ON AUTOMORPHISMS OF $\mathcal{P}(\lambda)/[\lambda]^{<\lambda}$.

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Abstract. We investigate the statement "all automorphisms of $\mathcal{P}(\lambda)/[\lambda]^{<\lambda}$ are trivial". We show that MA implies the statement for regular uncountable $\lambda < 2^{\aleph_0}$; that the statement is false for measurable λ if $2^\lambda = \lambda^+$; and that for "densely trivial" it can be forced (together with $2^\lambda = \lambda^{++}$) for inaccessible λ.

1. Introduction

The Boolean algebra $\mathcal{P}(\omega)/[\omega]^{<\omega}$ and its automorphisms have been studied extensively for many years. One can study variants for uncountable cardinals λ. Unsurprisingly, the behaviour here tends to be quite different to the countable case. One moderately popular such generalisation is $\mathcal{P}(\lambda)/[\lambda]^{<\omega}$. Here, we study another obvious generalization of the countable case:

$$\mathbb{B} := \mathcal{P}(\lambda)/[\lambda]^{<\lambda}$$

We study Boolean-algebra-automorphisms of $\mathbb{B}$. The main result of the paper is:

(T1, Thm. 5.2) It is equiconsistent with an inaccessible that λ is inaccessible, 2^λ is λ^{++} and all automorphisms of $\mathbb{B}$ are densely trivial.

Here, $2^\lambda > \lambda^{++}$ is necessary, at least for measurables:

(T2, Thm. 4.1) If λ is measurable and $2^\lambda = \lambda^+$, then there is a nontrivial automorphism of $\mathbb{B}$.

Remark 1.1. From [SS15, Lem. 3.2] it would follow that T2 holds even when "measurable" is replaced by just "inaccessible". However, the proof there turned out to be incorrect.[1]

Regarding λ that are not inaccessible, but below the continuum, we get the following result under Martin's Axiom:

(T3, Thm. 3.1) For $\aleph_0 < \kappa \leq \lambda < 2^{\aleph_0}$, κ regular, MA implies that every automorphism of $\mathcal{P}(\lambda)/[\lambda]^{<\kappa}$ is trivial.

(What we use is $\mathrm{MA}_{=\lambda}$ for σ-centered forcings.)

Larson and McKenney [LM16] showed the same under $\mathrm{MA}_{\aleph_1}$ for the case $\lambda = 2^{\aleph_0}$, and $\kappa = \aleph_1$.

Contrast this to the case $\lambda = \kappa = \omega$: Due to results of Veličković, Steprāns and the third author, "Every automorphisms of $\mathcal{P}(\omega)/[\omega]^{<\omega}$ is trivial" is implied by PFA [SS88], in fact even by MA+OCA [Vel93], but not by MA alone [Vel93] (not even for "somewhere trivial" [SS02]).

2. Definitions

We always assume:

- λ is an uncountable cardinal.
  (In Section 3 we will assume that λ is between $\aleph_1$ and the continuum; in Section 4 that it is measurable; and in Section 5 that it is inaccessible.)
- κ is regular, $\aleph_1 \leq \kappa \leq \lambda$.
  (Outside of Section 3 we always use $\kappa = \lambda$.)

Notation:

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[1] A corrected version has been submitted, see https://shelah.logic.at/papers/990a/. This version again establishes the result only assuming inaccessibility.

- We investigate the $<\kappa$-complete and λ^+-cc Boolean algebra (BA) $\mathbb{B} := \mathcal{P}(\lambda)/[\lambda]^{<\kappa}$.
- For $A \subseteq \lambda$, we denote the equivalence class of A with $[A] \in \mathbb{B}$.
- $A \subseteq^* B$ means $|B \setminus A| < \kappa$, analogously for $A =^* B$; and "for almost all α" means for all but $<\kappa$ many.

Note that any BA-automorphism of $\mathbb{B}$ is closed under $<\kappa$ unions.

f is a κ-almost permutation of λ, if there are subsets A, B of λ of size $<\kappa$ such that $f : \lambda \setminus A \to \lambda \setminus B$ is bijective.

For $\kappa = \lambda = \omega$, it is important to consider almost permutations; but in our settings they do not make any difference:

Lemma 2.1. *(Only requires $\mathrm{cf}(\kappa) > \omega$.) If $f : \lambda \setminus A \to \lambda \setminus B$ is a κ-almost permutation, then there is a $S \subseteq \lambda$ with $|\lambda \setminus S| < \kappa$ such that $f : S \to S$ is a permutation. (And thus there is a permutation $g : \lambda \to \lambda$ that agrees with f outside of a set of size $<\kappa$.)*

Proof. Set $X_0 := \lambda \setminus A = \mathrm{dom}(f)$, $X_{i+1} := X_i \cap f''X_i \cap f^{-1}X_i$, $S := \bigcap_{i\in\omega} X_i$.

The X_n are decreasing, and $|\lambda \setminus X_n| < \kappa$ and thus $|\lambda \setminus (f''X_n)| < \kappa$ for $n < \omega$. Accordingly, $|\lambda \setminus S| < \kappa$. We claim that $g := f \restriction S$ is a permutation of S. Clearly it is injective. If $\alpha \in S$ then $\alpha \in X_n$ for all $n \in \omega$, so $f(\alpha) \in X_{n+1}$ for all n. So $g : S \to S$. If $\alpha \in S$, then $\alpha \in X_{n+1}$ for all n, so $f^{-1}(\alpha)$ exists and is in X_n. □

Definition 2.2. Let $\phi : \mathbb{B} \to \mathbb{B}$ be a BA-automorphism.

- A κ-almost permutation $f : \lambda \setminus A \to \lambda \setminus B$ defines a BA-automorphism $\phi_f : \mathbb{B} \to \mathbb{B}$ by $\phi_f([X]) := [f''(X \setminus A)]$.
- ϕ is trivial, if $\phi = \phi_f$ for such an f.
  As we have seen, in our setting this implies that there is a "full" bijection $g : \lambda \to \lambda$ such that $\phi = \phi_g$.

For $\kappa = \lambda$, we will investigate somewhere and densely trivial automorphisms;

Definition 2.3.
- ϕ is somewhere trivial, if for some $B \in [\lambda]^\lambda$ there is an almost permutation f with $\phi([C]) = \phi_f([C])$ for all $C \subseteq B$.[2]
- ϕ is densely trivial, if for all $A \in [\lambda]^{<\lambda}$ there is a $B \subseteq A$ as above.

Lemma 2.4. *Assume $\kappa = \lambda$. If every automorphism is somewhere trivial, then every automorphism is densely trivial.*

Proof. Assume π is an automorphism, and fix $A \in [\lambda]^\lambda$. If $|\lambda \setminus A| < \lambda$, and π trivial on B, then is π trivial on $B \cap A \subseteq A$, so we are done. So assume otherwise. Pick some representative $\pi^* : \mathcal{P}(\lambda) \to \mathcal{P}(\lambda)$ of π such that $\pi^*(A)$ and $\pi^*(\lambda \setminus A)$ partition λ, and such that $\pi^*(C) \subseteq \pi^*(A)$ for every $C \subseteq A$. Let $i : \lambda \setminus A \to A$ and $j : \pi(\lambda \setminus A) \to \pi(A)$ both be bijective. Let π' map $[C]$ to $[\pi^*(C \cap A) \cup j^{-1}\pi^*(i''(C \setminus A))]$. This is an automorphism. Assume π' is trivial on C. If $|C \cap A| = \lambda$, we are done. So assume otherwise, i.e., π' is trivial on the large set $C \setminus A$. Then π is trivial on $i''(C \setminus A) \subseteq A$. □

3. Under MA, every automorphism is trivial for $\omega_1 \leq \lambda < 2^{\aleph_0}$

Theorem 3.1. *Assume $\aleph_0 < \kappa \leq \lambda < 2^{\aleph_0}$, κ regular, and $\mathrm{MA}_{(=\lambda)}(\sigma\text{-centered})$ holds. Then every automorphism of $\mathcal{P}(\lambda)/[\lambda]^{<\kappa}$ is trivial.*

For the proof we will use the following:

Lemma 3.2. *Assume $\aleph_0 < \kappa \leq \lambda < 2^{\aleph_0}$, $\mathrm{cf}(\kappa) > \aleph_0$, and $\mathrm{MA}_{(=\lambda)}(\sigma\text{-centered})$ holds. Assume A_0, A_1 are disjoint subsets of 2^ω of size $\leq \lambda$; $|A_0| \geq \kappa$. Then there is a tree T_0 in $2^{<\omega}$ such that $|A_0 \cap \lim(T_0)| \geq \kappa$ and $A_1 \cap \lim(T_0) = \emptyset$.*

If additionally $|A_1| \geq \kappa$, we get an additional tree T_1 such that $|A_1 \cap \lim(T_1)| \geq \kappa$, $A_0 \cap \lim(T_1) = \emptyset$, and $T_0 \cap T_1 \subseteq 2^n$ for some n.

[2]It is easy to see that equivalently we can require f to just be an injection from C to λ (or from $C \setminus N$ with $|N| < \lambda$), or that f is even a full permutation of λ to λ.

Proof of the lemma. In the following we identify an $x \in 2^\omega$ with the according (infinite) branch b in the tree $2^{<\omega}$. So a branch b can be in A_0 or in A_1 (but not both, as A_0 and A_1 are disjoint).

We define a poset Q as follows: A condition $q \in Q$ is a triple (n_q, S_q, f_q), where

- $n_q \in \omega$,
- S_q is a tree $2^{<\omega}$ of the following form: S_q is the union of $2^{\leq n_q}$ and finitely many (infinite) branches $\{b_j : j \in m\}$, each $b_j \in A_0 \cup A_1$, and $b_j \restriction n_q = b_k \restriction n_q$ implies ($b_j \in A_i$ iff $b_k \in A_i$).

  So every $s \in S_q$ with $|s| > n_q$ is either "in A_0-branches" (i.e., there is one or more $b_j \in A_0$ with $s \in b_j$), or "in A_1-branches", but not both. And an $s \in S_{\cdot}$' of length n_q is either in A_0-branches, or in A_1-branches, or neither.
- $f_q : S_q \to 2$ such that $f_q(s) = i$ whenever $s \in S_q$, $|s| \geq n$ and s is in A_i-branches.

The order on Q is the natural one: $q \leq p$ if $n_q \geq n_p$, $S_q \supseteq S_p$ and f_q extends f_p.

Q is σ-centered witnessed by $(n_q, S_q, f_q) \mapsto (n_q, f_q \restriction 2^{\leq n_q})$.

For $x \in A_i$, the set D_x of conditions containing x as branch is dense: Given $p \in Q$, let $n_q \geq n_p$ be such that all A_{1-i}-branches in p split off x below n_q; set $S_q := S_p \cup 2^{\leq n_q} \cup x$; and set $F_q(s) = i$ for $s \in S_q \setminus S_p$.

And obviously for all $n \in \omega$, the set D^*_n of conditions q with $n_q \geq n$ is dense as well.

By $\mathrm{MA}_{(=\lambda)}(\sigma\text{-centered})$ and $|A_i| \leq \lambda$, we can find a filter G which has nonempty intersection with each D_x for $x \in A_0 \cup A_1$ as well as for each D^*_n. So $F := \bigcup_{p \in G} f_p$ is a total function from $2^{<\omega}$ to 2; and for all $x \in A_i$ there is an $n_x \in \omega$ such that $m \geq n_x$ implies $F(x \restriction m) = i$.

As $|A_0| \geq \kappa$ and $\mathrm{cf}(\kappa) > \aleph_0$ we can assume that there is an n^*_0 such that $n_x = n^*_0$ for κ many $x \in A_0$. If additionally $|A_1| \geq \kappa$, we analogously get an n^*_1 and set $n^* := \max(n^*_0, n^*_1)$; otherwise we set $n^* := n^*_0$. We set $T^*_i := \{s \in 2^{<\omega} : |s| \geq n^*, (\forall n^* \leq k \leq |s|)\, F(s \restriction k) = i\}$ and generate a tree from it; i.e., we set $T_i := T^*_i \cup \{s \restriction m : m < n^*, s \in T^*_i\}$. As we have seen above, $\lim(T_i) \cap A_i \geq \kappa$ for $i = 0$ (and, if $|A_1| \geq \kappa$, for $i = 1$ as well). Clearly $T_0 \cap T_1 \subseteq 2^{n^*}$; and $\lim(T_i) \cap A_{i-1}$ is empty, as for any $x \in A_{i-1}$, cofinally many n satisfy $F(x \restriction n) = i - 1$. $\square$

Proof of the theorem. Fix an automorphism π of $\mathcal{P}(\lambda)/[\lambda]^{<\kappa}$ represented by some $\pi^* : \mathcal{P}(\lambda) \to \mathcal{P}(\lambda)$, and let π^{-1*} represent π^{-1}. We have to show that π is trivial.

Fix an injective function $\eta : \lambda \to 2^\omega$. Set $C_n := \{x \in 2^\omega : x(n) = 0\}$ and $\Lambda_n := \eta^{-1} C_n = \{\alpha < \lambda : \eta(\alpha)(n) = 0\}$.

Define $\nu : \lambda \to 2^\omega$ by $\nu(\beta)(n) = 0$ iff $\beta \in \pi^*(\Lambda_n)$. I.e., $\pi^*(\eta^{-1} C_n) =^* \nu^{-1} C_n$. As π honors $<\kappa$-unions and complements and as the A_n generate the open sets, this equation holds whenever C is generated by $<\kappa$-unions and complements from the open sets (in particular, if C is closed):

$$(*_1) \qquad \pi^*(\eta^{-1} C) =^* \nu^{-1} C.$$

In the following, "large" means "of cardinality $\geq\kappa$", and "small" means not large.

If A_0, A_1 are disjoint subsets of 2^ω, A_0 large, then A_0 and A_1 can be separated by a tree T_0 as in the previous lemma. I.e., $\lim(T_0) \cap A_1 = \emptyset$ and $X := \lim(T_0) \cap A_0$ is large, and thus also $\nu^{-1} X = \nu^{-1} \lim(T_0) \cap \nu^{-1} A_0 =^* \pi^*(\eta^{-1} \lim(T_0)) \cap \nu^{-1} A_0$, the last equation by $(*_1)$. This implies $\eta^{-1} \lim(T_0) \cap \pi^{-1*}(\nu^{-1} A_0)$ is large, and so $\pi^{-1*}(\nu^{-1} A_0) \setminus \eta^{-1} A_1$ is large. We get an analogous result when interchanging ν and η and using π^* instead of π^{-1*}. To summarize: If A_0, A_1 are disjoint subsets of 2^ω, A_0 large, then

$$(*_2) \qquad \pi^{-1*}(\nu^{-1} A_0) \setminus \eta^{-1} A_1 \text{ and } \pi^*(\eta^{-1} A_0) \setminus \nu^{-1} A_1 \text{ are both large.}$$

We will also need the following (which is the only place where we use that κ is regular):

$$(*_3) \qquad Y \subseteq \lambda \text{ and } |Y| \geq \kappa \text{ implies } |\nu'' Y| \geq \kappa$$

(Fix $x \in 2^\omega$. Then $\eta^{-1}\{x\}$ has at most one element (as η is injective), and $\eta^{-1}\{x\} =^* \pi^{-1*} \nu^{-1}\{x\}$ by $(*_1)$. I.e., $\nu^{-1}\{x\}$ is small. And $Y \subseteq \bigcup_{x \in \nu'' Y} \nu^{-1}\{x\}$, so as κ is regular we get $|\nu'' Y| \geq \kappa$.)

We claim that the following sets N_i are all small:

(1) $N_1 := \{\alpha \in \lambda : (\neg \exists \beta \in \lambda)\, \eta(\alpha) = \nu(\beta)\}$.
(2) $N_2 := \{\alpha \in \lambda : (\exists^{(\geq 2)} \beta \in \lambda)\, \eta(\alpha) = \nu(\beta)\}$.
(3) $N_3 := \{\beta \in \lambda : (\neg \exists \alpha \in \lambda)\, \eta(\alpha) = \nu(\beta)\}$.

Proof of (3): Assume N_3 is large. Set $A_0 := \nu'' N_3$, which is large by $(*_3)$; and $A_1 := \eta''\lambda$. So A_0 and A_1 are disjoint, and by $(*_1)$ $\pi^{-1*}\nu^{-1}A_0 \setminus \eta^{-1}A_1$ is large, but $\eta^{-1}A_1 = \lambda$.

Proof of (1) works analogously: Assume N_1 is large. Set $A_0 = \eta'' N_1$ (large, as $\underset{\sim}{\eta}$ is injective) and $A_1 := \nu''\lambda$. So A_0 and A_1 are disjoint, and so $\pi^*(\eta^{-1}A_0) \setminus \nu^{-1}A_1$ is large, but $\nu^{-1}A_1 = \lambda$.

