

GRAPHS REPRESENTED BY EXT

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ABSTRACT. This paper opens and discusses the question originally due to Daniel Herden, who asked for which graph (μ, R) we can find a family $\{\mathbb{G}_\alpha : \alpha < \mu\}$ of abelian groups such that for each $\alpha, \beta \in \mu$

$$\text{Ext}(\mathbb{G}_\alpha, \mathbb{G}_\beta) = 0 \text{ iff } (\alpha, \beta) \in R.$$

In this regard, we present four results. First, we give a connection to Quillen's small object argument which helps Ext vanishes and uses to present a useful criteria to the question. Suppose $\lambda = \aleph_0$ and $\mu = 2^\lambda$. We apply Jensen's diamond principle along with the criteria to present λ -free abelian groups representing bipartite graphs. Third, we use a version of black box to construct in ZFC, a family of \aleph_1 -free abelian groups representing bipartite graphs. Finally, applying forcing techniques, we present a consistent positive answer for general graphs.

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§ 1. INTRODUCTION

The vanishing and non-vanishing properties of $\text{Ext}(-, \sim)$ are useful tools, see for instance the book [18]. Here, we assume the objects $-, \sim$ are not necessarily noetherian, so the corresponding Ext-family becomes more mysterious. Despite its ubiquity, there is very little known about the correspondence between graph theory and the Ext-family. Our aim in this paper is to present a sample of such connection by coding graphs using the vanishing property of Ext of a family of almost free abelian groups.

Recall the following achievements from literature. In his seminal paper [13], Shelah proved that freeness of Whitehead groups, that is an abelian \mathbb{G} satisfying the vanishing property $\text{Ext}(\mathbb{G}, \mathbb{Z}) = 0$, is undecidable in ZFC. After this, there has been a considerable amount of work to understand set theoretical methods in algebra. Göbel and Shelah [6] introduced a method to construct splitters, that is groups \mathbb{G} satisfying $\text{Ext}(\mathbb{G}, \mathbb{G}) = 0$. They applied their method to prove the existence of enough projective and injective objects in the rational cotorsion pairs. Cotorsion pairs were introduced by Salce [12] in 1979. Combining with the splitters, this theory has a lot of applications, not only in group theory but also in the theory of rings and modules. For instance, see the book [7]. Göbel, Shelah and Wallutis, proved in [5] that any poset embeds into the lattice of cotorsion pairs of abelian groups. To be more explicit, let I be any set, and look at the power set $\mathcal{P}(I)$ of I . For any $X \in \mathcal{P}(I)$ they construct \aleph_1 -free abelian groups $\mathbb{G}_X, \mathbb{H}^X$ such that for all $X, Y \subseteq I$,

$$\text{Ext}(\mathbb{G}_Y, \mathbb{H}^X) = 0 \iff Y \subseteq X \quad (*).$$

Also, there are some restrictions on the size of Ext-groups. As a sample, suppose \mathbb{G} is countable torsion-free, then $\text{Ext}(\mathbb{G}, \mathbb{Z})$ is divisible and hence determined up to isomorphism by its torsion-free rank and its p -rank. Shelah and Strüngmann proved that the p -rank of $\text{Ext}(\mathbb{G}, \mathbb{Z})$ is either countable or 2^{\aleph_0} . For this and its generalization, see [17].

Definition 1.1. Given a directed graph $G = (\mu, R)$, we say that G is realized as an Ext-graph of groups, provided that there exists a family of groups $\{\mathbb{G}_\alpha : \alpha < \mu\}$ such that for each $\alpha, \beta \in \mu$

$$\text{Ext}(\mathbb{G}_\alpha, \mathbb{G}_\beta) = 0 \text{ iff } (\alpha, \beta) \in R.$$

Given the success of representing a wide range of rings as endomorphism rings of abelian groups (see [2] and [7]), and useful constructions of a rigid system consisting of 2^κ abelian groups (see [3]), it is a natural and interesting question of what can be represented as extension groups of abelian groups. In fact, Daniel Herden asked the following question:

Question 1.2. Which graphs (μ, R) can be realized as an Ext-graph of abelian groups.

Our aim in this paper is to partially answer Question 1.2.

The organization of the paper is as follows. Section 2 contains the preliminaries and basic notations that we need. In Section 3 we apply some ideas similar to Quillen's small object argument from model category (see [4, Theorem 2.1.14]) to present a general criteria for representing bipartite graphs as an Ext-graph of λ -free abelian groups:

Theorem (A). Let $\mu_1, \mu_2 \leq 2^\lambda$ be such that $\text{cf}(\mu_2) > \lambda$ and let $R \subseteq \mu_1 \times \mu_2$. Suppose the following assumptions are satisfied:

- (a) $\bar{\mathbb{G}} = \langle \mathbb{G}_\alpha : \alpha < \mu_1 \rangle$ and $\bar{\mathbb{G}}^\iota = \langle \mathbb{G}_\alpha^\iota : \alpha < \mu_1 \rangle$ for $\iota = 1, 2$, are sequences of abelian groups,
- (b) for each α , \mathbb{G}_α is an \aleph_1 -free abelian group of cardinality λ and $\mathbb{G}_\alpha = \mathbb{G}_\alpha^2 / \mathbb{G}_\alpha^1$, where $\mathbb{G}_\alpha^1 \subseteq \mathbb{G}_\alpha^2$ are free abelian groups of cardinality λ ,
- (c) if $\alpha < \mu_1$ and \mathbb{L} is constructible by $\{\mathbb{G}_\gamma : \gamma \in \mu_1 \setminus \alpha\}$ over \mathbb{G}_α^1 then, there is no homomorphism \mathbf{g} from \mathbb{G}_α^2 into \mathbb{L} extending $\text{id}_{\mathbb{G}_\alpha^1}$.

Then there exists a sequence $\bar{\mathbb{K}} = \langle \mathbb{K}_\beta : \beta < \mu_2 \rangle$ equipped with the following three properties:

- (α) \mathbb{K}_β is an \aleph_1 -free abelian group of cardinality 2^λ for each $\beta < \mu_2$,

(β) $\text{Ext}(\mathbb{G}_\alpha, \mathbb{K}_\beta) = 0$ iff $\alpha R \beta$,

(γ) if every \mathbb{G}_α is λ -free then every \mathbb{K}_α is λ -free as well.

Theorem (A) is one of the main technical results of the paper and plays an essential role in the sequel. Given cardinals $\mu < \lambda$ with μ regular we set

$$S_\mu^\lambda = \{\alpha < \lambda : \text{cf}(\alpha) = \mu\}.$$

In Section 4 we apply Jensen's diamond principle \diamond_S along with Theorem (A), and show the following:

Theorem (B). Let $S \subseteq S_{\aleph_0}^\lambda$ be a non-reflecting stationary subset of λ and suppose \diamond_S holds. Suppose $\lambda = \lambda^{\aleph_0}$, $\mu = 2^\lambda$ and let $R \subseteq \mu \times \mu$ be a relation. Then there are sequences $\langle \mathbb{G}_\alpha : \alpha < \mu \rangle$ and $\langle \mathbb{K}_\alpha : \alpha < \mu \rangle$ of λ -free abelian groups such that for all $\alpha < \mu$, $|\mathbb{G}_\alpha| = \lambda$, $|\mathbb{K}_\alpha| = 2^\lambda$ and for all $\alpha, \beta < \mu$,

$$\text{Ext}(\mathbb{G}_\alpha, \mathbb{K}_\beta) = 0 \iff \alpha R \beta.$$

The new advantage we have is that we work with λ -free abelian groups with a control on their size. Recall that Jensen's diamond principle is a kind of prediction principle whose truth is independent of ZFC.

In Section 5, we descend from Section 4 to the ordinary ZFC set theory and as another application of Theorem (A), we prove the following theorem, where instead of using the diamond principle we use some variant of "*Shelah's black box*":

Theorem (C). Let $\lambda = \lambda^{\aleph_0}$, $\mu = 2^\lambda$ and let $R \subseteq \mu \times \mu$ be a relation. Then there are families $\langle \mathbb{G}_\alpha : \alpha < \mu \rangle$, and $\langle \mathbb{K}_\alpha : \alpha < \mu \rangle$ of \aleph_1 -free abelian groups equipped with the following properties:

- (1) for all $\alpha < \mu$, \mathbb{G}_α has size λ and \mathbb{K}_α has size 2^λ ,
- (2) for all $\alpha, \beta < \mu$,

$$\text{Ext}(\mathbb{G}_\alpha, \mathbb{K}_\beta) = 0 \iff \alpha R \beta.$$

Here, we lose the λ -freeness from Theorem (B), the groups are just \aleph_1 -free, and this is the price that Theorem (B) should pay to be in ZFC. The black boxes were introduced by Shelah in [14], where he proved that they can be considered as a general method to generate a class of diamond-like principles provable in ZFC.

In particular, Question 1.2 has a positive answer for the case of bipartite graphs, where:

Definition 1.3. A graph (μ, R) is called bipartite if the vertex set can be decomposed as $V_1 \cup V_2$ such that all edges go between V_1 and V_2 .

Concerning Theorem (C), we can realize bipartite graphs as the Ext-graph of \aleph_1 -free abelian groups. Nevertheless, it is easy to see that bipartite graphs fit in the situation of (*). In particular, we recover the main result of [5] by a new argument, see Lemma 5.7. It may be worth to mention that Theorem (C) slightly improves [5] via computing the size of objects, namely $|\mathbb{G}_\alpha| = \lambda$ and $|\mathbb{K}_\alpha| = 2^\lambda$ for all $\alpha < \mu$.

In the final section we prove the following theorem:

Theorem (D). Suppose GCH holds and the pair (S, R) is a graph where $R \subseteq S \times S$, and let $\lambda > |S|$ be an uncountable regular cardinal. Then there exists a cardinal preserving generic extension of the universe, and there is a family $\{\mathbb{G}_s : s \in S\}$ of λ -free abelian groups such that

$$\text{Ext}(\mathbb{G}_s, \mathbb{G}_t) = 0 \iff sRt.$$

The strategy of the proof of Theorem (D) is given in Discussion 6.2. However, there are some details to be checked, and this is our task in §6. Here, the new advantage we have is that we work with a general graph and also we rely on forcing techniques. This theorem gives a consistent positive answer to Question 1.2. Despite this, we think Herden's question has a positive answer in ZFC. Namely, we present the following conjecture:

Conjecture 1.4. *Any graph can be realized as an Ext-graph of groups.*

We hope our results will shed more light on interplay between homological algebra and graph theory.

For all unexplained definitions from algebra see the books by Eklof-Mekler [2] and Göbel-Trlifaj [7]. Also, for unexplained definitions from the theory of forcing see the books of Jech [9] and Kunen [10].

§ 2. PRELIMINARY NOTATION

In this section, we set out our notation and discuss some facts that will be used throughout the paper and refer to the book of Eklof and Mekler [2] for more information. We restrict our discussion to the category $\text{Mod-}\mathbb{Z}$ of abelian groups, though most of the notions and results can be extended to module categories over more general rings. For abelian groups \mathbb{G} and \mathbb{H} , we set $\text{Ext}(\mathbb{G}, \mathbb{H}) := \text{Ext}_{\mathbb{Z}}^1(\mathbb{G}, \mathbb{H})$ and similarly, $\text{Hom}(\mathbb{G}, \mathbb{H}) := \text{Hom}_{\mathbb{Z}}(\mathbb{G}, \mathbb{H})$. We need the following well-known fact (see e.g. the book [18, Page 77]):

Fact 2.1. (Baer and Yoneda) Let $\zeta_i := 0 \longrightarrow \mathbb{B} \xrightarrow{g_i} \mathbb{C}_i \xrightarrow{f_i} \mathbb{A} \longrightarrow 0$ be two short exact sequences of abelian groups. We say ζ_1 is equivalent to ζ_2 if there is a commutative diagram:

$$\begin{array}{ccccccccc} \zeta_2 = 0 & \longrightarrow & \mathbb{B} & \xrightarrow{g_2} & \mathbb{C}_2 & \xrightarrow{f_2} & \mathbb{A} & \longrightarrow & 0 \\ & & = \uparrow & & \cong \uparrow & & = \uparrow & & \\ \zeta_1 = 0 & \longrightarrow & \mathbb{B} & \xrightarrow{g_1} & \mathbb{C}_1 & \xrightarrow{f_1} & \mathbb{A} & \longrightarrow & 0 \end{array}$$

Indeed, this is an equivalent relation, and there is a 1-1 correspondence between the equivalent class of these short exact sequences and $\text{Ext}(\mathbb{A}, \mathbb{B})$. In addition, $[\zeta_1] = 0 \in \text{Ext}(\mathbb{A}, \mathbb{B})$ iff ζ_1 splits.

Definition 2.2. An abelian group \mathbb{G} is called \aleph_1 -free if every subgroup of \mathbb{G} of cardinality $< \aleph_1$, i.e., every countable subgroup, is free. More generally, an abelian group \mathbb{G} is called λ -free if every subgroup of \mathbb{G} of cardinality $< \lambda$ is free.

Definition 2.3. Let κ be a regular cardinal. An abelian group \mathbb{G} is said to be strongly κ -free if there is a set \mathcal{S} of $< \kappa$ -generated free subgroups of \mathbb{G} containing 0 such that for any subset S of \mathbb{G} of cardinality $< \kappa$ and any $\mathbb{N} \in \mathcal{S}$, there is $\mathbb{L} \in \mathcal{S}$ such that $S \cup \mathbb{N} \subset \mathbb{L}$ and \mathbb{L}/\mathbb{N} is free.

Also, by a club subset of an uncountable regular cardinal κ we mean a closed and unbounded subset of κ .

Definition 2.4. Suppose κ is an uncountable regular cardinal. Let \mathcal{D}_κ denote the club filter on κ , i.e.,

$$\mathcal{D}_\kappa = \{A \subseteq \kappa : A \text{ contains a club subset of } \kappa\}.$$

Let also $\mathcal{P}(\kappa)/\mathcal{D}_\kappa$ denote the resulting quotient Boolean algebra.

It is easily seen that \mathcal{D}_κ is a normal κ -complete filter on κ and that it is closed under diagonal intersections, i.e., if $A_i \in \mathcal{D}_\kappa$, for $i < \kappa$, then their diagonal intersection

$$\Delta_{i < \kappa} A_i = \{\xi < \kappa : \forall i < \xi, \xi \in A_i\}$$

is also in \mathcal{D}_κ . This can be used to prove the following easy lemma.

Lemma 2.5. *Suppose κ is a regular uncountable cardinal, $\delta \leq \kappa$ and let $\Gamma_i \in \mathcal{P}(\kappa)/\mathcal{D}_\kappa$ for $i < \delta$. Then in the Boolean Algebra $\mathcal{P}(\kappa)/\mathcal{D}_\kappa$, the sequence $\{\Gamma_i : i < \delta\}$ has a lub (least upper bound) Γ .*

Proof. For each $i < \delta$ let $A_i \subseteq \kappa$ be such that $\Gamma_i = A_i/\mathcal{D}_\kappa$. If $\delta < \kappa$, then $\Gamma = A/\mathcal{D}_\kappa$ is as required where $A = \bigcup_{i < \delta} A_i$, and if $\delta = \kappa$, then $\Gamma = A/\mathcal{D}_\kappa$ is as required where $A = \Delta_{i < \kappa} A_i$ is the diagonal intersection of the sets $A_i, i < \kappa$. \square

The following definition plays an important role in the sequel.

Definition 2.6. Let κ be a regular cardinal. If \mathbb{G} is a $\leq \kappa$ -generated abelian group, a κ -filtration of \mathbb{G} is a sequence $\{\mathbb{G}_\nu : \nu < \kappa\}$ of subgroups of \mathbb{G} whose union is \mathbb{G} such that for all $\nu < \kappa$:

- (a) \mathbb{G}_ν is a $< \kappa$ -generated subgroup of \mathbb{G} ;
- (b) if $\mu < \nu$, then $\mathbb{G}_\mu \subset \mathbb{G}_\nu$;
- (c) if ν is a limit ordinal, then $\mathbb{G}_\nu = \bigcup_{\mu < \nu} \mathbb{G}_\mu$ i.e., the sequence is continuous.

It is easily seen that if $\{\mathbb{G}_\nu : \nu < \kappa\}$ and $\{\mathbb{H}_\nu : \nu < \kappa\}$ are two κ -filtrations of a group \mathbb{G} , then the set

$$\{\nu < \kappa : \mathbb{G}_\nu = \mathbb{H}_\nu\}$$

contains a club subset of κ , in particular, modulo the club filter \mathcal{D}_κ , the choice of the κ -filtration does not matter. This observation makes the following definition well-defined.

Definition 2.7. Let λ be an uncountable regular cardinal.

- (1) If \mathbb{G} is an abelian group of cardinality λ and $\langle \mathbb{G}_\alpha : \alpha < \lambda \rangle$ is a filtration of \mathbb{G} , then

$$\Gamma(\mathbb{G}, \bar{\mathbb{G}}) = \{\delta < \lambda : \mathbb{G}/\mathbb{G}_\delta \text{ is not } \lambda\text{-free}\}.$$

- (2) Let

$$\Gamma(\mathbb{G}) = \Gamma(\mathbb{G}, \bar{\mathbb{G}})/\mathcal{D}_\lambda$$

for some (and hence every) filtration $\bar{\mathbb{G}}$ of \mathbb{G} .

We recall that $\Gamma(\mathbb{G})$ is called the Γ -invariant of the group \mathbb{G} , and refer to [2, §IV.1] for more details and properties of this invariant. The following lemma gives a combinatorial characterization for λ -free groups to be free.

Lemma 2.8. ([2, Ch IV, Proposition 1.7]) *Let λ be an uncountable regular cardinal and let \mathbb{G} be a λ -free abelian group of cardinality λ . The following are equivalent:*

- (1) \mathbb{G} is free,
- (2) \mathbb{G} has a filtration $\langle \mathbb{G}_\alpha : \alpha < \lambda \rangle$ such that for all $\alpha < \lambda$, $\mathbb{G}_{\alpha+1}/\mathbb{G}_\alpha$ is free,
- (3) $\Gamma(\mathbb{G}) = \emptyset/\mathcal{D}_\lambda$.

These mean that $\Gamma(\mathbb{G}, \bar{\mathbb{G}})$ is non-stationary for some (and hence every) filtration $\langle \mathbb{G}_\alpha : \alpha < \lambda \rangle$ of \mathbb{G} .

