct primes s.t.:*

$$(\oplus) \qquad a = \sum_{\ell<k} q_\ell x_\ell,\ q_\ell \in \mathbb{Z}^+,\ (x_\ell : \ell < k) \in \mathrm{seq}_k(X) \Rightarrow p_a \not| \, q_\ell.$$

(1) For $a \in G^+_0$, let:

$$\mathbb{P}^{\leqslant_*}_a = \{p_b : b \in G^+_0, b \leqslant_* a\} \text{ and } \mathbb{P}^{\geqslant_*}_a = \{p_b : b \in G^+_0, a \leqslant_* b\}.$$

(2) Let $G_1 = G_1[\mathfrak{m}] = G_1[\mathfrak{m}(T)] = G_1[T]$ be the subgroup of G_2 generated by:

$\{m^{-1}a : a \in G^+_0,\ m \in \omega \setminus \{0\}$ a product of primes from $\mathbb{P}^{\leqslant_*}_a$, poss. with repetitions$\}$.

(3) For a prime p, let $G_{(1,p)} = \{a \in G_1 : a$ is divisible by p^m, for every $0 < m < \omega\}$ (notice that, by Observation 2.5, $G_{(1,p)}$ is always a pure subgroup of G_1).

(4) For $b \in G^+_1$, let $\mathbb{P}_b = \{p_a : a \in G^+_0, G_1 \models \bigwedge_{m<\omega} p^m_a \,|\, b\}$.

(5) For $t \in T$ and $\ell \in \{0,1,2\}$, let:

$$H_{(\ell,t)} = \langle x : x \in \mathrm{dom}(f_t)\rangle^*_{G_\ell} \quad \text{and} \quad I_{(\ell,t)} = \langle x : x \in \mathrm{ran}(f_t)\rangle^*_{G_\ell}$$

Remark 5.14. *(1) If $a, b \in G^+_1$ and $\mathbb{Q}a = \mathbb{Q}b \subseteq G_2$, then $\mathbb{P}_a = \mathbb{P}_b$.*

(2) If $b \in G^+_1$, then $\mathbb{P}_b$ is infinite.

(3) If $a \leqslant_* b$, then $\mathbb{P}_a \subseteq \mathbb{P}_b$.

Proof. Essentially as in Observation 2.5. ■

Claim 5.15. *Let $a \in G^+_0$ and $p = p_a$.*

(1) $W_p := \{b \in G^+_0 : a \leqslant_* b\}$ is a linearly indep. subset of G_2 as a $\mathbb{Q}$-vector space.

(2) We can choose a sequence $(b_{(a,n)} : n < \omega)$ listing W_p such that for all $n < \omega$:

$$\mathrm{supp}(b_{(a,n)}) \not\subseteq \bigcup \{\mathrm{supp}(b_{(a,k)}) : k < n\}.$$

Proof. (1) follows from (2) and (2) is clear from the definition of $\mathfrak{m}$ from 5.2, that is, by 5.11(3) we can find a list $(b_{(a,n)} : n < \omega)$ such that $b_{(a,k)} <_* b_{(a,n)}$ implies $k < n$. So for such an n, $b_{(a,n)}$ is $<_*$-maximal in $\{b_{(a,k)} : k \leqslant n\}$, and, as in the proof of 5.12, we have $\mathrm{supp}(b_{(a,n)}) \not\subseteq \bigcup \{\mathrm{supp}(b_{(a,k)}) : k < n\}$. ■

Choice 5.16. *For every $a \in G^+_0$ choose $(b_{(a,n)} : n < \omega)$ as in 5.15(2).*

Lemma 5.17. *(1) If $p = p_a$, $a \in G^+_0$, then:*

$$G_{(1,p)} = \langle b \in G^+_0 : a \leqslant_* b\rangle^*_{G_1}.$$

(2) For $t \in T$, $H_{(1,t)} := H_{(2,t)} \cap G_1$ and $I_{(1,t)} := I_{(2,t)} \cap G_1$ are pure in G_1.

(3) For $\hat{f}_t^{(i,2)}$ as in Definition 5.10(3c), $\hat{f}_t^{(i,2)}[H_{(1,t)}] \subseteq I_{(1,t)}$. We define:

$$\hat{f}_t^{(i,1)} = \hat{f}_t^{(i,2)} \restriction H_{(1,t)},$$

and for $\bar{t}$ a finite sequence of members of T we let:

$$\hat{f}_{\bar{t}}^{(i,1)} = (\cdots \circ \hat{f}_{t_\ell}^{(i,1)} \circ \cdots).$$

(4) $\hat{f}_t^2 \restriction H_{(1,t)}$ is not onto $I_{(1,t)}$.

(5) Assume $a = \sum_{\ell<k} q_\ell x_\ell \in G_0$, $k > 0$, $x_\ell \in X$, $q_\ell \in \mathbb{Q}^+$, $\bar{x} = (x_\ell : \ell < k) \in \mathrm{seq}_k(X)$ and $p = p_a$. If $b \in G_{(1,p)}$, then there are $j > 0$, $m > 0$, and, for $i < j$, $\bar{y}^i$, b_i and $q'_i \in \mathbb{Q}^+$ such that the following conditions are verified:
   - *(a) for $i < j$, $\bar{x} \leqslant_X^k \bar{y}^i$;*
   - *(b) $(b_i = \sum_{\ell<k} q_\ell y_\ell^i : i < j)$ is linearly independent;*
   - *(c) $b = \sum\{q'_i b_i : i < j\}$;*
   - *(d) for $i < j$, $ma \leqslant_* mb_i$.*

Proof. We prove item (1). The RTL inclusion is clear by 5.13(2). We prove the other implication. To this extent, recall that:

$(*_1)$ $p = p_a$ and we let $W_p := \{b \in G_0^+ : a \leqslant_* b\}$.

Now:

$(*_2)$ For the sake of contradiction, suppose that there is $c \in G_{(1,p)} \setminus \langle W_p \rangle_{G_1}^*$.