Proof of (2): Assume that N_2 is large. For every $\alpha \in N_2$, let $\beta^0_\alpha \neq \beta^1_\alpha$ in λ be such that $\eta(\alpha) = \nu(\beta^0_\alpha) = \nu(\beta^1_\alpha)$. For $i \in \{0,1\}$, set $Y_i := \{\beta^i_\alpha : \alpha \in N_2\}$ and $X_i := \pi^{-1*}(Y_i)$ (without loss of generality disjoint), and $A_i := \eta'' X_i$. So the A_i are large and disjoint, and we can find a tree T_0 such that $A_0 \cap \lim(T_0)$ is large, and $A_1 \cap \lim(T_0)$ is empty. So using $(*_1)$, we see that $\eta^{-1}A_0 \cap \eta^{-1}\lim(T_0) =^* X_0 \cap \pi^{-1*}\nu^{-1}\lim(T_0)$ is large, and therefore also $Y_0 \cap \nu^{-1}\lim(T_0)$ and $\nu''Y_0 \cap \lim(T_0)$ as well, by $(*_3)$. On the other hand $X := \lim(T_0) \cap A_1$ is empty, so $\pi^*(\eta^{-1}X) =^* \nu^{-1}\lim(T_0) \cap Y_1$ is small, and so $\lim(T_0) \cap \nu''Y_1$ is small, contradicting $\nu''Y_0 = \nu''Y_1$.

Note that this implies

$$(*_4) \qquad X \cap Y \text{ small implies } \nu''X \cap \nu''Y \text{ small.}$$

(Assume otherwise. Without loss of generality we can assume that X and Y are disjoint, and by (3) that $\nu''X$ and $\nu''Y$ both are subsets of $\eta''\lambda$. Then $\nu''X \cap \nu''Y \subseteq \eta''N_2$ is small.)

Set $D := \lambda \setminus (N_1 \cup N_2)$ and define $e : D \to \lambda$ such that $e(\alpha)$ is the (unique) $\beta \in \lambda$ with $\eta(\alpha) = \nu(\beta)$. Clearly e is injective. We claim that e generates π, i.e., that the following are small (where we can assume $X \subseteq D$):

(4) $N_4 := \pi^*(X) \setminus e''X$ for $X \subseteq \lambda$.
(5) $N_5 := e''X \setminus \pi^*(X)$ for $X \subseteq \lambda$.

Proof of (4): Assume that N_4 is large. Set $Y = \pi^{-1*}(N_4)$, without loss of generality $Y \subseteq X$ and $\pi^*(Y) = N_4$. So $\pi^*(Y)$ is disjoint from $e''Y$ (as it is even disjoint from $e''X$). We set $A_0 := \nu''\pi^*(Y)$ and $A_1 := \nu''e''Y$, by $(*_4)$ we can assume they are disjoint, and by $(*_3)$ both are large. $\eta^{-1}(A_1) = Y$, as $\nu(e(\alpha)) = \eta(\alpha)$ for all $\alpha \in D$. And $\pi^{-1*}(\nu^{-1}A_0) =^* Y$ by definition. This contradicts $(*_2)$.

Proof of (5) works analogously: If N_5 is large, we again get a large Y such that $\pi^*(Y)$ and $e''Y$ are disjoint (this time $Y =^* e^{-1}N_5$). □

4. For measureables, GCH implies a nontrivial automorphism

Theorem 4.1. *If λ is measurable and $2^\lambda = \lambda^+$, then there is a nontrivial automorphism of $\mathcal{P}(\lambda)/[\lambda]^{<\lambda}$.*

Proof. Let $\mathcal{D}$ be a normal ultrafilter on λ and denote by $\mathcal{I} := [\lambda]^\lambda \setminus \mathcal{D}$ its dual ideal restricted to sets of size λ.

Since $2^\lambda = \lambda^+$, we can list all permutations of λ as $\{e_\alpha : \alpha < \lambda^+\}$; and analogously all elements of $\mathcal{I}$ as $\{i_\alpha : \alpha < \lambda^+\}$.

We will construct, by induction on $\alpha < \lambda^+$ a set $A_\alpha \in \mathcal{I}$ and an almost permutation f_α of A_α, such that for $\alpha < \beta$:

(1) $A_\alpha \subseteq^* A_\beta$.
(2) $i_\alpha \subseteq A_{\alpha+1}$,
(3) $f_\alpha(x) = f_\beta(x)$ for almost all $x \in A_\alpha \cap A_\beta$.
(4) There is some $X \subseteq A_{\alpha+1}$ of size λ such that $e''_\alpha X$ and $f''_{\alpha+1}X$ are disjoint.

(Note that by $x \subseteq^* y$ we mean $|y \setminus x| = \lambda$, not $y \setminus x \in \mathcal{I}$; and the same for 'almost all".)

Successor stages $\alpha + 1$: Fix any $B \in \mathcal{I}$ disjoint to A_α such that $A_\alpha \cup B \supseteq i_\alpha$. Set $C := e''_\alpha B \cap A_\alpha$. First assume that $|C| = \lambda$. Then set $A_{\alpha+1} = A_\alpha \cup B$ and let $f_{\alpha+1}$ extend f_α by the identity on B. Then (4) is witnessed by $X := e^{-1}_\alpha C$. So we assume $|C| < \lambda$. Partition B into large sets B_0, B_1, B_2 such that $e''B_i$ is disjoint to A_α for $i = 0, 1$. Set $A_{i+1} := A_\alpha \cup B \cup e''B$, and define $f_\alpha + 1$ on B such that $f_\alpha \restriction B_i \to e''B_{1-i}$ is a bijection for $i = 0, 1$ and f_α restricted to $(B \cup e''B) \setminus (B_0 \cup B_1)$ is a bijection to $(B \cup e''B) \setminus e''(B_0 \cup B_1)$. Then (4) is witnessed by $X := B_0$.

Limit stages δ of cofinality less than λ: Let $\xi := \mathrm{cf}(\delta)$ and choose $\langle \alpha_i : i < \xi \rangle$ a cofinal increasing sequence converging to δ. The union $\bigcup_{i<\xi} A_{\alpha_i}$ is, by $<\lambda$ completeness, in $\mathcal{I}$ and f_δ

defined as $f_\delta(x) = f_{\alpha_i}(x)$, where α_i is least such that $x \in A_{\alpha_i}$, is an almost permutation of A_δ and f_δ satisfies (3).

limit stages δ of cofinality λ: We choose an increasing cofinal sequence $\langle \alpha_i : i < \lambda \rangle$ converging to δ. By induction on $i \in \lambda$ we construct $A'_i =^* A_{\alpha_i}$, such that

- $A'_i \cap i = \emptyset$,
- f_{α_i}'s fully extend each other on the A'_i's, i.e., if $x \in A'_i \cap A'_j$ then $f_{\alpha_i}(x) = f_{\alpha_j}(x)$,
- $f_{\alpha_i} : A'_i \to A'_i$ is a "full" permutation.

We can do this by removing from A_{α_i}: the points less than i, the points where f_{α_i} disagrees with some previous f_{α_j} for any $j < i$; and by removing $<\lambda$ many points to get a full permutation as in Lemma 2.1.

Now we can set A_δ and f_δ to be the unions of A'_i and f_{α_i}, respectively, for $i < \delta$. Note that A_δ is in $\mathcal{I}$ (as it is equal to the diagonal union: $A'_i \cap i = \emptyset$); and f_δ is a permutation of A_δ satisfying (3).

The automorphism π is defined as follows: For $X \in [\lambda]^\lambda$,

$$\pi([X]) := \begin{cases} [f''_\alpha X] & \text{if } X \in \mathcal{I}, X \subseteq^* A_\alpha \text{ for some } \alpha < \lambda^+ \text{ (Case 1)} \\ [\lambda \setminus f''_\alpha(\lambda \setminus X)] & \text{if } X \notin \mathcal{I}, \lambda \setminus X \subseteq^* A_\alpha \text{ for some } \alpha < \lambda^+ \text{ (Case 2)}. \end{cases}$$

Note that in Case 2, $\pi([X]) = [(\lambda \setminus A_\alpha) \cup (A_\alpha \setminus f''_\alpha(A_\alpha \setminus X))] = [(\lambda \setminus A_\alpha) \cup f''_\alpha(X \cap A_\alpha)]$, as $f''_\alpha A_\alpha =^* A_\alpha$.

π is well defined on $[\lambda]^\lambda$, as exactly one of X or $\lambda \setminus X$ will eventually be $\subseteq^* A_\alpha$.

π is an automorphism: $\pi([\emptyset]) = \emptyset$. π honors complements: If X is Case 1, then $\pi([\lambda \setminus X])$ is by definition (Case 2) $[\lambda \setminus f''_\alpha(X)]$. π honors intersections $X \cap Y$: This is clear if either both sets are the same Case. Assume that X is Case 1 and Y Case 2. Then $X \cap Y \subseteq X$ is Case 1, and for any α suitable for both X and Y we have

$$\pi([X]) \wedge \pi([Y]) = [f''_\alpha X \cap ((\lambda \setminus A_\alpha) \cup f''_\alpha(Y \cap A_\alpha)] = [f''_\alpha X \cap f''_\alpha(Y \cap A_\alpha)] = [f''_\alpha(X \cap Y))].$$

π is not trivial: Every automorphism e is an e_α for some $\alpha \in \lambda^+$; and according to (4) there is some $X_\alpha \subseteq A_{\alpha+1}$ (and therefore in $\mathcal{I}$) of size λ such that $e''_\alpha X_\alpha$ is disjoint to $f''_{\alpha+1} X_\alpha$, a representative of $\pi([X_\alpha])$. □

5. For inaccessible λ, all automorphisms can be densely trivial

In this section, we always assume the following (in the ground model):

Assumption 5.1. λ is inaccessible and $2^\lambda = \lambda^+$. We set $\mu := \lambda^{++}$.

In the rest of the paper, we will show the following:

Theorem 5.2. *There is a λ-proper, $<\lambda$-closed, λ^{++}-cc poset P forcing that $2^\lambda = \lambda^{++}$ and that every automorphism of $\mathcal{P}(\lambda)/[\lambda]^{<\lambda}$ is densely trivial.*

By Lemma 2.4, it is enough to show that every automorphism is somewhere trivial.

5.1. The single forcing Q.

Definition 5.3.

- Fix a strictly increasing sequence $(\theta^*_\zeta)_{\zeta<\lambda}$ with $\theta^*_\zeta < \lambda$ regular and $\theta^*_\zeta > 2^{|\zeta|}$.

  Let $(I^*_\zeta)_{\zeta \in \lambda}$ be an increasing interval partition of λ such that I^*_ζ has size $2^{\theta^*_\zeta}$; and fix a bijection of I^*_ζ and $2^{\theta^*_\zeta}$. Using this (unnamed) bijection, we set $[s] := \{\ell \in I^*_\zeta : \ell > s\}$ for $s \in 2^{<\theta^*_\zeta}$.

  We set $I^*(<\zeta) := \bigcup_{\ell<\zeta} I^*_\ell$, and analogously $I^*(\leq\zeta) := I^*(<\zeta+1)$, $I^*(\geq\zeta) := \lambda \setminus I^*(<\zeta)$, and $I^*(\geq\zeta, <\xi) := I^*(\geq\zeta) \cap I^*(<\xi)$.
- A condition q of the forcing notion Q is a function with domain λ such that, for all $\zeta \in \lambda$, $q(\zeta)$ is a partial function from I^*_ζ to 2, and such that for a club-set $C^q \subseteq \lambda$
  - if $\zeta \notin C^q$, then $q(\zeta)$ is total,
  - otherwise, the domain of $q(\zeta)$ is $I^*_\zeta \setminus [s^q_\zeta]$ for some $s^q_\zeta \in 2^{<\theta^*_\alpha}$.

C^q and s^q_ζ are uniquely determined by q; and q is uniquely determined by the partial function $\eta^q : \lambda \to 2$ defined as $\bigcup_{\zeta\in\lambda} q(\zeta)$.

- q is stronger than p, if η^q extends η^p.
  (This implies that $C^q \subseteq C^p$, and that s^q_ζ extends s^p_ζ for all $\zeta \in C^q$.)

The following is straightforward:

Lemma 5.4. *Q has size 2^λ, is $<\lambda$-closed and adds a generic real $\underset{\sim}{\eta} := \bigcup_{q\in G} \eta^q$ in 2^λ.*

Proof. $<\lambda$-closure is obvious, but for later reference we would like to point out the "problematic cases:"

Let $(p_i)_{i<\delta}$ be decreasing for a limit ordinal $\delta < \lambda$.

As first approximation, set $\eta^* := \bigcup_{i<\delta} \eta^{p_i}$ (a partial function) and $C^* := \bigcap_{i<\delta} C^{p_i}$ (a club set) and $s^*_\zeta := \bigcup_{i<\delta} s^{p_i}_\zeta \in 2^{\leq\theta^*_\zeta}$ for $s \in C^*$. For $\zeta \notin C^*$, η^* is indeed total on I^*_ζ, and for $\zeta \in C^*$ the domain in I^*_ζ is $I^*_\zeta \setminus [s^*_\zeta]$.

The **problematic case** is when s^*_ζ is unbounded in θ^*_ζ. (This can only happen if $\mathrm{cf}(\delta) = \theta^*_\zeta$, in particular for at most one ζ.) In this case we can just pick any extension η^q of η^* by filling all values in $I^*_{\leq\zeta}$. This gives the desired q, with $C^{q_\delta} = C^* \setminus \zeta + 1$. □

Remarks.
- The limits of $<\lambda$-sequences of conditions are not "canonical" if there are problematic ζ's, as we have to fill in arbitrary values.
- $\underset{\sim}{\eta}$ determines the generic filter, by $G = \{p \in Q : \eta^p \subseteq \underset{\sim}{\eta}\}$. This follows from the following facts:
  - p and q are compatible (as conditions in Q) iff η^p and η^q are compatible as partial functions and $X_{p,q} := \{\zeta \in C^p : s^p_\zeta \text{ and } s^q_\zeta \text{ are incomparable}\}$ is non-stationary.
  - If p, q are such that $X_{p,q}$ is stationary, then the set of conditions r such that η^r and η^q are incompatible (as partial functions) is dense below p.

5.2. Properness of Q: Fusion and pure decision.

Definition 5.5. We say $q \leq_\xi p$, if $q \leq p$, $\xi \in C^q$ and $q \restriction \xi = p \restriction \xi$.
$q \leq^+_\xi p$ means $q \leq_\xi p$ and $q(\xi) = p(\xi)$.

(Note the difference between $q \leq^+_\xi p$ and $q \leq_{\xi+1} p$: The former does not require $\xi + 1 \in C^q$.)

Lemma 5.6. *Let $\delta \leq \lambda$ be a limit ordinal, $\xi \in \lambda$ and $(q_i)_{i<\delta}$ a sequence in Q.*

(1) If $\delta < \lambda$ and $q_j <^+_\xi q_i$ for all $i < j < \delta$, then there is a q_∞ such that $q_\infty <^+_\xi q_i$ for all i.
(2) If $q_j <_{\xi_i} q_i$ for $i < j < \delta$, where $(\xi_i)_{i\in\delta}$ is a strictly increasing[3] sequence in λ, then there is a (canonical) limit q_∞ such that $q_\infty <_{\xi_i} q_i$ for all i.

Proof. (1): We perform the same construction as in the proof of Lemma 5.4. If there is a problematic case ζ, then $\zeta > \xi$ (as for $\zeta' \leq \xi$ the conditions $q_i(\zeta')$ are constant). We can then make η^* total on $I^*(>\xi, \leq\zeta)$. (It may not be enough to make it total on I^*_ζ, as $C^* \setminus \{\zeta\}$ might not be club.)

(2): Define $q_\infty(\zeta) := \bigcup_{i\in\delta} q_i(\zeta)$ for $\zeta \in \lambda$.

This is a non-total function (on I^*_ζ) iff $\zeta \in C^{q_\infty} := \bigcap_{i<\delta} C^{q_i}$, which is closed as intersection of closed sets, and also unbounded: If $\delta < \lambda$ because we have a small intersections of clubs, if $\delta = \lambda$ as it contains each ξ_i.

There are no problematic cases: If ζ is below some ξ_i, then $q_j(\zeta)$ is eventually constant. If ζ is above all ξ_i, which can only happen if $\delta < \lambda$, then $\mathrm{cf}(\delta) \leq \delta \leq \sup(\xi_i) \leq \zeta < \theta^*_\zeta$. □

So Q satisfies fusion; and we will now show that it also satisfies "pure decision"; standard arguments then imply that Q is λ-proper and λ^λ-bounding.

Definition 5.7. Let $\xi \in \lambda$, $q \in Q$.

[3]For $\delta = \lambda$, it is enough that the ξ_i converge to λ. For $\delta < \lambda$, we use that the ξ_i are increasing and that $\sup(\xi_i) \geq \mathrm{cf}(\delta)$.

- $\mathrm{POSS}^Q(\xi) := 2^{I^*(<\xi)}$. So in the extension $V[G]$, for each ξ there will be exactly one $x \in \mathrm{POSS}^Q(\xi)$ compatible with (or equivalently: initial segment of) $\underset{\sim}{\eta}$. We write "$x \subseteq \underset{\sim}{\eta}$" or "$G$ chooses x" for this x.
- $\mathrm{poss}(q,\xi)$ is the set of $x \in \mathrm{POSS}^Q(\xi)$ compatible with η^q (as partial functions), or equivalently: $x \in \mathrm{poss}(q,\xi)$ iff $\neg q \Vdash x \not\subseteq \underset{\sim}{\eta}$. So q forces that exactly one $x \in \mathrm{poss}(q,\xi)$ is chosen by G.
- Let $\underset{\sim}{\tau}$ be a name for an ordinal. We say that q ξ-decides $\underset{\sim}{\tau}$, if there is for all $x \in \mathrm{poss}(q,\xi)$ an ordinal τ^x such that q forces $x \subseteq \underset{\sim}{\eta} \rightarrow \underset{\sim}{\tau} = \tau^x$.