§ 3. A REALIZATION CRITERIA

Our main result in this section is Theorem 3.5. Let us start by some lemmas and definitions.

Lemma 3.1. (See [2, Ex. IV.22]) *If \mathbb{G} is the union of a continuous chain $\{\mathbb{G}_\alpha : \alpha \leq \beta\}$ of abelian groups such that \mathbb{G}_0 and $\mathbb{G}_{\alpha+1}/\mathbb{G}_\alpha$ are \aleph_1 -free for all $\alpha + 1 < \beta$, then \mathbb{G} is \aleph_1 -free.*

More generally, the following holds:

Lemma 3.2. (See [3, Pages 112-113]) *Let $\kappa < \lambda$ be infinite cardinals.*

- (i) λ -free implies κ -free.
- (ii) Subgroups and direct sums of κ -free are κ -free
- (iii) Extension of κ -free group is κ -free.

(iv) Let $0 = \mathbb{G}_0 \subseteq \dots \subseteq \mathbb{G}_i \subseteq \dots$ be a smooth chain of groups with union \mathbb{G} such that all $\mathbb{G}_i/\mathbb{G}_{i+1}$ are κ -free. Then \mathbb{G} is κ -free.

Definition 3.3. Let \mathcal{G} be a set or class of abelian groups.

- (1) We say $\bar{\mathbb{L}}$ is a construction by \mathcal{G} over \mathbb{G} when:
 - (a) $\bar{\mathbb{L}} = \langle \mathbb{L}_\varepsilon : \varepsilon \leq \varepsilon(*) \rangle$ is a \subseteq -increasing and continuous sequence of abelian groups,
 - (b) $\mathbb{L}_0 = \mathbb{G}$,
 - (c) for every $\varepsilon < \varepsilon(*)$, $\mathbb{L}_{\varepsilon+1}/\mathbb{L}_\varepsilon$ is free or is isomorphic to some member of \mathcal{G} .
- (2) Omitting “over \mathbb{G} ” means for $\mathbb{G} = \{0\}$.
- (3) We say \mathbb{L} is constructible by \mathcal{G} (over \mathbb{G}) when for some $\bar{\mathbb{L}} = \langle \mathbb{L}_\varepsilon : \varepsilon \leq \varepsilon(*) \rangle$, $\bar{\mathbb{L}}$ is a construction by \mathcal{G} (over \mathbb{G}) and $\mathbb{L} = \mathbb{L}_{\varepsilon(*)}$.

Notation 3.4. Suppose we have the following data of abelian groups and homomorphisms:

$$\begin{array}{ccc} & \mathbb{A} & \\ & \uparrow & \\ \mathbb{B} & \longrightarrow & \mathbb{C}, \end{array}$$

We denote the corresponding pushout by $\mathbb{A} \oplus_{\mathbb{B}} \mathbb{C}$.

The next result gives sufficient conditions for representing bipartite graphs using the functor Ext.

Theorem 3.5. Let $\mu_1, \mu_2 \leq 2^\lambda$ be such that $\text{cf}(\mu_2) > \lambda$ and let $R \subseteq \mu_1 \times \mu_2$. Suppose the following assumptions are satisfied:

- $\boxtimes_{\lambda, \mu_1}^1$ (a) $\bar{\mathbb{G}} = \langle \mathbb{G}_\alpha : \alpha < \mu_1 \rangle$ and $\bar{\mathbb{G}}^\iota = \langle \mathbb{G}_\alpha^\iota : \alpha < \mu_1 \rangle$ for $\iota = 1, 2$, are sequences of abelian groups,
- (b) for each α , \mathbb{G}_α is an \aleph_1 -free abelian group of cardinality λ and $\mathbb{G}_\alpha = \mathbb{G}_\alpha^2/\mathbb{G}_\alpha^1$, where $\mathbb{G}_\alpha^1 \subseteq \mathbb{G}_\alpha^2$ are free abelian groups cardinality λ ,

(c) if $\alpha < \mu_1$ and \mathbb{L} is constructible by $\{\mathbb{G}_\gamma : \gamma \in \mu_1 \setminus \alpha\}$ over \mathbb{G}_α^1 , then there is no homomorphism $\mathbf{g} : \mathbb{G}_\alpha^2 \rightarrow \mathbb{L}$ extending $\text{id}_{\mathbb{G}_\alpha^1}$:

$$\begin{array}{ccc} \mathbb{G}_\alpha^1 & \xrightarrow{\subseteq} & \mathbb{G}_\alpha^2 \\ \text{id} \downarrow & & \downarrow \neq \mathbf{g} \\ \mathbb{G}_\alpha^1 & \xrightarrow{\subseteq} & \mathbb{L} \end{array}$$

Then there exists a sequence $\bar{\mathbb{K}} = \langle \mathbb{K}_\beta : \beta < \mu_2 \rangle$ equipped with the following three properties:

- $\boxtimes_{\lambda, \mu_1}^2$ (α) \mathbb{K}_β is an \aleph_1 -free abelian group of cardinality 2^λ for each $\beta < \mu_2$,
- (β) $\text{Ext}(\mathbb{G}_\alpha, \mathbb{K}_\beta) = 0$ iff $\alpha R \beta$,
- (γ) if every \mathbb{G}_α is λ -free then every \mathbb{K}_α is λ -free as well.

Proof. Let $\langle \mathcal{U}_\varepsilon : \varepsilon < 2^\lambda \rangle$ be a partition of 2^λ such that $\mathcal{U}_\varepsilon \subseteq [\varepsilon, 2^\lambda)$ has cardinality 2^λ . We claim that there are sequences $\bar{\mathbb{K}}_\varepsilon$, $\bar{\mathbf{H}}_{\varepsilon, \beta}$ and $\bar{\mathbf{h}}_{\varepsilon, \beta}^*$, for $\varepsilon \leq 2^\lambda$ and $\beta < \mu_2$ with the following properties:

- \oplus_ε (a) $\bar{\mathbb{K}}_\varepsilon = \langle \mathbb{K}_{\varepsilon, \beta} : \beta < \mu_2 \rangle$ is a sequence of abelian groups,
- (b) for each $\beta < \mu_2$ the sequence $\langle \mathbb{K}_{\zeta, \beta} : \zeta \leq 2^\lambda \rangle$ is \subseteq -increasing and continuous,
- (c) $\mathbb{K}_{\varepsilon, \beta}$ has cardinality $\leq 2^\lambda$,
- (d) $\bar{\mathbf{H}}_{\varepsilon, \beta} = \langle (\alpha_{\beta, \zeta}, h_{\beta, \zeta}) : \zeta \in \mathcal{U}_\varepsilon \rangle$ lists all the pairs (α, h) where $\alpha < \mu_1$, $\alpha R \beta$ and $h \in \text{Hom}(\mathbb{G}_\alpha^1, \mathbb{K}_{\varepsilon, \beta})$,
- (e) $\bar{\mathbf{h}}_{\varepsilon, \beta}^* = \langle h_{\beta, \zeta}^* : \zeta \in \mathcal{U}_\varepsilon \rangle$,
- (f) if $\varepsilon \leq \zeta \in \mathcal{U}_\varepsilon$, then $h_{\beta, \zeta}^* \in \text{Hom}(\mathbb{G}_{\alpha_{\beta, \zeta}}^2, \mathbb{K}_{\zeta+1, \beta})$ extends $h_{\beta, \zeta}$. The property is conveniently summarized by the subjoined diagram:

$$\begin{array}{ccc} \mathbb{G}_{\alpha_{\beta, \zeta}}^1 & \xrightarrow{\subseteq} & \mathbb{G}_{\alpha_{\beta, \zeta}}^2 \\ h_{\beta, \zeta} \downarrow & & \downarrow h_{\beta, \zeta}^* \\ \mathbb{K}_{\zeta, \beta} & \xrightarrow{\subseteq} & \mathbb{K}_{\zeta+1, \beta} \end{array}$$

We proceed by a double induction on ζ and β to construct such a sequence. For $\zeta = 0$ and for all $\beta < \mu_2$ set

$$\mathbb{K}_{0,\beta} := \bigoplus \{\mathbb{G}_\alpha^1 : \alpha < \mu_1, \neg(\alpha R\beta)\}.$$

By Fact 3.2(ii), $\mathbb{K}_{0,\beta}$ is free, and so λ -free.

For ζ a limit ordinal and $\beta < \mu_2$, we set

$$\mathbb{K}_{\zeta,\beta} = \bigcup_{\varepsilon < \zeta} \mathbb{K}_{\varepsilon,\beta}.$$

Now suppose that the groups $K_{\varepsilon,\gamma}$ are defined for all $\varepsilon < \zeta + 1$ and $\gamma < \mu_2$. We define $\mathbb{K}_{\zeta+1,\beta}$ for $\beta < \mu_2$ as follows.

Let $\bar{\mathbf{H}}_{\varepsilon,\beta} = \langle (\alpha_{\beta,\zeta}, h_{\beta,\zeta}) : \zeta \in \mathcal{U}_\varepsilon \rangle$ be as in clause (d). We look at the following diagram

$$\begin{array}{ccc} & \mathbb{K}_{\zeta,\beta} & \\ & \uparrow h_{\beta,\zeta} & \\ \mathbb{G}_{\alpha_{\beta,\zeta}}^1 & \xrightarrow{\subseteq} & \mathbb{G}_{\alpha_{\beta,\zeta}}^2 \end{array}$$

and set

$$\mathbb{K}_{\zeta+1,\beta} := \mathbb{K}_{\zeta,\beta} \oplus_{\mathbb{G}_{\alpha_{\beta,\zeta}}^1} \mathbb{G}_{\alpha_{\beta,\zeta}}^2.$$

In particular, there is a homomorphism f induced from $h_{\beta,\zeta}$ which commutes the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{K}_{\zeta,\beta} & \xrightarrow{g} & \mathbb{K}_{\zeta+1,\beta} & \longrightarrow & \frac{\mathbb{G}_{\alpha_{\beta,\zeta}}^2}{\mathbb{G}_{\alpha_{\beta,\zeta}}^1} \longrightarrow 0 \\ & & \uparrow h_{\beta,\zeta} & & \uparrow f & & = \uparrow \\ 0 & \longrightarrow & \mathbb{G}_{\alpha_{\beta,\zeta}}^1 & \longrightarrow & \mathbb{G}_{\alpha_{\beta,\zeta}}^2 & \longrightarrow & \frac{\mathbb{G}_{\alpha_{\beta,\zeta}}^2}{\mathbb{G}_{\alpha_{\beta,\zeta}}^1} \longrightarrow 0 \end{array}$$

We set $h_{\beta,\zeta}^* := f$. By a diagram chasing, $h_{\beta,\zeta}^*$ extends $h_{\beta,\zeta}$, and recall that $g : \mathbb{K}_{\zeta,\beta} \hookrightarrow \mathbb{K}_{\zeta+1,\beta}$ is an embedding. This completes the inductive construction and hence proves the existence of the sequences claimed to exist above, i.e., the construction of \bigoplus_ε is now complete. Suppose now that every group \mathbb{G}_α is λ -free. By Fact 3.2(iii) it follows that $\mathbb{K}_{\zeta,\beta}$ is λ -free as well.

For $\beta < \mu_2$ let

$$\mathbb{K}_\beta := \mathbb{K}_{2^\lambda,\beta}.$$

In view of Lemma 3.1, the above short exact sequence and the hypotheses, each \mathbb{K}_β is an \aleph_1 -free abelian group of size 2^λ . Recall that λ -freeness of \mathbb{K}_α follows by λ -freeness of \mathbb{G}_α , as desired by $\boxtimes_{\lambda, \mu_1}^2(\gamma)$.

Let us check the item $\boxtimes_{\lambda, \mu_1}^2(\beta)$. First, suppose that $\alpha R \beta$ and let $h \in \text{Hom}(\mathbb{G}_\alpha^1, \mathbb{K}_\beta)$. Since $|\mathbb{G}_\alpha^1| \leq \lambda < \text{cf}(\mu_2)$, for some $\varepsilon < 2^\lambda$ we have $\text{Rang}(h) \subseteq K_{\varepsilon, \beta}$, and hence we can assume that $h \in \text{Hom}(\mathbb{G}_\alpha^1, \mathbb{K}_{\varepsilon, \beta})$. By the definition of $\bar{\mathbf{H}}_{\varepsilon, \beta}$, for some $\zeta \in \mathcal{U}_\varepsilon$ we have $(\alpha, h) = (\alpha_{\beta, \zeta}, h_{\beta, \zeta})$. Then by clause (f) of the construction, $h_{\beta, \zeta}^* \in \text{Hom}(\mathbb{G}_\alpha^2, \mathbb{K}_{\zeta+1, \beta})$ extends $h_{\beta, \zeta}$. There is a natural embedding map from $\text{Hom}(\mathbb{G}_\alpha^2, \mathbb{G}_{\zeta+1, \beta})$ into $\text{Hom}(\mathbb{G}_\alpha^2, \mathbb{K}_\beta)$, so without loss of generality, we may and do assume that

$$h_{\beta, \zeta}^* \in \text{Hom}(\mathbb{G}_\alpha^2, \mathbb{K}_\beta).$$

Now, we look at the exact sequence

$$(+):= 0 \longrightarrow \mathbb{G}_\alpha^1 \longrightarrow \mathbb{G}_\alpha^2 \longrightarrow \mathbb{G}_\alpha \longrightarrow 0.$$

Applying $\text{Hom}(-, \mathbb{K}_\beta)$ to (+), it induces the following long exact sequence

$$\text{Hom}(\mathbb{G}_\alpha^2, \mathbb{K}_\beta) \xrightarrow{\mathbf{f}} \text{Hom}(\mathbb{G}_\alpha^1, \mathbb{K}_\beta) \longrightarrow \text{Ext}(\mathbb{G}_\alpha, \mathbb{K}_\beta) \longrightarrow \text{Ext}(\mathbb{G}_\alpha^2, \mathbb{K}_\beta) = 0,$$

where the last vanishing $\text{Ext}(\mathbb{G}_\alpha^2, \mathbb{K}_\beta) = 0$ holds as \mathbb{G}_α^2 is free. From this,

$$\text{Ext}(\mathbb{G}_\alpha, \mathbb{K}_\beta) = \frac{\text{Hom}(\mathbb{G}_\alpha^1, \mathbb{K}_\beta)}{\text{Rang}(f)}.$$

Thus, $\text{Ext}(\mathbb{G}_\alpha, \mathbb{K}_\beta) = 0$ if and only if \mathbf{f} is surjective. But as we observed above, \mathbf{f} is onto, and hence we conclude that $\text{Ext}(\mathbb{G}_\alpha, \mathbb{K}_\beta) = 0$.

Let $(\alpha, \beta) \in (\mu_1 \times \mu_2) \setminus R$. We need to show $\text{Ext}(\mathbb{G}_\alpha, \mathbb{K}_\beta) \neq 0$. Suppose on the contrary that $\text{Ext}(\mathbb{G}_\alpha, \mathbb{K}_\beta) = 0$, and search for a contradiction. Since \mathbb{G}_α^1 is a direct summand of $\mathbb{K}_{0, \beta}$, we have $\mathbb{G}_\alpha^1 \oplus X = \mathbb{K}_{0, \beta}$. Let $\rho: \mathbb{G}_\alpha^1 \rightarrow \mathbb{G}_\alpha^1 \oplus X$ be the natural map, and look at

$$g_\alpha^1 := \mathbb{G}_\alpha^1 \xrightarrow{\rho} \mathbb{G}_\alpha^1 \oplus X \xrightarrow{=} \mathbb{K}_{0, \beta} \xrightarrow{\subseteq} \mathbb{K}_\beta$$

So by our assumption, there is $g_\alpha^2 : \mathbb{G}_\alpha^2 \rightarrow \mathbb{K}_\beta$ extending g_α^1 :

$$\begin{array}{ccc} \mathbb{G}_\alpha^1 & \xrightarrow{\subseteq} & \mathbb{G}_\alpha^2 \\ g_\alpha^1 \downarrow & \swarrow \exists g_\alpha^2 & \\ \mathbb{K}_\beta & & \end{array}$$

We now show that there exists $u \subseteq 2^\lambda$ equipped with:

- (a) $|u| = \lambda$,
- (b) $0, \alpha, \beta \in u$,
- (c) $\varepsilon \in u \Rightarrow \text{Rang}(h_{\beta, \varepsilon}) \subseteq \sum_{\zeta \in u \cap \varepsilon} \text{Rang}(h_{\beta, \zeta}^*) + \sum_{\gamma \in u} \mathbb{G}_\gamma^1$,
- (d) $\text{Rang}(g_\alpha^2) \subseteq \sum_{\varepsilon \in u} \text{Rang}(h_{\beta, \varepsilon}^*) + \sum_{\gamma \in u} \mathbb{G}_\gamma^1$.

Indeed, by induction on $i < \lambda$ we define an increasing and continuous sequence $\langle u_i : i < \lambda \rangle$ of subsets of 2^λ such that for each $i < \lambda$, the following properties are valid:

- $|u_i| \leq \lambda$,
- $0, \alpha, \beta \in u_0$,
- If i is limit ordinal, then $u_i = \bigcup_{j < i} u_j$,
- $\text{Rang}(g_\alpha^2) \subseteq \sum_{\varepsilon \in u_0} \text{Rang}(h_{\beta, \varepsilon}^*) + \sum_{\gamma \in u_0} \mathbb{G}_\gamma^1$,
- If $\varepsilon \in u_i$ then

$$\text{Rang}(h_{\beta, \varepsilon}) \subseteq \sum_{\zeta \in u_{i+1} \cap \varepsilon} \text{Rang}(h_{\beta, \zeta}^*) + \sum_{\gamma \in u_{i+1}} \mathbb{G}_\gamma^1.$$

Set $u := \bigcup_{i < \lambda} u_i$. It is easily seen that u satisfies the required properties. Let $\varepsilon(*) = \text{otp}(u)$, where $\text{otp}(u)$ denotes the order type of the set u and let $\langle \gamma_\varepsilon : \varepsilon < \varepsilon(*) \rangle$ be the increasing enumeration of u .

