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Classification Theory for Abstract Elementary Classes

Volume 2

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ANNOTATED CONTENTS

Annotated Content for Ch.N (E53): Introduction

(This chapter appeared in book 1.)

Abstract

§1 Introduction for model theorists,

(A) Why to be interested in dividing lines,

(B) Historical comments on non-elementary classes,

§2 Introduction for the logically challenged,

(A) What are we after?

> [We first explain by examples and then give a full definition of an a.e.c. (abstract elementary class), central in our context, $\mathfrak{K} = (K, \le_{\mathfrak{K}})$, with K a class of models (= structures), $\le_{\mathfrak{K}}$ a special notion of being a submodel, it means having only the quite few of the properties of an elementary class (like closure under direct limit). Such a class is $(\mathrm{Mod}_T, \prec)$ with $M \prec N$ meaning "being an elementary submodel"; but also the class of locally finite groups with $\subseteq$ is O.K. Second, we explain what is a superlimit model (meaning mainly that a $\le_{\mathfrak{K}}$-increasing chain of models isomorphic to it has a union isomorphic to it (if not of larger cardinality). We can define "an a.e.c. is superstable" if it has a superlimit model in every large enough cardinality. For first order class this is an equivalent definition. A stronger condition (still equivalent

for elementary classes) is being solvable: there is a $\mathrm{PC}_{\lambda,\lambda}$-class, i.e the class of reducts of some $\psi \in \mathbb{L}_{\lambda^+,\omega}$ which, in large enough cardinality, is the class of superlimit models; similarly we define being (μ,λ)-solvable. Of course we investigate the one cardinal version (hoping for equivalent behaviour) in all large enough cardinals, etc. We state the problem of the categoricity spectrum and the solvability spectrum. We finish explaining the parallel situation for first order classes and explain "dividing lines".]

(B) The structure/non-structure dichotomy,

[We define the function $\dot{I}(\lambda, K)$ counting the number of non-isomorphic models from K of cardinality λ, define the main gap conjecture, phrase and discuss some thesis explaining an outlook and intention. We then explain the main gap conjecture and the case it was proved and list the possible reasons for having many models. We then discuss dividing lines and their relevance to our problems.]

(C) Abstract elementary classes,

[We shall deal with a.e.c., good λ-frames and beautiful λ-frames. The first is very wide so we have to justify it by showing that we can say something about them, that there is a theory; the last has excellent theory and we have to justify it by showing that it arises from assumptions like few non-isomorphic models (and help prove theorems not mentioning it); the middle one needs justifications of both kinds. In this part, we concentrate on the first, a.e.c., explain the meaning of the definition, discuss examples, phrase our opinion on its place as a thesis, and present two theorems showing the function $\dot{I}(\lambda, \kappa)$ is not "arbitrary" under mild set theoretic conditions.]

(D) Toward good λ-frames,

[We explain how we arrive to "good λ-frame $\mathfrak{s}$" mentioned above, which is our central notion; it may be considered a "bare bone case of superstable class in one cardinal". We choose to concentrate on one cardinal λ, so $K_{\mathfrak{s}} = K_\lambda$. Also we may assume $\mathfrak{K}_\lambda$ has a superlimit model, and that it has amalgamation and the joint embedding property, so only in λ! Amalgamation is an "expensive" assumption, but amalgamation in one cardinal is much less so. This crucial difference holds because it is much easier to prove amalgamation in one cardinality (e.g. follows from having one model in λ (or a superlimit one) and few models in λ^+ up to isomorphism and mild set theoretic assumptions). We are interested in something like "M_1, M_2 are in non-forking (= free) amalgamation over M_0 inside M_3". But in the axioms we only have "an element a and model M_1 are in non-forking amalgamation over M_0 inside M_3, equivalently $\mathbf{tp}_{\mathfrak{s}}(a, M_1, M_3)$ does not fork over M_0", however the type is orbital, i.e. defined by the existence mapping and not by formulas. There are some further demands saying non-forking behave reasonably (mainly: existence/uniqueness of extensions, transitivity and a kind of symmetry). So far we have described a good λ-frame. Now we consider a dividing line - density of the class of appropriate triples (M, N, a) with unique amalgamation. Failure of this gives $\dot{I}(\lambda^{++}, K^{\mathfrak{s}})$ is large if $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$, from success (i.e. density) we derive the existence of non-forking amalgamation of models in $K_{\mathfrak{s}}$. After considering a further dividing line we get $\mathfrak{s}^+$, a good λ^+-frame such that $K^{\mathfrak{s}^+}_\mu \subseteq K^{\mathfrak{s}}_\mu$ for $\mu \geq \lambda^+$. All this (in Chapter II) gives the theorem: if $2^\lambda < 2^{\lambda^+} < \ldots < 2^{\lambda^{+n}}$, $\mathrm{LS}(\mathfrak{K}) \leq \lambda, \mathfrak{K}$ categorical in λ, λ^+, has a model in λ^{+2} and has not too many models in $\lambda^{+2}, \ldots, \lambda^{+n}$ then K has a model in λ^{+n+1}. If this holds for every n, we get categoricity in all cardinals $\mu \geq \lambda$. For the first result (from Chapter II) we just need to go from $\mathfrak{s}$ to $\mathfrak{s}^+$, for the second (from Chapter III) need considerably more.]

§3 Good λ-frames,

(A) getting a good λ-frame,

[We deal more elaborately on how to get a good λ-frame starting with few non-isomorphic models in some cardinals. If $\mathfrak{K}$ is categorical in λ, λ^+ and $2^\lambda < 2^{\lambda^+}$ we know that $\mathfrak{K}$ has amalgamation in λ. Now we define the (orbital) type $\mathbf{tp}_{\mathfrak{K}_\lambda}(a, M, N)$ for $M \le_{\mathfrak{K}} N, a \in N$. Instead of dealing with $\mathscr{S}_{\mathfrak{K}_\lambda}(M)$, the set of such types, we deal with $K^{3,\mathrm{na}}_\lambda = \{(M, N, a) : M \le_{\mathfrak{K}_\lambda} N \text{ and } a \in N \backslash M\}$, ordered naturally (fixing a!) The point is of dealing with triples, not just types, is the closureness under increasing unions, so existence of limit. Now we ask: are there enough minimal triples? (which means with no two contradictory extensions). If no, we have a non-structure result. If yes, we can deduce more and eventually get a good λ-frame. Here we consider $K^{3,\mathrm{bs}}_{\mathfrak{s}} = \{(M, N, a) : M \le_{\mathfrak{K}_{\mathfrak{s}}} N, \mathbf{tp}(a, M, N) \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$, i.e. is a basic type$\}$ (this is part of the basic notions of a good λ-frame $\mathfrak{s}$).]

(B) the successor of a good λ-frame,

[We elaborate the use of successive good frames in Chapter II. If $\mathfrak{s}$ is a good λ-frame, we investigate "N is a brimmed extension of M in $\mathfrak{K}_{\mathfrak{s}} = \mathfrak{K}^{\mathfrak{s}}_\lambda$", it is used here instead of saturated models, noting that as $K^{\mathfrak{s}}_{<\lambda}$ may be empty we cannot define saturated models. We now consider the class $K^{3,\mathrm{uq}}_{\mathfrak{s}}$ of triples $(M, N, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ such that if $M \le_{\mathfrak{K}} M^+$ then M^+, N can be $\le_{\mathfrak{K}}$-amalgamated uniquely over M as long as the type of a over M^+ does not fork over M. If the class of uniqueness triples (M, N, a) is not dense (in $K^{3,\mathrm{bs}}_{\mathfrak{s}}$) we get a non-structure result. Otherwise (assuming categoricity in λ, a soft assumption here) we can define $\mathrm{NF}_{\mathfrak{s}}$, non-forking amalgamation of models. We then investigate $K^{\mathfrak{s}}_{\lambda^+}$, more exactly the models there which are saturated. Either we get a non-structure result or our frame $\mathfrak{s}$ is successful and then we get a successor, a good λ^+-frame, $\mathfrak{s}^+ = \mathfrak{s}(*)$. Now $K_{\mathfrak{s}(+)} \subseteq K^{\mathfrak{s}}_{\lambda^+}$, but $\le_{\mathfrak{s}(+)}$ is only $\subseteq \le_{\mathfrak{K}^{\mathfrak{s}}} \restriction K_{\mathfrak{s}(+)}$.]

(C) the beauty of ω successive good λ-frames,

[Here we describe Chapter III. Assume for simplicity that letting $\mathfrak{s}^0 = \mathfrak{s}, \mathfrak{s}^{n+1} = (\mathfrak{s}^n)^+$ our assumption means that:

each $\mathfrak{s}^{+n}$ is a (well defined) successful good λ^{+n}-frame. We first try to understand better what occurs for each $\mathfrak{s}^n$ (at least when n is not too small). But to understand models of larger cardinalities we have to connect better the situation in the various cardinals, for this we use $(\lambda, \mathscr{P}^{(-)}(n))$-systems of models, particularly stable ones and in general properties for (λ, n) are connected to properties of $(\mu, n+1)$ for every large enough $\mu < \lambda$.]

§4 Appetite comes with eating

(A) The empty half of the glass,

[Here we try to see what is lacking in the present book.]

(B) The full half and half baked,

[Here we review Chapter IV which deals with abstract elementary classes which are catogorical (or just solvable) in some large enough μ. We also review Chapter VII which do the non-structure in particular eliminating the "weak diamond ideal on λ^+ is not λ^{++}-saturated" (but also do some positive theory on almost good λ-frames). We also discuss further works, which in general gives partial positive answer to the lackings in the previous subsection.]

(C) The white part of the map,

[We state conjectures and discuss them.]

§5 Basic knowledge,

(A) knowledge needed and dependency of chapters,

(B) Some basic definitions and notation,

[We review the basic set theory required for the reader and then review the model theoretic notation. Some parts need more - mainly Chapter I, Chapter IV.]

§6 Symbols,

Annotated Content for Ch.I (88r): A.e.c. near $\aleph_1$

(This chapter appeared in book 1.)

I.§0 Introduction

[We explain the background, the aims and what is done concerning the number of models of $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ in $\aleph_1$ and in $\aleph_2$; here $\mathbf{Q}$ is the quantifier there are uncountably many. Also several necessary definitions and theorems are quoted. We justify dealing with a.e.c. (abstract elementary classes). The original aim had been to make a natural, not arbitrary choice of the context ($\psi \in \mathbb{L}_{\omega_1,\omega}$ or $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$?, see [Sh 48]). The net result is a context related to, but different than, the axioms of Jónson for the existence of universal homogeneous models. One difference is that the notion of a submodel is abstract rather than a submodel; this forces us to formalize properties of being submodels and decide which we adopt, mainly AxV, (if $M_1 \subseteq M_2$ are $\leq_{\mathfrak{K}}$-submodels of N then $M_1 \leq_{\mathfrak{K}} M_2$). Another serious difference is the omission of the amalgamation property. So they are more like a class of models of $\psi \in \mathbb{L}_{\omega_1,\omega}$, recalling (as a background) that amalgamation and compactness are almost equivalent as properties of logics but formulas are not involved in the definition here.]

I.§1 Axioms and simple properties for classes of models

[We define the a.e.c. and deal with their basic properties, the classical examples being, of course, $(\mathrm{Mod}_T, \prec), T$ a first order theory, but also $(\mathrm{Mod}_\psi, \prec_{\mathrm{sub}(\psi)}), \psi \in \mathbb{L}_{\lambda^+,\omega}(\tau)$. Surprisingly (but not complicatedly) it is proved that every such class $\mathfrak{K}$ can be represented as a $\mathrm{PC}_{\lambda,2^\lambda}$, i.e. the class of $\tau_{\mathfrak{K}}$-reducts of models of a first order T omitting every type $p \in \Gamma$, where $|\Gamma| \leq 2^\lambda$ and the vocabulary has cardinality $\leq \lambda$. So though a wider context than $\mathrm{Mod}(\psi), \psi \in \mathbb{L}_{\lambda^+,\omega}$, it is not totally detached from it by the representation theorem just mentioned above. A particular consequence is the existence of relatively low Hanf numbers.]

I.§2 Amalgamation properties and homogeneity

[We present (D, λ)-sequence homogeneous and $(\mathbb{D}, \lambda)$-model homogeneous, various amalgamation properties and basic properties, in particular the existence and uniqueness of homogeneous models. Those are important properties but here they are usually unreasonable to assume; we have to console ourselves in proving them under strong assumptions (like categoricity) and after working we get the weak version.]

I.§3 Limit models and other results

[We introduce and investigate (several variants of) "limit models in $\mathfrak{K}_\lambda$", the most important one is superlimit. Ignoring the case "M_* is $<_{\mathfrak{K}}$-maximal", M_* is superlimit in $\mathfrak{K}_\lambda$ means that if $\langle M_i : i \le \delta \rangle$ is $\le_{\mathfrak{K}_\lambda}$-increasing continuous, and $i < \delta \Rightarrow M_i \cong M_*$ then $M_\delta \cong M_*$ and another formulation is "$\mathfrak{K}_\lambda \restriction \{M : M \cong M_*\}$ is a λ-a.e.c.". Note that if $\mathfrak{K}$ is categorical in λ, any $M \in \mathfrak{K}_\lambda$ is trivially superlimit. The main results use this to investigate the number of non-isomorphic models. We get amalgamation in $\mathfrak{K}_\lambda$ if $\mathfrak{K}$ has superlimit (or just so called λ^+-limit) models in $\lambda, 1 \le \dot{I}(\lambda^+, K) < 2^{\lambda^+}$ and $2^\lambda < 2^{\lambda^+}$. We at last resolve the Baldwin problem in ZFC: if $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ is categorical in $\aleph_1$ then it has a model in $\aleph_2$. In fact, the solution is in considerable more general context.]

I.§4 Forcing and Categoricity

[We assume $\mathfrak{K}$ is a $\mathrm{PC}_{\aleph_0}$-a.e.c. and it has at least one but less than the maximal number of models in $\aleph_1$, we would like to deduce as much as we can on $\mathfrak{K}$ or at least on some $\mathfrak{K}' = \mathfrak{K} \restriction K'$, which is still an a.e.c. and has models of cardinality $\aleph_1$. Toward this we build a "generic enough" model $M \in \mathfrak{K}_{\aleph_1}$ by an $\le_{\mathfrak{K}}$-increasing ω_1-sequence of models in $\mathfrak{K}_{\aleph_0}$ so define $N \Vdash^{\aleph_1}_{\mathfrak{K}} \varphi(\bar{a})$ for suitable $N \in \mathfrak{K}_{\aleph_0}$, i.e.

countable. This is reasonable for φ a formula in $\mathbb{L}_{\omega_1,\omega}(\tau_{\mathfrak{K}})$ or even $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})(\tau_{\mathfrak{K}})$. Now using $\mathbb{L}_{\omega_1,\omega_1}$ seems too strong. But we can do it over a fix $N \in K_{\aleph_0}$, so $N \leq_{\mathfrak{K}} M$. What does this mean? We have a choice: should we fix N pointwise (so adding an individual constant for each $c \in N$) or as a set (so adding a unary predicate always interpreted as N). The former makes sense only if $2^{\aleph_0} < 2^{\aleph_1}$ as is the case in §5, so in the present section we concentrate on the second. By the "not many models in $\aleph_1$" we deduce that fixing N, for a "dense" family of M satisfying $N \leq_{\mathfrak{K}} M \in K_{\aleph_0}$ we have: $(M, N) \Vdash^{\aleph_1}_{\mathfrak{K}}$ decides everything. So we know what type $p_{\bar{a}}$ each $\bar{a} \in M$ realizes in any generic enough M^+ when $M \leq_{\mathfrak{K}} M^+ \in K_{\aleph_1}$. But in general the sequence $\bar{a}$ does not realize the type $p_{\bar{a}}$ in M itself (e.g., this phenomena necessarily occurs if the formula really involves $\mathbf{Q}$). So we say $\bar{a}$ materializes the type in (M, N) and we play between some relevant languages (the logics are mainly $\mathbb{L}^{-1}_{\omega_1,\omega}$ which is without $\mathbf{Q}, \mathbb{L}^0_{\omega_1,\omega} = \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$, the vocabulary is $\tau = \tau_{\mathfrak{K}}$ or $\tau^{+0} = \tau_{\mathfrak{K}} \cup \{P\}, P$ predicate for N; and more cases). If we restrict the depth of the formulas by some countable ordinal, then the number of complete types is countable. We have to work in order to show that the number of complete $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$-types realized in quite generic models in $\mathfrak{K}_{\aleph_1}$ is $\leq \aleph_1$ (recalling that there may be Kurepa trees). We end commenting on further more complicated such results and the relevant logics.]

I.§5 There is a superlimit model in $\aleph_1$

[Here we add to §4 the assumption $2^{\aleph_0} < 2^{\aleph_1}$ hence we prove amalgamation of $\mathfrak{K}_{\aleph_0}$ (or get a non-structure result). Someone may say something like §1-§3 are conceptual and rich, I.§4-§5 are technicalities. I rather think that §1,§2,§3 are the preliminaries to the heart of the matter which is §4 and mainly §5. Assuming properties implying non-structure results in ($\aleph_1$ and) $\aleph_2$ fails, we understand models in $\mathfrak{K}_{\aleph_0}$ and $\mathfrak{K}_{\aleph_1}$ better. In particular we get for countable N that the number of types realized in some generic enough $M \in \mathfrak{K}_{\aleph_1}$, so called $\mathbf{D}(N)$ which $\leq_{\mathfrak{K}}$-extend N, is $\leq \aleph_1$, and we can restrict ourselves to subclasses with strong notion of elementary submodel such that each $\mathbf{D}(N)$ is countable. A central question is the existence of amalgamations which are stable, definable in a suitable sense of countable models trying to prove symmetry, equivalently some variants and eventually uniqueness. The culmination is proving the existence of a superlimit model in $\aleph_1$, though this is more than necessary for the continuation (see II§3).]

I.§6 Counterexamples

[Some of our results (in previous sections) were gotten in ZFC, but mostly we used $2^{\aleph_0} < 2^{\aleph_1}$. We show here that this is not incidental. Assuming $\mathrm{MA}_{\aleph_1}$, there is an a.e.c. $\mathfrak{K}$ which is $\mathrm{PC}_{\aleph_0}$, categorical in $\aleph_0$ and in $\aleph_1$, but fails the amalgamation property. We can further have that it is axiomatized by some $\psi \in \mathbb{L}_{\omega,\omega}(\mathbf{Q})$, and we deal with some related examples.]

Annotated Content for Ch.II (600): Categoricity in a.e.c.: going up inductive steps

(This chapter appeared in book 1.)

II.§0 Introduction

[We present the results on good λ-frames and explain the relationship with [Sh 576] that is Chapter VI and with Chapter I. We then suggest some reading plans and some old definitions.]

II.§1 Abstract elementary classes

[First we recall the definition and some claims. In particular we define types (reasonable over models which are amalgamation basis), and we prove some basic properties, in particular, model homogeneity - saturativity lemma II.1.14 which relate realizing types of <u>singleton</u> elements to finding copies of models. We also define "N is (λ, θ)-brimmed over M", etc., and their basic properties. Then we prove that we could have restricted our class $\mathfrak{K}$ to cardinality λ without any real loss, i.e., any λ-a.e.c. can be blown up to an a.e.c. with LS-number λ and any a.e.c. with LS-number $\leq \lambda$ can be restricted to cardinality λ and as long as we ignore the models of cardinality $< \lambda$, this correspondence is one to one (see II.1.23, II.1.24); reading those proofs is a good exercise in understanding what is an a.e.c.]

II.§2 Good frames

[We introduce the central axiomatic framework called "good λ-frames", $\mathfrak{s} = (K_{\mathfrak{s}}, \leq_{\mathfrak{s}}, \mathrm{NF}_{\mathfrak{s}})$. The axiomatization gives the class $K_{\mathfrak{s}}$ of models and a partial order $\leq_{\mathfrak{s}}$ on it, forming an a.e.c., $\mathfrak{K}_{\mathfrak{s}} = (K_{\mathfrak{s}}, \leq_{\mathfrak{s}})$, a set $\mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ of "basic" types over any model $M \in K_{\mathfrak{s}}$, the ones for which we have a non-forking notion. A (too good) example is regular types for superstable first order theories. We also check how can the non-forking of types be lifted up to higher cardinals or fewer models; but unlike the lifting of λ-a.e.c. in §1 in this lifting we lose some essential properties; in particular uniqueness and existence. We end noting some implications between axioms of good λ-frames.]

II.§3 Examples

[We prove here that cases treated in earlier relevant works fit the framework from §2. This refers to [Sh 576], Chapter I and also to [Sh 87a], [Sh 87b], [Sh 48].]

II.§4 Inside the frame

[We prove some claims used later, in particular stability in λ, sufficient condition for M_δ being $(\lambda, \mathrm{cf}(\delta))$-brimmed over

M_0 for a chain $\langle M_i : i \le \delta \rangle$ and the uniqueness of the $(\lambda, *)$-brimmed model over $M_0 \in K_\lambda$. We deal (for those results but also for later uses) with non-forking rectangles and triangles. An easy (but needed in the end) consequence is that $K^{\mathfrak{s}}_{\lambda^{++}}$ is not empty.]

II.§5 Non-structure or some unique amalgamations

[We prove that we have strong non-structure in $K^{\mathfrak{s}}_{\lambda^{++}}$ or for enough triples $(M_0, M_1, a) \in K^{3,\mathrm{bs}}$ we have unique amalgamation of M_1, M_2 over M_0 when $M_0 \le_{\mathfrak{K}} M_2 \le_{\mathfrak{K}} M_3, M_0 \le_{\mathfrak{K}} M_1 \le_{\mathfrak{K}} M_3$ and we demand that $\mathbf{tp}(a, M_2, M_3)$ does not fork over M_0. Naturally, we use the framework of [Sh 576, §3] or better Chapter VII and we do the model theoretic work required to be able to apply it. More explicitly, from the non-density of such triples with uniqueness we prove a non-structure theorem in λ^{++}. A major point in proving this dichotomy is to guarantee that $\bigcup_{\alpha<\delta} M_\alpha \in K_{\lambda^+}$ is saturated, when $\delta < \lambda^{++}$ and each $M_\alpha \in K_{\lambda^+}$ is saturated at least when $\langle M_\alpha : \alpha < \delta \rangle$ appears in our constructions. For this we use M_α which is $\le_{\mathfrak{K}}$-represented by $\langle M^\alpha_i : i < \lambda^+ \rangle$ so $M_\alpha = \bigcup_{i<\lambda^+} M^\alpha_i$ and $\langle \langle M^\alpha_i : i < \lambda^+ \rangle : \alpha < \lambda^{++} \rangle$ is used with extra promises on non-forking of types, which are preserved in limits of small cofinality. Note that we know that in $\mathfrak{K}^{\mathfrak{s}}_{\lambda^+}$ there is a model saturated above λ but we do not know that it is superlimit.]

II.§6 Non-forking amalgamation in $\mathfrak{K}_\lambda$

[Our aim is to define the relation of non-forking amalgamations for models in K_λ and prove the desired properties promised by the name. What we do is to start with the cases which §5 provides us with a unique amalgamation modulo non-forking of a type of an element, and "close" them by iterations arriving to a (λ, θ)-brimmed extension. This defines non-forking amalgamation in the brimmed case, and then by closing under the submodels we get the notion itself. Now we have to work on getting the properties we hope for. To clarify, we prove that "a non-forking relation with the reasonable nice properties" is unique. A consequence of all this is that we can change $\mathfrak{s}$ retaining $\mathfrak{K}_{\mathfrak{s}}$ such that it is type-full, i.e., every non-algebraic type (in $\mathscr{S}_{\mathfrak{K}_\lambda}(M)$ is basic for $\mathfrak{s}$. (This is nice and eventually needed.)]

II.§7 Nice extensions in K_{λ^+}

[Using the non-forking amalgamation from §6, we define nice models ($K^{\rm nice}_{\lambda^+}$) and "nice" extensions in $\lambda^+(\le^*_{\lambda^+})$, and prove on them nice properties. In particular K_{λ^+} with the nice extension relation has a superlimit model - the saturated one.]

II.§8 Is $K^{\rm nice}_{\lambda^+}$ with $\le^*_{\lambda^+}$ a λ^+-a.e.c.?

[We prove that $\mathfrak{K}^{\rm nice}_{\lambda^+} = (K^{\rm nice}_{\lambda^+} \le^*_{\lambda^+})$ is an a.e.c. under an additional assumption but we prove that the failure of this extra assumption implies a non-structure theorem. We then prove that there is a good λ^+-frame $\mathfrak{t}$ with $\mathfrak{K}_{\mathfrak{t}} = \mathfrak{K}^{\rm nice}_{\lambda^+}$ and prove that it relates well to the original $\mathfrak{s}$, e.g. we have locality of types.]

II.§9 Final conclusions

[We reach our main conclusions (like II.0.1) in the various settings.]

Annotated Content for Ch.III (705): Toward classification theory of good λ-frames and a.e.c.

(This chapter appeared in book 1.)

III.§0 Introduction

III.§1 Good$^+$ Frames

[We define when a good λ-frame is successful (III.1.1) and when it is good$^+$ (III.1.3). There are quite many good$^+$ frames $\mathfrak{s}$: the cases of good λ-frames we get in II§3 all are good$^+$ and further, if $\mathfrak{s}$ is successful good λ-frame (not necessarily good$^+$!) then $\mathfrak{s}^+$ is good$^+$ (see III.1.5, III.1.9). Moreover, if $\mathfrak{s}$ is a good$^+$ successful λ-frame, then $\mathfrak{s}^+ = \mathfrak{s}(+)$ satisfies $\leq_{\mathfrak{s}(+)} = \leq_{\mathfrak{K}[\mathfrak{s}]} \restriction K_{\mathfrak{s}(+)}$ (see Definition III.1.7 and Claim III.1.8), and we can continue and deal with $\mathfrak{s}^{+\ell} = \mathfrak{s}(+\ell)$, (see III.1.14). We define naturally "$\mathfrak{s}$ is n-successful" and look at some basic properties. We end recalling some things from Chapter II which are used often and add some. We prove locality for basic types and types for $\mathfrak{s}(+)$, see III.1.10, III.1.11. In III.1.21 we show that if $M_1 \leq_{\mathfrak{s}} M_2$ are brimmed and the type $p_2 \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_2)$ does not fork over M_1 then some isomorphism from M_2 to M_1 maps p_2 to $p_2 \restriction M_1$, similarly with $< \lambda$ types. In III.1.16-III.1.20 we essentially say to what we use on $\mathrm{NF}_{\mathfrak{s}}$, assuming $\mathfrak{s}$ is weakly successful; this is the part most used later.]

III.§2 Uni-dimensionality and non-splitting

[We are interested not only in the parallel of being superstable but also of being categorical, which under natural assumptions is closely related to being uni-dimensional. We now define (the parallel of) uni-dimensional, more exactly some variants including non-multi-dimensionality (in III.2.2, III.2.13). We then note when our examples are like that; we show that $\mathfrak{s}(+)$ satisfies such properties when (even iff) $\mathfrak{s}$ does (III.2.6, III.2.10, III.2.17 and more in III.2.12). Of course we show the close connection between uni-dimensionality and categoricity in λ^+ (see III.2.11). Next we deal with minimal types and with good λ-frames for minimals (III.2.13 - III.2.17). We then look at splitting, relevant ranks and connection to non-forking (from III.2.18 on). We also know what occurs if we make $\mathfrak{s}$ type-full (III.2.7) and we then consider frames where the basic types are the minimal types (III.2.15 - III.2.17). We then recall splitting.]

III.§3 Prime triples

[We define $K_{\mathfrak{s}}^{3,\mathrm{pr}}$, the family of prime triples (M, N, a), the family of minimal triples and "$\mathfrak{s}$ has primes" (Definition III.3.2). We look at the basic properties (III.3.5,III.3.8), connection to $K_{\mathfrak{s}}^{3,\mathrm{uq}}$ (III.3.7) and x-decompositions for $x = $ pr,uq,bs in Definition III.3.3. In particular if $\mathfrak{s}$ has primes then any pair $M <_{\mathfrak{s}} N$ has a pr-decomposition (see III.3.11). We prove the symmetry for "the type of a_ℓ over $M_{3-\ell}$ does not fork over M_0 wherever $M_{3-\ell}$ is prime over $M_0 \cup \{a_{3-\ell}\}$" (III.3.9, III.3.12); note that the symmetry axiom say "for some $M_{3-\ell} \ldots$".]

III.§4 Prime existence

[We deal with good$^+$ successful λ^+-frame $\mathfrak{s}$. We recall the definition of $\leq_{\mathrm{bs}}$ and variants, and prove that $\mathfrak{s}^+$ has primes (III.4.9). For this we prove in III.4.9 that a suitable condition is sufficient for (M, N, a) to belong to $K_{\mathfrak{s}(+)}^{3,\mathrm{pr}}$, proving it occurs (in III.4.3), and more in III.4.5, III.4.14, III.4.20. We use for it $\leq_{\mathrm{bs}}$ (defined with the variants $<_{\mathrm{bs}}^*, <_{\mathrm{bs}}^{**}$ in III.4.2), the relevant properties in III.4.6. We then investigate more on how properties for $\mathfrak{s}^+$ reflects to $\lambda_{\mathfrak{s}}$, for $\mathrm{NF}_{\mathfrak{s}(+)}$ in III.4.15 also in III.4.13(2). Also we consider other sufficient conditions for III.3.9's conclusion in III.4.13(1). Lastly, III.4.20 deals with the examples.]

III.§5 Independence

[We define $\mathbf{I}_{M,N}$ and define when $\mathbf{J} \subseteq \mathbf{I}_{M,N}$ is independent in (M, N), (see Definition III.5.2). In III.5.4 + III.5.5 + III.5.6 + III.5.8(2) we prove fundamental equivalences and properties, including M_0-based pr/uq-decomposition in N/of N and that "independent in (M, N)" has finitary character. We also define "N is prime over $M \cup \mathbf{J}$" denoted by $(M, N, \mathbf{J}) \in K_{\mathfrak{s}}^{3,\mathrm{qr}}$ (Definition III.5.7). We note existence and basic properties (claim III.5.8). We show embedding existence (III.5.9(1)) and how this implies NF (see III.5.9(2)). We show that "normally" independence satisfies continuity (III.5.10) and reflect from $\mathfrak{s}^+$ to $\mathfrak{s}$ (III.5.11). Using this we prove the basic claims on dimension for non-regular types, (see III.5.12, III.5.13 + III.5.14).

We generalize $K^{3,\mathrm{uq}}_{\mathfrak{s}}$, the class of uniqueness triples (M,N,a), to $K^{3,\mathrm{vq}}_{\mathfrak{s}}$, the class of uniqueness triples $(M,N,\mathbf{J}),\mathbf{J}$ independent in (M,N), Definition III.5.15(1). We then define when $(M,N,\mathbf{J}) \in K^{3,\mathrm{vq}}_{\mathfrak{s}}$ is thick (Definition III.5.15) and prove their basic properties, in particular $K^{3,\mathrm{qr}}_{\mathfrak{s}} \subseteq K^{3,\mathrm{vq}}_{\mathfrak{s}}$ (see III.5.16, III.5.16(3)). When $\mathfrak{s} = \mathfrak{t}^+$ we "reflect" $K^{3,\mathrm{qr}}_{\mathfrak{s}}$ to cases of $K^{3,\mathrm{vq}}_{\mathfrak{t}}$ (see III.5.22). Lastly, every triple in $K^{3,\mathrm{bs}}_{\mathfrak{s}}$ can be extended to one in $K^{3,\mathrm{vq}}_{\mathfrak{s}}$ (with the same $\mathbf{J}$, Claim III.5.24).]

III.§6 Orthogonality

[We define when $p,q \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ are weakly orthogonal/orthogonal, (Definition III.6.2), show that "for every $(M,N,a) \in K^{3,\mathrm{uq}}_{\mathfrak{s}}$..." can be replaced "for some ...", (III.6.3) and prove basic properties (III.6.4, III.6.7), and define parallelism (see III.6.5,III.6.6). We define "a type p is orthogonal/super-orthogonal to a model" (Definition III.6.9, the "super" say preservation under NF amalgamation), prove basic properties (III.6.10), and how we reflect from $\mathfrak{s}^+$ to $\mathfrak{s}$ (see III.6.11 concerning $p\perp q, p\perp M$). Orthogonality helps to preserve independence (III.6.12). We investigate decompositions of tower with orthogonality conditions. If $(M,N,a) \in K^{3,\mathrm{uq}}_{\mathfrak{s}}, M \cup \{a\} \subseteq N' < N$ and $p = \mathbf{tp}_{\mathfrak{s}}(b,N',N)$ then p is weakly orthogonal to M (see III.6.14(1),III.6.14(2)), and decompose such triples by it (III.6.14(2)), look at an improvement (III.6.15(1)) and reflection from $\mathfrak{s}^+$ (in III.6.15(2)), how we can use independence, $K^{3,\mathrm{vq/qr}}_{\mathfrak{s}}$ and orthogonality (III.6.16, III.6.18, III.6.20, III.6.22). In particular by III.6.20(2) if $(M_n,M_n,\mathbf{J}_n) \in K^{3,\mathrm{uq}}_{\mathfrak{s}}$ for $n<\omega$ and $c \in \mathbf{J}_{n+1} \Rightarrow \mathbf{tp}_{\mathfrak{s}}(c,M_{n+1},M_{n+2})\perp M_0$ then $(M_0,\bigcup\limits_n M_n,\mathbf{J}_0) \in K^{3,\mathrm{vq}}_{\mathfrak{s}}$. From pairwise orthogonality we can get independence (III.6.21), and one p cannot be non-orthogonal to infinitely many pairwise orthogonal types (III.6.22).]

III.§7 Understanding $K_{\mathfrak{s}}^{3,\mathrm{uq}}$

[In III.7.2, we define $\mathscr{W}, \leq_{\mathscr{W}}$ (weak form of decompositions of triples from $K_{\mathfrak{s}}^{3,\mathrm{vq}}$) and related objects, in III.7.3 we prove basic properties. In III.7.4 we define $K_{\mathfrak{s}}^{x}$ for $x =$ or,ar,br, decompositions of length $\leq \omega$ of triples in $K_{\mathfrak{s}}^{3,\mathrm{or}}$ with various orthogonality conditions (why of length $\leq \omega$? so that in inductive proof when we arrive to a limit case we are already done). We also define fat, related to thick, (III.5.8(5)) and we prove in III.7.6 some properties. In III.7.5 we define "$\mathfrak{s}$ weakly has regulars" and later, in III.7.18, define "almost has regulars". Existence for $K_{\mathfrak{s}}^{3,\mathrm{or}}, K_{\mathfrak{s}}^{3,\mathrm{ar}}$ (assuming enough regulars) are investigated (in III.7.7, III.7.8). We characterize being in $K_{\mathfrak{s}}^{3,\mathrm{uq}}$ in III.7.9, this is the main result of the section. We then deal with universality and uniqueness for fat uq/vq triples (see III.7.11 - III.7.13). We also deal with hereditary and limits of uq/vq triples in III.7.15, III.7.16.]