Note that for $p \in Q$ and $\zeta \in C^p$, $q \leq^+_\zeta p$ is equivalent to $\mathrm{poss}(q,\zeta+1) = \mathrm{poss}(p,\zeta+1)$, while $q \leq_\zeta p$ is equivalent to $\zeta \in C^q$ and $\mathrm{poss}(q,\zeta) = \mathrm{poss}(p,\zeta)$.

Lemma 5.8. *Assume p in Q, $\zeta \in C^p$, $x \in \mathrm{poss}(p,\zeta+1)$, and $r \leq p$ extends*[4] *x. Then there is a $q \leq^+_\zeta p$ forcing: $x \subseteq \underset{\sim}{\eta} \rightarrow r \in G$. This condition is denoted by $r \vee (p \restriction \zeta + 1)$.*

Proof. We set $q(\ell)$ to be $p(\ell)$ for $\ell \leq \zeta$, and $r(\ell)$ otherwise. If $q' \leq q$ forces $x \subseteq \underset{\sim}{\eta}$ then q' extends x and thus $q' \leq r$. □

Corollary 5.9. *("Pure decision") Let $\underset{\sim}{\tau}$ be a name for an ordinal, $p \in Q$, and $\zeta \in C^p$. Then there is a $q \leq^+_\zeta p$ which $(\zeta+1)$-decides $\underset{\sim}{\tau}$.*

Proof. Let $(x_i)_{i\in\delta}$ enumerate $\mathrm{poss}(p,\zeta+1)$, for some $\delta < \lambda$. Set $p_0 = p$, and define a $\leq^+_\zeta$-decreasing sequence p_j by induction on $j \leq \delta$: For limits use Lemma 5.6(1), and for successors choose some $r \leq p_i$ deciding $\underset{\sim}{\tau}$ with a stem extending x_i and set p_{i+1} to $r \vee p_i \restriction (\zeta+1)$. □

From fusion and pure decision we get bounding and λ-proper, via "continuous reading". This is a standard argument, and we will not give it here; we will anyway prove a more "general" variant (for an iteration of Q's), in Lemmas 5.25 and 5.27.

Fact 5.10.
- Q has "continuous reading of names": If $\underset{\sim}{\sigma}$ is a Q-name for a λ-sequence of ordinals, and $p \in Q$, then there is a $q \leq p$ and there are $\xi_i \in \lambda$ such that q ξ_i-decides $\underset{\sim}{\sigma}(i)$ for all $i \in \lambda$.
- Q is λ^λ-bounding. I.e., for every name $\underset{\sim}{\sigma} \in \lambda^\lambda$ and $p \in Q$ there is an $f \in \lambda^\lambda$ and $q \leq p$ such that q forces $f(i) > \underset{\sim}{\sigma}(i)$ for all $i \in \lambda$.
- Q is λ-proper. This means: If N is a $<\lambda$-closed elementary submodel of $H(\chi)$ of size λ containing Q, with χ sufficently large and regular, and if $p \in Q \cap N$, then there is a $q \leq p$ N-generic (i.e., forcing that each name of an ordinal which is in N is evaluated to an ordinal in N).

For completeness, we also mention the following well-known fact (the proof is straightforward):

Fact 5.11. Assume κ is regular, and that the forcing notion R is κ^κ-bounding. Then R preserves the regularity of κ, and every club-subset of κ in the extension contains a ground model club-set.

Remark. Another way to see that an iteration P of copies of Q is proper is to use the framework of [RaS11]. However, we will need an explicit form of continuous reading for P anyway, which in turn gives properness for free.

5.3. **The proper iteration P.** Let us first recall some well-known facts:

Facts 5.12. A $<\lambda$-closed forcing preserves cofinalities $\leq\lambda$ and also the inaccessibilty of λ.
The $\leq\lambda$-support iteration of $<\lambda$-closed forcings is $<\lambda$-closed.

We will iterate the forcings Q from the previous section in a $<\lambda$-closed $\leq\lambda$-support iteration of length $\mu := \lambda^{++}$:

Definition 5.13. Let $(P_\alpha, Q_\alpha)_{\alpha<\mu}$ be the $\leq\lambda$-support iteration such that each Q_α is the forcing Q (evaluated in the P_α-extension). We will write P to denote the limit.

[4] By which we mean $x \subseteq \eta^r$.

Definition 5.14. Assume that $w \in [\mu]^{<\lambda}$ and $\xi \in \lambda$.

- $\bar{\underset{\sim}{\eta}} = (\underset{\sim}{\eta}_\alpha)_{\alpha\in\mu}$ is the sequence of Q-generic reals added by P.
- $\mathrm{POSS}(w,\xi) := 2^{w\times I^*(<\xi)}$. Exactly one $x \in \mathrm{POSS}(w,\xi)$ is extended by $\bar{\underset{\sim}{\eta}}$, we write "$x$ is selected by G," or "$x \triangleleft G$."
- $\mathrm{poss}(p,w,\xi) := \{x \in \mathrm{POSS}(w,\xi) : \neg p \Vdash \neg x \triangleleft G\}$.
- Let $\underset{\sim}{\tau}$ be a name of an ordinal. $\underset{\sim}{\tau}$ is (w,ξ)-decided by q, if there are $(\tau^x)_{x\in \mathrm{poss}(q,w,\xi)}$ such that q forces $x \triangleleft G \rightarrow \underset{\sim}{\tau} = \tau^x$.

Clearly, if $\underset{\sim}{\tau}$ is (w,ξ)-decided by q, and if $q' \leq q$, $w' \supseteq w$ and $\xi' \geq \xi$, then $\underset{\sim}{\tau}$ is (w',ξ')-decided by q'.

Remark. If $q \in P$ (w,ζ)-decides some P_α-name $\underset{\sim}{\tau}$, then the *same* q will generally *not* $(w\cap\alpha,\xi)$-decide $\underset{\sim}{\tau}$ for any ξ.[5]

In the following, whenever we say that q (w,ζ)-decides something, we implicitly assume that $w \in [\mu]^{<\lambda}$ and $\zeta \in \lambda$.

Definition 5.15. Let $\underset{\sim}{\sigma}$ be a P-name for a λ-sequence of ordinals.

- q continuously reads $\underset{\sim}{\sigma}$, if there are $(w_i,\xi_i)_{i\in\lambda}$ such that q (w_i,ξ_i)-decides $\underset{\sim}{\sigma}(i)$ for each $i \in \lambda$.
- P has continuous reading, if for each such $\underset{\sim}{\sigma}$ and $p \in P$ there is some $q \leq p$ continuously reading $\underset{\sim}{\sigma}$.

The following is a straightforward standard argument:

Fact 5.16. If P has continuous reading, then it is λ^λ-bounding.

As a first step towards pure decision, let us generalize the $\leq_\zeta$-notation we defined for Q:

Definition 5.17.
- p fits (w,ξ), if $w \subseteq \mathrm{dom}(p)$ and $p \restriction \alpha \Vdash \xi \in C^{p(\alpha)}$ for all $\alpha \in w$.
- $q \leq_{w,\xi} p$ means: $q \leq p$, and for all $\alpha \in w$, $q \restriction \alpha$ forces $q(\alpha) <_\xi p(\alpha)$.
- $q \leq^+_{w,\xi} p$ is defined analogously using $<^+_\xi$ instead of $<_\xi$.

Obviously $q \leq^+_{w,\xi} p$ implies $q \leq_{w,\xi} p$; and $q \leq_{w,\xi} p$ implies that both p and q fit (w,ξ).

Remark. In contrast to the single forcing (or a product of such forcings), $q \leq_{w,\xi} p$ (or $q \leq^+_{w,\xi} p$) does *not* imply $\mathrm{poss}(q,w,\xi) = \mathrm{poss}(p,w,\xi)$.[6] More explicitly, setting $w = \{0,1\}$, it is possible that $x \in \mathrm{poss}(p,w,\xi)$ but p does not force that $x(0) \subseteq \underset{\sim}{\eta}_0$ implies $x(1) \in \mathrm{poss}(p(1),\xi)$. (But see Section 5.4.)

Lemma 5.18. *If q_i is a $\leq^+_{w,\zeta}$-decreasing sequence of length $\delta < \lambda$, then there is an $r \leq^+_{w,\zeta} q_i$ for all $i < \delta$.*

Proof. Set $\mathrm{dom}(r) := \bigcup_{i\in\delta} \mathrm{dom}(q_i)$, without loss of generality closed under limits. By induction on $\alpha \in \mathrm{dom}(r)$ we know that $r \restriction \alpha \leq q_i \restriction \alpha$ for all i, and define $r(\alpha)$ as follows: If $\alpha \in w$, we know that the $q_i(\alpha)$ are $\leq^+_\zeta$-increasing. Using Lemma 5.6(1), we pick some $r(\alpha)$ such that $r(\alpha) \leq^+_\zeta q_i(\alpha)$ for all i. If $\alpha \notin w$, we just pick any $r(\alpha) \leq q_i(\alpha)$ for all i. $\square$

It is easy to see that P satisfies a version of fusion:

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[5] For example: For a p-condition Q, let ODD^p be the set of odd elements of C^p (or any other unbounded subset X of C^p such that $C^p \setminus X$ is still club), and set $\mathrm{ODD}^p_? := \bigcup_{\zeta\in\mathrm{ODD}^p} I^*_\zeta \setminus \mathrm{dom}(\eta^p)$. Note that for any $x : \mathrm{ODD}^p_? \rightarrow 2$, $\eta^p \cup x$ defines a condition in Q (stronger than p). So if we fix any $p(0) \in P_1$, and define the P_1-name $\underset{\sim}{\tau} \in \{0,1\}$ to be 0 iff $\underset{\sim}{\eta}_0 \restriction \mathrm{ODD}^{p(0)}_?$ is eventually constant to 0, then $\underset{\sim}{\tau}$ cannot be $(\{0\},\zeta)$-decided by $p(0)$ for any ζ. And if $p(1)$ is any condition with $p(0) \Vdash \eta^{p(1)}(0) = \underset{\sim}{\tau}$, then $\underset{\sim}{\tau}$ is $(\{1\},1)$-decided by $q := (p(0),p(1))$.

[6] An example: $\mathrm{dom}(p) = \mathrm{dom}(q) = w = \{0,1\}$, $\min(C^{p(0)}) = \min(C^{q(0)}) = \xi$, and both $p(0)$ and $q(0)$ have trunk $a \in \mathrm{POSS}^Q(\xi)$. $p(0)$ forces that $p(1) = q(1)$, that $\min(C^{p(1)}) = \xi$ and that the trunk of $p(1)$ is either b or c (elements of $\mathrm{POSS}^Q(\xi)$); both are possible with $p(0)$. Now $q(0) \leq^+_\xi p(0)$ decides that the trunk of $p(1)$ is b. Then $q \leq^+_{w,\xi} p$, and (a,c) is in $\mathrm{poss}(p,w,\xi) \setminus \mathrm{poss}(q,w,\xi)$. In particular $(a,c) \in \mathrm{poss}(p,w,\xi)$ but p does not force that $a \subseteq \underset{\sim}{\eta}_0$ implies $c \in \mathrm{poss}(p(1),\xi)$.

Lemma 5.19. *Assume $(p_i)_{i<\delta}$ is a sequence of length $\delta \le \lambda$, such that $p_j \le_{w_i,\xi_i} p_i$ for $i \le j < \delta$, $w_i \in [\mu]^{<\lambda}$ increasing, $\xi_i \in \lambda$ strictly increasing. Set $w_\infty := \bigcup_{i<\delta} w_i$, $\mathrm{dom}_\infty := \bigcup_{i<\delta} \mathrm{dom}(p_i)$ and $\xi_\infty := \sup_{i<\delta} \xi_i$. If $\delta = \lambda$, we additionally assume $w_\infty = \mathrm{dom}_\infty$.*

Then there is a limit q_∞ with $\mathrm{dom}(q_\infty) = \mathrm{dom}_\infty$ such that $q_\infty \le_{w_i,\xi_i} p_i$ for all $i < \delta$.

If $\delta < \lambda$, then q_∞ fits (w_∞, ξ_∞).

(If $w_\infty = \mathrm{dom}_\infty$, then the limit q_∞ is "canonical".)

Proof. We define $q_\infty(\alpha)$ by induction on dom_∞. We assume that we already have $q' := q_\infty \restriction \alpha$ which satisfies $q' \le_{w_i \cap \alpha, \xi_i} p_i$ for all $i < \delta$.

Case 1: $\alpha \notin w_\infty$ (this can only happen if $\delta < \lambda$): We know that q' forces that $(p_i(\alpha))_{i<\delta}$ is a decreasing sequence, and we just pick some $q_\infty(\alpha)$ stronger then all of them.

Case 2: $\alpha \in w_\infty$: Let i^* be minimal such that $\alpha \in w_{i^*}$. We know that q' forces for all $i^* \le i < j < \delta$ that $p_j(\alpha) <_{\xi_i} p_i(\alpha)$, so according to Lemma 5.6(2) there is a limit $q_\infty(\alpha) <_{\zeta_i} p_i(\alpha)$ (so in particular $q' \Vdash \zeta_i \in C^{q_\infty(\alpha)}$ for all $i \ge i^*$).

Now assume $\delta < \lambda$. If $\alpha \in w_\infty$, then it is in w_i for coboundedly many $i < \delta$. In other words, $p_j \restriction \alpha \Vdash \zeta_i \in C^{p_j(\alpha)}$ for coboundedly many $i \in \delta$ and all $j > i$, which implies $q_\infty \restriction \alpha \Vdash \xi_\infty \in C^{q_\infty(\alpha)}$. □

Preliminary Lemma 5.20. *Let p fit (w, ζ), $x \in \mathrm{poss}(p, w, \zeta + 1)$, and let $r \le p$ extend x, i.e., $r \Vdash x \triangleleft G$. Then there is a $q \le^+_{w,\zeta} p$ forcing that $x \triangleleft G$ implies $r \in G$.*

Proof. Set $\mathrm{dom}(q) := \mathrm{dom}(r)$. We define $q(\alpha)$ by induction on $\alpha \in \mathrm{dom}(q)$ and show inductively:

- $q \restriction \alpha \le^+_{w\cap\alpha,\zeta} p \restriction \alpha$.
- $q \restriction \alpha \Vdash (x \restriction \alpha \triangleleft G_\alpha \to r \restriction \alpha \in G_\alpha)$.

For notational convenience, we assume $\mathrm{dom}(p) = \mathrm{dom}(r)$ (by setting $p(\alpha) = \mathbb{1}_Q$ for any α outside the original domain of p).

Assume we already have constructed $q_0 = q \restriction \alpha$. Work in the P_α-extension $V[G_\alpha]$ with $q_0 \in G$.
Case 1: $r \restriction \alpha \notin G_\alpha$. Set $q(\alpha) := p(\alpha)$.
Case 2: $r \restriction \alpha \in G_\alpha$. Then $r(\alpha) \le p(\alpha)$. If $\alpha \notin w$, we set $q(\alpha) := r(\alpha)$; otherwise we set $q(\alpha)$ to be $r(\alpha) \vee (p(\alpha) \restriction \zeta + 1)$ as in Lemma 5.8.

If $\alpha \in w$, then in both cases we get $q \restriction \alpha \vDash q(\alpha) \le^+_\zeta p(\alpha)$. Also, if $G_{\alpha+1}$ selects $x \restriction (\alpha + 1)$, then at stage α we used, by induction, Case 2; so then $r(\alpha) \in G(\alpha)$ as $x(\alpha) \subseteq \underset{\sim}{\eta}_\alpha$. □

We can iterate the construction for all elements of $\mathrm{poss}(w, \zeta + 1)$, which gives us:

Lemma 5.21. *If p fits (w, ζ) and $\underset{\sim}{\tau}$ is a name for an ordinal, then there is a $q \le^+_{w,\zeta} p$ which $(w, \zeta + 1)$-decides $\underset{\sim}{\tau}$.*

Proof. We enumerate $\mathrm{poss}(p, w, \zeta + 1)$ as $(x_i)_{i\in\delta}$. We start with $p_0 := p$. Inductively we construct p_ℓ: If at step ℓ, if x_ℓ is not in $\mathrm{poss}(p_\ell, w, \zeta + 1)$ any more, then we set $p_{\ell+1} := p_\ell$. Otherwise, pick an $r \le p_\ell$ that decides $\underset{\sim}{\tau}$ to be some τ^{x_ℓ} and extends x_ℓ. Then apply 5.20 to get $p_{\ell+1} \le^+_{w,\zeta} p_\ell$ which forces that $x_\ell \triangleleft G$ implies $\underset{\sim}{\tau} = \tau^{x_\ell}$. At limits use Lemma 5.18. □

For the proof of Lemma 5.23 we will need a variant where the "height" ζ is not the same for all elements of w, more specifically:

Preliminary Lemma 5.22. *Assume that p fits (w, ζ) and $p \restriction \alpha^* \Vdash \zeta^* \in C^{p(\alpha^*)}$, and that $\underset{\sim}{\tau}$ is a name for an ordinal. Then there is a $q \le^+_{w,\zeta} p$ such that $q \restriction \alpha^* \Vdash q(\alpha^*) \le^+_{\zeta^*} p(\alpha^*)$ and there is a (ground model) set A of size $<\lambda$ such that $q \Vdash \underset{\sim}{\tau} \in A$.*

Proof. This is just a notational variation of the previous proof. For notational simplicity we assume $\alpha^* \notin w$.

