# STRONG PARTITION RELATIONS BELOW THE POWER SET: CONSISTENCY; WAS SIERPINSKI RIGHT? II.

SAHARON SHELAH

Abstract. We continue here [She88] (see the introduction there) but we do not relay on it. The motivation was a conjecture of Galvin stating that $2^\omega \geq \omega_2 + \omega_2 \to [\omega_1]^n_{h(n)}$ is consistent for a suitable $h : \omega \to \omega$. In section 5 we disprove this and give similar negative results. In section 3 we prove the consistency of the conjecture replacing $\omega_2$ by $2^\omega$, which is quite large, starting with an Erdős cardinal. In section 1 we present iteration lemmas which needs when we replace $\omega$ by a larger $\lambda$ and in section 4 we generalize a theorem of Halpern and Lauchli replacing $\omega$ by a larger $\lambda$.

## § 0. Preliminaries

Let $<^*_\chi$ be a well ordering of $\mathcal{H}(\chi)$, where

$$\mathcal{H}(\chi) = \{x : \text{ the transitive closure of } x \text{ has cardinality} < \chi\}$$

agreeing with the usual well-ordering of the ordinals. $P$ (and $Q$, $R$) will denote forcing notions, i.e. partial orders with a minimal element $\varnothing = \varnothing_P$.

A forcing notion $P$ is $\lambda$-closed if every increasing sequence of members of $P$ of length less than $\lambda$ has an upper bound.

If $P \in \mathcal{H}(\chi)$, then for a sequence $\bar{p} = \langle p_i : i < \gamma\rangle$ of members of $P$ let

$$\underset{\sim}{\alpha} = \underset{\sim}{\alpha}_{\bar{p}} = \sup\left\{\underset{\sim}{j} : \{\beta_j : j < \underset{\sim}{j}\} \text{ has an upper bound in} P\right\}$$

and define $\&\bar{p}$, the *canonical upper bound of* $\bar{p}$, as follows:

(a) the least upper bound of $\{p_i : i < \underset{\sim}{\alpha}\}$ in $P$ if there exists such an element,
(b) the $<^*_\chi$-first upper bound of $\bar{p}$ if (a) can't be applied, but there is such,
(c) $p_0$ if (a) and (b) fail and $\gamma > 0$,
(d) $\varnothing_P$ if $\gamma = 0$.

Let $p_0 \& p_1$ be the canonical upper bound of $\langle p_\ell : \ell < 2\rangle$.

Take $[a]^\kappa = \{b \subseteq a : |b| = \kappa\}$ and $[a]^{<\kappa} = \bigcup\limits_{\theta<\kappa} [a]^\theta$.

For sets of ordinals, $A$ and $B$, define $H^{\rm OP}_{A,B}$ as the maximal order preserving bijection between initial segments of $A$ and $B$, i.e, it is the function with domain $\{\alpha \in A : \operatorname{otp}(\alpha \cap A) < \operatorname{otp}(B)\}$, and $H^{\rm OP}_{A,B}(\alpha) = \beta$ iff $\alpha \in A$, $\beta \in B$ and $\operatorname{otp}(\alpha \cap A) = \operatorname{otp}(\beta \cap B)$.

**Definition 0.1.** $\lambda \to^+ (\alpha)^{<\omega}_\mu$ holds provided whenever $F$ is a function from $[\lambda]^{<\omega}$ to $\mu$, $C \subseteq \lambda$ is a club then there is $A \subseteq C$ of order type $\alpha$ such that for any $w_1, w_2 \in [A]^{<\omega}$, $|w_1| = |w_2| \Rightarrow F(w_1) = F(w_2)$.

**Definition 0.2.** $\lambda \to [\alpha]^n_{\kappa,\theta}$ if for every function $F$ from $[\lambda]^n$ to $\kappa$ there is $A \subseteq \lambda$ of order type $\alpha$ such that $\{F(w) : w \in [A]^n\}$ has power $\leq \theta$.

*Date*: 2022-07-03.
2020 *Mathematics Subject Classification.* FILL.
Publication no 288, summer '86. The author would like to thank the United States – Israel Binational Science Foundation for partially supporting this research.

**Definition 0.3.** A forcing notion $P$ satisfies the Knaster condition (has property K) if for any $\{p_i : i < \omega_1\} \subset P$ there is an uncountable $A \subset \omega_1$ such that the conditions $p_i$ and $p_j$ are compatible whenever $i, j \in A$.

## § 1. Introduction

Concerning 1.1–1.3 see Shelah [She78], Shelah and Stanley [SS82], [SS86].

**Definition 1.1.** A forcing notion $Q$ satisfies $*^{\varepsilon}_{\mu}$ where $\varepsilon$ is a limit ordinal $< \mu$, if player **I** has a winning strategy in the following game:

Playing: the play finishes after $\varepsilon$ moves.
in the $\alpha^{\text{th}}$ move:
Player I – if $\alpha \neq 0$ he chooses $\langle q^{\alpha}_{\zeta} : \zeta < \mu^+\rangle$ such that $q^{\alpha}_{\zeta} \in Q$ and

$$(\forall \beta < \alpha)(\forall \zeta < \mu^+)\ p^{\beta}_{\zeta} \leq q^{\alpha}_{\zeta}$$

and he chooses a regressive function $f_\alpha : \mu^+ \to \mu^+$ (i.e. $f_\alpha(i) < 1 + i$). If $\alpha = 0$ let $q^{\alpha}_{\zeta} = \varnothing_Q$, $f_\alpha = \varnothing$.
Player II – he chooses $\langle p^{\alpha}_{\zeta} : \zeta < \mu^+\rangle$ such that $q^{\alpha}_{\zeta} \leq p^{\alpha}_{\zeta} \in Q$.
The outcome: Player I wins provided whenever $\mu < \zeta < \xi < \mu^+$, $\mathrm{cf}(\zeta) = \mathrm{cf}(\xi) = \mu$ and $\wedge_{\beta<\varepsilon} f_\beta(\zeta) = f_\beta(\xi)$ the set $\{p^{\alpha}_{\zeta} : \alpha < \varepsilon\} \cup \{p^{\alpha}_{\xi} : \alpha < \varepsilon\}$ has an upper bound in $Q$.

**Definition 1.2.** We call $\langle P_i, Q_j : i \leq i(*),\ j < i(*)\rangle$ a $*^{\varepsilon}_{\mu}$-iteration provided that:
(a) it is a $(< \mu)$-support iteration ($\mu$ is a regular cardinal)
(b) if $i_1 < i_2 \leq i(*)$, $\mathrm{cf}(i_1) \neq \mu$ then $P_{i_2}/P_{i_1}$ satisfies $*^{\varepsilon}_{\mu}$.

**Lemma 1.3.** *If $\bar{Q} = \langle P_i, Q_j : i \leq i(*),\ j < i(*)\rangle$ is a $(< \mu)$-support iteration, (a) or (b) or (c) below hold, then it is a $*^{\varepsilon}_{\mu}$-iteration.*
(a) *$i(*)$ is limit and $\bar{Q} \restriction j(*)$ is a $*^{\varepsilon}_{\mu}$-iteration for every $j(*) < i(*)$.*
(b) *$i(*) = j(*) + 1$, $\bar{Q} \restriction j(*)$ is a $*^{\varepsilon}_{\mu}$-iteration and $Q_{j(*)}$ satisfies $*^{\varepsilon}_{\mu}$ in $V^{P_{j(*)}}$.*
(c) *$i(*) = j(*) + 1$, $\mathrm{cf} j(*) = \mu^+$, $\bar{Q} \restriction j(*)$ is a $*^{\varepsilon}_{\mu}$-iteration and for every successor $i < j(*)$, $P_{i(*)}/P_i$ satisfies $*^{\varepsilon}_{\mu}$.*

*Proof.* Left to the reader (after reading [Sh80] or [ShSt154a]).

**Theorem 1.4.** *Suppose $\mu = \mu^{<\mu} < \chi < \lambda$, and $\lambda$ is a strongly inaccessible $k^2_2$-Mahlo cardinal, where $k^2_2$ is a suitable natural number (see 3.6(2) of [Sh289]), and assume $V = L$ for the simplicity. Then for some forcing notion $P$:*
(a) *$P$ is $\mu$-complete, satisfies the $\mu^+$-c.c., has cardinality $\lambda$, and $V^P \models$ "$2^\mu = \lambda$".*
(b) *$\Vdash_P \lambda \to [\mu^+]^2_3$ and even $\lambda \to [\mu^+]^2_{\kappa,2}$ for $\kappa < \mu$.*
(c) *if $\mu = \aleph_0$ then $\Vdash$ "$MA_\chi$".*
(d) *if $\mu > \aleph_0$ then: $\Vdash_P$ "for every forcing notion $Q$ of cardinality $\leq \chi$, $\mu$-complete satisfying $*^{\varepsilon}_{\mu}$, and for any dense sets $D_i \subseteq Q$ for $i < i_0 < \lambda$, there is a directed $G \subseteq Q$, $\wedge_i G \cap D_i \neq \varnothing$".*

*As the proof is very similar to* [She88], *(particularly after reading section 3) we do not give details. We shall define below just the systems needed to complete the proof. More general ones are implicit in [Sh289].*

**Convention 1.5.** *We fix a one to one function $Cd = Cd_{\lambda,\mu}$ from ${}^{\mu>}\lambda$ onto $\lambda$.*

*Remark* 1.6. Below we could have $\mathrm{otp}(B_x) = \mu^+ + 1$ with little change.

*Definition* 1.7. Let $\mu < \chi < \kappa \leq \lambda$, $\lambda = \lambda^{<\mu}$, $\chi = \chi^{<\mu}$, $\mu = \mu^{<\mu}$.

1) We call $x$ a $(\lambda, \kappa, \chi, \mu)$-precandidate if $x = \langle a^x_u : u \in I_x \rangle$ where for some set $B_x$ (unique, in fact):
   (i) $I_x = \{s : s \subseteq B_x, |x| \leq 2\}$,
   (ii) $B_x$ is a subset of $\kappa$ of order type $\mu^+$,
   (iii) $a^x_u$ is a subset of $\lambda$ of cardinality $\leq \chi$ closed under $Cd$,
   (iv) $a^x_u \cap B_x = u$,
   (v) $a^x_u \cap a^x_v \subseteq a^x_{u \cap v}$,
   (vi) if $u, v \in I_x$, $|u| = |v|$ then $a^x_u$ and $a^x_v$ have the same order type (and so $H^{\mathrm{OP}}_{a^x_u, a^x_v}$ maps $a^x_u$ onto $a^x_v$),
   (vii) if $u_\ell, v_\ell \in I_x$ for $\ell = 1, 2$, $|u_1| = |v_1|$, $|u_2| = |v_2|$, $|u_1 \cup u_2| = |v_1 \cup v_2|$, $H^{\mathrm{OP}}_{a^x_{u_1} \cup a^x_{u_2}, a^x_{v_1} \cup a^x_{v_2}}$ maps $u_\ell$ onto $v_\ell$ for $\ell = 1, 2$ then $H^{\mathrm{OP}}_{a^x_{u_1}, a^x_{v_1}}$ and $H^{\mathrm{OP}}_{a^x_{u_2}, a^x_{v_2}}$ are compatible.
2) We say $x$ is a $(\lambda, \kappa, \chi, \mu)$-candidate if it has the form $\langle M^x_u : u \in I_x \rangle$ where
   ($\alpha$) (i) $\langle |M^x_u| : u \in I_x \rangle$ is a $(\lambda, \kappa, \chi, \mu)$-precandidate (with $B_x$ defined as $\bigcup I_x$ )
   (ii) $L_x$ is a vocabulary with $(\leq \chi)$-many $(< \mu)$-ary place predicates and function symbols,
   (iii) each $M^x_u$ is an $L_x$-model,
   (iv) for $u, v \in I_x$, $|u| = |v|$, $M^x_u \restriction (|M^x_u| \cap |M^x_v|)$ is a model, and in fact an elementary submodel of $M^x_v$, $M^x_u$ and $M^x_{u \cap v}$.
   ($\beta$) for $u, v \in I_x$, $|u| = |v|$, the function $H^{\mathrm{OP}}_{|M^x_u|, |M^x_v|}$ is an isomorphism from $M^x_u$ onto $M^x_v$.
3) The set $\mathfrak{A}$ is a $(\lambda, \kappa, \chi, \mu)$-system if
   (A) each $x \in \mathfrak{A}$ is a $(\lambda, \kappa, \chi, \mu)$-candidate,
   (B) guessing: if $L$ is as in (2)($\alpha$)(ii), $M^*$ is an $L$-model with universe $\lambda$ then for some $x \in \mathfrak{A}$, $s \in B_x \Rightarrow M^x_s \prec M^*$.

**Definition 1.8.** 1) We call the system $\mathfrak{A}$ disjoint when:

($*$) if $x \neq y$ are from $\mathfrak{A}$ and $\mathrm{otp}(|M^x_\varnothing|) \leq \mathrm{otp}(|M^y_\varnothing|)$ then for some $B_1 \subseteq B_x$, $B_2 \subseteq B_y$ we have
   a) $|B_1| + |B_2| < \mu^+$
   b) the sets

$$\bigcup\{|M^x_s| : s \in [B_x \setminus B_1]^{\leq 2}\} \text{ and } \bigcup\{|M^y_s| : s \in [B_y \setminus B_2]^{\leq 2}\}$$

   have intersection $\subseteq M^y_\varnothing$.

2) We call the system $\mathfrak{A}$ almost disjoint when:

($**$) if $x, y \in \mathfrak{A}$, $\mathrm{otp}(|M^x_\varnothing|) \leq \mathrm{otp}(|M^y_\varnothing|)$ then for some $B_1 \subseteq B_x$, $B_2 \subseteq B_y$ we have:
   a) $|B_1| + |B_2| < \mu^+$,
   b) if $s \in [B_x \setminus B_1]^{\leq 2}$, $t \in [B_y \setminus B_2]^{\leq 2}$ then $|M^x_s| \cap |M^x_t| \subseteq |M^y_\varnothing|$.

## § 2. Introducing the partition on trees

**Definition 2.1.** Let

1) $\mathrm{Per}({}^{\mu >}2)$ be the set of $T$ such that
   (a) $T \subseteq {}^{\mu >}2$, $\langle\rangle \in T$,
   (b) $(\forall \eta \in T)\, (\forall \alpha < \ell g(\eta))\, \eta \restriction \alpha \in T$,
   (c) if $\eta \in T \cap {}^{\alpha}2$, $\alpha < \beta < \mu$ then for some $\nu \in T \cap {}^{\beta}2$, $\eta \triangleleft \nu$,
   (d) if $\eta \in T$ then for some $\nu$ we have $\eta \triangleleft \nu$, $\nu\hat{\ }\langle 0\rangle \in T$, $\nu\hat{\ }\langle 1\rangle \in T$
   (e) if $\eta \in {}^{\delta}2$, $\delta < \mu$ is a limit ordinal and $\{\eta \restriction \alpha : \alpha < \delta\} \subseteq T$ then $\eta \in T$.