Set

$$\mathcal{G} := \langle \mathbb{G}_\gamma : \gamma \in u \setminus \alpha \rangle.$$

We now define a construction $\bar{\mathbb{L}} := \langle \mathbb{L}_\varepsilon : \varepsilon \leq \varepsilon(*) \rangle$ by \mathcal{G} over $\mathbb{L}_0 = \mathbb{G}_\alpha^1$. To achieve this, we set

$$\mathbb{L}_{\varepsilon+1} := \sum_{\zeta < \varepsilon} \text{Rang}(h_{\beta, \zeta}^*) + \sum_{\gamma \in u} \mathbb{G}_\gamma^1.$$

Let $A := \text{coker}(h_{\beta,\zeta}^*)$. Let put this into the previous diagram and obtain the following:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 & & A & \xrightarrow{=} & A & & \\
 & & \pi \uparrow & & \pi \uparrow & & \\
 0 & \longrightarrow & \mathbb{K}_{\zeta,\beta} & \longrightarrow & \mathbb{K}_{\zeta+1,\beta} & \longrightarrow & \mathbb{G}_{\alpha\beta,\epsilon} \longrightarrow 0 \\
 & & h_{\beta,\zeta} \uparrow & & h_{\beta,\zeta}^* \uparrow & & = \uparrow \\
 0 & \longrightarrow & \mathbb{G}_{\alpha\beta,\zeta}^1 & \longrightarrow & \mathbb{G}_{\alpha\beta,\zeta}^2 & \longrightarrow & \mathbb{G}_{\alpha\beta,\epsilon} \longrightarrow 0
 \end{array}$$

This yields that:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Rang}(h_{\beta,\zeta}^*) & \longrightarrow & \mathbb{K}_{\zeta+1,\beta} & \longrightarrow & A \longrightarrow 0 \\
 & & f_1 \uparrow & & f_2 \uparrow & & = \uparrow \\
 0 & \longrightarrow & \text{Rang}(h_{\beta,\zeta}) & \longrightarrow & \mathbb{K}_{\zeta,\beta} & \longrightarrow & A \longrightarrow 0,
 \end{array}$$

where f_i are the natural inclusions. Now, we use the Ker – coker exact sequence:

$$0 \rightarrow \text{Ker}(f_1) \rightarrow \text{Ker}(f_2) \rightarrow \text{Ker}(=) \rightarrow \text{coker}(f_1) \rightarrow \text{coker}(f_2) \rightarrow \text{coker}(=) \rightarrow 0$$

Recall that $\text{Ker}(=) = \text{coker}(=) = 0$, $\text{coker}(f_1) = \frac{\text{Rang}(h_{\beta,\zeta}^*)}{\text{Rang}(h_{\beta,\zeta})}$ and $\text{coker}(f_2) = \mathbb{G}_{\alpha\beta,\epsilon}$.

So,

$$\text{Rang}(h_{\beta,\zeta}^*)/\text{Rang}(h_{\beta,\zeta}) \cong \mathbb{G}_{\alpha\beta,\epsilon} \quad (+)$$

For each $\epsilon < \epsilon(\star)$ we have

$$\frac{\mathbb{L}_{\epsilon+1}}{\mathbb{L}_{\epsilon}} \cong \text{Rang}(h_{\beta,\zeta}^*)/\text{Rang}(h_{\beta,\zeta}) \cong \mathbb{G}_{\alpha\beta,\epsilon} \in \mathcal{G}.$$

In view of Definition 3.3, $\bar{\mathbb{L}}$ is indeed a construction. Recall from (d) that $\text{Rang}(g_{\alpha}^2) \subset \mathbb{L}_{\epsilon(\star)}$. Hence $g_{\alpha}^2 : \mathbb{G}_{\alpha}^2 \rightarrow \mathbb{L}_{\epsilon(\star)}$ extends the identity function over \mathbb{G}_{α}^1 . This is in contradiction with $\boxtimes_{\lambda,\mu_1}^1(c)$. \square

§ 4. FROM DIAMOND TO EXT-GRAPH OF QUITE FREE GROUPS

In this section we are going to prove Theorem (B) from the introduction.

Notation 4.1. Let \mathbb{G} be a reduced torsion free abelian group, e.g., a free group. The notation \mathbb{G} stands for the \mathbb{Z} -adic completion of \mathbb{G} .

Remark 4.2. i) \mathbb{Z} -adic topology of a free abelian group is Hausdorff.

ii) For example, each element of $\widehat{\mathbb{Z}}$ can be represented as $\sum_n k_n$ where $n!|k_n$.

Discussion 4.3. Recall that for a stationary set $S \subseteq \lambda$, Jensen's diamond $\diamond_\lambda(S)$ asserts the existence of a sequence $\langle S_\alpha : \alpha \in S \rangle$ such that for every $X \subseteq \lambda$ the set $\{\alpha \in S : X \cap \alpha = S_\alpha\}$ is stationary.

For simplicity of the reader we cite the following result:

Fact 4.4. If $\diamond_\lambda(S)$ holds, then there is a decomposition $S = \bigcup_{\beta < \lambda} S_\beta$ such that $\diamond_\lambda(S_\beta)$ holds for all $\beta < \lambda$.

Proof. See [7, Theorem 9.1.17]. □

The following lemma plays a key role in our construction.

Lemma 4.5. *Let $\iota \in \{1, 2\}$. Suppose \mathbb{G}_n^ι are free abelian groups of the same size such that for all $n < \omega$ the following conditions are satisfied:*

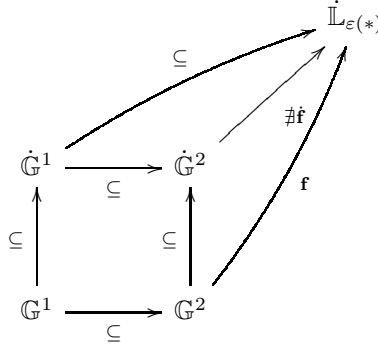
- (1) $\mathbb{G}_n^1 \subseteq \mathbb{G}_n^2$,
- (2) $\mathbb{G}_n^\iota \subseteq \mathbb{G}_{n+1}^\iota$,
- (3) $\mathbb{G}_{n+1}^\iota / \mathbb{G}_n^\iota$ is free,
- (4) $\mathbb{G}_{n+1}^2 / (\mathbb{G}_{n+1}^1 + \mathbb{G}_n^2)$ is free,
- (5) $\mathbb{G}_{n+1}^2 = (\mathbb{G}_{n+1}^1 \oplus_{\mathbb{G}_n^1} \mathbb{G}_n^2) \oplus \bigoplus_{m < \omega} y_m \mathbb{Z}$,
- (6) $\bar{\mathbb{L}} = \langle \mathbb{L}_\varepsilon : \varepsilon \leq \varepsilon(*) \rangle$ is a construction,
- (7) \mathbb{L}_ε and $\mathbb{L}_{\varepsilon+1} / \mathbb{L}_\varepsilon$ are \aleph_1 -free.

Let $\mathbb{G}^\iota = \bigcup_{n < \omega} \mathbb{G}_n^\iota$, so that $\mathbb{G}^1 \subseteq \mathbb{G}^2$ are free abelian groups and let $\mathbf{f} \in \text{Hom}(\mathbb{G}^2, \bar{\mathbb{L}}_{\varepsilon(*)})$.

Then there are free abelian groups $\dot{\mathbb{G}}^1 \subseteq \dot{\mathbb{G}}^2$ such that:

- (a) $\mathbb{G}^1 \subseteq \dot{\mathbb{G}}^1$,
- (b) $\mathbb{G}^2 \subseteq \dot{\mathbb{G}}^2$,
- (c) $\mathbb{G}^1 = \mathbb{G}^2 \cap \dot{\mathbb{G}}^1$,
- (d) $\dot{\mathbb{G}}^\iota / \mathbb{G}^\iota$ is free,
- (e) $\dot{\mathbb{G}}^2 / (\dot{\mathbb{G}}^1 + \mathbb{G}^2)$ is torsion free,
- (f) for all $n < \omega$, $\dot{\mathbb{G}}^2 / (\dot{\mathbb{G}}^1 + \mathbb{G}_n^2)$ is free,
- (g) there are no $\dot{\mathbf{f}}$ and $\langle \dot{\mathbb{L}}_\varepsilon : \varepsilon \leq \varepsilon(*) \rangle$ equipped with the following properties:

- (g1) $\langle \dot{\mathbb{L}}_\varepsilon : \varepsilon \leq \varepsilon(*) \rangle$ is a nice construction over $\dot{\mathbb{G}}^1$,
- (g2) $\dot{\mathbf{f}} \in \text{Hom}(\dot{\mathbb{G}}^2, \dot{\mathbb{L}}_{\varepsilon(*)})$,
- (g3) $\dot{\mathbf{f}}$ extends $\mathbf{f} \cup \text{id}_{\dot{\mathbb{G}}^1}$. This property is conveniently summarized by the subjoined diagram:



- (g4) $\mathbb{L}_\varepsilon \subseteq \dot{\mathbb{L}}_\varepsilon$ for $\varepsilon \leq \varepsilon(*)$,
- (g5) $\mathbb{L}_\varepsilon = \mathbb{L}_{\varepsilon+1} \cap \dot{\mathbb{L}}_\varepsilon$,
- (g6) $\dot{\mathbb{L}}_{\varepsilon+1}/(\mathbb{L}_{\varepsilon+1} + \dot{\mathbb{L}}_\varepsilon)$ is free.

Proof. Set $\dot{\mathbb{G}}^1 := \mathbb{G}^1 \oplus \bigoplus_{n < \omega} z_n \mathbb{Z}$, where $\bigoplus_{n < \omega} z_n \mathbb{Z}$ is a free abelian group with a base given by $\{z_n : n < \omega\}$. To define $\dot{\mathbb{G}}^2$, let us first set

$$\mathbb{G}^{2,*} := \dot{\mathbb{G}}^1 \oplus_{\mathbb{G}^1} \mathbb{G}^2.$$

For any infinite sequence $\vec{a} = \langle a_n : n < \omega \rangle \in \prod_{\omega} 2 = {}^\omega 2$, we look at the following

$$\mathbb{G}^{2,\vec{a}} := \langle \mathbb{G}^{2,*} \cup \{ \sum_{n < \omega} n!(y_n + a_n z_n) \} \rangle,$$

i.e., the subgroup of $\widehat{\mathbb{G}^{2,*}}$ generated by $\mathbb{G}^{2,*}$ and the distinguished element

$$\sum_{n < \omega} n!(y_n + a_n z_n) \in \widehat{\mathbb{G}^{2,*}}.$$

It is routine to see that the properties (a)–(f) are satisfied with $\mathbb{G}^{2,\vec{a}}$. We are going to show that $\dot{\mathbb{G}}^2 = \mathbb{G}^{2,\vec{a}}$ satisfies in the remaining property (g) for a suitable choice of \vec{a} . Suppose on the way of contradiction that for each $\vec{a} = \langle a_n : n < \omega \rangle \in {}^\omega 2$ there is a counterexample $\dot{\mathbf{f}}_{\vec{a}}$ and $\langle \dot{\mathbb{L}}_\varepsilon^{\vec{a}} : \varepsilon \leq \varepsilon(*) \rangle$ to clause (g). In particular, $\dot{\mathbf{f}}_{\vec{a}} \in \text{Hom}(\mathbb{G}^{2,\vec{a}}, \dot{\mathbb{L}}_{\varepsilon(*)})$, and it extends $\mathbf{f} \cup \text{id}_{\dot{\mathbb{G}}^1}$.

For $\vec{a} \in {}^\omega 2$ and $m < \omega$ set

$$(+) \quad y_m^{\vec{a}} := \sum_{n \geq m} \frac{n!}{m!} (y_n + a_n z_n).$$

Claim (A): Let \vec{a}, \vec{b} and m be such that

- $m < \omega$,
- $a_\ell = b_\ell$ for all $\ell \leq m$,
- $\dot{\mathbf{f}}_{\vec{a}}(y_\ell^{\vec{a}}) = \dot{\mathbf{f}}_{\vec{b}}(y_\ell^{\vec{b}})$ for all $\ell \leq n$.

Then $\vec{a} = \vec{b}$.

To prove the claim, we proceed by induction on $\ell < \omega$ and show that:

- (i) $a_\ell = b_\ell$,
- (ii) $\dot{\mathbf{f}}_{\vec{a}}(y_\ell^{\vec{a}}) = \dot{\mathbf{f}}_{\vec{b}}(y_\ell^{\vec{b}})$.

The case $\ell \leq m$ follows from the assumption, so we may assume that $\ell > m$. Now suppose that $\ell = k+1$ and the result is true for all natural numbers less or equal to k . Let us evaluate $\dot{\mathbf{f}}_{\vec{a}}$ at $y_m^{\vec{a}}$ from (+), and recall that $\dot{\mathbf{f}}_{\vec{a}}$ extends $\mathbf{f} \cup \text{id}_{\mathbb{C}^1}$. It turns out that:

$$\dot{\mathbf{f}}_{\vec{a}}(y_m^{\vec{a}}) = \sum_{n \geq m} \left(\frac{n!}{m!} \dot{\mathbf{f}}_{\vec{a}}(y_n) + a_n \dot{\mathbf{f}}_{\vec{a}}(z_n) \right) = \sum_{n \geq m} \frac{n!}{m!} (y_n + a_n z_n).$$

It immediately follows that for all $i < \omega$

$$(*) \quad \dot{\mathbf{f}}_{\vec{a}}(y_i^{\vec{a}}) = (i+1) \dot{\mathbf{f}}_{\vec{a}}(y_{i+1}^{\vec{a}}) - (y_i + a_i z_i).$$

We apply (*) at level $i = k$ and combine it with inductive hypothesis to observe that

$$\begin{aligned} 0 &= \dot{\mathbf{f}}_{\vec{a}}(y_k^{\vec{a}}) - \dot{\mathbf{f}}_{\vec{b}}(y_k^{\vec{b}}) \\ &= \ell \dot{\mathbf{f}}_{\vec{a}}(y_\ell^{\vec{a}}) - (y_{\ell-1} + a_{\ell-1} z_{\ell-1}) - \ell \dot{\mathbf{f}}_{\vec{a}}(y_\ell^{\vec{a}}) + (y_{\ell-1} + a_{\ell-1} z_{\ell-1}) \\ &= \ell (\dot{\mathbf{f}}_{\vec{a}}(y_\ell^{\vec{a}}) - \dot{\mathbf{f}}_{\vec{b}}(y_\ell^{\vec{b}})). \end{aligned}$$

Since $\mathbb{L}_{\varepsilon(*)}$ is torsion-free, it follows that $\dot{\mathbf{f}}_{\vec{a}}(y_\ell^{\vec{a}}) - \dot{\mathbf{f}}_{\vec{b}}(y_\ell^{\vec{b}}) = 0$, so (ii) is valid. In order to show (i) we apply (*) at level $i = \ell$:

$$0 = \dot{\mathbf{f}}_{\vec{a}}(y_\ell^{\vec{a}}) - \dot{\mathbf{f}}_{\vec{b}}(y_\ell^{\vec{b}}) = (\ell+1) (\dot{\mathbf{f}}_{\vec{a}}(y_{\ell+1}^{\vec{a}}) - \dot{\mathbf{f}}_{\vec{b}}(y_{\ell+1}^{\vec{b}})) - (a_\ell - b_\ell) z_\ell,$$

i.e.,

$$(a_\ell - b_\ell)z_\ell = (\ell + 1)(\dot{\mathbf{f}}_{\vec{a}}(y_\ell^{\vec{a}}) - \dot{\mathbf{f}}_{\vec{b}}(y_\ell^{\vec{b}})).$$

It follows that $(\ell + 1)|(a_\ell - b_\ell)$. Since $a_i, b_i \in \{0, 1\}$, we must have $a_\ell - b_\ell = 0$. This completes the proof of the desired claim. Let \vec{a} and \vec{b} be distinct in ${}^\omega 2$ such that $a_0 = b_0$ and $\dot{\mathbf{f}}_{\vec{a}}(y_0^{\vec{a}}) = \dot{\mathbf{f}}_{\vec{b}}(y_0^{\vec{b}})$. This contradicts Claim(A), so we are done. \square

Remark 4.6. Adopt the notation of Lemma 4.5, and only replace “free” by “ \aleph_1 -free”. Then the same claim, as Lemma 4.5 indicates, is valid. We leave its straightforward modification to the reader.

Recall that $S_{\aleph_0}^\lambda = \{\alpha < \lambda : \text{cf}(\alpha) = \aleph_0\}$. The main point in the development of Theorem B) is to introduce $\boxtimes_{\lambda, \mu}^1$ from Theorem 3.5. Here, we present a variation of it:

Theorem 4.7. *Let $S \subseteq S_{\aleph_0}^\lambda$ be stationary non-reflecting and suppose \diamond_S holds.*

Suppose one of the followings:

- (1) $\mu = \lambda = \text{cf}(\lambda) > \aleph_0$, or
- (2) $\mu = 2^\lambda, \lambda = \text{cf}(\lambda) > \aleph_0$.

Let $\iota = 1, 2$. Then there are sequences $\bar{\mathbb{G}}^\iota = \langle \mathbb{G}_\alpha^\iota : \alpha < \mu \rangle$ of abelian groups such that the following assertions are valid:

- (i) $\mathbb{G}_\alpha^1 \subseteq \mathbb{G}_\alpha^2$ are free abelian groups cardinality λ ,
- (ii) for each α , $\mathbb{G}_\alpha := \mathbb{G}_\alpha^2 / \mathbb{G}_\alpha^1$ is a strongly λ -free abelian group of cardinality λ ,
- (iii) if $\alpha < \mu$ and \mathbb{L} is constructible by $\{\mathbb{G}_\gamma : \gamma \in \mu \setminus \alpha\}$ over \mathbb{G}_α^1 , then there is no homomorphism $\mathbf{g} : \mathbb{G}_\alpha^2 \rightarrow \mathbb{L}$ extending $\text{id}_{\mathbb{G}_\alpha^1}$.

Proof. (1): Since S is non-reflecting, given any limit ordinal $\delta < \lambda$, there exists a club $C_\delta \subseteq \delta$ such that $S \cap C_\delta = \emptyset$. Furthermore as \diamond_S holds, and in the light of Fact 4.4, we can find a sequence $\langle S_\varepsilon : \varepsilon < \lambda \rangle$ of subsets of S such that the following two properties are satisfied:

- the sets S_ε are pairwise almost disjoint, i.e., for all $\varepsilon < \zeta < \lambda$, $S_\varepsilon \cap S_\zeta$ is a bounded subset of λ ,

- \diamond_{S_ε} holds for $\varepsilon < \lambda$.