First we have to modify 5.20: A candidate is a pair (x, a) where $x \in \mathrm{POSS}(w, \zeta)$ and $a^* \in \mathrm{POSS}^Q(\zeta^*)$. Assume that (x, a) is a candidate, that $p \in P$ fits (w, ζ) and that $p \restriction \alpha^* \Vdash \zeta^* \in C^{p(\alpha^*)}$, and assume that $r \le p$ extends (x, a), i.e., $r \Vdash (x \triangleleft G \,\&\, a^* \subseteq \underset{\sim}{\eta}_{\alpha^*})$. Then there is a q such that

$$(*) \qquad q \le^+_{w,\zeta} p, \quad q \restriction \alpha^* \Vdash q(\alpha^*) \le^+_{\zeta^*} p(\alpha^*), \text{ and } \quad q \Vdash \big((x \triangleleft G \,\&\, a^* \subseteq \underset{\sim}{\eta}_{\alpha^*}) \to r \in G\big).$$

The same proof works, with the obvious modifications:

When define $q(\alpha)$, we inductively show:

- $q \restriction \alpha \leq^+_{w\cap\alpha,\zeta} p \restriction \alpha$ and if $\alpha > \alpha^*$ then $q \restriction \alpha^* \Vdash q(\alpha^*) \leq^+_{\zeta^*} p(\alpha^*)$,
- $q \restriction \alpha \Vdash \big((x \restriction \alpha \lhd G_\alpha \,\&\, a^* \subseteq \underset{\sim}{\eta}_{\alpha^*}) \to r \restriction \alpha \in G_\alpha\big)$, unless $\alpha \leq \alpha^*$ in which case we omit the clause about α^*.

Again, in the P_α-extension we have:
Case 1: $r \restriction \alpha \notin G_\alpha$. Set $q(\alpha) := p(\alpha)$.
Case 2: $r \restriction \alpha \in G_\alpha$. Then $r(\alpha) \leq p(\alpha)$. If $\alpha \notin w \cup \{\alpha^*\}$, we set $q(\alpha) := r(\alpha)$; otherwise we set $q(\alpha)$ to be ‘ $r(\alpha) \vee (p(\alpha) \restriction \zeta + 1)$ as in Lemma 5.8.

Then we can show $(*)$ as before.

We then enumerate all candidates (there are $<\lambda$ many) as (x_ℓ, a_ℓ), and at step ℓ, if (x_ℓ, a_ℓ) is compatible with p_ℓ, use $(*)$ to decide $\underset{\sim}{\tau}$ to be some $\underset{\sim}{\tau}^\ell$. □

We will now show that P is λ^λ-bounding and proper. We first give two preliminary lemmas that assume this is already the case for all $P_{\beta'}$ with $\beta' < \beta$.

Preliminary Lemma 5.23. *Let $\beta \leq \mu$, and assume that $P_{\beta'}$ is λ^λ-bounding for all $\beta' < \beta$.*

Assume $p \in P_\beta$ fits (w, ζ), $\tilde{C} \subseteq \lambda$ is club, and $\alpha^* < \beta$.

Then there is a $q \leq^+_{w,\zeta} p$ and a $\xi \in \tilde{C}$ such that q fits $(w \cup \{\alpha^*\}, \xi)$.

If additionally $\alpha^* \in \mathrm{dom}(p)$ and $p \restriction \alpha^* \Vdash \zeta^* \in C^{p(\alpha^*)}$ for some $\xi^* \in \lambda$, then we can additionally assume $q \restriction \alpha^* \Vdash q(\alpha^*) \leq^+_{\zeta^*} p(\alpha^*)$.

Proof. For notational simplicity assume $\alpha^* \notin w$ and $\min(\tilde{C}) > \max(\zeta, \zeta^*)$. By induction on $\alpha \leq \beta$ we show that the result holds for all w, α^* with $w \cup \{\alpha^*\} \subseteq \alpha$.

Successor case $\alpha + 1$: Set $w_0 := w \cap \alpha$.

By our assumption P_α is λ^λ-bounding, so every club-set in the P_α-extension contains a ground-model club (see Fact 5.11). In particular, $C^{p(\alpha)}$ contains some ground-model C^*. By Lemma 5.21 (or 5.22, if $\alpha^* < \alpha$) there is a $p' \leq^+_{w_0,\zeta} p \restriction \alpha$ (also dealing with α^*, if $\alpha^* < \alpha$) leaving only $<\lambda$ many possibilities for C^*. So we can intersect them all, resulting in C'. Set $C'' := C' \cap \tilde{C}$. Apply the induction hypothesis in P_α to get $q' \leq^+_{w_0,\zeta} p'$ and ξ in C'' such that q' fits (w_0, ξ) (and also $(\{\alpha^*\}, \xi)$, if $\alpha^* < \alpha$). Set $q := q' \cup \{(\alpha, p(\alpha))\}$, so trivially $q \leq^+_{w,\zeta} p$ (and, if $\alpha = \alpha^*$, then $q \restriction \alpha \Vdash q(\alpha) \leq^+_{\zeta^*} p(\alpha)$), and q fits $(w \cup \{\alpha\}, \xi)$.

Limit case: If w is bounded in α there is nothing to do. So assume w is cofinal.

Set $\alpha_0 := \min(w \setminus \alpha^*)$ and $w_0 := (w \cap \alpha_0) \cup \{\alpha^*\}$. Use the induction hypothesis in P_{α_0} using $(p \restriction \alpha_0, w_0, \zeta, \alpha^*, \zeta^*)$ as $(p, w, \zeta, \alpha^*, \zeta^*)$. This gives us some $p'_0 \leq^+_{w\cap\alpha_0,\zeta} p \restriction \alpha_0$ fitting (w_0, ζ_0) and dealing with α^*, for some $\zeta_0 \in \tilde{C}$. Set $p_0 := p' \wedge p$.

Enumerate $w \setminus w_0$ increasingly as $(\alpha_i)_{i<\delta}$, and set $w_j := w_0 \cup \{\alpha_i : \ i < j\}$ for $j \leq \delta$.

We will construct p'_i in P_{α_i} and $(\zeta_i)_{i<\delta}$ a strictly increasing sequence in $\tilde{C}$, and we set $p_j := p'_j \wedge p$ and will get: p_ℓ fits (w_ℓ, ζ_ℓ),and $p_\ell \leq^+_{w_i,\zeta_i} p_i$ for all $i < \ell \leq j$.

For successors $\ell = i + 1$, we use the induction hypothesis in $P_{\alpha_{i+1}}$, using $(p_i \restriction \alpha_{i+1}, w_i, \zeta_i, \alpha_i, \zeta)$ as $(p, w, \zeta, \alpha^*, \zeta^*)$. This gives us $p'_{i+1} \leq^+_{w_i,\zeta_i} p_i \restriction \alpha_{i+1}$ and some $\zeta_{i+1} > \zeta_i$ in $\tilde{C}$ such that p_{i+1} fits (w_{i+1}, ζ_{i+1}) and $p_{i+1} \restriction \alpha_i \Vdash p_{i+1}(\alpha_i) \leq^+_\zeta p_i(\alpha_i)$.

For j limit, we set $\zeta_j := \sup_{i<j} \zeta_i$ (which is in $\tilde{C}$), and let p_j be a limit of the $(p_i)_{i<j}$. I.e., $\mathrm{dom}(p_j) = \bigcup_{i<j} \mathrm{dom}(p_i)$, and for $\beta \in \mathrm{dom}(p_j)$ let $p_j(\beta)$ be as follows: If $\beta \notin w$, fix some condition $p_j(\beta)$ stronger than all $p_i(\beta)$. Otherwise, there is a minimal $i_0 < j$ such that $\beta \in w_{i_0}$, and $p_\ell(\beta) <^+_{\zeta_i} p_i(\beta)$ for all $i_0 \leq i < \ell < j$. In that case let $p_j(\beta)$ be the (canonical) limit of the $(p_i(\beta))_{i_0 \leq i < j}$, and note that $\zeta_j \in C^{p_j(\beta)}$. □

Preliminary Lemma 5.24. *Let $\beta \leq \mu$, and assume that $P_{\beta'}$ is λ^λ-bounding for all $\beta' < \beta$.*

Assume that $p \in P_\beta$ fits (w, ζ), and $\underset{\sim}{\sigma}$ is a P_β-name for a λ-sequence of ordinals. Then there is a $q \leq^+_{w,\zeta} p$ continuously reading $\underset{\sim}{\sigma}$.

Proof. Set $p_0 := p$, $\zeta_0 := \zeta$, $w_0 := w$. We construct by induction on $i < \lambda$ p'_i, p_i, ζ_i, α_i and w_i as follows:

- Given p_j, w_j, and ζ_j, pick $\alpha_j \in \mathrm{dom}(p_j) \setminus w_j$ by bookkeeping (so that in the end the domains of all conditions will be covered).
- Successor $j = i+1$: Set $w_{i+1} := w_i \cup \{\alpha_i\}$. Find $p'_{i+1} \leq^+_{w_i,\zeta_i} p_i$ and $\zeta_{i+1} > \zeta_i$ such that p'_{i+1} fits (w_{i+1}, ζ_{i+1}) (using the previous preliminary lemma).
- Limit j: Let p'_j be the canonical limit of the $(p_i)_{i<j}$, $\zeta_j := \sup_{i<j}(\zeta_i)$, and $w_j := \bigcup_{i<j} w_i$. Note that p'_j fits (w_j, ζ_j).
- In any case, given p'_j we pick some $p_j \leq^+_{w_j,\zeta_j} p'_j$ which (w_j, ζ_j+1)-decides $\underset{\sim}{\sigma}(\zeta_j)$.

Then the limit q of the p_i continuously reads $\underset{\sim}{\sigma}$. □

Lemma 5.25. *P has continuous reading (and in particular is λ^λ-bounding).*

Proof. Assume by induction that $P_{\beta'}$ is λ^λ-bounding for all $\beta < \beta'$. Then the previous lemma gives us that P_β has continuous reading of names, and thus is λ^λ-bounding. □

The same construction shows λ-properness:

Definition 5.26. Let $\chi \gg \mu$ be sufficiently large and regular. An "elementary model" is an $M \preceq H(\chi)$ of size λ which is $<\lambda$-closed and contains λ and μ (and thus P).

Lemma 5.27. *If M is an elementary model containing $p \in P$, then there is a $q \leq p$ which is strongly M-generic in the following sense: For each P-name $\underset{\sim}{\tau}$ in M for an ordinal, q (w,ζ)-decides $\underset{\sim}{\tau}$ via a decision function in M (so in particular $q \Vdash \underset{\sim}{\tau} \in M$).*

(The decision function being in M is equivalent to $w \subseteq M$, as M is $<\lambda$ closed.)

Proof. Let $\underset{\sim}{\sigma}$ be a sequence of all P-names for ordinals that are in M. Starting with $p_0 \in M$, perform the successor step of the previous construction within M; as M is closed the limits at steps $<\lambda$ are in M as well. Then the λ-limit is M-generic. □

5.4. **Canonical conditions.** We will use conditions that "continuously read themselves."

Definition 5.28. $p \in P$ is (w,ζ)-canonical if p fits (w,ζ) and $p(\alpha) \restriction (\zeta+1)$ is $(w \cap \alpha, \zeta+1)$-decided by $p \restriction \alpha$ for all $\alpha \in w$.

Facts 5.29. Let p be canonical for (w,ζ).

1. If $q \leq^+_{w,\zeta} p$, then q is canonical for (w,ζ) and $\mathrm{poss}(p,w,\zeta+1) = \mathrm{poss}(q,w,\zeta+1)$
2. Let $x \in \mathrm{poss}(p,w,\zeta+1)$. There is a naturally defined $p \wedge x \leq p$ such that $p \Vdash (p \wedge x \in G \leftrightarrow x \lhd G)$. $\{p \wedge x : x \in \mathrm{poss}(p,w,\zeta+1)\}$ is a maximal antichain below p.
3. Let $x \in \mathrm{poss}(p,w,\zeta+1)$. In an intermediate P_α-extension $V[G_\alpha]$ with $x \restriction \alpha \lhd G_\alpha$ the rest of x, i.e., $x \restriction [\alpha,\mu]$, is compatible with p/G_α in the quotient forcing.
   Or equivalently: If $r_0 \leq p \restriction \alpha$ in P_α extends $x \restriction \alpha$, then there is an $r \leq r_0$ extending x.

Definition 5.30. Assume $p \in P$, and $\underset{\sim}{\sigma}$ is a P-name for a λ-sequence of ordinals. Let $E \subseteq \lambda$ be a club-set and $\bar{w} = (w_\zeta)_{\zeta \in E}$ an increasing sequence in $[\mu]^{<\lambda}$.

p canonically reads $\underset{\sim}{\sigma}$ as witnessed by $\bar{w}$ if the following holds:

- $\mathrm{dom}(p) = \bigcup_{\zeta \in E} w_\zeta$.
- p is (w_ζ, ζ)-canonical for all $\zeta \in E$.
- $p \restriction \alpha \Vdash C^{p(\alpha)} = E \setminus (\zeta'_\alpha)$ for some (ground model) ζ'_α.
- $\underset{\sim}{\sigma} \restriction I^*(\leq\zeta+1)$ is $(w_\zeta, \zeta+1)$-decided by p for all $\zeta \in E$.

If σ is the constant 0 sequence (or any sequence in V), we just say "p is canonical" (as witnessed by $\bar{w}$).

Lemma 5.31. *For p, $\underset{\sim}{\sigma}$ as above, there is a $q \leq p$ canonically reading $\underset{\sim}{\sigma}$.*

Proof. We just have to slightly modify the proof of Lemma 5.24.

We will construct p_j, ξ_j and α_j by induction on $j \in \lambda$, setting $w_j := \{\alpha_i : i < j\}$, such that for $0 < j < k$ the following holds:

- $p_k \leq^+_{w_j,\xi_j} p_j$.
- p_j is (w_j, ξ_j)-canonical.
- p_j $(w_j, \xi_j + 1)$-decides $\underset{\sim}{\sigma} \restriction I^*(\leq\xi_j + 1)$.
- In p_k, for $\alpha_j \in w_k$, $\{\zeta_i : j < i < k\}$ is (forced to be) an initial segment of $C^{p_k(\alpha_j)}$.
- The α_j are chosen (by some book-keeping) so that $\{\alpha_i : i \in \lambda\} = \bigcup_{i\in\lambda} \mathrm{dom}(p_i)$.

Then the limit of the p_j is as required, with $E = \{\xi_i : i \in \lambda\}$ and, for $\zeta = \xi_j$ in E, we use w_j as w_ζ.

Set $p_0 \leq p$ such that $|\mathrm{dom}(p_0)| = \lambda$, and set $\xi_0 := 0$. Assume we already have p_i, α_i for $i < j$ (so we also have w_j).

- For j limit, let s be a limit of $(p_i)_{i<j}$, and set $\xi_j := \sup_{i<j} \xi_i$. Note that s fits (w_j, ξ).
- Successor case $j = i + 1$: Find $s_0 \leq^+_{w_i,\xi_i} p_i$ and $\xi_j > \xi_i$ such that s fits (w_j, ξ_j). (As in Lemma 5.23. Recall that $w_j = w_i \cup \{\alpha_i\}$.)
  Strengthen s_0 to $s \leq^+_{w_i,\xi_i}$ so that:
  - s still fits (w_j, ξ_j),
  - the trunk at α_i has length ξ_j, i.e., $s \restriction \alpha_i \Vdash \min(C^{s(\alpha_i)}) = \xi_j)$,
  - for $\alpha_{i'}$, $i' < i$, there are no elements in $C^{s(\alpha_{i'})}$ between ξ_i and ξ_j.
- Construct $s^* \restriction \alpha$ by recursion on $\alpha \in w_j$, such that $s^* \restriction \alpha \leq^+_{w_j\cap\alpha,\xi_j} s \restriction \alpha$ and $s^* \restriction \alpha$ $(w_j \cap \alpha, \xi_j + 1)$-decides $s(\alpha) \restriction (\xi_j + 1)$ (which is the same as $s^*(\alpha) \restriction (\xi_j + 1)$). This gives $s^* \leq^+_{w_j,\xi_j} s$.
- Find $p_j \leq^+_{w_j,\xi_j} s^*$ which $(w_j, \xi_j + 1)$ decides $\underset{\sim}{\sigma} \restriction I^*(\leq\xi + 1)$.
- Choose $\alpha_j \in \mathrm{dom}(p_j) \setminus w_j$ by bookkeeping. □

5.5. **The stationary system.** From now on, assume

$$p_* \Vdash \underset{\sim}{\pi} : \mathcal{P}(\lambda) \to \mathcal{P}(\lambda) \text{ represents the automorphism } \underset{\sim}{\phi} : \mathcal{P}(\lambda)/[\lambda]^{<\lambda} \to \mathcal{P}(\lambda)/[\lambda]^{<\lambda},$$

and we set, for $\beta \in \mu$,

$$\underset{\sim}{a}_\beta := \underset{\sim}{\pi}(\underset{\sim}{\eta}_\beta),$$

where we identify $\underset{\sim}{\eta}_\beta \in 2^\lambda$ with $\underset{\sim}{\eta}_\beta^{-1}\{1\} \subseteq \lambda$. (In the following we sometimes use this identification of 2^λ and $\mathcal{P}(\lambda)$ without explicitly mentioning it.)