III.§8 Tries to decompose and independence of sequences of models

[We define and prove existence of x-decompositions $(\bar{M}, \bar{a})$ with $\mathbf{tp}_{\mathfrak{s}}(a_i, M_i, M_{i+1})$ does not fork over some M_j but is orthogonal to M_ζ when $\zeta < j$ and show that $(M_0, M_\alpha, \{a_i : \mathbf{tp}_{\mathfrak{s}}(a_i, M_i, M_{i+1})$ does not fork over $M_0\}) \in K_{\mathfrak{s}}^{3,\mathrm{vq}}$ and also revisit existence for $K_{\mathfrak{s}}^{3,\mathrm{vq}}$ (see III.8.2, III.8.3, III.8.6). We define and investigate when $\langle M_i : i < \alpha\rangle$ is $\mathfrak{s}$-independent over M inside N with witness $\bar{N} = \langle N_i : i \leq \alpha\rangle$ (see III.8.8 - III.8.18). In III.8.19 we return to investigating $\mathrm{NF}_{\mathfrak{s}}$, prove that it is preserved under reasonable limits and by III.8.21 this holds for $K_{\mathfrak{s}}^{3,\mathrm{vq}}$. We also further deal with $K_{\mathfrak{s}}^{3,\mathrm{vq}}$.]

III.§9 Between cardinals, non-splitting and getting fullness

[We deal mainly with varying $\mathfrak{s}$. We fulfill a promise, proving that a weakly successful good λ-frame $\mathfrak{s}$ can be doctored to be full (see III.9.5 - III.9.6. Also we show that if $\mathfrak{s}$ is a successful λ-good$^+$ frame, <u>then</u> we can define a λ^+-good$^+$ successor $\mathfrak{s}^{\mathrm{nf}}$ with $\mathfrak{K}_{\mathfrak{s}^{\mathrm{nf}}} = \mathfrak{K}_{\mathfrak{s}}$ and $\mathfrak{s}^{\mathrm{nf}}$ is full, i.e. $\mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}(+)} = \mathscr{S}^{\mathrm{na}}_{\mathfrak{s}(+)}$; moreover if $\mathfrak{s}$ is categorical and successful.]

III.§10 Regular types

[We deal mainly with type-full $\mathfrak{s}$. We define regular and regular$^+$ (Definition III.10.2),prove some basic equivalences (III.10.4) and prove that the set of regular types is "dense" (III.10.5). To prove that for regular type p, non-orthogonality, $(p \pm q)$ is equivalent to being dominated, $(p \trianglelefteq q)$ (in III.10.8), we prove a series of statements on regular and regular$^+$ types (in III.10.6). We prove e.g. that if $\langle M_i : i \leq \delta + 1 \rangle$ is increasing continuous, $M_\delta \neq M_{\delta+1}$ then some $c \in M_{\delta+1} \backslash M_\delta$ realizes a regular type over M_δ which does not fork over M_j but is orthogonal to M_{j-1} if $j > 0$, for some j, which necessarily is a successor ordinal (III.10.9(3)) that is, we prove that $\mathfrak{s}$ almost has regulars. Hence weakly has regulars as expected from the names we choose. Using this, we revisit decompositions (III.10.12).]

III.§11 DOP

[We deal with the dimensional order property.]

III.§12 Brimmed Systems

[This is the crux of the matter. We deal with systems $\mathbf{m} = \langle M_u : u \in \mathscr{P} \rangle$, $\mathscr{P}$ usually is $\mathscr{P}(n)$ or $\mathscr{P}^-(n)$, which are "stable", as witnessed by various maximal independent sets. A parameter $\ell = 1, 2, 3$ measure how brimmed is $\mathbf{m}$, presently the central one is $\ell = 3$. We then phrase properties related to such stable system, e.g. the weak (λ, n)-existence say every such $(\lambda, \mathscr{P}^-(n))$-system can be completed to a $(\lambda, \mathscr{P}(n))$-system; the strong (λ, n)-existence property says that we can do it "economically", by a "small M_n". We also define weak/strong uniqueness, weak/strong primeness and weak/strong prime existence. The main work is proving the relevant implications. The culmination is proving that if $\mathfrak{s}$ is ω-successful and $\langle 2^{\lambda_{\mathfrak{s}}^{+n}} : n < \omega \rangle$ is increasing, <u>then</u> all positive properties holds and so can understand, e.g. categoricity spectrum (and superlimit models).]

Annotated Content for Ch.IV (734): Categoricity and Solvability of a.e.c., quite highly

(This chapter appeared in book 1.)

IV.§0 Introduction

[Our polar star is: if an a.e.c. is categorical in arbitrarily large cardinals then it is categorical in every large enough cardinal. We make some progress getting some good λ-frames; and to point to a more provable advancement, confirm this conjecture (and even a reasonable bound on starting) for a.e.c. with amalgamation (as promised in [Sh:E36]). In fact we put forward solvability as the true parallel to superstability.]

IV.§1 Amalgamation in K^*_λ

[We assume $\mathfrak{K}$ is categorical in μ (or less-solvable in μ); and the best results are on λ such that $\mu > \lambda = \beth_\lambda > \text{LS}(\mathfrak{K})$ (i.e. λ is a fix point in the beth sequence) and λ has cofinality $\aleph_0$; we fix suitable $\Phi \in \Upsilon^{\text{or}}[\mathfrak{K}]$. We mostly assume $\mu = \mu^\lambda$.

First we investigate $K^*_\theta = \{M : M \cong \text{EM}(I, \Phi)$ for some linear order I of cardinality $\theta\}$, which is in general not an a.e.c. under $\leq_{\mathfrak{K}}$, but in our μ it is. We investigate such models in the logic $\mathbb{L}_{\infty,\partial}$, particularly when θ is large enough than $\partial, \partial > \text{LS}(\mathfrak{K})$ (mainly $\theta \geq \beth_{1,1}(\partial)$). We get more and more cases when $M \prec_{\mathbb{L}_{\infty,\partial}[\mathfrak{K}]} N$ follows from $M \leq_{\mathfrak{K}} N+$ additional assumptions. An evidence of our having gained understanding is proving the amalgamation theorem IV.1.29: the class $(K^*_\lambda, \leq_{\mathfrak{K}})$ has the amalgamation property. In the end we prove that if $\lambda = \Sigma\{\lambda_n : n < \omega\} < \mu$ each λ_n is as above and $< \lambda_{n+1}$ and is μ as above then $\mathfrak{K}_\lambda$ has a local superlimit model, see IV.1.38, in fact we get a version of solvability in λ, see IV.1.41.]

IV.§2 Trying to Eliminate $\mu = \mu^{<\lambda}$

[In §1 essentially (in the previous section) the first step in our ladder was proving $M \prec_{\mathbb{L}_{\infty,\theta}} N$ for $M \leq_{\mathfrak{K}} N$ from K_μ but we have to assume $\mu = \mu^{<\theta}$. As we use it for many $\theta < \lambda$, the investigation does not even start without assuming $\mu = \mu^{<\lambda}$. We eliminate this assumption <u>except</u> "few" exceptions (i.e., for a given $\mathfrak{K}$ and θ).]

IV.§3 Categoricity for cardinals in a club

[We assume $\mathfrak{K}$ is categorical in unbounded many cardinals. We show that for some closed and unbounded class $\mathbf{C}$ of cardinals, $\mathfrak{K}$ is categorical in μ for every $\mu \in \mathbf{C}$ of cofinality $\aleph_0$ (or $\aleph_1$). This is a weak theorem still show that the categoricity spectrum is far from being "random" (as is, e.g. the rigidity spectrum is by [Sh 56]).]

IV.§4 Good frames

[Assume for simplicity that $\mathfrak{K}$ is categorical in arbitrarily large cardinals μ. Then for every $\lambda = \Sigma\{\lambda_n : n < \omega\}, \lambda_n = \beth_{\lambda_n} > \mathrm{LS}(\mathfrak{K})$ there is a superlimit model in $\mathfrak{K}_\lambda$, and even a version of solvability. Moreover there is a good λ-frame $\mathfrak{s}_\lambda$ such that $K_{\mathfrak{s}_\lambda} \subseteq \mathfrak{K}_\lambda, \leq_{\mathfrak{s}_\lambda} = \leq_{\mathfrak{K}} \restriction K_{\mathfrak{s}_\lambda}$. Other works, in particular Chapter III, are a strong indication that this puts us on our way for proving the goal from §0.]

IV.§5 Homogeneous enough linear orders

[We construct linear order I of any cardinality $\lambda > \mu$ such that there are few $J \in [I]^\mu$ up to an automorphism of I and more. This helps when analyzing EM models using the skeleton I. Used only in §2 and §7. The proof is totally direct: we give a very explicit definition of I, though the checking turns out to be cumbersome.]

IV.§6 Linear orders and equivalence relations

[For a "small" linear order J and a linear order I, mainly well ordered we investigate equivalence relations $\mathscr{E}$ on $\mathrm{inc}_J(I) = \{h : h \text{ embed } J \text{ into } I\}$ which are invariant, i.e., defined by a quantifier free (infinitary) formula, hence can (under reasonable conditions) be defined on every I'. We are interested mainly to find when $\mathscr{E}$ has $> |I|$ equivalence classes; and for "there is a suitable I of cardinality λ". The expected answer is a simple question on λ: is $\lambda > \lambda^{|J|}/D$ for some suitable filter D? but we just prove enough for the application in §7, dealing with the case $\lambda > \lambda^{|J|}/D$ holds for some non-principal ultrafilter on $|J|$.]

IV.§7 Categoricity spectrum for a.e.c. with bounded amalgamation

[Let $\mathfrak{K}$ be a.e.c. categorical in μ (or less, $\Phi \in \Upsilon^{\rm or}_{{\rm LS}[\mathfrak{K}]}[\mathfrak{K}]$, if $\lambda > \mu \geq {\rm cf}(\mu) > {\rm LS}(\mathfrak{K})$ and $\mathfrak{K}_{<\mu}$ has amalgamation. Then for $\mu_* < \mu$, every saturated $M \in \mathfrak{K}$ of cardinality $\in [\mu_*, \mu)$ is μ_*-local, i.e., any type $p \in \mathscr{S}_{\mathfrak{K}}(M)$ is determined by its restriction to model $N \leq_{\mathfrak{K}} M$ of cardinality μ_*. Also $M \in K$ is (χ, μ)-saturated, e.g., if $2^{2^\chi} < \mu$. Then we prove that if $\mathfrak{K}$ is an a.e.c. categorical in a not too small cardinal μ and has amalgamation up to μ or less) <u>then</u> it is categorical in every not too small cardinal. We delay the improvements concerning solvability spectrum and saying more in the case $\mathfrak{K} = ({\rm Mod}_T, \prec_{\mathbb{L}_{\kappa,\omega}})$, where $T \subseteq \mathbb{L}_{\kappa,\omega}, \kappa$ measurable. In all cases we eliminate the restriction of starting with "μ successor" and having the upward directions, too.]

Annotated Content for Ch.V.A (300a): Stability theory for a model

V.A.§0 Introduction

[Introduction and notation.]

V.A.§1 The order property revisited

[We define some basic properties. First a model M has the $(\varphi(\bar{x};\bar{y};\bar{z}),\mu)$-order property (= there are $\bar{a}_\alpha,\bar{b}_\alpha,\bar{c}$ for $\alpha<\mu$ such that $\varphi(\bar{a}_\alpha;\bar{b}_\beta,\bar{c})$ is satisfied iff $\alpha<\beta$) and the non-order property is its negation. Also indiscernibility (of a set and of a sequence), and non-splitting. We then prove the non-splitting/order dichotomy: if M is an elementary submodel of N in a strong enough way related to χ and κ and $\bar{a}\in{}^\kappa N$ <u>then</u> either $\mathrm{tp}_\Delta(\bar{a},M,N)$ is definable in an appropriate way (i.e., does not split over some set $\leq\chi$ relevant formulas) <u>or</u> N has (ψ,χ^+)-order for a formula ψ related to Δ. Lastly, we prove that (Δ,χ^+)-non-order implies (μ,Δ)-stability if for appropriate χ,μ. We also define various sets of formulas Δ^x derived from Δ.]

V.A.§2 Convergent indiscernible sets

[For stable first order theory, an indiscernible set $\mathbf{I}\subseteq M$ define its average type over M: the set of $\varphi(\bar{x},\bar{b})$ satisfied by all but finitely many $\bar{c}\in\mathbf{I}$. In general not every indiscernible set $\mathbf{I}$ has an average, so we say $\mathbf{I}$ is (Δ,χ)-<u>convergent</u> if any formula $\varphi(\bar{x},\bar{b})$ where $\varphi\in\Delta$ and $\bar{b}$ is from M, divide $\mathbf{I}$ to two sets, exactly one of which has $<\chi$ members. We prove that convergent sets exists (V.A.2.8) under reasonable conditions (mainly non-order). We also prove that convergent sets contain indiscernible ones. Toward the existence we give a sufficient condition in V.A.2.10 for a sequence $\langle\bar{c}_i:i<\mu^+\rangle$ being (Δ,χ^+)-convergent including the (Δ,χ^+)-non-order property which is easy to obtain.]

V.A.§3 Symmetry and indiscernibility

[We prove a symmetry lemma (V.A.3.1), give sufficient conditions for being an indiscernible sequence (V.A.3.2), and when an indiscernible sequence is an indiscernible set (V.A.3.5), and on getting an indiscernible set from a convergent set.]

V.A.§4 What is the appropriate notion of a submodel

[We define $M \leq^{\kappa}_{\Delta,\mu,\chi} N$ which says that for $\bar{c} \in {}^{\kappa>}N$, the Δ-type which it realizes over M inside N is the average of some (Δ, χ^+)-convergent set of cardinality μ^+ inside M. We give an alternative definition of being a submodel (in V.A.4.4) when M has an appropriate non-order property, prove their equivalence and note some basic properties supporting the thesis that this is a reasonable notion of being a submodel. We then define "stable amalgamation of M_1, M_2 over M_0 inside M_3" and investigate it to some extent.]

V.A.§5 On the non-order implying the existence of indiscernibility

[We give a sufficient condition for the existence of "large" indiscernible set $\mathbf{J} \subseteq \mathbf{I}$, in which $|\mathbf{J}| < |\mathbf{I}|$, but the demand on the non-order property is weaker than in V.A.§2 speaking only on non-order among singletons. Even for some first order T which are unstable, this gives new cases e.g. for $\Delta =$ the set of quantifier free formulas.]

Annotated Content for Ch.V.B (300b): Axiomatic framework

V.B.§0 Introduction

[Rather than continuing to deal with universal classes per se, we introduce some frameworks, deal with them a little and show that universal classes with the $(\chi, < \aleph_0)$-non-order property fit some of them (for suitable choices of the extra relations). In the rest of Chapter II almost always we deal with AxFr_1 only.]

V.B.§1 The Framework

[We suggest several axiomatizations of being "a class of models K with partial order $\leq_{\mathfrak{K}}$ with non-forking and possibly the submodel generated by a subset" (so being a submodel, non-forking and $\langle A \rangle^{\rm gn}_M$ for $A \subseteq M$ are abstract notions). The main one here, AxFr_1 is satisfied by any universal class with $(\chi, < \aleph_0)$-non-order; (see §2). For AxFr_1 if M_1, M_2 are in non-forking amalgamation over M_0 inside M_3 then the union $M_1 \cup M_2$ generate a $\leq_{\mathfrak{K}}$-submodel of M_3. In such contexts we define a type as an orbit, i.e. by arrows (without formulas or logic); to distinguish we write **tp** (rather than tp_Δ) for such types. Also "Tarski-Vaught theorem" is divided to components. On the one hand we consider union existence Ax(A4) which says that: the union of an $\leq_{\mathfrak{K}}$-increasing chain belongs to the class and is $\leq_{\mathfrak{K}}$-above each member. On the other hand we consider smoothness which says that any $\leq_{\mathfrak{K}}$-upper bound is $\leq_{\mathfrak{K}}$-above the union.]

V.B.§2 The Main Example

[We consider a universal class K with no "long" linear orders, e.g. by quantifier free formulas (on χ-tuples), we investigate the class K with a submodel notion introduced in V.A§4, and a notion of non-forking, and prove that it falls under the main case of the previous section. We also show how the first order case fits in and how (D, λ)-homogeneous models does.]

V.B.§3 Existence/Uniqueness of Homogeneous quite Universal Models

[We investigate a model homogeneity, toward this we define $\mathbb{D}_\chi(M), \mathbb{D}_\chi(\mathfrak{K}), \mathbb{D}'_{\mathfrak{K},\chi}$ and define "M is $(\mathbb{D}, \lambda)$-model homogeneous". We show that being λ^+-homogeneous λ-universal model in $\mathfrak{K}$ can be characterized by the realization of types of singletons over models (as in the first order case) so having "the best of both worlds".]

Annotated Content for Ch.V.C (300c): A frame is not smooth or not χ-based

V.C.§0 Introduction

[The two dividing lines dealt with here have no parallel in the first order case, or you may say they are further parallels to stable/unstable, i.e. stability "suffer from schizophrenia", there are distinctions between versions which disappear in the first order case, but still are interesting dividing lines.]

V.C.§1 Non-smooth stability

[This section deals with proving basic facts inside AxFr_1. On the one hand we assume we are hampered by the possible lack of smoothness, on the other hand the properties of $\langle - \rangle^{\mathrm{gn}}_M$ are helpful. These claims usually say that specific cases of smoothness, continuity and non-forking hold. So it deals with the (meagre) positive theory in this restrictive context.]

V.C.§2 Non-smoothness implies non-structure

[We start with a case of failure of κ-smoothness, copy it many times on a tree $\mathscr{T} \subseteq {}^{\kappa \geq}\lambda$; for each $i < \kappa$ for every $\eta \in \mathscr{T} \cap {}^i\lambda$ we copy the same things while for $\eta \in \mathscr{T} \cap {}^\kappa\lambda$ we have a free choice. This is the cause of non-structure, but to prove this we have to rely heavily on §1. If we assume the existence of unions, for any $<_{\mathfrak{s}}$-increasing sequence, i.e. Ax(A4), the non-structure (in many cardinals), is proved in ZFC, but using weaker versions we need more.]

V.C.§3 Non χ-based

[We note some basic properties about directed systems and how much they depend on smootheness. We then define when $\mathfrak{s}$ is χ-based: if $M \leq_{\mathfrak{s}} N$ and $A \subseteq N$ has cardinality $\leq \chi$ then for some M_1, N_1 of cardinality $\leq \chi$ we have $\mathrm{NF}_{\mathfrak{s}}(M_1, N_1, M, N)$ and $A \subseteq N_1$. This is a way to say that $\mathbf{tp}_{\mathfrak{s}}(N_1, N)$ does not fork over M_1, so being χ-based is a relative of being stable, and when it fails, a very explicit counterexample.]

V.C.§4 Stable construction

[We generalize [Sh:c, IV] to this context. That is we deal with constructions: in each stage we add a "small" set which realizes over what was constructed so far a type which does not fork over their intersection. We define and investigate the basic properties of such constructions.]

V.C.§5 Non-structure from "NF is not χ-based"

[Assuming the explicit failure of "χ-based over models of cardinality χ^+", and using the existence of good stationary subsets S^* of regular $\lambda > \chi^{++}$ of cofinality χ^+, we build a model in $\mathfrak{K}^{\mathfrak{s}}_\lambda$ which codes any subset S of S^* (modulo the club filter) hence get a non-structure theorem. Naturally we use the stable constructions from the previous section, §4 and have some relatives.]

Annotated Content for Ch.V.D (300d): Non-forking and prime models

V.D.§0 Introduction

[Here we deal with types of models (rather than types of single elements). This is O.K. for parallel to some properties of stable first order theories T, mainly dealing with $|T|^+$-saturated models.]

V.D.§1 Being smooth and based propagate up

[By Chapter V.C we know that failure of smoothness and failure of being χ-based are non-structure properties, but they may fail only for some large cardinal. We certainly prefer to be able to prove that faillure, if it happens at all, happens for some quite small cardinal; we do not know how to do it for each property separately. But we show that if $\mathfrak{s}$ is $(\leq \chi, \leq \chi^+)$-smooth and (χ^+, χ)-based and LSP(χ) then for every $\mu \geq \chi, \mathfrak{s}$ is $(\leq \mu, \leq \mu)$-based, and $(\leq \mu, \leq \mu)$-smoothed and has the LSP(μ). So it is enough to look at what occurs in cardinality LS$(\mathfrak{K}_\mathfrak{s})$ for the non-structure possibility (rather than "for some χ"). We then by Chapter V.C get a non-structure result from the failure of the assumption above. We also investigate when $\mathfrak{K}_\mathfrak{s}$ has arbitrarily large models. So being "$(\leq \chi, \leq \chi^+)$-smooth, (χ^+, χ)-based, LSP(χ)" is a good dividing line.]

V.D.§2 Primeness

[We define prime models (over A), isolation (for types of the form $N/M + c$) and primary models. We prove the existence of enough isolated types; the difference with the first order case is that we need to deal with $M <_\mathfrak{s} \mathfrak{C}$ even if we start with a singleton. From this we deduce the existence of primary models over $A <_\mathfrak{s} \mathfrak{C}$ hence primes.]

V.D.§3 Theory of types of models

[We look at TP(N, M) when $N \cap M, N, M$ are in stable amalgamation. The set of such types is called $\mathscr{S}_c^\alpha(M)$ if $\langle a_i : i < \alpha \rangle$ list the elements of N. For such types we can define non-forking, stationarization and prove properties parallel to the first order case of stable first order classes.]

V.D.§4 Orthogonality

[For types in $\mathscr{S}_c^{<\infty}(M)$ we can define weak orthogonality and orthogonality of types and orthogonality of a type to a model and prove expected claims.]

V.D.§5 Uniqueness of $(\mathbb{D}_\mathfrak{s}, \mu)$-primary models

[We prove that the non-forking restriction of an isolated type is isolated. We then prove the uniqueness of the primary model.]

V.D.§6 Uniqueness of $(\mathbb{D}_{\mathfrak{s}}, \mu)$-prime models

[We deal with the uniqueness of prime models and only comment on $\mathfrak{C}^{\rm eq}$.]

Annotated Content for Ch.V.E (300e): Types of finite sequences

V.E.§0 Introduction

[The investigations in Chapter V.C, Chapter V.D do not suggest a parallel to superstable. For this we have to look at types of singletons, and the picture is more complicated, but a very reasonable parallel exist.]

V.E.§1 Forking over models of types of sequences

[We define when $\mathbf{tp}(\bar{c}, N)$ does not fork over $M \leq_{\mathfrak{s}} N$ even for sequences $\bar{c}$ not enumerating any appropriate $N' <_{\mathfrak{s}} \mathfrak{C}$ and investigate the properties.]

V.E.§2 Forking over sets

[We define when $\mathbf{tp}(\bar{c}, B)$ does not fork over A, show the equivalence and compatibility of several variants; we define when $\mathbf{tp}(\bar{c}, B)$ is stationary over A and investigate the basic properties (including symmetry). Compared to the first order stable case there may be "bad types", e.g. there may be no "small" $A \subseteq B$ such that "$\mathbf{tp}(\bar{c}, B)$ does not fork over A". We also define strong splitting in this context and convergence, independence and parallelism.]

V.E.§3 Defining superstability and $\kappa(\mathfrak{s})$

[We define $\kappa(\mathfrak{s})$, a set of regular cardinals, which replace $\{\theta : \theta = \operatorname{cf}(\theta) < \kappa_r(T)\}$ for stable first order T; (superstability means $\kappa(\mathfrak{s}) = \emptyset$) and get a non-structure theorem for unsuperstable $\mathfrak{s}$. We connect $\kappa(\mathfrak{s})$, the existence of $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous model in λ and the behaviour of a directed union of quite homogeneous models. For a regular cardinality θ, we have: $\theta \in \kappa(\mathfrak{s})$ iff there is a $\leq_{\mathfrak{s}}$-increasing sequence $\langle M_i : i \leq \theta \rangle$ of models and $p \in \mathscr{S}^1(M_\theta)$ such that for each $i < \theta$ the type p forks over M_i (but not necessarily $p \restriction M_{i+1}$ forks over M_i!). This is related to the existence of (λ, κ)-brimmed models.]

V.E.§4 Orthogonality

[We generalize the orthogonality calculus to the present context.]

V.E.§5 Niceness of types

[In general here we do not know that not all types behave "nicely". But for some we can translate problems about them to problems of types in $\mathscr{S}_c^\alpha(M)$ from Chapter V.D. This motivates the definition of nice and prenice types over models. The prenice ones behave as in stable theories. But without existence of pre-nice types this is of limited interest. However, there are quite many of them and in particular see §6 below.]

V.E.§6 Superstable frames

[We deal with rank of types. For superstable $\mathfrak{s}$, the rank is $< \infty$ and then we show that every $p \in \mathscr{S}^{<\omega}(M)$ is prenice fulfilling a promise from §5. The notion of rank is less central than in the first order case as "every $p \in \mathscr{S}(M)$ has rank $< \infty$" is not equivalent to $\kappa(\mathfrak{s}) = \emptyset$ but to a failure of a weak version of $\aleph_0 \in \kappa(\mathfrak{s})$.]

V.E.§7 Regular types and weight

[We generalize regular types and weight to this context. We delay dealing with **P**-simple, **P**-hereditarily orthogonal to **P** and $w_{\mathbf{P}}$ to [Sh 839].]

V.E.§8 Trivial regular types

[We deal with trivial regular types, the ones where depending on a set is equivalent to depending on some member.]

Annotated Content for Ch.V.F (300f): The Heart of the Matter

V.F.§0 Introduction

[We show that if $\mathfrak{s}$ falls under the high side of some dividing lines, it has many complicated models. If it falls under the low side, we can find $\mathfrak{s}^+ = \mathfrak{s}(+)$ with a stronger $\leq_{\mathfrak{s}(+)}$ which also satisfies AxFr_1.]

V.F.§1 More on indiscernibility

[In our context and in particular for stable theories we can combine getting indiscernibles and Erdös-Rado theorem. E.g. if M is a model of a (first order complete) stable T and $a_{\{\alpha,\beta\}} \in M$ for $\alpha < \beta < (2^\lambda)^+, \lambda \geq |T|$, then we can find $u \in [(2^\lambda)^+]^{\lambda^+}$ such that $\langle a_{\{\alpha,\beta\}} : \alpha < \beta$ are from $u\rangle$ is indiscernible, not just $\langle a_{\{\alpha\}} : \alpha \in u\rangle$ is 2-indiscernible in M.

The point is that we define when $\langle M_u : u \in [\lambda]^{\leq n}\rangle$ in independent (this applies even to $M_u \prec \mathfrak{C}, \mathfrak{C}$ a model of a stable theory). We prove existence of such systems parallel to Erdös-Rado theorem. We then turn to other cases.]

V.F.§2 Order properties considered again

[We start with non-order for infinitary formulas and get a non-structure result. This will justify the concentration on the case we have the relevant non-order property.]

V.F.§3 Strengthening the order $\leq_{\mathfrak{s}}$

[Assuming enough non-order, we derive from the framework $\mathfrak{s}$ a framework $\mathfrak{s}^+ = \mathfrak{s}(+)$ satisfying AxFr_1^*, the order letting $M \leq_{\mathfrak{s}(+)} N$ mean ($\leq_{\mathfrak{s}}$ and) preservation of the satisfaction of some infinitary universal formulas.]

V.F.§4 Regaining existence of ω-unions

[We investigate and get non-structure from failure of the existence of ω-limits for the new notion of being a sub-model, $\leq_{\mathfrak{s}(+)}$. The main point is investigation in the ranks of a tree of the form $\{f : f \text{ is a } \leq_{\mathfrak{s}}\text{-embedding of } M_n \text{ into } N\}$ ordered by $\subseteq$ where $\langle M_n : n < \omega \rangle$ is $\leq_{\mathfrak{s}}$-increasing. We conclude (in Conclusion V.F.4.9) that non-structure follows from failure of $\mathrm{Ax}(\mathrm{A4})_\theta$ for $\theta = \aleph_0$ but get only $\dot{I}(\mu, K_{\mathfrak{s}}) \geq \mu^+$ for many μ's.]

V.F.§5 Non-existence of union implies non-structure

[This section is complementary to the previous one getting non-structure from non-existence of an $\leq_{\mathfrak{s}(+)}$-upper bound of an $\leq_{\mathfrak{s}(+)}$-increasing continuous δ-chain also when $\theta = \mathrm{cf}(\delta)$ is minimal and $\theta > \aleph_0$. So the counterexample is less easily manipulated, and the rank from §4 is meaningless. But by the amount of existence which follows by the minimality of θ (and free amalgamation of families of models), we know more how to construct non-forking trees of models and this enables us to prove non-structure.]

Annotated Content for Ch.V.G (300g): Changing the framework

V.G.§0 Introduction

[We hope that by repeating the operation $\mathfrak{s} \mapsto \mathfrak{s}^+$ up to some limit ordinal δ we get an $\mathfrak{s}^{+\delta}$ for which we can prove the main gap. Here we take some steps toward this.]

V.G.§1 On the family of $\mathfrak{s}$'s

[We define a natural (partial) order on the family of reasonable frameworks $\mathfrak{s}$, and prove its basic properties. In particular, increasing sequences has a limit.]

V.G.§2 From large enough $\mathrm{rk}^{\mathrm{emb},2}_{\bar{M}}(f, N)$ to every ordinal

[This continues V.F§1. Here we are interested in well-founded trees, and so if we start with a well-founded tree of $\leq_{\mathfrak{s}}$-submodels of N of cardinality $\leq \mu$ which form a tree of large enough rank, then there is a large enough subtree which is "free enough" so we can "blow it up" to larger ordinals. This is applied to the case $\mathrm{rk}^{\mathrm{emb},2}_{\bar{M}}(f, N)$ is large enough.]

Annotated Content for Ch.VI, (E46): Categoricity of an abstract elementary class in two successive cardinals revisited

VI.§0 Introduction

VI.§1 Basic properties

[We look at an a.e.c. $\mathfrak{K}$ with $\mathrm{LS}(\mathfrak{K}) \leq \lambda$, with assumptions as in the abstract and, by Chapter I, deduce amalgamation (in $\mathfrak{K}_\lambda$ and $\mathfrak{K}_{\lambda^+}$). We define the class $K^{3,\mathrm{na}}_\lambda$ of triples (M, N, a) ordered by $\leq = \leq_{\mathrm{na}}$ representing (orbital) types in $\mathscr{S}^{\mathrm{na}}(M)$ for $M \in K_\lambda$, and start to investigate it, dealing with the weak extension property, the extension property, minimality, reduced triples and types (except for minimality, in the first order case, these hold trivially). Our aims are to have the extension property or at least the weak extension property for all triples in $K^{3,\mathrm{na}}_\lambda$, and the density of minimal triples. The first property makes the model theory more like the first order case, and the second is connected with categoricity. We start by proving the weak extension property under reasonable assumptions and a consequence of having too many types, reminding the Δ-system lemma.]

VI.§2 The extension property and toward density of minimal types

[We deal with triples from $K_\lambda^{3,\mathrm{na}}$. Under "expensive" assumptions (mainly categoricity in λ^+) we prove that all triples have the extension property and that we have disjoint amalgamation in K_λ. We prove the density of minimal triples under the strong assumptions: $K_{\lambda^{+3}} = \emptyset$ and an extra cardinal arithmetic assumption ($2^{\lambda^+} > \lambda^{++}$). Now the assumption $K_{\lambda^{+3}} = \emptyset$ does no harm if we just intend to prove Theorem VI.0.2(1),(2)(a), i.e. $K_{\lambda^{+3}} \neq \emptyset$ but is a disaster if we would like to continue as in Chapter II or try to get an almost good λ-frame from the present assumptions (without $K_{\lambda^{+3}} = \emptyset$), i.e. VI.0.2(2)(b). The reader willing to accept these assumptions may skip some proofs later.]

VI.§3 On UQ from non-density of minimal (assuming weak extensions)

[Assume ($\mathfrak{K}_\lambda$ has amalgamation and) the minimal types are not dense in $K_\lambda^{3,\mathrm{na}}$, we define and investigate UQ, the class of triples of models with unique amalgamation. So we have some positive model theoretic consequences from what is a non-structure assumption. We get some non-structure results relying on Chapter VII.]

VI.§4 Density of minimal types

[We continue §3 getting the promised results, relying on Chapter VII.]

VI.§5 Inevitable types and stability in λ

[We continue to "climb the ladder", using the amount of structure we already have (and sometimes categoricity) to get more. We start by assuming there are minimal types, and show that some minimal types are inevitable. We construct $p_i \in \mathscr{S}(N_i)$ minimal ($i \leq \lambda^+$) both strictly increasing continuous and with p_0, p_δ inevitable, and then as in the proof of the equivalence of saturativity and model homogeneity, we show N_δ is universal over N_0. We can then deduce stability in λ, so the model in λ^+ is saturated. Then we note that we have disjoint amalgamation in K_λ.]

VI.§6 Density of uniqueness and proving for $\mathfrak{K}$ categorical in λ^{+2}

[We give a shortcut to proving the main theorem by using stronger assumptions (may be useful in categoricity theorems). For this we first look at uniqueness triples. If $\dot{I}(\lambda^{+2}, K) = 1$ and $\dot{I}(\lambda^{+3}, K) = 0$ then for some triple $(M, N, a) \in K_{\lambda^+}$, a is "1-algebraic" over M, i.e. this is a maximal triple. Now first assuming for some pair $M_0 \leq_{\mathfrak{K}} M_2$ in K_λ we have unique (disjoint) amalgamation for every possible M_1 with $M_0 \leq_{\mathfrak{K}} M_1 \in K_\lambda$ (and using stability), we get a pair of models in λ^+ which contradicts the existence of maximal triples. We then rely on Chapter VII to prove that there are enough cases of unique amalgamation.]

VI.§7 Extensions and Conjugacy

[We investigate types. We prove that in $\mathscr{S}(N), N \in K_\lambda$ the following: reduced implies inevitable, and non-algebraic extensions preserve the conjugacy classes for minimal reduced types (so solving parallel to the realize/materialize problem from Chapter I, see in particular Definition I.4.3(5), the discussion in the beginning of I§5 just after I.5.1 and Claim I.5.23).]

VI.§8 Almost good frame

[We prove the main theorem in particular find an almost good λ-frame $\mathfrak{s}$ with $\mathfrak{K}_{\mathfrak{s}} = \mathfrak{K}_\lambda$.]

Annotated Content for Chapter VII (838): Non-structure in λ^{++} using instances of WGCH

VII.§0 Introduction

[In addition to explaining what we are doing, we quote some definitions (and results) on the weak diamond.]