$(*_3)$ For $d \in G_1$, let:

$$n(d) = \min\{k < \omega : \mathrm{supp}(d) \cap \bigcup_{\ell<\omega} \mathrm{supp}(b_{(a,\ell)}) \subseteq \bigcup_{\ell<k} \mathrm{supp}(b_{(a,\ell)})\}.$$

$(*_4)$ W.l.o.g. we have:
   - (a) $n(c) = \min\{n(d) : d \in G_{(1,p)} \setminus \langle W_p \rangle_{G_1}^*\}$;
   - (b) if $n(c) > 0$, then $|\mathrm{supp}(c) \cap \mathrm{supp}(b_{(a,n(c))})|$ equals the following:

$$\min\{|\mathrm{supp}(d) \cap \mathrm{supp}(b_{(a,n))})| : d \in G_{(1,p)} \setminus \langle W_p \rangle_{G_1}^*, n(d) = n(c)\}.$$

As $G_0 \leqslant G_1 = \langle G_0 \rangle_{G_1}^*$, there is $m \in \mathbb{Z}^+$ such that $mc \in G_0$ and, as $c \neq 0$, we have that $mc \in G_0^+$. Now:

$(*_5)$ For some $0 \leqslant k < \omega$ we have:
   - (a) $c_1 := p^{-k} mc \in G_0 \setminus pG_0$;
   - (b) $c_1 \in G_{(1,p)} \setminus \langle W_p \rangle_{G_1}^*$;
   - (c) w.l.o.g. $c_1 = c$, i.e., $c \in G_0 \setminus pG_0$.

[Why (b)? As $c \in G_{(1,p)}$, $\langle W_p \rangle_{G_1}^* \leqslant_* G_1$, so $\langle W_p \rangle_{G_1}^* \leqslant_* G_{(1,p)}, \mathrm{supp}(c_1) = \mathrm{supp}(c)$.]

Now, we claim:

$(\star)$ Recalling the def. of $\mathbb{Q}_p$ from 2.6, there is an homomorphism $h_0 : G_0 \to \mathbb{Q}_p$ s.t.:
   - (a) $h_0(b_{(a,n)}) = 0$, for all $n < \omega$;
   - (b) $h_0(c) \neq 0$.

We first show why $(\star)$ is enough and then we show why $(\star)$ is true.

$(*_6)$ (a) Let $G_* = \langle W_p^+ \rangle_{G_1}$, where:

$$W_p^+ = \{p^{-m} b_{(a,n)} : m, n < \omega\};$$

   - (b) there is $h_* : G_* \to \mathbb{Q}_p$ extending $h_0 \restriction (G_* \cap G_0)$ which is constantly 0;
   - (c) there is $h_{0.5} : G_0 + G_* \to \mathbb{Q}_p$ extending h_0 and h_*, and let $G_{0.5} = G_0 + G_*$.

Why $(*_6)$? Concerning clause (b), notice first that:

$(*_{6.1})$ $G_* \cap G_0 = \langle b_{(a,\ell)} : \ell < \omega \rangle_{G_0}$.

Why $(*_{6.1})$? Letting $H = \langle b_{(a,\ell)} : \ell < \omega \rangle_{G_0}$, by 5.16 and 5.17, we have that $\{b_{(a,\ell)} : \ell < \omega\}$ is a free basis of H (that is $H = \bigoplus_{\ell<\omega} \mathbb{Z} b_{(a,\ell)}$), $H \leqslant G_0$ and $H \leqslant G_*$. Now, if $d \in G_* \setminus H$, then necessarily $d = \sum\{r_i b_{(a,i)} : i \in u\}$, where:

$(\cdot_1)$ $u \subseteq_\omega \omega$;
$(\cdot_2)$ $r_i \in \mathbb{Q}^+$;
$(\cdot_3)$ $r_i = p^{-m(i)} r^i$, $r^i \in \mathbb{Z}^+$;
$(\cdot_4)$ w.l.o.g. $m(i) > 0$ implies $p \not| \, r^i$.

Let:

$$Y = \bigcup\{\mathrm{supp}(b_{(a,\ell)}) : \ell < n(c)\} \quad \text{and} \quad q'_y = \sum\{r_i q_\ell : y^i_\ell = y\} \text{ (for } y \in Y),$$

where:

$(*_{6.2})$ $a = \sum\{q_\ell x_\ell : \ell < k\}$, $q_\ell \in \mathbb{Z}^+$ and $(x_\ell : \ell < k) \in \mathrm{seq}_k(X)$.

By $(\cdot_1)$-$(\cdot_4)$, necessarily $d \in p^{-m} G_0$, for some $m < \omega$; as $d \notin H$, necessarily for some $i \in u$, $p^{m(i)} r^i \notin \mathbb{Z}$, so necessarily $m(i) > 0$, hence $p \not| \, r^i$. Choose a maximal such $i \in u$ and call it $i(*)$. So there is $y \in \mathrm{supp}(b_{(a,i(*))}) \setminus \bigcup\{\mathrm{supp}(b_{(a,j)}) : j < i(*)\}$. Hence $q'_y \notin \mathbb{Z}$, recalling that by $\oplus$ from 5.13 we have that:

$$a = \sum_{\ell<k} q_\ell x_\ell,\ q_\ell \in \mathbb{Z}^+,\ (x_\ell : \ell < k) \in \mathrm{seq}_k(X) \Rightarrow p_a \not| \, q_\ell.$$

So $d \notin G_0$. As d was any member of $G_* \setminus H$ we get that $G_* \setminus H$ is disjoint from G_0, but $H \subseteq G_0$ and $H \leqslant G_*$, so $G_* \cap G_0 = H$, i.e., $(*_{6.1})$ holds indeed.

Coming back to the proof of $(*_6)$, $h_0 \restriction (G_* \cap G_0)$ is the zero homomorphism from $H = \langle b_{(a,\ell)} : \ell < \omega \rangle_{G_0}$ into $\mathbb{Q}_p$, by $(\star)$ (which we are assuming to be true). Hence, the zero homomorphism from G_* into $\mathbb{Q}_p$ extends $h_0 \restriction (G_* \cap G_0)$, as desired in $(*_6)$(b). Concerning clause (c), as $h_0 \in \mathrm{Hom}(G_0, \mathbb{Q}_p)$ and $h_* \in \mathrm{Hom}(G_*, \mathbb{Q}_p)$ agree on $G_* \cap G_0$ it is easy to see that there is one and only one $h_{0.5} \in \mathrm{Hom}(G_* + G_0, \mathbb{Q}_p)$ extending both homomorphisms (using classical abelian group theory).