2) $\mathrm{Per}_f({}^{\mu>}2) =$

$$\Big\{T \in \mathrm{Per}({}^{\mu>}2) : \alpha < \mu,\ \nu_1,\, \nu_2 \in {}^{\alpha}2 \cap T \Rightarrow \Big[\bigwedge_{\ell=0}^{1} \nu_1{}^\frown\langle \ell\rangle \in T \Longleftrightarrow \bigwedge_{\ell=0}^{1} \nu_2{}^\frown\langle \ell\rangle \in T\Big]\Big\}.$$

3) $\mathrm{Per}_u({}^{\mu>}2) =$

$$\Big\{T \in \mathrm{Per}({}^{\mu>}2) : \alpha < \mu,\ \nu_1 \neq \nu_2 \text{ from } {}^{\alpha}2 \cap T,\ \text{then } \bigvee_{\ell=0}^{1}\bigvee_{m=1}^{2} \nu_m{}^\frown\langle \ell\rangle \notin T\Big\}$$

4) For $T \in \mathrm{Per}({}^{\mu>}2)$ let $\lim T = \big\{\eta \in {}^{\mu}2 : (\forall \alpha < \mu)\, \eta \restriction \alpha \in T\big\}$.
5) For $T \in \mathrm{Per}_f({}^{\mu>}2)$ let $\mathrm{clp}_T : T \to {}^{\mu>}2$ be the unique one-to-one function from $\mathrm{sp}(T) = \{\eta \in T : \eta{}^\frown\langle 0\rangle, \eta{}^\frown\langle 1\rangle \in T\}$ onto ${}^{\mu>}2$, which preserves $\triangleleft$ and lexicographic order.
6) Let $\mathrm{SP}(T) = \{\ell g(\eta) : \eta \in \mathrm{sp}(T)\}$, where

$$\mathrm{sp}(\eta, \nu) = \min\{i : \eta(i) \neq \nu(i) \vee i = \ell g(\eta) \vee i = \ell g(\nu)\}$$

**Definition 2.2.** 1) For cardinals $\mu, \sigma$ and $n < \omega$ and $T \in \mathrm{Per}({}^{\mu>}2)$ let $\mathrm{Col}^n_\sigma(T) = \big\{d : d \text{ is a function from } \bigcup_{\alpha<\mu} [{}^{\alpha}2]^n \cap T \text{ to } \sigma\big\}$. We will write $d(\nu_0, \dots, \nu_{n-1})$ for $d\big(\{\nu_0, \dots, \nu_{n-1}\}\big)$.
2) Let $<^*_\alpha$ denote a well ordering of ${}^{\alpha}2$ (in this section it is arbitrary). We call $d \in \mathrm{Col}^n_\sigma(T)$ end-homogeneous for $\big\langle <^*_\alpha : \alpha < \mu\big\rangle$ provided that: if $\alpha < \beta$ are from $\mathrm{SP}(T)$, $\{\nu_0, \dots, \nu_{n-1}\} \subseteq {}^{\beta}2 \cap T$, $\langle \nu_\ell \restriction \alpha : \ell < n\rangle$ are pairwise distinct and $\bigwedge_{\ell,m} [\nu_\ell <^*_\beta \nu_m \Longleftrightarrow \nu_\ell \restriction \alpha <^*_\alpha \nu_m \restriction \alpha]$ then

$$d(\nu_0, \dots, \nu_{n-1}) = d(\nu_0 \restriction \alpha, \dots, \nu_{n-1} \restriction \alpha).$$

3) Let $\mathrm{EhCol}^n_\sigma(T) = \big\{d \in \mathrm{Col}^n_\sigma(T) : d \text{ is end-homogeneous}\big\}$ (for some $\langle <^*_\alpha : \alpha < \mu\rangle$).
4) For $\nu_0, \dots, \nu_{n-1}, \eta_0, \dots, \eta_{n-1}$ from ${}^{\mu>}2$, we say $\bar\nu = \langle \nu_0, \dots, \nu_{n-1}\rangle$ and $\bar\eta = \langle \eta_o, \dots, \eta_{n-1}\rangle$ are strongly similar for $\big\langle <^*_\alpha : \alpha < \mu\big\rangle$ if:
   (i) $\ell g(\nu_\ell) = \ell g(\eta_\ell)$
   (ii) $\mathrm{sp}(\nu_\ell, \nu_m) = \mathrm{sp}(\eta_\ell, \eta_m)\ (= \eta_\ell \cap \eta_m)$
   (iii) if $\ell_1, \ell_2, \ell_3, \ell_4 < n$ and $\alpha = \mathrm{sp}(\nu_{\ell_1}, \nu_{\ell_2})$ then

$$\nu_{\ell_3} \restriction \alpha <^*_\alpha \nu_{\ell_4} \restriction \alpha \Longleftrightarrow \eta_{\ell_3} \restriction \alpha <^*_\alpha \eta_{\ell_4} \restriction \alpha \text{ and } \nu_{\ell_3}(\alpha) = \eta_{\ell_3}(\alpha)$$

5) For $\nu^a_0, \dots, \nu^a_{n-1}$, $\nu^b_0, \dots, \nu^b_{n-1}$ from ${}^{\mu>}2$ we say $\bar\nu^a = \langle \nu^a_0, \dots, \nu^a_{n-1}\rangle$ and $\bar\nu^b = \langle \nu^b_0, \dots, \nu^b_{n-1}\rangle$ are similar if the truth values of (i)–(iii) below do not depend on $t \in \{a, b\}$ for any $\ell(1), \ell(2), \ell(3), \ell(4) < n$:
   (i) $\ell g(\nu^t_{\ell(1)}) < \ell g(\nu^t_{\ell(2)})$
   (ii) $\mathrm{sp}(\nu^t_{\ell(1)}, \nu^t_{\ell(2)}) < \mathrm{sp}(\nu^t_{\ell(3)}, \nu^t_{\ell(4)})$
   (iii) for $\alpha = \mathrm{sp}(\nu^t_{\ell(1)}, \nu^t_{\ell(2)})$, the truth value of the following does not depend on $\ell$:

$$\nu^t_{\ell(3)} \restriction \alpha <^*_\alpha \nu^t_{\ell(4)} \restriction \alpha \text{ and } \nu^t_{\ell(3)}(\alpha) = 0.$$

6) We say $d \in \mathrm{Col}^n_\sigma(T)$ is almost homogeneous [homogeneous] on $T_1 \subseteq T$ (for $\big\langle <^*_\alpha : \alpha < \mu\rangle\big)$ if for every $\alpha \in \mathrm{SP}(T_1)$, $\bar\nu$, $\bar\eta \in [{}^{\alpha}2]^n \cap T_1$ which are strongly similar [similar] we have $d(\bar\nu) = d(\bar\eta)$.
7) We say $\big\langle <^*_\alpha : \alpha < \mu\big\rangle$ is nice to $T \in \mathrm{Per}({}^{\mu>}2)$, provided that: if $\alpha < \beta$ are from $\mathrm{SP}(T)$, $(\alpha, \beta) \cap \mathrm{SP}(T) = \varnothing$, $\eta_1 \neq \eta_2 \in {}^{\beta}2 \cap T$, $\big[\eta_1 \restriction \alpha <^*_\alpha \eta_2 \restriction \alpha$ *or* $\eta_1 \restriction \alpha = \eta_2 \restriction \alpha,\ \eta_1(\alpha) < \eta_2(\alpha)\big]$ then $\eta_1 <^*_\beta \eta_2$.

**Definition 2.3.** 1) $\mathrm{Pr}_{eht}(\mu, n, \sigma)$ means: for every $d \in \mathrm{Col}^n_\sigma({}^{\mu>}2)$ for some $T \in \mathrm{Per}({}^{\mu>}2)$, $d$ is end homogeneous on $T$.

2) $\mathrm{Pr}_{aht}(\mu, n, \sigma)$ means for every $d \in \mathrm{Col}^n_\sigma({}^{\mu>}2)$ for some $T \in \mathrm{Per}({}^{\mu>}2)$, $d$ is almost homogeneous on $T$.
3) $\mathrm{Pr}_{ht}(\mu, n, \sigma)$ means for every $d \in \mathrm{Col}^n_\sigma({}^{\mu>}2)$ for some $T \in \mathrm{Per}({}^{\mu>}2)$, $d$ is homogeneous on $T$.
4) For $x \in \{eht, aht, ht\}$, $\mathrm{Pr}^f_x(\mu, n, \sigma)$ is defined like $\mathrm{Pr}_x(\mu, n, \sigma)$ but we demand $T \in \mathrm{Per}_f({}^{\mu>}2)$.
5) If above we replace $eht$, $aht$, $ht$ by $ehtn$, $ahtn$, $htn$, respectively, this means $\langle <^*_\alpha : \alpha < \mu \rangle$ is fixed a priori.
6) Replacing $n$ by "$< \kappa$", $\sigma$ by $\bar\sigma = \langle \sigma_\ell : \ell < \kappa \rangle$ for $\kappa \le \aleph_0$, means that $\langle d_n : n < \kappa \rangle$ are given, $d_n \in \mathrm{Col}^n_\sigma({}^{\mu>}2)$ and the conclusion holds for all $d_n$ with $n < \kappa$ simultaneously. Replacing "$\sigma$" by "$< \sigma$" means that the assertion holds for every $\sigma_1 < \sigma$.

**Definition 2.4.** 1) $\mathrm{Pr}_{aht}(\mu, n, \sigma(1), \sigma(2))$ means: for every $d \in \mathrm{Col}^n_{\sigma(1)}({}^{\mu>}2)$, for some $T \in \mathrm{Per}({}^{\mu>}2)$ and $\langle <^*_\alpha : \alpha < \mu \rangle$ for every $\bar\eta \in \bigcup \{ [{}^\alpha 2]^n \cap T : \alpha \in \mathrm{SP}(T) \}$,

$$\Big\{ d(\bar\nu) : \bar\nu \in \bigcup \{ [{}^\alpha 2]^n \cap T_1 : \alpha \in \mathrm{SP}((T_1) \},\ \bar\eta \text{ and } \bar\nu \text{ are strongly similar for } \langle <^*_\alpha : \alpha < \mu \rangle \Big\}$$

has cardinality $< \sigma(2)$.
2) $\mathrm{Pr}_{ht}(\mu, n, \sigma(1), \sigma(2))$ is defined similarly with "similar" instead of "strongly similar".
3) $\mathrm{Pr}_x\Big(\mu, < \kappa, \langle \sigma^1_\ell : \ell < \kappa \rangle, \langle \sigma^2_\ell : \ell < \kappa \rangle \Big)$, $\mathrm{Pr}^f_x(\mu, n, \sigma(1), \sigma(2))$, $\mathrm{Pr}^f_x(\mu, < \aleph_0, \bar\sigma^1, \bar\sigma^2)$ are defined in the same way.

There are many obvious implications.

**Fact 2.5.** 1) For every $T \in \mathrm{Per}({}^{\mu>}2)$ there is a $T_1 \subseteq T$, $T_1 \in \mathrm{Per}_u({}^{\mu>}2)$.
2) In defining $\mathrm{Pr}^f_x(\mu, n, \sigma)$ we can demand $T \subseteq T_0$ for any $T_0 \in \mathrm{Per}_f({}^{\mu>}2)$, similarly for $\mathrm{Pr}^f_x(\mu, < \kappa, \sigma)$.
3) The obvious monotonicity holds.

**Claim 2.6.** *1) Suppose $\mu$ is regular, $\sigma \ge \aleph_0$ and $\mathrm{Pr}^f_{eht}(\mu, n, < \sigma)$. Then $\mathrm{Pr}^f_{aht}(\mu, n, < \sigma)$ holds.*
*2) If $\mu$ is weakly compact and $\mathrm{Pr}^f_{aht}(\mu, n, < \sigma)$, $\sigma < \mu$, then $\mathrm{Pr}^f_{ht}(\mu, n, < \sigma)$ holds.*
*3) If $\mu$ is Ramsey and $\mathrm{Pr}^f_{aht}(\mu, < \aleph_0, < \sigma)$, $\sigma < \mu$, then $\mathrm{Pr}^f_{ht}(\mu, < \aleph_0, < \sigma)$.*
*4) If $\mu = \omega$, in the "nice" version, the orders $\langle <^*_\alpha : \alpha < \mu \rangle$ disappear.*

*Proof.* We induct on $n$; for $n+1$ and given $d_{n+1} : \bigcup \{ [{}^\alpha 2]^{n+1} : \alpha < \mu \} \to \sigma$ and $\bar<^{n+1} = \langle <^{n+1}_\alpha : \alpha < \mu \rangle$, we apply $\mathrm{Pr}^f_{eht}(\mu, n, < \sigma)$. We get $T$.

Let $f = \mathrm{clp}_T : T \to {}^{\mu>}2$ be as in 2.1(5). Define $\bar<^* = \langle <^{*n}_\alpha : \alpha < \mu \rangle$ and $d_n$ as follows:

(A) for $\alpha < \mu$ and $\eta_0, \eta_1 \in {}^\alpha 2$, $\mathrm{clp}_T(\nu_\ell) = \eta_\ell$, $\ell g(\nu_\ell) = \beta$ <u>then</u>
$$\eta_0 <^n_\alpha \eta_1 \Longleftrightarrow \nu_0 <^{n+1}_\alpha \nu_1$$
(B) for $\alpha < \mu$ and $\eta_0 <^n_\alpha \ldots <^n_\alpha \eta_{n-1}$, $\mathrm{clp}_T(\nu_\ell) = \eta_\ell$, $\ell g(\nu_\ell) = \beta$ and for $k < n$, $\rho < 2$ we have $\nu_k{}^\frown\langle \ell \rangle \lhd \rho_{k,\ell} \in \mathrm{sp}(T_{n+1}) \cap {}^\gamma 2$. If $\gamma$ minimal then $d_n(\{\eta_0, \ldots \eta_{n-1}\})$ codes the set of the following objects $\mathbf{t}$:
- For some $\gamma > \alpha$ there are $\rho_{k,\ell} \in \mathrm{sp}(T_{n+1}) \cap {}^\gamma 2$ such that $\nu_k{}^\frown\langle \ell \rangle \trianglelefteq \rho_{k,\ell}$ for $k < n$, $\ell < 2$ and $\mathbf{t}$ codes all the information on the sequence $\langle \rho_{k,\ell} : k < n, \ell < 2 \rangle$ (i.e. the order $<^{n+1}_\gamma$ and instances of $\mathbf{d}_{n+1}$). $\square$

*The following theorem is a quite strong positive result for $\mu = \omega$. Halpern Lauchli proved 2.7(1), Laver proved 2.7(2) (and hence (3)), Pincus pointed out that Halpern Lauchli's proof can be modified to get 2.7(2), and then $\mathrm{Pr}^f_{eht}(\omega, n, < \sigma)$ and (by it) $\mathrm{Pr}^f_{ht}(\omega, n, < \sigma)$ are easy. [No idea why this is all in italics]*

**Theorem 2.7.** *1) If $d \in \mathrm{Col}^n_\sigma({}^{\omega>}2)$, $\sigma < \aleph_0$, then there are $T_0, \dots, T_{n-1} \in \mathrm{Per}_f({}^{\omega>}2)$ and $k_0 < k_1 < \dots < k_\ell < \dots$ and $s < \sigma$ such that for every $\ell < \omega$, if $\mu_0 \in T_0, \mu_1 \in T_1, \dots, \nu_{n-1} \in T_{n-1}$, $\bigwedge\limits_{m<n} \ell g(\nu_m) = k_\ell$, then $d(\nu_0, \dots, \nu_{n-1}) = s$.*
*2) We can demand in 1) that*

$$\mathrm{SP}(T_\ell) = \{k_0, k_1, \dots\}$$

*3) $\mathrm{Pr}^f_{htn}(\omega, n, \sigma)$ for $\sigma < \aleph_0$.*
*4) $\mathrm{Pr}^f_{htn}\big(\omega, < \aleph_0, \langle \sigma^1_n : n < \omega\rangle, \langle \sigma^2_n : n < \omega\rangle\big)$ if $\sigma^1_n < \aleph_0$ and $\langle \sigma^2_n : n < \omega\rangle$ diverge to infinity.*

**Definition 2.8.** Let $d$ be a function with domain $\supseteq [A]^n$, $A$ be a set of ordinals, $F$ be a one-to-one function from $A$ to ${}^{\alpha(*)}2$, $<^*_\alpha$ be a well ordering of ${}^\alpha 2$ for $\alpha \le \alpha(*)$ such that $F(\alpha) <^*_\alpha F(\beta) \iff \alpha < \beta$, and $\sigma$ be a cardinal.
1) We say $d$ is $(F,\sigma)$-canonical on $A$ if for any $\alpha_1 < \dots < \alpha_n \in A$,

$$\Big|\Big\{d(\beta_1, \dots, \beta_n) : \big\langle F(\beta_1), \dots, F(\beta_n)\big\rangle \text{ similar to } \big\langle F(\alpha_1), \dots, F(\alpha_n)\big\rangle\Big\}\Big| \le \sigma$$

2) We define "almost $(F,\sigma)$-canonical" similarly using strongly similar instead of "similar".