VII.§1 Nice construction framework

[The intention is to build (many complicated) models of cardinality ∂^+ by approximations of cardinality $< \partial$. We give the basic definitions: of $\mathfrak{u}$ being a nice construction framework (consisting of a $(< \partial)$-a.e.c. $\mathfrak{K}_{\mathfrak{u}}$, the class of approximations to the desired $M \in K^{\mathfrak{u}}_{\partial^+}$, classes FR_ℓ of triples $(M, N, \mathbf{J})$ for $\ell = 1, 2$ and some relations on $K_{\mathfrak{u}}$) and of $\mathfrak{u}$-free rectangles and triangles. We define approximations of size ∂, i.e. the class of triples $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f})$ from $K^{\mathrm{qt}}_{\mathfrak{u}}$ and some quasi orders on them. We prove some basic properties and define what is meant by: almost$_\ell$ all such triples has a property; this will many times mean $M = \cup\{M_\alpha : \alpha < \partial\} \in K^{\mathfrak{u}}_{\partial}$ is saturated.]

VII.§2 Coding properties and non-structure

[the coding properties are sufficient conditions on $\mathfrak{u}$ for finding many non-isomorphic models in $K^{\mathfrak{u},*}_{\partial^+}$. They have the form that $\mathfrak{K}_{\mathfrak{u}}$ has strong forms of failure of amalgamation of two members of $\mathfrak{K}_{\mathfrak{u}}$, so of cardinality $< \partial$ over a third using FR_1, FR_2]

VII.§3 Invariant coding

[We deal with some further coding properties; the invariant meaning that the relevant isomorphisms (which we demand does not exist) fix some models setwise rather than pointwise.]

VII.§4 Straight Applications of codings properties

[We mainly deal with theorems using the weak coding property of a suitable $\mathfrak{u}$ derived from an a.e.c. with $\partial_{\mathfrak{u}} = \lambda^+$ when $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ so assuming WDmId_∂ is not λ^{++}-saturated. The first case (in §4(A)) deals with the density of minimal types for $\mathfrak{K}_\lambda$ when $\mathfrak{K}$ is categorical in λ, λ^+ and has a medium number of models in λ^{++} and $\mathrm{LS}(\mathfrak{K}) \leq \lambda$; this is promised in VI§4. The second case (in §4(C)) deals with an a.e.c. which is $\mathrm{PC}_{\aleph_0}$ and has a medium number of models in $\aleph_1$ and not too many models in $\aleph_2$ and derive uniqueness of one sided stable amalgamation (promised in Chapter I). The third case (in §4(D)) continues the first, proving the density of uniqueness triples (M, N, a) in $K^{3,\mathrm{na}}_\lambda$ under the same assumptions, as promised in VI§6. The fourth case (in §4(E)) proves the density of uniqueness triples in $K^{3,\mathrm{bs}}_{\mathfrak{s}}$, for $\mathfrak{s}$ a good λ-frame as promised in II§5. In addition, concerning the first case we eliminate the use of "$\mathrm{WDmId}_{\lambda^+}$ is λ^{++}-saturated" by using $\mathfrak{u}$ with the vertical coding property, this is done in §4(B); this redo [Sh 603]. Finally in §4(F) we do the full versions of the theorems, assuming only the relevant cases of the WGCH, but relying on the results of the subsequent sections §5-§8.]

VII.§5 On almost good λ-frames

[We say some basic things on <u>almost</u> good λ-frames $\mathfrak{s}$; they arise in Chapter VI. E.g. we prove that "N is brimmed over M" is unique up to isomorphism over M (i.e. if N_ℓ is $(\lambda_{\mathfrak{s}}, \kappa_\ell)$-brimmed over M for $\ell = 1, 2$ then N_1, N_2 are isomorphic over M). This is a consequence of analyzing full and brimmed $\mathfrak{u}$-free rectangles and triangles for some nice construction framework $\mathfrak{u}$ derived from $\mathfrak{s}$.]

VII.§6 Density of weak versions of uniqueness

[For a good λ-frame, for any $\xi < \lambda^+$ we prove that either $K^{\mathfrak{s}}$ has non-structure in λ^{++} by getting vertical uq-invariant coding, from §3, or prove density for $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$, a quite weak form of uniqueness of triples, i.e. of a kind of uniqueness for a suitable form of amalgamation. As we like to deal also with almost good λ-frames, we rely on §5. This relates to §4(D),§4(E).]

VII.§7 Pseudo uniqueness

[From existence for $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ for $\xi = \lambda^+$ we define $\mathrm{WNF}_{\mathfrak{s}}$, a weak form of the class of quadruples $\langle M_\ell : \ell < 4\rangle$ of models from $K_{\mathfrak{s}}$ with M_1, M_2 amalgamated in a non-forking way over M_0 inside M_3. We prove that $\mathrm{WNF}_{\mathfrak{s}}$ is a weak $\mathfrak{s}$-non-forking relation which respects $\mathfrak{s}$.]

VII.§8 Density of $K^{3,\mathrm{uq}}_{\mathfrak{s}}$

[We try to prove non-structure in λ^{++} from failure of density of $K^{3,\mathrm{uq}}_{\mathfrak{s}}$. By §6 we justify assuming existence for $K^{3,\mathrm{up}}_{\mathfrak{s}}$, so by §7 the relation $\mathrm{WNF}_{\mathfrak{s}}$ is a well defined weak $\mathfrak{s}$-non-forking relation on $\mathfrak{K}_{\mathfrak{s}}$ (respecting $\mathfrak{s}$). So we can define $\mathfrak{u}$ such that $(M_0, N_0, a) \le^\ell_{\mathfrak{u}} (M_1, N_1, a)$ implies $\mathrm{WNF}(M_0, N_0, M_1, N_1)$. We also show that it is enough to show $K^{3,\mathrm{up}}_{\mathfrak{s}} \subseteq K^{3,\mathrm{uq}}_{\mathfrak{s}}$. Now the proof splits to two cases. In the first we assume wnf-delayed uniqueness fails and get vertical coding. In the second we assume wnf-delayed uniqueness holds but density of uniqueness triples fail and get horizontal coding (using the properties of WNF).]

VII.§9 The combinatorial part

[We first quote; central in justifying our results is $\mu_{\mathrm{unif}}(\partial^+, 2^\partial)$ which "usually" is 2^{∂^+}, (in VII.9.4). We show that building an appropriate tree $\langle M_\eta : \eta \in {}^{\partial^+ \ge}(2^\partial)\rangle$ is enough (in VII.9.1). We present building $\langle \bar{M}_{\eta\hat{}<\alpha>} : \alpha < 2^\partial\rangle$ as above (in VII.9.3); as well as the "universal case", i.e. when $M_\eta(\eta \in {}^\partial 2)$ are pairwise non-isomorphic of $M_{<>}$. Also we deal with the results on having many models in ∂ (when $\emptyset \in \mathrm{WDmId}_\partial$) and mention the case in each step $\alpha < \partial^+$ we use, e.g. ∂ sub-steps.]

VII.§10 Proofs of the non-structure theorems, with choice functions

[This has a somewhat more set theoretic character compared to, and fulfills promises from §2,§3. We prove various coding theorems saying that there are many non-isomorphic models in ∂^+. In particular we prove this for nice construction frameworks in cases in which we need amalgamation choice functions.]

VII.§11 Remarks on pcf

[We prove things in pcf relevant to non-structure in a reasonably self contained way. One is a relative of Hajnal free subset theorem. The main other says that if $2^\lambda < 2^{\lambda^+}$ then one of three cases occurs, each helpful in proof of non-structure and some related results. This is a revised version of part of [Sh 603].]

UNIVERSAL CLASSES: STABILITY THEORY FOR A MODEL SH300A

§0 Introduction

(A) Introduction to Chapter V

We have been interested in classifying first order theories, not in the sense of finite group theory, i.e. explicit list of families but like biology - find main taxonomies, dividing lines. See Chapter N.

We try here to develop the case of universal classes (see below). In fact we develop it less concretely, more abstractly, both per se, and as by our program we think that we shall need eventually to define inductively a sequence of such frameworks; and we had thought such proofs necessary and intriguing.

0.1 Convention.

(*a*) Let τ be a vocabulary (= signature)
(*b*) K will denote a class of $\tau(K)$-models (= structures).

0.2 Definition. K is universal if K is closed under submodels, increasing chains and under isomorphisms.

Note:

0.3 Observation. 1) Not every elementary class is universal, but many universal classes are not elementary, e.g. the locally finite groups.
2) If K is universal, $\tau(M) = \tau(K)$ then $M \in K$ if and only if every finitely generated submodel of M belongs to K (see V.B.2.5).

It may be instructive to remember in this context the following theorem of Tarski:

0.4 Theorem. *For a finite relational vocabulary (i.e., with predicates only so no function symbols) K is universal if and only if K is the class of models of a universal first order theory.*

A strong motivation for developing a classification for universal classes was [Sh 155], which proves the main gap for universal first order theories. It was not superceded by [Sh:c, XIII] as it does not assume "the vocabulary is countable", and moreover gives additional information.

The main point there is that

0.5 Theorem. *Assume T is a universal first order theory which is stable, i.e., every completion T' of T (i.e. complete first order $T' \supseteq T$, in $\mathbb{L}(\tau_T)$ of course) is stable.*
1) (a) $\Rightarrow$ (b) where

- *(a) $n < \omega, \langle M_u : u \in \mathscr{P}(n)\rangle$ is a stable system of models of T (so $u \subseteq n \Rightarrow M_u \prec M_n$ and $\mathrm{tp}_*(M_u, \cup\{M_v : v \subseteq n, u \nsubseteq v\}, M_n)$ does not fork over $\cup\{M_v : v \subset u\}$ for every $u \subset n$; (see [Sh:c, XII,§4])*
- *(b) $M_n \restriction A \prec M$ where A is the closure of $\cup\{M_u : u \subset n\}$ in M_n.*

2) So every completion of T has a (strong) variant of the (λ, n)-existence property, see [Sh:c, Ch.XII,Definition 5.2,5.4,pg.616].

Note that this work contains several results on the existence of indiscernibility which are meaningful also for first order logic. This contains finding for any given $\langle a_\alpha : \alpha < \lambda\rangle$ a long indiscrenible of subsequence if no $\varphi(x, y, \bar{c})$ has the μ-order property (note: here finite sequences are not the same as elements), see 5.1. Also we can find a relative to Erdös-Rado theorem, e.g. given $\langle \bar{a}_{\{\alpha,\beta\}} : \alpha < \beta < \lambda\rangle$, for a large $\mathscr{U} \subseteq \lambda, \langle \bar{a}_{\alpha,\beta} : \alpha < \beta$ from $\mathscr{U}\rangle$ is indiscernible (not just 2-indiscernible, see V.F§1).

Lastly, we similarly deal with tree with ω-levels (end of V.F§1) and with trees like $\mathrm{des}(\alpha) = \{\eta : \eta$ a decreasing sequence of ordinals $< \alpha\}$ (see V.G.2.7).

In [Sh:c, VIII] we have worked hard to get $\dot{I}(\lambda, T)$ is maximal in all relevant cases. But now we will have less strict "rules of the game", accepting as sufficient, results negating a structure theory (e.g. getting $\dot{I}(\lambda, \kappa)$ for many cardinals), unfortunately sometimes we get only consistency (see V.F§5).

In the framework of Chapter V.B the main role of Chapter V.A is §4. If a universal class K fails (relevant cases of) the order property, we would like to make it a class with amalgamation. Toward this we define some appropriate partial orders $\leq_{\mathfrak{K}}$. We would like $(K, \leq_{\mathfrak{K}})$ to be an a.e.c. (see Chapter I) with amalgamation. In §4 we prove the equivalence of two versions and define stable amalgamation and prove the LS property and closure under increasing unions.

In V.B§1 we introduce our main framework, AxFr_1, and deal somewhat with relatives; compared to a.e.c. such $\mathfrak{s}$ has "explicite" amalgamation, but may lack smoothness, see below; so this choice is incomparable with a.e.c. We then prove in V.B§2 that if K is a universal class with suitable non-order property, then we can find $\mathfrak{s}$ satisfying AxFr_1 such that $K_{\mathfrak{s}}$ is the original class K (and $\mathrm{LS}(K_{\mathfrak{s}})$ is not too large). We also say how first order stable T fits and where the class of (D, μ)-homogeneous models fit. In V.B§3 we sort out sequence-homogeneity = (D, λ)-homogeneity and model-homogeneity = $(\mathbb{D}, \lambda)$-homogeneity and saturation. In particular we prove "λ-saturated is equivalent to λ-model-homogeneous" where M is λ-saturated when: if $N \leq_{\mathfrak{s}} M \wedge \|N\| < \lambda \wedge p \in \mathscr{S}^1(N) \Rightarrow p$ realizes in M. Note that we do not use types which are sets of formulas, but define them by "orbits over the model".

If we look at $\mathfrak{s}$ satisfying AxFr_1 as a generalization of first order theory much is lacking. In particular we point two dividing lines which do not appear in the first order case: being smooth and being χ-based. Smoothness is the property "if $\langle M_\alpha : \alpha \leq \delta \rangle$ is $\leq_{\mathfrak{s}}$-increasing then $\cup\{M_\alpha : \alpha < \delta\} \leq_{\mathfrak{s}} M_\delta$"; it is what $\mathfrak{K}_{\mathfrak{s}}$ is missing to be an a.e.c. To get non-structure from it we have to know something on constructing models (so developing the positive theory); this is done in V.C§1, using $< - >^{\mathrm{gn}}$ heavily, and subsequently, in V.C§2 we get the non-stucture. We then deal in V.C§3-§5 with the failure of "$\mathfrak{s}$ is χ-based" which means that: if $M \le_{\mathfrak{s}} N$ and $A \subseteq N$ has cardinality $\le \chi$ then for some M_1, N_1 of cardinality $\le \chi$ we have $A \subseteq N_1$ and $\mathrm{NF}_{\mathfrak{s}}(M_1, N_1, M, N)$.

Now the failures imply non-structures from the "cardinality of the failure up". Naturally we would like to say that if there is a failure then it occurs in reasonably small cardinals (e.g. like $(\beth_2((2^{\mathrm{LS}(\mathfrak{K})})^+)$ rather than the first weakly compact cardinal or whatever). We do not know this for each of those properties separately but we know it when we put them together, e.g. if we have smoothness in $\mathfrak{K}_{\le\chi^+}, \mathfrak{s}$ is χ-based and $\mathrm{LSP}(\mathfrak{s}) \le \chi$ holds then this holds in all $\chi' \ge \chi$; this is done in V.D§1 and is sufficient for our program.

The rest of Chapter V.D tries to understand the class of $(\mathbb{D}_{\mathfrak{s}}, \kappa)$-homogeneous models, the parallel of the class of κ-saturated model of a stable elementary class Mod_T with $\kappa \ge \kappa_r(T)$. A crucial point is defining isolated types and proving their density. However, the types are not of singletons (or finite sequences) but of submodels, i.e. $p \in \mathscr{S}_0^\alpha(M)$ and we use "orbital" types (not set of formulas); we shall use the stable construction from V.C§4.

We deal with the basic properties, of such types, the existence (and uniqueness) of $(\mathbb{D}_{\mathfrak{s}}, \mu)$-prime models and orthogonality of types. This is fine when we consider frames including "λ-saturated models in a strictly stable elementary class". But if we like to include properties of superstable such classes this will not help. So in Chapter V.E we investigate types of singletons (or finite sequences). We have a theory parallel to that of superstable elementary class. While we have no problem with the parallel of DOP, as we like to analyze $M \in K_{\mathfrak{s}}$ (not just $(\mathbb{D}_{\mathfrak{s}}, \chi_{\mathfrak{s}}^+)$-homogeneous ones) we have to consider order properties defined as follows. Assume $\langle M_{\{\alpha\}} : \alpha < \alpha^*\}$ are independent over $M_\emptyset$ inside $\mathfrak{C}, \bar{a}_\alpha$ list the elements of $M_{\{\alpha\}}, a_{\alpha,i} \mapsto a_{\beta,i}\,(i < \ell g(\bar{a}_0))$ is an isomorphism from $M_{\{\alpha\}}$ onto $M_{\{\beta\}}$ over $M_\emptyset$ called $h_{\beta,\alpha}$ and $M_{\{\alpha\}} \cup M_{\{\beta\}} \subseteq M_{\{\alpha,\beta\}}$ and $h_{\beta_1,\alpha_1} \cup h_{\beta_2,\alpha_2}$ can be extended to an isomorphism from $M_{\{\alpha_1,\alpha_2\}}$ onto $M_{\{\beta_1,\beta_2\}}$. Varying $R \subseteq [\alpha(*)]^2$ we ask: does the model $M_R = \langle \cup\{M_{\{\alpha\}} : \alpha < \lambda\} \cup \{M_{\{\alpha,\beta\}} : \{\alpha,\beta\} \in R\}\rangle^{\mathrm{gn}}$ represent (up to isomorphisms) somewhat R, i.e. if $R_1, R_2 \subseteq [\alpha(*)]^2$ are "different enough" then $M_{R_1} \not\cong M_{R_2}$. In a sense our problem is that

$M \leq_{\mathfrak{s}} N$ does not imply that M is a Σ_1-submodel of N in an appropriate sense. So we define the appropriate order property (V.F§2) which is enough for a non-structure result. If the non-order properties hold we would like to define a "successor" $\mathfrak{s}^+ = \mathfrak{s}(+)$ to $\mathfrak{s}$ using $\leq_{\mathfrak{s}(+)}$ as a strong submodel. (Σ_1-submodel for formulas with χ variable for appropriate χ). However, it is far from clear that the $\mathfrak{s}^+$ we have gotten satisfies AxFr_1. Not only smoothness is missing for the intended $\leq_{\mathfrak{s}(+)}$, but even the existence of union, Ax(A3). That is, a $\leq_{\mathfrak{s}(+)}$-increasing sequence $\langle M_\alpha : \alpha < \delta \rangle$ have a union $M_\delta := \cup\{M_\alpha : \alpha < \delta\}$ and it is in $K_{\mathfrak{s}(+)} = K_{\mathfrak{s}}$, but do we have $\alpha < \delta \Rightarrow M_\alpha \leq_{\mathfrak{s}(+)} M_\delta$? We use this as another dividing line. But to prove non-structure we have, as in Chapter V.C, to do some constructions in such a weak frame. We use weaker forms of (A4) which holds: first $M = \langle \cup\{M_{\{\alpha\}} : \alpha < \alpha(*)\}\rangle^{\mathrm{gn}}$ when $M_\emptyset \leq_{\mathfrak{s}(+)} M_{\{\alpha\}}$ are so called independent over $M_\emptyset$ inside some N and second if $\langle M_\alpha : \alpha \leq \delta \rangle$ is $\leq_{\mathfrak{s}(+)}$-increasing then $M_\alpha \leq_{\mathfrak{s}(+)} M^-_\delta := \cup\{M_\alpha : \alpha < \delta\}$ though we do not know if $M^-_\delta \leq_{\mathfrak{s}(+)} M_\delta$. The non-structure uses minimal δ (so $\delta = \mathrm{cf}(\delta)$) and the case $\delta = \omega$ has a different character (we get $\geq \lambda^+$ models in λ rather than 2^λ).

In Chapter V.G we actually deal with the successor framework $\mathfrak{s}^+$. There is a natural order such that $\mathfrak{s} \leq \mathfrak{s}^+$. Also if $\langle \mathfrak{s}_\alpha : \alpha < \delta \rangle$ is increasing continuous then a natural limit $\mathfrak{s}_\delta$ is well defined and satisfies $(\mathrm{AxFr})_1$ (provided that along the way, our class falls on the structure side). So we can iterate the operation $\mathfrak{s}^+$. See on improvements and comments in [Sh:E54] (on all subchapters).

Our expectations were (and still are) that for δ large enough, $\mathfrak{s}_\delta$ is similar enough to first order to enable us to prove a main gap theorem.

This work was mainly done on 6-12/85 and lectured on at Rutgers. Chapter V.D, Chapter V.E exists but were not in good enough form for publication in [Sh 300], but the situation concerning the rest turns out to be more complicated than what seemed at first. So Chapter V.F, Chapter V.G were done later in the eighties and present in the Helsinki '90 meeting. Because of hopes to continue and technical reasons the appearance was delayed.

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(B) Notation

There is some repetitions on N§5(B).

0.6 Set Theoretic Notation.

1) $\lambda, \mu, \chi, \kappa$ denote cardinals (usually infinite).
2) $\alpha, \beta, \gamma, i, j, \varepsilon, \zeta, \xi$ denote ordinals.
3) δ denotes a limit ordinal.
4) $\mathscr{H}(\lambda)$ denote the family of sets with transitive closure of cardinality $< \lambda$.

Note that 0.7(8)-(12) are not used often.

0.7 Model Theoretic Notation. 1) τ denotes a vocabulary, i.e., set of predicates and function symbols, each with a designated fixed (finite) arity.
2) M a model (= a structure), $\tau(M)$ its vocabulary, for $\tau = \tau(M)$ we say M is a τ-model, $|M|$ the universe of M.
3) K a class of models all with the same vocabulary $\tau(K)$, for $\tau = \tau(K)$ we say K is a τ-class.
4) $\bar{a}, \bar{b}, \bar{c}$ denote sequences of elements from a model, not necessarily finite. The length of a sequence $\bar{a}$ is denoted by $\ell g(\bar{a})$. Instead $\bar{a} \in {}^{\ell g(\bar{a})}(|M|)$ we may write $\bar{a} \in {}^{\ell g(\bar{a})}M$ or $\bar{a} \in M$.
5) $\mathscr{L}$ a logic, i.e. for every vocabulary τ, $\mathscr{L}(\tau)$ is a set of formulas $\varphi(\bar{x})$ not necessarily first order; ($\bar{x}$ is a possibly infinite sequence of variables including all free variables of φ) and we assume always $[\tau_1 \subseteq \tau_2 \Rightarrow \mathscr{L}(\tau_1) \subseteq \mathscr{L}(\tau_2)]$, $[\varphi \in \mathscr{L}(\tau_1)$ and $\varphi \in \mathscr{L}(\tau_2)$ implies $\varphi \in \mathscr{L}(\tau_1 \cap \tau_2)]$; if M is a τ-model, $\bar{c} \in {}^{\ell g(\bar{x})}|M|$, the truth value of "$M \models \varphi[\bar{c}]$" is defined.
6) For every sentence ψ which is in some $\mathscr{L}(\tau)$, there is a vocabulary $\tau(\psi)$ such that for any vocabulary τ', $\psi \in \mathscr{L}(\tau') \Leftrightarrow \tau \subseteq \tau'$. For a theory T (i.e. set of sentences) let $\tau(T) = \cup\{\tau(\psi) : \psi \in \mathscr{L}\}$ and the

truth of $M \models \varphi[\bar{a}]$ depends only on $M \restriction \tau$ if $\varphi \in \mathscr{L}(\tau)$.
7) Let φ, ψ, ϑ denote formulas, on $\varphi(\bar{x})$ see above; $\varphi, \varphi(\bar{x}), \varphi(\bar{x}; \bar{y})$ may be treated as objects of a different kind (see below) but $\varphi(\bar{x}, \bar{y})$ is really $\varphi(\bar{x}\char`\^\bar{y})$ and we shall not be pedantic concerning the difference when clear from the context. We sometimes separate "type", "free" variables from "parameter" variables. $\mathbb{L}_{\lambda,\kappa}$ is the logic such that $\mathbb{L}_{\lambda,\kappa}(\tau)$ is the set of formulas we get from the atomic formulas by closing under $\neg\varphi$ (negation), $\bigwedge\limits_{i<\alpha} \varphi_i$ (where $\alpha < \lambda$, conjunction) and $(\exists x_0, \dots, x_i, \cdots)_{i<\alpha}\varphi$ (where $\alpha < \kappa$, existential quantification), but for $\varphi(\bar{x}) \in \mathbb{L}_{\lambda,\kappa}(\tau)$ we demand $\ell g(\bar{x}) < \kappa$. (So $\mathbb{L}_{\lambda,\kappa}$ is a logic, $\mathbb{L}_{\omega,\omega}$ first order logic).
8) A class K of τ-models is a $\mathrm{PC}_{<\lambda}$, so $\mathrm{PC}_\lambda = \mathrm{PC}_{<\lambda^+}$, <u>when</u> for some vocabulary $\tau_1, \tau \subseteq \tau_1, |\tau_1| < \lambda$ and $\psi \in \mathbb{L}_{\lambda,\omega}(\tau_1)$ we have $K = \{M \restriction \tau : M \models \psi\}$. $\mathrm{PC}(T_1, T)$ is the class of $\tau(T)$-reducts of models of T_1.
9) A class K of models is $\mathrm{PC}_{\lambda,\kappa}$ if for some $\tau_1, \tau \subseteq \tau_1$, $|\tau_1| = \kappa$, first order theory $T_1 \subseteq \mathbb{L}_{\omega,\omega}(\tau_1)$ and set Γ of λ $(< \omega)$-types in $\mathbb{L}_{\omega,\omega}(\tau_1)$, $K = \{M \restriction \tau : M$ a model of T_1 omitting every $p \in \Gamma\}$. If $\kappa = \lambda$ we may omit it. We know that $\mathrm{PC}_\lambda = \mathrm{PC}_{\lambda,1} = \mathrm{PC}_{\lambda,\lambda}$.
10) Δ, Λ, Γ will denote sets of formulas of the form $\varphi(\bar{x}, \bar{y})$ or $\varphi(\bar{x})$. If $\varphi(\bar{x}) \in \Delta$ this means $\varphi(\bar{x}^1, \bar{x}^2) \in \Delta$ when $\bar{x} = \bar{x}^1 {}^\frown \bar{x}^2$. These formulas may have parameters. Let $\mathrm{arity}(\Delta) = \sup\{|\ell g(\bar{x})|^+ : \varphi(\bar{x}) \in \Delta$ or $\varphi(\bar{x}, \bar{a}) \in \Delta\}$. Usually for $\bar{x}, \bar{y}$ of length α with no repetitions, there is no difference between $\varphi(\bar{x})$ and $\varphi(\bar{y})$.
11) Let $\Delta^{\mathrm{qf}}_\tau$ be the set of quantifier free formulas in $\mathbb{L}_{\omega,\omega}(\tau)$, we may write just qf.
12) For a logic $\mathscr{L}$ we define the logic $\triangle(\mathscr{L})$ as follows: for a vocabulary τ a sentence is a pair $\psi = (\psi_1, \psi_2)$ where $\psi_\ell \in \mathscr{L}(\tau_\ell), \tau_\ell$ a vocabulary extending τ such that:

- (a) for every τ-model M, M has a τ_1-expansion M_1 to a model satisfying ψ_1 iff M has no τ_2-expansion M_2 to a model satisfying ψ_2
- (b) naturally for a τ-model M, we let $M \models \psi$ iff M has an expansion to a τ_1-model satisfying ψ_1.

For a vocabulary τ let $\Delta_{\mathrm{qf}}(\tau)$ be the set of finite Boolean combinations of atomic formulas, i.e., first order quantifier free formulas. We may write qf intead $\Delta_{\rm qf}(\tau)$ assuming τ is clear from the context.

0.8 Definition. 1) We say that p is a (formal) type inside M if p is a set of $\tau(M)$-formulas with parameters from M and a fixed set of variables $\bar{x}$; so we have no requirement like "finitely satisfiable"! unlike the first order case.
We say a Δ-type when each member has the form $\varphi(\bar{x},\bar{a}), \varphi(\bar{x},\bar{y}) \in \Delta$. We say over A if $\varphi(\bar{x},\bar{a}) \in p \Rightarrow \bar{a} \subseteq A$. If Δ is closed under negations (but we do not distinguish between φ and $\neg\neg\varphi$), we say a complete Δ-type over A inside M <u>when</u> for every $\varphi(\bar{x},\bar{y}) \in \Delta$ and $\bar{a} \in {}^{\ell g(\bar{y})}A$ we have $\varphi(\bar{x},\bar{a}) \in p$ or $\neg\varphi(\bar{x},\bar{a}) \in p$.
2) $\mathrm{tp}_{\varphi(\bar{x};\bar{y})}(\bar{a},A,M) = \{\varphi(\bar{x};\bar{b}) : \bar{b} \in {}^{\ell g(\bar{y})}A \text{ and } M \models \varphi[\bar{a};\bar{b}]\}$ where $\varphi(\bar{x};\bar{y}) \in \mathscr{L}(\tau(M))$ for some $\mathscr{L}$ and $A \subseteq |M|$.

Notation for such types is needed when a monster model ($\mathfrak{C}$) is absent (or still is absent), (otherwise we can omit M).
3) If $p \subseteq \mathrm{tp}_{\varphi(\bar{x};\bar{y})}(\bar{a},A,M)$ then we say that p is a $\varphi(\bar{x}\hat{\ }\bar{y})$-type (or $\varphi(\bar{x};\bar{y})$-type) which $\bar{a}$ realizes over A inside M so $\ell g(\bar{a}) = \ell g(\bar{x})$.
4) Similarly for the following variants

- (a) $\mathrm{tp}_{\varphi(\bar{x})}(\bar{a},A,M) = \mathrm{tp}_{\varphi(\bar{x}_1;\bar{x}_2)}(\bar{a},A,M)$ where $\bar{x} = \bar{x}_1\hat{\ }\bar{x}_2, \ell g(\bar{a}) = \ell g(\bar{x}_1)$
- (b) $\mathrm{tp}_{\{\varphi\}}(\bar{a},A,M) = \mathrm{tp}_{\varphi}(\bar{a},A,M)$
- (c) $\mathrm{tp}_\Delta(\bar{a},A,M) = \bigcup\limits_{\varphi\in\Delta} \mathrm{tp}_\varphi(\bar{a},A,M)$
  and $\mathrm{tp}_{\pm\varphi}(\bar{a},A,M) = \mathrm{tp}_{\{\varphi,\neg\varphi\}}(\bar{a},A,M)$
- (d) $\mathbf{S}^\alpha_\Delta(A,M) = \{\mathrm{tp}_\Delta(\bar{a},A,M) : \bar{a} \in {}^\alpha|M|\}$
- (e) now $\mathrm{Sfr}^\alpha_\Delta(A,M)$ is the set of formal complete Δ-type over A, i.e.

$$\begin{aligned}\mathrm{Sfr}^\alpha_\Delta(A,M) = \big\{p :\, & p \subseteq \{\varphi(\bar{x},\bar{a}),\neg\varphi(x,\bar{a}) : \varphi(\bar{x},\bar{y}) \in \Delta, \\ & \ell g(\bar{x}) = \alpha \text{ and } \bar{a} \in {}^{(\ell g(\bar{y}))}A\} \\ & \text{ and for any such } \varphi(\bar{x},\bar{a}) \text{ we have} \\ & \varphi(\bar{x},\bar{a}) \in p \Leftrightarrow \neg\varphi(\bar{x},\bar{a}) \notin p\big\}\end{aligned}$$

(used in 1.10(2),(4) 2.7, and the proof of 4.4 only, clearly $|\mathbf{S}^\alpha_\Delta(A,M)| \le |\mathrm{Sfr}^\alpha_\Delta(A,M)|$ and actually in $\mathrm{Sfr}^\alpha_\Delta(A,M), M$ is not necessary)

(*f*) if $A = |M|$ we may write $\mathrm{Sfr}^\alpha_\Delta(M)$.

5) We can replace A by $\mathbf{J}$, a family of sequences, e.g.

$$\mathrm{tp}_{\varphi(\bar{x};\bar{y})}(\bar{a},\mathbf{J},M) = \{\varphi(\bar{x};\bar{b}_1,\ldots,\bar{b}_n) : n < \omega, M \models \varphi[\bar{a},\bar{b}_1,\ldots,\bar{b}_n], \bar{b}_\ell \in \mathbf{J} \text{ for } \ell = 1,\ldots,n\}$$

or by a set of formulas with parameters, e.g.

$$\mathrm{tp}_{\varphi(\bar{x};\bar{y})}(\bar{a},\Gamma,M) = \{\varphi(\bar{x};\bar{c}) : M \models \varphi(\bar{a};\bar{c}) \text{ and } \varphi(\bar{x};\bar{c}) \in \Gamma\}.$$

We then say "type over Γ" or "type over $\mathbf{J}$".
6) $M \le_\Delta N$ <u>means</u> that $M \subseteq N$ and for $\varphi(\bar{x}) \in \Delta$ and $\bar{a} \in {}^{\ell g(\bar{x})}M$ we have:

$$M \models \varphi[\bar{a}] \text{ if and only if } N \models \varphi[\bar{a}]$$

7) Let $\sum_{\lambda,\chi,\kappa}(\Delta)$ be the set of formulas of the form

$$\psi(\bar{y},\bar{z}) := (\exists \bar{x}) \bigwedge_{\alpha<\lambda} \varphi_\alpha(\bar{x},\bar{y},\bar{z} \restriction w_\alpha)^{\eta(\alpha)}$$

<u>where</u> $\eta \in {}^\lambda 2, \bar{x} = \langle x_i : i < \alpha^* \rangle, \alpha^* \le \kappa$ (or just $|\alpha^*| \le \kappa$), $\ell g(\bar{y}) \le \chi$ or just $|\ell g(\bar{y})| \le \chi, w_\alpha \subseteq \ell g(\bar{z}), |w_\alpha| \le \chi$, but $\bar{z}$ of any length $\le \lambda$ and even $< \lambda^+$ is allowed, $\varphi_\alpha(\bar{x},\bar{y},\bar{z} \restriction w_\alpha) \in \Delta$ for each $\alpha < \lambda$ and recall $\varphi^1 = \varphi, \varphi^0 = \neg\varphi$; if $\chi = \kappa$ we may omit it. So $\sum_{\lambda,\chi,\kappa}(\Delta)$ includes every $\varphi(\bar{y}) \in \Delta$ and its negation, when $\ell g(\bar{y}) \le \chi$ [used in 4.3, we may consider bounding w_α by a different cardinal but we have enough parameters].