$(*_7)$ There is $h_1 : G_1 \to \mathbb{Q}_p$ extending $h_{0.5}$.

Why $(*_7)$? A set of generators for G_1 is $\bigcup\{Z_{p_b} : b \in G_0^+\} \cup G_0$ where:

$$Z_{p_b} = \{p_b^{-m} d : b \leqslant_* d, m \geqslant 0\}.$$

Also, G_1 is generated over G_* by $\bigcup\{Z_{p_b} : b \in G_0^+ \setminus \{a\}\}$. We define:

$$h_{0.7} : \bigcup\{Z_{p_b} : b \in G_0^+ \setminus \{a\}\} \cup G_* \to \mathbb{Q}_p$$

by the following conditions:

$(\cdot_1)$ $h_{0.7} \restriction G_{0.5} = h_{0.5}$;
$(\cdot_2)$ if $b \in G_0 \setminus \{a\}$, $b \leqslant_* d$ and $m \geqslant 0$, then:

$$h_{0.7}(p_b^{-m} d) = p_b^{-m}(h_{0.7}(d)) = p_b^{-m}(h_0(d)).$$

Why $h_{0.7}$ is well-defined? Because for $b \in G_0 \setminus \{a\}$, $p_b \neq p = p_a$, hence $\mathbb{Q}_p$ is p_b-divisible and torsion-free (of course).

$(*_8)$ $(*_7)$ gives a contradiction.

Why? So $h_1 : G_1 \to \mathbb{Q}_p$ is such that $h_1(c) \neq 0$, but

$$r \in \mathbb{Q}_p^+ \Rightarrow \exists m \geqslant 0, \mathbb{Q}_p \models p^m \not| \, r.$$

Hence, there is $m \geqslant 0$ such that $G_1 \models p^m \nmid c$, contradicting the fact that $c \in G_{(1,p)}$. We finally show why $(\star)$ above is true. To this extent, let:

$$B = \bigcup_{\ell < \omega} \mathrm{supp}(b_{(a,\ell)}) \text{ and, for } n < \omega,\ B_n = \bigcup_{\ell < n} \mathrm{supp}(b_{(a,\ell)}).$$

We distinguish two cases:
Case 1. $\mathrm{supp}(c) \not\subseteq B$.
Choose $x \in \mathrm{supp}(c) \not\subseteq B$ and choose $h_{-1} : X \to \mathbb{Z}$ by:

$(\cdot)$ $h_{-1} \restriction (X \setminus \{x\})$ is constantly 0;
$(\cdot)$ $h_{-1}(x) = 1 \in \mathbb{Z}^+$.

Clearly h_{-1} extends to an homomorphism h_0 of G_0 into $\mathbb{Z} \leqslant \mathbb{Q}_p$, as promised in $(\star)$.
Case 2. $\mathrm{supp}(c) \subseteq B$.
Let n_* be minimal such that $\mathrm{supp}(c) \subseteq B_{n_*}$. Now, $n_* = 0$ is impossible, as $c \neq 0$. Consider now the free abelian group $K = \bigoplus_{x \in B_{n_*}} \mathbb{Z}x$. So $c \in K$ and $c \in G_{(1,p)} \setminus \langle W_p \rangle^*_{G_1}$. By $(*_5)$, clearly $c \in K \setminus \langle b_{(a,\ell)} : \ell < n_* \rangle^*_K$, hence $\exists g \in \mathrm{Hom}(K, \mathbb{Q}_p)$ s.t.:

$(\cdot)$ $g(b_{(a,\ell)}) = 0$, for $\ell < n_*$;
$(\cdot)$ $g(c) \neq 0$.

For $i < \omega$, let $K_i = \langle B_{n_*+i} \rangle_{G_1}$. Now, we choose $h_{(0,i)} : \langle B_i \cup (X \setminus B) \rangle_{G_1} \to \mathbb{Q}_p$ by induction on $i < \omega$. If $i = 0$, we choose $h_{(0,i)}$ such that $g \subseteq h_{(0,i)}$ and $h_{(0,i)} \restriction (X \setminus B)$ is zero. So suppose that $i = j + 1$. Now, first recall that $B_{n_*+i} \setminus B_{n_*+j} \neq \emptyset$, by 5.16. Let then $x \in B_{n_*+i} \setminus B_{n_*+j}$. Hence $x \in \mathrm{supp}(b_{(a,j)}) \setminus B_{n_*+j}$.
Now, recalling $(*_{6.2})$ and 5.16 we have:

$$b_{(a,j)} = \sum \{q_\ell y_\ell : \ell < k = |\mathrm{supp}(a)|\},$$

for some pairwise distinct $y_\ell \in X$ and some $q_\ell \in \mathbb{Q}_p^{\odot}$. W.l.o.g., $x = y_0$. Let $h^+_{(0,j)} \in \mathrm{Hom}(\langle K_j \cup \{y_\ell : \ell \in (0,k)\} \rangle_{G_1}, \mathbb{Q}_p)$ extend $h_{(0,j)}$ in such a way that it is zero on y_ℓ if $\ell > 0$ and $y_\ell \notin K_j$. We want to choose $h \in \mathrm{Hom}(K_i, \mathbb{Q}_p)$ extending $h^+_{(0,i)}$ such that $h_{(0,i)}(b_{(a,n_*+j)}) = 0$. Now, $h^+_{(0,j)}(\sum\{q_\ell y_\ell : \ell \in (0,k)\})$ is well-defined and belongs to $\mathbb{Q}_p$, hence it is divisible by q_0 (as $q_0 \in \mathbb{Q}_p^{\odot}$). So we choose $h_{(0,i)}(y_0) = -h^+_{(0,j)}(\sum\{q_\ell y_\ell : \ell \in (0,k_*)\})/q_0$. So we can carry the induction and finally letting $h_0 = \bigcup_{i<\omega} h_{(0,i)}$ we fulfill the request in $(\star)$.
This concludes the proof of item (1). Concerning item (2), notice:

$$H_{(1,t)} = \langle \mathbb{Z}x : x \in \mathrm{dom}(f_t) \rangle^*_{G_1},$$

$$I_{(1,t)} = \langle \mathbb{Z}x : x \in \mathrm{ran}(f_t) \rangle^*_{G_1}.$$

Item (3) is by item (2) and the following observation, if $f_t(x) = y$, then we have $x \leqslant_* y$ (recall 5.7(2)), and so $\mathbb{P}_x \subseteq \mathbb{P}_y$ (cf. 5.14(3)). Concerning item (4), assume that $0 < n < \omega$, $t \in T_n \setminus T_{<n}$, $x \in X_{n-1} \setminus X_{<n-1}$ and let $y = f_t(x) \in X_n \setminus X_{<n}$ (cf. Observation 5.5), notice that in particular $x <_* y$. So p_y is well-defined, since $y \in G_0^+$, and we have the following:

(a) $G_1 \models p_y \nmid x$, and so $H_{(1,t)} \models p_y \nmid x$ (as $H_{(1,t)}$ is pure in G_1, cf. item (2));
(b) $G_1 \models \bigwedge_{m<\omega} p_y^m \,|\, y$.

[Why (b)? By the definition of G_1. Why (b)? Recalling that $x <_* y$.]
But then, since by item (3), $\hat{f}_t \restriction H_{(1,t)}$ is an embedding of $H_{(1,t)}$ into $I_{(1,t)}$ we have that $\hat{f}_t[H_{(1,t)}] \models p_y \nmid f(x) \wedge f(x) = y$. On the other hand, since $I_{(1,t)}$ is pure in

G_1 (cf. (2) of this lemma) we have that for every $m < \omega$, $p_y^{-m} y \in I_{(t,1)}$ (cf. 2.4). Finally, item (5) is by clause (1) and unraveling definitions. ∎

Theorem 5.18. *Let $\mathfrak{m}(T) \in \mathrm{K}_1^{\mathrm{ri}}(T)$.*

(1) We can modify the construction so that $G_1[\mathfrak{m}(T)] = G_1[T]$ has domain ω and the function $T \mapsto G_1[T]$ is Borel (for T a tree with domain ω).

(2) If T is not well-founded, then $G_1[T] = G_1$ has a 1-to-1 $f \in \mathrm{End}(G_1)$ which is not multiplication by an integer and such that $G_1/f[G_1]$ is not torsion.

(3) If T is well-founded, then $G_1[T]$ is endorigid.

Proof. Item (1) is easy. We prove item (2). Let $(t_n : n < \omega)$ be a strictly increasing infinite branch of T. By Lemma 5.11(4), $(\hat{f}^2_{t_n} : n < \omega)$ is increasing, by Definition 5.10(3c), $\hat{f}^2_{t_n}$ embeds $H_{(2,t_n)}$ into $I_{(2,t_n)}$, thus $\hat{f}^2 = \bigcup_{n<\omega} \hat{f}^2_{t_n}$ is an embedding of G_2 into G_2, since $G_2 = \bigcup_{n<\omega} H_{(2,t_n)}$, where $(H_{(2,t_n)} : n < \omega)$ is a chain of pure subgroups of G_2 with limit G_2, because, recalling 5.2(e), we have that:

$$H_{(2,t_n)} \supseteq \mathrm{dom}(f_{t_n}) \subseteq \mathrm{dom}(f_{t_{n+1}}) \subseteq H_{(2,t_{n+1})}$$

and by 5.2(c) we have that $\bigcup_{n<\omega} H_{(2,t_n)} = G_2$. Thus $\hat{f}^1 := \hat{f} \restriction G_1 = \bigcup_{n<\omega} \hat{f}^1_{t_n} = \bigcup_{n<\omega} \hat{f}^2_{t_n} \restriction H_{(1,t_n)}$ is an embedding of G_1 into G_1 (cf. Lemma 5.17(3)), in fact we have that $\mathrm{dom}(\hat{f}_{t_n}) = H_{(1,t_n)}$ (cf. Lemma 5.17(3)) and $G_1 = \bigcup_{n<\omega} H_{(1,t_n)}$, where $(H_{(1,t_n)} : n < \omega)$ is chain of pure subgroups of G_1 with limit G_1. Clearly $\hat{f}^1$ is not of the form $g \mapsto mg$ for some $m \in \mathbb{Z} \setminus \{0\}$, since for every $x \in \mathrm{dom}(f_t)$ we have $x \neq f_t(x)$ (cf. Obs. 5.5(2)). We claim that $G_1/\hat{f}^1[G_1]$ is not torsion. To this extent, first of all notice that $X_0 \neq \emptyset$ (by Definition 5.2(a)) and $X_0 \cap \mathrm{ran}(f_{t_n}) = \emptyset$ (by Definition 5.2(d)). Thus:

$$\mathrm{ran}(\hat{f}^1) \subseteq G^2_{X \setminus X_0} := \sum \{\mathbb{Q}x : x \in X \setminus X_0\} = \langle X \setminus X_0 \rangle^*_{G_2}.$$

Now, let $x \in X_0$, then $x \in G_1 \setminus \mathrm{ran}(\hat{f}^1)$, moreover, for $q \in \mathbb{Q} \setminus \{0\}$:

$$qx \notin G^2_{X \setminus X_0} \text{ and so } qx \notin \mathrm{ran}(\hat{f}^1),$$

and so in particular, for every $0 < n < \omega$ we have that $nx \notin \mathrm{ran}(\hat{f}^1)$, hence $n(x/(\mathrm{ran}(\hat{f}^1)) \neq 0$. This concludes the proof of item (2).

We now prove item (3). To this extent, suppose that $(T, <_T)$ is well-founded and, letting $G_1 = G_1[T]$, suppose that $\pi \in \mathrm{End}(G_1)$. We shall show that there is $m \in \mathbb{Z}$ such that, for every $a \in G_1$, $\pi(a) = ma$, i.e., G_1 is endorigid.