## § 3. Consistency of a strong partition below the continuum; Irrelevant

This section is dedicated to the proof of

**Theorem 3.1.** *Suppose $\lambda$ is the first Erdős cardinal, i.e. the first such that $\lambda \to (\omega_1)^{<\omega}_2$. Then, if $A$ is a Cohen subset of $\lambda$, in $V[A]$ for some $\aleph_1$–c.c. forcing notion $P$ of cardinality $\lambda$, $\Vdash_P$ "$MA_{\aleph_1}(Knaster) + 2^{\aleph_0} = \lambda$" and:*
*1.) $\Vdash_P$ "$\lambda \to [\aleph_1]^n_{h(n)}$" for suitable $h : \omega \mapsto \omega$ (explicitly defined below).*
*2.) In $V^P$, for any colorings $d_n$ of $\lambda$ where $d_n$ is $n$-place, and for any divergent $\langle \sigma_n : n < \omega\rangle$ (see below), there is a $W \subseteq \lambda$, $|W| = \aleph_1$ and a function $F : W \to {}^\omega 2$ such that $d_n$ is $(F, \sigma_n)$-canonical on $W$ for each $n$. (See definition 2.8 above.)*

*Remark* 3.2. $h(n)$ is $n!$ times the number of $u \in [{}^\omega 2]^n$ satisfying (if $\eta_1, \eta_2, \eta_3, \eta_4 \in u$ are distinct then $\mathrm{sp}(\eta_1, \eta_2)$, $\mathrm{sp}(\eta_3, \eta_4)$ are distinct) up to strong similarity for any nice $\big\langle <^*_\alpha : \alpha < \omega\big\rangle$.
2) A sequence $\langle \sigma_n : n < \omega\rangle$ is divergent if $\forall m : \exists k : \forall n \ge k : \sigma_n \ge m$.

*Notation* 3.3. For a sequence $a = \langle \alpha_i, e^*_i : i < \alpha\rangle$, we call $b \subseteq \alpha$ closed if
(i) $i \in b \Rightarrow a_i \subseteq b$
(ii) if $i < \alpha$, $e^*_i = 1$ and $\sup(b \cap i) = i$ then $i \in b$.

*Definition* 3.4. Let $\mathfrak{K}$ be the family of $\bar Q = \langle P_i, \underset{\sim}{Q}_j, a_j, e^*_j : j < \alpha,\ i \le \alpha\rangle$ such that
(a) $a_i \subseteq i$, $|a_i| \le \aleph_1$,
(b) $a_i$ is closed for $\langle a_j, e^*_j : j < i\rangle$, $e^*_i \in \{0,1\}$, and $[e^*_i = 1 \Rightarrow \mathrm{cf} i = \aleph_1]$
(c) $P_i$ is a forcing notion, $\underset{\sim}{Q}_j$ is a $P_j$-name of a forcing notion of power $\aleph_1$ with minimal element $\varnothing$ or $\varnothing_j$ and for simplicity the underlying set of $\underset{\sim}{Q}_j$ is $\subseteq [\omega_1]^{<\aleph_0}$ (we do not lose by this).
(d) $P_\beta = \big\{p : p$ is a function whose domain is a finite subset of $\beta$ and for $i \in \mathrm{dom}(p)$, $\Vdash_{P_i}$ "$f(i) \in \underset{\sim}{Q}_i$"$\big\}$ with the order $p \le q$ if and only if for $i \in \mathrm{dom}(p)$, $q \restriction i \Vdash_{P_i}$ "$p(i) \le q(i)$".
(e) for $j < i$, $\underset{\sim}{Q}_j$ is a $P_j$-name involving only antichains contained in $\{p \in P_j : \mathrm{dom}(p) \subseteq a_j\}$.

For $p \in P_i$, $j < i$, $j \notin \mathrm{dom} p$ we let $p(j) = \varnothing$. Note for $p \in P_i$, $j \le i$, $p \restriction j \in P_j$

*Definition* 3.5. For $\bar{Q} \in \mathfrak{K}$ as above (so $\alpha = \ell g(\bar{Q})$):
1) for any $b \subseteq \beta \leq \alpha$ closed for $\langle a_i, e_i^* : i < \beta\rangle$ we define $P_b^{\mathrm{cn}}$ [by simultaneous induction on $\beta$]:

$$P_b^{\mathrm{cn}} = \{p \in P_\beta : \mathrm{dom} p \subseteq b, \textit{ and for } i \in \mathrm{dom} p,\ p(i) \textit{ is a canonical name}\}$$

i.e., for any $x$, $\{p \in P_{a_i}^{\mathrm{cn}} : p \Vdash_{P_i} \text{“}p(i) = x\text{”}$ or $p \Vdash_{P_i} \text{“}p(i) \neq x\text{”}\ \}$ is a predense subset of $P_i$.
2) For $\bar{Q}$ as above, $\alpha = \ell g(\bar{Q})$, take $\bar{Q} \restriction \beta = \langle P_i, Q_j, a_j : i \leq \beta,\ j < \beta\rangle$ for $\beta \leq \alpha$ and the order is the order in $P_\alpha$ (if $\beta \geq \alpha$, $\bar{Q} \restriction \beta = \bar{Q}$).
3) “$b$ *closed for* $\bar{Q}$ means “$b$ closed for $\langle a_i, e_i^* : i < \ell g\bar{Q}\rangle$”.

*Fact* 3.6. 1) if $\bar{Q} \in \mathfrak{K}$ then $\bar{Q} \restriction \beta \in \mathfrak{K}$.
2) Suppose $b \subseteq c \subseteq \beta \leq \ell g(\bar{\theta})$, $b$ and $c$ are closed for $\bar{Q} \in \mathfrak{K}$.
(i) If $p \in P_c^{\mathrm{cn}}$ then $p \restriction b \in P_b^{\mathrm{cn}}$.
(ii) If $p, q \in P_c^{\mathrm{cn}}$ and $p \leq q$ then $p \restriction b \leq q \restriction c$.
(iii) $P_c^{\mathrm{cn}} \lessdot P_\beta$. 3) $\ell g\bar{Q}$ is closed for $\bar{Q}$.
4) if $\bar{Q} \in \mathfrak{K}$, $\alpha = \ell g\bar{Q}$ then $P_\alpha^{\mathrm{cn}}$ is a dense subset of $P_\alpha$.
5) If $b$ is closed for $\bar{Q}$, $p, q \in P_{\ell g\bar{Q}}^{\mathrm{cn}}$, $p \leq q$ in $P_{\ell g\bar{Q}}$ and $i \in \mathrm{dom} p$ then $q \restriction a_i \Vdash_{P_i}$ “$p(i) \leq q(i)$” hence $\Vdash_{P_{a_i}^{\mathrm{cn}}}$ “$p(i) \leq_{Q_i} q(i)$”.

*Definition* 3.7. Suppose $W = (W, \leq)$ is a finite partial order and $\bar{Q} \in \mathfrak{K}$.
1) $IN_W(\bar{Q})$ is the set of $\bar{b}$-s satisfying $(\alpha)$–$(\gamma)$ below:
$(\alpha)$ $\bar{b} = \langle b_w : w \in W\rangle$ is an indexed set of $\bar{Q}$-closed subsets of $\ell g(\bar{Q})$,
$(\beta)$ $W \models$ “$w_1 \leq w_2$” $\Rightarrow b_{w_1} \subseteq b_{w_2}$,
$(\gamma)$ $\zeta \in b_{w_1} \cap b_{w_2}$, $w_1 \leq w$, $w_2 \leq w$ then $(\exists u \in W)\zeta \in b_u \wedge u \leq w_1 \wedge u \leq w_2$.
We assume $\bar{b}$ codes $(W, \leq)$.
2) For $\bar{b} \in IN_W(\bar{Q})$, let

$$\bar{Q}[\bar{b}] = \big\{\langle p_w : w \in W\rangle : p_w \in P_{b_w}^{\mathrm{cn}}, [W \models w_1 \leq w_2 \Rightarrow p_{w_2} \restriction b_{w_1} = p_{w_1}]\big\}$$

with ordering $\bar{Q}[\bar{b}] \models \bar{p}^1 \leq \bar{p}^2$ iff $\bigwedge_{w\in W} p_w^1 \leq p_w^2$.
3) Let $\mathfrak{K}^1$ be the family of $\bar{Q} \in \mathfrak{K}$ such that for every $\beta \leq \ell g(\bar{Q})$ and $(\bar{Q} \restriction \beta)$-closed $b$, $P_\beta$ and $P_\beta / P_b^{\mathrm{cn}}$ satisfy the Knaster condition.

*Fact* 3.8. Suppose $\bar{Q} \in \mathfrak{K}^1$, $(W, \leq)$ is a finite partial order, $\bar{b} \in IN_W(\bar{Q})$ and $\bar{p} \in \bar{Q}[\bar{b}]$.
1) If $w \in W$, $p_w \leq q \in P_{b_w}^{\mathrm{cn}}$ then there is $\bar{r} \in \bar{Q}[\bar{b}]$, $q \leq r_w$, $\bar{p} \leq \bar{r}$, in fact

$$r_u(\gamma) = \begin{cases} p_u(\gamma) & \text{if } \gamma \in \mathrm{Dom}\ p_u \setminus \mathrm{Dom}\ q \\ p_u(\gamma)\ \&\ q(\gamma) & \text{if } \gamma \in b_u \cap \mathrm{Dom}\ q \text{ and for some } v \in W, v \leq u, \\ & \quad v \leq w \text{ and } \gamma \in b_v \\ p_u(\gamma) & \text{if } \gamma \in b_u \cap \mathrm{dom}\ q \text{ but the previous case fails} \end{cases}$$

2) Suppose $(W_1, \leq)$ is a submodel of $(W_2, \leq)$, both finite partial orders, $\bar{b}^l \in IN_{W_l}(\bar{Q})$, $\bar{b}_w^1 = \bar{b}_w^2$ for $w \in W_1$.
$(\alpha)$ If $\bar{q} \in \bar{Q}[\bar{b}^2]$ then $\langle q_w : w \in W_1\rangle \in \bar{Q}[\bar{b}^1]$.
$(\beta)$ If $\bar{p} \in \bar{Q}[\bar{b}^1]$ then there is $\bar{q} \in \bar{Q}[\bar{b}^2]$, $\bar{q} \restriction W_1 = \bar{p}$, in fact $q_w(\gamma)$ is $p_u(\gamma)$ if $u \in W_1$, $\gamma \in b_u$, $u \leq w$, provided that
$(**)$ if $w_1, w_2 \in W_1$, $w \in W_2$, $w_1 \leq w$, $w_2 \leq w$ and $\zeta \in b_{w_1} \cap b_{w_2}$ then for some $v \in W_1$, $\zeta \in b_v$, $v \leq w_1$, $v \leq w_2$.
(this guarantees that if there are several u's as above we shall get the same value).
3) If $\bar{Q} \in \mathfrak{K}^1$ then $\bar{Q}[\bar{b}]$ satisfies the Knaster condition. If $\varnothing$ is the minimal element of $W$ (i.e. $u \in W \Rightarrow W \models \varnothing \leq u$) then $\bar{Q}[\bar{b}]/P_{b_\varnothing}^{\mathrm{cn}}$ also satisfies the Knaster condition and so $\lessdot\bar{Q}[\bar{b}]$, when we identify $p \in P_b^{\mathrm{cn}}$ with $\langle p : w \in W\rangle$.

*Proof.* 1) It is easy to check that each $r_u(\gamma)$ is in $P^{\rm cn}_{b_u}$. So, in order to prove $\bar r \in \bar Q[\bar b]$, we assume $W \models u_1 \le u_2$ and has to prove that $r_{u_2} \restriction b_{u_1} = r_{u_1}$. Let $\zeta \in b_{u_1}$.

First case: $\zeta \notin \mathrm{Dom}(p_{u_1}) \cup \mathrm{Dom} q$.

So $\zeta \notin \mathrm{Dom}(r_{u_1})$ (by the definition of $r_{u_1}$) and $\zeta \notin \mathrm{Dom} p_{u_2}$ (as $\bar p \in \bar Q[\bar b]$) hence $\zeta \notin (\mathrm{Dom} p_{u_2}) \cup (\mathrm{Dom} q)$ hence $\zeta \notin \mathrm{Dom}(r_{u_2})$ by the choice of $r_{u_2}$, so we have finished.

Second case: $\zeta \in \mathrm{Dom} p_{u_1} \setminus \mathrm{Dom} q$.

As $\bar p \in \bar Q[\bar b]$ we have $p_{u_1}(\zeta) = p_{u_2}(\zeta)$, and by their definition, $r_{u_1}(\zeta) = p_{u_1}(\zeta)$, $r_{u_2}(\zeta) = p_{u_2}(\zeta)$.

Third case: $\zeta \in \mathrm{Dom} q$ and $(\exists v \in W)$ $(\zeta \in b_v \wedge v \le u_1 \wedge v \le w)$. By the definition of $r_{u_1}(\zeta)$, we have $r_{u_1}(\zeta) = p_{u_1}(\zeta) \& q(\zeta)$, also the same $v$ witnesses $r_{u_2}(\zeta) = p_{u_2}(\zeta) \& q(\zeta)$, (as $\zeta \in b_v \wedge v \le u_1 \wedge v \le w \Rightarrow \zeta \in b_v \wedge v \le u_2 \wedge v \le w$) and of course $p_{u_1}(\zeta) = p_{u_2}(\zeta)$ (as $\bar p \in \bar Q[\bar b]$).

Fourth case: $\zeta \in \mathrm{Dom} q$ and $\neg(\exists v \in W)$ $(\zeta \in b_v \wedge v \le u_1 \wedge v \le w)$.

By the definition of $r_{u_1}(\zeta)$ we have $r_{u_1}(\zeta) = p_{u_1}(\zeta)$. It is enough to prove that $r_{u_2}(\zeta) = p_{u_2}(\zeta)$ as we know that $p_{u_1}(\zeta) = p_{u_2}(\zeta)$ (because $\bar p \in \bar Q[\bar b]$, $u_1 \le u_2$). If not, then for some $v_0 \in W$, $\zeta \in b_{v_0} \wedge v_0 \le u_2 \wedge v_0 \le w$. But $\bar b \in \mathrm{IN}_W(\bar Q)$, hence (see Def. 3.7(1) condition $(\gamma)$ applied with $\zeta$, $w_1$, $w_2$, $w$ there standing for $\zeta$, $v_0$, $u_1$, $u_2$ here) we know that for some $v \in W$, $\zeta \in v \wedge v \le v_0 \wedge v \le u_1$. As $(W, \le)$ is a partial order, $v \le v_0$ and $v_0 \le w$, we can conclude $v \le w$. So $v$ contradicts our being in the fourth case. So we have finished the fourth case.

Hence we have finished proving $\bar r \in \bar Q[\bar b]$. We also have to prove $q \le r_w$, but for $\zeta \in \mathrm{Dom} q$ we have $\zeta \in b_w$ (as $q \in P^{\rm cn}_w$ is on assumption) and $r_w(\zeta) = q(\zeta)$ because $r_w(\zeta)$ is defined by the second case of the definition as $(\exists v \in W)$ $(\zeta \in b_w \wedge v \le w \wedge v \le w)$, i.e. $v = w$.

Lastly we have to prove that $\bar p \le \bar r$ (in $\bar Q[\bar b]$). So let $u \in W$, $\zeta \in \mathrm{Dom} p_u$ and we have to prove $r_u \restriction \zeta \Vdash_{P_\zeta}$ "$p_u(\zeta) \le_{P_\zeta} r_u(\zeta)$". As $r_u(\zeta)$ is $p_u(\zeta)$ or $p_u(\zeta) \& q(\zeta)$ this is obvious.

2) Immediate.

3) We prove this by induction on $|W|$.
For $|W| = 0$ this is totally trivial.
For $|W| = 1, 2$ this is assumed.
For $|W| > 2$ fix $\bar p^i \in \bar Q[\bar b]$ for $i < \omega_1$. Choose a maximal element $v \in W$ and let $c = \bigcup\{b_w : W \models w < v\}$. Clearly $c$ is closed for $\bar Q$.