Above we do not distinguish between Δ and $\{\varphi, \neg\varphi : \varphi \in \Delta\}$, and so define

0.9 Definition. 1) $M \leq^{\rm pos}_{\Delta} N$ <u>iff</u> ($M \subseteq N$ and) for every $\varphi(\bar{x}) \in \Delta$ and
$\bar{a} \in {}^{\ell g(\bar{x})}M$ we have $M \models \varphi[\bar{a}] \Rightarrow N \models \varphi[\bar{a}]$. We define $M \leq^{\rm neg}_{\Delta} N$ iff ($M \subseteq N$) and for every $\varphi(\bar{x}) \in \Delta$ and $a \in {}^{\ell g(\bar{x})}M$ we have $N \models \varphi[\bar{a}] \Rightarrow M \models \varphi[\bar{a}]$.
2)

$$\begin{aligned}\Sigma^{\rm pos}_{\lambda,\chi,\kappa}(\Delta) = \{\varphi(\bar{y},\bar{z}) : \ell g(\bar{y}) \leq \chi, \ell g(\bar{z}) \leq \lambda \text{ and} \\ \varphi(\bar{y},\bar{z}) = (\exists x_0,\dots,x_i,\dots)_{i<\kappa} \bigwedge_{\alpha<\lambda} \varphi_\alpha(\bar{x},\bar{y},\bar{z}{\restriction}w_\alpha), \\ \text{where } w_\alpha \subseteq \ell g(\bar{z}), |w_\alpha| \leq \chi \text{ and } \varphi_\alpha(\bar{x},\bar{y},\bar{z}{\restriction}w_\alpha) \in \Delta\}.\end{aligned}$$

[Used in 1.12 with $\chi = \kappa$.]
3) Above writing "$< \kappa$" instead κ means having $< \kappa$ instead of $\leq \kappa$, similarly for $< \chi, < \lambda$. If $\kappa = \chi$ we may omit κ.
4) Let $M \leq^*_{\Sigma_{\lambda,\chi,\kappa}(\Delta)} N$ means that: ($M \subseteq N$) and for every $\psi(\bar{y},\bar{z}) = (\exists \bar{x}) \bigwedge_{\alpha<\lambda} \varphi_\alpha(\bar{x},\bar{y},\bar{z} \restriction w_\alpha)^{\eta(\alpha)}$ as in Definition 0.8(7), if $\bar{c} \in {}^{\ell g(\bar{z})}M, \bar{y} \in {}^{\ell g(\bar{y})}M$ and $N \models \psi[b,\bar{a}]$ then for some $\bar{a} \in {}^{\ell g(\bar{x})}M$ we have $N \models \bigwedge_{\alpha<\lambda} \varphi_\alpha(\bar{a},\bar{b},\bar{c} \restriction w_\alpha)^{\eta(\alpha)}$; note that satisfaction in M does not appear unlike part (1)!.
5) We define $M \leq^*_{\Sigma^{\rm pos}_{\lambda,\chi,\kappa}(\Delta)} N$ parallely.

Remark. Why, in Definition 0.9(4), do we not just say: "without loss of generality Δ is a set of quantifier free formulas"?, e.g. as then the arity of the vocabulary is $> \aleph_0$. Really not "the end of the world", but not yet. You can say that we pretend Δ is a set of quantifier free formulas.

0.10 Observation. 1) $M \leq_{\Sigma_{\lambda,\chi,\kappa}(\Delta)} N$ <u>iff</u> $M \leq_\Delta N$ and $M \leq^*_{\Sigma_{\lambda,\chi,\kappa}(\Delta)} N$.
2) $M \leq_{\Sigma^{\rm pos}_{\lambda,\chi,\kappa}(\Delta)} N$ <u>iff</u> $M \leq^{\rm pos}_{\Delta} N$ and $M \leq^*_{\Sigma^{\rm pos}_{\lambda,\chi,\kappa}(\Delta)} N$.

(C) Introduction to Chapter V.A: Stability Theory for a Model

In [Sh:a, Ch.I,§2] little stability theory was developed for one (arbitrary) model; quite naturally as this was peripheral there. More attention was given to non-structure theorems for infinitary logics (see [Sh 16, §2] and Grossberg and Shelah [GrSh 222], [GrSh 238], [GrSh 259] and concerning applications, see Macintyre and Shelah [McSh 55], Grossberg and Shelah [GrSh 174]).

However, in our present framework it is important to get results on infinitary languages. As we have fewer transfer theorems, it is natural to concentrate on one model.

Surprisingly we have something to say, most of it was in some form in [Sh:a, Ch.1,§2]: the theorems that non-stability implies order (i.e. existence of quite long set of sequences, linearly order by a formula), that non-order implies the existence of indiscernibles and that we can average types, all have reasonable analogs. Toward this end we introduce here "convergent sets", which capture a crucial property of indiscernible sets in the stable case. In particular the difference between 2.8 and the results in [Sh:c, Ch.I,§2] is the use of convergence which essentially strengthens indiscernibility and proves existence for it. In §3 we note symmetry for averages (when we do not have order). In §4 we investigate some derived partial order.

Lastly, we prove (in §5) that in order to get just indiscernible sets, less "non-order" is needed, and this gives new information even on first order theories. E.g. <u>if</u> T is first order, there is no quantifier free formula $\varphi(x, y; \bar{z})$ such that some model M of T has the $(\varphi(x, y; \bar{z}), \aleph_0)$-order property (note that x, y are <u>not</u> sequences), M a model of T, $a_i \in M$ for $i < (2^\lambda)^+$, $\lambda \geq |T|$, <u>then</u> for some $w \subseteq (2^\lambda)^+, |w| > \lambda$, the set $\{a_i : i \in w\}$ is an indiscernible <u>set</u> in M. For first order theories with elimination of quantifiers this does not add anything new (see [Sh 715, 1.37]) but in general it does, see 5.6. Compared to [Sh:c] we get somewhat less good cardinal bound but weaker non-order demand (see [Sh:c, I,2.12,2.18], the result of [Sh:c] is represented here in 2.13, and not in §5 as had confused some readers).

Splitting and strong splitting are from [Sh 3].

§1 The order property revisited

The main results of this section are Theorem 1.3 and 1.19. We begin by recounting the appropriate definition of the order property in this context. We note in Theorem 1.3 (proved in [Sh:e, Ch.III,§3]) that this relevant order property implies the existence of many non-isomorphic models.

These notions have two parameters: a formula and a cardinal. As we no longer are attached to first order logic, the formula (or set of formulas) as a parameter is even more important than in [Sh:c]. As we assume generally no closure properties for the set of formulas, we have to be more explicit in asserting "there is a formula". (Note that we may have to consider several logics, simultaneously, as in [MaSh 285], and that usually we use weaker closure properties).

A new parameter is a cardinal (the length of the order). Its presence is desirable as we no longer assume compactness, so not necessarily all infinite cardinals give equivalent definitions.

Then we describe the notions of "indiscernible" and "splitting" appropriate for this context. In Theorem 1.12 we show that either for each type we can find a "base" over which it does not split or the order property holds. In Theorem 1.20 we show that for appropriate μ if the number of Δ-types over a set of cardinality μ which are realized in M is not bounded by μ then there is a Δ^* (closely associated with Δ) such that M has the (Δ^*, κ^+)-order property. We could have replaced M by ${}^{\kappa>}M$ and thus avoide infinary sequences, but the existing presentation is closer to our intentions.

1.1 Definition.
1) M has the $(\varphi(\bar{x};\bar{y};\bar{z}),\mu)$-order property <u>if</u> there are sequences $\bar{c},\bar{a}_\alpha,\bar{b}_\alpha$ (for $\alpha<\mu$) from M (with $\ell g(\bar{c})=\ell g(\bar{z})$, $\ell g(\bar{a}_\alpha)=\ell g(\bar{x})$, $\ell g(\bar{b}_\alpha)=\ell g(\bar{y})$), such that for $\alpha,\beta<\mu$:

$$M \models \varphi[\bar{a}_\alpha,\bar{b}_\beta,\bar{c}] \text{ if and only if } \alpha<\beta.$$

We extend this notion to sets (or classes) of formulas and classes of models as follows.
2) M has the (Δ,μ)-order property <u>if</u> for some $\varphi(\bar{x};\bar{y};\bar{z})\in\Delta$ the model M has the

$(\varphi(\bar{x};\bar{y};\bar{z}),\mu)$-order property.
3) K has the (Δ,μ)-order property <u>if</u> for some $M \in K, M$ has the (Δ,μ)-order property.
4) M [or K] has the $(\Delta,<\mu)$-order property <u>if</u> M [or K] has the (Δ,μ_1)-order property for every $\mu_1 < \mu$.
5) Replacing "order" by "non-order" is just the negation. We may replace μ by an ordinal (but this is not central here).
6) M has the $(\pm\varphi,\mu)$-order property <u>if</u> it has the (φ,μ)-order property or the $(\neg\varphi,\mu)$-order property; similarly for the other definitions.
7) Let "$(\varphi(\bar{x},\bar{y}),\mu)$-order" mean "$(\varphi(\bar{x};\bar{y};\bar{z}),\mu)$-order" for $\bar{z}$ the empty sequence, and $(\varphi(\bar{x}),\mu)$-order" means $(\varphi(\bar{x}_1;\bar{x}_2;\bar{x}_3),\mu)$-order" where $\bar{x} = \bar{x}_1{}^\frown\bar{x}_2{}^\frown\bar{x}_3$ for some $\bar{x}_1,\bar{x}_2,\bar{x}_3$.

1.2 Remark. 1) Usually $\Delta \subseteq \mathbb{L}_{\infty,\omega}$, but sometimes $\Delta \subseteq \triangle(\mathbb{L}_{\infty,\omega})$ (the $\triangle$-closure of $\mathbb{L}_{\infty,\omega}$, see Definition 0.7(12) which means that every sentence ψ from Δ, it and its negation are defined as $\{M \restriction \tau_\Delta : M \models \psi\}$ for some $\psi \in \mathbb{L}_{\infty,\omega}$).
2) On the other hand for universal K (see §2) we may well use $\Delta =$ set of quantifier free finite formulas.
3) Note that if M has the $(\varphi(\bar{x};\bar{y};\bar{z}),\mu)$-order property, then it has the $(\varphi(\bar{x};\bar{y}{}^\frown\bar{z}),\mu)$-order property and the $(\psi(\bar{x}',\bar{y}),\mu)$-order property where $\bar{x}' = \bar{x}{}^\frown\bar{z};\psi(\bar{x}{}^\frown\bar{z},y) = \varphi(\bar{x};\bar{y};\bar{z})$.
4) In Definition 1.1(1) we can ignore the case $\alpha = \beta$ as we can use $\bar{a}'_\alpha = a_{2\alpha+1}, \bar{b}'_\alpha = b_{2\alpha}$. This may be a better definition - see some exercises.
5) We prove in [Sh 300, Ch.III] and better in [Sh:e, Ch.III] (and in [Sh 220]) that order implies complexity, "non-structure".
6) If M has the $(\varphi(\bar{x};\bar{y};\bar{z}),\gamma)$-order property, then letting $\bar{z}_\ell = \bar{x}_\ell{}^\frown\bar{y}_\ell$, $\ell g(\bar{x}_\ell) = \ell g(\bar{x})$, $\ell g(\bar{y}_\ell) = \ell g(\bar{y})$ for $\ell = 1,2$ and we let $\varphi'(\bar{z}_1;\bar{z}_2;\bar{z}) = \varphi(\bar{x}_2;\bar{y}_2,\bar{z})$, <u>then</u> for some $\bar{c}_\beta \in {}^{\ell g(\bar{z}_\ell)}M$ for $\beta < \gamma$ and $\bar{d} \in {}^{\ell g(\bar{z})}M$ we have: if $\alpha,\beta < \gamma$ then $M \models \varphi'[\bar{c}_\alpha,\bar{c}_\beta,\bar{d}]$ iff $\alpha < \beta$.

Very relevant is (but our proof consists of quoting):

1.3 Theorem. *0) If K is a PC_κ class and there is $M \in K$ of cardinality $\geq \beth_{\delta(\kappa)}$ (see Definition 1.4(1) below noting that $\delta(\kappa) < (2^\kappa)^+$) or just there is $M \in K$ of cardinality $\geq \beth_\alpha$ for every $\alpha < \delta(\kappa)$, <u>then</u>*

K has members in every cardinal $\mu \geq \kappa$. If K is a $\mathrm{PC}_{\kappa,2^\kappa}$ class, similarly replacing $\delta(\kappa)$ by $(2^\kappa)^+$.
0A) Similarly if K is definable by some sentence from $\triangle(\mathbb{L}_{\kappa,\omega})$.
1) If K is a $\mathrm{PC}_{\lambda,2^\lambda}$ class or is definable by a sentence in $\triangle(\mathbb{L}_{\lambda^+,\omega})$, and it has the $(\varphi(\bar{x};\bar{y}),<\infty)$-order property for some $\varphi(\bar{x};\bar{y}) \in \triangle(\mathbb{L}_{\lambda^+,\omega})$, see Definition 0.7(12) but we allow $\ell g(\bar{x}) = \ell g(\bar{y})$ to be infinite __then__:

(a) *for every $\mu > \lambda + |\ell g(\bar{x}\hat{\ }\bar{y})|^+$ the class K has 2^μ non-isomorphic members of cardinality μ*
(b) *if $\mu > \lambda^+$, $\mathrm{cf}(\mu) > \lambda$ and μ is regular or strong limit, __then__ K has 2^μ non-isomorphic members of cardinality μ which are $\mathbb{L}_{\infty,\mu}$-equivalent*
(c) *if $\mu > \lambda$ is regular, $\mu = \mu^{\ell g(\bar{x}\hat{\ }\bar{y})}$ __then__ K has 2^μ members of cardinality μ, no one embeddable into another by an embedding preserving $\pm\varphi(\bar{x},\bar{y})$.*

2) If K is a PC_λ class or is definable by a sentence from $\triangle(\mathbb{L}_{\kappa^+,\omega})$ and it has the $(\varphi(\bar{x};\bar{y}),\lambda)$-order property, $\varphi(\bar{x},\bar{y}) \in \mathbb{L}_{\mu^+,\omega}$ but we allow $\ell g(\bar{x}) = \ell g(\bar{y})$ to be infinite __then__ (see below):

(a) *if $\lambda \geq \beth_{\delta(\mu+\kappa)}$ __then__ K has the $(\varphi(\bar{x};\bar{y}),<\infty)$-order property*
(b) *if $\lambda \geq \beth_{\delta(\mu+\kappa)}$ __then__ for some $\varphi'(\bar{x}';\bar{y}') \in \mathbb{L}_{\kappa^+,\omega}$ the class K has the $(\varphi'(\bar{x}',\bar{y}'),<\infty)$-order property and φ' "inherits all relevant properties" of φ. More exactly[1], for some $\lambda, \varphi \in \mathscr{H}(\lambda)$, and for some elementary submodel N of $(\mathscr{H}(\lambda),\in)$ of cardinality κ, φ' is the image of φ under the Mostowski Collapse of N)*
(c) *if $\lambda \geq \beth_{\delta_{\mathrm{wo}}(\mu,\kappa)}$ __then__ (see Definition 1.4(2) below) the conclusion of clause (b) holds.*

3) Similar conclusions holds for $\varphi(\bar{x};\bar{y};\bar{z})$.

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[1]recalling $\mathscr{H}(\lambda)$ is the family of sets with transitive closure of cardinality $< \lambda$

1.4 Definition. 1) For a cardinal κ let $\delta(\kappa)$ be the first ordinal δ such that: if τ is a vocabulary of cardinality $\leq \kappa, \psi \in \mathbb{L}_{\kappa^+,\omega}(\tau)$ and the predicate $<$ belongs to the vocabulary $\tau, <$ is binary, $P \in \tau$ is unary and for every $\alpha < \delta$, the sentence ψ has a model M such that $(P^M, <^M)$ is a well ordering of order type $\geq \alpha$ <u>then</u> for some model M of $\psi, (P^M, <^M)$ is not well ordered.
2) Let $\delta_{\text{wo}}(\mu, \kappa)$ is the first ordinal $\delta > \mu$ such that: if $\psi \in \mathbb{L}_{\kappa^+,\omega}(\tau), <$ is a binary predicate from τ and P is a unary predicates from τ and for every $\alpha < \delta$ the sentence ψ has a model M such that $(P^M, <)$ is a well ordering of order type $\geq \alpha$ and Q^M consists of the first λ members of $(P^M, <)$ <u>then</u> there are models $N_1 \prec N_2$ of ψ such that $N_1 \prec M$ and $(P^{N_2}, <^{N_2})$ is not well ordered and $Q^{N_1} = Q^{N_2}$.

Remark. 1) For a proof of more than 1.3(1) see [Sh:e, Ch.III,§3].
2) On the subject and proof of 1.3(2), 1.3(3) see Shelah [Sh 16] and Grossberg and Shelah [GrSh 222], [GrSh 259].
3) We do not try to get the optimal results, just state what previous proofs obviously give. E.g. we ignore the slightly stronger versions we can get by replacing μ by a limit cardinal.

Proof of 1.3. 0) By Morley and improvements of Chang, see e.g. [Sh:c, VIII,§5].
0A) Similarly.
1) <u>Clause (a)</u>.
By [Sh 300, III,§3] or better [Sh:e, III,3.4] using [Sh:e, 1.11](3).

<u>Clause (b)</u>:
If μ is regular by [Sh 220, §2] using [Sh:e, III,1.11](3).
If μ is strong limit by [Sh 220, §3] using [Sh:e, III,1.11](3).

<u>Clause (c)</u>:
By [Sh:e, III,§3].

2),3) <u>Clause (a)</u>:
Similarly.

<u>Clause (b)</u>:
By [Sh:c] or use clause (c).

Clause (c):

By Grossberg-Shelah [GrSh 222], [GrSh 259]. $\square_{1.3}$

1.5 Definition. 1) The sequence $\langle \bar{a}_t : t \in I \rangle$, where I is a linear order and for some ordinal α we have $\bar{a}_t \in {}^\alpha M$ for $t \in I$, is a (Δ, n)-indiscernible sequence inside M over A if: for all $t_1 <_I \ldots <_I t_n \in I, \bar{a}_{t_n} \hat{} \cdots \hat{} \bar{a}_{t_1}$ realize the same Δ-type inside M over A.
2) Writing Δ instead (Δ, n) means "for all $n < \omega$". If we omit A we mean A is empty.
3) Instead "over A" we may say "over $\mathbf{J}$", for $\mathbf{J}$ a set of sequences from M, meaning that: if $\varphi(\bar{x}_n, \ldots, \bar{x}_1, \bar{y}_1, \ldots, \bar{y}_k) \in \Delta, \ell g(\bar{x}_\ell) = \alpha$ and $\bar{c}_1, \ldots, \bar{c}_k \in \mathbf{J}$ satisfying $\ell g(\bar{c}_\ell) = \ell g(\bar{y}_\ell)$ for $\ell = 1, \ldots, k$ then for any $s_1 <_I \ldots <_I s_n$ and $t_1 <_I \ldots <_I t_n$ we have $M \models$ "$\varphi[\bar{a}_{t_n}, \ldots, \bar{a}_{t_1}, \bar{c}_1, \ldots, \bar{c}_k] \equiv \varphi[\bar{a}_{s_1}, \ldots, \bar{a}_{s_n}, \bar{c}_1, \ldots, \bar{c}_k]$".
4) We say "strictly over $\mathbf{J}$" when in part (3) we demand $k \leq 1$.

1.6 Remark. Note that the sequences may have infinite length but still in 1.5(1) we have $n < \omega$. I.e. we use only finitely many sequences at a time. This should not be surprising, as $\lambda \rightarrow (\mu)^\omega_\chi$ is much more difficult to have than $\lambda \rightarrow (\mu)^{<\omega}_\chi$.

1.7 Definition. 1) $\{\bar{a}_t : t \in I\}$ is a (Δ, n)-indiscernible set inside M (over A) if for all pairwise distinct $t_1, \ldots, t_n \in I$

$$(\bar{a}_{t_1} \hat{} \ldots \hat{} \bar{a}_{t_n}) \text{ realizes the same } \Delta\text{-type in } M \text{ (over } A).$$

2) The parallels of Definition 1.5(2),(3),(4) for indiscernible set.

Remark. Applying Definition 1.7 we shall not always pedantically say "$\{\bar{a}_t : t \in I\}$ as an indexed set is ...".

* * *

We define here the notion "the type p does not (Δ, Λ)-split over A" (inside M). This says that in some weak sense, $p \restriction \Delta$ is definable over

A. More specifically the Λ-type of the parameters over A, separates between the $\bar{b}$ such that $\varphi(\bar{x}, \bar{b}) \in p$ and the $\bar{b}$ such that $\neg\varphi(\bar{x}, \bar{b}) \in p$. In Definition 1.8(2) we replace the pair (Λ, A) by a collection of formulas Γ.

1.8 Definition. 1) For a type $p = p(\bar{x})$ inside M we say that it (Δ, Λ)-splits over A <u>if</u> there are $\bar{b}, \bar{c} \in M$, and $\varphi(\bar{x}; \bar{y}) \in \Delta$ such that:

- (i) $\varphi(\bar{x}, \bar{b}), \neg\varphi(\bar{x}, \bar{c}) \in p$
- (ii) in appropriate sense, $\bar{b}$ and $\bar{c}$ realize the same Λ-type over A inside M; more exactly, for every $\psi(\bar{x}; \bar{y}; \bar{z}) \in \Lambda$ satisfying $\ell g(\bar{b}) = \ell g(\bar{c}) = \ell g(\bar{y})$ and $\bar{e} \in {}^{\ell g(\bar{z})}A$, $\bar{a}' \in {}^{\ell g(\bar{x})}A$ we have: $M \models \psi[\bar{a}', \bar{b}, \bar{e}]$ if and only if $M \models \psi[\bar{a}', \bar{c}, \bar{e}]$; note that $\bar{x}$ is determined by $p(\bar{x})$.

2) For a type $p = p(\bar{x})$ inside M, we say that it (Δ, Λ)-splits over Γ where Γ consisting of Λ-formulas with parameters from M, see below <u>when</u> there are $\bar{b}, \bar{c} \in M$ of equal length and $\varphi(\bar{x}; \bar{y}) \in \Delta$ such that:

- (i) $\varphi(\bar{x}; \bar{b}), \neg\varphi(\bar{x}, \bar{c}) \in p$,
- (ii) if $\psi(\bar{x}; \bar{y}; \bar{z}) \in \Lambda, \ell g(\bar{y}) = \ell g(\bar{c}) = \ell g(\bar{b})$ and $\bar{a}' \in {}^{\ell g(\bar{x})}A, \bar{e} \in {}^{\ell g(\bar{z})}A$ and $\psi(\bar{a}', \bar{y}, \bar{e}) \in \Gamma$ <u>then</u> $M \models \psi[\bar{a}', \bar{b}, \bar{e}]$ if and only if $M \models \psi[\bar{a}', \bar{c}, \bar{e}]$.

3) We define "p does Δ-split over Γ" similarly, omitting "Γ consist of Λ-formulas with parameters from M".

1.9 Remark. Clearly 1.8(1) is an instance of 1.8(2).

1.10 Fact. 1) If $p = p(\bar{x})$ is a type inside M, which (Δ, Λ)-splits over A inside M and $p \subseteq q(\bar{x})$, with $q(\bar{x})$ a type inside $M, \Delta \subseteq \Delta_1, \Lambda_1 \subseteq \Lambda$ <u>then</u> $q(\bar{x})$ does (Δ_1, Λ_1)-splits over A inside M.
2) Suppose for $\ell = 1, 2$ that $p_\ell(\bar{x})$ does not (Δ, Λ)-split over Γ, Γ a set of formulas over $A, p_\ell \in \mathrm{Sfr}^{\ell g(\bar{x})}_\Delta(C, M)$ recalling Definition 0.8(4)(e), and $A \subseteq B \subseteq C \subseteq |M|$. If for every $\bar{b} \in C$ there is $\bar{b}' \in B$ such that for every $\varphi(\bar{a}, \bar{y}, \bar{e}) \in \Gamma$ we have $M \models \varphi[\bar{a}, \bar{b}, \bar{e}] \equiv \varphi[\bar{a}, \bar{b}', \bar{e}]$ provided

that $\ell g(\bar b) = \ell g(\bar b') = \ell g(\bar y)$ and $\ell g(\bar a) = \ell g(\bar x)$ <u>then</u> $p_1 \restriction B = p_2 \restriction B$ implies $p_1 = p_2$. We can replace A by any set including the sequences appearing as $\bar a$ in any $\psi(\bar a, \bar y, \bar e) \in \Gamma$.
3) Suppose $p(\bar x)$ is a type inside $M, A \subseteq M$ and $\Gamma = \{\psi(\bar x; \bar y; \bar e) : \bar e \in {}^{\ell g(\bar z)}A$ and $\psi(\bar x; \bar y; \bar z) \in \Lambda\}$.
<u>Then</u>: $p(\bar x)$ does not (Δ, Λ)-split over A <u>if and only if</u> $p(\bar x)$ does not (Δ, Λ)-split over Γ.
4) If $B \subseteq |M|$ and $M, \Delta, \Lambda, \Gamma$ are as in 1.8(2) and α is an ordinal, <u>then</u> $\mathbf{P} = \{p \in \mathrm{Sfr}^\alpha_\Delta(B, M) : p$ does not (Δ, Λ)-split over $\Gamma\}$ has cardinality $\le 2^{(2^{|\Gamma|} \times |\Delta|)}$ and so $\le 2^{(2^{|\Gamma|} + |\Delta|)}$ if p, Δ are not both finite.
5) If $A \subseteq M$, <u>then</u> $|\mathrm{Sfr}^\alpha_\Delta(A, M)| \le \Pi\{|\mathrm{Sfr}^\alpha_{\varphi(\bar x;\bar y)}(A, M)| : \ell g(\bar x) = \alpha, \varphi(\bar x; \bar y) \in \Delta\}$. Similarly for $\mathbf{S}$ instead Sfr recalling Definition 0.8(4)(d).

1.11 Remark. We can systematically replace sets of elements by sets of formulas.

Proof. Easy, e.g.
4) For each $p \in \mathrm{Sfr}^\alpha_\Delta(A, M)$ and $\varphi(\bar x, \bar y) \in \Delta$ and Γ let $F^p_{\varphi(\bar x, \bar y)}$ be the following function. Its domain is $\mathrm{Sfr}^{\ell g(\bar y)}_\psi(\Gamma \restriction \psi)$ (where $\psi(\bar y, \bar x) = \varphi(\bar x, \bar y)$ and $\Gamma \restriction \psi = \{\psi(\bar y, \bar a) : \psi(\bar y, \bar a) \in \Gamma\}$), its range is {true,false} and $F^p_\varphi(q) =$ true iff for some $\eta \in {}^{\Gamma \restriction \psi}\{0, 1\}$ we have: $q = \{\psi(\bar y, \bar a)^{\eta(\psi(\bar y, \bar a))} : \psi(\bar y, \bar a) \in \Gamma\}$ and: for some (if there are such $\bar b$, equivalently every) $\bar b \in {}^{\ell g(\bar y)}(B)$ realizing q we have $\varphi(\bar x, \bar b) \in p$. The "some, equivalently every" holds by the definition of non-splitting (and $F^p_{\varphi(\bar x, \bar y)} \restriction \{q \in \mathrm{Sfr}^{\ell g(\bar y)}_\psi(\Gamma \restriction \psi) : q$ is realized in $M\}$ suffice). So if $\Delta = \{\varphi(\bar x, \bar y)\}$ then $|\mathbf{P}| \le |\{F^p_\varphi : p \in S\}| \le 2^{2^{|\Gamma|}}$. By part (5) we can finish the general case. Note that there was no need to deal with $\varphi(\bar x; \bar y; \bar z)$ as we use Γ and in 1.8(2) the formulas in Γ were allowed to have parameters from M.
5) We define below a one to one map, H with domain $\mathrm{Sfr}^\alpha_\Delta(A, M)$ and
range included in $\Pi\{\mathrm{Sfr}^\alpha_{\varphi(\bar x, \bar y)}(A, M) : \ell(\bar x) = \alpha, \varphi(\bar x, \bar y) \in \Delta\}$, this clearly suffices. Now H is defined by $H(p) = \langle p \restriction \varphi : \varphi \in \Delta, \varphi = \varphi(\bar x, \bar y)\rangle$. $\square_{1.10}$

1.12 The non-splitting/order dichotomy theorem. *Suppose* $M \leq^*_{\sum^{\rm pos}_{\chi,\kappa}(\{\neg\psi(\bar x;\bar y_1 {}^\frown \bar y_2)\})} N$ *recalling Definition 0.9(4) and* $\varphi(\bar x;\bar y)$ *is a* $\tau(N)$*-formula,* $\ell g(\bar x) \leq \kappa, \ell g(\bar y) = \ell g(\bar y_1) = \ell g(\bar y_2) \leq \kappa$ *and* $\psi(\bar x;\bar y_1 {}^\frown \bar y_2) := [\varphi(\bar x;\bar y_1) \equiv \varphi(\bar x;\bar y_2)]$. *Then (i) or (ii) (or both) hold where:*

- (i) *for every* $\bar c \in |N|$ *satisfying* $\ell g(\bar c) = \ell g(\bar x)$ *for some* $\Gamma \subseteq \{\varphi(\bar d;\bar y) : \bar d \in {}^{\ell g(\bar x)}|M|\}$, $|\Gamma| \leq \chi$ *and* ${\rm tp}_{\varphi(\bar x;\bar y)}(\bar c, |M|, N)$ *does not* $(\varphi(\bar x;\bar y), \varphi(\bar x;\bar y))$*-split over* Γ
- (ii) N *has the* (ψ, χ^+)*-order property (in fact, exemplified by sequences from* M*).*

Note that

1.13 Fact. Assume $\varphi = \varphi(\bar x, \bar y)$ and $\ell g(\bar x) \leq \kappa$ and $\ell g(\bar y) \leq \kappa$ are in the vocabulary of $M \subseteq N$.
1) When $\chi = \chi^\kappa$ (just combining definitions) $M \leq^*_{\sum_{\chi,\kappa}(\{\varphi\})} N$ means the following: for every $\bar c \in {}^\kappa N$ and $A \subseteq |M|$ satisfying $|A| \leq \chi$, there is $\bar c' \in {}^\kappa|M|$ realizing the type ${\rm tp}_{\{\varphi\}}(\bar c, A, N)$ in the model N.
2) $M \leq^*_{\Sigma_{\chi,\kappa}(\{\varphi\})} N$ iff: for every $\bar c \in {}^{\ell g(\bar x)}N$ and $\mathbf{I} \subseteq {}^{\ell g(\bar y)}M$ of cardinality $\leq \chi$ some $\bar c' \in {}^{\ell g(\bar x)}M$ we have ${\rm tp}_{\{\varphi\}}(\bar c, \mathbf{I}, N) = {\rm tp}_{\{\varphi\}}(\bar c', \mathbf{I}, N)$, see Definition 0.8(5).
3) If $M \leq^*_{\Sigma_{\chi,\kappa}(\{\varphi\})} N$ then $M \leq^*_{\Sigma^{\rm pos}_{\chi,\kappa}(\{\neg\psi(\bar x;\bar y_1 {}^\frown \bar y_2)\})} N$ where $\psi(\bar x;\bar y_1 {}^\frown \bar y_2)$ is defined as in 1.12.

Proof of 1.13. Easy. $\square_{1.13}$

1.14 Remark. In 1.12 we essentially contrast $(\varphi(\bar x;\bar y), \varphi(\bar x;\bar y))$-splitting with the $(\pm\varphi(\bar x;\bar y), \chi^+)$-order property where $\chi = \ell g(\bar y)$, (see 1.15 below). This χ is the crucial parameter because it governs our ability to continue to choose $\bar a_i, \bar b_i$.

Proof of 1.12. Assume that ${\rm tp}_{\varphi(\bar x;\bar y)}(\bar c, |M|, N)$ fails clause (i) and we shall prove clause (ii). We choose by induction on $i < \chi^+$ the sequences $\bar{a}_i, \bar{b}_i, \bar{c}_i$ from M with $\ell g(\bar{c}_i) = \ell g(\bar{x}), \ell g(\bar{b}_i) = \ell g(\bar{a}_i) = \ell g(\bar{y})$; such that:

(a) $N \models$ "$[\varphi(\bar{c};\bar{a}_i) \equiv \neg\varphi(\bar{c};\bar{b}_i)]$"

(b) for $j < i, N \models$ "$\varphi[\bar{c}_j, \bar{a}_i] \equiv \varphi[\bar{c}_j, \bar{b}_i]$"

(c) $\bar{c}_i$ realizes $\{\varphi(\bar{x}, \bar{a}_j) \equiv \neg\varphi(\bar{x}, \bar{b}_j) : j \leq i\}$ inside N.

Note: Clauses (a) and (b) say exactly: $\varphi(\bar{x}, \bar{y}), \bar{a}_i, \bar{b}_i$ exemplify that $\mathrm{tp}_{\varphi(\bar{x};\bar{y})}(\bar{c}, |M|, N)$ does $(\varphi(\bar{x};\bar{y}), \varphi(\bar{x};\bar{y}))$-split over $\{\varphi(\bar{c}_j;\bar{y}) : j < i\}$. Hence as we are assuming the failure of clause (i), for $i < \chi^+$ if $\bar{c}_j, \bar{b}_j, \bar{a}_j$ $(j < i)$ are defined, we can choose $\bar{a}_i, \bar{b}_i$ as required in clauses (a) + (b); then using $M \leq^*_{\Sigma^{\mathrm{pos}}_{\chi,\kappa}(\neg\psi)} N$ we can choose $\bar{c}_i$ as required in clause (c).

Having defined all $\bar{a}_j, \bar{b}_j, \bar{c}_j$ (for $j < \chi^+$), clearly $N \models$ "$\varphi(\bar{c}_\alpha, \bar{b}_\beta) \equiv \varphi(\bar{c}_\alpha, \bar{a}_\beta)$" if and only if $\alpha < \beta$. So $\langle \bar{c}_\alpha : \alpha < \chi^+ \rangle$ and $\langle \bar{b}_\beta\char`^\bar{a}_\beta : \beta < \chi^+ \rangle$ exemplify clause (ii).

$\square_{1.12}$

1.15 Observation. 1) Suppose φ, ψ are as in 1.12, and N has the (ψ, μ_1)-order property and $\mu_1 \to (\mu_2)^2_4$. Then N has the $(\pm\varphi(\bar{x};\bar{y}), \mu_2)$-order property.
2) For μ_1, μ_2 as above and Δ, if N has the $(\Delta^{\mathrm{eb}}, \mu_1)$-order property then it has the $(\Delta^{\mathrm{i,r}}, \mu_2)$-order property.

Proof. 1) So there is $\langle (\bar{a}_\alpha\char`^b_\alpha, \bar{c}_\alpha) : \alpha < \mu_1 \rangle$ such that $N \models$ "$\varphi[\bar{c}_\alpha; \bar{c}_\beta] = \varphi[\bar{c}_\alpha; \bar{b}_\beta]$" iff $\alpha < \beta$. We choose a function $\mathbf{f}$ from $[\mu_1]^2$ to $\{0,1,2,3\}$ as follows: if $\alpha < \beta$ then $\mathbf{f}(\{\alpha, \beta\}) = i_0 + 2i_1$ where i_0 is 1 if $N \models \varphi[\bar{c}_\beta, \bar{a}_\alpha]$ and 0 otherwise, and $i_1 = 1$ if $N \models \varphi[\bar{c}_\alpha, \bar{a}_\beta]$ and 0 otherwise. Let $Y \subseteq \mu_1$ be of order type μ_2 such that $\mathbf{f} \restriction [Y]^2$ is constantly $i_0 + 2i_1$. Now check all the possibilities: if $i_0 = 0$ & $i_1 = 1$ then $\langle (\bar{c}_{2\alpha+1}, \bar{a}_{2\alpha}) : \alpha \in Y \rangle$ exemplifies that M has the $(\varphi(\bar{x};\bar{y}), \mu_2)$-order property; if $i_0 = 1$ & $i_1 = 0$ then $\langle (\bar{c}_{2\alpha+1}, \bar{a}_{2\alpha}) : \alpha \in Y \rangle$ exemplies that M has the $(\neg\varphi(\bar{x};\bar{y}), \mu_2)$-order property; and if $i_0 = i_1$ we replace $\bar{a}_\alpha$ by $\bar{b}_\alpha$ above.
2) Follows. $\square_{1.15}$

1.16 Remark. Using this, and only $(\pm\varphi(\bar{x};\bar{y}),\lambda)$-order properties, the formulation of theorems in this section becomes nicer. I.e. we lose some sharpness in cardinality bounds, but we use only $\pm\varphi$-order and φ-unstability properties.