<u>Case 1</u>. The set $Y = \{x/E_1 : \text{ for some } y \in x/E_1, \pi(y) \notin \mathbb{Q}y\}$ is infinite.

$(*_1)$ Choose x_i, n_i, for $i < \omega$, such that:

- (a) n_i is increasing with i;
- (b) $x_i \in X_{n_{i+1}} \setminus X_{n_i}$;
- (c) $\pi(x_i) \notin \mathbb{Q}x_i$, $\mathrm{supp}(\pi(x_i)) \subseteq X_{n_{i+1}}$, $\mathrm{supp}(\pi(x_i)) \not\subseteq X_{n_i}$, $x_i/E_1 \cap X_{n_i} = \emptyset$;
- (d) $(x_i/E_1 : i < \omega)$ are pairwise distinct (this actually follows).

Note that, for $i < \omega$, we have (possible by 5.9(2)):

$(*_2)$ $\mathrm{supp}(\pi(x_i)) \subseteq x_i/E_1$.

[Why? As usual, by 5.17(5).]

Clearly $(\mathrm{supp}(x_\ell) : \ell \leqslant r)$ is a sequence of non-empty sets and $\mathrm{supp}(x_\ell) \subseteq X_{n_{\ell+1}} \setminus X_{n_\ell}$, so it is a sequence of pairwise disjoint non-empty sets. Now, for each $r < \omega$, let $x^+_r = \sum_{\ell \leqslant r} x_\ell$, $p_r = p_{x^+_r}$ and $\bar{x}_r = (x_\ell : \ell \leqslant r)$.

As $\pi \in \mathrm{End}(G_1)$, clearly $\pi(x_r^+) \in G_{(1,p_r)}$, hence by 5.17(5) applied to x_r^+, $\pi(x_r^+)$ here standing for a, b there we can find $j_r, m_r > 0$, and, for $j < j_r$, $\bar{y}^{(r,j)}$, b_j^r, $q_j^r \in \mathbb{Q}^+$ such that the following holds:

$(*_3)$ (a) for $j < j_r$, $\bar{x}_r \leqslant_X^{r+1} \bar{y}^{(r,j)}$;
(b) $(b_j^r = \sum_{\ell \leqslant r} y_\ell^{(r,j)} : j < j_r)$ is linearly independent;
(c) $\pi(x_r^+) = \sum\{q_j^r b_j^r : j < j_r\}$;
(d) for $j < j_r$, $m_r x_r^+ \leqslant_* m_r b_j^r$ (and $m_r x_r^+ \in G_0^+$).

As $\bar{x}_r \leqslant_X^{r+1} \bar{y}^{(r,j)}$ we can apply 5.8 and find a finite sequence $\bar{t}_j^r \in T^{<\omega}$ s.t.:

$(*_4)$ (A) if $\bar{x}_r <_X^{r+1} \bar{y}^{(r,j)}$, then:
(a) $f_{\bar{t}_j^r}^{(j,1)}(\bar{x}_r) = \bar{y}^{(r,j)}$;
(b) $\hat{f}_{\bar{t}_j^r}^{(j,1)}(x_r^+) = b_j^r$;
(c) for $\ell \leqslant r$ and $j < j_r$ we have $f_{\bar{t}_j^r}^{(j,1)}(x_\ell) \neq x_\ell$ and $\lg(\bar{t}_j^r) = 0$;
(d) $\bar{t}_j^r = (t_{(j,\ell)}^r : \ell < \lg(\bar{t}_j^r))$;
(e) abusing our notation we let $f_{\bar{t}_j^r \restriction 0}$ be $\mathrm{id}_{X_{n(r,j)}}$ for:

$$n(r,j) = \min\{n \in (n_j, n_{j+1}) : x_r \in X_n\};$$

(B) if $\bar{x}_r = \bar{y}^{(r,j)}$, then let $\bar{t}_j^r = ()$ and abuse our notation as in (A)(e).

$(*_5)$ (a) If $\ell \leqslant r$ and $j < j_r$, then $y_\ell^{(r,j)} \in x_\ell/E_1$;
(b) $\pi(x_\ell) = \sum\{q_j^r y_\ell^{(r,j)} : j < j_r\}$.

$(*_6)$ Letting $Y_\ell = \mathrm{supp}(\pi(x_\ell))$, we have:
(a) $(\cdot_1)$ Y_ℓ is finite and non-empty;
$(\cdot_2)$ moreover, $Y_\ell \not\subseteq \{x_\ell\}$ (cf. $(*_1)(c)$);
$(\cdot_3)$ $Y_\ell \subseteq x_\ell/E_1$.
(b) for $m \neq \ell$ we have $Y_m \cap Y_\ell = \emptyset$;
(c) if $\ell \leqslant r$, then:
$(\cdot_1)$ $Y_\ell \subseteq \{y_\ell^{(r,j)} : j < j_r\}$;
$(\cdot_2)$ if $y \in Y_\ell$, then $\sum\{q_j^r : j < j_r, y_\ell^{(r,j)} = y\} \neq 0$;
(d) for $\ell \leqslant r$ and $y \in Y_\ell$ let:

$$\mathcal{U}_{(r,\ell,y)} = \{j < j_r : f_{\bar{t}_j^r}(x_\ell) = y_\ell^{(r,j)} = y\};$$

(e) for $\ell \leqslant r$, we let $\mathcal{U}_{(r,\ell)} = \bigcup\{\mathcal{U}_{(r,\ell,y)} : y \in Y_\ell \setminus \{x_\ell\}\} \neq \emptyset$ (by (a)$(\cdot_2)$).

[Why $(*_6)$(a)$(\cdot_3)$? As usual by 5.17(5). Why $(*_6)$(c)? As the x_ℓ/E_1's are pairwise disjoint and $(*_6)$(a)$(\cdot_3)$.]

$(*_7)$ If $\ell + 1 \leqslant r$, $j \in \mathcal{U}_{(r,\ell)}$, then $\lg(\bar{t}_j^r) \leqslant n_{\ell+1}$.

[Why? As if $f_{\bar{t}_j^r}(x) = y$ and $x \in X_{n+1} \setminus X_n$, then $y \notin X_{n+\lg(\bar{t}_j^r)}$, recalling $\mathrm{supp}(\pi(x_\ell)) \subseteq X_{n_{\ell+1}}$.]