Note that, other than $\underset{\sim}{\eta}_\beta$, $\underset{\sim}{a}_\beta$ is a priori not a $P_{\beta+1}$-name (but see Section 5.8).

With $S^\mu_{\lambda^+}$ we denote the stationary subset of μ consisting of ordinals with cofinality λ^+.

Definition 5.32. Let $S \subseteq S^\mu_{\lambda^+}$ be stationary, $\chi \gg \mu$ sufficiently large and regular, and $z \in H(\chi)$. "An elementary S-system" (using parameter z) is a sequence $(M_\beta, p_\beta)_{\beta\in S}$ such that, for each $\beta \in S$, M_β is elementary (as in Definition 5.26) an contains z, β, p_*, $\underset{\sim}{\phi}$ and $\underset{\sim}{\pi}$, and $p_\beta \in P \cap M_\beta$ canonically reads $\underset{\sim}{a}_\beta$ witnessed by some $(w^p_{\beta,\zeta})_{\zeta\in E^p_\beta}$.

By a simple Δ-system argument we can make an S-system homogeneous:

Definition 5.33. $(M_\beta, p_\beta)_{\beta\in S}$ forms a "Δ-system", if $\bar{M}, \bar{p}$ is an elementary S-system with parameter z, and the system is a homogeneous Δ-system in the following sense: For β and $\beta_1 < \beta_2$ in S, we get:

1. $M_{\beta_1} \cap M_{\beta_2} \cap \mu$ is constant. We call this set the "heart" and, abusing notation, denote it with Δ. Obviously $\Delta \supseteq \lambda$ and $\Delta \supseteq \mathrm{dom}(p_*)$.
2. $M_\beta \cap \beta = \Delta$. So in particular β is the minimal element of M_β above Δ. All the non-heart elements of M_{β_2} are above all elements of M_{β_1}. I.e., $\sup(M_{\beta_1} \cap \mu) < \beta_2$.
3. There is an $\in$-isomorphism $h^*_{\beta_1,\beta_2} : M_{\beta_1} \to M_{\beta_2}$, mapping β_1 to β_2, p_{β_1} to p_{β_2}, $\underset{\sim}{a}_{\beta_1}$ to $\underset{\sim}{a}_{\beta_2}$ and fixing $\lambda, \mu, \underset{\sim}{\phi}, \underset{\sim}{\pi}$ as well as each α in Δ.

Note that this implies that the continuous reading of $\underset{\sim}{a}_\beta$ works the same way for all β. In particular the E^{p_β} are that same for all β; and if F^β_ζ is the function mapping $\mathrm{POSS}(w^p_{\beta,\zeta}, \zeta + 1)$ to the value of $\underset{\sim}{a}_\beta \restriction I^*(\leq\zeta + 1)$ (for $\zeta \in E^{p_\beta}$), then $h^*_{\beta_1,\beta_2}(F^{\beta_1}_\zeta) = F^{\beta_2}_\zeta$ and in particular $h^*_{\beta_1,\beta_2}(w^p_{\beta_1,\zeta}) = w^p_{\beta_2,\zeta}$; i.e., they are the same apart from shifting coordinates above Δ.

Lemma 5.34.
- *For every $z \in H(\chi)$, $S \subseteq S^{\mu}_{\lambda^+}$ and $(p'_\beta)_{\beta\in S}$ there are M_β and $p_\beta \leq p'_\beta$ such that $\bar{M}, \bar{p}$ is an S-system with paramter z.*
- *If $\bar{M}, \bar{p}$ is an S-system then there is an $S' \subseteq S$ stationary such that $(M_\beta, p_\beta)_{\beta\in S'}$ is a Δ-system on S'.*

Proof. The first item is trivial, using the fact that everything can be read canonically.

Using $2^\lambda = \lambda^+$, a standard Δ-system argument (or: Fodor's Lemma argument) lets us thin out S so that $(M_\beta \cap \mu)_{\beta\in S}$ satisfies (1–3). Assign to each $\beta \in S^3$ the ι_β-images of M_β, p_β, $\underset{\sim}{a}_\beta$, μ, $\underset{\sim}{\phi}$, $\underset{\sim}{\pi}$ and, for $\zeta \in E$ and $\gamma \in w^\beta_\zeta$, F^β_γ, where $\iota_\beta : M_\beta \cup \{M_\beta\} \to H(\lambda^+)$ is the transitive collapse. Again, there are $|H(\lambda^+)|^\lambda < \mu$ many possibilities, so the objects are constant on a stationary $S' \subseteq S^3$. For $\alpha < \beta$ in S', we can define $h^*_{\beta_1,\beta_2} := \iota^{-1}_{\beta_2} \circ \iota_{\beta_1}$. (Note that $\iota_{\beta_1}(\alpha) = \iota_{\beta_2}(\alpha)$ for $\alpha \in \Delta$.) □

So in particular if we have a Δ-system on S, then $p_\beta \restriction \sup(\Delta) = p_\beta \restriction \beta \in M_\beta$ is the same for all $\beta \in S$, and outside of Δ the domains of the p_β are disjoint for $\beta \in S$. In particular we get:

Fact 5.35. For a Δ-system with domain S, and $A \subseteq S$ of size $\leq\lambda$, the union of the $(p_\beta)_{\beta\in A}$ is a condition in P (and stronger than each p_β).

It is important to note that all $p_\beta(\beta)$ are the same as well; we give it a name:

$$\tilde{p} := p_\beta(\beta),$$

which is a P_β-name independent of $\beta \in S$.

"Independent" might sound a bit vague, but as we assume that the p_β are canonical, we can formulate it in the following form: $p_\beta(\beta) \restriction \zeta + 1$ is $(w_{\beta,\zeta}, \zeta + 1)$-determined for cofinally many ζE and some $w_{\beta,\zeta} \in [\beta]^{<\lambda}$ which is a subset of M_β. So $w_{\beta,\zeta} \subseteq \Delta$, and the isomorphisms between the M_β guarantee that each $w_{\beta,\zeta}$ is the same, and that $p_\beta(\beta) \restriction \zeta + 1$ is decided the same way. So $\tilde{p}$ is a P_γ-name for $\gamma = \sup(w_{\beta,\zeta})_{\zeta\in\lambda}$ (which is independent of λ, and in $\Delta \cap M$). So $\tilde{p}$ is actually a P_α-name for some $\alpha \in \Delta$.

For later reference we note:

Lemma 5.36. *For all but non-stationary many β, p_* forces $\underset{\sim}{a}_\beta \notin V_\beta$.*

Proof. Assume that $p_\beta \leq p_*$ forces that $\underset{\sim}{a}_\beta = \underset{\sim}{x}_\beta$ for a P_β-name $\underset{\sim}{x}_\beta$ for all $\beta \in S^*$ stationary. We can also assume that p_β canonically reads $\underset{\sim}{a}_\alpha$. Pick M_β containung p_β and $S \subseteq S^*$ such that $(M_\beta, p_\beta)_{\beta\in S}$ is a Δ-system, where we can assume (or get from homogeneity) that $h^*_{\beta_0,\beta_1}(\underset{\sim}{x}_{\beta_0}) = \underset{\sim}{x}_{\beta_1}$. So as the $\underset{\sim}{x}_\beta$ are P_β and therefore $P_{\sup(\Delta)}$-names, they are the same for all β. Choose $\beta_1 > \beta_0$ in S. So $p_{\beta_0} \wedge p_{\beta_1}$ force that $\underset{\sim}{a}_{\beta_0} = \underset{\sim}{x} = \underset{\sim}{a}_{\beta_1}$, which contradicts the injectivity of $\underset{\sim}{\phi}$. □

5.6. Preservation of cofinalities.

Corollary 5.37. *P preserves all cofinalities.*

Proof. Cofinalities $\leq\lambda$ are preserved as P is $<\lambda$-closed.

Cofinality λ^+ is preserved by properness: If it is forced that κ has a cofinal λ-sequence, then there is an elementary model containing the names for all elements of the sequence, so each element will be evaluated to an ordinal in M, but $|M| = \lambda$, so κ has cofinality $\leq \lambda$ in the ground model.

Cofinality $\geq \lambda^{++}$ is preserved as P has the λ^{++}-cc, which we have shown in a very roundabout way with the fact about Δ-systems: If $(p'_\alpha)_{\alpha\in\mu}$ are arbitrary conditions, then (M_β, p_β) form a Δ-system from some $p_\beta < p'_\beta$ and stationary S, and any two p_β are compatible for $\beta \in S$. □

5.7. **Majority decisions.** For any (a_1, a_2, a_3) with $a_i \in \{0, 1\}$ there is a $b \in \{0, 1\}$ such that $b = a_i$ for at least two $i \in \{1, 2, 3\}$. We write $b = \text{major}_{i=1,2,3}(a_i)$.

Similarly, if f_1, f_2, f_3 are functions $A \to 2$ we write $\text{major}_{i=1,2,3}(f_i)$ for the function $A \to 2$ that maps ℓ to $\text{major}_{i=1,2,3}(f_i(\ell))$.

The following is a central point of the whole construction:

Lemma 5.38. *Let $(M_\alpha, p_\alpha)_{\alpha\in S}$ be a Δ-system. Pick $\beta_0 < \beta_1 < \beta_3 < \beta_4$ in S.*

(1) *p_* forces: If $\underset{\sim}{\eta}_{\beta_0} =^* \text{major}_{i=1,2,3}(\underset{\sim}{\eta}_{\beta_i})$, then $\underset{\sim}{a}_{\beta_0} =^* \text{major}_{i=1,2,3}(\underset{\sim}{a}_{\beta_i})$.*

(2) *Let $s = \bigwedge_{i<4} p_i$. Recall that $s(\beta_i)$ is the same P_{β_i}-name called $\tilde{p}$ for all i. We can strengthen s by strengthening, for $i = 1,2,3$, the condition $s(\beta_i) = \tilde{p}$ to some P_{β_0+1}-conditions $r_i \leq \tilde{p}$ (without changing $C^{\tilde{p}}$) such that the resulting condition forces $\underset{\sim}{\eta}_{\beta_0} = \mathrm{major}_{i=1,2,3}(\underset{\sim}{\eta}_{\beta_i})$.*

Recall that $\nu_1 =^* \nu_2$ denotes that $\nu_1(\ell) = \nu_2(\ell)$ for all but $<\lambda$ many $\ell \in \lambda$.

Proof. (1) Identifying 2^λ with $P(\lambda)$, we have $\mathrm{major}_{i=1,2,3} f_i = (f_1 \cap f_2) \cup (f_2 \cap f_3) \cup (f_1 \cap f_3)$ for any f_i. As $\underset{\sim}{\pi}$ represents an automorphism, we get $\underset{\sim}{\pi}(\mathrm{major}_{i=1,2,3}(f_i)) =^* \mathrm{major}_{i=1,2,3}(\underset{\sim}{\pi}(f_i))$. Apply this to $f_i := \underset{\sim}{\eta}_{\beta_i}$.

(2) Work in the P_{β_0+1}-extension. So $\tilde{p} = p_{\beta_0}(\beta_0)$ and $\underset{\sim}{\eta}_{\beta_0}$ are already determined, and $\underset{\sim}{\eta}_{\beta_0}$ extends $\eta^{\tilde{p}}$. Set $r_0 := \tilde{p}$.

Set $s_1 := (0,0)$, $s_2 := (0,1)$, $s_3 := (1,0)$. For $\zeta \in C^{\tilde{p}}$ and $i = 1,2,3$, we define $r_i(\zeta) \supseteq \tilde{p}(\zeta)$ as follows:

(5.39) $$\text{Extend } s^{\tilde{p}}_\zeta \text{ by } s_i: \ s^{r_i}_\zeta := (s^{\tilde{p}})^\frown s_i; \text{ and set } r_i(\zeta)(\ell) = \underset{\sim}{\eta}_\beta(\ell) \text{ for } \ell \in [s^{\tilde{p}}_\zeta] \setminus [s^{r_i}_\zeta].$$

So η^{r_i} agrees with $\underset{\sim}{\eta}_{\beta_0}$ on its domain, and each $\ell \in \lambda$ is in $\mathrm{dom}(\eta^{r_i})$ for at least two $i \in \{1,2,3\}$. Accordingly, an extension containing all r_i will satisfy $\underset{\sim}{\eta}_{\beta_0} = \mathrm{major}_{i=1,2,3}(\underset{\sim}{\eta}_{\beta_i})$. □

We describe this by "$(r_i)_{i<4}$ honors majority".

The same proof works if we do not start with the p_β but with any stronger conditions, as long as they still "cohere":

Lemma 5.40. *Let β_i $(i<4)$ be as above, and $s_i \leq p_{\beta_i}$ with $\mathrm{dom}(s_i) \subseteq M_{\beta_i}$ and $s^* := s_i \restriction \beta_i$ is the same for all i, and s^* forces that the $s_i(\beta_i)$ are the same for all i.*

Then there is condition stronger than all s_i forcing that $\underset{\sim}{\eta}_{\beta_0} = \mathrm{major}_{i=1,2,3}(\underset{\sim}{\eta}_{\beta_i})$ and thus $\underset{\sim}{a}_{\beta_0} =^* \mathrm{major}_{i=1,2,3}(\underset{\sim}{a}_{\beta_i})$.

5.8. **$\underset{\sim}{a}_\beta$ is in the $\beta+1$-extension.** We now show that $\underset{\sim}{a}_\beta$ can be assumed to be a P_β-name.

The following definitions, in particular everything concerning the notion of coherence, is used only in this section, for the proof of Lemma 5.48.

Let $(M_\beta, p_\beta)_{\beta\in S}$ be a Δ-system, and $\beta_0 < \beta_1 < \beta_2 < \beta_3$ in S.

Definition 5.41.
- $\bar{q} = (q_i)_{i<4}$ is β-coherent, if each q_i is stronger than p_β and $q_i \restriction (\beta+1)$ is the same for all $i<4$.
- If $\bar{q} \in M_{\beta_0}$ is β_0-coherent, then the union of $h^*_{\beta_0,\beta_i}(q_i)$ (i.e., of the copies of q_i in M_{β_i}) is a valid condition in P, call it q^*, which is stronger than $\bigwedge_{i<4} p_{\beta_i}$.

  $q \in P$ is called coherent, if it is the q^* of some coherent $\bar{q} \in M_{\beta_0}$.

Note that the p_{β_i} are coherent, or rather: $\bigwedge_{i\in 4} p_{\beta_i}$ is coherent; equivalently: $(h^{*-1}_{\beta_0,\beta_i}(p_{\beta_i}))_{i<4}$ is coherent.

Also note that $\bar{q} \in M_{\beta_0}$ coherent and $r_i \leq q_i$ in M_{β_0}, then $h^*_{\beta_0,\beta_i}(r_i)$ is compatible with q^*.

Remark. $q \in P$ is coherent iff: $\mathrm{dom}(q) \subseteq \bigcup M_{\beta_i}$, $q \restriction M_{\beta_i} \in M_{\beta_i}$ is stronger than p_{β_i}, and each $q(\beta_i)$ is forced to be the same condition. (Then we can use $q_i := h^{*-1}_{\beta_0,\beta_i}(q \restriction M_{\beta_i})$.)

Lemma 5.42. *If $r^* \leq \bigwedge_{i<4} p_{\beta_i}$ is coherent, then it can be strengthened to force $\underset{\sim}{a}_{\beta_0} = \mathrm{major}_{i=1,2,3}\, \underset{\sim}{a}_{\beta_i}$.*

Proof. This follows from Lemma 5.40. □

Definition 5.43.
- $\bar{w} = (w_i)_{i<4}$ is β-coherent, if $w_i \cap (\beta+1)$ is independent of i.

  In the following we always assume that $\bar{w}$ is coherent.
- $\bar{q}$ fits $(\bar{w},\zeta)$, if each q_i fits (w_i,ζ).
- $\bar{q}$ is $(\bar{w},\zeta)$-canonical, if each q_i is (w_i,ζ)-canonical.
- $\bar{r} \leq^+_{\bar{w},\zeta} \bar{q}$ means: $\bar{r}$ is coherent, and $r_i \leq^+_{w_i,\zeta} q_i$ for all $i<4$.
- $\bar{x} = (x_i)_{i<4}$ is defined to be in $\mathrm{poss}(\bar{q},\bar{w},\zeta)$ if $x_i \in \mathrm{poss}(q_i,w_i,\zeta)$ and $x_i \restriction \beta$ is independent of i. Such a $\bar{x}$ will be called coherent possibility.

The $x_i(\beta)$ in a coherent possibility can be different.

Note that if $\bar r \leq^+_{\bar w,\zeta} \bar q$ and $\bar q$ is $(\bar w,\zeta)$-canonical, then $\bar r$ and $\bar q$ have the same coherent $(\bar w,\zeta+1)$-possibilities (see Fact 5.29).

Several of the previous constructions result in coherent 4-tuples when applied to coherent 4-tuples. In particular:

Lemma 5.44. *(1) Assume $(\bar q^j)_{j\in\delta}$ is a sequence of coherent 4-tuples such that, for each $i<4$, the i-part $(q^j_i)_{j\in\delta}$ satisfies the assumptions of Lemma 5.18 Then the limits q^δ_i of the lemma can be chosen so that they form a coherent 4-tuple.*

(2) The same applies to Lemma 5.19. I.e., we can get a coherent fusion limit from a λ-sequence of coherent tuples.