We know that $P^{\rm cn}_c$, $P^{\rm cn}_{b_v}/P^{\rm cn}_c$ are Knaster by the induction hypothesis. We also know that $p^i_v \restriction c \in P^{\rm cn}_c$ for $i < \omega_1$, hence for some $r \in P^{\rm cn}_c$,

$$r \Vdash \text{“}\underset{\sim}{A} = \big\{i < \omega_1 : p^i_v \restriction c \in \underset{\sim}{G}_{P^{\rm cn}_c}\big\} \quad \text{is uncountable”}$$

hence

$$\Vdash \text{“there is an uncountable } A^1 \subseteq \underset{\sim}{A} \text{ such that } \big[i, j \in A^1 \Rightarrow p^i_v,\, p^j_v \text{ are compatible in } P^{\rm cn}_{b_v}/\underset{\sim}{G}_{P^{\rm cn}_c}\big]\text{”}.$$

Fix a $P^{\rm cn}_c$-name $\underset{\sim}{A}^1$ for such an $A^1$.

Let $A^2 = \big\{i < \omega_1 : \exists q \in P^{\rm cn}_c,\ q \Vdash i \in \underset{\sim}{A}^1\big\}$. Necessarily $|A_2| = \aleph_1$, and for $i \in A^2$ there is $q^i \in P^{\rm cn}_c$, $q^i \Vdash i \in A^1$, and w.l.o.g. $p^i_v \restriction c \le q^i$. Note that $p^i_v \& q^i \in P^{\rm cn}_c$.

For $i \in A^2$ let, $\bar r^i$ be defined using 3.8(1) (with $\bar p^i$, $p^i_v \& q^i$). Let $W_1 = W \setminus \{v\}$, $\bar b' = \langle b_w : w \in W_1\rangle$.

By the induction hypothesis applied to $W_1$, $\bar b'$, $\bar r^i \restriction W_1$, for $i \in A^2$ there is an uncountable $A^3 \subseteq A^2$ and for $i < j$ in $A^3$, there is $\bar r^{i,j} \in \bar Q[\bar b']$, $\bar r^i \restriction W_1 \le \bar r^{i,j}$, and $\bar r^j \restriction W_1 \le \bar r^{i,j}$. Now define $r^{i,j}_c \in P^{\rm cn}_c$ as follows: its domain is $\bigcup\big\{\mathrm{dom} r^{i,j}_w : W \models w < v\big\}$, $r^{i,j}_c \restriction (\mathrm{dom} r^{i,j}_w) = r^{i,j}_w$ whenever $W \models w < v$. Why is this a definition? As if $W \models w_1 \le v \wedge w_2 \le v$, $\zeta \in b_{w_1} \wedge \zeta \in b_{w_2}$ then for some $u \in W$, $u \le w_1 \wedge u \le w_2$ and $\zeta \in u$. It is easy to check that $r^{i,j}_c \in P^{\rm cn}_c$. Now $r^{i,j}_c \Vdash_{P^{\rm cn}_c}$ "$p^i_{b_v}$, $p^j_{b_v}$ are compatible in $P^{\rm cn}_{b_v}/P^{\rm cn}_c$".

So there is $r \in P^{\rm cn}_{b_v}$such that $r^{i,j}_c \le r$, $p^i_{b_v} \le r$, $p^j_{b_v} \le r$. As in part (1) of 3.8 we can combine $r$ and $\bar r^{i,j}$ to a common upper bound of $\bar p^i$, $\bar p^j$ in $\bar Q[\bar b]$. $\square$

*Claim* 3.9. *3.9.If $e = 0, 1$ and $\delta$ is a limit ordinal, and $P_i, \underset{\sim}{Q}_i, \alpha_i, e^*_i (i < \delta)$ are such that for each $\alpha < \delta$, $\bar Q^\alpha = \langle P_i, \underset{\sim}{Q}_j, \alpha_j, e^*_j : i \le \alpha,\ j < \alpha\rangle$ belongs to $\mathfrak{K}^\ell$, then for a unique $P_\delta$, $\bar Q = \langle P_i, \underset{\sim}{Q}_j, \alpha_j, e^*_j : i \le \delta,\ j < \delta\rangle$ belongs to $\mathfrak{K}^\ell$.*

*Proof.* We define $P_\delta$ by (d) of Definition 3.4. The least easy problem is to verify the Knaster conditions (for $\bar Q \in \mathfrak{K}^1$). The proof is like the preservation of the c.c.c. under iteration for limit stages. $\square$

**Convention 3.9A.** *By 3.9 we shall not distinguish strictly between $\langle P_i, \underset{\sim}{Q}_j, \alpha_j, e^*_j : i \le \delta,\ j < \delta\rangle$ and $\langle P_i, \underset{\sim}{Q}_i, \alpha_i, e^*_i : i < \delta\rangle$.*

*Claim* 3.10. *If $\bar Q \in \mathfrak{K}^\ell$, $\alpha = \ell g(\bar Q)$, $a \subset \alpha$ is closed for $\bar Q$, $|a| \le \aleph_1$, $\underset{\sim}{Q}_1$ is a $P^{\rm cn}_a$-name of a forcing notion satisfying (in $V^{P_\alpha}$) the Knaster condition, its underlying set is a subset of $[\omega_1]^{<\aleph_0}$ then there is a unique $\bar Q^1 \in \mathfrak{K}^\ell$, $\ell g(\bar Q_1) = \alpha + 1$, $Q^1_\alpha = \underset{\sim}{Q}$, $\bar Q \restriction \alpha = \bar Q$.*

*Proof.* Left to the reader. $\square$

*Proof.* 3.1 **A Stage:** We force by $\mathfrak{K}^1_{<\lambda} = \{\bar Q \in \mathfrak{K}^1 : \ell g(\bar Q) < \lambda, \bar Q \in H(\lambda)\}$ ordered by being an initial segment (which is equivalent to forcing a Cohen subset of $\lambda$). The generic object is essentially $\bar Q^* \in \mathfrak{K}^1_\lambda$, $\ell g(\bar Q^*) = \lambda$, and then we force by $P_\lambda = \lim \bar Q^*$. Clearly $\mathfrak{K}^\ell_{<\lambda}$ is a $\lambda$-complete forcing notion of cardinality $\lambda$, and $P_\lambda$ satisfies the c.c.c. Clearly it suffices to prove part (2) of 3.1.

Suppose $\underset{\sim}{d}_n$ is a name of a function from $[\lambda]^n$ to $\underset{\sim}{k}_n$ for $n < \omega$, $\underset{\sim}{\sigma}_n < \omega$, $\langle \sigma_n : n < \omega\rangle$ diverges (i.e. $\forall m\ \exists k\ \forall n \ge k\ \sigma_n \ge m$) and for some $\bar Q^0 \in \mathfrak{K}^1_{<\lambda}$.

$$\bar Q^0 \Vdash_{\mathfrak{K}^1_{<\lambda}} \text{“there is } p \in \underset{\sim}{P}_\lambda\big[p \Vdash_{P_\lambda} \langle \underset{\sim}{d}_n : n < \omega\rangle \text{ is a counterexample to (2) of 3.1”}\big].$$

In $V$ we can define $\langle \bar Q^\zeta : \zeta < \lambda\rangle$, $\bar Q^\zeta \in \mathfrak{K}^1_{<\lambda}$, $\zeta < \xi \Rightarrow \bar Q^\zeta = \bar Q^\xi \restriction \ell g(\bar Q^\zeta)$, in $\bar Q^{\zeta+1}$, $e^*_{\ell g(\bar Q_\zeta)} = 1$, $\bar Q^{\zeta+1}$ forces (in $\mathfrak{K}^1_{<\lambda}$) a value to $p$ and the $\underset{\sim}{P}_\lambda$-names $\underset{\sim}{d}_n \restriction \zeta$, $\underset{\sim}{\sigma}_n$, $\underset{\sim}{k}_n$ for $n < \omega$, i.e. the values here are still $P_\lambda$-names. Let $\bar Q^*$ be the limit of the $\bar Q^\xi$-s. So $\bar Q^* \in \mathfrak{K}^1$, $\ell g(\bar Q^*) = \lambda$, $\bar Q^* = \langle P^*_i, \underset{\sim}{Q}^*_j, \alpha^*_j, e^*_j : i \le \lambda, j < \lambda\rangle$, and the $P^*_\lambda$-names $\underset{\sim}{d}_n$, $\underset{\sim}{\sigma}_n$, $\underset{\sim}{k}_n$ are defined such that in $V^{\tilde P^*_\lambda}$, $\underset{\sim}{d}_n$, $\underset{\sim}{\sigma}_n$, $\underset{\sim}{k}_n$ contradict (2) (as any $P^*_\lambda$-name of a bounded subset of $\lambda$ is a $P^*_{\ell g(\bar Q^\xi)}$-name for some $\xi < \lambda$). **B Stage:** Let $\chi = \kappa^+$and $<^*_\chi$ be a well-ordering of $\mathcal{H}(\chi)$. Now we can apply $\lambda \to (\omega_1)^{<\omega}_2$ to get $\delta, B, N_s$ (for $s \in [B]^{<\aleph_0}$) and $\mathbf{h}_{s,t}$ (for $s, t \in [B]^{<\aleph_0}$, $|s| = |t|$) such that:

(a) $B \subseteq \lambda$, $\mathrm{otp}(B) = \omega_1$, $\sup B = \delta$,
(b) $N_s \prec (H(\chi), \in, <^*_\chi)$, $\bar Q^* \in N_s$, $\langle \underset{\sim}{d}, \underset{\sim}{\sigma}_n, \underset{\sim}{k}_n : n < \omega\rangle \in N_s$,
(c) $N_s \cap N_t = N_{s \cap t}$,
(d) $N_s \cap B = s$,
(e) if $s = t \cap \alpha$, $t \in [B]^{<\aleph_0}$ then $N_s \cap \lambda$ is an initial segment of $N_t$,
(f) $\mathbf{h}_{s,t}$ is an isomorphism from $N_t$ onto $N_s$ (when defined)
(g) $\mathbf{h}_{t,s} = \mathbf{h}^{-1}_{s,t}$
(h) $p_0 \in N_s$, $p_0 \Vdash_{P_\lambda}$ “$\langle \underset{\sim}{d}_n, \underset{\sim}{\sigma}_n, \underset{\sim}{k}_n : n <\rangle$ is a counterexample”,
(i) $\omega_1 \subseteq N_s$, $|N_s| = \aleph_1$ and if $\gamma \in N_s$, $\mathrm{cf}\gamma > \aleph_1$ then $\mathrm{cf}(\sup(\gamma \cap N_s)) = \omega_1$.

Let $\bar Q = \bar Q^* \restriction \delta$, $P = P^*_\delta$ and $P_a = P^{\rm cn}_a$ (for $\bar Q$), where $a$ is closed for $\bar Q$. Note: $P^*_\lambda \cap N_s = P^*_\delta \cap N_s = P_{\sup \lambda \cap N_s} \cap N_s = P_s \cap N_s$. Note also $\gamma \in \lambda \cap N_s \Rightarrow a^*_\gamma \subseteq \lambda \cap N_s$.

**C Stage:** It suffices to show that we can define $\underset{\sim}{Q}_\delta$ in $V^{P_\delta}$ which forces a subset $W$ of $B$ of cardinality $\aleph_1$ and $\underset{\sim}{F} : W \to {}^\omega 2$ which exemplify the desired conclusion in

(2), and prove that $\underset{\sim}{Q}_\delta$ satisfies the $\aleph_1$-c.c.c. (in $V^{P_\delta}$ (and has cardinality $\aleph_1$)) and moreover (see Definitions 3.4 and 3.7(3)) we also define $a_\delta = \bigcup_{s\in[B]^{<\aleph_0}} N_s$, $e_\delta = 1$, $\bar{Q}' = \bar{Q}\,\hat{}\,\langle P^*_\delta, \underset{\sim}{Q}_\delta, a_\delta, e_\delta\rangle$ and prove $\bar{Q}' \in \mathfrak{K}^1$.
We let $\underset{\sim}{d}(u) = \underset{\sim}{d}_{|u|}(u)$.

Let $F : \omega_1 \to {}^\omega 2$ be one-to-one such that $[\forall \eta \in {}^{\omega >}2][\exists^{\aleph_1}\alpha < \omega_1][\eta \triangleleft F(\alpha)]$. (This will not be the needed $\underset{\sim}{F}$, just notation).

For $s,t \in [B]^{<\aleph_0}$, we say $s \equiv^n_F t$ if $|s| = |t|$ and $\forall \xi \in s$, $\forall \zeta \in t[\xi = \mathbf{h}_{s,t}(\zeta) \Rightarrow F(\xi) \restriction n = F(\zeta) \restriction n]$. Let

$$I_n = I_n(F) = \big\{s \in [B]^{<\aleph_0} : (\forall \zeta \neq \xi \in s),\ [F(\zeta) \restriction n \neq F(\xi) \restriction n]\big\}.$$

We define $R_n$ as follows: a sequence $\langle p_s : s \in I_n\rangle \in R_n$ if and only if

(i) for $s \in I_n$, $p_s \in P^*_\lambda \cap N_s$,
(ii) for some $c_s$ we have $p_s \Vdash$ "$\underset{\sim}{d}(s) = c_s$",
(iii) for $s,t \in I_n$, $s \equiv^n_F t \Rightarrow \mathbf{h}_{s,t}(p_t) = p_s$,
(iv) for $s,t \in I_n$, $p_s \restriction N_{s\cap t} = p_t \restriction N_{s\cap t}$.

$R^-_n$ is defined similarly omitting (ii).

For $x = \langle p_s : s \in I_n\rangle$ let $n(x) = n$, $p^x_s = p_s$, and (if defined) $c^x_s = c_s$. Note that we could replace $x \in R_n$ by a finite subsequence. Let $R = \bigcup_{n<\omega} R_n$, $R^- = \bigcup_{n<\omega} R^-_n$. We define an order on $R^-$ : $x \leq y$ if and only if $n(x) \leq n(y)$, and $[s \in I_{n(x)} \wedge t \in I_{n(y)} \wedge s \subseteq t \Rightarrow p^x_s \leq p^y_t]$.

**D Stage:** Note the following facts:
**D$(\alpha)$ Subfact:** If $x \in R^-_n$, $t \in I_n$ and $p^x_t \leq p^1 \in P^*_\delta \cap N_t$, then there is $y$ such that $x \leq y \in R^-_n$, $p^y_t = p^1$.

*Proof.* We let for $s \in I_n$

$$p^y_s = \&\big\{\mathbf{h}_{s_1,t_1}(p^1 \restriction N_{t_1}) : s_1 \subseteq s, t_1 \subseteq t, s_1 \equiv^n_F t_1\big\}\,\& p^x_s.$$

(This notation means that $p^y_s$ is a function whose domain is the union of the domains of the conditions mentioned, and for each coordinate we take the canonical upper bound, see preliminaries.) Why is $p^y_s$ well defined? Suppose $\beta \in N_s \cap \lambda$ (for $\beta \in \lambda \setminus N_s$, clearly $p^y_s(\beta) = \varnothing_\beta$), $s_\ell \subseteq s$, $t_\ell \subseteq t$, $s_\ell \equiv^n_F t_\ell$ for $\ell = 1,2$ and $\beta \in \mathrm{Dom}\Big[\mathbf{h}_{s_\ell,t_\ell}(p^1 \restriction N_{t_\ell})\Big]$, and it suffices to show that $p^x_s(\beta)$, $\mathbf{h}_{s_1,t_1}(p^1 \restriction N_{t_1})(\beta)$, $\mathbf{h}_{s_2,t_2}(p^1 \restriction N_{t_2})(\beta)$ are pairwise comparable. Let $u = \bigcap\big\{v \in [B]^{<\aleph_0} : \beta \in N_v\big\}$, necessarily $u \subseteq s_1 \cap s_2$, and let $u_\ell = \mathbf{h}^{-1}_{s_\ell,t_\ell}(u)$. As $s_\ell, t_\ell, t \in I_n$, $s_\ell \equiv^n_F t_\ell$ and $u_\ell \subseteq t_\ell \subseteq t$, necessarily $u_1 = u_2$. Thus $\gamma = \mathbf{h}^{-1}_{u,v}(\beta) = \mathbf{h}^{-1}_{s_\ell,t_\ell}(\beta)$ and so the last two conditions are equal.