For $\varepsilon < \lambda$ let

$$\langle (f_\alpha^\varepsilon, \mathbb{H}_\alpha^{1,\varepsilon}, \mathbb{H}_\alpha^{2,\varepsilon}, \bar{\mathbb{L}}_\alpha) : \alpha \in S_\varepsilon \rangle$$

be such that:

- $f_\alpha^\varepsilon : \alpha \rightarrow \alpha$ is a function,
- $\mathbb{H}_\alpha^{1,\varepsilon} \subseteq \mathbb{H}_\alpha^{2,\varepsilon}$ are abelian groups,
- the universe of $\mathbb{H}_\alpha^{2,\varepsilon}$ is α ,
- $\bar{\mathbb{L}}_\alpha = \langle \mathbb{L}_\varepsilon^\alpha : \varepsilon \leq \alpha \rangle$ is a construction such that \mathbb{L}_α has universe α ,
- each \mathbb{L}_ε and $\mathbb{L}_{\varepsilon+1}/\mathbb{L}_\varepsilon$ is \aleph_1 -free,
- for any tuple $(f, \mathbb{H}^1, \mathbb{H}^2, \bar{\mathbb{L}})$, where $f : \lambda \rightarrow \lambda$ is a function, $\mathbb{H}^1 \subseteq \mathbb{H}^2$ are abelian groups where the universe of \mathbb{H}^2 is λ and $\bar{\mathbb{L}} = \langle \mathbb{L}_\varepsilon : \varepsilon \leq \lambda \rangle$ is a construction such that the universe of \mathbb{L}_λ is λ and \mathbb{L}_ε and $\mathbb{L}_{\varepsilon+1}/\mathbb{L}_\varepsilon$ are \aleph_1 -free, then the following set

$$\{\alpha \in S_\varepsilon : (f \upharpoonright \alpha, \mathbb{H}^1 \upharpoonright \alpha, \mathbb{H}^2 \upharpoonright \alpha, \bar{\mathbb{L}} \upharpoonright \alpha) = (f_\alpha^\varepsilon, \mathbb{H}_\alpha^{1,\varepsilon}, \mathbb{H}_\alpha^{2,\varepsilon}, \bar{\mathbb{L}}_\alpha)\}$$

is stationary.

Given $\varepsilon < \lambda$ by induction on $\alpha < \lambda$, we choose the sequences $\bar{\mathbb{G}}_\varepsilon^\iota = \langle \mathbb{G}_{\alpha,\varepsilon}^\iota : \alpha < \lambda \rangle$, for $\iota = 1, 2$ such that:

- ⊛ (a) $\bar{\mathbb{G}}_\varepsilon^\iota$ is an increasing and continuous sequence of abelian groups $\mathbb{G}_{\alpha,\varepsilon}^\iota = (G_{\alpha,\varepsilon}^\iota, +, 0)$.
- (b) $\mathbb{G}_{0,\varepsilon}^\iota = \{0\}$ and for $\alpha < \lambda$, the universe of $\mathbb{G}_{\alpha,\varepsilon}^2$, namely $G_{\alpha,\varepsilon}^2$ is an ordinal $\gamma_{\alpha,\varepsilon} < \lambda$.
- (c) The following holds:
 - (c1) : $\mathbb{G}_{\alpha,\varepsilon}^1 \subseteq \mathbb{G}_{\alpha,\varepsilon}^2$ are free.
 - (c2) : If $\alpha < \beta$ then $\mathbb{G}_{\alpha,\varepsilon}^2 \cap \mathbb{G}_{\beta,\varepsilon}^1 = \mathbb{G}_{\alpha,\varepsilon}^1$.
 - (c3) : If $\alpha < \beta$ and $\alpha \notin S_\varepsilon$, then $\mathbb{G}_{\beta,\varepsilon}^2 / (\mathbb{G}_\beta^1 + \mathbb{G}_\alpha^2)$ is free.
- (d) If $\alpha < \beta$ and $\alpha \notin S_\varepsilon$, then $\mathbb{G}_{\beta,\varepsilon}^1 / \mathbb{G}_{\alpha,\varepsilon}^1$ and $\mathbb{G}_{\beta,\varepsilon}^2 / \mathbb{G}_{\alpha,\varepsilon}^2$ are free.
- (e) if $\delta \in S_\varepsilon$, then there are no $\dot{\mathbb{f}}$ and $\langle \dot{\mathbb{L}}_\zeta : \zeta \leq \delta \rangle$ equipped with the following properties:
 - (e1) $\langle \dot{\mathbb{L}}_\zeta : \zeta \leq \delta \rangle$ is a construction over $\mathbb{G}_{\delta+1,\varepsilon}^1$,

$$(e2) \dot{\mathbf{f}} \in \text{Hom}(\mathbb{G}_{\delta+1,\varepsilon}^2, \dot{\mathbb{L}}_\delta),$$

(e3) $\dot{\mathbf{f}}$ extends $\mathbf{f}_\delta^\varepsilon \cup \text{id}_{\mathbb{G}_{\delta+1,\varepsilon}^1}$. This means that $\mathbb{G}_{\delta,\varepsilon}^2 \subseteq \mathbb{H}_\delta^{2,\varepsilon}$, and also:

$$\begin{array}{ccc}
 & & \dot{\mathbb{L}}_\delta \\
 & \searrow^{\subseteq} & \nearrow^{\dot{\mathbf{f}}} \\
 \mathbb{G}_{\delta+1,\varepsilon}^1 & \xrightarrow{\subseteq} & \mathbb{G}_{\delta+1,\varepsilon}^2 \\
 \uparrow^{\subseteq} & & \uparrow^{\subseteq} \\
 \mathbb{G}_{\delta,\varepsilon}^1 & \xrightarrow{\subseteq} & \mathbb{G}_{\delta,\varepsilon}^2 \\
 & & \nearrow^{\mathbf{f}_\delta^\varepsilon}
 \end{array}$$

$$(e4) \mathbb{L}_\zeta^\delta \subseteq \dot{\mathbb{L}}_\zeta \text{ for } \zeta \leq \delta,$$

$$(e5) \mathbb{L}_\zeta^\delta = \mathbb{L}_{\zeta+1}^\delta \cap \dot{\mathbb{L}}_\zeta,$$

$$(e6) \dot{\mathbb{L}}_{\zeta+1}/(\mathbb{L}_{\zeta+1}^\delta + \dot{\mathbb{L}}_\zeta) \text{ is free.}$$

For $\alpha = 0$, set $\mathbb{G}_{0,\varepsilon}^1 = \mathbb{G}_{0,\varepsilon}^2 = \{0\}$. For limit ordinal δ , set $\mathbb{G}_{\delta,\varepsilon}^\iota = \bigcup_{\alpha < \delta} \mathbb{G}_{\alpha,\varepsilon}^\iota$. Let us show that items (c) and (d) continue to hold. By the induction hypothesis and for all $\alpha < \beta < \delta$ with $\alpha \notin S$, $\mathbb{G}_{\beta,\varepsilon}^\iota/\mathbb{G}_{\alpha,\varepsilon}^\iota$ and $\mathbb{G}_{\beta,\varepsilon}^2/(\mathbb{G}_{\beta,\varepsilon}^1 + \mathbb{G}_{\alpha,\varepsilon}^2)$ are free, and since we have a club C_δ of δ which is disjoint to S , it immediately follows that the groups $\mathbb{G}_{\delta,\varepsilon}^\iota$, $\mathbb{G}_{\delta,\varepsilon}^\iota/\mathbb{G}_{\alpha,\varepsilon}^\iota$ and $\mathbb{G}_{\delta,\varepsilon}^2/(\mathbb{G}_{\delta,\varepsilon}^1 + \mathbb{G}_{\alpha,\varepsilon}^2)$ are free for all $\alpha < \delta$ with $\alpha \notin S$.

Let $\iota = 1, 2$. Now suppose that $\delta < \lambda$ and we have defined the groups $\mathbb{G}_{\alpha,\varepsilon}^\iota$ and ordinals $\alpha \leq \delta$. We would like to define the groups $\mathbb{G}_{\delta+1,\varepsilon}^1$ and $\mathbb{G}_{\delta+1,\varepsilon}^2$ so that the (a)-(e) continue to hold.

In the case $\delta \notin S_\varepsilon$, we set

$$\mathbb{G}_{\delta+1,\varepsilon}^1 := \mathbb{G}_{\delta,\varepsilon}^1$$

and

$$\mathbb{G}_{\delta+1,\varepsilon}^2 := (\mathbb{G}_{\delta+1,\varepsilon}^1 \oplus_{\mathbb{G}_{\delta+1,\varepsilon}^1} \mathbb{G}_{\delta,\varepsilon}^2) \oplus \bigoplus_{n < \omega} y_{\delta,n} \mathbb{Z}.$$

It is not difficult to show that items (a)-(d) continue to hold and there is nothing to prove for case (e).

Now suppose that $\delta \in S_\varepsilon$. We define the groups $\mathbb{G}_{\delta+1,\varepsilon}^1$ and $\mathbb{G}_{\delta+1,\varepsilon}^2$ such that items (a)-(e) above continue to hold, and further we have:

- (f) $\mathbb{G}_{\delta+1,\varepsilon}^2/(\mathbb{G}_{\delta,\varepsilon}^1 + \mathbb{G}_{\delta,\varepsilon}^2)$ is not free,
 (g) if $\gamma \in \delta \setminus S_\varepsilon$, then $\mathbb{G}_{\delta+1,\varepsilon}^2/(\mathbb{G}_{\delta,\varepsilon}^1 + \mathbb{G}_{\gamma,\varepsilon}^2)$ is free.

As $\delta \in S_\varepsilon$, we have $\text{cf}(\delta) = \aleph_0$, so let $\langle \gamma_{\delta,n} : n < \omega \rangle$ be an increasing sequence of successor ordinals $< \delta$ with limit δ .

For any $n < \omega$, $\gamma_{\delta,n} \notin S_\varepsilon$, it is easily seen that the abelian groups $\mathbb{G}_{\gamma_{\delta,n},\varepsilon}^1 \subseteq \mathbb{G}_{\gamma_{\delta,n},\varepsilon}^2$ satisfy the hypotheses of Lemma 4.5, hence by the lemma, we can find the groups $\mathbb{G}_{\delta+1,\varepsilon}^1 \subseteq \mathbb{G}_{\delta+1,\varepsilon}^2$ such that there for all constructions $\mathbb{L} \supseteq \mathbb{G}_{\delta+1,\varepsilon}^1$ as described in (e), there is no $\mathbf{f} \in \text{Hom}(\mathbb{G}_{\delta+1,\varepsilon}^2, \mathbb{L})$ such that $\mathbf{f} \supseteq \mathbf{f}_\delta^\varepsilon \cup \text{id}_{\mathbb{G}_{\delta+1,\varepsilon}^1}$.

This finishes our inductive construction. For $\iota = 1, 2$ and $\varepsilon < \lambda$, we define:

$$\mathbb{G}_\varepsilon^\iota = \bigcup_{\alpha < \lambda} \mathbb{G}_{\alpha,\varepsilon}^\iota.$$

Set also

- (h1): $\mathbb{G}_{\alpha,\varepsilon} = \mathbb{G}_{\alpha,\varepsilon}^2/\mathbb{G}_{\alpha,\varepsilon}^1$,
 (h2): $\mathbb{G}_\varepsilon = \bigcup_{\alpha < \lambda} \mathbb{G}_{\alpha,\varepsilon}$,
 (h3): $\bar{\mathbb{G}} = \langle \mathbb{G}_\varepsilon : \varepsilon < \lambda \rangle$.

Let us show that the properties.

(i): This is clear.

(ii): We apply the properties taken from items (c) and (d) of the construction, along with freeness $\mathbb{G}_\varepsilon^1 \subseteq \mathbb{G}_\varepsilon^2$ to deduce \mathbb{G}_ε is strongly λ -free as witnessed by the sequence

$$\mathcal{S}_\varepsilon = \{\mathbb{G}_{\alpha,\varepsilon} : \alpha \in \lambda \setminus S_\varepsilon\},$$

for more details, see [2, IV.1.11].

(iii): Suppose by contradiction that there are $\varepsilon < \lambda$, a construction $\bar{\mathbb{L}} = \langle \mathbb{L}_\varepsilon : \varepsilon \leq \lambda \rangle$ by $\{\mathbb{G}_\alpha : \alpha \in \lambda \setminus \varepsilon\}$ over \mathbb{G}_ε^1 and there is a homomorphism \mathbf{g} from \mathbb{G}_ε^2 into \mathbb{L}_λ which extends $\text{id}_{\mathbb{G}_\varepsilon^1}$:

$$\begin{array}{ccc} \mathbb{G}_\alpha^1 & \xrightarrow{\subseteq} & \mathbb{G}_\alpha^2 \\ \text{id} \downarrow & & \downarrow \mathbf{g} \\ \mathbb{G}_\alpha^1 & \xrightarrow{\subseteq} & \mathbb{L} \end{array}$$

Without loss of generality we can assume that \mathbb{L}_λ has size λ and that its universe is λ . The set

$$E = \{\delta < \lambda : \mathbf{g} \upharpoonright \delta : \delta \rightarrow \delta \text{ and } \bar{\mathbb{L}} \upharpoonright \delta = \bar{\mathbb{L}}_\delta \upharpoonright \delta \text{ and } \mathbb{L}_\lambda \upharpoonright \delta \text{ is a subgroup of } \mathbb{L}\}$$

is a club, thus we can find some $\delta \in E \cap S_\varepsilon$ such that:

- (i1): $\mathbf{g} \upharpoonright \delta = f_\delta^\varepsilon$,
- (i2): $\mathbb{L}_\kappa \upharpoonright \delta = \mathbb{H}_\delta^{2,\varepsilon}$,
- (i3): $\mathbb{H}_\delta^{1,\varepsilon} = \mathbb{G}_{\delta,\varepsilon}^1$,
- (i4): $\mathbb{G}_\varepsilon^2 \cap \delta = \mathbb{G}_{\delta,\varepsilon}^2$.

Now note that $f = \mathbf{g} \upharpoonright \mathbb{G}_{\delta+1,\varepsilon}^2 : \mathbb{G}_{\delta+1,\varepsilon}^2 \rightarrow \mathbb{L}_\lambda$ is such that $f_\delta^\varepsilon \cup \text{id}_{\mathbb{G}_{\delta+1,\varepsilon}^1} \subseteq f$, which is in contradiction with clause (e) of the construction.

(2): The proof is similar to the proof of (1), this time, we find the following family

$$\langle S_\varepsilon : \varepsilon < 2^\lambda \rangle$$

of almost disjoint subsets of λ such that \diamond_{S_ε} holds for all ε . By [1], such a sequence exists. \square

Now, we are ready to prove Theorem (B):

Theorem 4.8. *Let $S \subseteq S_{\aleph_0}^\lambda$ be stationary non-reflecting and suppose \diamond_S holds.*

Suppose one of the followings:

- (1) $\mu = \lambda = \text{cf}(\lambda) > \aleph_0$, or
- (2) $\mu = 2^\lambda, \lambda = \text{cf}(\lambda) > \aleph_0$.

Then there are sequences $\langle \mathbb{G}_\alpha : \alpha < \mu \rangle$ and $\langle \mathbb{K}_\alpha : \alpha < \mu \rangle$ of λ -free abelian groups such that for all $\alpha < \mu$, $|\mathbb{G}_\alpha| = \lambda$, $|\mathbb{K}_\alpha| = 2^\lambda$ and for all $\alpha, \beta < \mu$,

$$\text{Ext}(\mathbb{G}_\alpha, \mathbb{K}_\beta) = 0 \Leftrightarrow \alpha R \beta.$$

Proof. This follows from Theorem 3.5 and Theorem 4.7. \square

§ 5. REPRESENTING A BIPARTITE GRAPH BY EXT IN ZFC

In this section we show that it is possible to remove the diamond principle from the construction of Section 4 and get ZFC result. The main result is Theorem 5.5. This answers Herden's question for the case of bipartite graphs. We will do this by using a simple version of Shelah's black box. In this case, the groups \mathbb{G}_α that we construct are not λ -free, but just \aleph_1 -free.

Notation 5.1. Let χ be infinite cardinal. By $\mathcal{H}(\chi)$ we mean the collection of sets of hereditary cardinality less than χ .

Let us start by stating the version of the black box we are using in this paper.

Theorem 5.2. *Let χ , λ and μ be infinite cardinals such that $\lambda = \mu^+$, $\mu^{\aleph_0} = \mu$, and E_0, \dots, E_{m-1} are pairwise disjoint stationary subsets of λ consisting of ordinals of cofinality ω , and $\chi > \lambda$. Let N be an expansion in a countable language of $(\mathcal{H}(\chi), \in, \triangleleft, \lambda)$ where \triangleleft is a well ordering of $\mathcal{H}(\chi)$. Then there is a family of countable sets $\{(M_i, X_i) : i \in I\}$ such that the following properties hold:*

- (a) $M_i \prec N$ and $X_i \subset \lambda$.
- (b) Let $\delta(i) := \sup(M_i \cap \lambda)$. If $\delta(i) = \delta(j)$, then $(M_i, X_i) \cong (M_j, X_j)$ and $M_i \cap M_j \cap \lambda$ is an initial segment of $M_i \cap \lambda$.
- (c) For all $X \subset \lambda$, all $\ell < m$, the following set

$$\{\delta \in E_\ell : \exists i \text{ such that } \delta(i) = \delta \text{ and } (M_i, X_i) \equiv_{M_i \cap \lambda} (N, X)\}$$

is stationary in λ .

Proof. See [2, Page 444]. □

Notation 5.3. Given a torsion-free group \mathbb{G} and a subgroup $\mathbb{H} \subseteq \mathbb{G}$. By \mathbb{H}_* we mean the pure-closure, i.e., the smallest pure subgroup of \mathbb{G} containing \mathbb{H} . In fact,

$$\mathbb{H}_* = \{g \in \mathbb{G} : ng \in \mathbb{H} \text{ for some nonzero } n \in \mathbb{Z}\}.$$

The next task is to construct $\boxtimes_{\lambda, \mu}^1$ from Theorem 3.5 in ZFC:

Theorem 5.4. *Adopt one of the following assumptions:*

- (1) If $\lambda = \aleph_0, \mu = \lambda$,
(2) If $\lambda = \aleph_0, \mu = 2^\lambda$.