1.17 Observation. 1) Theorem 1.12 has an obvious version for (Δ,Δ)-splitting and the (Δ,χ^+)-order property. We just have to demand $|\Delta| \le \chi$ and in the proof replace (a), (b), (c) by:

$(a)'$ $N \models$ "$\varphi_i[c;\bar{a}_i] \equiv \neg\varphi_i[\bar{c},\bar{b}_i]$"
$(b)'$ for $j < i, N \models$ "$\varphi_j[\bar{c}_j,\bar{a}_i] \equiv \varphi_j[\bar{c}_j,\bar{b}_i]$"
$(c)'$ $\bar{c}_i$ realizes $\{\varphi_j(\bar{x};\bar{a}_j) \equiv \neg\varphi_j(\bar{x};\bar{b}_j) : j \le i\}$.

In the end for some $\varphi(\bar{x};\bar{y}) \in \Delta$ the set $S = \{i < \chi^+ : \varphi_i(\bar{x};\bar{y}) = \varphi(\bar{x},\bar{y})\}$ is an unbounded subset of χ^+ and we use $\langle(\bar{a}_i{}^\frown\bar{b}_i,\bar{c}_i) : i \in S\rangle$.
2) Assume that we strengthen the assumption of 1.12 to $M \le^*_{\Sigma_{\chi,\kappa}(\varphi(\bar{x},\bar{y}))}$ N. Then in the proof of 1.12 we can strengthen clauses (a),(b),(c) to:

$(a)^+$ $N \models$ "$\varphi[\bar{c};\bar{a}_i]$ & $\neg\varphi[\bar{c},\bar{b}_i]$"
$(c)^+$ $\bar{c}_i$ realizes $\{\varphi(\bar{x};\bar{a}_j) \wedge \neg\varphi(\bar{x};\bar{b}_j) : j \le i\}$.

Then in 1.15 it is enough to demand $\mu_1 \to (\mu_2)^2_2$.
3) We could have replaced χ^+ by a limit cardinal (sometimes of large cofinality or regular and/or uncountable).

Proof. Straight, e.g.
2) The proof works because in stage i we first choose $\bar{a}_i, b_i$ such that clause (a) holds and then, if clause (a)$^+$ fails we interchange $\bar{a}_i$ with $\bar{b}_i$. $\square_{1.17}$

Remark. 1) We have remarked above that for non-first order logics we must be careful about closure properties of sets of formulas. The following notation permits us to take this care.
2) The operations (i.e., $\Delta \mapsto \Delta^x$) defined in Definition 1.18 below are the ones used explicitly or implicitly in this section, e.g., Δ^{es} is used in 1.19 below.

1.18 Definition. 1) Let (for Δ a set of formulas with the variables divided to two, i.e. $\varphi(\bar{x};\bar{y})$):

(*a*) $\Delta^{\rm i} = \Delta$

(*b*) $\Delta^{\rm cn} = \{\neg\varphi : \varphi \in \Delta\}$

(*c*) $\Delta^{\rm nc} = \Delta \cup \Delta^{\rm cn}$

(*d*) $\Delta^{\rm es} = \{\psi(\bar{x};\bar{y}_1,\bar{y}_2) : \psi(\bar{x};\bar{y}_1,\bar{y}_2) := [\varphi(\bar{x};\bar{y}_1) \equiv \varphi(\bar{x};\bar{y}_2)]$ for some $\varphi(\bar{x};\bar{y}) \in \Delta\}$

(*e*) $\Delta^{\rm r} = \{\psi(\bar{y};\bar{x}) : \psi(\bar{y};\bar{x}) = \varphi(\bar{x};\bar{y}) \in \Delta\}$

(*f*) $\Delta^{\rm rs} = ((\Delta)^r)^{\rm es} = \{\psi(\bar{y};\bar{x}^1{}^\frown\bar{x}^2) : \psi(\bar{y};\bar{x}^1{}^\frown\bar{x}^2) := [\varphi(\bar{x}^1;\bar{y}) \equiv (\bar{x}^2;\bar{y})]\}$

(*g*) $\Delta^{\rm eb} = \Delta^{\rm es} \cup \Delta^{\rm rs}$.

2) If $x_1,\dots,x_m \in \{$cn,nc,es,r,rs,eb,i$\}$ then we let $\Delta^{x_1,\dots,x_m} = \bigcup\limits_{\ell=1}^{m} \Delta^{x_\ell}$ and $\Delta^{x_1,\dots,x_k;x_{k+1},\dots,x_m} = (\Delta^{x_1,\dots,x_k})^{x_{k+1},\dots,x_n}$ and $z = x * y$ means $\Delta^z = (\Delta^x)^y$.

3) We say that Δ is $(<\kappa)$-variable closed when: if $\varphi(\bar{x}) \in \Delta, \bar{x} = \bar{x}'{}^\frown\bar{x}'', \ell g(\bar{x}') < \kappa, \bar{x}' = \bar{x}'_0{}^\frown \dots {}^\frown\bar{x}'_{n-1}, \bar{y} = \bar{y}'{}^\frown\bar{y}'', \ell g(\bar{y}'') = \ell g(\bar{x}''), \bar{y}' = \bar{y}'_0{}^\frown \dots {}^\frown\bar{y}'_{n-1}, \pi$ is a permutation of $\{0,\dots,n-1\}, \ell g(\bar{y}'_{\pi(\ell)}) = \ell g(\bar{x}_\ell)$ and $\varphi'(\bar{y}) := \varphi(y'_{\pi(0)}{}^\frown\bar{y}'_{\pi(1)}{}^\frown \dots {}^\frown\bar{y}'_{\pi(n-1)}{}^\frown\bar{y}'')$ <u>then</u> $\varphi'(\bar{y}) \in \Delta$.

The next theorem connects non-order and stability.

1.19 The Stability Theorem. *Suppose M has the $(\Delta^{\rm es},\chi^+)$-non-order property, $\mu = \mu^\chi + 2^{2^\chi}$, $|\Delta| \le \chi$ and $[\varphi(\bar{x}) \in \Delta \Rightarrow |\ell g(\bar{x})| \le \chi]$. <u>Then</u> for $A \subseteq M, |A| \le \mu$ implies $\mathbf{S}^\kappa_\Delta(A,M) = \{{\rm tp}_\Delta(\bar{a},A,M) : \bar{a} \in {}^\kappa|M|\}$ has cardinality $\le \mu$.*

Proof. We would like to say that clearly there is M_1 satisfying $A \subseteq M_1 \subseteq M, |M_1| \le \mu$ such that $M_1 \le_{\sum_{\chi,\chi}(\Delta)} M$. But this is problematic, maybe even there is no $M_1 \le_\Delta M, M_1 \ne M$. However, there is $M_1 \subseteq M$ of cardinality $\le \mu$ such that $M_1 \le^*_{\Sigma_{\chi,\kappa}(\Delta)} M$ and $A \subseteq M$ so if $M \models (\exists\bar{x}) \bigwedge\limits_{i<\chi} \varphi_i(\bar{x},\bar{a}_i)^{\eta(i)}$ where $\varphi_i = \varphi_i(\bar{x},\bar{y}_i) \in \Delta$

and $\bar{a}_i \in {}^{\ell g(\bar{y}_i)}(M_1)$ for $i < \chi$ then for some $\bar{b} \in {}^{\ell g(\bar{x})}(M_1)$ we have $M \models \bigwedge_{i<\chi} \varphi_i(\bar{b}, \bar{a}_i)^{\eta(i)}$. (Note that it does not matter if we use $\bigwedge_{i<i(*)}$ with $i(*) < \chi^+$.)

Without loss of generality replace A by M_1 and assume Δ is $\{\varphi(\bar{x};\bar{y})\}$ (by 1.10(5)). Now by the assumption of our present theorem, clause (ii) of Theorem 1.12 fails, hence clause (i) of Theorem 1.12 holds. So every $p = \text{tp}_\Delta(\bar{a}, M_1, M) \in \mathbf{S}^\kappa_\Delta(M_1, M)$ does not $(\varphi(\bar{x};\bar{y}), \varphi(\bar{x};\bar{y}))$-split over some $\Gamma_p \subseteq \{\varphi(\bar{c}';\bar{y}) : \bar{c}' \in |M_1|, \ell g(\bar{c}') = \ell g(\bar{x})\}$ which has cardinality $\leq \chi$. There are at most $\|M_1\|^\chi \leq \mu$ such sets Γ_p. So if the conclusion fails then for some Γ the set $\{p \in \mathbf{S}^\kappa_\Delta(M_1, M) : \Gamma_p = \Gamma\}|$ has cardinality $> \mu$ and so necessarily $|\Gamma| \leq \chi$. Hence $\{p \in \mathbf{S}^\kappa_\Delta(M_1, M) : p$ does not (Δ, Δ)-split over $\Gamma\}$ has cardinality $> \mu$. But it has cardinality $\leq 2^{2^\chi}$ (by 1.10(4)), contradiction. $\square_{1.19}$

1.20 Conclusion. Suppose M has the $(\Delta^{\text{nc}}, \chi^+)$-non-order property, $\mu = \mu^{2^\chi} + \beth_3(\chi)$, $|\Delta| \leq 2^\chi$ and $[\varphi(\bar{x}) \in \Delta \Rightarrow \ell g(\bar{x}) \leq \chi]$. Then for $A \subseteq M, |A| \leq \mu$ implies $\mathbf{S}^\kappa_\Delta(A, M)$ has cardinality $\leq \mu$.

Proof. By Observation 1.15

(∗) M has the $(2^\chi)^+$-non-order property.

Now apply 1.19 with 2^χ here standing for χ there. $\square_{1.20}$

1.21 Exercise. 1) $|\mathbf{S}^\alpha_{\Delta^x}(A, M)| \leq |\mathbf{S}^\alpha_\Delta(A, M)|$ for $x =$ cn,nc,es,i. Similarly for Sfr when the right side is infinite (or we restrict ourselves to formal types "respecting" the connectives in the natural sense).
2) The $(\{\varphi(\bar{x};\bar{y})\}^r, \lambda)$-order property is equivalent to the $(\neg\varphi(\bar{x},\bar{y}), \lambda)$-order property.

1.22 Exercise: In 1.19 + 1.20 replace χ^+ by a limit cardinal (e.g. $\chi = \aleph_0$).

<u>1.23 Fact</u>: 1) If M has the (Δ^x, χ)-non-order property, <u>then</u> M has the (Δ^y, χ)-non-order property when $(x, y) \in \{$(eb,es),(eb,rs),(es $*$ r,i),(r $*$ es $*$ r,i),(es,r),(rs,i)$\}$.
2) If M has the (Δ^x, χ)-non-order property <u>then</u> M has the (Δ^y, χ^+)-non-order property when $(x, y) \in \{$(es,i),(rs,r)$\}$.
3) If $\psi = \psi(\bar{x}; \bar{y}; \bar{z}) = \varphi = \varphi(\bar{y}; \bar{x}; \bar{z})$, then M has the (ψ, χ)-order property iff M has the $(\neg\varphi, \chi)$-order property, so M has the $(\{\varphi(\bar{x}; \bar{y})\}, \chi)$-non-order property iff M has the $(\{\neg\varphi(\bar{x}; \bar{y})\}^r, \chi)$-non-order property.

Proof. Easy, and it seems more convenient to start with the order property.
1) <u>Case 1</u>: $(x, y) =$ (rs,i).

Without loss of generality $\Delta = \{\varphi(\bar{x}, \bar{y})\}$. So assume $\langle(\bar{a}_\alpha, \bar{b}_\alpha) : \alpha < \chi\rangle$ exemplifies "M has the $(\{\varphi(\bar{x}, \bar{y})\}, \chi)$-order property", so $M \models \varphi[\bar{a}_\alpha, \bar{b}_\beta]$ iff $\alpha < \beta$. Let $\psi(\bar{y}; \bar{x}_1 \hat{\ } \bar{x}_2) := [\varphi(\bar{x}_1, \bar{y}) \equiv \varphi(\bar{x}_2, \bar{y})] \in \{\varphi(\bar{x}, \bar{y})\}^{\mathrm{rs}}$.

Let $\bar{a}'_\alpha = \bar{b}_{1+\alpha}, \bar{b}'_\alpha = \bar{a}_{1+\alpha} \hat{\ } \bar{a}_0$. So $M \models \psi[\bar{a}'_\alpha, \bar{b}'_\beta]$ iff $M \models$ "$\varphi[\bar{a}_{1+\beta}, \bar{b}_{1+\alpha}] \equiv \varphi[\bar{a}_0, \bar{b}_{1+\alpha}]$". As $0 < 1 + \alpha$ clearly $M \models \varphi[\bar{a}_0, \bar{b}_{1+\alpha}]$ hence $M \models \psi[\bar{a}'_\alpha, \bar{b}'_\beta]$ iff $M \models \varphi[\bar{a}_{1+\beta}, \bar{b}_{1+\alpha}]$. But $M \models \varphi[\bar{a}_{1+\beta}, \bar{b}_{1+\alpha}]$ iff $1 + \beta < 1 + \alpha$ iff $\alpha < \beta$. So $M \models \psi[\bar{a}'_\alpha, \bar{b}'_\beta]$ iff $\alpha < \beta$, so we are done.

<u>Case 2</u>: $(x, y) =$ (es,r).

Without loss of generality $\Delta = \{\varphi(\bar{x}; \bar{y})\}$. Let $\vartheta(\bar{y}; \bar{x}) = \varphi(\bar{x}; \bar{y})$, so we assume M has the $(\{\vartheta(\bar{y}; \bar{x})\}, \chi)$-order property so we can find $\langle(\bar{a}_\alpha, \bar{b}_\alpha) : \alpha < \chi\rangle$ such that $M \models \vartheta[\bar{a}_\alpha, \bar{b}_\beta]$ iff $\alpha < \beta$.

So

$(*)_1$ $M \models \varphi[\bar{b}_\beta, \bar{a}_\alpha]$ iff $\alpha < \beta$.

Let $\bar{a}'_\alpha = \bar{b}_{1+\alpha}, \bar{b}'_\beta = \bar{a}_{1+\beta} \hat{\ } \bar{a}_0$ and let $\psi(\bar{x}, \bar{y}_1 \hat{\ } \bar{y}_2) = [\varphi(\bar{x}, \bar{y}_1) \equiv \varphi(\bar{x}, \bar{y}_2)]$.

Now $M \models \psi[\bar{a}'_\alpha, \bar{b}'_\beta]$ iff $M \models$ "$\varphi[\bar{b}_{1+\alpha}, \bar{a}_{1+\beta}] \equiv \varphi[\bar{b}_{1+\beta}, \bar{a}_0]$". But $M \models \varphi[b_{1+\beta}, \bar{a}_0]$ as $0 < 1 + \beta$ hence $M \models \psi[\bar{a}'_\alpha, \bar{b}'_\alpha]$ iff $M \models \varphi[\bar{b}_{1+\beta}, \bar{a}_{1+\alpha}]$ iff $(1 + \alpha < 1 + \beta)$ iff $\alpha < \beta$. So $\langle(\bar{a}'_\alpha, \bar{b}'_\alpha) : \alpha < \chi\rangle$ exemplies that M has the $(\{\varphi(\bar{x}, \bar{y})\}^{\mathrm{es}}, \chi)$-order property.

Case 3: $(x, y) =$ (es $*$ r,i).

As $\Delta^i = \Delta$ let $\langle (\bar{a}_\alpha, \bar{b}_\alpha) : \alpha < \chi \rangle$ exemplify the (Δ, χ)-order property for $\varphi(\bar{x}, \bar{y}) \in \Delta$. Let $\bar{b}'_\alpha = \bar{a}_{1+2\alpha+1}, \bar{a}'_\alpha = \bar{b}_0 \hat{\ } \bar{b}_{1+2\alpha}$ and let $\psi(\bar{y}_1 \hat{\ } \bar{y}_2, \bar{x}) = [\varphi(\bar{x}, \bar{y}_1) \equiv \varphi(\bar{x}, \bar{y}_2)]$, so it belongs to $\Delta^{\text{es*r}}$. So for $\alpha, \beta < \chi$, clearly

$(*)_1$ $M \models \psi[\bar{a}'_\alpha, \bar{b}'_\beta]$ iff $M \models$ "$\varphi[\bar{a}_{1+2\beta+1}, \bar{b}_0] \equiv \varphi[a_{1+2\beta+1}, \bar{b}_{1+2\alpha}]$".

But $1 + 2\beta + 1 \not< 0$ hence $M \models \neg\varphi[\bar{a}_{1+2\beta+1}, \bar{b}_0]$ so

$(*)_2$ $M \models \psi[\bar{a}'_\alpha, \bar{b}'_\beta]$ iff $M \models \neg\varphi[\bar{a}_{1+2\beta+1}, \bar{b}_{1+2\alpha}]$
but

$(*)_3$ $M \models \neg\varphi[\bar{a}_{1+2\beta+1}, \bar{b}_{1+2\alpha}]$ iff $\neg(1+2\beta+1 < 1+2\alpha)$ iff $\alpha < \beta$.

Together we are done.

Case 4: $(x, y) =$ (r $*$ es $*$ r,i).

Let $\langle (\bar{a}_\alpha, \bar{b}_\alpha) : \alpha < \chi^+ \rangle$ and $\varphi(\bar{x}, \bar{y})$ be as in case 1.

Let $\psi = \psi(\bar{x}', \bar{y}) = [\varphi(\bar{x}_1, \bar{y}) \equiv \varphi(\bar{x}_2, \bar{y})]$ so $\bar{x}' = \bar{x}_1 \hat{\ } \bar{x}_2, \ell g(\bar{x}_1) = \ell g(\bar{x}_2) = \ell g(\bar{x})$ so $\psi(\bar{x}', \bar{y}) \in \Delta^{\text{r*es*r}}$ and any member of $\Delta^{\text{r*es*r}}$ has such form.

Let $\bar{a}'_\alpha = \bar{a}_{1+\alpha} \hat{\ } \bar{a}_0$ and let $\bar{b}'_\alpha = \bar{b}_{1+\alpha}$. We leave the checking to the readers.

Case 5: $(x, y) \in \{$(eb,es),(eb,rs)$\}$.

Note that $\Delta^x \supseteq \Delta^y$ so the desired implication is trivial.
2),3) Left to the reader. $\square_{1.23}$

1.24 Exercise: Why does Fact 1.23(1) ignore, e.g. the pair (es,i)?

Let $\chi \geq \aleph_0$ be regular (for simplicity).

Let $\tau = \{R\}, R$ a two-place predicate. Let M be the τ-model with universe χ and $R^M = \{(2\alpha, 2\beta + 1) : \alpha < \beta\}$ and $\varphi(x, y) = R(x, y)$. Then

(a) M has the $\{\{\varphi\}, \chi)$-order property

(b) M fails the $(\{\varphi\}^{\text{es}}, \chi)$-non-order property.

[Hint: Clause (a): Let $a_\alpha = 2\alpha, b_\alpha = 2\alpha + 1$.

Clause (b): So toward contradiction assume M has the (ψ,χ)-order property where $\psi = \psi(x;y_1,y_2) = [\varphi(x,y_1) \equiv \varphi(x,y_2)]$ and let $\langle(a_\alpha,\langle b_\alpha,c_\alpha\rangle) : \alpha < \chi\rangle$ exemplifies it. Clearly $\langle a_\alpha : \alpha < \chi\rangle$ is without repetitions.

Let $(a_\alpha,b_\alpha,c_\alpha) = (\gamma_1(\alpha),\gamma_2(\alpha),\gamma_3(\alpha)\rangle$ for $\alpha < \chi$; easily as χ is regular, without loss of generality $\langle(\gamma_1(\alpha),\gamma_2(\alpha),\gamma_3(\alpha)) : \alpha < \chi\rangle$ is an indiscernible sequence of triples in the model $(\chi,<)$.

Clearly for $\alpha > \beta, \neg(\gamma_1(\alpha)R\gamma_2(\beta) \equiv \gamma_1(\alpha)R\gamma_3(\beta))$ so necessarily $(\gamma_1(\alpha)R\gamma_2(\beta)) \vee (\gamma_1(\alpha)R\gamma_3(\beta))$ so: $\gamma_1(\alpha)$ is even and $\gamma_2(\beta)$ is odd or $\gamma_3(\beta)$ is odd (but not both). So without loss of generality (as we can interchange $\gamma_2(\beta),\gamma_3(\beta)$ for every β) for every $\alpha < \chi$ we have $\gamma_1(\alpha)$ is even, $\gamma_2(\alpha)$ is odd and $\gamma_3(\alpha)$ is even.

Then for $\alpha,\beta < \lambda$ we have $M \models$ "$\neg\gamma_1(\alpha)R\gamma_3(\beta)$", i.e. $M \models$ "$\neg\varphi[a_\alpha,c_\beta]$" hence $M \models$ "$\psi[a_\alpha,b_\beta,c_\beta] \equiv [\varphi(a_\alpha,b_\beta) \equiv \varphi(a_\alpha,c_\beta)] \equiv \neg\varphi[a_\alpha,b_\beta]$". So as we start with a counterexample, $\alpha < \beta \Leftrightarrow (\gamma_1(\alpha),\gamma_2(\alpha)) \notin R$. Hence by the indiscernibility $\langle b_\beta : \beta < \chi\rangle$ is without repetitions hence it is increasing hence $\sup\{\gamma_2(\beta) : \beta < \chi\} = \chi$ hence for some $\alpha < \beta$ we have $\gamma_1(\alpha) < \gamma_2(\beta)$ hence $(a_\alpha,b_\beta) = (\gamma_1(\alpha),\gamma_2(\beta)) \in R^M$, contradiction.]

1.25 Exercise: 0) If M has the $(\varphi(\bar{x};\bar{y};\bar{z}),\beta)$-order property and $\alpha < \beta$ then M has the $(\varphi(\bar{x};\bar{y},\bar{z}),\alpha)$-order property.

1) In 1.15 replace the cardinals μ_1,μ_2 by ordinals.

2) Assume $(x,y) \in \{$(es,r),(rs,i)$\}$ and $\beta \geq 1+\alpha$,
   or $(x,y) \in \{$(eb,es),(eb,rs)$\}$ and $\beta \geq \alpha$,
   or $(x,y) \in \{$(es $*$ r,i), (r $*$ es $*$ r,i) and $\beta \leq 1+2\alpha\}$.

If M has the $(\Delta^x,\alpha+\beta)$-order property, then M has the (Δ^y,β)-order property.

[Hint: see 1.23(1)'s proof.]

3) Assume $(x,y) \in \{$(r,i),(i,r)$\}$, $\beta \geq \alpha+1$ or $(x,y) \in \{$(es,i),(rs,r)$\}$, $\beta \geq 1+2\alpha+1\}$. If M has the (Δ^x,α)-non-order property then M has the $(\Delta^y;\beta)$-non-order property.

4) Assume $\psi = \psi(\bar{x};\bar{y};\bar{z}) \equiv \varphi = \varphi(\bar{y};\bar{x};\bar{z})$; if M has the $(\psi,2\alpha+1)$-order property then M has the $(\neg\varphi,\alpha)$-order property.

1.26 Exercise Prove that under the following definition we have better bounds in the parallel to 1.25.

We say that M has the $(\varphi(\bar{x},\bar{y},\bar{z}),\gamma)$-order$'$ property when: there are sequences $\bar{c}, \bar{a}_\alpha, \bar{b}_\alpha$ (for $\alpha < \gamma$) from M (with $\ell g(\bar{c}) = \ell g(\bar{z}), \ell g(\bar{a}_\alpha) = \ell g(\bar{x}), \ell g(\bar{b}_\alpha) = \ell g(\bar{y})$ such that for $\alpha, \beta < \gamma$ we have:

(a) $M \models \varphi[\bar{a}_\alpha, \bar{b}_\beta, \bar{c}]$ if $\alpha < \beta < \gamma$
(b) $M \models \neg\varphi[\bar{a}_\alpha, \bar{b}_\beta, \bar{c}]$ if $\beta < \alpha < \gamma$.

Remark. This (1.26) is an alternative to Definition 1.1(1) and imitate the other parts of Definition 1.1.

§2 Convergent and Indiscernible Sets

2.1 Definition. We say that $\{\bar{a}_t : t \in I\}$ is a (Δ, χ)-convergent set inside M <u>when</u> for every $\bar{c} \in M$ (of suitable length, i.e., $\ell g(\bar{a}_t) + \ell g(\bar{c}) = \ell g(\bar{x})$ for some $\varphi(\bar{x}) \in \Delta$ or $\varphi(\bar{x}, \bar{y}) \in \Delta$, $\ell g(\bar{a}_t) = \ell g(\bar{x})$, $\ell g(\bar{c}) = \ell g(\bar{y})$), for all but $< \chi$ members $t \in I$ the type $\mathrm{tp}_\Delta(\bar{a}_t \hat{\ } \bar{c}, \Delta, M)$ (i.e., the Δ-type which $\bar{a}_t \hat{\ } \bar{c}$ realizes inside M) is constant (so of course we demand that all $\bar{a}_t$ have the same length). We also demand, of course $|I| \geq \chi$.

2.2 Remark. 1) In the first order case we were able to show that if T is stable and I is an infinite set of indiscernibles then I admits an average. Here, we do not know this. Fortunately we have a reasonable replacement: we show in 2.8 below that if M does not have the $(\Delta^{\mathrm{eb}}, \chi^+)$-order property then each sufficiently long indiscernible sequence from M contains a (Δ, χ^+)-convergent subsequence. Originally in the first order case we were interested in the existence of indiscernible sets, but in fact we use quite extensively their being convergent. So we will be more interested in the existence of convergent sets here.
2) We can in Definition 2.1 demand that all but $< \chi$ for each formula separately; no real difference here.

2.3 Claim. *1) Assume that Δ is $(< \kappa)$-variable closed (see Definition 1.18(3)) and $\mathbf{I} = \{\bar{a}_t : t \in I\}$ is (Δ, χ)-convergent in the model $M, \ell g(\bar{a}_t) < \kappa, |I| > \chi, \chi$ regular, $|\Delta| < \chi, A \subseteq M$ and*

$[\varphi(\bar{x}) \in \Delta \Rightarrow |A|^{|\ell g\, \bar{x}|} < \chi]$. Then there is $J \subseteq I, |J| = |I|$ such that $\{\bar{a}_t : t \in J\}$ is a Δ-indiscernible set over A in the model M. Moreover, if $I_0 \subseteq I, |I_0| < |I|$ then we can demand that $\langle \bar{a}_t : t \in J\rangle$ is Δ-indiscernible over $A \cup \{\bar{a}_s : s \in u\}$ for every finite $u \subseteq I_0$.
2) Assume that $n(*) \leq \omega, \Delta$ is a set of formulas $\varphi(\bar{x}), \alpha$ an ordinal and for $n < n(*)$ we have $[\bigwedge_{\ell=1}^{n} \ell g(\bar{x}_\ell) = \alpha \ \& \ \varphi(\bar{x}_n, \dots, \bar{x}_1, \bar{y}) \in \Delta \Rightarrow \varphi(\bar{x}_{n-1}, \dots, \bar{x}_1, \bar{x}_n, \bar{y}) \in \Delta\}]$.

If $\{\bar{a}_t : t \in I\}$ is (Δ, χ)-convergent in the model $M, \ell g(\bar{a}_t) = \alpha, |I| > \chi, \chi$ regular, $|I| > \chi, \chi$ regular, $|\Delta| < \chi$ and $\mathbf{J}$ a set of sequences from M each of length α, and $\mathbf{J}$ of cardinality $\leq \chi$ then for some subset J of I of cardinality $|I|$ the set $\{\bar{a}_t : t \in J\}$ is $(\Delta, < n(*))$-indiscernible over $\mathbf{J}$ inside M.

Remark. 1) Why do we require "finite u"? As maybe $|I_0|^{\aleph_0} \geq |I|$.
2) Assume Δ is $(< \kappa)$-variable closed (see 1.18(3)), $\kappa > \aleph_0$ and $\langle \bar{a}_t : t \in I\rangle$ is Δ-indiscernible over A in M and $\alpha = \ell g(\bar{a}_t) \geq \omega$ for $t \in I$. In this case if $I_0 \subseteq I$ is infinite it does not follow that $\langle \bar{a}_t : t \in I \backslash I_0\rangle$ is indiscernible over $\cup\{\bar{a}_t : t \in I_0\}$. The reason is that there may be a formula $\varphi(x_0, \bar{x}_1, \dots) \in \Delta$ with $\ell g(\bar{x}_n) = \alpha$ such that $\varphi(-, \bar{a}_{t_0}, \bar{a}_{t_1}, \dots)$ divide $\{\bar{a}_t : t \in I \backslash I_0\}$ to two infinite sets. The point is that "Δ-indiscernible" means "(Δ, n)-indiscernible for every n".

Proof. 1) Let $\lambda = |I|$ and so by renaming without loss of generality $I = \lambda$. First we consider the case λ is regular.

For any finite $u \subseteq \lambda$ let $\bar{c}_u = \bar{a}_{\alpha_0} \hat{} \dots \hat{} \bar{a}_{\alpha_{|u|-1}}$ when $\{\alpha_0, \dots, \alpha_{|u|-1}\}$ list u in increasing order and let $w_u \in [\lambda]^{<\chi}$ be such that the sequence $\langle \mathrm{tp}_\Delta(\bar{a}_t \hat{} \bar{c}_u, A, M) : t \in \lambda \backslash w_u\rangle$ is constant (exists by Definition 2.1 and the assumptions). Let $E = \{\delta < \lambda : \delta$ is a limit ordinal such that $u \in [\delta]^{<\aleph_0} \Rightarrow w_u \subset \delta\}$, it is a club of λ. Clearly

$(*)_1$ if $\delta \in E, \delta \leq \alpha_\ell < \lambda$ for $\ell = 1, 2$ and $u \subseteq \delta$ is finite then $\bar{a}_{\alpha_1} \hat{} \bar{c}_u, \bar{a}_{\alpha_2} \hat{} \bar{c}_u$ realize the same Δ-type over A for every finite $u \subseteq \delta$.

Now
Question: Is there an $\alpha^* < \lambda$ such that $\langle \bar{a}_\alpha : \alpha \in E \backslash \alpha^* \rangle$ is an indiscernible set over $A \cup \bar{c}_u$ for every finite $u \subseteq \alpha^*$?

If yes, then we are done (also for the "Moreover"), so assume that the answer is no. So for every $\delta \in E$ there are $n = n(\delta)$ and $\beta_0^\delta < \ldots < \beta_{n-1}^\delta, \gamma_0^\delta < \ldots < \gamma_{n-1}^\delta$ from $E \backslash \delta$ and $k = k_\delta = k(\delta) < \omega, \alpha_0^\delta < \ldots < \alpha_{k-1}^\delta < \delta$ (e.g., $k = 0$) such that $\bar{a}_{\alpha_0^\delta} \hat{} \ldots \hat{} \bar{a}_{\alpha_{k-1}^\delta} \hat{} \bar{a}_{\beta_0^\delta} \hat{} \ldots \hat{} \bar{a}_{\beta_{n-1}^\delta}$ and $\bar{a}_{\alpha_0^\delta} \hat{} \ldots \hat{} \bar{a}_{\alpha_{k-1}^\delta} \hat{} \bar{a}_{\gamma_0^\delta} \hat{} \ldots \hat{} \bar{a}_{\gamma_{n-1}^\delta}$ does not realize the same Δ-type over A. Without loss of generality we choose an example with minimal $n(\delta)$.

By the choice of E

$(*)_2$ (a) if $\beta_{n-1}^\delta \leq \beta < \lambda$ then $\bar{a}_{\alpha_0^\delta} \hat{} \ldots \hat{} \bar{a}_{\alpha_{k-1}^\delta} \hat{} \bar{a}_{\beta_0^\delta} \hat{} \ldots \hat{} \bar{a}_{\beta_{n-2}^\delta} \hat{} \bar{a}_{\beta_{n-1}^\delta}$ and
$\bar{a}_{\alpha_0^\delta} \hat{} \ldots \hat{} \bar{a}_{\alpha_{k-1}^\delta} \hat{} \bar{a}_{\beta_0^\delta} \hat{} \ldots \hat{} \bar{a}_{\beta_{n-2}^\delta} \hat{} \bar{a}_\beta$ realizes the same Δ-type over A
(b) similarly replacing β_ℓ^δ by γ_ℓ^δ.

Hence without loss of generality

$(*)_3$ $\beta_{n(\delta)-1}^\delta = \gamma_{n(\delta)-1}^\delta$
and similarly

$(*)_4$ if $\beta_{n(\delta)-1}^\delta \leq \beta < \lambda \wedge \gamma_{n(\delta)-1}^\delta \leq \beta$ then the tuples
$\bar{a}_{\alpha_0^\delta} \hat{} \ldots \bar{a}_{\alpha_{k(\delta)-1}^\delta} \hat{} \bar{a}_{\beta_0^\delta} \hat{} \ldots \hat{} \bar{a}_{\beta_{n(\delta)-2}^\delta} \hat{} \bar{a}_\beta$ and
$\bar{a}_{\alpha_0^\delta} \hat{} \ldots \hat{} \bar{a}_{\alpha_{k(\delta)-1}^\delta}, \bar{a}_{\gamma_0^\delta} \hat{} \ldots \hat{} \bar{a}_{\gamma_{n(\delta)-2}^\delta} \hat{} \bar{a}_\beta$ realize different Δ-types over A.

For some stationary $S \subseteq E$ we have: $\delta \in S \Rightarrow n(\delta) = n(*)$ & $k_\delta = k(*)$ & $\bigwedge\limits_{\ell < n(*)} \alpha_\ell^\delta = \alpha_\ell^*$ and $\delta \in S \Rightarrow \beta_{n(*)-1}^\delta <$ Min$(S \backslash (\delta + 1))$. For $i < \lambda$ let $\xi(i)$ be the i-th member of S and let $u = \{\alpha_\ell^{\xi(\chi)}, \beta_m^{\xi(\chi)}, \gamma_m^{\xi(\chi)} : \ell < k(*), m < n(*) - 1\}$ so we can find $i < \chi$ such that $[\xi(i), \xi(i+1)) \cap w_u = \emptyset$. By $(*)_3$ clearly

$$\bar{a}_{\alpha_0^{\xi(\chi)}} \hat{} \ldots \hat{} \bar{a}_{\alpha_{k(*)-1}^{\xi(\chi)}} \hat{} \bar{a}_{\beta_0^{\xi(\chi)}} \hat{} \ldots \hat{} \bar{a}_{\beta_{n(*)-2}^{\xi(\chi)}} \hat{} \bar{a}_{\xi(\chi+1)}$$

and

$$\bar{a}_{\alpha_0^{\xi(\chi)}} \hat{} \ldots \hat{} \bar{a}_{\alpha_{k(*)-1}^{\xi(\chi)}} \hat{} \bar{a}_{\gamma_0^{\xi(\chi)}} \hat{} \ldots \hat{} \bar{a}_{\gamma_{n(*)-2}^{\xi(\chi)}} \hat{} \bar{a}_{\xi(\chi+1)}$$

realize different Δ-types over A hence this holds also if we replace $\xi(\chi+1)$ by $\xi(i)$, but this contradicts the choice on $n(\xi(\chi))$. So we have proved the claim in the case λ is regular.