(3) Assume $\bar p$ fits $(\bar w,\zeta)$, and $\alpha_i\in\mu$ such that $w'_i := w_i\cup\{\alpha_i\}$ is coherent. Then there is a $\xi>\xi_0$ and a $\bar q \leq^+_{\bar w,\zeta} \bar p$ which fits $(\bar w',\xi)$ and is $(\bar w',\xi)$-canonical.

(4) Assume $\bar q$ is β-coherent and (for simplicity) $(\bar w,\zeta)$-canonical with $\beta\in w^*$, and $\underset{\sim}{\tau}_i$ are names of ordinals. Then there is an $\bar r \leq^+_{\bar w,\zeta} \bar q$ such that $\underset{\sim}{\bar\tau}$ is $(\bar w,\zeta+1)$-decided by $\bar r$.

By this we mean that $\underset{\sim}{\tau}_i$ is $(w_i,\zeta+1)$-decided by r_i for all $i<4$.

Proof. For the first items, we just have to look at the proofs of the according lemmas (For (3) this is Lemmas 5.23 and 5.24) and note that coherent input gives us coherent output. In the following we will prove (4).

Enumerate all coherent possibilities as $(\bar x_k)_{k\in K}$. Set $\bar r^0 := \bar q$. We now construct $\bar r^{k+1}$ from $\bar r := \bar r^k$ where we assume $\bar r^k \leq^+_{\bar w,\zeta} \bar q$.

- Find s_0 stronger than r_0 and extending x_0, deciding $\underset{\sim}{\tau}_0$.
- $s^* := (s_0\restriction\beta)\wedge r_1$ is stronger than r_1, as $\bar r$ is coherent. Strengthen $s^*(\beta)=r_1(\beta)=r_0(\beta)$ to $s_0(\beta)$, but replace the trunk with $x_1(\beta)$. Then $s^*\restriction\beta$ forces that $s^*(\beta)\leq r_1(\beta)$, as $x_1\restriction\beta = x_0\restriction\beta$ and as $x_1(\beta)$ is guaranteed to be possible, because r_1 is canonical. Further strengthen s^* (above β) to extend (the rest of) x_1; and then strengthen the whole condition once more to decide $\underset{\sim}{\tau}_1$. Call the result s_1.
- Do the same for $i=2$, starting with s_1, resulting in s_2, and then for $i=3$, starting with s_2, resulting in some s_3.

  So $s_i\leq r_i$ extends x_i and decides $\underset{\sim}{\tau}_i$, and $s_3\restriction\beta\leq s_i\restriction\beta$ and $s_3(\beta)$ is stronger than $s_i(\beta)$ "above $\zeta+1$".
- We define $r'_i\leq r_i$ as follows: $\mathrm{dom}(r'_i) = (\mathrm{dom}(s_3)\cap\beta)\cup\mathrm{dom}(s_i)$. We define $r'_i(\alpha)$ inductively such that $r'_i\restriction\alpha \leq^+_{w_i\cap\alpha,\zeta} r_i$ forces that $x_i\restriction\alpha\preceq G$ implies $s_i\restriction\alpha\in G$.
  - For $\alpha\leq\beta$:
    If $s_3\restriction\alpha\notin G_\alpha$, set $r'_i(\alpha)=r_i(\alpha)$. Assume otherwise. So $s_3(\alpha)$ is defined and stronger than $r_i(\alpha)=r_3(\alpha)$. If $\alpha\notin w_i$ (which implies $\alpha<\beta$), set $r'_i(\alpha)=s_3(\alpha)$. Otherwise, use $s_3(\alpha)\vee(r_3(\alpha)\restriction\zeta+1)$, as in Lemma 5.8.
  - For $\alpha>\beta$, we do the same but we use s_i instead of s_3. In more detail:
    If $s_i\restriction\alpha\notin G_\alpha$, set $r'_i(\alpha)=r_i(\alpha)$. Assume otherwise. If $\alpha\notin w_i$, set $r'_i(\alpha)=s_i(\alpha)$. Otherwise, use $s_i(\alpha)\vee(r_i(\alpha)\restriction\zeta+1)$.

  We can use this $\bar r'$ as $\bar r^{k+1}$: It is coherent, $\bar r' \leq^+_{\bar w,\zeta} \bar r^k$, and r'_i decides $\underset{\sim}{\tau}_i$ assuming $x_i\lhd G$. □

Assuming that everything lives in M_{β_0}, then the notion we just defined for coherent tuples $\bar q$ correspond to the natural nations of the P-conditions q^*. However we have to assume that $\bar q$ is canonical to guarantee that coherent possibilities correspond to q^*-possibilities:

Lemma 5.45. *Assume $\bar q$ and $\bar w$ are both in M_{β_0} and are both β_0-coherent. We set $w^* := \bigcup_{i<4} h^*_{\beta_0,\beta_i}(w_i)$. Let $\bar x$ in $\mathrm{poss}(\bar q,\bar w,\zeta+1)$ (and therefore in M_{β_0}).*

(1) $\bar q$ fits $(\bar w,\zeta)$ iff q^* fits (w^*,ζ).

(2) $\bar r \leq^+_{\bar w,\zeta} \bar q$ iff $r^* \leq^+_{w^*,\zeta} q^*$.

(3) Assume $\bar q$ fits $(\bar w,\zeta)$. Then $\bar q$ is $(\bar w,\zeta)$-canonical iff q^* is (w^*,ζ)-canonical.

(4) Assume that $\bar q$ is $(\bar w,\zeta)$-canonical. Let x^* be the union of the $h^*_{\beta_0,\beta_i}(x_i)$. Then $x^*\in \mathrm{poss}(q^*,w^*,\zeta+1)$; and every element of $\mathrm{poss}(q^*,w^*,\zeta+1)$ has this form for some $\bar x\in \mathrm{poss}(\bar q,\bar w,\zeta+1)$.

(5) *Assume that $\bar q$ is $(\bar w,\zeta)$-canonical. Then $\bar q$ $(\bar w,\zeta+1)$-decides some $\underset{\sim}{\tau}_i$ for all $i<4$ iff q^* $(w^*,\zeta+1)$-decides all $h^*_{\beta_0,\beta_i}(\underset{\sim}{\tau}_i)$.*

Proof. Assume $\alpha\in w_i$. Set $\alpha' := h^*_{\beta_0,\beta_i}(\alpha)\in w^*$ and $q' := h^*_{\beta_0,\beta_i}(q_i)$.

(1) Assume Assume q_i,α satisfy $q_i\restriction\alpha\Vdash\zeta\in C^{q_i(\alpha)}$. By absoluteness they satisfy it in M_{β_0}, so the $h^*_{\beta_0,\beta_1}$-images q',α' satisfy it in M_{β_i}, which again is absolute; and $q^*\restriction\alpha'\le q'\restriction\alpha'$ forces that $q^*(\alpha')=q'(\alpha')$. For the other direction, assume (in M_{β_0}) some $s\le q_i\restriction\alpha$ forces $\zeta\notin C^{q_i(\alpha)}$. Then $h^*_{\beta_0,\beta_i}(s)$ is compatible with q^* and forces $\zeta\notin C^{q'_i(\alpha')}=C^{q^*(\alpha')}$.

In the same way we can show (2), as well as the trivial directions of (3,4), and (5): E.g., If $\bar q$ is $(\bar w,\zeta)$-canonical, then q^* is (w^*,ζ)-canonical. For this, use the fact that every element $y^*\in\operatorname{poss}(q^*,w^*,\zeta+1)$ "induces" a coherent possibility $\bar y$ (which is true whether $\bar q$ is canonical or not). And if additionally $\bar x\in\operatorname{poss}(\bar q,\bar w,\zeta+1)$, then $x^*\in\operatorname{poss}(q^*,w^*,\zeta+1)$; and if each q_i forces that $x_i\triangleleft G$ implies $\underset{\sim}{\tau}_i=x^i$, then q^* forces that $x^*\triangleleft G$ implies $h^*_{\beta_0,\beta_1}(\underset{\sim}{\tau}_i)=h^*_{\beta_0,\beta_1}(x^i)$.

We omit the (also straightforward) proofs of the other directions of (3,4) (which we do not need in this paper). □

In the following, whenever we mention q^* or w^*, we assume $\bar w$, $\bar q$ to be coherent and in M_{β_0}. We will (and can) use x^* only if $\bar q$ additionally is canonical (otherwise x^* will generally not be a possibility for q^*). In this case, every P-generic filter containing q^* will select an x^* for some coherent possibility $\bar x$.

Lemma 5.46. *Assume $\bar q$ is coherent, $\underset{\sim}{\sigma}_i$ are P-names for an elements of 2^λ, and $q_0\Vdash\underset{\sim}{\sigma}_0\notin V_{\beta+1}$. Then there is a coherent $\bar r\le\bar q$, and sequences $(\zeta^j)_{j\in\lambda}$ and $(\bar w^j)_{j\in\lambda}$ such that $\bar r$ is $(\bar w^j,\zeta^j)$-canonical for all j, and for all $\bar x\in\operatorname{poss}(\bar r,\bar w^j,\zeta^j+1)$ there is some $\ell\in I^*(>\zeta^j,<\zeta^{j+1})$ and $\bar b\in 2$ violating majority[7] such that for all $i<4$*

$$r_i\Vdash x_i\triangleleft G\rightarrow\underset{\sim}{\sigma}_i(\ell)=b_i.$$

As the p_{β_i} are coherent, we can apply the lemma to $\beta=\beta_0$ and $\underset{\sim}{\sigma}_i=\underset{\sim}{a}_{\beta_0}$ (independent of i) and get:

Corollary 5.47. *If $p_{\beta_0}\Vdash\underset{\sim}{a}_{\beta_0}\notin V_{\beta_0+1}$, then there is a coherent $r^*\le\bigwedge_{i<4}p_i$ forcing that*

$$\neg\left(\underset{\sim}{a}_{\beta_0}=^*\operatorname{major}_{i=1,2,3}(\underset{\sim}{a}_{\beta_i})\right).$$

Proof of the lemma. We will construct, by induction on $j\in\lambda$, ζ^j, $\bar w^j$ and $\bar r^j$ with $r^0_i=q_i$, such that the following holds:

1. $\bar r^j$ is coherent.
2. $\bar w^j$ is coherent, for each $i<4$ the w^j_i are increasing with j, and their union covers $\bigcup_{j\in\lambda}\operatorname{dom}(r^j_i)$.
3. $\bar r^j$ is $(\bar w^j,\zeta^j)$-canonical.
4. $\bar r^k\le^+_{\bar w^j,\zeta^j}\bar r^j$ for $j<k$.
5. If $\bar x\in\operatorname{poss}(\bar r^j,\bar w^j,\zeta^j+1)$, then there is an $\ell\in I^*(>\zeta^j,<\zeta^{j+1})$ and a $b\in 2$ such that for at least two i_1,i_2 in $\{1,2,3\}$, r^{j+1}_i forces that $x_i\triangleleft G$ implies

$$(*)\qquad \underset{\sim}{\sigma}_0(\ell)=1-b,\quad \underset{\sim}{\sigma}_{i_1}(\ell)=b,\quad \underset{\sim}{\sigma}_{i_2}(\ell)=b.$$

Then we take the usual fusion limits, as in Lemma 5.44(2), and are done.

For limits j, let $\bar r'$ be a (coherent) limit of $(\bar r^{j'})_{j'<j}$, and set $\zeta^*:=\sup_{j'<j}(\zeta^{j'})$ and $w^*_i:=\bigcup_{j'<j}w^{j'}_i$ for each $i<4$. Note that $\bar r'$ fits $(\bar w^*,\zeta^*)$. Then we can find coherent $\bar r^*\le^+_{\bar w^*,\zeta^*}\bar r'$ which is $(\bar w^*,\zeta^*)$-canonical, as in Lemma 5.44(3).

In successor cases $j=j'+1$ set $(\bar r^*,\bar w^*,\zeta^*):=(\bar r^{j'},\bar w^{j'},\zeta^{j'})$.

In any case we want to construct $\bar r^j$, $\bar w^j$, and ζ^j.

Enumerate $\operatorname{poss}(\bar r^*,\bar w^*,\zeta^*+1)$ as $(\bar x^k)_{k\in K}$.

We define $\bar s^k$ for $k\le K$, with $\bar s^0:=\bar r^*$ and, as usual, taking (coherent) limits at limits, such that:

[7] I.e., $b_0=1-\operatorname{major}_{i=1,2,3}(b_i)$.

- $\bar{s}^k$ is coherent.
- $\bar{s}^\ell \leq^+_{\bar{w}^*,\zeta^*} \bar{s}^k$ for $k<\ell<K$. (This implies that $\bar{s}^k$ is $(\bar{w}^*,\zeta^*)$-canonical.)
- There is a ξ^k and an $\ell \in I^*(>\zeta^*,<\xi^k)$ and a $b\in 2$ such that

$$(**)\qquad s_0^{k+1} \Vdash x_0^k \lhd G \to \underset{\sim}{\tau}_0(\ell)=1-b \quad\text{and}\quad (\exists^{\geq 2} i\in\{1,2,3\})\ s_i^{k+1}\Vdash x_i^k \lhd G \to \underset{\sim}{\tau}_i(\ell)=b.$$

Assume we can construct these $\bar{s}^k$, ξ^k for all $k\in K$, then let $\bar{s}^K$ be again a (coherent) limit. We set $w_i^j := w_i^* \cup \{\alpha_j\}$ such that $\bar{w}^j$ is coherent (and such that, by bookkeeping, all elements of of $\mathrm{dom}(p_i^j)$ will be eventually covered), and find some $\zeta^j > \sup_{k\in K}(\xi^k)$ and $\bar{r}^j \leq^+_{\bar{w}^*,\zeta^*} r^*$ which is $(\bar{w}^j,\zeta^j)$-canonical, again as in Lemma 5.44(3). Then $\bar{r}^j$, $\bar{w}^j$ and ζ^j are as required.

So it remains to construct, for $k\in K$, $\bar{s}^{k+1}$ and ξ^k, which we will do in the rest of the proof. Set $\bar{s}:=\bar{s}^k$, $\bar{x}:=\bar{x}^k$, $\bar{w}:=\bar{w}^*$ and $\zeta:=\zeta^*$. Recall that $\bar{s}$ is $(\bar{w},\zeta)$-canonical, $\bar{x}\in \mathrm{poss}(\bar{s},\bar{w},\zeta)$, and we are looking for $\bar{s}^{k+1}\leq^+_{\bar{w},\zeta}\bar{s}$ which satisfies $(**)$ for $\bar{x}$.

Set $s_i' := s_i \wedge x_i$. It is enough to construct $t_i \leq s_i'$ such that:

- Both $t_i \restriction \beta$ and $t_i(\beta)\restriction(\lambda\setminus\zeta+1)$ are independend of i.
- $t_0 \Vdash \underset{\sim}{\tau}_0(\ell)=1-b$ and $(\exists^{\geq 2} i\in\{1,2,3\})\ t_i\Vdash \underset{\sim}{\tau}_i(\ell)=b$.

Then we can define $\bar{s}^{k+1}$ in the usual way: $\mathrm{dom}(s_i^{k+1})=\mathrm{dom}(t_i)$ (and we can assume $\mathrm{dom}(s_i)=\mathrm{dom}(t_i)$, by using trivial conditions). For $\alpha\in\mathrm{dom}(t_i)$, if $t_i\restriction\alpha\notin G_\alpha$ then set $s_i^{k+1}(\alpha)$ to be $s_i(\alpha)$, otherwise $t_i(\alpha)\vee(s_i(\alpha)\restriction\zeta+1)$ if $\alpha\in w_i$ and $t_i(\alpha)$ otherwise. The resulting $\bar{s}^{k+1}\leq^+_{\bar{w},\zeta}\bar{s}$ is coherent and s_i^{k+1} forces that $x_i\lhd G$ implies $t_i\in G$.

We have to introduce more notation: Fix $j\neq i$, and $a\leq s_j'$ and $b\leq s_i'\restriction\beta+1$ (in $P_{\beta+1}$) such that $b\restriction\beta\leq a$ and $b\restriction\beta$ forces that $b(\beta)$ is stronger than $a(\beta)$ above ζ (i.e., $b\restriction\beta\Vdash(\forall\xi>\zeta)\,b(\beta)(\xi)\supseteq a(\beta)(\xi)$). Then we define $b^{[j]}\wedge a$ by

$$(b^{[j]}\wedge a)(\alpha)(\xi)=\begin{cases} b(\alpha)(\xi) & \text{if } \alpha<\beta,\\ x_j(\beta)(\xi) & \text{if } \alpha=\beta \text{ and } \xi\leq\zeta,\\ b(\beta)(\xi) & \text{if } \alpha=\beta \text{ and } \xi>\zeta,\\ a(\alpha)(\xi) & \text{otherwise.}\end{cases}$$

Note that $b^{[j]}\wedge a$ is stronger than a, but generally not stronger than b.