Then $\boxtimes_{\lambda, \mu}^1$ from Lemma 3.5 holds.

Proof. (1): By a result of Solovay (see [2, Corollary II.4.9]) we can find a partition $\langle S_\varepsilon : \varepsilon < \lambda \rangle$ of λ into λ many disjoint stationary sets. Let

$$\mathbb{C}_\varepsilon := \bigoplus_{\nu < \lambda, n < \omega} (y_{\varepsilon, \nu, n} \mathbb{Z} \oplus z_{\varepsilon, \nu, n} \mathbb{Z})$$

and recall that its \mathbb{Z} -adic completion is denoted by $\widehat{\mathbb{C}}_\varepsilon$. Clearly, $\widehat{\mathbb{C}}_\varepsilon$ has cardinality λ , so we identify it with λ . Let also

$$Y_\varepsilon := \{y_{\varepsilon, \nu, n} : \nu < \lambda, n < \omega\} \cup \{z_{\varepsilon, \nu, n} : \nu < \lambda, n < \omega\}.$$

Set $\chi = \lambda^{+3}$. The initial structure for the Black box, corresponding to the stationary set S_ε is as follows:

$$N_\varepsilon := (\mathcal{H}(\chi), \in, \triangleleft, \lambda, \widehat{\mathbb{C}}_\varepsilon, Y_\varepsilon),$$

where $\widehat{\mathbb{C}}_\varepsilon$ denotes the 3-ary relation on λ which is the graph of the addition operation on the group $\widehat{\mathbb{C}}_\varepsilon$. We take the first bijection $\mathbf{g}_\varepsilon : \lambda \times \lambda \rightarrow \lambda$ with respect to \triangleleft , and use it to identify each $X \subseteq \lambda$ with a subset of $\widehat{\mathbb{C}}_\varepsilon \times \widehat{\mathbb{C}}_\varepsilon$. Let $\{(M_i^\varepsilon, X_i^\varepsilon) : i \in I_\varepsilon\}$ be as in the statement of Theorem 5.2 when $m = 1$ and $E_0 = S_\varepsilon$, and note that for each $i \in I_\varepsilon$ and $\varepsilon < \lambda$, $\mathbf{g}_\varepsilon \in M_i^\varepsilon$.

Let $\iota = 1, 2$. We proceed as in the previous section and for a given $\varepsilon < \lambda$, by induction on $\alpha < \lambda$, we choose the sequences $\bar{\mathbb{G}}_\varepsilon^\iota = \langle \mathbb{G}_{\alpha, \varepsilon}^\iota : \alpha < \lambda \rangle$ equipped with the following five items:

- (a) $\bar{\mathbb{G}}_\varepsilon^\iota$ is an increasing and continuous sequence of abelian groups $\mathbb{G}_{\alpha, \varepsilon}^\iota = (G_{\alpha, \varepsilon}^\iota, +, 0)$.
(b) $\mathbb{G}_{0, \varepsilon}^\iota = \{0\}$ and for $\alpha < \lambda$, the universe of $\mathbb{G}_{\alpha, \varepsilon}^2$, namely $G_{\alpha, \varepsilon}^2$ is an ordinal $\gamma_{\alpha, \varepsilon} < \lambda$.
(c) The following three properties hold:
(c1): $\mathbb{G}_{\alpha, \varepsilon}^1 \subseteq \mathbb{G}_{\alpha, \varepsilon}^2$ are \aleph_1 -free,
(c2): if $\alpha < \beta$ then $\mathbb{G}_{\alpha, \varepsilon}^2 \cap \mathbb{G}_{\beta, \varepsilon}^1 = \mathbb{G}_{\alpha, \varepsilon}^1$,

- (c3): if $\alpha < \beta$ and $\alpha \notin S_\varepsilon$, then $\mathbb{G}_\beta^2 / (\mathbb{G}_\beta^1 + \mathbb{G}_\alpha^2)$ is \aleph_1 -free.
- (d) If $\alpha < \beta$ and $\alpha \notin S_\varepsilon$, then $\mathbb{G}_{\beta,\varepsilon}^1 / \mathbb{G}_{\alpha,\varepsilon}^1$ and $\mathbb{G}_{\beta,\varepsilon}^2 / \mathbb{G}_{\alpha,\varepsilon}^2$ are \aleph_1 -free.
- (e) If $\delta \in S_\varepsilon$, then there are no f and $\langle \dot{\mathbb{L}}_\zeta : \zeta \leq \delta \rangle$ equipped with the following properties:
- (e1) $\langle \dot{\mathbb{L}}_\zeta : \zeta \leq \delta \rangle$ is a construction over $\mathbb{G}_{\delta+1,\varepsilon}^1$,
 - (e2) $f \in \text{Hom}(\mathbb{G}_{\delta+1,\varepsilon}^2, \dot{\mathbb{L}}_\delta)$,
 - (e3) f extends $f_{\delta,i}^\varepsilon \cup \text{id}_{\mathbb{G}_{\delta+1,\varepsilon}^1}$ where $i \in I_\varepsilon$ and

$$\langle f_{\delta,i}^\varepsilon, \bar{\mathbb{L}}_{\delta,i}^\varepsilon = \langle \mathbb{L}_{\delta,i,\zeta}^\varepsilon : \zeta \leq \delta \rangle \rangle$$

is coded by X_i^ε , under the identification given by \mathbf{g}_ε . Also, $f_{\delta,i}^\varepsilon$ is in $\text{Hom}(\mathbb{G}_{\delta,\varepsilon}^2, \mathbb{L}_{\delta,i,\delta}^\varepsilon)$ and $\bar{\mathbb{L}}_{\delta,i}^\varepsilon$ is a construction:

$$\begin{array}{ccccc}
 & & \dot{\mathbb{L}}_\delta & & \\
 & \subseteq & \uparrow & \swarrow f_{\delta,i}^\varepsilon & \\
 \mathbb{G}_{\delta+1,\varepsilon}^1 & \xrightarrow{\subseteq} & \mathbb{G}_{\delta+1,\varepsilon}^2 & \xleftarrow{\subseteq} & \mathbb{G}_{\delta,\varepsilon}^2 \\
 & & \uparrow \neq f & & \\
 & & \dot{\mathbb{L}}_\delta & &
 \end{array}$$

- (e4) $\mathbb{L}_\zeta^\delta \subseteq \dot{\mathbb{L}}_\zeta$ for $\zeta \leq \delta$,
- (e5) $\mathbb{L}_\zeta^\delta = \mathbb{L}_{\zeta+1}^\delta \cap \dot{\mathbb{L}}_\zeta$,
- (e6) $\dot{\mathbb{L}}_{\zeta+1} / (\mathbb{L}_{\zeta+1}^\delta + \dot{\mathbb{L}}_\zeta)$ is free.

For $\alpha = 0$, set $\mathbb{G}_{0,\varepsilon}^1 = \mathbb{G}_{0,\varepsilon}^2 = \{0\}$. For the limit ordinal δ , we set $\mathbb{G}_{\delta,\varepsilon}^\iota := \bigcup_{\alpha < \delta} \mathbb{G}_{\alpha,\varepsilon}^\iota$. Let us show that items (c) and (d) continue to hold. By the induction hypothesis and for all $\alpha < \beta < \delta$ with $\alpha \notin S$, $\mathbb{G}_{\beta,\varepsilon}^\iota / \mathbb{G}_{\alpha,\varepsilon}^\iota$ and $\mathbb{G}_\beta^2 / (\mathbb{G}_\beta^1 + \mathbb{G}_\alpha^2)$ are \aleph_1 -free, and since we have a club C_δ of δ which is disjoint to S , it immediately follows from Lemma 3.1 that the groups $\mathbb{G}_{\delta,\varepsilon}^\iota$, $\mathbb{G}_{\delta,\varepsilon}^\iota / \mathbb{G}_{\alpha,\varepsilon}^\iota$ and $\mathbb{G}_{\delta,\varepsilon}^2 / (\mathbb{G}_{\delta,\varepsilon}^1 + \mathbb{G}_{\alpha,\varepsilon}^2)$ are \aleph_1 -free for all $\alpha < \delta$ with $\alpha \notin S_\varepsilon$.

Now suppose that $\delta < \lambda$ and we have defined the groups $\mathbb{G}_{\alpha,\varepsilon}^\iota$ for $\iota = 1, 2$ and ordinals $\alpha \leq \delta$. We would like to define the groups $\mathbb{G}_{\delta+1,\varepsilon}^1$ and $\mathbb{G}_{\delta+1,\varepsilon}^2$ so that the (a)-(e) continue to hold.

If $\delta \notin S_\varepsilon$, then we set

$$\mathbb{G}_{\delta+1,\varepsilon}^1 := \mathbb{G}_{\delta,\varepsilon}^1$$

and

$$\mathbb{G}_{\delta+1,\varepsilon}^2 := (\mathbb{G}_{\delta+1,\varepsilon}^1 \oplus_{\mathbb{G}_{\delta+1,\varepsilon}^1} \mathbb{G}_{\delta,\varepsilon}^2) \oplus \bigoplus_{n < \omega} y_{\varepsilon,\delta,n} \mathbb{Z}.$$

We leave to the reader to check that the items presented from (a) to (d) all are valid, and recall that there is nothing to prove for case (e).

Now suppose that $\delta \in S_\varepsilon$. We define the groups $\mathbb{G}_{\delta+1,\varepsilon}^1$ and $\mathbb{G}_{\delta+1,\varepsilon}^2$ such that items (a)-(e) above continue to hold, and further we have:

- (f) $\mathbb{G}_{\delta+1,\varepsilon}^2 / (\mathbb{G}_{\delta,\varepsilon}^1 + \mathbb{G}_{\delta,\varepsilon}^2)$ is not free,
- (g) if $\gamma \in \delta \setminus S_\varepsilon$, then $\mathbb{G}_{\delta+1,\varepsilon}^2 / (\mathbb{G}_{\delta,\varepsilon}^1 + \mathbb{G}_{\gamma,\varepsilon}^2)$ is \aleph_1 -free.

As $\delta \in S_\varepsilon$, we have $\text{cf}(\delta) = \aleph_0$, so let $\langle \gamma_{\delta,n} : n < \omega \rangle$ be an increasing sequence of successor ordinals $< \delta$ with limit δ .

Let $n < \omega$ be such that $\gamma_{\delta,n} \notin S_\varepsilon$. It turns out that the abelian groups $\mathbb{G}_{\gamma_{\delta,n},\varepsilon}^1 \subseteq \mathbb{G}_{\gamma_{\delta,n},\varepsilon}^2$ are suited well in the hypotheses of Lemma 4.5.

The notation $\Sigma_{\delta,\varepsilon}$ stands for the following set:

$$\{i \in I_\varepsilon : \delta(i) = \delta \text{ and } X_i^\varepsilon \text{ codes } \langle f_{\delta,i}^\varepsilon, \bar{\mathbb{L}}_{\delta,i}^\varepsilon = \langle \mathbb{L}_{\delta,i,\zeta}^\varepsilon : \zeta \leq \delta \rangle \rangle \text{ as in (e3)}\}.$$

Given any $i \in \Sigma_{\delta,\varepsilon}$, and according to Lemma 4.5, we can find the \aleph_1 -free groups $\mathbb{G}_{\delta+1,\varepsilon}^{1,i} \subseteq \mathbb{G}_{\delta+1,\varepsilon}^{2,i}$ such that there for all constructions $\langle \mathbb{L}_\zeta : \zeta \leq \delta \rangle$ over $\mathbb{G}_{\delta+1,\varepsilon}^{1,i}$ as in item (e), if we set $\mathbb{L} = \dot{\mathbb{L}}_\delta$, then there is no $\mathbf{f} \in \text{Hom}(\mathbb{G}_{\delta+1,\varepsilon}^{2,i}, \mathbb{L})$ such that $\mathbf{f} \supseteq f_{\delta,i}^\varepsilon \cup \text{id}_{\mathbb{G}_{\delta+1,\varepsilon}^{1,i}}$. For $\iota = 1, 2$ we look at

$$\mathbb{G}_{\delta+1,\varepsilon}^\iota := \langle \mathbb{G}_{\delta,\varepsilon}^\iota \cup \bigcup \{ \mathbb{G}_{\delta+1,\varepsilon}^{\iota,i} : i \in \Sigma_{\delta,\varepsilon} \} \rangle_*$$

i.e., the pure closure of $\langle \mathbb{G}_{\delta,\varepsilon}^\iota \cup \bigcup \{ \mathbb{G}_{\delta+1,\varepsilon}^{\iota,i} : i \in \Sigma_{\delta,\varepsilon} \} \rangle$ in $\widehat{\mathbb{C}}_\varepsilon$. Let us show that the hypothesis (a)-(e) hold. We first show that the group $\frac{\mathbb{G}_{\delta+1,\varepsilon}^2}{\mathbb{G}_{\alpha,\varepsilon}^2}$ is \aleph_1 -free, provided that α is a successor ordinal. Let \mathbb{K} be any countable subgroup of $\mathbb{G}_{\delta+1,\varepsilon}^2$. We are going to show that $\frac{\mathbb{K} + \mathbb{G}_{\delta+1,\varepsilon}^2}{\mathbb{G}_{\alpha,\varepsilon}^2}$ is free. There is an ω -sequence $\{i_m : m \in \omega\}$ together with a countable subgroup $\mathbb{I} \subset \mathbb{G}_{\delta,\varepsilon}^2$ such that \mathbb{K} is the subgroup generated by \mathbb{I} together with some countable subset $\{w_{n,i_m} : n, m \in \omega\}$ of $\bigcup_{i \in \Sigma_{\delta,\varepsilon}} \mathbb{G}_{\delta+1,\varepsilon}^{\iota,i}$. We can assume that for all n, m , $y_{\varepsilon,\alpha_n,i_m} \in \mathbb{I}$. Choose an increasing sequence of ordinals

$\{\alpha_k : k < \omega\}$ with limit δ such that $\alpha_0 = \alpha$ and for all $m \in \omega$ and all but finitely many of n , $\alpha_{n, i_m} \in \{\alpha_k : k < \omega\}$. Notice that for any successor ordinal $\gamma < \delta$

$$\mathbb{I}/(\mathbb{I} \cap \mathbb{G}_{\gamma, \varepsilon}^2) \cong (\mathbb{I} + \mathbb{G}_{\gamma, \varepsilon}^2)/\mathbb{G}_{\gamma, \varepsilon}^2$$

which is free by the induction hypothesis. So for such γ , $\mathbb{I} \cap \mathbb{G}_{\gamma, \varepsilon}^2$ is a direct summand of \mathbb{I} . Inductively choose subgroups \mathbb{I}_k so that:

$$\mathbb{I} \cap \mathbb{G}_{\alpha_{k+1}, \varepsilon}^2 \oplus \mathbb{I}_k = \mathbb{I} \cap \mathbb{G}_{\alpha_{k+1}, \varepsilon}^2$$

for all k . Hence

$$\mathbb{I} = \bigoplus_k \mathbb{I}_k \oplus \bigoplus \mathbb{Z}y_{\varepsilon, \alpha_k, n}.$$

In view of Theorem 5.2(b), we are able to choose $\{n(m) : m < \omega\}$ so that:

- the collections $\{\alpha_{n, i_m} : n(m) < n\}$ are pairwise disjoint.
- $\{\alpha_{n, i_m} : n(m) < n\} \subset \{\alpha_k : k < \omega\}$.

We observe that $\frac{\mathbb{K} + \mathbb{G}_{\delta, \varepsilon}^2}{\mathbb{G}_{\delta, \varepsilon}^2}$ is isomorphic to the direct sum of $\bigoplus_k \mathbb{I}_k$ together with the group freely generated by

$$\{w_{n, i_m} : n(m) \leq n \text{ and } m \in \omega\}$$

and

$$\{y_{\varepsilon, \alpha_k, n} : \forall m < \omega \text{ and } n(m) \leq n, \alpha_k \neq \alpha_{n, i_m}\}.$$

From this, the claim follows. By a similar argument, the group $\frac{\mathbb{G}_{\delta+1, \varepsilon}^1}{\mathbb{G}_{\alpha, \varepsilon}^1}$ is \aleph_1 -free, provided that α is a successor ordinal.

In the same vein, we also observe that the group $\mathbb{G}_{\delta+1, \varepsilon}^1$ is \aleph_1 -free and the hypotheses (a)-(e) continue to hold.