Second, if λ is singular let $\lambda = \Sigma\{\lambda_i : i < \text{cf}(\lambda)\}$ such that $i < \text{cf}(\lambda) \Rightarrow \chi^+ + \text{cf}(\lambda) + \sum_{j<i} \lambda_j < \lambda_i$, and for each $i < \text{cf}(\lambda)$ we can find $J_i \subseteq [\lambda_i, \lambda_i^+)$ of cardinality λ_i^+ such that $\langle \bar{a}_t : t \in J_i\rangle$ is Δ-indiscernible over $A \cup \bar{c}_u$ for every finite $u \subseteq \lambda_i$ (by the case of the claim we have already proved).

Let $I'_i \subseteq J_i$ be of cardinality χ, so $\cup\{I'_i : i < \text{cf}(\kappa)\} \leq \text{cf}(\lambda)+\chi < \lambda_0$. Now we can find $J'_i \subseteq J_i$ of cardinality λ_i^+ such that $\langle \bar{a}_t : t \in J'_i\rangle$ is Δ-indiscernible over $A \cup \bar{c}_u$ for every finite $u \subseteq \cup\{I'_j : j < \text{cf}(\lambda)\} \cup \lambda_i$ (by the case of the claim we have already proved). Now it is easy to check that for each $i < \text{cf}(\lambda)$ the set $\{\bar{a}_t : t \in J'_i\}$ is Δ-indiscernible over $\cup\{\bar{a}_t : t \in u\} \cup A$ for every finite $u \subseteq \cup\{J'_j : j < \text{cf}(\lambda)$ and $j \neq i\}$. Now recalling that $\{\bar{a}_t : t \in I\}$ is Δ-convergent we see that $J = \cup\{J'_i : i < \text{cf}(\lambda)\}$ is as required.

2) Left to the reader. $\square_{2.3}$

2.4 Remark. If $\mathbf{I}$ is a (Δ_i, χ)-convergent inside M for $i < \alpha$, and $\text{cf}(\chi) > |\alpha|$ <u>then</u> $\mathbf{I}$ is a $(\bigcup_{i<\alpha} \Delta_i, \chi)$-convergent inside M. Also obvious monotonicity holds, and (Δ, χ)-convergence implies $(\Delta^{\text{i,es,cn}}, \chi)$-convergence.

2.5 Remark. We can define something similar to 2.1 for sequences instead of sets (so we have that $\text{tp}_\Delta(\bar{a}_t{}^\frown\bar{c}, \emptyset, M)$, divide $\mathbf{I}$ into $< \chi$ convex subsets); but no need arises here.

2.6 Definition. For $\mathbf{I}$ which is a (Δ, χ)-convergent set inside M, and $A \subseteq |M|$, define $\text{Av}_\Delta(\mathbf{I}, A, M) = \{\varphi(\bar{x}, \bar{c}) : \bar{c} \in A$ and $\varphi(\bar{x}, \bar{y}) \in \Delta$ are such that for at least $|\mathbf{I}|$ sequences $\bar{a} \in \mathbf{I}$ we have $M \models \varphi[\bar{a}, \bar{c}]\}$. Of course, all members of $\mathbf{I}$ have the same length.

Note that the definition of the average does not depend on χ.

2.7 Fact. If $\mathbf{I}$ is (Δ, χ)-convergent inside $M, A \subseteq M$ and $[\bar{a} \in \mathbf{I} \Rightarrow \ell g(\bar{a}) = \alpha]$ <u>then</u> $p := \text{Av}_\Delta(\mathbf{I}, A, M)$ satisfies the requirements on being from $\mathrm{Sfr}^{\alpha}_{\Delta}(A, M)$ but may not be realized in M. However, if Λ is a set of formulas of the form $\varphi(\bar{x}, \bar{a}), \ell g(\bar{x}) = \alpha, \bar{a} \in {}^{\ell g(\bar{y})}A$ and $\varphi(\bar{x}, \bar{y}) \in \Delta$ and $|\Lambda| <$ cf$(|I|)$ or $|\Lambda| < \chi^+ + |I|$ <u>then</u> $p \cap \Lambda \in \mathbf{S}^{\alpha}_{\Delta}(\Lambda, M) := \{p \cap \Lambda : p \in \mathbf{S}^{\alpha}_{\Delta}(M, M)\}$ and $p = \cup\{p \cap \Lambda : \Lambda \text{ as above}\}$.

Proof. By the assumption on $\mathbf{I}$, if $\varphi(\bar{x}; \bar{y}) \in \Delta$ and $\bar{c} \in A$ then exactly one of $\varphi(\bar{x}; \bar{c}), \neg\varphi(\bar{x}; \bar{c})$ belongs to $\mathrm{Av}_{\Delta}(\mathbf{I}, A, M)$. Also the second sentence is easy. $\square_{2.7}$

2.8 The convergent set existence theorem. *Suppose M has the $(\Delta^{\mathrm{eb}}, \chi^+)$-non-order property, $\mu = \mu^{\chi} + 2^{2^{\chi}}$ and $|\Delta| \leq \chi$ and* arity$(\Delta) \leq \chi^+$, *see 0.7(10).*

1) Let $\mathbf{I}$ be a family of β^*-sequences from M, $\beta^* \leq \kappa\,(\leq \chi)$ and $|\mathbf{I}| = \mu^+$; <u>then</u> there is $\mathbf{J} \subseteq \mathbf{I}$ such that:

- (i) $|\mathbf{J}| = \mu^+$
- (ii) $\mathbf{J}$ *is (Δ, χ^+)-convergent (and Δ-indiscernible sequence if Δ is $(< \kappa^+)$-variable closed).*

2) If $\mathbf{I} = \{\bar{a}_\alpha : \alpha < \mu^+\}$, <u>then</u> there is a closed unbounded $E \subseteq \mu^+$, and a function h on μ^+ which is regressive (i.e., $h(\alpha) < 1 + \alpha$) such that for every $i < \mu^+$ the set $\mathbf{J}_i = \{\bar{a}_\alpha : \alpha \in E, h(\alpha) = i, \mathrm{cf}(\alpha) > \chi\}$, when not empty, is (Δ, χ^+)-convergent (and Δ-indiscernible sequence if Δ is $(< \kappa^+)$-variable closed) .

3) If we replace "$|\Delta| \leq \chi$" by "$\mu^{|\Delta|} = \mu$", we still get a $(\Delta, \chi^+ + |\Delta|^+)$-convergent (and Δ-indiscernible) $\mathbf{J}$.

2.9 Remark. Alternatively we could have demanded just "M has $(\Delta^{\mathrm{i,r}}, \chi^+)$-non-order" when $\mu \geq \beth_3(\chi)$.

Proof. Let $\mathbf{I} = \{\bar{a}_\alpha : \alpha < \mu^+\}$. The Δ-indiscernibility in clause (ii) of part (1) can be gotten by applying 2.3 with $\Delta, \mu^+, \chi^+, \kappa^+$ here standing for $\Delta, |I|, \chi, \kappa$ there and similarly for the "and Δ-indiscernibles" in parts (2),(3) hence we can ignore them. Also part (2) implies part (1) and we leave part (3) to the reader, so henceforth we shall deal only with part (2). Clearly it suffices to prove (2) for

Δ a singleton, (by Fodor's lemma as $\mu = \mu^{|\Delta|}$) hence without loss of generality $\Delta = \{\varphi(\bar{x};\bar{y})\}$. Let $\psi = \psi(\bar{y};\bar{x}) := \varphi(\bar{x};\bar{y})$. We choose by induction on $\alpha < \mu^+$ a submodel M_α of M such that:

(a) M_α is increasing continuously (in α), $\bar{a}_\alpha \in M_{\alpha+1}$ and M_α has cardinality $\leq \mu$

(b) Every $p \in \mathbf{S}^{\ell g(x)}_\varphi(M_\alpha, M) \cup \mathbf{S}^{\ell g(\bar{y})}_\psi(M_\alpha, M)$ is realized in M by some sequence from $M_{\alpha+1}$.

This is possible - for clause (b) use 1.19 because by Fact 1.23 the model M has the $(\{\varphi\}^{\rm es}, \chi^+)$-non-order property and the $(\{\varphi\}^{\rm rs}, \chi^+)$-non-order property hence by 1.19, clause (b) above deals with $\leq \mu$ types. Now for every $\alpha < \mu^+$, if $\mathrm{cf}(\alpha) > \chi$ then (by (a),(b) and 1.13(2)) we have $M_\alpha \leq^*_{\sum_{\chi,\chi}(\varphi)} M$. So by 1.12 there is $\Gamma_\alpha \subseteq \{\varphi(\bar{a},\bar{y}) : \bar{a} \in |M_\alpha|, \ell g(\bar{x}) = \ell g(\bar{a})\}$ of cardinality $\leq \chi$ such that $\mathrm{tp}_\varphi(\bar{a}_\alpha, M_\alpha, M)$ does not $(\varphi(\bar{x};\bar{y}), \varphi(\bar{x};\bar{y}))$-split over Γ_α. As $\mathrm{cf}(\alpha) > \chi$, there is $h_0(\alpha) < \alpha$ such that $\Gamma_\alpha \subseteq \{\varphi(\bar{c},\bar{y}) : \bar{c} \in M_{h_0(\alpha)}\}$. Now (by straightforward coding and 1.19) for some closed unbounded subset E of μ^+ and regressive h_1, for every $\alpha \in E$ satisfying $\mathrm{cf}(\alpha) > \chi$ the type $\mathrm{tp}_\varphi(\bar{a}_\alpha, M_{h_0(\alpha)+1}, M)$ is determined by $h_1(\alpha)$, and also $h_0(\alpha)$ is determined by $h_1(\alpha)$. Without loss of generality for $\alpha \in E$, if $\mathrm{cf}(\alpha) > \chi$ then $\{\delta : h_1(\delta) = \alpha \text{ and } \mathrm{cf}(\delta) > \chi\}$ is a stationary subset of μ^+.

Now suppose $S \subseteq \{\delta \in E : \mathrm{cf}(\delta) > \chi\}, S \neq \emptyset$ and h_1 is constant on S. We shall prove

$$(*) \qquad \{\bar{a}_\alpha : \alpha \in S\} \text{ is } (\varphi(\bar{x},\bar{y}), \chi^+) - \text{convergent inside } M.$$

Being an indiscernible sequence follows by 3.2 below (and then, e.g. by being convergent it is an indiscernible set).
It is enough for proving $(*)$ to prove the Claim 2.10 below [just define by induction on $i < \mu^+$, $\alpha_0 = \mathrm{Min}(S), \beta_i = \mathrm{Min}(S \backslash \alpha_i), \alpha_{i+1} = \mathrm{Min}(S \backslash (\beta_i + 1))$ (so $\alpha_{i+1} = \beta_{i+1}$), $\alpha_\delta = \bigcup_{i<\delta} \alpha_i$ when δ is a limit ordinal, $M'_i = M_{\alpha_i}$, $\bar{a}'_i = \bar{a}_{\beta_i}$. Clearly $\langle \beta_i : i < \mu^+ \rangle$ list S and $\langle \alpha_i : i < \mu^+ \rangle$ is increasing continuous; $\beta_i \in S$; now we apply 2.10 to the sequence $\langle M'_i, \bar{a}'_i : i < \mu^+ \rangle$ clearly the assumptions of 2.10 hold].

2.10 Claim. *Suppose*

(a) $\mu = \mu^\chi + 2^{2^\chi}$ *and* $\ell g(\bar{x}\hat{\ }\bar{y}) \le \chi$
(b) M *has the* $(\{\varphi(\bar{x},\bar{y})\}^{\text{eb}}, \chi^+)$*-non-order property*
(c) $M_i, i < \mu^+$ *is increasing continuous,* $M_i \subseteq M$
(d) $\bar{a}_i \in M_{i+1}$ *and* $M_{i+1} \le^*_{\sum_{\chi,\chi}(\varphi)} M$
(e) $\psi(\bar{y},\bar{x}) = \varphi(\bar{x},\bar{y})$ *and* $\vartheta(\bar{x},\bar{y}_1\hat{\ }\bar{y}_2) = [\varphi(\bar{x},\bar{y}_1) \equiv \varphi(\bar{x},\bar{y}_2)]$
(f) *every* $p \in \mathbf{S}^{\ell g(\bar{x})}_{\{\varphi(\bar{x},\bar{y})\}}(M_i, M) \cup \mathbf{S}^{\ell g(\bar{y})}_{\{\psi(\bar{y},\bar{x})\}}(M_i, M)$ *is realized in* M *by a sequence from* M_{i+1}
(g) $\|M_i\| \le \mu$ *for* $i < \mu^+$
(h) $\text{tp}_\varphi(\bar{a}_i, M_i, M)$ *does not* (φ,φ)*-split over* Γ *and*

$$\Gamma \subseteq \{\varphi(\bar{c},\bar{y}) : \bar{c} \in M_0\}$$

(i) $\text{tp}_\varphi(\bar{a}_i, M_1, M)$ *is constant for* $i \ge 1$.

<u>*Then*</u> $\{\bar{a}_i : i < \mu^+\}$ *is* $(\{\varphi\}, \chi^+)$*-convergent inside* M

Proof of 2.10. Let $\bar{c} \in M$ be such that $\ell g(\bar{c}) = \ell g(\bar{y})$. We would like to prove that

$$|\{i < \mu^+ : M \models \varphi[\bar{a}_i, \bar{c}]\}| \le \chi$$

<u>or</u>

$$|\{i < \mu^+ : M \models \neg\varphi[\bar{a}_i, \bar{c}]\}| \le \chi.$$

Let $M_{\mu^+} = \bigcup_{i<\mu^+} M_i$.

Now

2.11 Fact. There is a set A of elements and a set E of ordinals such that:

(i) $A \subseteq M_{\mu^+}, E \subseteq \mu^+ + 1$, and $|A| \leq \chi, |E| \leq \chi$

(ii) (a) $i + 1 \in E \Rightarrow i \in E$

(b) if $\delta \in E$ and $\mathrm{cf}(\delta) \leq \chi$ then $\delta = \sup(E \cap \delta)$

(iii) if $\delta \in E$ and $\mathrm{cf}(\delta) > \chi$ then $\mathrm{tp}_\psi(\bar{c}, M_\delta, M)$ does not (ψ, ψ)-split over $A \cap M_\delta$ and $A \cap M_\delta \subseteq M_{\sup(E \cap \delta)}$

(iv) $\mu^+ \in E$.

Proof of 2.11. To see this, choose E_n, A_n by induction on $n < \omega$, both increasing with n as follows:

⊛ (α) $E_0 = \{\mu^+\}$

(β) $i + 1 \in E_n \Rightarrow i \in E_{n+1}$

(γ) if $\delta \in E_n$ and $\mathrm{cf}(\delta) \leq \chi$ then $\delta = \sup(E_{n+1} \cap \delta)$

(δ) if $\delta \in E_n$ and $\mathrm{cf}(\delta) > \chi$ then $\mathrm{tp}_\psi(\bar{c}, M_\delta, M)$ does not (ψ, ψ)-split over $A_{n+1} \cap M_\delta$

(ε) $A_n \cap (M_{i+1} \backslash M_i) \neq \emptyset \Rightarrow i, i + 1 \in E_{n+1}$

(ζ) $A_n \subseteq A_{n+1}$

(η) $E_n \subseteq E_{n+1}$

(θ) $|E_n| + |A_n| \leq \chi$.

For $n = 0$ use clause (α). For $n + 1$, clauses $(\beta) - (\theta)$ tells you to throw in $\leq \chi$ sets, each of cardinality $\leq \chi$. For clause (δ) use Theorem 1.12 and clause (b) of the assumption of 2.10 but note that $\psi(\bar{y}, \bar{x}), \vartheta(\bar{y}, \bar{x}_2{}^\frown\bar{x}_2) := [\psi(\bar{y}, \bar{x}_1) \equiv \psi(\bar{y}, \bar{x}_2)]$ here stand for φ, ψ in 1.12. Now $\bigcup\limits_{n<\omega} E_n$, $\bigcup\limits_{n<\omega} A_n$ are as required in Fact 2.11. E.g. for clause (iii), if $\delta \in E$ and $\mathrm{cf}(\delta) \geq \chi$ and we let $\alpha = \sup(\delta \cap E)$, it is $< \delta$ as $\mathrm{cf}(\delta) > \chi$ and the non-splitting demands holds by clause (δ) of ⊛, and "$A \cap M_\delta \subseteq M_{\sup(E \cap \delta)}$ holds by clause (ε) of ⊛. $\square_{2.11}$

Continuation of the Proof of 2.10. By clause (f) of the assumption of 2.10 for each $i < \mu^+$ we can choose $\bar{c}_i \in M_{i+1}$ realizing $\mathrm{tp}_\psi(\bar{c}, M_i, M)$ inside M; so if $\bar{c} \subseteq M_i$ then $\bar{c}_i = \bar{c}$.

Now E divides $(\mu^+ + 1)\backslash E$ naturally into $\leq \chi$ intervals. That is for $\alpha \in E$ we let $I_\alpha := \{i < \mu^+ : i \notin E \text{ and } \alpha = \text{Min}\{j : i < j \in E\}\}$. We first show that "$M \models \varphi[\bar{a}_i, \bar{c}]$" has constant truth value on each interval, then we prove that all intervals give the same answer. Note that $I_\alpha \neq \emptyset$ implies that α is a limit ordinal of cofinality greater than χ.

First part.

Let $\delta_1 \in E$ be such that $\text{cf}(\delta_1) > \chi$ and $\delta_0 = \min\{\gamma : [\gamma, \delta_1) \cap E = \emptyset\}$. Now δ_0 is not necessarily a limit ordinal (but have we added in $\circledast$ above also $0 \in E_0, [i \in E_n \Rightarrow i + 1 \in E_{n+1})$ it would be). So $I_{\delta_1} = \{i : \delta_0 \leq i < \delta_1\}$.

Remember:

(A) $\text{tp}_\psi(\bar{c}, M_{\delta_1}, M)$ does not (ψ, ψ)-split over $A \cap M_{\delta_1}$
[Why? By clause (iii) of 2.11.]

(B) $A \cap M_{\delta_1} \subseteq M_{\delta_0}$
[Why? By clause (iii) of 2.11 as δ_1 is $\geq \sup(E \cap \delta_1)$.]

(C) $\langle \text{tp}_\varphi \langle (\bar{a}_i, M_i, M) : i < \mu^+ \rangle$ is increasing with i
[Why? By (h) + (i) + (f) of 2.10 by 1.10(2).]
hence

(D) $\delta_0 \leq i, j < \delta_1 \Rightarrow \text{tp}_\varphi(\bar{a}_i, M_{\delta_0}, M) = \text{tp}_\varphi(\bar{a}_j, M_{\delta_0}, M)$

Together $\varphi(\bar{a}_i, \bar{c}) \equiv \varphi(\bar{a}_j, \bar{c})$, we mean of course $M \models$ "...".

Second part.

Let $\delta_0 < \delta_1$ and $\delta_2 < \delta_3$ be such that $\delta_1, \delta_3 \in E$, $\text{cf}(\delta_1)$,$\text{cf}(\delta_3) > \chi, \delta_0 = \min\{\gamma < \delta_1 : [\gamma, \delta_1) \cap E = \emptyset\}$ and $\delta_2 = \min\{\gamma : [\gamma, \delta_3) \cap E = \emptyset\}$. We would like to prove $\varphi(\bar{a}_{\delta_0}, \bar{c}) \equiv \varphi(\bar{a}_{\delta_2}, \bar{c})$. Suppose not and, possibly exchanging (δ_0, δ_1) with (δ_2, δ_3) we have

(α) $\varphi(\bar{a}_{\delta_0}, \bar{c})$ & $\neg\varphi(\bar{a}_{\delta_2}, \bar{c})$.

Then

(β) $i \in [\delta_0, \delta_1) \Rightarrow \varphi(\bar{a}_i, \bar{c})$

[Why? By the first part]

(γ) if $i < j$ are both in $[\delta_0, \delta_1)$ then $\varphi(\bar{a}_i, \bar{c}_j)$

[Why? By the choice of $\bar{c}_j$ and clause (β)].

(δ) $j < \alpha \leq \beta < \mu^+ \Rightarrow \varphi(\bar{a}_\alpha, \bar{c}_j) \equiv \varphi(\bar{a}_\beta, \bar{c}_j)$.

[Why? As $\mathrm{tp}_\varphi(\bar{a}_\alpha, M_\alpha, M)$ is increasing with α by clauses (h) + (i) and (f) of the assumptions of Claim 2.10, by Fact 1.10(2)].

(ε) $j_1, j_2 < \alpha < \mu^+ \Rightarrow \varphi(\bar{a}_\alpha, \bar{c}_{j_1}) \equiv \varphi(\bar{a}_\alpha, \bar{c}_{j_2})$

[Why? As $\mathrm{tp}_\varphi(\bar{a}_\alpha, M_\alpha, M)$ does not (φ, φ)-split over M_0 and $\bar{c}_{j_1}, \bar{c}_{j_2}$ realize $\mathrm{tp}_\psi(\bar{c}, M_0, M)$].

(ζ) if $j_1 < \alpha_1 < \mu^+, j_2 < \alpha_2 < \mu^+$ then $\varphi(\bar{a}_{\alpha_1}, \bar{c}_{j_1}) \equiv \varphi(\bar{a}_{\alpha_2}, \bar{c}_{j_2})$

[Why? Combine (δ) and (ε) using $\varphi(\bar{a}_{\max\{\alpha_1,\alpha_2\}}, \bar{c}_{j_\ell})$ for $\ell = 1, 2$ as intermediates]

(η) $j \in [\delta_2, \delta_3) \Rightarrow \neg\varphi(\bar{a}_j, \bar{c})$

[Why? By first part and the assumption (see clause (α)) that $\neg\varphi(\bar{a}_{\delta_2}, \bar{c})$]

(θ) if $j < \alpha$ and both are in $[\delta_2, \delta_3)$ then $\neg\varphi(\bar{a}_j, \bar{c}_\alpha)$

[Why? By combining (η) and "$\bar{c}_\alpha$ realizes $\mathrm{tp}_\psi(\bar{c}, M_j, M)$"].

Now if $\models \varphi[\bar{a}_1, \bar{c}_0]$ then by clauses (ζ) and (θ) the model M has the $(\neg\varphi(\bar{x}, \bar{y}), \chi^+)$-order property as exemplified by $\langle \bar{a}_{\delta_2+j}, \bar{c}_{\delta_2+j} : j < \chi^+ \rangle$ hence by 1.23(3) the model M has the (ψ, χ^+)-order property; pedantically we should have considered $\langle \bar{a}_{\delta_2+2j+1}, \bar{c}_{\delta_2+2j} : j < \chi^+ \rangle$, similarly below. Second, if $\models \neg\varphi[\bar{a}_1, \bar{c}_0]$ then by clauses (ζ) and (γ) the model M has the (φ, χ^+)-order property exemplified by $\langle \bar{a}_{\delta_0+j}, \bar{c}_{\delta_0+j} : j < \chi^+ \rangle$. As both intervals has cardinality $> \chi$ we get a contradiction as $\varphi, \psi \in \{\varphi\}^{\mathrm{i,r}}$ and apply Fact 1.23(1).]

This completes the proof of the second part. So $\varphi(\bar{a}_j, \bar{c})$ has the same truth value for all $j \in \mu^+ \backslash E$, but $|E| \leq \chi$ so we have finished. $\square_{2.10}$

2.12 Exercise. In Theorem 2.8, replace μ^+ by a (possibly weakly) inaccessible cardinal μ.

2.13 Conclusion. 1) Assume $\mu = \mu^\chi + 2^{2^\chi}$, the model M has the $\{\Delta^{\mathrm{eb}}, \chi^+\}$-non-order property, $|\Delta| \leq \chi$ (or just $\mu^{|\Delta|} = \mu$) and

$\ell g(\bar{x}), \ell g(\bar{y}) < \chi^+$. If $\mathbf{I} \subseteq M$ has cardinality μ^+ <u>then</u> some $\mathbf{J} \subseteq \mathbf{I}$ of cardinality μ^+ is Δ-indiscernible set and (Δ, χ^+)-convergent. Also in 2.8(2) we get an indiscernible set.
2) If in addition $\mu \geq \beth_3|\chi|$ <u>then</u> it is enough in part (1) to demand "M has the $(\Delta^{\mathrm{i,r}}, \chi^+)$-non-order property.

Proof. Just put together 2.8 + 2.3 + 1.15(2) + 3.5 + 3.6. $\square_{2.13}$

<u>2.14 Exercise</u>: Assume that (a)-(i) of Claim 2.10 and $M \leq_\Delta N$ and $\mathrm{tp}_{\{\varphi\}}(\bar{c}, M, N)$ does not (φ, φ)-split over Γ and extend $\mathrm{tp}_{\{\varphi\}}(\bar{c}_i, M_i, M)$ for $i < \mu^+$. <u>Then</u>

- (a) $\mathrm{Av}_{\{\varphi\}}(\{\bar{a}_i : i < \mu^+\}, M, N) = \mathrm{tp}_{\{\varphi\}}(\bar{c}, M, N)$
- (b) $\mathrm{Av}_{\{\varphi(\bar{x},\bar{y})\}}(\{\bar{a}_i : i < \mu^+\}, M, N)$ does not (φ, φ)-split over Γ.

[Hint:

<u>Clause (a)</u>: As in the proof of 2.10, that is assume toward contradiction that $\bar{b} \in {}^{\ell g(\bar{y})}M$ and $N \models$ "$\varphi[\bar{c}, \bar{b}] \equiv \neg\varphi[\bar{c}_i, \bar{b}]$" for every $i < \mu^+$ large enough say $\geq i(*)$. For $i < \mu^+$ choose $\bar{b}_i \in {}^{\ell g(\bar{y})}(M_{i+1})$ realizing $\mathbf{tp}_\psi(\bar{b}, M_i, N)$. Now if $i(*) < i < j < \mu^+$ then $N \models$ "$\varphi[\bar{c}, \bar{b}] \equiv \varphi[\bar{c}, \bar{b}_i]$" because $\mathrm{tp}_{\{\varphi\}}(\bar{c}, M, N)$ does not (φ, φ)-split over Γ and $\mathrm{tp}_{\{\psi\}}(\bar{b}, M_i, M)$ include Γ. Also for $i < j$ from $(i(*), \mu^+)$ we have $N \models$ "$\varphi[\bar{c}, \bar{b}_i] \equiv \varphi[\bar{c}_j, \bar{b}_i]$" as $\bar{c}_j$ realizes $\mathrm{tp}_{\{\varphi\}}(\bar{c}, M_j, N)$. But if $i(*) \leq i < j < \mu^+$ then $N \models$ "$\varphi[\bar{c}_i, \bar{b}] \equiv \varphi[\bar{c}_i, \bar{b}_j]$" by the choice of $\bar{b}_j$. Together M has the $(\{\varphi\}^{\mathrm{i,r}}, \mu^+)$-order property, contradiction by clause (b) of 2.10 and 1.23(1).

<u>Clause (b)</u>: By clause (a) and 1.10(2) (used in the proof of 4.4.]

§3 Symmetry and indiscernibility

3.1 The Symmetry Lemma. *1) Assume M has $(\{\varphi, \neg\varphi\}, \mu)$-non-order, $\ell = 1, 2$, $\mu \leq \mu_\ell \leq \mu'_\ell$, all regular cardinals. Suppose $\mathbf{I}_\ell = \{\bar{a}^\ell_\alpha : \alpha < \mu'_\ell\}$ is (φ_ℓ, μ_ℓ)-convergent inside M where*

$$\varphi = \varphi(\bar{x}; \bar{y}; \bar{z})$$

$$\varphi_1(\bar{x};\bar{y};\bar{z}) = \varphi(\bar{x};\bar{y};\bar{z})$$

$$\varphi_2(\bar{y};\bar{x};\bar{z}) = \varphi(\bar{x};\bar{y};\bar{z})$$

$$\ell g(\bar{a}^1_\alpha) = \ell g(\bar{x}), \ell g(\bar{a}^2_\alpha) = \ell g(\bar{y}).$$

Then for $\bar{c} \in M$; taking the averages of the $\bar{a}^1_\alpha$'s and of the $\bar{a}^2_\beta$'s in $\varphi(\bar{a}^1_\alpha, \bar{a}^2_\beta, \bar{c})$ commutes, that is

$$(\exists^{\geq \mu_1} \alpha < \mu'_1)(\exists^{\geq \mu_2} \beta < \mu'_2)\varphi(\bar{a}^1_\alpha, \bar{a}^2_\beta, \bar{c})$$

if and only if

$$(\exists^{\geq \mu_2} \beta < \mu'_2)(\exists^{\geq \mu_1} \alpha < \mu'_1)\varphi(\bar{a}^1_\alpha, \bar{a}^2_\beta, \bar{c})$$

2) We can omit the assumption "M has the $(\{\varphi, \neg\varphi\}, \mu)$-non-order" if $\min\{\mu_1, \mu_2\} < \max\{\mu'_1, \mu'_2\}$.

Proof. 1) So assume that the conclusion fails. First assume that $(\exists^{\geq \mu_1} \alpha < \mu'_1)(\exists^{\geq \mu_2} \beta < \mu'_2)\varphi(\bar{a}^1_\alpha, \bar{a}^2_\beta, \bar{c})$ holds so $\mathscr{U}_1 := \{\alpha < \mu'_1 : (\exists^{\geq \mu_2} \beta < \mu'_2)\varphi(\bar{a}^1_\alpha, \bar{a}^2_\beta, \bar{c})\}$ has cardinality $\geq \mu_1$. So the other assertion fails, so as $\mu_2 \leq \mu'_2$ the set

$$\mathscr{U}_2 := \{\beta < \mu'_2 : \neg(\exists^{\geq \mu_1} \alpha < \mu'_1)\varphi(\bar{a}^1_\alpha, \bar{a}^2_\beta, \bar{c})\}$$

has cardinality μ'_2. Now we choose a pair $(\alpha_\varepsilon, \beta_\varepsilon)$ by induction on $\varepsilon < \mu$ such that

$\circledast_1$ (*a*) $\alpha_\varepsilon \in \mathscr{U}_1 \backslash \{\alpha_\zeta : \zeta < \varepsilon\}$

(*b*) $\beta_\varepsilon \in \mathscr{U}_2 \backslash \{\beta_\zeta : \zeta < \varepsilon\}$

(*c*) if $\zeta < \varepsilon$ then $\neg\varphi(\bar{a}^1_{\alpha_\varepsilon}, \bar{a}^2_{\beta_\zeta}, \bar{c})$

(*d*) if $\zeta \leq \varepsilon$ then $\varphi(\bar{a}^1_{\alpha_\zeta}, \bar{a}^2_{\beta_\varepsilon}, \bar{c})$.

In stage ε, we are looking for $\alpha_\varepsilon \in \mathscr{U}_1$ and recall $|\mathscr{U}_1| \geq \mu_1$. The demand (a) excludes $|\varepsilon| < \mu \leq \mu_1$ ordinals: $\{\alpha_\zeta : \zeta < \varepsilon\}$. Also for each $\zeta < \varepsilon$, the demand (c) excludes $< \mu_1$ ordinals as for each $\beta \in \mathscr{U}_2$ by the definition of $\mathscr{U}_2$ we have $\neg(\exists^{\geq \mu_1} \alpha < \mu'_1)\varphi(\bar{a}^1_\alpha, \bar{a}^2_\beta, \bar{c})$ hence by $\langle \bar{a}^1_\alpha : \alpha < \mu'_1\rangle$ being (φ_1, μ_1)-convergent we exclude $< \mu_1$. As μ_1 is regular and each $\zeta < \varepsilon$ excludes $< \mu_1$ and $|\varepsilon| < \mu \leq \mu_1$, we exclude $< \mu_1$ members of $\mathscr{U}_1$. So we can choose α_ε.

Similarly we can choose β_ε (we have $|\mathscr{U}_2| = \mu'_2 \geq \mu_2 \geq \mu$ candidates, clause (b) excludes $|\varepsilon| < \mu \leq \mu_1$ candidates and for each $\zeta \leq \varepsilon$ clause (d) excludes $< \mu_2$ candidates (by $\langle \bar{a}^2_\beta : \beta < \mu'_2\rangle$ being (φ_2, μ_2)-convergent). So for $\varepsilon, \zeta < \mu$ we have $\models \varphi[\bar{a}^1_{\alpha_\varepsilon}, \bar{a}^2_{\beta_\zeta}, \bar{c}]$ iff $\varepsilon \leq \zeta$, so $\langle (\bar{a}^1_{\alpha_\varepsilon}, \bar{a}^2_{\beta_\varepsilon} : \varepsilon < \mu\rangle$ exemplify "M has the $(\varphi(\bar{x}, \bar{y}, \bar{c}), \mu)$-order property", contradicting an assumption.

But we have another case: when

$$(\exists^{\geq \mu_2} \beta < \mu'_2)(\exists^{\geq \mu_1} \alpha < \mu'_1)\varphi(\bar{a}^1_\alpha, \bar{a}^2_\beta, \bar{c})$$

holds and so $\neg(\exists^{\geq \mu_1} \alpha < \mu'_1)(\exists^{\geq \mu_2} \beta < \mu'_2)\varphi(\bar{a}^1_\alpha, \bar{a}^2_\beta, \bar{c})$ and we let

$$\mathscr{U}_1 = \{\alpha < \mu'_1 : \neg(\exists^{\geq \mu_2} \beta < \mu'_2)\varphi(\bar{a}^1_\alpha, \bar{a}^2_\beta, \bar{c})\}$$

$$\mathscr{U}_2 = \{\beta < \mu'_2 : (\exists^{\geq \mu_1} \alpha < \mu'_1)\varphi(\bar{a}^1_\alpha, \bar{a}^2_\beta, \bar{c})\}.$$

So by induction on $\varepsilon < \mu$ we choose $\alpha_\varepsilon, \beta_\varepsilon$ such that

$\circledast$ (a) $\alpha_\varepsilon \in \mathscr{U}_1 \backslash \{\alpha_\zeta : \zeta < \varepsilon\}$

(b) $\beta_\varepsilon \in \mathscr{U}_2 \backslash \{\beta_\zeta : \zeta < \varepsilon\}$

(c) if $\zeta < \varepsilon$ then $\varphi(\bar{a}^1_{\alpha_\varepsilon}, \bar{a}^2_{\beta_\zeta}, \bar{c})$

(d) if $\zeta \leq \varepsilon$ then $\neg\varphi(\bar{a}^1_{\alpha_\zeta}, \bar{a}^2_{\beta_\varepsilon}, \bar{c})$.