By our assumption, q_0 and therefore s_0' forces $\underset{\sim}{\sigma}_0\notin V_{\beta+1}$. So in an intermediate model $V[G_{\beta+1}]$, there is some $\ell\in I^*(>\zeta)$ such that $s_0'/G_{\beta+1}$ does not decide $\underset{\sim}{\sigma}_0(\ell)$. Back in V, fix some $b_0\leq s_0'\restriction(\beta+1)$ in $P_{\beta+1}$ which determines this ℓ.

Find $r_1'\leq b_0^{[1]}\wedge s_1'$ which determines $\underset{\sim}{\sigma}_1(\ell)$ to be j_1 for some $j_1\in 2$. Find $r_2'\leq(r_1'\restriction\beta+1)^{[2]}\wedge s_2'$ which determines $\underset{\sim}{\sigma}_2(\ell)$ to be j_2; analogously find $r_3'\leq(r_2'\restriction\beta+1)^{[3]}\wedge s_3'$ which determines $\underset{\sim}{a}(\ell)$ to be some j_3. Let $j\in 2$ be equal to at least two of j_1,j_2,j_3.

Set $p:=(r_3'\restriction\beta+1)^{[0]}\wedge s_0'$. In any $P_{\beta+1}$-extension honoring $p\restriction\beta+1$, $\underset{\sim}{\sigma}_0(\ell)$ is not determined by $p/G_{\beta+1}$, i.e., there is an $t_0\leq p$ forcing that $\underset{\sim}{a}(\ell)=1-j$.

We now set and $t_i:=(t_0\restriction\beta+1)^{[i]}\wedge r_i'$ for $i=1,2,3$. Note that $t_i\leq r_i'\leq s_i'$ extends x_i and forces $\underset{\sim}{\sigma}_i(\ell)$ to be $1-j$ if $i=0$ and to be j for at least two i in $\{1,2,3\}$. □

We can now easily show:

Lemma 5.48. *For all but non-stationary many $\beta\in S^\mu_{\lambda^+}$*

$$p_* \Vdash \underset{\sim}{a}_\beta \in V_{\beta+1}$$

Proof. Assume towards a a contradiction that on a non-stationary set S' there are $p_\beta\leq p_*$ forcing $\underset{\sim}{a}_\beta\notin V_{\beta+1}$; by strengthening we can assume that p_β canonically reads $\underset{\sim}{a}_\beta$. Let M_β contain p_β and let $S\subseteq S'$ be such that $(M_\beta,p_\beta)_{\beta\in S}$ is a Δ-system. Fix $\beta_0<\beta_1<\beta_2<\beta_3$ in S. By Corollary 5.47 we get a coherent $\bar{r}$ stronger than $\bar{p}$ such that $r^*\Vdash\neg\big(\underset{\sim}{a}_{\beta_0} =^* \mathrm{major}_{i=1,2,3}(\underset{\sim}{a}_{\beta_i})\big)$. This contradicts Lemma 5.42. □

5.9. **Local reading.** So apart from a non-stationary set, $\underset{\sim}{a}_\beta$ is forced (by p_*) to be in $V_{\beta+1}$ but not in V_β (see Lemma 5.36). So there is a $p'_\beta \le p_*$ forcing that $\underset{\sim}{a}_\beta$ is equal to some $P_{\beta+1}$-name (which we just call $\underset{\sim}{a}_\beta$ again); and we choose $p_\beta \le p'_\beta$ which canonically reads this name "below $\beta+1$".[8]

Then we pick M_β containing p_β and choose S such that $(M_\beta, p_\beta)_{\beta\in S}$ is a Δ-system.

So $\underset{\sim}{a}_\beta$ is read continuously by p_β from $(w'_\zeta)_{\zeta\in E'}$ for some $E' \subseteq \lambda$ club, with $w'_\zeta \subseteq \beta+1$. Moreover, the w'_ζ and the decision functions that map $\text{poss}(p_\zeta, w'_\zeta, \zeta+1)$ to $\underset{\sim}{a}_\beta \restriction I^*(<\zeta+1)$ are independent of β (modulo the shifting of the coordinate β via the mapping $h^*_{\beta_0,\beta_1}$.)

Let E be the limit points of E', and set $w_\zeta := \bigcup_{\nu<\zeta} w_\nu$. Then $\underset{\sim}{a}_\beta \restriction I^*(<\xi)$ is (w_ξ, ξ)-determined by p_β for all $\xi \in E$.

In the P_β-extension V_β (i.e., in $V[G_\beta]$, where we assume that $p_\beta \restriction \beta$ is in G_β), only $\underset{\sim}{\eta}_\beta$ remains undetermined, i.e., there are F_ξ for $\xi \in E$ such that p_β/G_β forces

$$\underset{\sim}{a}_\beta \restriction I^*(<\xi) = F_\zeta(\underset{\sim}{\eta}_\beta \restriction I^*(<\xi)).$$

(And F_ξ is independent of β.)

Recall that $\tilde{p} := p_\beta(\beta)$ (independent of β) and that $x \in \text{poss}(\tilde{p}, \xi)$ iff $x \in 2^{I^*(<\xi)}$ and x extends $\eta^{\tilde{p}} \restriction I^*(<\xi)$.

The following definition (the notion of candidate) is only used in this section, for the proof of Lemma 5.51.

Definition 5.49. (In V_β)

- For $A \subseteq \lambda$ and $\bar{x} = (x_i)_{i<4}$, $x_i : A \to 2$, we say $\bar{x}$ honors majority above ζ, if
$$x_0(\ell) = \text{major}_{i=1,2,3}\, x_i(\ell) \text{ for all } \ell \in A \cap I^*(\ge\zeta).$$
We say $\bar{x}$ honors $\tilde{p}$, if each x_i is compatible (as partial function) with $\eta^{\tilde{p}}$.
- A (ζ_0, ζ_1)-candidate is an $(x_i)_{i<4}$ with $x_i \in \text{poss}(\tilde{p}, \zeta_1)$ honoring majority above ζ_0.
- If $\bar{x}$ is a (ζ_0, ζ_1)-candidate, we say "$\bar{y}$ extends $\bar{x}$" if $\bar{y}$ is a (ζ_1, ζ_2)-candidate[9] for some $\zeta_2 \ge \zeta_1$ and each y_i extends x_i.
  Equivalently, $\bar{y} = \bar{x}^\frown \bar{b}$ for some $\bar{b}$ with domain $I^*(\ge\zeta_1, <\zeta_2) \to 2$ which honors majority and $\tilde{p}$.
- A (ζ_0, ζ_1)-candidate $\bar{y}$ is "good", if for every candidate $\bar{z}$ of height $\xi > \zeta_1$ that extends $\bar{y}$ we have:

$$(*_1) \qquad F_\xi(z_0)(\ell) = \text{major}_{i=1,2,3}\, F_\xi(z_i)(\ell) \text{ for all } \ell \in I^*(\ge\zeta_1, <\xi).$$

Preliminary Lemma 5.50. *(In V_β.) Every candidate can be extended to a good candidate.*

Proof. Assume otherwise, i.e.: There is ζ_0 and a (ζ_0, ζ_0)-candidate $\bar{x}$ such that:

$(*_2)$ Whenever $\bar{y}$ is a (ζ_0, ζ_1)-candidate extending $\bar{x}$ then there is a (ζ_1, ξ)-candidate $\bar{z}$ extending $\bar{y}$ which violates $(*_1)$.

We now construct $r_0 \le \tilde{p}$ and, for $i = 1, 2, 3$, Q-names $r_i \le \tilde{p}$. All these conditions live on the same $C^* \subseteq E$ with $\min(C^*) = \zeta_0$. The trunk of r_i is x_i.

We now construct inductively $C^* \restriction \zeta$ and $r_i \restriction \zeta$.

Assume we have determined that $\zeta \in C^*$ and we have constructed each r_i below ζ. Set $r_0(\zeta) := \tilde{p}(\zeta)$ and pick $r_i(\zeta)$ as in (5.39), i.e., they have majority $\underset{\sim}{\eta}_\beta$ and leave enough freedom to form a valid condition.

We will now construct the C^*-successor ξ of ζ, together with r_i on $I^*(>\zeta, <\xi)$.

Enumerate all $(\zeta_0, \zeta+1)$-candidates extending $\bar{x}$ as $(\bar{y}^k)_{k\in K}$.

Let $\bar{a}^0$ be the empty 4-tuple and set $\xi_0 := \zeta+1$. We will construct, for $k \in K$, ξ_k and some $\bar{a}^k$ that honors majority and $\tilde{p}$, where a^k_i has domain $I^*(\ge\zeta+1, <\xi_k)$ and extends a^j_i if $j < k$.

If k is a limit, let $\bar{a}^x$ be the (pointwise) union of $\bar{a}^j$ with $j < k$, and set $\xi_k := \sup_{j<k}(\xi_j)$.

[8]But generally $p_\beta \restriction \beta+1$ is not enough to force that the object read this way is $\underset{\sim}{a}_\beta$, for this the whole p is required.

[9]or equivalently, a (ζ_0, ζ_2)-candidate

Assume we already have $\bar a^j$. Extend $\bar y^j{}^\frown\bar a^j$ to some candidate $\bar{y^j}^\frown\bar a^{j+1}$ of some height ξ_{j+1} in E such that

$$(*_3)\qquad \bar y^j{}^\frown\bar a^{j+1} \text{ violates } (*_1) \text{ for some } \ell\in I^*(\geq\xi_j,<\xi_{j+1}).$$

We can due that due to $(*_2)$.

So in the end we get some $\xi>\zeta$ in E and $\bar b^\zeta$ with domain $I^*(>\zeta,<\xi)$ honoring majority and $\tilde p$ such that

$(*_4)$ for every $(\zeta_0,\zeta+1)$-candidate $\bar y$ extending $\bar x$, $\bar y^\frown\bar b^\zeta$ is a (ζ_0,ξ)-candidate violating $(*_1)$ for some $\ell\in I^*(>\zeta,<\xi)$.

We then define C^* below $\xi+1$ by adding only ξ, i.e., ξ is the C^*-successor of ζ. We extend the conditions r_i by b^ζ_i for $i<4$. I.e., we have $\eta^{r_i}(\ell)=b^\zeta_i(\ell)$. This ends the construction of $r_i\leq\tilde p$.

Assume that $(*_2)$ is forced by some $q'\leq p_\beta\restriction\beta$. Pick an increasing sequence β_i $(i<4)$ in S. We take the union of q' and the p_{β_i}, call it s, and strengthen $s(\beta_i)=\tilde p$ to r_i. The resulting condition s' forces the following:

- $\underset{\sim}{a}_{\beta_i}\restriction I^*(<\xi)=F_\xi(\underset{\sim}{\eta}_{\beta_i}\restriction I^*(<\xi))$. This is because $s'\leq p_{\beta_i}$.
- The $\underset{\sim}{\eta}_{\beta_i}$ honor majority above ζ_0. This is because for all $\zeta\in C^*$, $r_i(\zeta)$ are chosen as in (5.39) and therefore honor majority; and for $\zeta\in\lambda\setminus(C^*\cup\zeta_0)$ we use values $\bar b$ which honor majority.
- Accordingly, the $\underset{\sim}{a}_{\beta_i}$ honor majority above some γ. See Lemma 5.38(1). Pick ζ_1 such that $\sup(I^*(<\zeta_1))>\gamma$.
- So for all $\xi>\zeta_1$ the $F_\xi(\underset{\sim}{\eta}_{\beta_i}\restriction I^*(<\xi))$ honor majority above ζ_1.
- Pick some $\zeta>\zeta_0,\zeta_1$ in C^* with C^*-successor ξ. By construction of the r_i, $\underset{\sim}{\eta}_{\beta_i}\restriction I^*(\geq\zeta+1,<\xi)$ is b^ζ_i. As r_i extends x_i, $\bar y:=\underset{\sim}{\eta}_{\beta_i}\restriction I^*(<\zeta+1)$ is a $(\zeta_0,\zeta+1)$-candidate extending $\bar x$. So by $(*_4)$, the $\underset{\sim}{\eta}_{\beta_i}\restriction I^*(<\xi)$ violate $(*_1)$ at some $\ell\in I^*(>\zeta,<\xi)$, a contradiction. $\square$

Let $U\subseteq\lambda$ be club. Set U^{ODD} to be the odd elements of U.[10] For $\xi\in U^{\text{ODD}}$ with U-successor ν, set

$$A^U_\xi:=I^*(\geq\xi,<\nu)$$

Lemma 5.51. *(In V_β.) There is an $r_0\leq\tilde p$, a club $U\subseteq C^{r_0}$ and, for $\xi\in U^{\text{ODD}}$, an $H_\xi:2^{A^U_\xi}\to2^{A^U_\xi}$ such that*

- *r_0 forces that $H_\xi(\underset{\sim}{\eta}_\beta\restriction A^U_\xi)=\underset{\sim}{a}_\beta\restriction A^U_\xi$.*
- *H_ξ is not constant: There are, for $k=1,2$, z^k_ξ in $\text{poss}(r_0,I^*(<\nu))$ and $\ell_\xi\in A^U_\xi$ such that $H_\xi(z^k_\xi\restriction A^U_\xi)(\ell_\xi)=k$. (Again, ν is the U-successor of ξ.)*

(Only those elements of $2^{A^U_\xi}$ that are compatible with r_0 are relevant as arguments for H_ξ.)

Proof. We construct r_i for $i<4$ and U iteratively: C^{r_i} will be independent of i, call it C. All r_i have the same trunk as $\tilde p$; i.e., $\min(C)=\min(C^{\tilde p})=:\zeta_0$ and $r_i\restriction\zeta_0:=\tilde p\restriction\zeta_0$. We also set $\min(U)=\zeta_0$.

Assume we already know that $\zeta\in U\subseteq C$, and we know $r_i\restriction\zeta$.

For all $\zeta\in C$, we will choose $r_i(\zeta)$ as in (5.39), i.e., $r_0(\zeta)=\tilde p(\zeta)$ and the $r_i(\zeta)$ are such that the majority of their generics will be the $r_0(\zeta)$-generic.

- Even case: If ζ is an even element of U, we add a "shield", or "isolator" above ζ: As in the previous proof, we iterate over all $\zeta+1$-candidates $\bar y^j$, but but in $(*_3)$, instead of violating $(*_1)$ for some ℓ, we demand that $\bar y^j{}^\frown\bar z^{j+1}$ is good. (We already know that every candidate can be extended to a good one.) Accordingly, we get some $\xi>\zeta$ and $\bar b^\zeta$

[10] I.e., if $(u_\alpha)_{\alpha<\lambda}$ is the canonical enumeration of U, then $\zeta\in U$ is in U^{ODD} if $\zeta=u_{\delta+2n+1}$ for δ a limit (or 0) and $n\in\omega$.

with domain $I^*(>\zeta, <\xi)$ (and honoring majority and $\tilde{p}$) such that $\bar{y}^\frown \bar{b}^\zeta$ is good for every candidate $\bar{y}$ of height $\zeta+1$; i.e.:

$(*_4')$ If $\bar{z}$ is a $(\zeta+1,\nu)$-candidate whose restriction to $I^*(>\zeta,<\xi)$ is $\bar{b}^\zeta$, then the $F_\nu(z_i)$ honor majority above ξ.

We now let ξ be the successor of ζ in both C and U. (So ξ will be an odd element of U.)

- Odd case: Let ξ be odd in U. We now choose some $\nu > \xi$ in C^{p_0} large enough such that there are, for $k=1,2$, z^k_ξ in $\mathrm{poss}(\tilde{p},\nu)$ compatible with the r_0 constructed so far, such that the $F_\nu(z^k_\nu)(\ell) = k$ for some $\ell > I^*(<\xi)$. (Such ℓ has to exist as $\underset{\sim}{a}_\beta$ is not in V_β.)

  We let C restricted to $[\xi,\nu]$ be the same as $C^{\tilde{p}}$. We already know how to construct $r_i(\zeta)$ for $\zeta \in C$, and there is no freedom left outside of C (the $r_i(\zeta)$ have to be $\tilde{p}(\zeta)$). So this defines all r_i on $[\xi,\nu]$. We let ν be the U-successor of ξ.

This ends the construction of U and r_i.