The rest of the argument is similar to Theorem 4.7. Let us elaborate the main idea of the proof. For $\iota = 1, 2$ and $\varepsilon < \lambda$, we define:

$$\mathbb{G}_{\varepsilon}^{\iota} = \bigcup_{\alpha < \lambda} \mathbb{G}_{\alpha, \varepsilon}^{\iota}.$$

Set also

$$(h1): \mathbb{G}_{\alpha, \varepsilon} = \mathbb{G}_{\alpha, \varepsilon}^2 / \mathbb{G}_{\alpha, \varepsilon}^1,$$

$$(h2): \mathbb{G}_{\varepsilon} = \bigcup_{\alpha < \lambda} \mathbb{G}_{\alpha, \varepsilon},$$

$$(h3): \bar{\mathbb{G}} = \langle \mathbb{G}_{\varepsilon} : \varepsilon < \lambda \rangle.$$

Let us show that $\boxtimes_{\lambda, \mu_1}^1$ is satisfied. By items (c) and (d) of the construction, $\mathbb{G}_\varepsilon^1 \subseteq \mathbb{G}_\varepsilon^2$ are \aleph_1 -free and \mathbb{G}_ε is \aleph_1 -free as well. Let us now show that $\boxtimes_{\lambda, \mu_1}^1(c)$ is satisfied as well. Suppose by contradiction that there are $\varepsilon < \lambda$, a construction \mathbb{L} by $\{\mathbb{G}_\alpha : \alpha \in \lambda \setminus \varepsilon\}$ over \mathbb{G}_ε^1 , witnessed by $\langle \mathbb{L}_\varepsilon : \varepsilon \leq \lambda \rangle$, and there is a homomorphism \mathbf{g} from \mathbb{G}_ε^2 into \mathbb{L} which extends $\text{id}_{\mathbb{G}_\varepsilon^1}$:

$$\begin{array}{ccc} \mathbb{G}_\alpha^1 & \xrightarrow{\subseteq} & \mathbb{G}_\alpha^2 \\ \downarrow = & & \downarrow \mathbf{g} \\ \mathbb{G}_\alpha^1 & \xrightarrow{\subseteq} & \mathbb{L} \end{array}$$

Without loss of generality we can assume that \mathbb{L} has size λ and that its universe is λ . Let $X \subseteq \lambda$ be codes $\langle \mathbf{g}, \langle \mathbb{L}_\varepsilon : \varepsilon \leq \lambda \rangle \rangle$. The set

$$E := \{\delta < \lambda : \mathbf{g} \upharpoonright \delta : \delta \rightarrow \delta, X \cap \delta \text{ codes } \langle \mathbf{g} \upharpoonright \delta, \langle \mathbb{L}_\zeta : \zeta \leq \delta \rangle \rangle \text{ and } \mathbb{L} \upharpoonright \delta \subseteq_{\text{group}} \mathbb{L}\}$$

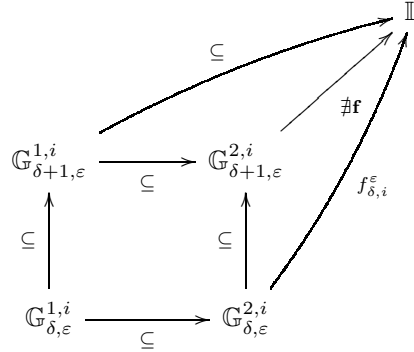
is a club, thus we can find some $\delta \in E \cap S_\varepsilon$ and some i with $\delta(i) = i$ such that $(M_i^\varepsilon, X_i^\varepsilon) \equiv_{M_i^\varepsilon \cap \lambda} (N_\varepsilon, X)$. It then follows that $M_i^\varepsilon \cap \mathbb{G}_\varepsilon^2 = M_i^\varepsilon \cap \mathbb{G}_{\delta, \varepsilon}^{2, i}$, and since $(M_i^\varepsilon, X_i^\varepsilon) \equiv_{M_i^\varepsilon \cap \lambda} (N_\varepsilon, X)$, we can easily observe that X_i^ε codes

$$\langle \mathbf{g} \upharpoonright M_i^\varepsilon \cap \mathbb{G}_{\delta, \varepsilon}^{2, i}, \langle \mathbb{L}_\zeta \cap M_i^\varepsilon : \zeta \leq \lambda \rangle \rangle.$$

Now by elementarily and the choice of \mathbf{g} ,

$$(M_i^\varepsilon, X_i^\varepsilon) \models \text{“} X_i^\varepsilon \text{ codes a homomorphism } f_{\delta, i}^\varepsilon \text{ from } \mathbb{G}_{\delta, \varepsilon}^{2, i} \text{”}$$

It follows that $i \in \Sigma_{\delta, \varepsilon}$, and hence by our construction, there is no $\mathbf{f} \in \text{Hom}(\mathbb{G}_{\delta+1, \varepsilon}^{2, i}, \mathbb{L})$ such that \mathbf{f} extends $f_{\delta, i}^\varepsilon \cup \text{id}_{\mathbb{G}_{\delta+1, \varepsilon}^{1, i}}$. This property is conveniently summarized by the subjoined diagram:



This is not possible, as \mathbf{g} is such an extension. This shows that there is no homomorphism \mathbf{g} as above and the result follows.

(2): This is similar to the proof of (1). Take a sequence $\langle S_\epsilon : \epsilon < 2^\lambda \rangle$ of almost disjoint subsets of λ such that each S_ϵ is stationary and proceed as before. \square

Now, we are ready to prove Theorem (C) from §1:

Theorem 5.5. *Let $\lambda = \lambda^{\aleph_0}$, $\mu = 2^\lambda$ and let $R \subseteq \mu \times \mu$ be a relation. Then there are families $\langle \mathbb{G}_\alpha : \alpha < \mu \rangle$, and $\langle \mathbb{K}_\alpha : \alpha < \mu \rangle$ of \aleph_1 -free abelian groups such that:*

- (1) *for all $\alpha < \mu$, \mathbb{G}_α has size λ and \mathbb{K}_α has size 2^λ ,*
- (2) *for all $\alpha, \beta < \mu$,*

$$\text{Ext}(\mathbb{G}_\alpha, \mathbb{K}_\beta) = 0 \iff \alpha R \beta.$$

Proof. This follows using Theorem 3.5 and Theorem 5.4. \square

Let us close this section by showing the relation between Theorem 5.5 and the result of Göbel, Shelah and Wallutis [5] stated in the introduction.

Lemma 5.6. *Let μ be an infinite cardinal. Then (1) implies (2), where:*

- (1) *There are abelian groups $\langle \mathbb{G}_X, \mathbb{H}^X : X \subseteq \mu \rangle$ such that for all $X, Y \subseteq \mu$,*

$$\text{Ext}(\mathbb{G}_Y, \mathbb{H}^X) = 0 \iff Y \subseteq X.$$

- (2) *If (μ, R) is a bipartite graph, then there are families $\langle \mathbb{G}_\alpha : \alpha < \mu \rangle$, and $\langle \mathbb{K}_\alpha : \alpha < \mu \rangle$ of abelian groups such that for all $\alpha, \beta < \mu$,*

$$\text{Ext}(\mathbb{G}_\alpha, \mathbb{K}_\beta) = 0 \iff \alpha R \beta.$$

Proof. Suppose (1) holds as witnessed by the sequence $\langle \mathbb{G}_X, \mathbb{H}^X : X \subseteq \mu \rangle$ and let (μ, R) be a bipartite graph. Given any $\alpha, \beta < \mu$ set $Y_\alpha = \{\alpha\}$ and $X_\beta = \{\alpha < \mu : \alpha R \beta\}$. Now define

- i) $\mathbb{G}_\alpha := \mathbb{G}_{Y_\alpha}$ and
- ii) $\mathbb{K}_\alpha := \mathbb{H}^{X_\alpha}$.

This is now straightforward to see

$$\text{Ext}(\mathbb{G}_\alpha, \mathbb{K}_\beta) = 0 \iff Y_\alpha \subseteq X_\beta \iff \alpha R \beta.$$

Thus, the family $\langle \mathbb{G}_\alpha, \mathbb{K}_\alpha : \alpha < \mu \rangle$ is as required. \square

The converse of the above lemma also holds in the following sense:

Lemma 5.7. *Let μ be an infinite cardinal and set $\lambda = 2^\mu$. Then (1) implies (2), where:*

- (1) *If (λ, R) is a bipartite graph, then there are families $\langle \mathbb{G}_\alpha : \alpha < \lambda \rangle$, and $\langle \mathbb{K}_\alpha : \alpha < \lambda \rangle$ of abelian groups such that for all $\alpha, \beta < \lambda$,*

$$\text{Ext}(\mathbb{G}_\alpha, \mathbb{K}_\beta) = 0 \iff \alpha R \beta.$$

- (2) *There are abelian groups $\langle \mathbb{G}_X, \mathbb{H}^X : X \subseteq \mu \rangle$ such that for all $X, Y \subseteq \mu$,*

$$\text{Ext}(\mathbb{G}_Y, \mathbb{H}^X) = 0 \iff Y \subseteq X.$$

Proof. Suppose (1) holds. Let $\langle X_\alpha : \alpha < \lambda \rangle$ and $\langle Y_\beta : \beta < \lambda \rangle$ be two enumerations of $\mathcal{P}(\mu)$. Let us define the bipartite graph (λ, R) as

$$\alpha R \beta \iff Y_\alpha \subseteq X_\beta.$$

By (1), there exists a family $\langle \mathbb{G}_\alpha, \mathbb{K}_\alpha : \alpha < \lambda \rangle$, of abelian groups such that for all $\alpha, \beta < \lambda$,

$$\text{Ext}(\mathbb{G}_\alpha, \mathbb{K}_\beta) = 0 \iff \alpha R \beta.$$

For $X, Y \subseteq \mu$, let $\alpha, \beta < \lambda$ be such that $X_\beta = X$ and $Y_\alpha = Y$ and set

- $\mathbb{G}_Y := \mathbb{G}_\alpha$;
- $\mathbb{H}^X := \mathbb{K}_\beta$.

Then

$$\text{Ext}(\mathbb{G}_Y, \mathbb{H}^X) = 0 \iff \text{Ext}(\mathbb{G}_\alpha, \mathbb{K}_\beta) = 0 \iff \alpha R \beta \iff Y \subseteq X.$$

The lemma follows. \square

§ 6. REPRESENTING A GENERAL GRAPH BY EXT

In this section we consider general graphs and discuss if they can be represented by Ext as before. We do not know the result in ZFC, but we show the following consistency result which shows that there are no restrictions on such graphs in ZFC, as promised by Theorem (D) from the introduction:

Theorem 6.1. *Suppose GCH holds and the pair (S, R) is a graph where $R \subseteq S \times S$, and let $\lambda > |S|$ be an uncountable regular cardinal. Then there exists a cardinal preserving generic extension of the universe, and there is a family $\{\mathbb{G}_s\}_{s \in S}$ of λ -free abelian groups such that*

$$\text{Ext}(\mathbb{G}_s, \mathbb{G}_t) = 0 \iff s R t.$$

The rest of this section is devoted to the proof of the above theorem. Before we go into the details, let us sketch the idea of the proof:

Discussion 6.2. We first define a forcing notion \mathbb{P}_* which adds a sequence $\langle \mathbb{G}_s : s \in S \rangle$ of λ -free abelian groups of size λ such that for each $s, t \in S$ if $(s, t) \notin R$, then for some abelian group $\mathbb{H}_{s,t}$ of size λ there exists an exact sequence

$$0 \longrightarrow \mathbb{G}_s \longrightarrow \mathbb{H}_{s,t} \longrightarrow \mathbb{G}_t \longrightarrow 0$$

which does not split. We apply this along with Fact 2.1 to deduce that $\text{Ext}(\mathbb{G}_s, \mathbb{G}_t) \neq 0$. Then working in the generic extension by \mathbb{P}_* , we define a cardinal preserving λ -support iteration forcing notion of length λ^+ , which makes $\text{Ext}(\mathbb{G}_s, \mathbb{G}_t) = 0$ for all $s, t \in S$ with $s R t$. This is done by adding a splitter for any exact sequence

$$0 \longrightarrow \mathbb{G}_s \longrightarrow \mathbb{H} \longrightarrow \mathbb{G}_t \longrightarrow 0,$$

where \mathbb{H} is an abelian group of size λ . By using a suitable book-keeping argument, we make sure that at the end all such exact sequences are considered for all pairs $(s, t) \in R$. We also show that the exact sequence

$$0 \longrightarrow \mathbb{G}_s \longrightarrow \mathbb{H}_{s,t} \longrightarrow \mathbb{G}_t \longrightarrow 0$$

still fails to split after the iteration, which will complete the proof.

Let us now go into the details of the proof. Recall from Notation 5.1 that $\mathcal{H}(\chi)$ is the collection of sets of hereditary cardinality less than χ .

Notation 6.3. Let $\Phi : \lambda^+ \rightarrow \mathcal{H}(\lambda^+)$ be such that $\Phi^{-1}[x] \subseteq \lambda^+$ is unbounded for all $x \in \mathcal{H}(\lambda^+)$.

The existence of Φ follows by the GCH assumption. We will use Φ as our book-keeping function. Here, we define the forcing notion \mathbb{P}_* .

Definition 6.4. Let $\lambda = \text{cf}(\lambda) > \aleph_0$ be an uncountable regular cardinal, and let S be a set of cardinality $< \lambda$. Finally, let $R \subseteq S \times S$.

(a) The forcing notion \mathbb{P}_* consists of conditions

$$p = \langle \langle \mathbb{G}_{s,\beta}^p : s \in S, \beta \leq \alpha_p \rangle, E_p, \langle \mathbf{x}_{s,t,\beta}^p : (s,t) \notin R, \beta \leq \alpha_p \rangle \rangle,$$

where

- (1) $\alpha_p < \lambda$ is an ordinal,
- (2) for each $s \in S$, $\langle \mathbb{G}_{s,\beta}^p : \beta \leq \alpha_p \rangle$ is an increasing and continuous sequence of free Abelian groups from $\mathcal{H}(\lambda)$,
- (3) $E_p \subseteq (\alpha_p + 1) \cap S_{\aleph_0}^\lambda$ does not reflect,
- (4) $\mathbb{G}_{s,\beta}^p / \mathbb{G}_{s,\gamma}^p$ is free when $\gamma < \beta < \alpha_p, \gamma \notin E_p$,
- (5) if $(s, t) \notin R$, then

$$\mathbf{x}_{s,t,\beta}^p := 0 \longrightarrow \mathbb{G}_{t,\beta}^p \xrightarrow{f_{s,t,\beta}^p} \mathbb{H}_{s,t,\beta}^p \xrightarrow{g_{s,t,\beta}^p} \mathbb{G}_{s,\beta}^p \longrightarrow 0$$

is an exact sequence, all increasing with $\beta \leq \alpha_p$, from $\mathcal{H}(\lambda)$:

$$\begin{array}{ccccccc}
\mathbf{x}_{s,t,\alpha_p}^p := 0 & \longrightarrow & \mathbb{G}_{t,\alpha_p}^p & \xrightarrow{f_{s,t,\alpha_p}^p} & \mathbb{H}_{s,t,\alpha_p}^p & \xrightarrow{g_{s,t,\alpha_p}^p} & \mathbb{G}_{s,\alpha_p}^p \longrightarrow 0 \\
& & \subseteq \uparrow & & \subseteq \uparrow & & \subseteq \uparrow \\
& & \vdots & & \vdots & & \vdots \\
& & \subseteq \uparrow & & \subseteq \uparrow & & \subseteq \uparrow \\
\mathbf{x}_{s,t,1}^p := 0 & \longrightarrow & \mathbb{G}_{t,1}^p & \xrightarrow{f_{s,t,1}^p} & \mathbb{H}_{s,t,1}^p & \xrightarrow{g_{s,t,1}^p} & \mathbb{G}_{s,1}^p \longrightarrow 0 \\
& & \subseteq \uparrow & & \subseteq \uparrow & & \subseteq \uparrow \\
\mathbf{x}_{s,t,0}^p := 0 & \longrightarrow & \mathbb{G}_{t,0}^p & \xrightarrow{f_{s,t,0}^p} & \mathbb{H}_{s,t,0}^p & \xrightarrow{g_{s,t,0}^p} & \mathbb{G}_{s,0}^p \longrightarrow 0.
\end{array}$$

(b) Given $p, q \in \mathbb{P}_*$, let $p \leq q$ (q is stronger than p) when

- (1) $\alpha_p \leq \alpha_q$,
- (2) for all $s \in S$ and $\beta \leq \alpha_p$, $\mathbb{G}_{s,\beta}^q = \mathbb{G}_{s,\beta}^p$,
- (3) $E^q \cap (\alpha_p + 1) = E^p$,
- (4) If $(s, t) \notin R$ and $\beta \leq \alpha_p$, then $\mathbf{x}_{s,t,\beta}^q = \mathbf{x}_{s,t,\beta}^p$.

For simplicity of the reader we recall:

Definition 6.5. Let \mathbb{P} be a forcing notion and κ be a regular cardinal.

- (a) \mathbb{P} is called κ -distributive if the intersection of less than κ -many dense open subsets of \mathbb{P} is dense, or equivalently, forcing with \mathbb{P} does not add any new sequences of ordinals of length less than κ .
- (b) \mathbb{P} is called κ -strategically closed if for every $\alpha < \kappa$, player II has a winning strategy in the following game:
 $\mathfrak{D}_{\mathbb{P}}(\alpha)$: the game has length α in which the players I and II take turns to play conditions from \mathbb{P} for α -many moves, with player I playing at odd stages and player II at even stages (including all limit stages). II must play $1_{\mathbb{P}}$ at move zero. Let p_β be the condition played at move β ; the player who played p_β loses immediately unless $p_\beta \geq p_\gamma$ for all $\gamma < \beta$.
If neither player loses at any stage $\beta < \alpha$, then player II wins.
- (c) \mathbb{P} is called κ -c.c. if any antichain $A \subseteq \mathbb{P}$ has size less than κ .

It is evident that if \mathbb{P} is κ -strategically closed, then it is κ -distributive.

Lemma 6.6. *Adopt the above notation. Then \mathbb{P}_* is λ -distributive and λ^+ -c.c. In particular, forcing with \mathbb{P}_* does not add any new sequences of ordinals of size less than λ and it preserves all cardinals.*

Proof. Let us first show that \mathbb{P}_* is λ -distributive. Thus suppose that $\mu < \lambda$ and let $f : \mu \rightarrow V$ be a function in the generic extension by \mathbb{P}_* . We will show that f is in V . Let \dot{f} be a \mathbb{P}_* -name for f and let $p \in \mathbb{P}_*$ be such that

$$p \Vdash \text{“}\dot{f} : \mu \rightarrow \check{V}\text{”}.$$

By induction on $\xi \leq \mu$ we define a sequence $\langle C_\xi : \xi \leq \mu \rangle$ of closed subsets of λ and an increasing sequence $\langle p_\xi : \xi \leq \mu \rangle$ of conditions in \mathbb{P}_* such that:

- (1) C_ξ is a closed subset of $\alpha_{p_\xi} + 1$ with $\alpha_{p_\xi} \in C$, $C_\xi \cap \alpha_{p_\xi}$ is unbounded in α_{p_ξ} and $C_\xi \cap E_{p_\xi} = \emptyset$,
- (2) $C_{\xi+1} \cap \alpha_{p_\xi} + 1 = C_\xi$,
- (3) if $\xi \leq \mu$ is a limit ordinal, then $C_\xi \cap \alpha_{p_\xi} = \bigcup_{\zeta < \xi} C_{p_\zeta}$,
- (4) $p_0 = p$,
- (5) for each $\xi < \mu$, $p_{\xi+1}$ decides $\dot{f}(\xi)$, say $p_{\xi+1} \Vdash \text{“}\dot{f}(\xi) = a_\xi\text{”}$.
- (6) If $\xi \leq \mu$ is a limit ordinal and $\langle p_\zeta : \zeta < \xi \rangle$ is defined, then p_ξ is the least upper bound of the sequence $\langle p_\zeta : \zeta < \xi \rangle$ defined as follows:
 - (a) $\alpha_{p_\xi} = \sup_{\zeta < \xi} \alpha_\zeta$,
 - (b) $E_{p_\xi} = \left(\bigcup_{\zeta < \xi} E_{p_\zeta} \cup \{\alpha_{p_\xi}\} \right) \cap S_{\aleph_0}^\lambda$,
 - (c) for $s \in S$ and $\beta < \alpha_{p_\xi}$, $G_{s,\beta}^{p_\xi} = G_{s,\beta}^{p_\zeta}$ for some, and hence all, $\zeta < \xi$ with $\beta < \alpha_{p_\zeta}$,
 - (d) if $(s, t) \notin R$ and $\beta < \alpha_{p_\xi}$, then $\mathbf{x}_{s,t,\beta}^{p_\xi} = \mathbf{x}_{s,t,\beta}^{p_\zeta}$ for some, and hence all, $\zeta < \xi$ with $\beta < \alpha_{p_\zeta}$,
 - (e) for $s \in S$, $G_{s,\alpha_{p_\xi}}^{p_\xi} = \bigcup_{\zeta < \xi} G_{s,\alpha_{p_\zeta}}^{p_\zeta}$,
 - (f) if $(s, t) \notin R$, then $\mathbf{x}_{s,t,\alpha_{p_\xi}}^{p_\xi} = \bigcup_{\zeta < \xi} \mathbf{x}_{s,t,\alpha_{p_\zeta}}^{p_\zeta}$, which is defined in the natural way.