$(*_8)$ If $j \in \mathcal{U}_{(r,\ell)}$ (so $\lg(\bar{t}_j^r) \geqslant 1$), $k \leqslant \lg(\bar{t}_j^r)$, then $X_{n_r} \subseteq \mathrm{dom}(f_{\bar{t}_j^r \restriction k})$ (cf. $(*_4)$(A)(e)).

[Why? Let $y_{(r,k)}^{(r,j)} = f_{\bar{t}_j^r \restriction k}(x_r)$, so $f_{t_{(j,k)}^r}(y_{(r,k)}^{(r,j)}) = y_{(r,k+1)}^{(r,j)}$, see $(*_4)$(A)(d). For $k \leqslant \lg(\bar{t}_j^r)$, so $y_{r,0}^{(r,j)} = x_r \notin X_{n_r}$, and, by induction on k, $y_{(r,k)}^{(r,j)} \notin X_{n_r}$, but $y_{(r,k)}^{(r,j)} \in \mathrm{dom}(f_{t_{(j,k)}^r})$, hence $X_{n_r} \subseteq \mathrm{dom}(f_{t_{(j,k)}^r})$ (and of course $X_{n_r} \subseteq \mathrm{dom}(f_{\bar{t}_j^r})$).]

$(*_9)$ For $\ell \leqslant r$, let:

$$E_{(r,\ell)} = \{(i_1, i_2) : i_1 < j_r, i_2 < j_r, f_{\bar{t}^r_{i_1}}(x_\ell) = f_{\bar{t}^r_{i_2}}(x_\ell)\}.$$

$(*_{10})$ (a) $E_{(r,\ell)}$ is an equivalence relation on j_r;
(b) if $\ell + 1 \leqslant r$, then $E_{(r,\ell+1)}$ refines $E_{(r,\ell)}$.

[Why $(*_{10})$(b)? By 5.2(j).]

$(*_{11})$ (a) $\mathcal{U}_{(r,\ell,y)}$ is $E_{(r,\ell)}$-closed;
(b) $\mathcal{U}_{(r,\ell)}$ is $E_{(r,\ell)}$-closed.

$(*_{12})$ The following are equivalent for $j < j_r$:
(a) $j/E_{(r,\ell)} \cap \mathcal{U}_{(r,\ell)} \neq \emptyset$;
(b) $j/E_{(r,\ell)} \subseteq \mathcal{U}_{(r,\ell)}$;
(c) $\sum\{q^r_i : i \in j/E_{(r,\ell)}\} \neq 0$ and $\bar{t}^r_j \neq ()$.

$(*_{13})$ If $\ell + 1 \leqslant r$, $j_1 \in \mathcal{U}_{(r,\ell)}$, then for some $j_2 \in \mathcal{U}_{(r,\ell+1)}$ we have:

$$j_2/E_{(r,\ell)} \subseteq j_1/E_{(r,\ell)}.$$

[Why? By $(*_{10})$ and $(*_{12})$.]

$(*_{14})$ For each r we can choose $\mathbf{i}(r)$ such that:
(a) $\mathbf{i}(r) < j_r$;
(b) If $\ell \leqslant r$, then $\mathbf{i}(r) \in \mathcal{U}_{(r,\ell)}$.

$(*_{15})$ $\lg(\bar{t}^r_{\mathbf{i}(r)}) \leqslant n_1$.

[Why? As $\mathbf{i}(r) \in \mathcal{U}_{(r,0)}$ by $(*_7)$ applied with $\mathbf{i}(r), 0$ here standing for j, ℓ there.]

$(*_{16})$ Let $\mathcal{D}$ be a non-principal ultrafilter on ω.

$(*_{17})$ For some $n_* \leqslant n_1$, $A_* = \{r : \lg(\bar{t}^r_{\mathbf{i}(r)}) = n_*\} \in \mathcal{D}$.

[Why? By $(*_{15})$+$(*_{16})$.]

$(*_{18})$ $n_* > 0$.

[Why? Because $\lg(\bar{t}^r_{\mathbf{i}(r)}) > 0$, given that $\mathbf{i}(r) \in \mathcal{U}_{(r,\ell)}$, recalling $(*_6)$(e) and $(*_4)$(A)(c).]

$(*_{19})$ For each ℓ there is a unique y_ℓ such that:
(a) $y_\ell \in Y_\ell \setminus \{x_\ell\}$;
(b) the set $A_\ell \in \mathcal{D}$, where:

$$A_\ell = \{r \in A_* : \ell < r \text{ and } \mathbf{i}(r) \in \mathcal{U}_{(r,\ell,y_\ell)}\}.$$

$(*_{20})$ If $\ell < r_1 < r_2$ and $r_1, r_2 \in A_\ell$, then:
(a) $r_1, r_2 \in A_{\ell'}$, for $\ell' < \ell$;
(b) $f_{\bar{t}^{r_1}_{\mathbf{i}(r_1)}}(x_\ell) = f_{\bar{t}^{r_2}_{\mathbf{i}(r_2)}}(x_\ell)$;
(c) $\lg(\bar{t}^{r_1}_{\mathbf{i}(r_1)}) = \lg(\bar{t}^{r_2}_{\mathbf{i}(r_2)})$ and, for $\ell_1 \leqslant \ell$ and $k \leqslant \lg(\bar{t}^{r_1}_{\mathbf{i}(r_1)})$, we have:

$$f_{\bar{t}^{r_1}_{\mathbf{i}(r_1)\restriction k}}(x_{\ell_1}) = f_{\bar{t}^{r_2}_{\mathbf{i}(r_2)\restriction k}}(x_{\ell_1}).$$

[Why? By our choices clause (b) holds. By clause (b) and 5.8 for $k = 1$, clause (c) holds for $\ell_1 = \ell$. Lastly, by induction on $k \leqslant \lg(\bar{t}^{r_1}_{\mathbf{i}(r_1)})$ we can prove that $f_{\bar{t}^{r_1}_{\mathbf{i}(r_1)\restriction k}}(x_{\ell_1}) = f_{\bar{t}^{r_2}_{\mathbf{i}(r_2)\restriction k}}(x_{\ell_1})$. For $k = \lg(\bar{t}^{r_1}_{\mathbf{i}(r_1)})$ we get clause (a).]