We continue as above and get that M has the $(\neg\varphi, \mu)$-order property hence by 1.23(3) has the (φ_2, μ)-order property, contradiction.
2) So $\mu_1 < \mu'_2$ or $\mu_2 < \mu'_1$. In each case it is trivial. $\square_{3.1}$

3.2 The indiscernibility/non-splitting Lemma. *Assume*

(a) *for* $i < i(*)$ *let* $\varphi_i(\bar{x}_{n_i};\ldots;\bar{x}_1;\bar{y}_i)$ *be a* $\tau(M)$*-formula,* $\alpha = \ell g(\bar{x}_\ell)$

(b) $\Delta_n = \{\varphi_i(\bar{x}_{n_i};\ldots;\bar{x}_1;\bar{y}) : i < i(*), n_i = n\}$*, and* $\Delta = \bigcup\limits_{m<\omega} \Delta_n$

(c) I *is a linear order*

(d) $\bar{a}_t \in {}^\alpha|M|$ *for* $t \in I$

(e) $A \subseteq |M|$

(f) *for* $t \in I$ *and* n *the type* $p_{t,n} := \mathrm{tp}_\Delta(\bar{a}_t, A \cup \bigcup\limits_{s<_I t} \bar{a}_s, M)$ *or just* $p_{t,n} = \{\varphi_i(\bar{x}, \bar{a}_{t_{n_i-1}},\ldots,\bar{a}_{t_1},\bar{c})^{\mathbf{t}} : i < i(*)$ *satisfy* $n_i = n, \bar{c} \in {}^{\ell g(\bar{y})}A$ *and* $t_1 <_I \ldots <_I t_{n_i-1} <_I t, \mathbf{t}$ *a truth value (or* $\in \{0,1\}$*) and* $M \models \varphi_i[\bar{a}_t;\bar{a}_{t_{n_i-1}};\ldots;\bar{a}_{\bar{t}_1};\bar{c}]^{\mathbf{t}}\}$ *does not split over* $\Gamma_{t,n}$*, where*

  (α) $\Gamma_{t,n} = \{\varphi_i(\bar{x}_{n_i};\ldots;\bar{x}_1;\bar{c}) : i < i(*)$ *satisfies* $n_i = n-1$ *and* $\bar{c} \in A\}$ *and*

  (β) $s <_I t \Rightarrow p_{s,n} \subseteq p_{t,n}$.

<u>Then</u> $\langle \bar{a}_t : t \in I\rangle$ *is a* Δ*-indiscernible sequence over* A.

Proof. Note that if $I' \subseteq I$ and we restrict ourselves to $\langle \bar{a}_t : t \in I'\rangle$ and redefine p_t for $t \in I'$ accordingly, all the assumptions still holds. Hence it suffices to deal with the case I is finite; and we prove this by induction on $|I|$. See the proof of [Sh:c, Lemma 2.5,p.11]. Note the order in Definition 1.7(2); note that if we make Δ close under permuting the variables (or the first n blocks of α variables), things may be clearer. $\square_{3.2}$

<u>3.3 Exercise</u>: In 3.2 we can conclude that $\{a_t : t \in I\}$ is $(\Delta, < n(*))$-indiscernible strictly over $\mathbf{J}$ inside M when:

$(a) - (d)$ as in 3.2

(e) $\mathbf{J}$ is a family of sequences from M

(f) for $n < n(*)$ and $t \in I$ the type $\{\varphi_i(x, \bar{a}_{t_{n_i-1}}, \dots, \bar{a}_{t_1}, \bar{c})^{\mathbf{t}} : i < i(*)$ satisfies $n_i = n; \bar{c} \in \mathbf{J}$ and $t_1 <_I \dots <_I t_{n_i-1} <_I t, \mathbf{t}$ a truth value and $M \models \varphi_i[\bar{a}_t, \bar{a}_{t_{n_i-1}}, \dots, \bar{a}_{t_1}, \bar{c}]^{\mathbf{t}}\}$ does not split over $\Gamma_{t,n}$ where (α, β) are as there.

3.4 Conclusion. Suppose $\varphi(\bar{x}_n, \dots, \bar{x}_1, \bar{y})$ is a $\tau(M)$-formula and for $\ell = 0, \dots, n-1$

$$\varphi_\ell(\bar{x}_n, \dots, \bar{x}_1, \bar{y}) := \varphi(\bar{x}_{n-\ell}, \bar{x}_{n-\ell-1}, \dots, \bar{x}_1, \bar{x}_n, \dots, \bar{x}_{n-\ell+1}, \bar{y})$$

and $\alpha = \ell g(\bar{x}_\ell)$ for $\ell = 1, \dots, n$ and let $\Delta = \{\varphi_\ell : \ell = 0, \dots, n-1\}$.

1) If I is a linear order, $\bar{a}_t \in {}^\alpha M$ for $t \in I, p_t = \text{tp}_\Delta(\bar{a}_t, A \cup \bigcup\limits_{s <_I t} \bar{a}_s, M)$ increases with t, and is finitely satisfiable in A <u>then</u> $\langle \bar{a}_t : t \in I \rangle$ is a Δ-indiscernible sequence over A.

2) Suppose I is a linear order and $\mathbf{J}$ is a family of sequences, $\bar{a}_t \in {}^\alpha|M|$ for $t \in I$ and let $\mathbf{J}_t = \mathbf{J} \cup \{\bar{a}_s : s <_I t\}$. Further assume

$$p_t = \text{tp}_\Delta(\bar{a}_t, \mathbf{J}_t, M) := \big\{\vartheta(\bar{x}, \bar{c}_1, \dots, \bar{c}_k) : \vartheta \in \Delta, \bar{c}_\ell \in \mathbf{J}_t \text{ for } \ell = 1, \dots, k \text{ and } M \models \vartheta[\bar{a}_t, \bar{c}_1, \dots, \bar{c}_k]\big\}$$

is increasing with t and is finitely satisfiable in $\mathbf{J}$. <u>Then</u> $\langle \bar{a}_t : t \in I \rangle$ is a Δ-indiscernible (sequence) over $\mathbf{J}$.

3) We can in (2) replace p_t by

$$p'_t = \{\varphi_\ell(\bar{x}, \bar{a}_{s_{n-1}}, \dots, \bar{a}_{s_1}; \bar{c})^{\mathbf{t}} : s_1 <_I \dots <_I s_{n-1} s \text{ and } \bar{c} \in \mathbf{J}, \mathbf{t} \text{ a truth value (or } \in \{0,1\}) \text{ and } M \models \varphi[\bar{a}_t, a_{s_{n-1}}, \dots, \bar{a}_{s_1}; \bar{c}]^{\mathbf{t}}\}.$$

4) We can replace Δ by a union of such sets, fixing α.

Remark. 1) Of course we can restrict p_t to the set of formulas used, this is done in part (3).

Proof. Easy. $\square_{3.4}$

3.5 Lemma. *Suppose $\langle \bar{a}_i : i < i(*) \rangle$ is a $(\varphi(\bar{x}_n, \dots, \bar{x}_1, \bar{c}), n)$-indiscernible sequence but not an $(\varphi(\bar{x}_n, \dots, \bar{x}_1, \bar{c}), n)$-indiscernible set so $n \geq 2$.*
1) For any permutation π of $\{1, \dots, n\}$, let
$\varphi_\pi(\bar{x}_n, \dots, \bar{x}_1, \bar{z}) := \varphi(\bar{x}_{\pi(n)}, \dots, \bar{x}_{\pi(1)}, \bar{z})$.
Then for some permutation π and $m \in \{0, 1, \dots, n-2\}$, for any j such that $m + j + (n - m - 2) \leq i(*)$ the model M has the $(\varphi_\pi(\bar{x}_m; \bar{x}_{m-1}; \bar{a}_0, \dots, \bar{a}_{m-1}, \bar{a}_{m+j}, \bar{a}_{m+j+1}, \dots, \bar{a}_{m+j+n-m-3}, \bar{c}), j)$-order property.
2) Let $\varphi'_{\pi,m}(\bar{x}^*, \bar{y}^*, \bar{z}^*) = \varphi'_{\pi,m}(\bar{x}_1, \dots, \bar{x}_m; \bar{y}_1, \dots, \bar{y}_m; \bar{z}_{m-1}, \dots, \bar{z}_n, \bar{z}) =$

$\varphi(\bar{x}_{\pi(1)}, \dots, \bar{x}_{\pi(m-1)}, \bar{y}_m; \bar{z}_{m+1}, \dots, \bar{z}_n, \bar{z})$. Then for some π and $m \in \{2, \dots, n\}$, the model M has the $(\varphi'_{\pi,m}(\bar{x}^*; \bar{y}^*; \bar{a}_{m-1}, \dots, \bar{a}_1, \bar{c})$, $(i(*) - m)/(n - m))$-order property.

Proof. By now left to the reader (really by Morley [Mo65] or see [Sh:c, AP 3.9]).

$\square_{3.5}$

3.6 Observation. 1) If $\langle \bar{a}_i : i < i(*) \rangle$ is a Δ-indiscernible sequence over A but not a Δ-indiscernible set over A, then for some formula $\varphi(\bar{x}_n, \dots, \bar{x}_1, \bar{y}) \in \Delta$ with $\ell g(\bar{x}_\ell) = \ell g(\bar{a}_i)$ and $\bar{c} \in {}^{\ell g(\bar{y})}A$ the assumption of 3.5 holds.
2) In (1) we can fix n and use (Δ, n)-indiscernibility.

3.7 Claim. *Suppose $\mathbf{I} = \{\bar{a}_i : i < \lambda\}$ is (Δ, χ^+)-convergent in M, $\ell g(\bar{a}_i) = \alpha$ for $i < \lambda$ and $\mathbf{J}$ is a set of sequences from M. Suppose further that M has (Δ, χ^+)-non-order property and Δ satisfies*

- (*) *if $\varphi(\bar{x}_n, \dots, \bar{x}_1, \bar{y}) \in \Delta$, $\ell g(\bar{x}_\ell) = \alpha$, π a permutation of $\{1, \dots, n\}$ then $\varphi_\pi(\bar{x}_n, \dots, \bar{x}_1, \bar{y}) := \varphi(\bar{x}_{\pi(n)}, \dots, \bar{x}_{\pi(1)}, \bar{y})$ belongs to Δ.*

1) If $\lambda = \mathrm{cf}(\lambda) > \chi + |\mathbf{J}| + \Delta$ <u>then</u> there is $\mathbf{I}' \subseteq \mathbf{I}, |\mathbf{I}'| = \lambda$ such that $\mathbf{I}'$ is a Δ-indiscernible set over $\mathbf{J}$ inside M.
2) In fact there is an algebra N with universe λ and $\leq |\mathbf{J}| + \chi + |\Delta|$ functions such that: if for $\zeta < \lambda, i_\zeta < \lambda, i_\zeta$ not in the N-closure of $\{i_\xi : \xi < \zeta\}$ then $\{a_{i_\zeta} : \zeta < \lambda\}$ is an Δ-indiscernible set over $\mathbf{J}$ inside M.

3.8 Remark. In 3.7:
1) If we assume "$\{a_{i_\zeta} : \zeta < \lambda\}$ is a Δ-indiscernible sequence over $\mathbf{J}$" in M we can weaken $(*)$ to $[\varphi \in \Delta \Rightarrow \varphi_\ell \in \Delta]$ for φ_ℓ as in 3.4.
2) This is a variant strengthening of 2.3.
3) We can replace the role of $(|\mathbf{J}| + \chi + |\Delta|)^+$ by regular χ^* or just $\mathrm{cf}(\chi^*)$ (so the N closure of a finite set has cardinality $< \chi^*$).
4) If we weaken the conclusion to "indiscernible sequence", then we can omit the "non-order" assumption.

Proof of 3.7. 1) by 2) as we can choose $i_\zeta < \lambda$ by induction on $\zeta < \lambda$ such that $i_\zeta \notin c\ell_N(\{i_\xi : \xi < \zeta\})$.
2) We define for $\psi = \psi(\bar{x}_n, \ldots, \bar{x}_1, \bar{z}_m, \ldots, \bar{z}_1) \in \Delta, \bar{c}_\ell \in \mathbf{J}$ (for $\ell = 1, m$), $\ell g(\bar{c}_\ell) = \ell g(\bar{z}_\ell)$ and $\gamma < \chi$ a function $F^\gamma = F^{\psi,\gamma}_{\bar{c}_m,\ldots,\bar{c}_1}$ such that

- $(*)$ for $i_1, \ldots, i_{n-1} < \lambda$ the set $\mathscr{U}_{i_1,\ldots,i_{n-1}} = \{F^\gamma(i_1, \ldots, i_{n-1}) : \gamma < \chi\}$ satisfies
  - (a) it includes $\{i : i < \chi\}$
  - (b) for any $j_1, j_2 \in \lambda \backslash \mathscr{U}_{i_1,\ldots,i_{n-1}}$, we have

$$M \models \text{“}\psi[\bar{a}_{j_1}, \bar{a}_{i_{n-1}}, ..., \bar{a}_{i_1}, \bar{c}_m, ..., \bar{c}_1] \equiv \psi[\bar{a}_{j_2}, \bar{a}_{i_{n-1}}, ..., \bar{a}_{i_1}, \bar{c}_m, ..., \bar{c}_1]\text{”}$$

[this is possible as $\mathbf{I}$ is $(\Delta, < \chi)$-convergent in M].

Now if $\langle i_\zeta : \zeta < \lambda \rangle$ are as in 3.7(2), by 3.4(2) (with $\mathbf{J} \cup \{\bar{a}_i; i < \lambda\}$ here standing for $\mathbf{J}$ there) $\langle \bar{a}_{i_\zeta} : \zeta < \lambda \rangle$ is a Δ-indiscernible sequence over $\mathbf{J}$ which suffices. By 3.5 the sequence $\langle \bar{a}_{i_\zeta} : \zeta < \lambda \rangle$ is a Δ-indiscernible sequence over $\mathbf{J}$. $\square_{3.7}$

§4 What is the appropriate notion of a submodel

We would like a context for which amalgamation exists preferably close to being a.e.c. and with a non-forking notion. For this we need a suitable notion of elementary submodel. Using $M \leq_{\mathscr{L}} N, \mathscr{L}$ a strong logic, is not good enough. For example, for δ a limit ordinal $M_\alpha \leq_{\mathscr{L}} M_\beta \leq_{\mathscr{L}} M$ for $\alpha < \beta < \delta$ does not necessarily imply $\bigcup_{\alpha<\delta} M_\alpha \leq_{\mathscr{L}} M$ and even $M_0 \leq_{\mathscr{L}} \bigcup_{\alpha<\delta} M_\alpha$, an undesirable phenomena. For δ of large cofinality (i.e. $\geq \kappa$) this holds, e.g., for $\mathscr{L} = \mathbb{L}_{\lambda,\kappa}$, but remember that if we can quantify over countable sets concepts become very dependent on the exact set theoretic hypothesis. Our problem is: Find a good notion of an elementary submodel.

We use the following relation: $M <^\kappa_{\Delta,\mu,\chi} N$ saying mainly that for $\alpha < \kappa$ types in $\mathbf{S}^\alpha_\Delta(M,N)$, i.e. the types from $\mathrm{Sfr}^\alpha_\Delta(M,N)$ realized in N are averages of convergent sets, (see 4.1). In Lemma 4.4 we show that if the suitable non-order property holds, then we are dealing with $\leq_{\Sigma_{\mu,(<\kappa)}}$.

4.1 Definition. $M \leq^\kappa_{\Delta,\mu,\chi} N$ <u>when</u> ($\mu > \chi \geq \kappa$ and we let $\Delta_{[<\theta]} := \{\varphi(\bar{x}) \in \Delta : \ell g(\bar{x}) < \theta\}, \Delta_{[\theta]} = \Delta_{[<\theta^+]}$ and):

- (a) $M \subseteq N$
- (b) $M \leq_{\Delta_{[\chi]}} N$, that is for every $\varphi(\bar{x}) \in \Delta_{[\chi]}$ and $\bar{c} \in {}^{\ell g(\bar{x})}M$ we have:
  $M \models \varphi[\bar{c}]$ if and only if $N \models \varphi[\bar{c}]$
- (c) for every $\bar{c} \in N$ satisfying $\ell g(\bar{c}) < \kappa$ there is $\mathbf{I} = \{\bar{c}_i : i < \mu^+\}$, which is (Δ,χ^+)-convergent inside M such that $\mathrm{tp}_\Delta(\bar{c},M,N) = \mathrm{Av}_\Delta(\mathbf{I},M,N)$.

4.2 Remark. 1) Our main case is:

> $\Delta =$ the set of finite quantifier free formulas, $\kappa = \aleph_0$ and μ,χ are related as in Theorem 2.8 and then we omit them and write just $\leq$.

2) We could separate the two roles of Δ, but we have already enough parameters.
3) Similarly we could use μ, χ instead μ^+, χ^+ gaining a little in generality.
4) Many of the "obvious" properties of a candidate for "elementary submodel" here are not so obvious. Some are proved, the failure of some is used in non-structure theorems.
5) We could have demanded $\ell g(\bar{y}) < \kappa$ below. Note that in some of the claims we can use μ^+-convergence, but not so in 4.4.
6) Note that χ below has some roles which we could separate:

(*a*) the part of $\bar{z}$ we can use for $\varphi(\bar{x}, \bar{y}, \bar{z}) \in \Delta$

(*b*) bound the length of $\bar{y}$ above

(*c*) the (Δ, χ^+)-non-order property

(*d*) say how good is the convergence (connected to (c)).

We could have separated.
7) Why do we usually demand $\mu = \mu^\chi$?

This is because the assumption we use is a suitable (Δ, χ^+)-non-order property so we use 1.20 and it requires $\mu = \mu^\chi$. This holds for 2.8, too, though there we could have redefined (Δ, μ)-stability as: if $\mathbf{J}$ is a set of $\leq \mu$ sequences then $|\mathbf{S}_\Delta(\mathbf{J}, M)| \leq \mu$. But reorganizing, so it seems, do not get material gains.

4.3 Observation. If $M \leq^\kappa_{\Delta,\mu,\chi} N$ <u>then</u> $M \leq_{\Sigma_{\mu,\chi,<\kappa}(\Delta)} N$; see Definition 0.8(7),(8).

Proof. Without loss of generality $\Delta = \Delta_{[\chi]}$, just check the definitions, so $M \leq_\Delta N$ by clause (b) of Definition 4.1.

Let $\varphi(\bar{y}, \bar{z}) = (\exists \bar{x}) \bigwedge\limits_{\alpha<\mu} \varphi_\alpha(\bar{x}, \bar{y}, \bar{z} \restriction w_\alpha)^{\eta(\alpha)}$ where $\varphi_\alpha \in \Delta, \eta \in {}^\mu 2, \ell g(\bar{y}) \leq \chi, \ell g(\bar{x}) < \kappa, w_\alpha \subseteq \ell g(\bar{z})$ such that $|w_\alpha| \leq \chi$ for every $\alpha < \mu$. First, if $M \models \varphi[\bar{b}, \bar{c}]$ then some $\bar{a} \in {}^{\ell g(\bar{x})}M$ witnesses it and it witnesses also $N \models \varphi[\bar{b}, \bar{c}]$. Second, assume that $N \models \varphi[\bar{b}, \bar{c}]$. Hence there is $\bar{a} \in {}^{\ell g(\bar{x})}N$ such that $N \models \bigwedge\limits_{\alpha<\mu} \varphi_\alpha(\bar{a}, \bar{b}, \bar{c} \restriction w_\alpha)^{\eta(\alpha)}$. Apply clause (c) of Definition 4.1 to $(M, N, \bar{a})$ so there is a set $\mathbf{I} = \{\bar{a}_i : i < \mu^+\}$ which is (Δ, χ^+)-convergent inside M and $\mathrm{tp}_\Delta(\bar{a}, M, N) = \mathrm{Av}_\Delta(\mathbf{I}, M, N)$. Now for each $\alpha < \mu$ there is $i_\alpha < \mu^+$ such that $i \in [i_\alpha, \mu^+) \Rightarrow N \models$ "$\varphi_\alpha[\bar{a}_i, \bar{b}, \bar{c} \restriction w_\alpha] \equiv \varphi_\alpha[\bar{a}, \bar{b}, \bar{c} \restriction w_\alpha]$".

So if $i = \cup\{i_\alpha : \alpha < \mu\}$, then $\bar{a}_i$ witness $N \models \varphi[\bar{b}, \bar{c}]$ but $\bar{a}_i \in {}^{\ell g(\bar{x})}M$ so as $M \leq_\Delta N$ we get that $\bar{a}_i$ witness $M \models \varphi[\bar{b}, \bar{c}]$. $\square_{4.3}$

4.4 Lemma. *Suppose $\mu = \mu^\chi + 2^{2^\chi}, \kappa \leq \chi^+, |\Delta| \leq \chi, [\varphi(\bar{x}) \in \Delta \Rightarrow \ell g(\bar{x}) \leq \chi]$, and M has the $(\Delta^{\mathrm{eb}}, \chi^+)$-non-order property.*
Then: $M \leq^\kappa_{\Delta,\mu,\chi} N$ if and only if $M \leq_{\sum_{\mu,\chi,(<\kappa)}(\Delta)} N$.

Remark. In $\exists \bar{x} \bigwedge_{\alpha<\mu} \varphi_\alpha(\bar{x}, \bar{y}, \bar{z} \restriction w_\alpha)$, the cardinal κ is used for $\ell g(\bar{x}) < \kappa$ and χ is used to bound the length of $\bar{y}$ and of the number of variables from $\bar{z}$ really appearing in φ_α.

Proof. The direction $\Rightarrow$ holds by 4.3. For the other direction, easily clauses (a),(b) of Definition 4.1 holds (and $\Delta = \Delta_{[\chi]}$), so it suffices to deal with clause (c). So let $\bar{c} \in {}^\alpha N$, $\alpha < \kappa$. Let $\Delta = \{\varphi_\varepsilon(\bar{x}; \bar{y}_\varepsilon) : \varepsilon < \varepsilon(*)\}$ and $\varepsilon(*) \leq \chi$ and let $\psi_\varepsilon(\bar{y}_\varepsilon, \bar{x}) = \varphi_\varepsilon(\bar{x}, \bar{y}_\varepsilon)$. By 1.12 for some $\Gamma_\varepsilon \subseteq \{\varphi_\varepsilon(\bar{a}, \bar{y}) : \bar{a} \in M\}, |\Gamma_\varepsilon| \leq \chi$ and $\mathrm{tp}_{\varphi_\varepsilon(\bar{x};\bar{y})}(\bar{c}, M, N)$ does not $(\varphi_\varepsilon(\bar{x}, \bar{y}), \varphi_\varepsilon(\bar{x}, \bar{y}))$-split over Γ_ε.

Let $\Gamma = \cup\{\Gamma_\varepsilon : \varepsilon < \varepsilon(*)\}$ so $\mathrm{tp}_\Delta(\bar{c}, M, N)$ does not (Δ, Δ)-split over Γ. By induction on $i < \mu^+$ choose $M_i, \bar{c}_i$ such that Γ is over $M_0, M_i \subseteq M, j < i \Rightarrow M_j \subseteq M_i, \|M_i\| \leq \mu$, for $\varepsilon < \varepsilon(*)$ every q such that $q \in \mathbf{S}^{\ell g(\bar{x})}_{\varphi(\bar{x},\bar{y})}(M_i, N)$ or $q \in \mathbf{S}^{\ell g(\bar{y}_\varepsilon)}_{\psi_\varepsilon(\bar{y},\bar{x})}(M_i, N)$ is realized inside M by some sequence from M_{i+1} and $\bar{c}_i \in M_{i+1}$ realizes $\mathrm{tp}_{\varphi_\varepsilon(\bar{x},\bar{y})}(\bar{c}, M_i, N)$ in N for each $\varepsilon < \varepsilon(*)$. This is clearly possible by 1.19. As $M \leq_\Delta N$, clearly $\mathrm{tp}_\Delta(\bar{c}_i, M_i, M) = \mathrm{tp}_\Delta(\bar{c}_i, M_i, N) \subseteq \mathrm{tp}_\Delta(\bar{c}, M, N)$ hence also $\mathrm{tp}_\Delta(\bar{c}_i, M_i, N)$ does not (Δ, Δ)-split over Γ. Now by 2.10 the set $\mathbf{I} := \{\bar{c}_\alpha : \alpha < \mu^+\}$ is $(\{\varphi_\varepsilon(\bar{x}, \bar{y})\}, \chi^+)$-convergent in M for each $\varepsilon < \varepsilon(*)$ hence recalling $\mu = \mu^\chi$ also is (Δ, χ^+)-convergent in M. Hence by Fact 2.7 the type $q = \mathrm{Av}_\Delta(\mathbf{I}, M, M)$ is well defined and by 1.10(2) is equal to $\mathrm{Av}_\Delta(\mathbf{I}, M, N)$ which belongs to $\mathrm{Sfr}^{\ell g(\bar{x})}_\Delta(M, N)$.

Now the types q and $\mathrm{tp}_\Delta(\bar{c}, M, N)$ are both from $\mathrm{Sfr}^{\ell g(\bar{x})}_\Delta(M, N)$, do not (Δ, Δ)-split over M_{χ^+}.
[Why? The type $\mathrm{tp}_\Delta(\bar{c}, M, N)$ by the choice of Γ, the type q by clause (a) of 2.14.] Also they have the same restriction to $M_{\chi^+ +1}$ which is $\mathrm{tp}_\Delta(c_i, M_{\chi^+ +1} N)$ for every $i \in (\chi^+ + 1, \mu^+)$. Hence by 1.10(2) they are equal. So we finish the second direction.

$\square_{4.4}$

4.5 Conclusion. For $\kappa, \Delta, \mu, \chi$ as in 4.4, on the class of models with the $(\Delta^{\mathrm{eb}}, \chi^+)$-non-order property, the relation $\leq^\kappa_{\Delta,\mu,\chi}$ is transitive. $\square_{4.5}$

Proof. Because $\leq_{\sum_{\mu,\chi,(<\kappa)}(\Delta)}$ is transitive. $\square_{4.5}$

4.6 Claim. *1) Assume $M_1 \leq^\kappa_{\Delta,\mu,\chi} M_2$ and $\alpha < \kappa$ and $\mathbf{I} \subseteq {}^\alpha(M_1)$ has cardinality μ^+. Then $\mathbf{I}$ is (Δ, χ^+)-convergent inside M_1 iff $\mathbf{I}$ is (Δ, χ^+)-convergent inside M_2.*
2) If $\mathbf{I}_1, \mathbf{I}_2$ are (Δ, χ^+)-convergent inside M_1, $M_1 \leq^\kappa_{\Delta,\mu,\chi} M_2$, $|\mathbf{I}_\ell| = \mu^+, \mathbf{I}_\ell \subseteq {}^{\kappa>}(M_1)$ then: $\mathrm{Av}_\Delta(\mathbf{I}_1, M_1) = \mathrm{Av}_\Delta(\mathbf{I}_2, M_1)$ *iff* $\mathrm{Av}_\Delta(\mathbf{I}_1, M_2) = \mathrm{Av}_\Delta(\mathbf{I}_2, M_2)$.
3) In part (1) we can replace χ^+ by λ when $\chi < \lambda \leq \mu$. Also we can replace χ^+ by $\lambda = \mu^+$ if M_1 has the $(\Delta^{i,\mathrm{cn}}, \mu^+)$-non-order property.
4) In part (2) we can replace χ^+ by $\chi < \lambda \leq \mu$. If M_1 has the $(\Delta^{i,\mathrm{cn}}, \mu^+)$-non-order property then in part (2) we can replace χ by $\lambda = \mu^+$.

Remark. We can replace $\Delta^{\mathrm{i,cn}}$ by $\Delta^{\mathrm{i,r}}$.

Proof. 1) Without loss of generality $\Delta = \Delta_{[\chi]}$. The "if" direction follows by $M_1 \leq_\Delta M_2$ recalling clause (b) of Definition 4.1, so we deal with the "only if" direction. Let $\bar{c} \in {}^{\kappa>}(M_2)$. By "$M_1 \leq^\kappa_{\Delta,\mu,\chi} M_2$" there is $\mathbf{J} \subseteq M_1$ of cardinality μ^+ which is (Δ, χ^+)-convergent inside M_1 such that $\mathrm{Av}_\Delta(\mathbf{J}, M_1, M_2) =$ $\mathrm{tp}_\Delta(\bar{c}, M_1, M_2)$ so if $\varphi(\bar{x}, \bar{c})$ divides $\mathbf{I}$ into two sets each of cardinality $> \chi$ then so does some $\bar{c}' \in \mathbf{J}$

by 3.1(2) (recalling that by Definition 4.1, we have $|\mathbf{I}| > \mu > \chi$), impossible so we are done.
2) Similar; alternatively use 4.3 and the definitions (see Example 4.11).
3) If $\chi < \lambda \leq \mu$ then the proof of part (1) holds so we are left with the case $\lambda = \mu^+$, i.e. the second sentence in part (3) so we are assuming that M_1 has the $(\Delta^{\mathrm{i,cn}}, \mu^+)$-non-order property. Assume toward contradiction that $\varphi(\bar{x}, \bar{y}) \in \Delta$, so $\ell g(\bar{x}) = \ell g(\bar{c}) < \kappa, \ell g(\bar{y}) \leq \chi$ and $\mathbf{I}_{\mathbf{t}} = \{\bar{a} \in \mathbf{I} : M_2 \models \varphi[\bar{c}, \bar{a}]^{\mathbf{t}}\}$ has cardinality μ^+ for $\mathbf{t} =$ true, false. For every $q \subseteq \mathrm{tp}_{\{\varphi(\bar{x},\bar{y})\}}(\bar{c}, M_1, M_2)$ of cardinality $\leq \mu$ some $\bar{c}_q \in {}^{\ell g(\bar{y})}(M_1)$ realizes it and as $\mathbf{I}$ is (Δ, λ)-convergent in M_1, for some $\mathbf{t}_q \in \{\text{true, false}\}$ for all but $\leq \mu$ members $\bar{a}$ of $\mathbf{I}$ we have $M_1 \models \varphi[\bar{c}_q, \bar{a}]^{\mathbf{t}_q}$. As $[q_1 \subseteq q_2 \Rightarrow \bar{c}_{q_2}$ can serve as $\bar{c}_{q_2}]$ without loss of generality for some $\mathbf{t}^*$ for every $q \subseteq \mathrm{tp}_{\{\varphi(\bar{x},\bar{y})\}}(\bar{c}, M_1, M_2)$ of cardinality $\leq \mu$ we choose $\bar{c}_q, \mathbf{t}_q$ such that $\mathbf{t}_q = \mathbf{t}(*)$. Now we choose $\bar{c}_\alpha \in {}^{\ell g(\bar{x})}(M_1)$ and $\bar{a}_\alpha \in \mathbf{I}$ by induction on $\alpha < \mu^+$ such that

$(*)_1$ $\bar{a}_\alpha \in \mathbf{I}_{\neg \mathbf{t}(*)}$

$(*)_2$ $\{\bar{a} \in \mathbf{I} : M_1 \models \varphi[\bar{c}_\alpha, \bar{a}]^{\neg \mathbf{t}(*)}\}$ has cardinality $\leq \mu$

$(*)_3$ $\beta < \alpha \Rightarrow M_1 \models \varphi[\bar{c}_\beta, \bar{a}_\alpha]^{\mathbf{t}(*)}$

$(*)_4$ $\beta \leq \alpha \Rightarrow \models \varphi[\bar{c}_\alpha, \bar{a}_\beta]^{\neg \mathbf{t}(*)}$.

In stage α, we first can choose $\bar{a}_\alpha \in \mathbf{I}_{\neg \mathbf{t}(*)}$ as required in $(*)_1 + (*)_3$ because $|\mathbf{I}_{\neg \mathbf{t}(*)}| = \mu^+$ and by $(*)_2$ each case of $(*)_3$ excludes $\leq \mu$ members. We then can choose $\bar{c}_\alpha$ as $\bar{c}_{q_\alpha}$ where $q_\alpha = \{\varphi(\bar{x}, \bar{a}_\beta)^{\neg \mathbf{t}(*)} : \beta \leq \alpha\}$; it is well defined as $q_\alpha \subseteq \mathrm{tp}_{\{\varphi(\bar{x},\bar{y})\}}(\bar{c}, M_1, M_2)$ by $(*)_1$ and q_α has cardinality $\leq \mu < \mu^+$.

We have gotten that M_1 has the $(\varphi(x, \bar{y})^{\mathbf{t}(*)}, \mu^+)$-order property, contradicting an assumption.
4) The proof is similar to that of part (3). $\square_{4.6}$

4.7 Union existence lemma. *Let $\Delta, \mu, \chi, \kappa$ be as in 4.4, each M_i with the $(\Delta^{\mathrm{eb}}, \chi^+)$-non-order property. If M_i is $\leq^\kappa_{\Delta,\mu,\chi}$-increasing for $i < \delta$ and $\mathrm{cf}(\delta) \geq \kappa$ <u>then</u> $M_i \leq^\kappa_{\Delta,\mu,\chi} \bigcup_{j<\delta} M_j$ provided that $i < \delta \Rightarrow M_i \leq_\Delta \bigcup_{j<\delta} M_j$.*

Remark. For our main case ($\kappa = \aleph_0$ and $\Delta =$ the set of quantifier free formulas) the demands in 4.9 are satisfied.

Proof. Straightforward recalling that $\mu > \chi$. $\square_{4.9}$

4.8 The Lowenheim-Skolem Lemma. *If Δ is a set of quantifier free formulas and $\Delta, \mu, \chi, \kappa$ are as in 4.4, and M has the $(\Delta^{\rm eb}, \chi^+)$-non-order property, $A \subseteq M$, $|A| \le \mu^+$ then there is M' satisfying $A \subseteq M' \le^\kappa_{\Delta,\mu,\chi} M$ and $\|M'\| \le \mu^+$.*

Proof. Trivial for $\le_{\sum_{\mu,\chi,(<\kappa)}}$ recalling that M is (μ, Δ)-stable by 1.19 which is applicable as $\Delta^{\rm es} \subseteq \Delta^{\rm eb}$, see Definition 1.18(1)(g)). Now use 4.4. $\square_{4.8}$

4.9 Definition. We say that M_0, M_1, M_2 are in $(\Delta, \mu, \chi, \kappa)$-stable (or stable for $<^\kappa_{\Delta,\mu,\chi}$) amalgamation inside M (or M_1, M_2 are in $(\Delta, \mu, \chi, \kappa)$-stable amalgamation over M_0 inside M) when

- (a) each M_ℓ has the (Δ, χ^+)-non-order property
- (b) $M_\ell \le_\Delta M$ for $\ell = 0, 1, 2$
- (c) $M_0 \le_\Delta M_\ell$ for $\ell = 1, 2$; actually $M_0 \subseteq M_\ell$ suffice
- (d) for every $\bar c \in M_1$ for some Δ-convergent $\mathbf{I} \subseteq M_0$, $|\mathbf{I}| = \mu^+$ we have $\mathrm{Av}_\Delta(\mathbf{I}, M_2, M_2) = \mathrm{tp}_\Delta(\bar c, M_2, M)$.