Given $\xi \leq \mu$, let us show that p_ξ as defined above is a condition, as then it is clear that p_ξ extends all p_ζ 's, for $\zeta < \xi$. Items (1) and (5) from Definition 6.4 are clearly satisfied.

In order to see clause (2), it suffices to show that for each $s \in S$, $G_{s, \alpha_{p_\xi}}^{p_\xi}$ is free. We have $C_\xi \cap \alpha_{p_\xi}$ is a club of α_{p_ξ} and

$$G_{s, \alpha_{p_\xi}}^{p_\xi} = \bigcup_{\gamma \in C_\xi \cap \alpha_{p_\xi}} G_{s, \gamma}^{p_\xi}.$$

Furthermore, the sequence $\langle G_{s, \gamma}^{p_\xi} : \gamma \in C_\xi \rangle$ is an increasing and continuous sequence of free groups, such that if $\gamma < \beta$ are successive points in C_ξ , then $G_{s, \beta}^{p_\xi} / G_{s, \gamma}^{p_\xi}$ is free. It follows that $G_{s, \alpha_{p_\xi}}^{p_\xi}$ is free as required.

The set E_{p_ξ} does not reflect as witnessed by C_ξ , hence clause (3) is satisfied.

To show that clause (4) holds, let $\gamma < \beta < \alpha_{p_\xi}$ with $\gamma \notin E_{p_\xi}$. Let $\zeta < \xi$ be such that $\beta < \alpha_{p_\zeta}$. This implies $\gamma \notin E_{p_\zeta}$, and consequently

$$G_{s, \beta}^{p_\xi} / G_{s, \gamma}^{p_\xi} = G_{s, \beta}^{p_\zeta} / G_{s, \gamma}^{p_\zeta}$$

is free.

Let $q = p_\mu$, and let $h = \langle a_\xi : \xi < \mu \rangle$. Then $h \in V$ and $q \Vdash \dot{f} = h$. As p was arbitrary, we are done.

Now as $\lambda^{<\lambda} = \lambda$ and $\mathbb{P}_* \subseteq \mathcal{H}(\lambda)$, we have $|\mathbb{P}_*| = \lambda$ and hence it clearly satisfies the λ^+ -c.c. \square

Remark 6.7. The above proof shows that the forcing notion \mathbb{P}_* is indeed λ -strategically closed.

Lemma 6.8. *Suppose $\mathbf{G}_* \subseteq \mathbb{P}_*$ is generic over V . Then $E := \bigcup_{p \in \mathbf{G}_*} E_p$ is a non-reflecting stationary subset of λ .*

Proof. This is standard, so we just sketch the proof. Let $p \in \mathbb{P}_*$ and suppose that

$$p \Vdash \dot{C} \subseteq \lambda \text{ is a club of } \lambda."$$

Let $\chi > \lambda^+$ be large enough regular, \triangleleft be a well-ordering of $\mathcal{H}(\chi)$ and let $M \prec (\mathcal{H}(\chi), \in, \triangleleft)$ be an elementary submodel of $\mathcal{H}(\chi)$ such that:

- (1) $|M| < \lambda$,
- (2) $p, \dot{C}, \mathbb{P}_*, \dots \in M$,
- (3) $M \cap \lambda = \delta$ for some $\delta \in S_{\aleph_0}^\lambda$.

By induction on $n < \omega$ we define an increasing sequence $\langle p_n : n < \omega \rangle$ of conditions, together with a sequence $\langle C_n : n < \omega \rangle$ satisfying the following:

- (4) $p \leq p_0$,
- (5) p_{n+1} decides $\dot{C} \cap \alpha_{p_n}$ to be C_n ,
- (6) $p_{n+2} \Vdash \dot{C} \cap (\alpha_{p_n}, \alpha_{p_{n+1}}) \neq \emptyset$,
- (7) $\sup_{n < \omega} \alpha_{p_n} = \delta$.

We are now ready to define

$$q := \langle \langle \mathbb{G}_{s,\beta}^q : s \in S, \beta \leq \delta \rangle, E_q, \langle \mathbf{x}_{s,t,\beta}^q : (s,t) \notin R, \beta \leq \delta \rangle \rangle,$$

as follows:

- (8) for $s \in S$ and $\beta < \delta$, $\mathbb{G}_{s,\beta}^q = \mathbb{G}_{s,\beta}^{p_n}$, where n is such that $\alpha_{p_n} > \beta$,
- (9) for $s \in S$, $\mathbb{G}_\delta^q = \bigcup_{n < \omega} \mathbb{G}_{s,\alpha_{p_n}}^{p_n}$,
- (10) $E_q = \bigcup_{n < \omega} E_{p_n} \cup \{\delta\}$,
- (11) if $(s,t) \notin R$ and $\beta < \delta$, $\mathbf{x}_{s,t,\beta}^q = \mathbf{x}_{s,t,\beta}^{p_n}$, for some n with $\alpha_{p_n} > \beta$,
- (12) $\mathbf{x}_{s,t,\delta}^q$ is the exact sequence that is the direct limit of the sequence $\mathbf{x}_{s,t,\alpha_{p_n}}^{p_n}$.

It is easily seen that $q \in \mathbb{P}_*$ is well-defined and it forces $\delta \in \dot{C} \cap \dot{E}$, which completes the proof. \square

Recall the following easy fact:

Fact 6.9. (See [3]) Any subgroup of a free abelian group is free.

For each $s \in S$ and $\beta < \lambda$, we set $\mathbb{G}_{s,\beta} = \mathbb{G}_{s,\beta}^p$ for some (and hence any) $p \in \mathbf{G}_*$ with $\alpha_p \geq \beta$. Also, we set

$$\mathbb{G}_s := \bigcup_{\beta < \lambda} \mathbb{G}_{s,\beta}.$$

Let us first show that \mathbb{G}_s is not free.

Lemma 6.10. *Suppose $s \in S$. Then \mathbb{G}_s is a non-free strongly λ -free abelian group of size λ .*

Proof. It is clear from Definition 6.4(a)(2) that \mathbb{G}_s is a strongly λ -free abelian group of size λ . Let us show that it is not free. Recall that we are in \mathbf{V} , and let \mathbb{A} be a

λ -free abelian group of size λ which is not free. We may assume that the universe of \mathbb{A} is λ . Let also $\langle A_\alpha : \alpha < \lambda \rangle$ be a filtration of \mathbb{A} . It follows that the set

$$E_{\mathbb{A}} := \{\alpha < \lambda : \mathbb{A}/\mathbb{A}_\alpha \text{ in not } \lambda\text{-free}\}$$

is stationary. We apply an argument similar to the proof of Lemma 6.8, and we are able to show $E_{\mathbb{A}}$ remains stationary in $\mathbf{V}[\mathbf{G}_*]$. In particular, \mathbb{A} remains non-free in the generic extension $\mathbf{V}[\mathbf{G}_*]$.

For each $\gamma < \lambda$ let D_γ be the set of all $p \in \mathbb{P}_*$ such that there exists β such that:

- $\alpha_p = \beta + 1 > \gamma$,
- $\mathbb{G}_{s, \beta+1}^p$ includes \mathbb{A}_β as a subgroup.

It is easily seen that each set D_γ is dense in \mathbb{P}_* . Now for each $\gamma < \lambda$ pick some $p_\gamma \in D_\gamma \cap \mathbf{G}_*$ and set $\alpha_{p_\gamma} = \beta_\gamma + 1 > \gamma$.

Now we look at the following commutative diagram:

$$\begin{array}{ccccccc} \dots & \xrightarrow{\subseteq} & \mathbb{A}_{\beta_\gamma} & \xrightarrow{\subseteq} & \mathbb{A}_{\beta_{\gamma+1}} & \xrightarrow{\subseteq} & \dots \\ \subseteq \downarrow & & \subseteq \downarrow & & \downarrow \subseteq & & \\ \dots & \xrightarrow{\subseteq} & \mathbb{G}_{s, \beta_\gamma+1}^{p_\gamma} & \xrightarrow{\subseteq} & \mathbb{G}_{s, \beta_{\gamma+1}+1}^{p_{\gamma+1}} & \xrightarrow{\subseteq} & \dots \end{array}$$

Taking direct limits of these directed systems, lead us to a natural inclusion map

$$\mathbb{G}_s = \bigcup_{\gamma < \lambda} \mathbb{G}_{s, \beta_\gamma+1}^{p_\gamma} \supseteq \bigcup_{\gamma < \lambda} \mathbb{A}_{\beta_\gamma} = \mathbb{A}.$$

As \mathbb{A} is non-free, it follows from Fact 6.9 that \mathbb{G}_s is also non-free. This completes the proof. \square

For each $(s, t) \in (S \times S) \setminus R$, we look at the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \subseteq \uparrow & & \subseteq \uparrow & & \subseteq \uparrow \\
 \mathbf{x}_{s,t,\beta} := 0 & \longrightarrow & \mathbb{G}_{t,\beta} & \xrightarrow{f_{s,t,\beta}} & \mathbb{H}_{s,t,\beta} & \xrightarrow{g_{s,t,\beta}} & \mathbb{G}_{s,\beta} \longrightarrow 0 \\
 & & \subseteq \uparrow & & \subseteq \uparrow & & \subseteq \uparrow \\
 & & \vdots & & \vdots & & \vdots \\
 & & \subseteq \uparrow & & \subseteq \uparrow & & \subseteq \uparrow \\
 \mathbf{x}_{s,t,1} := 0 & \longrightarrow & \mathbb{G}_{t,1} & \xrightarrow{f_{s,t,1}} & \mathbb{H}_{s,t,1} & \xrightarrow{g_{s,t,1}} & \mathbb{G}_{s,1} \longrightarrow 0 \\
 & & \subseteq \uparrow & & \subseteq \uparrow & & \subseteq \uparrow \\
 \mathbf{x}_{s,t,0} := 0 & \longrightarrow & \mathbb{G}_{t,0} & \xrightarrow{f_{s,t,0}} & \mathbb{H}_{s,t,0} & \xrightarrow{g_{s,t,0}} & \mathbb{G}_{s,0} \longrightarrow 0,
 \end{array}$$

where for each $\beta < \lambda$, $\mathbf{x}_{s,t,\beta} = \mathbf{x}_{s,t,\beta}^p$, for some and hence any $p \in \mathbf{G}_*$ with $\alpha_p \geq \beta$. By taking the corresponding inductive limit, we lead to the following short exact sequence

$$\mathbf{x}_{s,t} := 0 \longrightarrow \mathbb{G}_t \xrightarrow{f_{s,t}} \mathbb{H}_{s,t} \xrightarrow{g_{s,t}} \mathbb{G}_s \longrightarrow 0.$$

In other words, $\mathbf{x}_{s,t} := \lim_{\beta \rightarrow \lambda} \mathbf{x}_{s,t,\beta}^p$. The next lemma shows that $\mathbf{x}_{s,t}$ does not split.

Lemma 6.11. *Suppose $s_*, t_* \in S$ and $(s_*, t_*) \notin R$. Then in $V[\mathbf{G}_*]$ the exact sequence \mathbf{x}_{s_*, t_*} does not split. In particular, $\text{Ext}(\mathbb{G}_{s_*}, \mathbb{G}_{t_*}) \neq 0$.*

Proof. Suppose towards contradiction that the exact sequence \mathbf{x}_{s_*, t_*} splits and let $p \in \mathbb{P}_*$ and \underline{h} be such that

$$p \Vdash \text{“} \underline{h} : \mathbb{G}_{s_*} \rightarrow \mathbb{H}_{s_*, t_*} \text{ is such that } \underline{h} \circ g_{s_*, t_*} = \text{id}_{\mathbb{G}_{s_*}} \text{”}.$$

As in the proof of Lemma 6.8, we can find an extension $q \leq p$ such that

- (1) $\alpha_q \in E_q \cap S_{\aleph_0}^\lambda$,
- (2) q decides $\underline{h} \upharpoonright \mathbb{G}_{s_*, \alpha_q}^q$, say

$$q \Vdash \text{“} \underline{h} \upharpoonright \mathbb{G}_{s_*, \alpha_q}^q = h_* \text{”}.$$

We will need the following claim.

Claim 6.12. *Let*

$$\mathbf{x}_{s_*, t_*, \alpha_q} := 0 \longrightarrow \mathbb{G}_{t_*, \alpha_q}^q \xrightarrow{f_{s_*, t_*, \alpha_q}^q} \mathbb{H}_{s_*, t_*, \alpha_q}^q \xrightarrow{g_{s_*, t_*, \alpha_q}} \mathbb{G}_{s_*, \alpha_q}^q \longrightarrow 0$$

be given and suppose h_* is an splitting of g_{s_*, t_*, α_q} . Then there exists an exact sequence \mathbf{x}_{s_*, t_*} that fits in the following commutative diagram

$$\begin{array}{ccccccccc} \mathbf{x}_{s_*, t_*} := 0 & \longrightarrow & \mathbb{G}'_{t_*} & \xrightarrow{f'} & \mathbb{H}'_{s_*, t_*} & \xrightarrow{g'} & \mathbb{G}'_{s_*} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ \mathbf{x}_{s_*, t_*, \alpha_q} := 0 & \longrightarrow & \mathbb{G}_{t_*, \alpha_q}^q & \xrightarrow{f_{s_*, t_*, \alpha_q}^q} & \mathbb{H}_{s_*, t_*, \alpha_q}^q & \xrightarrow{g_{s_*, t_*, \alpha_q}} & \mathbb{G}_{s_*, \alpha_q}^q & \longrightarrow & 0 \end{array}$$

such that there exists no splitting h' of g' extending h_* . Furthermore, there is an extension r of q such that $\alpha_r = \alpha_q + 1$ and $\mathbf{x}_{s_*, t_*, \alpha_r}^r = \mathbf{x}_{s_*, t_*}$.

Proof. Without loss of generality, either α_p is a cardinal, or $|\alpha_p| < \alpha_p$ and $\alpha_p^\omega = \alpha_p$. Since $\text{cf}(\alpha_p) = \omega$, there is an increasing sequence $\langle \gamma_m : m < n \rangle$ which is cofinal in α_p with $\gamma_m \notin E_q$. To simplicity, let $(s_1, s_2) = (t_*, s_*)$ and for $\ell = 1, 2$ we set $\mathbb{G}_{s_\ell} := \mathbb{G}_{s_\ell, \alpha_q}^q$. We have

$$\mathbb{G}_{s_\ell} = \bigcup_{n < \omega} \mathbb{G}_{s_\ell, \gamma_n}^q,$$

where for each n , both $\mathbb{G}_{s_\ell, \gamma_n}^q$ and $\mathbb{G}_{s_\ell, \gamma_{n+1}}^q / \mathbb{G}_{s_\ell, \gamma_n}^q$ are free, so we can find a free basis $\vec{x}_\ell := \langle x_\alpha^\ell : \alpha < \alpha_q \rangle$ of \mathbb{G}_{s_ℓ} such that $\vec{x}_\ell \upharpoonright \gamma_n$ is a free basis of $\mathbb{G}_{s_\ell, \gamma_n}^q$. Without loss of generality, we may and do assume that f_{s, t, α_q}^q is the natural inclusion morphism:

$$\mathbb{G}_{s_1} \xrightarrow{f_{s, t, \alpha_q}^q} \mathbb{H}_{s_2, s_1, \alpha_q}^q,$$

and $h_*(x_\alpha^2) = x_\alpha^2$ for any $x_\alpha^2 \in \mathbb{G}_{s_2}$.

We define the abelian group \mathbb{G}'_{s_ℓ} to be generated by

$$X_\ell := \mathbb{G}_{s_\ell} \cup \{z_n^\ell : n < \omega\}$$

freely except the equations below which \mathbb{G}'_{s_ℓ} satisfy:

$$(*)_1 : n!z_{n-1}^\ell = z_n^\ell + x_{\gamma_n}^\ell.$$

Recall that $\langle x_\alpha^1, x_\alpha^2 : \alpha < \alpha_q \rangle$ is a free basis of \mathbb{H}_{s_1, s_2} . We let \mathbb{H}'_{s_1, s_2} be the abelian group generated by

$$\mathbb{H}_{s_1, s_2} \cup \{z_n : n < \omega\},$$

freely except the equations below which \mathbb{H}_{s_1, s_2} should satisfy:

$$\odot_1 : n!z_{n-1} = z_n + x_{\gamma_n}^1 + x_{\gamma_n}^2.$$

First, we show that these new abelian groups $\{\mathbb{G}'_{s_1}, \mathbb{H}'_{s_1, s_2}, \mathbb{G}'_{s_2}\}$ are free, by presenting their bases:

(b)₁ : The group \mathbb{G}'_{s_1} is free. Indeed, in view of $(*)_1$ we observe that $z_n^1 = n!z_{n-1}^1 - x_{\gamma_n}^1$. An easy inductive argument on ℓ yields that $z_n^1 \in \langle x_{\gamma_n}^\ell, z_0^1 \rangle$, and so the set

$$B_1 := \{x_\alpha^1 : \alpha < \alpha_q\} \cup \{z_0^1\}$$

generates \mathbb{G}'_{s_1} . Since there is no relation involved in B_1 we deduce that it is a free base of \mathbb{G}'_{s_1} . Thus, \mathbb{G}'_{s_1} is free, as claimed.