$(*_{21})$ $(f_{t^r_{(\mathbf{i}(r),0)}}(x_i) : i < \ell)$ is the same for all $r \in A_\ell$.

Lastly, choose $S_1 \in [\omega]^{\aleph_0}$ such that $r_1 < r_2 \in S_1$ implies $r_2 \in A_{r_1}$. How? We choose r_i by induction on $i < \omega$ with r_i being the first r such that $j < i$ implies $r_i \in A_j$. Why r_i is well-defined? We have that $j < r$ implies $A_j \in \mathcal{D}$, hence $\bigcap_{j<i} A_j \in \mathcal{D}$, hence $\bigcap_{j<i} A_j$ is infinite and so r_i is well-defined.

Now, by $(*_{21})$, the sequence $(f_{t^r_{(\mathbf{i}(r),0)}} \restriction X_{n_r} : r \in S_1)$ is $\subseteq$-increasing. Also, there is $S_2 \in [S_1]^{\aleph_0}$ such that for each $r(*) \in S_2$, the sequence $(t^{r(*)}_{(\mathbf{i}(r(*)),0)}) \wedge t^r_{(\mathbf{i}(r),0)} : r \in S_2, r > r(*))$ is constant, say it is $t_{r(*)}$.

$(*_{22})$ For $r \in S_2$, let t^*_r be such that $r < r' \in S_2 \Rightarrow \bar{t}^r_{(\mathbf{i}(r),0)} \wedge \bar{t}^{r'}_{(\mathbf{i}(r'),0)} = t^*_r$.

[Why well-defined? By the choice of S_2 above.]

$(*_{23})$ For $r(*) \in S_2$, $t^*_{r(*)} \notin T_{n_r-2}$.

[Why? As $(f_{\bar{t}^r_{(\mathbf{i}(r),0)}}(x_{r-1}) : r \in S_2, r > r(*))$ is constant.]

Easily $(t^*_{r(*)} : r(*) \in S_2)$ is either constant or $<_T$-increasing. In the first case we get a contradiction to $(*)_{23}$. In the second case T has an infinite branch contradicting our assumption that T is well-founded. This concludes the analysis of Case 1.

Case 2. The set $Y = \{x/E_1 :$ for some $y \in x/E_1, \pi(y) \notin \mathbb{Q}y\}$ is finite and $\neq \emptyset$.

Choose $x_0 \in X$ such that $\pi(x_0) \notin \mathbb{Q}x_0$. Let $n < \omega$ be such that $x_0 \in X_n$ and choose x_1 such that:

$(\oplus_1)$ (a) $x_1 \in X \setminus \bigcup\{y/E_1 : y \in X_n\}$;
(b) $\pi(x_1) \in \mathbb{Q}x_1$.

[Why possible? By the assumption in Case 2.] Notice that:

$(\oplus_2)$ $\mathrm{supp}(x_\ell) \subseteq x_\ell/E_1$.

$(\oplus_3)$ By 5.17(5), there are $(\bar{t}_j, q_j : j < j_*)$ such that:
(a) $\bar{t}_j$ $(j < j_*)$ are with no repetitions and $q_j \in \mathbb{Q}^+$;
(b) $\pi(x_0 + x_1) = \sum\{q_j f_{\bar{t}_j}(x_0 + x_1) : j < j_*\}$.

$(\oplus_4)$ As in $(*_5)$ in Case 1, we have:
(a) $\pi(x_0) = \sum\{q_j f_{\bar{t}_j}(x_0) : j < j_*\}$;
(b) $\pi(x_1) = \sum\{q_j f_{\bar{t}_j}(x_1) : j < j_*\}$.

However $\pi(x_1) \in \mathbb{Q}x_1$ by choice, and so:

$(\oplus_5)$ (a) for some $j < j_*$, $\bar{t}_j = ()$, w.l.o.g. for $j = 0$;
(b) for $0 < j < j_*$, $\lg(\bar{t}_j) > 0$ (by $(\oplus_3)$(a)).

$(\oplus_6)$ For $i = 0, 1$, let $\mathcal{E}_i$ be the following equivalence relation on j_*:

$$\{(j_1, j_2) : f_{\bar{t}_{j_1}}(x_i) = f_{\bar{t}_{j_2}}(x_i)\}.$$

$(\oplus_7)$ $0/\mathcal{E}_1 = \{0\}$ and if $0 < j < j_*$, then:
(a) $\sum\{q_\ell : \ell \in j/\mathcal{E}_1\} = 0$;
(b) $\sum\{q_\ell f_{\bar{t}_\ell}(x_1) : \ell \in j/\mathcal{E}_1\} = 0$.

[Why? Note that:

$$\sum\{q_\ell f_{\bar{t}_\ell}(x_1) : \ell \in j/\mathcal{E}_1\} = \sum\{q_\ell : \ell \in j/\mathcal{E}_1\} f_{\bar{t}_j}(x_1).$$

So if $\sum\{q_\ell : \ell \in j/\mathcal{E}_1\} \neq 0$, then $f_{\bar{t}_j}(x_1)$ belongs to the support of the RHS of $(\oplus_4)(b)$ but the support of this object is $\{x_1\}$ (by $(\oplus_1)$(b)) and $x_1 \neq f_{\bar{t}_j}(x_1)$, as $\bar{t}_j \neq ()$, together we reach a contradiction, and so we have $(\oplus_7)$(a)(b).]

$(\oplus_8)$ E_1 refines E_0.

[Why? Assume that $j_1, j_2 < j_*$ and $j_1 \mathcal{E}_1 j_2$, this means that $f_{\bar{t}_{j_1}}(x_1) = f_{\bar{t}_{j_2}}(x_1)$. By 5.2(j), as $x_1 \notin X_n$ we have that $X_n \subseteq \mathrm{dom}(f_{\bar{t}_{j_1}}) \cap \mathrm{dom}(f_{\bar{t}_{j_2}})$ and $f_{\bar{t}_{j_1}} \restriction X_n = f_{\bar{t}_{j_2}} \restriction X_n$. As $x_0 \in X_n$ we get that $f_{\bar{t}_{j_1}}(x_0) = f_{\bar{t}_{j_2}}(x_0)$, which means $j_1 \mathcal{E}_0 j_2$, as desired.]