Now $p^x_s(\beta) = p^x_u(\beta) = \mathbf{h}_{u,v}(p^x_s(\gamma)) \leq \mathbf{h}_{s_\ell,t_\ell}\big((p^x_t \restriction N_{t_\ell})(\gamma)\big) = \Big(\mathbf{h}_{s_\ell,t_\ell}(p^x_t \restriction N_{t_\ell})\Big)(\beta)$.

We leave to the reader checking the other requirements. $\square$

**D$(\beta)$ Subfact:** If $x \in R^-_n$, $t \in I$ then $\bigcup\big\{p^x_s : s \in I_n,\ s \subseteq t\big\}$ (as union of functions) exists and belongs to $P^*_\lambda \cap N_t$.

*Proof.* See (iv) in the definition of $R^-_n$. $\square$

**D$(\gamma)$ Subfact:** If $x \leq y$, $x \in R_n$, $y \in R^-_n$, *then* $y \in R_n$.

*Proof.* Check it. $\square$

**D$(\delta)$ Subfact:** If $x \in R^-_n$, $n < m$, *then* there is $y \in R_m$, $x \leq y$.

*Proof.* By subfact D($\beta$) we can find $x^1 = \langle p^1_t : t \in I_m\rangle \in inR^-_m$ with $x \le x^1$. Using repeatedly subfact D($\alpha$) we can increase $x^1$ (finitely many times) to get $y \in R_m$. □

**D($\varepsilon$) Subfact:** If $x \in R^-_n$, $s, t \in I_n$, $s \equiv^n_F t$, $p^x_s \le r_1 \in P^*_\lambda \cap N_s$, $p^x_t \le r_2 \in P^*_\lambda \cap N_t$, $(\forall \zeta \in t)\big[F(\zeta)(n) \ne \big(F\big(\mathbf{h}_{s,t}(\zeta)\big)\big)(n)\big]$ ( or just $p^x_{s_1} \restriction s_1 = \mathbf{h}_{s,t}(p^x_{t_1} \restriction t_1)$ where $t_1 = \{\xi \in t : F(\xi)(n) = (F(\mathbf{h}_{s,t}(\xi)))(n)\}$, $s_1 = \{\mathbf{h}_{s,t}(\xi) : \xi \in t_1\}$), then there is $y \in R_{n+1}$, $x \le y$ such that $r_1 = p^y_s$ and $r_2 = p^y_t$.

*Proof.* Left to the reader. □

**E Stage:** [1]

We define: $T^*_k \subseteq {}^{2^k \ge}2$ by induction on $k$ as follows:

$$\begin{aligned} T^*_0 =& \{\langle\rangle, \langle 1\rangle\} \\ T^*_{k+1} =& \big\{\nu : \nu \in T^*_k \text{ or } 2^k < \ell g(\nu) \le 2^{k+1} \ , \nu \restriction 2^k \in T^*_k \text{ and} \\ & [2^k \le i < 2^{k+1} \wedge \nu(i) = 1] \Rightarrow i = 2^k + (\sum_{m<2^k} \nu(i)2^m)]\big\}. \end{aligned}$$

We define

$$\begin{aligned} \text{Tr Emb}(k,n) = \Big\{ & h : h \text{ a is function from } T^*_k \text{ into } {}^{n\ge}2 \text{ such that} \\ & \text{for } \nu, \rho \in T^*_k : \\ & [\eta = \nu \Leftrightarrow h(\eta) = h(\nu)] \\ & [\eta \lhd \nu \Leftrightarrow h(\eta) \lhd h(\nu)] \\ & [\ell g(\eta) = \ell g(\nu) \Rightarrow \ell g(h(\eta) = \ell g(h(\nu))] \\ & \big[\nu = \eta\hat{\ }\langle i\rangle \Rightarrow (h(\nu))[\ell g(h(\eta))] = i\big] \\ & \big[\ell g(\eta) = {}^k 2 \Rightarrow \ell g(h(\eta)) = n\big] \Big\}. \end{aligned}$$

$$\begin{aligned} \mathbf{T}(k,n) =& \big\{\text{Rang } h : h \in \text{Tr Emb}(k,n)\big\}, \\ \mathbf{T}(*,n) =& \bigcup_k \mathbf{T}(k,n), \\ \mathbf{T}(k,*) =& \bigcup_k \mathbf{T}(k,n). \end{aligned}$$

For $T \in \mathbf{T}(k,*)$ let $n(T)$ be the unique $n$ such that $T \in \mathbf{T}(k,n)$ and let

$$B_T = \big\{\alpha \in B : F(\alpha) \restriction n(T) \text{ is a maximal member of } T\big\},$$

$$fs_T = \Big\{t \subseteq B_T : \eta \in t \wedge \nu \in t \wedge \eta \ne \nu \Rightarrow \eta \restriction n(T) \ne \nu \restriction n(T)\big\},$$

$$\Theta_T = \Big\{\langle p_s : s \in fs_T\rangle : p_s \in P \cap N_s, \big[s \subseteq t \wedge \{s,t\} \subseteq fs_T \Rightarrow p_s = p_t \restriction N_s\big]\Big\}.$$

Let further

$$\begin{aligned} \Theta_k =& \bigcup \{\Theta_T : T \in \mathbf{T}(k,*)\} \\ \Theta =& \bigcup_k \Theta_k. \end{aligned}$$

For $\bar p \in \Theta$, $\mathbf{n}_{\bar p} = \mathbf{n}(\bar p)$, $T_{\bar p}$ are defined naturally.

For $\bar p, \bar q \in \Theta$, $\bar p \le \bar q$ iff $\mathbf{n}_{\bar p} \le \mathbf{n}_{\bar q}$ and for every $s \in fs_{T_{\bar p}}$ we have $p_s \le q_s$. **F Stage:** Let $\underset{\sim}{g} : \omega \to \omega$, $\underset{\sim}{g} \in N_s$, $\underset{\sim}{g}$ grows fast enough relative $\langle \sigma_n : n < \omega\rangle$. We

[1] We will have $T \subset {}^{\omega >}2$ gotten by 2.7(2) and then want to get a subtree with as few as possible colors, we can find one isomorphic to ${}^{\omega >}2$, and there restrict ourselves to $\cup_n T^*_n$.

define a game Gm. A play of the game lasts after $\omega$ moves, in the $n^{\text{th}}$ move player I chooses $\bar{p}^n \in \Theta_n$ and a function $h_n$ satisfying the restrictions below and then player II chooses $\bar{q}_n \in \Theta_n$, such that $\bar{p}_n \le \bar{q}_n$ (so $T_{\bar{p}_n} = T_{\bar{q}_n}$). Player I loses the play if sometimes he has no legal move; if he never loses, he wins. The restrictions player I has to satisfy are:
(a) for $m < n$, $\bar{q}_m \le \bar{p}_n$, $p^n_s$ forces a value to $\underset{\sim}{g} \restriction (n+1)$,
(b) $h_n$ is a function from $[B_{T_{\bar{p}_n}}]^{\le g(n)}$ to $\omega$,
(c) if $m < n \Rightarrow h_n, h_m$ are compatible,
(d) If $m < n$, $\ell < g(m)$, $s \in [B_{T_{\bar{p}_n}}]^\ell$, then $p^n_s \Vdash \underset{\sim}{d}(s) = h_n(s)$,
(e) Let $s_1, s_2 \in \mathrm{Dom} h_n$. Then $h_n(s_1) = h_n(s_2)$ whenever $s_1, s_2$ are similar over $n$ which means:
(i) $\Big(F\Big(H^{\mathrm{OP}}_{s_2,s_1}(\zeta)\Big)\Big) \restriction \mathbf{n}[\bar{p}^n] = \Big(F(\zeta)\Big) \restriction \mathbf{n}[\bar{p}^n]$ for $\zeta \in s_1$,
(ii) $H^{\mathrm{OP}}_{s_2,s_1}$ preserves the relations $\Big(F(\zeta_1), F(\zeta_2)\Big) < \Big(F(\zeta_3), F(\zeta_4)\Big)$ and $F(\zeta_3)\Big(F(\zeta_1), F(\zeta_2)\Big) = i$ (in the interesting case $\zeta_3 \ne \zeta_1, \zeta_2$ implies i = 0).

**G Stage/Claim:** Player I has a winning strategy in this game.

*Proof.* As the game is closed, it is determined, so we assume player II has a winning strategy , and eventually we shall get a contradiction. We define by induction on $n$, $\bar{r}^n$ and $\Phi^n$ such that
(a) $\bar{r}^n \in R_n$, $\bar{r}^n \le \bar{r}^{n+1}$,
(b) $\Phi^n$ is a finite set of initial segments of plays of the game,
(c) in each member of $\Phi^n$ player II uses his winning strategy,
(d) if $y$ belongs to $\Phi^n$ then it has the form $\langle \bar{p}^{y,\ell}, h^{y,\ell}, \bar{q}^{y,\ell} : \ell \le m(y)\rangle$; let $h_y = h^{y,n_y}$ and $T_y = T_{\bar{q}^{y,m(y)}}$; also $T_y \subseteq^{n\ge} 2$, $q^{y,\ell}_s \le r^n_s$ for $s \in fs_{T_y}$.
(e) $\Phi_n \subseteq \Phi_{n+1}$, $\Phi_n$ is closed under taking the initial segments and the empty sequence (which too is an initial segment of a play) belongs to $\Phi_0$.
(f) For any $y \in \Phi_n$ and $T, h$ *either* for some $z \in \Phi_{n+1}$, $n_z = n_y + 1$, $y = z \restriction (n_y + 1)$, $T_z = T$ and $h_z = h$ *or* player I has no legal $(n_y+1)^{\text{th}}$ move $\bar{p}^n, h^n$ (after $y$ was played) such that $T_{\bar{p}^n} = T$, $h^n = h$, and $p^n_s = r^n_s$ for $s \in fs_T$ (or always $\le$ or always $\ge$).

There is no problem to carry the definition. Now $\langle \bar{r}^n_s : n < \omega\rangle$ define a function $d^*$: if $\eta_1, \dots, \eta_k \in^m 2$ are distinct then $d^*\big(\langle \eta_1, \dots, \eta_k\rangle\big) = c$ *iff* for every (equivalently some) $\zeta_1 < \cdots < \zeta_k$ from $B$, $\eta_\ell \lhd F(\zeta_\ell)$ and $r^k_{\{\zeta_1,\dots,\zeta_k\}} \Vdash$ "$\underset{\sim}{d}_k\big(\{\zeta_1, \dots, \zeta_k\}\big) = c$".

Now apply 2.7(2) to this coloring, get $T^* \subseteq^{\omega>} 2$ as there. Now player I could have chosen initial segments of this $T^*$ (in the $n^{\text{th}}$ move in $\Phi_n$) and we get easily a contradiction. $\square$

**H Stage:** We fix a winning strategy for player I (whose existence is guaranteed by stage G).

We define a forcing notion $Q^*$. We have $(r, y, f) \in Q^*$ *iff*
(i) $r \in P^{\mathrm{cn}}_{a_\delta}$
(ii) $y = \langle \bar{p}^\ell, h^\ell, \bar{q}^\ell : \ell \le m(y)\rangle$ is an initial segment of a play of Gm in which player I uses his winning strategy
(iii) $f$ is a finite function from $B$ to $\{0,1\}$ such that $f^{-1}(\{1\}) \in fs_{T_y}$ (where $T_y = T_{\bar{q}^{m(y)}}$ ).
(iv) $r = q^{y,m(y)}_{f^{-1}(\{1\})}$.

The *Order* is the natural one.

**I Stage:** If $\underset{\sim}{J} \subseteq P^{\mathrm{cn}}_{a_\delta}$ is dense open then $\big\{(r, y, f) \in Q^* : r \in \underset{\sim}{J}\big\}$ is dense in $Q^*$.

*Proof.* By 3.8(1) (by the appropriate renaming). $\square$

**J Stage:** We define $Q_\delta$ in $V^{P_\delta}$ as $\{(r,y,f) \in Q^* : r \in \underset{\sim}{G}_{P_\delta}\}$, the order is as in $Q^*$.

The main point left is to prove the Knaster condition for the partial ordered set $\bar{Q}^* = \bar{Q}\hat{\ }\langle P_\delta, \underset{\sim}{Q}_\delta, a_\delta, e_\delta\rangle$ demanded in the definition of $\mathfrak{K}^1$. This will follow by 3.8(3) (after you choose meaning and renamings) as done in stages K,L below.

**K Stage:** So let $i < \delta$, $\mathrm{cf}(i) \neq \aleph_1$, and we shall prove that $P^+_{\delta+1}/P_i$ satisfies the Knaster condition. Let $p_\alpha \in P^*_{\delta+1}$ for $\alpha < \omega_1$, and we should find $p \in P_i$, $p \Vdash_{P_i}$ "*there is an unbounded* $A \subseteq \{\alpha : p_\alpha \restriction i \in \underset{\sim}{G}_{P_i}\}$ *such that for any* $\alpha, \beta \in A$, $p_\alpha, p_\beta$ *are compatible in* $P^*_{\delta+1}/\underset{\sim}{G}_{P_i}$".

Without loss of generality:

(a) $p_\alpha \in P^{cn}_{\delta+1}$.

(b) for some $\langle i_\alpha : \alpha < \omega_1\rangle$ increasing continuous with limit $\delta$ we have: $i_0 > i$, $\mathrm{cf} i_\alpha \neq \aleph_1$, $p_\alpha \restriction \delta \in P_{i_{\alpha+1}}$, $p_\alpha \restriction i_\alpha \in P_{i_0}$.

Let $p^0_\alpha = p^\alpha \restriction i_0$, $p^1_\alpha = p_\alpha \restriction \delta = p_\alpha \restriction i_{\alpha+1}$, $p_\alpha(\delta) = (r_\alpha, y_\alpha, f_\alpha)$, so without loss of generality

(c) $r_\alpha \in P_{i_{\alpha+1}}$, $r_\alpha \restriction i_\alpha \in P_{i_0}$, $m(y_\alpha) = m^*$,

(d) $\mathrm{Dom} f_\alpha \subseteq i_0 \cup [i_\alpha, i_{\alpha+1})$,

(e) $f_\alpha \restriction i_0$ is constant (remember $\mathrm{otp}(B) = \omega_1$,

(f) if $\mathrm{Dom} f_\alpha = \{j^\alpha_0, \dots j^\alpha_{k_\alpha - 1}\}$ then $k_\alpha = k$, $[j^\alpha_\ell < i_\alpha \Leftrightarrow \ell < k^*]$, $\bigwedge_{\ell < k^*} j^\alpha_\ell = j^\ell$, $f(j^\alpha_\ell) = f(j^\beta_\ell)$, $F(j^\alpha_\ell)) \restriction m(y_\alpha) = F(j^\beta_\ell) \restriction m(y_\beta)$.