Pick $\xi \in U^{\mathrm{ODD}}$, let ζ be the U-predecessor and ν the U-successor. Fix any $z^\zeta_* \in \mathrm{poss}(r_0,\zeta+1)$. Let $x_0 \in \mathrm{poss}(r_0,\nu)$. In particular x_0 extends b^ζ_0. For $i=1,2,3$, let x_i be the copy of x_0 with the initial segment $x_0 \restriction \xi$ replaced by $z^\zeta_*{}^\frown b^\zeta_i$. Note that $\bar{x}$ is a candidate extending $\bar{b}^\zeta$. Accordingly the $F_\nu(x_i)$ honor majority above ξ. So we can define

$$H(x_0 \restriction A^U_\xi) := \mathrm{major}_{i=1,2,3} F_\nu(x_i) \restriction A^U_\xi = F_\nu(x_0) \restriction A^U_\xi.$$

This is well-defined,[11] and r_0 forces that $H(x_0 \restriction A^U_\xi) = \underset{\sim}{a}_\beta \restriction A^U_\xi$. □

5.10. **Finding the generator.** In this section we again work in V_β (in an extension $V[G_\beta]$ with $p_\beta \restriction \beta \in G$), and use the r_0, $U \subseteq C^{r_0}$, $(H_\xi)_{\xi\in U^{\mathrm{ODD}}}$ of the previous lemma. Let $\xi \in U^{\mathrm{ODD}}$ and ν its U-successor. Set

$$A_\xi := A^U_\xi = I^*(\geq\xi,<\nu), \qquad A^?_\xi := A_\zeta \setminus \mathrm{dom}(\eta^{r_0}),$$
$$\mathrm{ODD} := \bigcup_{\xi\in U^{\mathrm{ODD}}} A_\xi, \qquad \mathrm{ODD}^? := \bigcup_{\xi\in U^{\mathrm{ODD}}} A^?_\xi = \mathrm{ODD} \setminus \mathrm{dom}(\eta^{r_0}).$$

For H_ξ it is enough to use $\underset{\sim}{\eta}_\beta \restriction A^?_\xi$ as input (the part in $A_\xi \setminus A^?_\xi$ is determined anyway by r_0), and *every* element of $2^{A^?_\xi}$ is compatible with r_0 (and thus a possible input for H_ξ). Identifying 2^B and $\mathcal{P}(B)$ as usual, we get:

$$H_\xi : \mathcal{P}(A^?_\xi) \to \mathcal{P}(A_\xi)$$

is such that $r_0 \wedge p_\beta/G_\beta$ forces

$$H_\xi(\underset{\sim}{\eta}_\beta \cap A^?_\xi) = \underset{\sim}{a}_\beta \cap A_\xi,$$

We now define

$$H : \mathcal{P}(\mathrm{ODD}^?) \to \mathcal{P}(\mathrm{ODD}) \quad \text{by} \quad x \mapsto \bigcup_{\xi\in U^{\mathrm{ODD}}} H_\xi(x \cap A^?_\xi).$$

So in particular $r_0 \wedge p_\beta/G_\beta$ forces that

$$H(\underset{\sim}{\eta}_\beta \cap \mathrm{ODD}^?) = \underset{\sim}{a}_\beta \cap \mathrm{ODD}.$$

Note that for very $z \subseteq \mathrm{ODD}^?$ (in V_β that is) there is an $r' \leq r_0$ forcing that $\underset{\sim}{\eta}_\beta \cap \mathrm{ODD}^? = z$. ($C' := U \setminus U^{\mathrm{ODD}}$ is club, so it is enough to leave freedom at C' and assign arbitrary values everywhere else.)

We can argue that $\underset{\sim}{\pi}(\mathrm{ODD}^?)$ is a P_β-name; or we can just assume β is large enough so that $\underset{\sim}{\pi}(\mathrm{ODD}^?)$ is a P_β-name (as it is independent of $\beta \in S$). So the following can be formulated in V_β:

Lemma 5.52. *(In V_β.)* $|\underset{\sim}{\pi}(\mathrm{ODD}^?) \cap \mathrm{ODD}| = \lambda$.

[11] Assume y and x in $\mathrm{poss}(r_0,\nu)$ are the identical restricted to A^U_ξ. Then y defines the same $(x_i)_{i=1,2,3}$ and thus the same H.

Proof. Let $q \le p_\beta \restriction \beta$ be arbitrary. We have to show that q does not force $|\pi(\text{ODD}^?) \cap \text{ODD}| < \lambda$.

For $\xi \in U^{\text{ODD}}$, use r_0 and z^k_ξ and ℓ_ξ as in Lemma 5.51 and set $b^k_\xi := z^k_\xi \cap A^?_\xi$.

Set $B^k := \bigcup_{\xi \in U^{\text{ODD}}}(b^k_\xi)$. Note that $H(B^1) \setminus H(B^0)$ contains $\{\ell_\xi : \xi \in U^{\text{ODD}}\}$, a set of size λ.

Pick increasing $(\beta_i)_{i<4}$ in S with $\beta_0 = \beta$. Strengthen $q \wedge \bigwedge_{i<4} p_{\beta_i}$ to use $r_0 \le \tilde{p}$ at each β_i, resulting in s.

Now strengthen $s(\beta_i)$ at the even intervals to honor majority; and at the odd intervals to $B^{\text{sgn}(i)}$. Accordingly, we have

$$\pi(\eta_{\beta_i}) \cap \text{ODD} = H(\eta_{\beta_i} \cap \text{ODD}^?) = H(B^{\text{sgn}(i)}),$$

or, when we split $\pi(\eta_{\beta_i})$ into the parts in and out of $\pi(\text{ODD}^?)$:

$$\Big(\big(\pi(\eta_{\beta_i}) \setminus \pi(\text{ODD}^?)\big) \cap \text{ODD} \Big) \cup \Big(\pi(\eta_{\beta_i}) \cap \pi(\text{ODD}^?) \cap \text{ODD} \Big) =^* H(B^{\text{sgn}(i)})$$

Now assume towards a contradiction that $\pi(\text{ODD}^?) \cap \text{ODD} =^* \emptyset$. Then we get:

$$\big(\pi(\eta_{\beta_i}) \setminus \pi(\text{ODD}^?)\big) \cap \text{ODD} =^* H(B^{\text{sgn}(i)}), \text{ but}$$
$$\eta_{\beta_0} \setminus \text{ODD}^? = \text{major}_{i=1,2,3}(\eta_{\beta_i} \setminus \text{ODD}^?), \text{ so}$$
$$\big(\pi(\eta_{\beta_0}) \setminus \pi(\text{ODD}^?)\big) \cap \text{ODD} =^* \text{major}_{i=1,2,3}\Big(\big(\pi(\eta_{\beta_i}) \setminus \pi(\text{ODD}^?)\big) \cap \text{ODD} \Big), \text{ and thus}$$
$$H(B^0) =^* \text{major}_{i=1,2,3} H(B^1) = H(B^1), \text{ a contradiction.} \qquad \square$$

We fix a P-name for a representative of the Boolean Algebra automorphism ϕ^{-1} and, abusing notation, denote it by $\pi^{-1} : \mathcal{P}(\lambda) \to \mathcal{P}(\lambda)$. Set

$$X := \text{ODD}^? \cap \pi^{-1}(\text{ODD}).$$

We assume that the P-name π^{-1} is in M_β for each β; and as before, we can argue (or assume) that $\pi^{-1}(\text{ODD})$ and thus X is a P_β-name. (However, we do not assume that $\pi(z)$ is in V_β for all $z \in V_\beta$.)

Lemma 5.53. *(In V_β.) $|X| = \lambda$, and $r_0 \wedge p_\beta/G_\beta$ forces:*

$$z \subseteq X \quad \to \quad \pi(z) =^* H(z) \cap \pi(X).$$

Proof. $|X| = \lambda$ follows from Lemma 5.52.

Set $y := \eta_\beta \cap \text{ODD}^?$. So $r_0 \wedge p_\beta/G_\beta$ forces: $H(y) = \pi(\eta_\beta) \cap \text{ODD}$. As $\pi(X) \subseteq^* \text{ODD}$, we get $H(y) \cap \pi(X) =^* \pi(\eta_\beta) \cap \pi(X)$. Then $y \subseteq^* \pi^{-1}(\text{ODD})$ (or equivalently, $y \subseteq^* X$) implies $y =^* y \cap \pi^{-1}(\text{ODD}) = \eta_\beta \cap X$ and thus $\pi(y) =^* \pi(\eta_\beta) \cap \pi(X)$. To summarize:

$$\text{For } y := \eta_\beta \cap \text{ODD}^?,\ y \subseteq^* X \text{ implies } r_0 \wedge p_\beta/G_\beta \Vdash \pi(y) =^* H(y) \cap \pi(X).$$

Now assume towards a contradiction that some $q \le p_\beta \wedge r_0$ forces that the lemma fails, i.e., that $z \subseteq X$ is a counterexample as witnessed by q/G_β. We can assume that z and $\pi(z)$ are P_β-names for some $\beta \in S$ above $\text{dom}(q)$. In V_β we can strengthen r_0 to some r_1 that forces $\eta_\beta \cap \text{ODD}^? = z$. (Recall that we can fix the values in the odd intervals, as the even intervals still form a club). But then r_1 forces $\pi(z) = \pi(\eta_\beta \cap \text{ODD}^?) =^* H(z) \cap \pi(X)$, a contradiction. $\square$

For $\xi \in U^{\text{ODD}}$, set

$$x_\xi := A^?_\xi \cap X \qquad\qquad y_\xi := A_\xi \cap \pi(X).$$

Note that $\displaystyle\bigcup_{\xi \in U^{\text{ODD}}} x_\xi = X \qquad\qquad \bigcup_{\xi \in U^{\text{ODD}}} y_\xi = \text{ODD} \cap \pi(X) =^* \pi(X).$

We also define

$$H'_\xi : \mathcal{P}(\underset{\sim}{x}_\xi) \to \mathcal{P}(\underset{\sim}{y}_\xi) \quad \text{by} \quad a \mapsto H_\xi(a) \cap \underset{\sim}{\pi}(\underset{\sim}{X}),$$

$$\text{and} \quad H' : P(\underset{\sim}{X}) \to P(\underset{\sim}{\pi}(\underset{\sim}{X})) \quad \text{by} \quad z \mapsto \bigcup_{\xi \in U^{\text{ODD}}} H'_\xi(z \restriction \underset{\sim}{x}_\xi) = H(z) \cap \underset{\sim}{\pi}(\underset{\sim}{X}).$$

So it is forced that

$$H'(z) =^* \underset{\sim}{\pi}(z) \text{ for all } z \subseteq \underset{\sim}{X}.$$

Lemma 5.54. *(In V_β.) For almost all $\xi \in U^{\text{ODD}}$, H'_ξ is a Boolean algebra isomorphism from $P(\underset{\sim}{x}_\xi)$ to $P(\underset{\sim}{y}_\xi)$.*

Proof.

- All or nothing: We claim that for almost all ζ, $H'_\zeta(\underset{\sim}{x}_\zeta) = \underset{\sim}{y}_\zeta$. Assume that $\ell \in \underset{\sim}{y}_\zeta \setminus H'_\zeta(\underset{\sim}{x}_\zeta) \subseteq \underset{\sim}{\pi}(\underset{\sim}{X})$. Then $\ell \in \underset{\sim}{\pi}(\underset{\sim}{X})$, and ℓ is not in $H'(\underset{\sim}{X}) =^* \underset{\sim}{\pi}(\underset{\sim}{X})$, so there cannot be many such ℓ. Similarly $H'_\zeta(\emptyset) = \emptyset$ for almost all ζ .
- Unions: We claim that for almost all ζ, $H'_\zeta(a) \cup H'_\zeta(b) = H'_\zeta(a \cup b)$ for all subsets a, b of $\underset{\sim}{x}_\zeta$. Let $A \subseteq \lambda$ be the set of counterexamples, i.e., for $\xi \in A$ there are $\ell_\xi \in \underset{\sim}{y}_\xi$, and a_ξ b_ξ subsets of $\underset{\sim}{x}_\zeta$ such that $\ell_\xi \in \big(H'_\xi(a_\xi) \cup H'_\xi(b_\xi)\big) \Delta H'_\xi(a_\xi \cup b_\xi)$. Set $x := \bigcup_{\xi \in A} a_\xi$ and $y := \bigcup_{\xi \in A} b_\xi$. Then ℓ_ξ is in $\big(H'(x) \cup H'(y)\big) \Delta H'(x \cup y) =^* \emptyset$, so A cannot be large.
- Complements: We claim that for almost all ξ, $H'_\xi(a) \cap H'_\xi(\underset{\sim}{x}_\xi \setminus a) = \emptyset$. Let A be the set of counterexamples, i.e., for $\xi \in A$ there is an $a_\xi \subseteq \underset{\sim}{x}_\xi$ and $\ell \in \underset{\sim}{y}_\xi$ such that $\ell_\xi \in H'_\xi(a_\xi) \cap H'_\xi(\underset{\sim}{x}_\xi \setminus a_\xi)$. Then ℓ_ξ is in $H'(\bigcup_{\zeta \in A} a_\zeta) \cap H'(\bigcup_{\zeta \in A} \underset{\sim}{x}_\xi \setminus a_\zeta) =^* \emptyset$, so A cannot be large.
- Injectivity: We already know that union and complements (and thus disjointness) are preserved, so it is enough to show that a nonempty set is mapped to a nonempty set.
  Assume this fails often, then we get an $x \subseteq \underset{\sim}{X}$ of size λ such that $\emptyset = H'(x) =^* \underset{\sim}{\pi}(x)$, a contradiction.
- Surjectivity: Assume surjectivity fails often; i.e., there are many $b_\zeta \subseteq \underset{\sim}{\pi}(\underset{\sim}{X}) \cap \text{ODD}$ not in the range of H'_ζ. Let y be the union of those b_ζ. Pick $x \subseteq \lambda$ such that $\underset{\sim}{\pi}(x) =^* y \subseteq \underset{\sim}{\pi}(\underset{\sim}{X})$. So we can assume $x \subseteq \underset{\sim}{X}$ and so $H'(x) =^* y$, which implies that $H_\zeta(x \cap \underset{\sim}{x}_\zeta) = y \cap A_\zeta = b_\zeta$ for almost all ζ, a contradiction. □

Corollary 5.55. *(In V_β) There is a $G : \underset{\sim}{X} \to \underset{\sim}{\pi}(\underset{\sim}{X})$ bijective such that $\underset{\sim}{\pi}(z) =^* G''z$ for all $z \subseteq \underset{\sim}{X}$.*

Proof. Every Boolean algebra isomorphism from $P(A)$ to $P(B)$ is generated by a bijection from A to B (the restriction to the atoms). So there is an $U' \subseteq U^{\text{ODD}}$ with $|U^{\text{ODD}} \setminus U'| < \lambda$ such that $\zeta \in U'$ implies that H'_ζ is generated by some bijection $g_\zeta : \underset{\sim}{x}_\zeta \to \underset{\sim}{y}_\zeta$. So H' is generated by $g := \bigcup_{\zeta \in U'} g_\zeta$; and we can change g into a bijection from $\underset{\sim}{X}$ to $\underset{\sim}{\pi}(\underset{\sim}{X})$ by changing less than λ many values. □

5.11. Putting everything together.

Corollary 5.56. *P forces that every automorphism of $\mathcal{P}(\lambda)/[\lambda]^{<\lambda}$ is somewhere trivial.*

Proof. Start with an arbitrary p_* forcing that $\underset{\sim}{\phi}$ is an automorphism represented by $\underset{\sim}{\pi}$. As described above, we can find a Δ-system $(M_\beta, p_\beta)_{\beta \in S}$ and P_β-names $r_0 \le \tilde{p}$, $\underset{\sim}{X}$, and G (that are really all independent of β) such that $p_\beta \restriction \beta$ forces that $|\underset{\sim}{X}| = \lambda$, $G : \underset{\sim}{X} \to \underset{\sim}{\pi}(\underset{\sim}{X})$ and that for all $z \subseteq \underset{\sim}{X}$ $r_0 \wedge p_\beta / G_\beta$ forces that $\underset{\sim}{\pi}(z) =^* G''z$.

We claim that $p_\beta \restriction \beta$ (which is stronger than p_*) forces that G witnesses that $\underset{\sim}{\phi}$ is trivial on $\underset{\sim}{X}$: Let $q \le p_\beta \restriction \beta$ force that $x \subseteq \underset{\sim}{X}$. Pick $\beta' \in S$ above $\text{dom}(q)$ such that x is a $P_{\beta'}$-name. Then $q \wedge p_{\beta'} \wedge r_0$ (r_0 at position β', that is) forces that $\underset{\sim}{\pi}(z) =^* G''z$, as required. □

Further plans

Trivial instead of densely trivial. (Forcing) constructions that trivialize automorphisms (of $\mathcal{P}(\lambda)$/fin or similar quotients) always work in two "steps:" First it is forced that all automorphism are densely trivial. Then one combines witnesses of local triviality to a single witness of "global" triviality.

In this paper we have only done the first step: Making all automorphisms densely trivial.

We conjecture that we can make all automorphisms trivial by alternating the forcings Q with a λ-variant of random forcing (as in [She17, BGS21]). (In the countable case, something similar has been done in [FS14].)

Larger 2^λ. In this paper, the length of the iteration μ (and the size of 2^λ in the extension) is λ^{++}.

The whole construction works unchanged for larger μ as well, but then we only get that P is μ-cc; so cardinals between λ^+ and μ can be collapsed, and we may again end up with $2^\lambda = \mu = \lambda^{++}$ in the extension.

In the following we propose a potential λ^{++}-cc variant for $\mu > \lambda^{++}$; i.e., a P that is $<\lambda$-closed, λ-proper and λ^{++}-cc and thus preserves all cofinalities, and forces $2^\lambda = \mu > \lambda^{++}$. We have to assume μ to be regular, and $\kappa^\lambda < \mu$ for all $\kappa < \mu$ (which should be sufficient for the Δ-system argument). We would use the following variant of P:

Q_α is not Q evaluated in the P_α-extension $V[G_\alpha]$, but rather in an intermediate model $V'_\alpha := V[(\underset{\sim}{\eta}_\alpha)_{\alpha \in A_\alpha}]$ for some $A_\alpha \in [\alpha]^{\leq\lambda^+}$ (which we choose by bookkeeping).

Then most of the arguments should still work with minimal changes. In particular, P_α is still $<\lambda$-closed, etc. Of course, Lemma 5.6(2) now only holds for sequences in V'_α; accordingly Lemma 5.19 has to be modified and more care has to be taken to argue continuous reading.

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