$(\oplus_9)$ $0/E_0 = \{0\}$ and if $0 < j < j_*$, then:
(a) $\sum\{q_\ell : \ell \in j/\mathcal{E}_0\} = 0$;

(b) $\sum\{q_\ell f_{\bar{t}_0}(x_\ell) : \ell \in j/\mathcal{E}_0\} = 0$.

[Why? By $(\oplus_7)$+$(\oplus_8)$, recalling 5.2(j).]

$(\oplus_{10})$ $\pi(x_0) = q_0 x_0$ (follows by $(\oplus_9)$(b)).

But $(\oplus_{10})$ contradicts our choice of x_0, as $\pi(x_0) \notin \mathbb{Q}x_0$.

Case 3. The set $Y = \{x/E_1 :$ for some $y \in x/E_1, \pi(y) \notin \mathbb{Q}y\}$ is empty.

For $x \in X$, let $\pi(x) = q_x x$. Now:

$(\star_1)$ $(q_x : x \in X)$ is constant.

Why $(\star_1)$? Choose $x_0, x_1 \in X$ such that $q_{x_0} \neq q_{x_1}$ and, if possible, they are both $\neq 0$. Let $n < \omega$ be such that $x_0, x_1 \in X_n$ and choose a $<_X$-minimal $x_2 \in X_{n+1} \setminus X_n$, possible by 5.9(2). Let $a = x_0 + x_1 + x_2$, $p = p_a$ (cf. 5.13) and $\bar{x} = (x_0, x_1, x_2)$. As $a \in G_{(1,p)}$ and $\pi \in \mathrm{End}(G_1)$, clearly $\pi(a) = b \in G_{(1,p)}$ and so by 5.17(5) there are $j < \omega$ and, for $i < j$, $\bar{y}^i \in \mathrm{seq}_3(X)$ and $q^i \in \mathbb{Q}^+$ such that $\bar{x} \leqslant^3_X \bar{y}^i$ and:

$(\star_{1.1})$ $b = \sum_{i<j} q^i (\sum_{\ell<3} y^i_\ell)$.

Notice that $j > 0$ and w.l.o.g. we can assume that for $i < j-1$ we have $\bar{y}^{j-1} \nleqslant^3_X \bar{y}^i$ and also that $\bar{x}$ is reasonable (so the $\bar{y}^i$'s are also reasonable). Also:

$(\star_{1.2})$ $b = \pi(a) = q_{x_0} x_0 + q_{x_1} x_1 + q_{x_2} x_2$.

As $i < j-1$, implies $\bar{y}^{j-1} \nleqslant^3_X \bar{y}^i$, clearly $y^{j-1}_2 \notin \{y^i_\ell : i < j-1, \ell \leqslant 2\} \cup \{y^{j-1}_0, y^{j-1}_1\}$ (by 5.9(9) + $\bar{y}^{j-1} \in \mathrm{seq}_3(X)$), and so y^{j-1}_2 appears exactly once in the RHS of equation $(\star_{1.1})$, and so it appears in LHS of $(\star_{1.1})$, so $y^{j-1}_2 \in \mathrm{supp}(b) \subseteq \{x_0, x_1, x_2\}$. But $x_2 \notin x_0/E_1 \cup x_1/E_1$, as $x_0, x_1 \in X_n$ and $x_2 \in X_{n+1} \setminus X_n$ is $<_X$-minimal. On the other hand, clearly $y^i_\ell \in x_\ell/E_1$ for $\ell \leqslant 2$ and $i < j$. Hence, necessarily, $y^{j-1}_2 = x_2$. Finally, as x_2 is $<_X$-minimal and for some $\bar{t} \in T^{<\omega}$, $f_{\bar{t}}(\bar{x}) = \bar{y}^{j-1}$, necessarily, $f_{\bar{t}}(x_2) = y^{j-1}_2$, so clearly $\bar{t} = ()$. Hence, $\bar{y}^{j-1} = \bar{x}$ and of course $\bar{x} \leqslant^3_X \bar{y}$ implies $f_{\bar{t}}(\bar{x}) \leqslant^3_X \bar{y}$. Thus, by the statement after $(\star_{1.1})$, $j = 1$ and $\bar{y}^0 = \bar{x}$. So we have:

$(\star_{1.3})$ $q_{x_0} x_0 + q_{x_1} x_1 + q_{x_2} x_2 = q^0 (y^0_0 + y^0_1 + y^0_2) = q^0 (x_0 + x_1 + x_2)$.

Thus, $q_{x_0} = q^0 = q_{x_1}$, contradicting our assumption that $q_{x_0} \neq q_{x_1}$.

$(\star_2)$ Let $q_x = q_*$ for $x \in X$ (recalling $(\star_1)$).

$(\star_3)$ q_* is an integer.

Why $(\star_3)$? If not $q_* = \frac{m}{n}$, with $m, n \in \mathbb{Z}^+$, m and n coprimes. If p is a prime and $p | n$, then we get a contradiction as in 4.15. Hence, $n = 1$ and $(*_3)$ holds.

Hence, our proof is complete, as Cases 1 and 2 are contradictory, while in Case 3 we showed that the arbitrary $\pi \in \mathrm{End}(G_1)$ is indeed multiplication by an integer. ■

Remark 5.19. *Notice that, in the proof of 5.18, Cases 2 and 3 do not use the assumption that T is well-founded and so for an arbitrary tree T (as in 5.1) and $\pi \in \mathrm{End}(G_1[T])$ we have:*

(a) Case 1 happens if only if T is not well-founded;

(b) Case 2 never happens;

(c) if Case 3 happens, then π is multiplication by an integer.

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Department of Mathematics "Giuseppe Peano", University of Torino, Via Carlo Alberto 10, 10123, Italy.

Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Israel and Department of Mathematics, Rutgers University, U.S.A.