The main problem is the compatibility of the $q^{y_\alpha, m(y_\alpha)}$. Now by the definition $\Theta_\alpha$ (in stage E) and 3.8(3) this holds. $\square$

**L Stage:** *If* $c \subset \delta + 1$ *is closed for* $\bar{Q}^*$, *then* $P^*_{\delta+1}/P^{cn}_c$ *satisfies the Knaster condition.*

*If* $c$ *is bounded in* $\delta$, *choose a successor* $i \in (\sup c, \delta)$ *for* $\bar{Q} \restriction i \in \mathfrak{K}_1$. *We know that* $P_i/P^{cn}_c$ *satisfies the Knaster condition and by stage K,* $P^*_{\delta+1}/P_i$ *also satisfies the Knaster condition; as it is preserved by composition we have finished the stage.*

*So assume* $c$ *is unbounded in* $\delta$ *and it is easy too. So as seen in stage J, we have finished the proof of 3.1.*

*Theorem* 3.11. *If* $\lambda \geq \beth_\omega$, $P$ *is the forcing notion of adding* $\lambda$ *Cohen reals then*

$(*)_1$ *in* $V^P$, *if* $n < \omega$ $d : [\lambda]^{\leq n} \to \sigma$, $\sigma < \aleph_0$, *then for some c.c.c. forcing notion* $Q$ *we have* $\Vdash_Q$ "there are an uncountable $A \subseteq \lambda$ and an one-to-one $F : A \to {}^\omega 2$ such that $d$ is $F$-canonical on $A$" *(see notation in §2).*

$(*)_2$ *if in* $V$, $\lambda \geq \mu \to_{\mathrm{wsp}} (\kappa)_{\aleph_0}$ *(see [Sh289]) and in* $V^P$, $d : [\mu]^{\leq n} \to \sigma$, $\sigma < \aleph_0$ *then in* $V^P$ *for some c.c.c. forcing notion* $Q$ *we have* $\Vdash_Q$ "there are $A \in [\mu]^\kappa$ and one-to-one $F : A \to^\omega 2$ such that $d$ is $F$-canonical on $A$" *(see §2, ).*

$(*)_3$ *if in* $V$, $\lambda \geq \mu \to_{\mathrm{wsp}} (\aleph_1)^n_{\aleph_2}$ *and in* $V^P$ $d : [\mu]^{\leq n} \to \sigma$, $\sigma < \aleph_0$ *<u>then</u> in* $V^P$ *for every* $\alpha < \omega_1$ *and* $F : \alpha \to^\omega 2$ *for some* $A \subseteq \mu$ *of order type* $\alpha$ *and* $F' : A \to^\omega 2$, $F'(\beta) = F(\mathrm{otp}(A \cap \beta))$, $d$ *is* $F'$*-canonical on* $A$.

$(*)_4$ *in* $V^P$, $2^{\aleph_0} \to (\alpha, n)^3$ *for every* $\alpha < \omega_1$, $n < \omega$. *Really, assuming* $V \models$ *GCH, we have* $\aleph_{n^1_3} \to (\alpha, n)$ *see [Sh289].*

*Proof.* Similar to the proof of 3.1. Superficially we need more indiscernibility then we get, but getting $\langle M_u : u \in [B]^{\leq n}\rangle$ we ignore $d(\{\alpha, \beta\})$ when there is no $u$ with $\{\alpha, \beta\} \in M_u$. $\square$

*Theorem* 3.12. *If* $\lambda$ *is strongly inaccessible* $\omega$*-Mahlo,* $\mu < \lambda$, *then for some c.c.c. forcing notion* $P$ *of cardinality* $\lambda$, $V^P$ *satisfies*

*(a)* $MA_\mu$

*(b)* $2^{\aleph_0} = \lambda = 2^\kappa$ *for* $\kappa < \lambda$

*(c)* $\lambda \to [\aleph_1]^n_{\sigma, h(n)}$ *for* $n < \omega$, $\sigma < \aleph_0$, $h(n)$ *is as in 3.1.*

*Proof.* Again, like 3.1. $\square$

## § 4. Partition theorem for trees on large cardinals

*Lemma* 4.1. *Suppose $\mu > \sigma + \aleph_0$ and*
$(*)_\mu$ *for every $\mu$-complete forcing notion $P$, in $V^P$, $\mu$ is measurable.*
*Then*

*(1) for $n < \omega$, $Pr^f_{eht}(\mu, n, \sigma)$.*
*(2) $Pr^f_{eht}(\mu, < \aleph_0, \sigma)$, if there is $\lambda > \mu$, $\lambda \to \left(\mu^+\right)^{<\omega}_2$.*
*(3) In both cases we can have the $Pr^f_{ehtn}$ version, and even choose the $\langle <^*_\alpha : \alpha < \mu\rangle$ in any of the following ways.*
  *(a) We are given $\langle <^0_\alpha : \alpha < \mu\rangle$, and we let for $\eta, \nu \in^\alpha 2 \cap T$, $\alpha \in SP(T)$ ($T$ is the subtree we consider):*
    - *$\eta <^*_\alpha \nu$ if and only if $\mathrm{clp}_T(\eta) <^0_\beta \mathrm{clp}_T(\nu)$ where $\beta = \mathrm{otp}(\alpha \cap SP(T))$ and $\mathrm{clp}_T(\eta) = \langle \eta(j) : j \in \ell g(\eta), j \in \mathrm{SP}(T)\rangle$.*
  *(b) We are given $\langle <^0_\alpha : \alpha < \mu\rangle$, we let that for $\nu, \eta \in^\alpha 2 \cap T$, $\alpha \in SP(T)$: $\eta <^*_\alpha \nu$ if and only if $n \restriction (\beta + 1) <^0_{\beta+1} \nu \restriction (\beta + 1)$ where $\beta = \sup(\alpha \cap SP(T))$.*

*Remark* 4.2. 1) $(*)_\mu$ holds for a supercompact after Laver treatment. On hypermeasurable see Gitik Shelah [GS89].
2) We can in $(*)_\mu$ restrict ourselves to the forcing notion $P$ actually used. For it by Gitik [M. Gitik, Measurability preserved by $\kappa$-complete forcing notion] much smaller large cardinals suffice.
3) The proof of 4.1 is a generalization of a proof of Harrington to Halpern Lauchli theorem from 1978.

*Conclusion* 4.3. *In 4.1 we can get $Pr^f_{ht}(\mu, n, \sigma)$ (even with (3)).*

*Proof.* 4.3 We do the parallel to 4.1(1). By $(*)_\mu$, $\mu$ is weakly compact hence by 2.6(2) it is enough to prove $Pr^f_{aht}(\mu, n, \sigma)$. This follows from 4.1(1) by 2.6(1). $\square$

*Proof.* 4.1 1), 2). Let $\kappa \leq \omega$, $\sigma(n) < \mu$, $d_n \in \mathrm{Col}^n_{\sigma(n)}({}^{\mu>}2)$ for $n < \kappa$.

Choose $\lambda$ such that $\lambda \to (\mu^+)^{<2\kappa}_{2^\mu}$ (there is such a $\lambda$ by assumption for (2) and by $\kappa < \omega$ for (1)). Let $Q$ be the forcing notion $({}^{\mu>}2, \triangleleft)$, and $P = P_\lambda$ be $\{f : \mathrm{dom}(f)$ *is a subset of* $\lambda$ *of cardinality* $< \mu$, $f(i) \in Q\}$ ordered naturally. For $i \notin \mathrm{dom}(f)$, take $f(i) = \langle\ \rangle$. Let $\underset{\sim}{\eta}_i$ be the P-name for $\bigcup\{f(i) : f \in \underset{\sim}{G}_P\}$. Let $\underset{\sim}{D}$ be a P-name of a normal ultrafilter over $\mu$. For each $n < \omega$, $d \in \mathrm{Col}^n_{\sigma(n)}({}^{\mu>}2)$, $j < \sigma(n)$ and $u = \{\alpha_0, \ldots, \alpha_{n-1}\}$, where $\alpha_0 < \cdots < \alpha_{n-1} < \lambda$, let $\underset{\sim}{A}^j_d(u)$ be the $P_\lambda$-name of the set

$$A^j_d(u) = \left\{i < \mu : \langle \underset{\sim}{\eta}_{\alpha_\ell} \restriction i : \ell < n\rangle \text{ are pairwise distinct, } j = d(\eta_{\alpha_0} \restriction i, \ldots, \eta_{\alpha_{n-1}} \restriction i)\right\}.$$

So $\underset{\sim}{A}^j_d(u)$ is a $P_\lambda$-name of a subset of $\mu$, and for $j(1) < j(2) < \sigma(n)$ we have $\Vdash_{P_\lambda}$ "$\underset{\sim}{A}^{j(1)}_d(u) \cap \underset{\sim}{A}^{j(2)}_d(u) = \varnothing$, *and* $\bigcup_{j<\sigma(n)} \underset{\sim}{A}^j_d(u)$ *is a co-bounded subset of* $\mu$". As $\Vdash_P$ "$\mathfrak{D}$ is $\mu$-complete uniform ultrafilter on $\mu$", in $V^P$ there is exactly one $j < \sigma(n)$ with $A^j_d(u) \in \mathfrak{D}$. Let $\underset{\sim}{j}_d(u)$ be the $P$-name of this $j$.

Let $I_d(u) \subseteq P$ be a maximal antichain of $P$, each member of $I_d(u)$ forces a value to $\underset{\sim}{j}_d(u)$. Let $W_d(u) = \bigcup\{\mathrm{dom}(p) : p \in I_d(u)\}$ and $W(u) = \bigcup\{W_{d_n}(u) : n < \kappa\}$. So $W_d(u)$ is a subset of $\lambda$ of cardinaltiy $\leq \mu$ as well as $W(u)$ (as P satisfies the $\mu^+$-c.c. and $p \in P \Rightarrow |\mathrm{dom}(p)| < \mu$).

As $\lambda \to (\mu^{++})^{<2\kappa}_{2^\mu}$, $d_n \in \mathrm{Col}^n_{\sigma_n}({}^{\mu>}2)$ there is a subset $Z$ of $\lambda$ of cardinality $\mu^{++}$ and set $W^+(u)$ for each $u \in [Z]^{<\kappa}$ such that:

(i) $W^+(u_1) \cap W^+(u_2) = W^+(u_1 \cap u_2)$,

(ii) $W(u) \subseteq W^+(u)$ if $u \in [Z]^{<\kappa}$,

(iii) if $|u_1| = |u_2| < \kappa$ and $u_1, u_2 \subseteq Z$ then $W^+(u_1)$ and $W^+(u_2)$ have the same order type – and note that $H[u_1, u_2] = H^{\rm OP}_{W^+(u_1), W^+(u_2)}$ naturally induces a map from $P \restriction u_1 = \{p \in P : \mathrm{dom}(p) \subseteq W^+(u_1)\}$ to $P \restriction u_2 = \{p \in P : \mathrm{dom}(p) \subseteq W^+(u_2)\}$.

(iv) if $u_1, u_2 \in [Z]^{<\kappa}$, $|u_1| = |u_2|$ then $H[u_1, u_2]$ maps $I_{d_n}(u_1)$ onto $I_{d_n}(u_2)$ and

$$q \Vdash \text{“}\underset{\sim}{j}_d(u_1) = j\text{”} \Leftrightarrow H[u_1, u_2](q) \Vdash \text{“}\underset{\sim}{j}_d(u_2) = j\text{”}$$

(v) if $u_1 \subseteq u_2 \in [Z]^{<\kappa}$, $u_3 \subseteq u_4 \in [Z]^{<\kappa}$, $|u_4| = |u_2|$, $H^{\rm OP}_{u_2, u_4}$ maps $u_1$ onto $u_3$ <u>then</u> $H[u_1, u_3] \subseteq H[u_2, u_4]$.

Let $\gamma(i)$ be the $i^{\rm th}$ member of $Z$.

Let $s(m)$ be the set of the first $m$ members of $Z$ and

$$R_n = \left\{ p \in P : \mathrm{dom}(p) \subseteq W^+(s(n)) \setminus \bigcup_{t \subset s(n)} W^+(t) \right\}.$$

We define by induction on $\alpha < \mu$ a function $F_\alpha$ and $p_u \in R_{|u|}$ for $u \in \bigcup_{\beta<\alpha} [{}^\beta 2]^{<\kappa}$ where we let $\varnothing_\beta$ be the empty subset of $[{}^\beta 2]$ and we behave as if $[\beta \neq \gamma \Rightarrow \varnothing_\beta \neq \varnothing_\gamma]$ and we also define $\zeta(\beta) < \mu$, such that:

(i) $F_\alpha$ is a function from ${}^{\alpha>}2$ into ${}^{\mu>}2$, extending $F_\beta$ for $\beta < \alpha$,

(ii) $F_\alpha$ maps ${}^\beta 2$ to ${}^{\zeta(\beta)}2$ for some $\zeta(\beta) < \mu$ and $\beta_1 < \beta_2 < \alpha \Rightarrow \zeta(\beta_1) < \zeta(\beta_2)$,

(iii) $\eta \triangleleft \nu \in {}^{\alpha>} 2$ implies $F_\alpha(\eta) \triangleleft F_\alpha(\nu)$,

(iv) for $\eta \in {}^\beta 2$, $\beta + 1 < \alpha$ and $\ell < 2$ we have $F_\alpha(\eta)\hat{\ }\langle \ell \rangle \trianglelefteq F_\alpha(\eta\hat{\ }\langle \ell \rangle)$,

(v) $p_u \in R_m$ whenever $u \in [{}^\beta 2]^m$, $m < \kappa$, $\beta < \alpha$ and for $u(1) \in [Z]^m$ let $p_{u,u(1)} = H[s(|u|), u(1)](p_u)$.

(vi) $\eta \in {}^\beta 2$, $\beta < \alpha$, then $p_{\{\eta\}}(\min Z) = F_\alpha(\eta)$.

(vii) if $\beta < \alpha$, $u \in [{}^\beta 2]^n$, $n < \kappa$, $h : u \to s(n)$ one-to-one onto (not necessarily order preserving) <u>then</u> for some $c(u, h) < \sigma(n)$:

$$\bigcup_{t \subseteq u} p_{t, h''(t)} \Vdash_{P_\lambda} \text{“}\underset{\sim}{d}_n(\underset{\sim}{\eta}_{\gamma(0)}, \ldots, \underset{\sim}{\eta}_{\gamma(n-1)}) = c(u, h)\text{”},$$

(Note: as $p_u \in R_{|u|}$ the domains of the conditions in this union are pairwise disjoint.)

(viii) If $n, u, \beta, h$ are as in (vii), $u = \{\nu_0, \ldots, \nu_{n-1}\}$, $\nu_\ell \triangleleft \rho_\ell \in {}^\gamma 2$, $\beta \le \gamma < \alpha$ then $d_n(F_\alpha(\rho_0), \ldots, F_\alpha(\rho_{n-1})) = c(u, h)$ where $h$ is the unique function from $u$ onto $s(n)$ such that $[h(\nu_\ell) \le h(\nu_m) \Rightarrow \rho_\ell <^*_\gamma \rho_m]$.

(ix) if $\beta < \gamma < \alpha$, $\nu_1, \ldots, \nu_{n-1} \in {}^\gamma 2$, $n < \kappa$, and $\nu_0 \restriction \beta, \ldots, \nu_{n-1} \restriction \beta$ are pairwise distinct then:

$$p_{\{\nu_0 \restriction \beta, \ldots, \nu_n \restriction \beta\}} \subseteq p_{\{\nu_0, \ldots, \nu_{n-1}\}}.$$

<u>For $\alpha$ limit</u>: no problem. <u>For $\alpha + 1, \alpha$ limit</u>: we try to define $F_\alpha(\eta)$ for $\eta \in {}^\alpha 2$ such that $\bigcup_{\beta<\alpha} F_{\beta+1}(\eta \restriction \beta) \trianglelefteq F_\alpha(\eta)$ and (viii) holds. Let $\zeta = \bigcup_{\beta<\alpha} \zeta(\beta)$, and for $\eta \in {}^\alpha 2$, $F^0_\alpha(\eta) = \bigcup_{\beta<\alpha} F_\alpha(\eta \restriction \beta)$ and for $u \in [{}^\alpha 2]^{<\kappa}$, $p^0_u = \bigcup\{p^0_{\{\nu\restriction\beta:\nu\in u\}} : \beta < \alpha,\ |u| = |\{\nu \restriction \beta : \nu \in u\}|\}$. Clearly $p^0_u \in R_{|u|}$.

Then let $h :{}^\alpha 2 \to Z$ be one-to-one, such that $\eta <^*_\alpha \nu \Leftrightarrow h(\eta) < h(\nu)$ and let $p = \bigcup\{p^0_{u,u(1)} : u(1) \in [Z]^{<\kappa},\ u \in [{}^\alpha 2]^{<\kappa}, |u(1)| = |u|, h''(u) = u(1)\}$.