(b)₂ : The group \mathbb{G}'_{s_2} is free. Indeed, in (b)₁ replace B_1 with

$$B_2 := \{x_\alpha^1 : \alpha < \alpha_q\} \cup \{z_0^2\}$$

and conclude that \mathbb{G}'_{s_2} is free.

(b)₃ : The group \mathbb{H}'_{s_1, s_2} is free. Indeed, in view of \odot_1 the set

$$B_3 := \{x_\alpha^1, x_\alpha^2 : \alpha < \alpha_q\} \cup \{z_0\}$$

generates \mathbb{H}'_{s_1, s_2} . Since there is no relation involved in elements of B_3 we deduce that it is a free base of \mathbb{H}'_{s_1, s_2} . Thus, \mathbb{H}'_{s_1, s_2} is free, as claimed.

Now, we are going to define a map f' (resp. g') extending $\text{id}_{\mathbb{G}_{s_1}}$ (resp. $g = g_{s_1, s_2, \alpha_q}$) such that the following data becomes an exact sequence

$$0 \longrightarrow \mathbb{G}'_{s_1} \xrightarrow{f'} \mathbb{H}'_{s_1, s_2} \xrightarrow{g'} \mathbb{G}'_{s_2} \longrightarrow 0.$$

Since we need f' extends f , we define $f'(x_\alpha^1) := x_\alpha^1$. In order to complete the definition of f' , we proceed by induction on n to define $f'(z_n^1)$. When $n = 0$, we set

$$f'(z_0^1) := z_1 - z_0 - x_{\gamma_0}^2 \quad (\dagger).$$

Now suppose $f'(z_{n-1}^1)$ is defined. We use the equation $n!z_{n-1}^1 = z_n^\ell + x_{\gamma_n}^1$ and define

$$f'(z_n^1) := n!f'(z_{n-1}^1) - x_{\gamma_n}^1 \quad (+)$$

Also, the assignments $z_n \mapsto z_n^2$, $x_\alpha^1 \mapsto 0$ and $x_\alpha^2 \mapsto x_\alpha^2$ define a map $g' : \mathbb{H}'_{s_1, s_2} \rightarrow \mathbb{G}'_{s_2}$ which extends g . This is well-defined. Namely, it sends the relation

$$n!z_{n-1} = z_n + x_{\gamma_n}^1 + x_{\gamma_n}^2$$

from \odot_1 into the relation $n!z_{n-1}^2 = z_n^2 + x_{\gamma_n}^2$ from $(*)_1$.

Let us show that $\text{Rang}(f') \subseteq \ker(g')$. To see this, first note that $g'(f'(z_0^1)) = 0$. Now suppose by induction that $g'(f'(z_{n-1}^1)) = 0$. Then

$$g'(f'(z_n^1)) \stackrel{(+)}{=} n!g'(f'(z_{n-1}^1))n! - g'(x_{\gamma_n}^1) = -g'(x_{\gamma_n}^1) = 0.$$

We proved that

$$\text{Rang}(f') \subseteq \ker(g') \quad (\star)$$

To see the reverse inclusion, we revisit \odot_1 and note that $\ker(g')$ is generated by

$$\mathbb{G}_{s_1} \cup \{k_n := n!z_{n-1} - z_n - x_{\gamma_n}^2\}.$$

As g' extends g , we have

$$\mathbb{G}_{s_1} = \text{Rang}(f) \subseteq \text{Rang}(f') \stackrel{(\star)}{\subseteq} \ker(g').$$

By induction on n , we show that $k_n \in \text{Rang}(f')$. Following $(+)$ it is enough to deal with $n = 0$, and in this case $k_0 = f'(z_0^1)$. In sum, we proved that

$$0 \longrightarrow \mathbb{G}'_{s_1} \xrightarrow{f'} \mathbb{H}'_{s_1, s_2} \xrightarrow{g'} \mathbb{G}'_{s_2} \longrightarrow 0$$

is an exact sequence of free abelian groups.

Assume toward the contradiction that there exists $h' \in \text{Hom}(\mathbb{G}'_{s_2}, \mathbb{H}'_{s_1, s_2})$ such that $h' \supseteq h_*$. Let $z_n^* := h'(z_n^2) \in \mathbb{H}'_{s_1, s_2}$. As

$$\mathbb{G}'_{s_2} \models n!z_{n-1}^2 = z_n^2 + x_{\gamma_n}^2,$$

we have

$$\odot_2 : \mathbb{H}'_{s_1, s_2} \models n!z_{n-1}^* = z_n^* + h_*(x_{\gamma_n}^2).$$

Subtract $\odot_1 - \odot_2$ we get

\odot_3 :

$$\mathbb{H}'_{s_1, s_2} \models n!(z_{n-1} - z_{n-1}^*) = (z_n - z_n^*) + x_{\gamma_n}^1.$$

Noting that $g'(z_n - z_n^*) = z_n^2 - z_n^{*2} = 0$. From this, $z_n - z_n^* = f'(y_n)$ for some $y_n \in \mathbb{G}'_{s_1}$. Recall that f' is injective. Apply this along with \odot_3 and deduce that

$$\mathbb{G}'_{s_1} \models n!y_{n-1} = y_n + x_{\gamma_n}^1.$$

Due to the uniqueness of solutions of $(*)_1$ we conclude that

$$f'(z_n^1) = f'(y_n) = z_n - z_n^* = z_n - h'(z_n^2).$$

In other words, we determined h' , that is:

$$h'(z_n^2) = z_n - f'(z_n^1).$$

We substitute this from (\dagger) and evaluate g' on the both sides of that equation, then we observe that

$$\begin{aligned} z_0^2 &= g'(h'(z_0^2)) \\ &= g'(z_0 - f'(z_0^1)) \\ &= g'(z_0 - (z_1 - z_0 - x_{\gamma_0}^2)) \\ &= z_0^2 - z_1^2 + z_0^2 + x_{\gamma_0}^2. \end{aligned}$$

Thus

$$z_0^2 = z_1^2 - x_{\gamma_0}^2.$$

But recalling from $(*)_1$,

$$z_0^2 = z_1^2 + x_{\gamma_0}^2,$$

which imply $x_{\gamma_0}^2 = 0$, a contradiction. This contradiction shows that a such h' does not exist.

Finally let r be the extension of q such that:

$$r_1) : \alpha_r = \alpha_q + 1,$$

$$r_2) : \mathbb{G}_{s, \alpha_r}^r = \mathbb{G}_{\alpha_q}^q \text{ if } s \notin \{s_*, t_*\},$$

$$r_3) : \mathbf{x}_{s_*, t_*, \alpha_r}^r = \mathbf{x}_{s_*, t_*}, \text{ where } \mathbf{x}_{s_*, t_*} \text{ is the exact sequence defined above,}$$

$$r_4) : \mathbf{x}_{s, t, \alpha_r}^r = \mathbf{x}_{s, t, \alpha_q}^q \text{ for all } (s, t) \notin R \text{ such that } s \neq s_* \text{ and } t \neq t_*,$$

r_5) : if $s = s_*$ and $t \neq t_*$, then the desired sequence $\mathbf{x}_{s_*,t,\alpha_r}^r$ is defined as follows

$$0 \longrightarrow \mathbb{G}_{t,\alpha_r}^r \xrightarrow{f_{s_*,t,\alpha_r}^r} \mathbb{H}_{s_*,t,\alpha_r}^r \xrightarrow{g_{s_*,t,\alpha_r}^r} \mathbb{G}_{s_*,\alpha_r}^r \longrightarrow 0,$$

where

$$(r_5.i) : \mathbb{H}_{s_*,t,\alpha_r}^r = \frac{\mathbb{H}_{s_*,t,\alpha_q}^q \oplus \bigoplus_{n < \omega} z_n \mathbb{Z}}{\langle n!z_{n-1} - z_n - x_{\gamma_n}^2 \rangle},$$

$$(r_5.ii) : f_{s_*,t,\alpha_r}^r = f_{s_*,t,\alpha_q}^q,$$

$$(r_5.iii) : g_{s_*,t,\alpha_r}^r \upharpoonright \mathbb{H}_{s_*,t,\alpha_q}^q = g_{s_*,t,\alpha_q}^q \text{ and } g_{s_*,t,\alpha_r}^r(z_n) = z_n^2,$$

r_6) : In the case $s \neq s_*$ and $t = t_*$, the proposed sequence $\mathbf{x}_{s,t_*,\alpha_r}^r$ is defined as follows

$$0 \longrightarrow \mathbb{G}_{t_*,\alpha_r}^r \xrightarrow{f_{s,t_*,\alpha_r}^r} \mathbb{H}_{s,t_*,\alpha_r}^r \xrightarrow{g_{s,t_*,\alpha_r}^r} \mathbb{G}_{s,\alpha_r}^r \longrightarrow 0,$$

where

$$(r_6.i) : \mathbb{H}_{s,t_*,\alpha_r}^r = \frac{\mathbb{H}_{s,t_*,\alpha_q}^q \oplus \bigoplus_{n < \omega} z_n \mathbb{Z}}{\langle n!z_{n-1} - z_n - x_{\gamma_n}^1 \rangle},$$

$$(r_6.ii) : f_{s,t_*,\alpha_r}^r \upharpoonright \mathbb{G}_{t_*,\alpha_q}^q = f_{s,t_*,\alpha_q}^q \text{ and } f_{s,t_*,\alpha_r}^r(z_n^1) = 0,$$

$$(r_6.iii) : g_{s,t_*,\alpha_r}^r \upharpoonright \mathbb{H}_{s,t_*,\alpha_q}^q = g_{s,t_*,\alpha_q}^q \text{ and } g_{s,t_*,\alpha_r}^r(z_n) = 0.$$

It is easily seen that r as defined above is a condition extending q . □

Let us continue the proof of Lemma 6.11. By the way we defined r , see the above itemized properties $r_1), \dots, r_6)$, we have

$$r \Vdash \text{“there is no splitting map for } g_{s_*,t_*} \text{ extending } h_* \text{”}.$$

In order to see this, let $r' \leq r$ be such that r' decides $h \upharpoonright \mathbb{G}_{s_*,\alpha_r}^{r'}$. Then

$$r' \Vdash \text{“} h \upharpoonright \mathbb{G}_{s_*,\alpha_r}^{r'} = g' \text{”}.$$

This contradicts the choice of g' . The argument of Lemma 6.11 is now completed. □

Let us now work in the generic extension $V[\mathbf{G}_*]$. We define a λ -support iteration

$$\mathbb{P} = \langle \langle \mathbb{P}_\alpha : \alpha \leq \lambda^+ \rangle, \langle \dot{\mathbb{Q}}_\beta : \beta < \lambda^+ \rangle \rangle$$

of forcing notions which forces $\text{Ext}(\mathbb{G}_s, \mathbb{G}_t) = 0$ for all $s, t \in S$ with sRt . We first define the building blocks of this iteration.

Suppose $W \supseteq V[\mathbf{G}_*]$ is a forcing extension of $V[\mathbf{G}_*]$, $s, t \in S$ are such that sRt and suppose that

$$\mathbf{x} := 0 \longrightarrow \mathbb{G}_s \xrightarrow{f} \mathbb{H} \xrightarrow{g} \mathbb{G}_t \longrightarrow 0$$

is an exact sequence in W .

Definition 6.13. The forcing notion $\mathbb{Q}_{s,t,\mathbf{x}}^W$ consists of partial functions $q : \mathbb{G}_t \dashrightarrow \mathbb{H}$ with domain $\text{dom}(q)$ such that:

- (1) $\text{dom}(q) = \mathbb{G}_{s,\gamma}$, where $\gamma \notin E$,
- (2) $g \circ q = \text{id}_{\text{dom}(q)}$.

$\mathbb{Q}_{s,t,\mathbf{x}}^W$ is ordered by inclusion.

Thus, the forcing notion $\mathbb{Q}_{s,t,\mathbf{x}}^W$ aims to add a splitter for the exact sequence \mathbf{x} . The next lemma shows that this is indeed possible.

Lemma 6.14. *The forcing notion $\mathbb{Q}_{s,t,\mathbf{x}}^W$ is λ -closed, λ^+ -c.c., and forcing with it adds a function $h : \mathbb{G}_t \rightarrow \mathbb{H}$ such that $g \circ h = \text{id}_{\mathbb{G}_t}$.*

Proof. The fact that $\mathbb{Q}_{s,t,\mathbf{x}}^W$ is λ -closed follows from a combination of Definition 6.13(1) along with Definition 6.4(4). This allows us to take unions (or direct limits) at limit stages and still have a condition. According to Δ -system lemma, the forcing is λ^+ -c.c., and claimed. \square

We are finally ready to define our iteration. Let $\beta < \lambda^+$ and suppose that \mathbb{P}_β is defined. If $\Phi(\beta)$ is a $\mathbb{P}_* * \dot{\mathbb{P}}_\beta$ -name for a triple $(s, t, \dot{\mathbf{x}})$, where $s, t \in S$, sRt and $\dot{\mathbf{x}}$ is a name for an exact sequence

$$\mathbf{x} := 0 \longrightarrow \mathbb{G}_s \xrightarrow{\dot{f}} \dot{\mathbb{H}} \xrightarrow{\dot{g}} \mathbb{G}_t \longrightarrow 0,$$

in $\mathcal{H}(\lambda^+)$, then

$$\Vdash_{\mathbb{P}_\beta} \text{“}\dot{\mathbb{Q}}_\beta = \dot{\mathbb{Q}}_{s,t,\dot{\mathbf{x}}}^{V^{\mathbb{P}_* * \dot{\mathbb{P}}_\beta}}\text{”}.$$

Otherwise, let $\dot{\mathbb{Q}}_\beta$ be forced to be the trivial forcing notion.

The next lemma follows from [16].

Lemma 6.15. *Work in $V[\mathbf{G}_*]$. The forcing notion $\mathbb{P} = \mathbb{P}_{\lambda^+}$ is λ -complete and λ^+ -c.c.*

It follows from the above lemma that forcing with \mathbb{P} preserves all cardinals and adds no new sequences of ordinals of length less than λ . Suppose

$$\mathbf{G} = \langle \langle \mathbf{G}_\alpha : \alpha \leq \lambda^+ \rangle, \langle \mathbf{H}_\beta : \beta < \lambda^+ \rangle \rangle$$

is \mathbb{P} -generic over $V[\mathbf{G}_*]$.

Lemma 6.16. *Work in $V[\mathbf{G}_* * \mathbf{G}]$. If $s, t \in S$ and sRt , then $\text{Ext}(\mathbb{G}_s, \mathbb{G}_t) = 0$.*

Proof. It suffices to show that any exact sequence

$$\mathbf{x} := 0 \longrightarrow \mathbb{G}_s \xrightarrow{f} \mathbb{H} \xrightarrow{g} \mathbb{G}_t \longrightarrow 0$$

with \mathbb{H} of size λ splits. Let $\dot{\mathbf{x}}$ be a $\mathbb{P}_* * \mathbb{P}$ -name for \mathbf{x} which we may assume that $\dot{\mathbf{x}} \in \mathcal{H}(\lambda^+)$. Furthermore, we can find some $\alpha < \lambda^+$ such that $\dot{\mathbf{x}}$ is a $\mathbb{P}_* * \mathbb{P}_\alpha$ -name, and then by the choice of Φ , we may also assume that $\Phi(\alpha) = \dot{\mathbf{x}}$. By definition of forcing notion, we know

$$\Vdash_{\mathbb{P}_* * \mathbb{P}_{\alpha+1}} \text{“}\dot{\mathbf{x}} \text{ splits”},$$

and consequently

$$\Vdash_{\mathbb{P}_* * \mathbb{P}} \text{“}\dot{\mathbf{x}} \text{ splits”}.$$

This completes the proof. \square

Let $(s, t) \in (S \times S) \setminus R$. We now show that the iteration does not add splitters for $\mathbf{x}_{s,t}$,

Lemma 6.17. *Suppose $(s, t) \in (S \times S) \setminus R$. Then the exact sequence*

$$\mathbf{x}_{s,t} := 0 \longrightarrow \mathbb{G}_s \xrightarrow{f_{s,t}} \mathbb{H}_{s,t} \xrightarrow{g_{s,t}} \mathbb{G}_t \longrightarrow 0$$

does not split in $V[\mathbf{G}_ * \mathbf{G}]$.*

Proof. Since the argument is similar to the proof of Lemma 6.11, we just present the sketch of proof. Here, we work in V . Let us first combine Lemma 6.6 along with Lemma 6.15, and note that the set of conditions of the form

$$(p, \langle q_\alpha : \alpha < \lambda^+ \rangle) \in \mathbb{P}_* * \mathbb{P}_{\lambda^+}$$

such that $p, q_\alpha \in V$, is dense in $\mathbb{P}_* * \mathbb{P}_{\lambda^+}$. Suppose towards contradiction that the exact sequence $\mathbf{x}_{s,t}$ splits. This gives us

$$(p, \langle p_\alpha : \alpha < \lambda^+ \rangle) \in \mathbb{P}_* * \mathbb{P}_{\lambda^+}$$

and h satisfying:

$$(p, \langle p_\alpha : \alpha < \lambda^+ \rangle) \Vdash "h : \mathbb{G}_s \rightarrow \mathbb{H}_{s,t} \text{ is such that } h \circ g_{s,t} = \text{id}_{\mathbb{G}_s}."$$

As before, we can find an extension

$$(p, \langle p_\alpha : \alpha < \lambda^+ \rangle) \leq (q, \langle q_\alpha : \alpha < \lambda^+ \rangle)$$

equipped with the following two properties:

- (1) $\alpha_q \in E_q \cap S_{\aleph_0}^\lambda$,
- (2) $(q, \langle q_\alpha : \alpha < \lambda^+ \rangle)$ decides $h \upharpoonright \mathbb{G}_{s, \alpha_q}^q$, say

$$(q, \langle q_\alpha : \alpha < \lambda^+ \rangle) \Vdash "h \upharpoonright \mathbb{G}_{s, \alpha_q}^q = h_*."$$

In view of Claim 6.12 we can find an extension r of q such that $(r, \langle q_\alpha : \alpha < \lambda^+ \rangle)$ extends $(q, \langle q_\alpha : \alpha < \lambda^+ \rangle)$ and also, it forces

$$"there \text{ is no splitting map for } g_{s,t} \text{ extending } h_*."$$

We get a contradiction and the lemma follows. \square

This completes the proof of Theorem 6.1.

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