For any generic $G \subseteq P_\lambda$ to which $p$ belongs, $\beta < \alpha$ and ordinals $i_0 < \cdots < i_{n-1}$ from $Z$ such that $\langle h^{-1}(i_\ell) \restriction \beta : \ell < n\rangle$ are pairwise distinct we have that

$$B_{\{i_\ell:\ell<n\},\beta} = \left\{ \xi < \mu : d_n(\eta_{i_0} \restriction \xi, \ldots, \eta_{i_{n-1}} \restriction \xi) = c(u, h^*) \right\},$$

belongs to $\mathfrak{D}[G]$, where $u = \{h^{-1}(i_\ell) \restriction \beta : \ell < n\}$ and $h^* : u \to s(|u|)$ is defined by $h^*(h^{-1}(i_\ell) \restriction \beta) = H^{\mathrm{OP}}_{\{i_\ell:\ell<n\},s(n)}(i_\ell)$. Really every large enough $\beta < \mu$ can serve so we omit it. As $\mathfrak{D}[G]$ is $\mu$-complete uniform ultrafilter on $\mu$, we can find $\xi \in (\zeta, \kappa)$ such that $\xi \in B_u$ for every $u \in [{}^\alpha 2]^n$, $n < \kappa$. We let for $\nu \in^\alpha 2$, $F_\alpha(\nu) = \underset{\sim}{\eta}_{h(i)}[G] \restriction \xi$, and we let $p_u = p^0_u$ except when $u = \{\nu\}$, then:

$$p_u(i) = \begin{cases} p^0_u(i) & i \neq \gamma(0) \\ F_{\alpha+1}(\nu) & i = \gamma(0) \end{cases}$$

<u>For $\alpha + 1$, $\alpha$ is a successor</u>: First for $\eta \in {}^{\alpha-1}2$ define $F\big(\eta\hat{}\langle\ell\rangle\big) = F_\alpha(\eta)\hat{}\langle\ell\rangle$. Next we let $\big\{(u_i, h_i) : i < i^*\big\}$, list all pairs $(u, h)$, $u \in [{}^\alpha 2]^{\leq n}$, $h : u \to s(|u|)$, one-to-one onto. Now, we define by induction on $i \leq i^*$, $p^i_u\big(u \in [{}^\alpha 2]^{<\kappa}\big)$ such that:

(a) $p^i_u \in R_{|u|}$,
(b) $p^i_u$ increases with $i$,
(c) for $i + 1$, (vii) holds for $(u_i, h_i)$,
(d) if $\nu_m \in^\alpha 2$ for $m < n$, $n < \kappa$, $\langle \nu_m \restriction (\alpha - 1) : m < n\rangle$ are pairwise distinct, *then* $p_{\{\nu_m \restriction (\alpha-1)\ :m<n\}} \leq p^0_{\{\nu_m:m<n\}}$,
(e) if $\nu \in {}^\alpha 2$, $\nu(\alpha - 1) = \ell$ <u>then</u> $p^0_{\{\nu\}}(0) = F_\alpha(\nu \restriction (\alpha - 1))\hat{}\langle\ell\rangle$.

There is no problem to carry the induction.

Now $F_{\alpha+1} \restriction {}^\alpha 2$ is to be defined as in the second case, starting with $\eta \to p^{i^*}_{\{\eta\}}(\eta)$.

<u>For $\alpha = 0, 1$</u>: Left to the reader.

So we have finished the induction hence the proof of 4.1(1), (2).

3) Left to the reader ( the only influence is the choice of $h$ in stage of the induction). $\square$

## § 5. Somewhat complementary negative partition relation in ZFC

*The negative results here suffice to show that the value we have for $2^{\aleph_0}$ in §3 is reasonable. In particular the Galvin conjecture is wrong and that for every $n < \omega$ for some $m < \omega$, $\aleph_n \not\to [\aleph_1]^m_{\aleph_0}$.*

*See Erdos Hajnal Máté Rado* [EHMR84] *for*

*Fact* 5.1. If $2^{<\mu} < \lambda \leq 2^\mu$, $\mu \not\to [\mu]^n_\sigma$ then $\lambda \not\to [(2^{<\mu})^+]^{n+1}_\sigma$.

This shows that if e.g. in 1.4 we want to increase the exponents, to 3 (and still $\mu = \mu^{<\mu}$) e.g. $\mu$ cannot be successor (when $\sigma \leq \aleph_0$) (by [She88], 3.5(2)).

*Definition* 5.2. $Pr_{np}(\lambda, \mu, \bar{\sigma})$, where $\bar{\sigma} = \langle \sigma_n : n < \omega\rangle$, means that there are functions $F_n : [\lambda]^n \to \sigma_n$ such that for every $W \in [\lambda]^\mu$ for some $n$, $F''_n([W]^n) = \sigma(n)$. The negation of this property is denoted by $NPr_{np}(\lambda, \mu, \bar{\sigma})$.

If $\sigma_n = \sigma$ we write $\sigma$ instead of $\langle \sigma_n : n < \omega\rangle$.

*Remark* 5.3. 1) Note that $\lambda \to [\mu]^{<\omega}_\sigma$ means: if $F : [\lambda]^{<\omega} \to \sigma$ then for some $A \in [\lambda]^\mu$, $F''([A]^{<\omega}) \neq \sigma$. So for $\lambda \geq \mu \geq \sigma = \aleph_0$, $\lambda \not\to [\mu]^{<\omega}_\sigma$, (use $F : F(\alpha) = |\alpha|$) and $Pr_{np}(\lambda, \mu, \sigma)$ is stronger than $\lambda \not\to [\mu]^{<\omega}_\sigma$.
2) We do not write down the monotonicity properties of $Pr_{np}$ — they are obvious.

*Claim* 5.4. *5.31) We can (in 5.2) w.l.o.g. use $F_{n,m} : [\lambda]^n \to \sigma_n$ for $n, m < \omega$ and obvious monotonicity properties holds, and $\lambda \geq \mu \geq n$.*
*2) Suppose $NPr_{np}(\lambda, \mu, \kappa)$ and $\kappa \not\to [\kappa]^n_\sigma$ or even $\kappa \not\to [\kappa]^{<\omega}_\sigma$. Then the following case of Chang conjecture holds:*
*(*) for every model $M$ with universe $\lambda$ and countable vocabulary, there is an elementary submodel $N$ of $M$ of cardinality $\mu$,*

$$|N \cap \kappa| < \kappa$$

*3) If $NPr_{np}(\lambda, \aleph_1, \aleph_0)$ then $(\lambda, \aleph_1) \to (\aleph_1, \aleph_0)$.*

*Proof.* Easy. $\square$

*Theorem* 5.5. *Suppose $Pr_{np}(\lambda_0, \mu, \aleph_0)$, $\mu$ regular $> \aleph_0$ and $\lambda_1 \geq \lambda_0$, and no $\mu' \in (\lambda_0, \lambda_1)$ is $\mu'$-Mahlo. Then $Pr_{np}(\lambda_1, \mu, \aleph_0)$.*

*Proof.* Let $\chi = \beth_8(\lambda_1)^+$, let $\{F^0_{n,m} : m < \omega\}$ list the definable $n$-place functions in the model $(H(\chi), \in, <^*_\chi)$, with $\lambda_0, \mu, \lambda_1$ as parameters, let $F^1_{n,m}(\alpha_0, \dots, \alpha_{n-1})$ (for $\alpha_0, \dots, \alpha_{n-1} < \lambda_1$) be $F^0_{n,m}(\alpha_0, \dots, \alpha_{n-1})$ if it is an ordinal $< \lambda_1$ and zero otherwise. Let $F_{n,m}(\alpha_0, \dots, \alpha_{n-1})$ (for $\alpha_0, \dots, \alpha_{n-1} < \lambda_1$) be $F^0_{n,m}(\alpha_0, \dots, \alpha_{n-1})$ if it is an ordinal $< \omega$ and zero otherwise. We shall show that $F_{n,m}(n, m < \omega)$ exemplify $Pr_{np}(\lambda_1, \mu, \aleph_0)$ (see 5.3(1)).

So suppose $W \in [\lambda_1]^\mu$ is a counterexample to $Pr(\lambda_1, \mu, \aleph_0)$ i.e. for no $n, m$, $F''_{n,m}([W]^n) = \omega$. Let $W^*$ be the closure of $W$ under $F^1_{n,m}(n, m < \omega)$. Let $N$ be the Skolem Hull of $W$ in $(H(\chi), \in, <^*_\chi)$, so clearly $N \cap \lambda_1 = W^*$. Note $W^* \subseteq \lambda_1$, $|W^*| = \mu$. Also as $\mathrm{cf}(\mu) > \aleph_0$ if $A \subseteq W^*$, $|A| = \mu$ then for some $n, m < \omega$ and $u_i \in [W]^n$ (for $i < \mu$), $F^1_{n,m}(u_i) \in A$ and $[i < j < \mu \Rightarrow F^1_{n,m}(u_i) \neq F^1_{n,m}(u_i)]$. It is easy to check that also $W^1 = \{F^1_{n,m}(u_i) : i < \mu\}$ is a counterexample to $Pr(\lambda_1, \mu, \sigma)$. In particular, for $n, m < \omega$, $W_{n,m} = \{F^1_{n,m}(u) : u \in [W]^n\}$ is a counterexample if it has power $\mu$. W.l.o.g. $W$ is a counterexample with minimal $\delta = \sup(W) = \cup\{\alpha + 1 : \alpha \in W\}$. The above discussion shows that $|W^* \cap \alpha| < \mu$ for $\alpha < \delta$. Obviously $\mathrm{cf}\delta = \mu^+$. Let $\langle \alpha_i : i < \mu \rangle$ be a strictly increasing sequence of members of $W^*$, converging to $\delta$, such that for limit $i$ we have $\alpha_i = \min(W^* - \bigcup_{j<i}(\alpha_j + 1)$. Let $N = \bigcup_{i<\mu} N_i$, $N_i \prec N$, $|N_i| < \mu$, $N_i$ increasing continuous and w.l.o.g. $N_i \cap \delta = N \cap \alpha_i$. $\underline{\alpha \text{ Fact}}$: $\delta$ is $> \lambda_0$.

Proof. Otherwise we then get an easy contradiction to $Pr(\lambda_0, \mu, \sigma))$ as choosing the $F^0_{n,m}$ we allowed $\lambda_0$ as a parameter. $\underline{\beta \text{ Fact}}$: If $F$ is a unary function definable in $N$, $F(\alpha)$ is a club of $\alpha$ for every limit ordinal $\alpha(< \lambda_1)$ <u>then</u> for some club $C$ of $\mu$ we have

$$\big(\forall j \in C \setminus \{\min C\}\big)\big(\exists i_1 < j\big)\big(\forall i \in (i_1, j)\big)\big[i \in C \Rightarrow \alpha_i \in F(\alpha_j)\big].$$

Proof. For some club $C_0$ of $\mu$ we have $j \in C_0 \Rightarrow \big(N_j, \{\alpha_i : i < j\}, W\big) \prec \big(N, \{\alpha_i : i < \mu\}, W\big)$. We let $C = C_0' = \mathrm{acc}(C)$ (= set of accumulation points of $C_0$).

We check $C$ is as required; suppose $j$ is a counterexample. So $j = \sup(j \cap C)$ (otherwise choose $i_1 = \max(j \cap C)$). So we can define, by induction on $n$, $i_n$, such that:

(a) $i_n < i_{n+1} < j$
(b) $\alpha_{i_n} \notin F(\alpha_j)$
(c) $(\alpha_{i_n}, \alpha_{i_{n+1}}) \cap F(\alpha_j) \neq \varnothing$.

Why $(C_0')$? $\models$ "$F(\alpha_j)$ is unbounded below $\alpha_j$" hence $N \models$ "$F(\alpha_j)$ is unbounded below $\alpha_j$", but in $N$, $\{\alpha_i : i \in C_0,\ i < j\}$ is unbounded below $\alpha_j$.

Clearly for some $n, m, \alpha_j \in W_{n,m}$ (see above). Now we can repeat the proof of [She88, 3.3(2)] (see mainly the end) using only members of $W_{n,m}$.
Note: here we use the number of colors being $\aleph_0$. $\underline{\beta^+ \text{ Fact}}$: Wolog the $C$ in Fact $\beta$ is $\mu$.
Proof: Renaming. $\underline{\gamma \text{ Fact}}$: $\delta$ is a limit cardinal.

Proof: Suppose not. Now $\delta$ cannot be a successor cardinal (as $\mathrm{cf}\delta = \mu \leq \lambda_0 < \delta$) hence for every large enough $i$, $|\alpha_i| = |\delta|$, so $|\delta| \in W^* \subseteq N$ and $|\delta|^+ \in W^*$.

So $W^* \cap |\delta|$ has cardinality $< \mu$ hence order-type some $\gamma^* < \mu$. Choose $i^* < \mu$ limit such that $[j < i^* \Rightarrow j + \gamma^* < i^*]$. There is a definable function $F$ of $(H(\chi), \in$

$,<^*_\chi)$ such that for every limit ordinal $\alpha$, $F(\alpha)$ is a club of $\alpha$, $0 \in F(\alpha)$, if $|\alpha| < \alpha$, $F(\alpha) \cap |\alpha| = \varnothing$, $\mathrm{otp}(F(\alpha)) = \mathrm{cf}\alpha$.

So in $N$ there is a closed unbounded subset $C_{\alpha_j} = F(\alpha_j)$ of $\alpha_j$ of order type $\le \mathrm{cf}\alpha_j \le |\delta|$, hence $C_{\alpha_j} \cap N$ has order type $\le \gamma^*$, hence for $i^*$ chosen above unboundedly many $i < i^*$, $\alpha_i \notin C_{\alpha_{i^*}}$. We can finish by fact $\beta^+$. $\underline{\delta \text{ Fact}}$: For each $i < \mu$, $\alpha_i$ is a cardinal.

Proof: If $|\alpha_i| < i$ then $|\alpha_i| \in N_i$, but then $|\alpha_i|^+ \in N_i$ contradicting to Fact $\gamma$, by which $|\alpha_i|^+ < \delta$, as we have assumed $N_i \cap \delta = N \cap \alpha_i$. $\underline{\varepsilon \text{ Fact}}$: For a club of $i < \mu$, $\alpha_i$ is a regular cardinal.

(Proof: if $S = \{i : \alpha_i \text{ singular}\}$ is stationary, then the function $\alpha_i \to \mathrm{cf}(\alpha_i)$ is regressive on $S$. By Fodor lemma, for some $\alpha^* < \delta$, $\{i < \mu : \mathrm{cf}\alpha_i < \alpha^*\}$ is stationary. As $|N \cap \alpha^*| < \mu$ for some $\beta^*$, $\{i < \mu : \mathrm{cf}\alpha_i = \beta^*\}$ is stationary. Let $F_{1,m}(\alpha)$ be a club of $\alpha$ of order type $\mathrm{cf}(\alpha)$, and by fact $\beta$ we get a contradiction as in fact $\gamma$. $\underline{\zeta \text{ Fact}}$: For a club of $i < \mu$, $\alpha_i$ is Mahlo.

Proof: Use $F_{1,m}(\alpha) =$ a club of $\alpha$ which, if $\alpha$ is a successor cardinal or inaccessible not Mahlo, then it contains no inaccessible, and continue as in fact $\gamma$. $\underline{\xi \text{ Fact}}$: For a club of $i < \mu$, $\alpha_i$ is $\alpha_i$-Mahlo.

Proof: Let $F_{1,m(0)}(\alpha) = \sup\{\zeta : \alpha \text{ is } \zeta\text{-Mahlo}\}$. If the set $\{i < \mu : \alpha_i$ is not $\alpha_i$-Mahlo$\}$ is stationary then as before for some $\gamma \in N$, $\{i : F_{1,m(0)}(\alpha_i) = \gamma\}$ is stationary and let $F_{1,m(1)}(\alpha)$ — a club of $\alpha$ such that if $\alpha$ is not $(\gamma+1)$-Mahlo then the club has no $\gamma$-Mahlo member. Finish as in the proof of fact $\delta$. □

*Remark* 5.6. We can continue and say more.

*Lemma* 5.7. *1) Suppose $\lambda > \mu > \theta$ are regular cardinals, $n \ge 2$ and*
*(i) for every regular cardinal $\kappa$, if $\lambda > \kappa \ge \theta$ then $\kappa \not\to [\theta]^{<\omega}_{\sigma(1)}$.*
*(ii) for some $\alpha(*) < \mu$ for every regular $\kappa \in (\alpha(*), \lambda)$, $\kappa \not\to [\alpha(*)]^n_{\sigma(2)}$.*
Then
*(a) $\lambda \not\to [\mu]^{n+1}_\sigma$ where $\sigma = \min\{\sigma(1), \sigma(2)\}$,*
*(b) there are functions $d_2 : [\lambda]^{n+1} \to \sigma(2)$, $d_1 : [\lambda]^3 \to \sigma(1)$ such that for every $W \in [\lambda]^\mu$, $d_1''([W]^3) = \sigma(1)$ or $d_2''([W]^{n+1}) = \sigma(2)$.*
*2) Suppose $\lambda > \mu > \theta$ are regular cardinals, and*
*(i) for every regular $\kappa \in [\theta, \lambda)$, $\kappa \not\to [\theta]^{<\omega}_{\sigma(1)}$,*
*(ii) $\sup\{\kappa < \lambda : \kappa \text{ regular}\} \not\to [\mu]^n_{\sigma(2)}$.*
*Then*
*(a) $\lambda \not\to [\mu]^{2n}_\sigma$ where $\sigma = \min\{\sigma(1), \sigma(2)\}$*
*(b) there are functions $d_1 : [\lambda]^3 \to \sigma(1)$, $d_2 : [\lambda]^{2n} \to \sigma(2)$ such that for every $W \in [\lambda]^\mu$, $d_1''([W]^3) = \sigma(1)$ or $d_2''([W]^{2n} = \sigma(2)$.*

*The proof is similar to that of* [She88] *3.3,3.2.*

*Proof.* 1) We choose for each $i$, $0 < i < \lambda_i$, $C_i$ such that: if $i$ is a successor ordinal, $C_i = \{i-1, 0\}$; if $i$ is a limit ordinal, $C_i$ is a club of $i$ of order type $\mathrm{cf}i$, $0 \in C_i$, $[\mathrm{cf}i < i \Rightarrow \mathrm{cf}i < \min(C_i - \{0\})]$ and $C_i \setminus \mathrm{acc}(C_i)$ contains only successor ordinals.

Now for $\alpha < \beta$, $\alpha > 0$ we define by induction on $\ell$, $\gamma^+_\ell(\beta,\alpha)$, $\gamma^-_\ell(\beta,\alpha)$, and then $\kappa(\beta,\alpha)$, $\varepsilon(\beta,\alpha)$.

(A) $\gamma^+_0(\beta,\alpha) = \beta$, $\gamma^-_0(\beta,\alpha) = 0$.
(B) if $\gamma^+_\ell(\beta,\alpha)$ is defined and $> \alpha$ and $\alpha$ is not an accumulation point of $C_{\gamma^+_\ell(\beta,\alpha)}$ then we let $\gamma^-_{\ell+1}(\beta,\alpha)$ be the maximal member of $C_{\gamma^+_\ell(\beta,\alpha)}$ which is $< \alpha$ and $\gamma^+_{\ell+1}(\beta,\alpha)$ is the minimal member of $C_{\gamma^+_\ell(\beta,\alpha)}$ which is $\ge \alpha$ (by the choice of $C_{\gamma^+_\ell(\beta,\alpha)}$ and the demands on $\gamma^+_\ell(\beta,\alpha)$ they are well defined).
So

(B1) (a) $\gamma^-_\ell(\beta,\alpha) < \alpha \le \gamma^+_\ell(\beta,\alpha)$, and if the equality holds then $\gamma^+_{\ell+1}(\beta,\alpha)$ is not defined.
(b) $\gamma^+_{\ell+1}(\beta,\alpha) < \gamma^+_\ell(\beta,\alpha)$ when both are defined.

(C) Let $k = k(\beta,\alpha)$ be the maximal number $k$ such that $\gamma^+_k(\beta,\alpha)$ is defined (it is well defined as $\langle \gamma^+_\ell(\beta,\alpha) : \ell < \omega\rangle$ is strictly decreasing). So

(C1) $\gamma^+_{k(\beta,\alpha)}(\beta,\alpha) = \alpha$ or $\gamma^+_{k(\beta,\alpha)} > \alpha$, $\gamma^+_{k(\beta,\alpha)}$ is a limit ordinal and $\alpha$ is an accumulation point of $C_{\gamma^+_{k(\beta,\alpha)}}(\beta,\alpha)$.

(D) For $m \le k(\beta,\alpha)$ let us define

$$\varepsilon_m(\beta,\alpha) = \max\{\gamma^-_\ell(\beta,\alpha)+1 : \ell \le m\}.$$

Note

(D1) (a) $\varepsilon_m(\beta,\alpha) \le \alpha$ (if defined),
(b) if $\alpha$ is limit then $\varepsilon_m(\beta,\alpha) < \alpha$ (if defined),
(c) if $\varepsilon_m(\beta,\alpha) \le \xi \le \alpha$ then for every $\ell \le m$ we have

$$\gamma^+_\ell(\beta,\alpha) = \gamma^+_\ell(\beta,\xi), \quad \gamma^-_\ell(\beta,\alpha) = \gamma^-_\ell(\beta,\xi), \quad \varepsilon_\ell(\beta,\alpha) = \varepsilon_\ell(\beta,\xi).$$

(explanation for (c): if $\varepsilon_m(\beta,\alpha) < \alpha$ this is easy (check the definition) and if $\varepsilon_m(\beta,\alpha) = \alpha$, necessarily $\xi = \alpha$ and it is trivial).
(d) if $\ell \le m$ then $\varepsilon_\ell(\beta,\alpha) \le \varepsilon_m(\beta,\alpha)$

For a regular $\kappa \in (\alpha(*),\lambda)$ let $g^1_\kappa : [\kappa]^{<\omega} \to \sigma(2)$ exemplify $\kappa \not\to [\theta]^{<\omega}_{\sigma(1)}$ and for every regular cardinal $\kappa \in [\theta,\lambda)$ let $g^2_\kappa : [\kappa]^n \to \sigma(2)$ exemplify $\kappa \not\to [\alpha(*)]^n_{\sigma(2)}$. Let us define the colourings:

Let $\alpha_0 > \alpha_1 > \ldots > \alpha_n$. Remember $n \ge 2$.

Let $n = n(\alpha_0,\alpha_1,\alpha_2)$ be the maximal natural number such that:
(i) $\varepsilon_n(\alpha_0,\alpha_1) < \alpha_0$ is well defined,
(ii) for $\ell \le n$, $\gamma^-_\ell(\alpha_0,\alpha_1) = \gamma^-_\ell(\alpha_0,\alpha_2)$.

We define $d_2(\alpha_0,\alpha_1,\ldots,\alpha_n)$ as $g^2_\kappa(\beta_1,\ldots,\beta_n)$ where

$$\kappa = \mathrm{cf}\big(\gamma^+_{n(\alpha_0,\alpha_1,\alpha_2)}(\alpha_0,\alpha_1)\big),$$
$$\beta_\ell = \mathrm{otp}\Big[\alpha_\ell \cap C_{\gamma^+_{n(\alpha_0,\alpha_1,\alpha_2)}(\alpha_0,\alpha_1)}\Big].$$

Next we define $d_1(\alpha_0,\alpha_1,\alpha_2)$ .

Let $i(*) = \sup\Big[C_{\gamma^+_n(\alpha_0,\alpha_2)} \cap C_{\gamma^+_n(\alpha_1,\alpha_2)}\Big]$ where $n = n(\alpha_0,\alpha_1,\alpha_2)$, $E$ be the equivalence relation on $C_{\gamma^+_n(\alpha_0,\alpha_1)} \setminus i(*)$ defined by

$$\gamma_1 E \gamma_2 \Leftrightarrow \forall \gamma \in C_{\gamma^+_n(\alpha_0,\alpha_2)}[\gamma_1 < \gamma \leftrightarrow \gamma_2 < \gamma].$$

If the set $w = \Big\{\gamma \in C_{\gamma^+_n(\alpha_0,\alpha_1)} : \gamma > i(*),\ \gamma = \min \gamma/E\Big\}$ is finite, we let $d_1(\alpha_0,\alpha_1,\alpha_2)$ be $g^1_\kappa\big(\{\beta_\gamma : \gamma \in w\}\big)$ where $\kappa = \Big|C_{\gamma^+_n(\alpha_0,\alpha_1)}\Big|$, $\beta_\gamma = \mathrm{otp}\Big(\gamma \cap C_{\gamma^+_n(\alpha_0,\alpha_1)}\Big)$.

We have defined $d_1$, $d_2$ required in condition $(b)$ ( though have not yet proved that they work) We still have to define $d$ (exemplifying $\lambda \not\to [\mu]^{n+1}_\ell$). Let $n \ge 3$, for $\alpha_0 > \alpha_1 > \ldots > \alpha_n$, we let $d(\alpha_0,\ldots,\alpha_n)$ be $d_1(\alpha_0,\alpha_1,\alpha_2)$ if $w$ defined during the definition has odd number of members and $d_2(\alpha_0,\ldots,\alpha_n)$ otherwise.

Now suppose $Y$ is a subset of $\lambda$ of order type $\mu$, and let $\delta = \sup Y$. Let $M$ be a model with universe $\lambda$ and with relations $Y$ and $\{(i,j) : i \in C_j\}$. Let $\langle N_i : i < \mu\rangle$ be an increasing continuous sequence of elementary submodels of $M$ of cardinality $< \mu$ such that $\alpha(i) = \alpha_i = \min(Y \setminus N_i)$ belongs to $N_{i+1}$, $\sup(N \cap \alpha_i) = \sup(N \cap \delta)$. Let $N = \bigcup_{i<\mu} N_i$. Let $\delta(i) = \delta_i = \sup(N_i \cap \alpha_i)$, so $0 < \delta_i \le \alpha_i$, and let $n = n_i$ be the first natural number such that $\delta_i$ an accumulation point of $C^i = C_{\gamma^+_n(\alpha_i,\delta(i))}$, let

$\varepsilon_i = \varepsilon_{n(i)}(\alpha_i, \delta_i)$. Note that $\gamma_n^+(\alpha_i, \delta_i) = \gamma_n^+(\alpha_i, \varepsilon_i)$ hence it belongs to $N$. Case I: *For some (limit)* $i < \mu$, $\mathrm{cf}(i) \geq \theta$ *and* $(\forall \gamma < i)[\gamma + \alpha(*) < i]$ *such that for arbitrarily large* $j < i$, $C^i \cap N_j$ *is bounded in* $N_j \cap \delta = N_j \cap \delta_j$.
This is just like the last part in the proof of [She88], 3.3 using $g^1_\kappa$ and $d_1$ for $\kappa = \mathrm{cf}(\gamma^+_{n_i}(\alpha_i, \delta_i)$.
Case II: *Not case I.*

Let $S_0 = \{i < \mu : (\forall \alpha < i)[\gamma + \alpha(*) < i], \mathrm{cf}(i) = \theta\}$. So for every $i \in S_0$ for some $j(i) < i$, $(\forall j)\Big[j \in (j(i), i) \Rightarrow C^i \cap N_j$ is unbounded in $\delta_j\Big]$. But as $C^i \cap \delta_i$ is a club of $\delta_i$, clearly $(\forall j)\Big[j \in (j(i), i) \Rightarrow \delta_j \in C^i\Big]$.

We can also demand $j(i) > \varepsilon_{n(\alpha(i), \delta(i))}(\alpha(i), \delta(i))$.

As $S_0$ is stationary, (by not case I) for some stationary $S_1 \subseteq S_0$ and $n(*)$, $j(*)$ we have $(\forall i \in S_1)\Big[j(i) = j(*) \wedge n(\alpha(i), \delta_i) = n(*)\Big]$.

Choose $i(*) \in S_1$, $i(*) = \sup(i(*) \cap S_1)$, such that the order type of $S_1 \cap i(*)$ is $i(*) > \alpha(*)$. Now if $i_2 < i_1 \in S_1 \cap i(*)$ then $n(\alpha_{i(*)}, \alpha_{i_1}, \alpha_{i_2}) = n(*)$. Now $L_{i(*)} = \Big\{\mathrm{otp}(\alpha_i \cap C^{i(*)}) : i \in S_1 \cap i(*)\Big\}$ are pairwise distinct and are ordinals $< \kappa = |C^{i(*)}|$, and the set has order type $\alpha(*)$. Now apply the definitions of $d_2$ and $g^2_\kappa$ on $L_{i(*)}$. 2) The proof is like the proof of part (1) but for $\alpha_0 > \alpha_1 > \cdots$ we let $d_2(\alpha_0, \ldots, \alpha_{2n-1}) = g^2_\kappa(\beta_0, \ldots, \beta_n)$ where

$$\beta_\ell = \mathrm{otp}\big(C_{\gamma_n^+(\beta_{2\ell}, \beta_{2\ell+1})}(\beta_{2\ell}, \beta_{2\ell+1}) \cap \beta_{2\ell+1}\big)$$

and in case II note that the analysis gives $\mu$ possible $\beta_\ell$'s so that we can apply the definition of $g^2_\kappa$. $\square$

*Definition* 5.8. Let $\lambda \not\to_{\mathrm{stg}} [\mu]^n_\theta$ mean: if $d : [\lambda]^n \to \theta$, and $\langle \alpha_i : i < \mu \rangle$ is strictly increasingly continuous and for $i < j < \mu$, $\gamma_{i,j} \in [\alpha_i, \alpha_{i+1})$ then

$$\theta = \Big\{d(w) : \text{ for some } j < \mu,\ w \in \big[\{\gamma_{i,j} : i < j\}\big]^n\Big\}.$$

*Lemma* 5.9. *1)* $\aleph_t \not\to [\aleph_1]^{n+1}_{\aleph_0}$ *for* $n \geq 1$.
*2)* $\aleph_n \not\to_{\mathrm{stg}} [\aleph_1]^{n+1}_{\aleph_0}$ *for* $n \geq 1$.

*Proof.* 1) For $n = 2$ this is a theorem of Todorčevič [[Tod87]], and if it holds for $n \geq 2$ by 5.5(1) we get that it holds for n+1 (with $n$, $\lambda$, $\mu$, $\theta$, $\alpha(*)$, $\sigma(1)$, $\sigma(2)$ there corresponding to $n+1$, $\aleph_{n+1}$, $\aleph_1$, $\aleph_0$, $\aleph_0$, $\aleph_0$, $\aleph_0$ here).
2) Similar. $\square$

## References


[EHMR84] Paul Erdős, Andras Hajnal, A. Maté, and Richard Rado, *Combinatorial set theory: Partition relations for cardinals*, Studies in Logic and the Foundation of Math., vol. 106, North–Holland Publ. Co, Amsterdam, 1984.

[GS89] Moti Gitik and Saharon Shelah, *On certain indestructibility of strong cardinals and a question of Hajnal*, Arch. Math. Logic **28** (1989), no. 1, 35–42. MR 987765

[She78] Saharon Shelah, *A weak generalization of MA to higher cardinals*, Israel J. Math. **30** (1978), no. 4, 297–306. MR 0505492

[She88] ———, *Was Sierpiński right? I*, Israel J. Math. **62** (1988), no. 3, 355–380. MR 955139

[SS82] Saharon Shelah and Lee J. Stanley, *Generalized Martin's axiom and Souslin's hypothesis for higher cardinals*, Israel J. Math. **43** (1982), no. 3, 225–236. MR 689980

[SS86] ———, *Corrigendum to: "Generalized Martin's axiom and Souslin's hypothesis for higher cardinals" [Israel J. Math. 43 (1982), no. 3, 225–236]*, Israel J. Math. **53** (1986), no. 3, 304–314, corrigendum to [Sh:154]. MR 852482

[Tod87] Stevo Todorčević, *Partitioning pairs of countable ordinals*, Acta Math. **159** (1987), 261–294.

Einstein Institute of Mathematics, The Hebrew University of Jerusalem, 9190401, Jerusalem, Israel; and, Department of Mathematics, Rutgers University, Piscataway, NJ 08854-8019, USA

*URL:* *https://shelah.logic.at/*