4.10 Observation. In Definition 4.9 if we replace clause (d) by clause (d)$'$ we get an equivalence definition, where

- $(d)'$ for every $\bar c \in {}^{\kappa >}(M_1)$ and every (Δ, χ^+)-convergent $\mathbf{I} \subseteq M_0$, if $\mathrm{Av}_\Delta(\mathbf{I}, M_0, M) = \mathrm{tp}_\Delta(\bar c, M_0, M)$ then $\mathrm{Av}_\Delta(\mathbf{I}, M_2, M) = \mathrm{tp}_\Delta(\bar c, M_2, M)$, (see 4.6)).

Proof. By 4.6(2). $\square_{4.10}$

4.11 Exercise: Assume $\alpha < \kappa$ and $M \le_{\Sigma_{\mu,\chi,<\kappa}(\Delta)} N$ and $\mathbf{I}, \mathbf{J} \subseteq {}^\alpha M$ has cardinality $> \mu$. Then

(a) **I** is (Δ, χ^+)-convergent in M iff **I** is (Δ, χ^+)-convergent in N

(b) if **I**, **J** are (Δ, χ^+)-convergent in M then $\mathrm{Av}_\Delta(\mathbf{I}, M, M) = \mathrm{Av}(\mathbf{J}, M, M) \Leftrightarrow \mathrm{Av}_\Delta(\mathbf{I}, N, N) = \mathrm{Av}(\mathbf{I}, N, N)$.

§5 More on the non-order implying the existence of indiscernibles

The results here, 5.1, 5.3 improve in some respects the older results 2.13 (see [Sh:c, I,§1]) by weakening the demands on M, "M has a suitable non-order property". In detail, we are dealing with sets of singletons, the non-order property is for $\varphi = \varphi(x, y, \bar{c})$ with x, y singleton elements not finite sequences. (Alternatively deal with n-tuples but then deal with the non-order property for $\varphi(\bar{x}, \bar{y}, \bar{z})$ such that $\ell g(\bar{x}) = n = \ell g(\bar{y})$. Choosing $n = 1$ is just a notational restriction.)

Note that in 5.1, possibilities (B) and (C) give only that some $\varphi(\bar{x}, \bar{y}, \bar{z}) \in \Delta$ has the μ-order property in M with $\ell g(\bar{x}) = \ell g(\bar{y})$ for some $n < \omega$. However, if $\mu' < \mu$ then some $\varphi'(x, y, \bar{c})$ with $\varphi'(x, y, \bar{z}) \in \Delta$ has the μ'-order property; note that now x, y are singletons, (see 5.2(5)). To compare with earlier results note that if M has the $(\Delta^{\mathrm{eb}}, \mu)$-non-order property and for simplicity we assume $\mu = \chi^+$ then M is $(2^{2^\chi}, \Delta)$-stable (by 1.19, note that μ there is not the same as here) hence, e.g. for every $\mathbf{I} \subseteq M$ of cardinality $(2^{2^\chi})^+ = (2^{(2^{<\mu})})^+$ there is a (Δ, χ)-convergent Δ-indiscernible subset $\mathbf{J} \subseteq \mathbf{I}$ of the same cardinality. Compared to 2.13 (or [Sh:c]) the loss in 5.1 is

(α) we get in possibility (A) an indiscernible set $\mathbf{J} \subseteq \mathbf{I}$ of cardinality μ only.

The gains are

(β) being able to speak on singletons in possibilities (B), (C), i.e., in the variants of order

(γ) the "distance" between the place of non-order and the size of the indiscernible set is smaller.

So the result in this section and 2.13 (equivalently [Sh:c]) are incomparable.

5.1 Theorem. *Suppose μ is a regular uncountable cardinal, M is a τ-model, Δ a set of $< \mu$ quantifier free $\mathbb{L}_\tau$-formulas $\varphi = \varphi(\bar{x})$ closed under negation, adding dummy variables and permuting the variables (each formula from Δ is with finitely many variables).*
If $\mathbf{I} \subseteq M$ has cardinality $> 2^{<\mu}$ then at least one of the following possibilities holds.

Possibility A. There is a Δ-indiscernible set $\mathbf{J} \subseteq \mathbf{I}$ of cardinality μ.

Possibility B. There are distinct $a_i \in \mathbf{I}$ for $i \leq \mu$ and n, $2 \leq n < \omega$ and $\varphi = \varphi(\bar{z}, x_1, \ldots, x_n) \in \Delta$ and $\bar{c} \in {}^{\ell g(\bar{z})}M$ finite such that:

(a) $\langle a_i : i \leq \mu \rangle$ is $(< n)$-end-indiscernible which means that: if $m < n, k < \omega, \alpha_1 < \cdots < \alpha_k, \alpha_k < \beta_1 < \cdots < \beta_m \leq \mu, \alpha_k < \gamma_1 < \cdots < \gamma_m \leq \mu$ and $\psi(\bar{z}, y_1, \ldots, y_k, x_1, \ldots, x_m) \in \Delta$ then:

$$M \models \psi[\bar{c}, a_{\alpha_1}, \ldots, a_{\alpha_k}, a_{\beta_1}, \ldots, a_{\beta_m}] \text{ iff}$$

$$M \models \psi[\bar{c}, a_{\alpha_1}, \ldots, a_{\alpha_k}, a_{\gamma_1}, \ldots, a_{\gamma_m}]$$

moreover

$(a)^+$ $\langle a_i : i \leq \mu \rangle$ is an indiscernible sequence

(b) if $\beta_1 < \cdots < \beta_n \leq \mu$ and $\bar{d} = \langle a_{\beta_3}, a_{\beta_4}, \ldots, a_{\beta_n} \rangle$ then

$$M \models \varphi[\bar{c}, a_{\beta_1}, a_{\beta_2}, \bar{d}]$$

$$M \models \neg\varphi[\bar{c}, a_{\beta_2}, a_{\beta_1}, \bar{d}]$$

Possibility C. There are pairwise distinct $a_i \in \mathbf{I}$ for $i \le \mu$ and n, $2 \le n < \omega$ and $\varphi = \varphi(\bar{y}, x_1, \ldots, x_n) \in \Delta$ and $\bar{c} \in {}^{\ell g(\bar{y})}M$ so finite, such that:

(a) as in Possibility B (but not $(a)^+$! there)

(b) If $\alpha, \beta < \gamma_3 < \cdots < \gamma_n \le \mu$, $\alpha \neq \beta, \bar{d} = \langle a_{\gamma_3}, \ldots, \alpha_{\gamma_n} \rangle$ then $M \models \varphi[\bar{c}, a_\alpha, a_\beta, \bar{d}]$ *if and only if* $\mathrm{Min}\{\alpha, \beta\}$ is even.

Before we prove 5.1 we make some remarks, draw a conclusion for first order theories and give an example.

5.2 Remark. 0) Putting the parameters $(\bar{c})$ first in the formulas is accidental, also without loss of generality $\bar{c} \in {}^{\omega>}\mathbf{I}$.
1) From each of the clauses (B) and (C) of the theorem it follows that: for some $\varphi'(\bar{x}, \bar{y}, \bar{z}) \in \Delta, \ell g(\bar{x}) = \ell g(\bar{y}) = n$ gotten from φ by adding dummy variable, the model M has the (Δ, μ)-order property. If we define the (Δ, μ)-order property more liberally, we get the (Δ, μ)-order property for singletons, i.e. if we define it as in clauses (B),(C); see more in part (4),(5).
2) We can do everything over a set of $< \mu$ parameters - just expand M by individual constants or restrict its universe to $\mathbf{I}$.
3) We can deal instead of elements with m-tuples (or α-tuples) - replace M by an appropriate model with universe ${}^m\|M\|$.
4) Note that in possibility (C), if $\gamma < \mu$, letting $\theta(x_0, x_1)) =$

$$\varphi(\bar{c}, x_0, y_0, a_{2\gamma+3}, \ldots, a_{2\gamma+n}) \equiv \neg\varphi(\bar{c}, x_1, y_0, a_{2\gamma+3}, \ldots, a_{2\gamma+n})$$

is a formula which linearly orders $\langle \langle a_{2\alpha}, a_{2\alpha+1} \rangle : 2\alpha + 1 < \gamma \rangle$, because

$$M \models \varphi[\bar{c}, a_{2\alpha}, a_{2\beta+1}, a_{\gamma+3}, a_{\gamma+4}, \ldots, a_{\gamma+n}] \quad \text{iff} \quad (\alpha < \beta) \equiv \mathbf{t}.$$

A parallel statement for possibility (B) should be more transparent.
5) From each of the clauses (B) and (C) of the theorem it follows that for some $\varphi'(x_1, x_2; \bar{y}) \in \Delta$ for any $\mu' < \mu$ the model M has the $(\varphi'(x_1, x_2; \bar{y}), \mu')$-order property, see Definition 1.1(1). How do we get φ'? If clause (B) holds for $\bar{c}$ and $\varphi(\bar{y}', x_1, \ldots, x_n)$ let $\varphi' = \varphi'(x_1, x_2; \bar{y})$ be

$$\varphi(\bar{y}', x_1, x_2; z_1, \ldots, z_{n-2})$$

so $\bar{y} = \bar{y}'\hat{\ }\langle z_1, \ldots, z_{n-2}\rangle$.
For any ordinal $\gamma < \mu$ let $\bar{d}^\gamma = \bar{c}\hat{\ }\langle a_\gamma, a_{\gamma+1}, \ldots, a_{\gamma+n-3}\rangle$; so by clause (b) of Possibility B we have: for every $\beta_1 < \beta_2 < \gamma$

$$M \models \varphi'[a_{\beta_1}, a_{\beta_2}, \bar{d}^\gamma]$$

$$M \models \neg\varphi'[a_{\beta_2}, a_{\beta_1}, \bar{d}^\gamma].$$

If Possibility C holds as witnessed by $\varphi = \varphi(\bar{y}, x_1, \ldots, x_n) \in \Delta$ and $\bar{c} \in {}^{\ell g(\bar{y})}M$, then we let φ' be as above. For any $\gamma < \mu$ let $\bar{d}^\gamma = \bar{c}\hat{\ }\langle a_{2\gamma}, \ldots, a_{2\gamma+1}, \ldots, a_{2\gamma+n-3}\rangle$ and lastly for $\beta < \gamma$, let $a^\ell_\beta = a_{2\beta+\ell-1}$ for $\ell = 1, 2$. So by clause (b) of Possibility (C) of the theorem $(a) \Leftrightarrow (b) \Leftrightarrow (c)$ where

- (a) $M \models \varphi'[a^1_{\beta_1}, a^2_{\beta_2}, \bar{d}^\gamma]$
- (b) $M \models \varphi[\bar{c}, a_{2\beta_1}, a_{2\beta_2+1}, a_{2\gamma}, a_{2\gamma+1}, \ldots, a_{2\gamma+n-1}]$
- (c) $\text{Min}\{2\beta_1, 2\beta_2 + 1\}$ is even which holds iff $\beta_1 < \beta_2$.

5.3 Conclusion. Suppose T is first order and

- $(*)$ for no model M of T and quantifier free formula $\varphi(x, y, \bar{z})$ does M have the $(\varphi(x, y; \bar{z}))$-order property [i.e. for no $\bar{c}, a_n, b_n$ (for $n < \omega$) from M, $M \models \varphi[a_\ell, b_k, \bar{c}]$ if and only if $\ell < k$].

If N is a model of T, $\mu \geq |T|^+, \lambda$ is regular and A, B subsets of N such that $|A| < \mu, |B| > 2^{<\mu}$ <u>then</u> B has a subset of cardinality μ which is an indiscernible set for quantifier free formulas over A inside N.

5.4 Conclusion. Suppose

- $(*)$ the model M is such that for no quantifier free formula $\varphi(x, y, \bar{z})$ does M have the $(\varphi(x, y; \bar{z}), \kappa)$-order property.

If $\mu > \kappa + |\tau_\mu|$ is regular, $A \subseteq M, B \subseteq M$ and $|A| < \mu, |B| > 2^{<\mu}$ then B has a subset of cardinality μ which is an indiscernible set over A in M for quantifier free formulas.

5.5 Discussion: 1) Why in 5.3 we demand "quantifier free"?
Because of [Sh 715, 1.37=np1.11tex].
2) Is 5.3 non-empty? Yes, by the following example.

5.6 Example: Let $\tau = \{R\}, R$ a three-place relation.
Let M be the following τ-model:

- (a) its universe is $\{a^\ell_r : \ell < 3$ and r is rational (i.e. a number)$\}$ without repetitions
- (b) $R^M = \{(a^0_{r_1}, a^1_{r_2}, a^2_{r_2}) : r_1 < r_2$ are rationals$\}$.

Let $T = \text{Th}(M)$, now:

$\boxtimes$ T has an unstable quantifier free formula: $\varphi(x; \bar y) = R(x; y_0, y_1)$.

[Why? Because $M \models R[a^0_{r_1}, a^1_{r_2}, a^2_{r_2}]$ iff $r_1 < r_2$.]
But

$\boxtimes$ T is as in 5.3, i.e. satisfies $(*)$ there.

[Why? Consider a quantifier free formula $\varphi(x, y, \bar z)$. Clearly it is enough to find n such that

$\circledast_1$ for no $\bar d \in {}^{\ell g(\bar z)}M$ can we find $b_\ell, c_\ell \in M$ for $\ell < n$ such that $M \models \varphi[b_\ell, c_k, \bar d]^{\text{if}(\ell<k)}$ for $\ell, k < n$.

Why does $\circledast_1$ hold? Let $\bar d \in {}^{\ell g(\bar z)}M$, let u be a minimal set of rationals such that $\text{Rang}(\bar d) \subseteq \{a^\ell_r : \ell < 3$ and $r \in u\}$.

Let $k = |u|$, so $k \le \ell g(\bar z)$ and u divides the rationals minus u into $k+1$ open convex sets $I_0, I_1, \dots, I_k$. Now let $H = \{\bar h : \bar h = \langle h_m : m \le k\rangle, h_m$ a permutation of $I_m\}$ and for each $\bar h \in H$ we define a function $\pi_{\bar h}$ with domain the universe of M by

$$\pi_{\bar h}(a^\ell_r) \text{ is } : a^\ell_r \text{ if } r \in u$$
$$a^\ell_{h_m(r)} \text{ if } r \in I_m.$$

Let $A = \{a^\ell_r : r \in u \text{ and } \ell < 3\}$, so $A \subseteq M$ has $\le 3k \le 3\ell g(\bar{z})$ elements.

Clearly it is enough to prove

$\circledast_2$ for each $\bar{h} \in H, \pi_{\bar{h}}$ is a permutation of M such that:

- (α) for every $b \in M \backslash A$ the elements $b, \pi_{\bar{h}}(b)$ realize the same quantifier free type over A
- (β) for every $b \neq c \in M \backslash A$ the pairs $(b, c), (\pi_{\bar{h}}(b), \pi_{\bar{h}}(c))$ realize the same quantifier free type over A.

[Why $\circledast_2$ holds? This follows by $\circledast_3 + \circledast_4 + \circledast_5$ below. Why? Clause (α) of $\circledast_2$ holds mainly by $\circledast_3$ with a^i_r, a^j_s there standing for $b, \pi_{\bar{h}}(b)$ in (α) above, but we need also $\circledast_4 + \circledast_5$ in the case $r_1 = s_1, r_2 = s_2$. Then (knowing clause (α)) clause (β) of $\circledast_2$ holds by $\circledast_4 + \circledast_5$ with $a^i_{r_1}, a^j_{s_1}, a^i_{r_2}, a^j_{s_2}$ standing in $\circledast_4, \circledast_5$ for $b, c, \pi_{\bar{h}}(b), \pi_{\bar{h}}(c)$ in clause (β) above]

$\circledast_3$ if $b, c \in A$ and $m \le k$ and $r, s \in I_m$ and $i \in \{0, 1, 2\}$ <u>then</u>

- (a) $M \models R[a^i_r, b, c] \equiv R[a^i_s, b, c]$
- (b) $M \models R[b, a^i_r, c] \equiv R[b, a^i_s, c]$
- (c) $M \models R[b, c, a^i_r] \equiv R[b, c, a^i_s]$.

[Why this holds? As $(a', b', c') \in R^M \Rightarrow (b', c') \in \{(a^1_t, a^2_t) : t \in \mathbb{Q}\}$, by the choice of A clearly in clauses (b),(c) both sides fail. In clause (a), if at least one side holds then $i = 0$ and for some $t, (b, c) = (a^1_t, a^2_t)$ and the equivalence means $r < t \equiv s < t$ but $t \in u$, and $\{r, s\} \in I_m$ so the equivalence holds.]

$\circledast_4$ if $a \in A$ and $m \le k, n \le k$ and r_ℓ, s_ℓ satisfy $s_\ell \in I_m, r_\ell \in I_n$ for $\ell = 1, 2$ and $i, j \in \{0, 1, 2\}$ and $r_1 = s_1 \leftrightarrow r_2 = s_2$ <u>then</u>

- (a) $M \models R[a, a^i_{r_1}, a^j_{s_1}] \equiv R[a, a^i_{r_2}, a^j_{s_2}]$
- (b) $M \models R[a^i_{r_1}, a, a^j_{s_1}] \equiv R[a^i_{r_2}, a, a^j_{s_2}]$
- (c) $M \models R[a^i_{r_1}, a^j_{s_1}, a] \equiv R[a^i_{r_2}, a^j_{s_2}, a]$.

[Why this holds? In clauses (b),(c) both sides fail as $(a', b', c') \in R^M \Rightarrow (b', c') \in \{(a^1_t, a^2_t) : t \in \mathbb{Q}\}$, so we are left with clause (a). Assume that at least one side holds, so for some $\ell \in \{1, 2\}$ we have $M \models R[a, a^i_{r_\ell}, a^j_{s_\ell}]$, hence $i = 1, j = 2$ and $r_\ell = s_\ell$ hence $r_{3-\ell} = s_{3-\ell}$ and we continue as in the proof of $\circledast_3$.]

$\circledast_5$ in $\circledast_4$ we can add:

(a) $M \models R[a^j_{s_1}, a^i_{r_1}, a^i_{r_1}] \equiv R[a^j_{s_2}, a^i_{r_2}, a^i_{r_2}]$

(b) $M \models R[a^i_{r_1}, a^j_{s_1}, a^i_{r_1}] \equiv R[a^i_{r_2}, a^j_{s_2}, a^i_{r_2}]$

(c) $M \models R[a^i_{r_1}, a^i_{r_1}, a^j_{s_1}] \equiv R[a^i_{r_2}, a^i_{r_2}, a^j_{s_2}]$.

[Why? In clauses (a),(b),(c) both sides fail as $(a', b', c') \in R \Rightarrow (a', b', c') \in \{(a^1_{r_1}, a^2_{r_2}, a^3_{r_2}) : r_1, r_2 \in \mathbb{Q}\}$.]
So we are done. $\square_{5.6}$

Proof of 5.1. Without loss of generality the formulas in Δ are atomic and restrict the universe of M to $\mathbf{I}$. Let $A^* \subseteq M, |A^*| = 2^{<\mu}$ be such that:

$(*)$ if $A \subseteq A^*, |A| < \mu$ and $a \in M$ <u>then</u> some $a' \in A^* \backslash A$ realizes $\text{tp}_\Delta(a, A, M)$.

Such A^* exists as $(2^{<\mu})^{<\mu} = 2^{<\mu}$ which holds as μ is regular. Choose $a^* \in M \backslash A^*$.

Now for every $\bar{c} \in A^*$ and formula $\varphi = \varphi(\bar{c}, \bar{x}) = \varphi(\bar{c}, x_1, \dots, x_n)$ such that $\varphi(\bar{z}, x_1, \dots, x_n) \in \Delta$, (so we write $n = n(\varphi), \bar{c} = \bar{c}_\varphi$) we define a game $\partial_\varphi = \partial_{M, \varphi(\bar{c}, \bar{x})}$ as follows:

A play of the game lasts $n+1$ moves (numbered by $0, 1, 2, \dots, n$); in the ℓ-th move: player I chooses a set A_ℓ satisfying $A_\ell \subseteq A^*$, $[m < \ell \Rightarrow A_m \cup \{a_m\} \subseteq A_\ell]$ and $|A_\ell| < \mu$; player II chooses an element $a_\ell, a_\ell \in A^* \backslash A_\ell$ which realizes $\text{tp}_\Delta(a^*, A_\ell, M)$.

In the end player I wins if

$$M \models \text{“}\varphi[\bar{c}, a_1, a_2, a_3, \dots, a_n]\text{”} \Leftrightarrow M \models \text{“}\varphi[\bar{c}, a_0, a_2, a_3, \dots, a_n]\text{”}.$$

This game is clearly determined. So one of the players has a winning strategy $\bar{F}_\varphi = \langle F^\varphi_\ell : \ell \leq n \rangle$, F^φ_ℓ computes his ℓ-th move from the previous moves of his opponent. Without loss of generality if player I wins then for every $a_0, \dots, a_{\ell-1} \in A^*, F^\varphi_\ell(a_0, \dots, a_{\ell-1})$ is well defined and is a subset of A^* of cardinality $< \mu$, extending $F^\varphi_m(a_0, \dots, a_{m-1}) \cup \{a_0, \dots, a_m\}$ for each $m < \ell$, (of course, F^φ_ℓ depends on $\bar{c} = \bar{c}_\varphi$). Also if player II wins then without loss of generality F^φ_ℓ is such that for any $A_0 \subseteq \dots \subseteq A_\ell \subseteq A^*$ of cardinality $< \mu$ the set $F^\varphi_\ell(A_0, \dots, A_\ell)$ is a member of $A^* \backslash A_\ell$.

Case I. For every $\varphi(\bar{c}, \bar{x})$ as above, player I wins the game $\partial_{\varphi(\bar{c},\bar{x})}$, i.e. has a winning strategy.

We choose by induction on $\alpha < \mu$ a pair (a_α, A_α) such that:

(i) $\{a_\beta\} \cup A_\beta \subseteq A_\alpha \subseteq A^*$ for $\beta < \alpha$ and $|A_\alpha| < \mu$

(ii) $a_\alpha \in A^* \backslash A_\alpha$ realizes $\mathrm{tp}_\Delta(a^*, A_\alpha, M)$

(iii) if $\beta < \alpha, \bar{c} \in {}^{\omega>}(A_\beta \cup \{a_\beta\})$, $\varphi(\bar{y}, \bar{x}) \in \Delta, \ell g(\bar{y}) = \ell g(\bar{c})$, $x = \langle x_1, \dots, x_n \rangle, \ell \leq n$ and $b_0, \dots, b_{\ell-1} \in A_\beta \cup \{a_\beta\}$, <u>then</u> $F^{\varphi(\bar{c},\bar{x})}_\ell(b_0, \dots, b_{\ell-1}) \subseteq A_\alpha$
(we can further restrict $b_0, \dots, b_{\ell-1}, \bar{c}$)

(iv) if $\varphi(\bar{x}) \in \Delta$ and $\bar{x} = \langle x_1, \dots, x_n \rangle$ then $F^{\varphi(\bar{x})}_0() \subseteq A_0$.

There is no problem to do it. (In stage α, first choose A_α to satisfy clause (iv) if $\alpha = 0$, and to satisfy clauses (i) + (iii) if $\alpha > 0$, [exists as the value of $F^{\varphi(\bar{c},\bar{x})}_\ell$ is always a subset of A^* of cardinality $< \mu$, μ regular $> \aleph_0$]. Then choose a_α to satisfy (ii). [This is possible by the choice of A^*, a^*].

Now we can prove

$(*)_a$ if $n, k < \omega$ and $\alpha_1 < \dots < \alpha_k < \beta_0 < \beta_1 < \dots < \beta_n < \mu$,

$$\text{and } \varphi(y_1, \dots, y_k, x_1, \dots, x_n) \in \Delta$$

then

$$M \models \varphi[a_{\alpha_1}, \dots, a_{\alpha_k}, a_{\beta_1}, a_{\beta_2}, \dots, a_{\beta_n}] \text{ if and only if}$$
$$M \models \varphi[a_{\alpha_1}, \dots, a_{\alpha_k}, a_{\beta_0}, a_{\beta_2}, \dots, a_{\beta_n}]$$

$(*)_b$ if $\alpha_1 < \dots < \alpha_k < \mu, \alpha_k < \beta_1 < \dots < \beta_n < \mu$, $\alpha_k < \gamma_1 < \dots < \gamma_n < \mu$ and

$$\varphi(y_1, \dots, y_k, x_1, \dots, x_n) \in \Delta$$

then

$$M \models \varphi[a_{\alpha_1}, \dots, a_{\alpha_k}, a_{\beta_1}, \dots, a_{\beta_n}] \text{ if and only if } M \models \varphi[a_{\alpha_1}, \dots, a_{\alpha_k}, a_{\gamma_1}, \dots, a_{\gamma_n}].$$

Why this holds? As for $(*)_a$, let $\bar{c} = \langle a_{\alpha_1}, \dots, a_{\alpha_k} \rangle$, remember that player I wins the game $\partial_{\varphi(\bar{c},\bar{x})}$ and that $\langle F_\ell^{\varphi(\bar{c},\bar{x})} : \ell \le n \rangle$ is a winning strategy for him. Let $A^\ell = F_\ell^{\varphi(\bar{c},\bar{x})}(a_{\beta_0}, \dots, a_{\beta_{\ell-1}})$ for $\ell \le n$. By (iii) + (iv) above $A^\ell \subseteq A_{\beta_\ell}$ hence a_{β_ℓ} realizes $\text{tp}_\Delta(a^*, A^\ell, M)$ and $a_{\beta_\ell} \in A^* \backslash A^\ell$. So $A^0, a_{\beta_0}, A^1, a_{\beta_1}, \dots, A^n, a_{\beta_n}$ is a play of the game $\partial_{\varphi(\bar{c},\bar{x})}$ in which player I uses his winning strategy $\langle F_\ell^{\varphi(\bar{c},\bar{x})} : \ell \le n \rangle$, so he wins the play, i.e. the conclusion of $(*)_a$ holds.

By the transitivity of equivalence we can deduce $(*)_b$ proving it by induction on n.

So $\langle a_\alpha : \alpha < \mu \rangle$ is a Δ-indiscernible sequence.

If it is a Δ-indiscernible set, possibility (A) of the theorem holds. If it is not, then (see 3.5; by Morley's work [Mo65], see, e.g., [Sh:c, AP.3.9]) for some n, possibility (B) of the theorem holds (i.e. use again transitivity of equivalence to get the "good form") [we have to check that letting $a_\mu := a^*$ is O.K., but this is easy].

Case II. For some $\varphi(\bar{c}, \bar{x})$, player II wins in the game $\partial_{\varphi(\bar{c},\bar{x})}$. Choose such $\varphi_0 = \varphi_0(\bar{c}_0, x_1, \dots, x_{n(0)})$ with minimal $n(0)$. Necessarily $n(0) \ge 2$.

We now choose by induction on $\zeta < \mu$, for every $\alpha < (n(0)+1) \times \zeta$ the set A_α and element a_α such that:

(i) $\bar{c}_0 \cup \{a_\beta\} \cup A_\beta \subseteq A_\alpha \subseteq A^*$ for $\beta < \alpha$ and $|A_\alpha| < \mu$

(ii) $a_\alpha \in A^* \backslash A_\alpha$ realizes $\text{tp}_\Delta(a^*, A_\alpha, M)$

(iii) if $\beta < \alpha, \bar{c} \in {}^{\omega>}(A_\beta \cup \{a_\beta\})$, $\varphi(\bar{y}, \bar{x}) \in \Delta, \ell g(\bar{y}) = \ell g(\bar{c})$, $\bar{x} = \langle x_1, \dots, x_n \rangle, \ell \le n$ and $b_0, \dots, b_{\ell-1} \in A_\beta \cup \{a_\beta\}$ and player I wins the game $\partial_{\varphi(\bar{c},\bar{x})}$ (which occurs when $n < n(0)$), then

$$F_\ell^{\varphi(\bar{c},\bar{x})}(b_0, \dots, b_{\ell-1}) \subseteq A_\alpha$$

(iv) if $\alpha = (n(0)+1) \times \zeta$ and $\ell \leq n(0)$ then

$$a_{\alpha+\ell} = F_\ell^{\varphi_0(\bar{c}_0,\bar{x})}(A_\alpha, A_{\alpha+1}, \dots, A_{\alpha+\ell})$$

(v) if $\varphi(\bar{x}) \in \Delta$ and $\bar{x} = \langle x_1, \dots, x_n \rangle$ then $F_0^{\varphi(\bar{x})}() \subseteq A_0$.

There are no problems in carrying this out.

As in Case I we can prove

$(*)_c$ if $n < n(0), k < \omega, \bar{c} \in {}^{\omega>}(A_\alpha), \alpha < \mu,$

$$\alpha \leq \beta_1 < \dots < \beta_n < \mu, \alpha \leq \gamma_1 < \dots < \gamma_n < \mu$$
$$\text{and } \varphi(\bar{y}, x_1, \dots, x_n) \in \Delta$$

<u>then</u>

$M \models \varphi[\bar{c}, a_{\beta_1}, \dots, a_{\beta_n}]$ if and only if
$M \models \varphi[\bar{c}, a_{\gamma_1}, \dots, a_{\gamma_n}]$

and let the truth value be

$$\mathbf{t}_{\varphi(\bar{c},x_1,\dots,x_n)}.$$

Let $\mathbf{t}_{\zeta,\ell}$ be the truth value of $M \models \varphi_0[\bar{c}, a_{\alpha+\ell}, a_{\alpha+2}, a_{\alpha+3}, \dots, a_{\alpha+n}]$ for $\ell \in \{0,1\}$ where $\alpha = (n(0))+1) \times \zeta$.

There are truth values $\mathbf{t}_0, \mathbf{t}_1$ such that $S = \{\zeta < \mu : \mathbf{t}_{\zeta,0} = \mathbf{t}_0$ and $\mathbf{t}_{\zeta,1} = \mathbf{t}_1\}$ is an unbounded subset of μ.

By clause (iv) in the construction and the choice of $\langle F_\ell^{\varphi_0(\bar{c}_0,\bar{x})} : \ell \leq n\rangle$ clearly $\mathbf{t}_{\zeta,0} \neq \mathbf{t}_{\zeta,1}$ hence $\mathbf{t}_0 \neq \mathbf{t}_1$. By $(*)_c$ we have: if $\zeta \in S$ and $\alpha = (n(0)+1) \times \zeta$ and $\alpha + 1 < \alpha_1 < \alpha_2 \ldots < \alpha_{n(0)} < \mu$ and $\ell \in \{0,1\}$ <u>then</u> $M \models \varphi_0(\bar{c}_0, a_{\alpha+\ell}, a_{\alpha_1}, \dots, a_{\alpha_{n(0)}})$ iff $\mathbf{t}_{\zeta,\ell}$.
By renaming we have $S = \mu$. Let $a'_{2\zeta} = a_{\zeta \times (n(0)+1)}, a'_{2\zeta+1} = a_{\zeta \times (n(0))+1)+1}$ so $\langle a'_\zeta : \zeta < \mu\rangle, \varphi(\bar{c}_0, x_1, x_2, \dots, x_{n(0)})$ are as required in possibility (C). $\square_{5.1}$

Remark. 1) The example does not show that for first order theories with no quantifier free formula $\varphi(x; y; \bar{z})$ with the order property, the cardinality bound in 5.1 is optimal.
2) But "$|\mathbf{I}| > 2^{<\mu}$" cannot be improved in 5.1 at least if e.g. $\mu = \kappa^+ = 2^\kappa$.

UNIVERSAL CLASSES:
AXIOMATIC FRAMEWORK
SH300B

§0 Introduction

We give here (§1) an axiomatic framework for dealing with classes of models which have something like "free amalgamations". We give several versions, but we shall deal here mainly with the strongest one. (Somewhere else we have intended to concentrate on the "prime" framework for which we can repeat the development, see beginning Baldwin Shelah [BlSh 330], [BlSh 360], [BlSh 393] and much later in [Sh 839]). We show that it holds for two main examples: stable first order T (for simplicity T and $T^{\rm eq}$ has elimination of quantifiers; and we work in $\mathfrak{C}^{\rm eq}$ or assume we can eliminate imaginaries; here the models are algebraically closed subsets of $\mathfrak{C}^{\rm eq}$) and a universal class (with a special submodel relation as developed in I,§4 assuming some non-order property). So the main applications are the results for universal classes, whereas our guiding line is to make the theory similar to the one of stable first order T.

In the third section we deal with a weaker framework, but with smoothness (just as the "abstract elementary classes" of Chapter I). An easy theorem but with important consequences is the "model homogeneity equal to saturativity" lemma, saying that for a model to be $(\mathbb{D}, \lambda)$-model homogeneous, it is enough that all relevant 1-types are realized. This makes dealing with model-homogeneous models similar to saturated ones. Still $\mathbf{tp}_{\mathfrak{s}}(a, M, N)$ (for $M \le_{\mathfrak{s}} N, a \in N$) may not be determined by the collection of $\mathbf{tp}_{\mathfrak{s}}(a, M', N)$ for all "small" $M' \le_{\mathfrak{s}} M$, i.e., not be local (κ-local if "small" means of cardinality $\le \kappa$).

In the main framework, if M_1, M_2 are in stable amalgamation over M_0 inside $M, M_1 \cup M_2$ generates a "good" submodel of M_3; in a weaker variant there is over $M_1 \cup M_2$ a prime model, and similarly for union of increasing chains.

Now 1.2 through 1.6 describe the context for the rest of "universal classes". We then discuss some parallel sets of axioms of various order of strength. These are AxFr_1 in 1.6, the main framework, AxFr_1^+, a variant AxFr_2 in 1.11, the primal framework, and AxFr_3, in 1.9, the existential framework. The difference between these frameworks is the way in which a "cover" of a pair of models (extending a given one, i.e. we are amalgamating) or of an increasing sequence of models is described. In the main framework the axiom group C_{gn} express the idea that the "cover" is generated from the given models by functions. The existential framework simply demands the existence of a "cover". The primal framework expresses the idea that the "cover" is prime in the sense of a first order model theory.

These frameworks all avoid the introduction of element-types and deal only with models. In 1.13 we move in an orthogonal direction and describe axioms which generalize the idea of a non-forking type of element.

§1 The Framework

We introduce here the framework AxFr_1, which is the main (for analyzing universal classes), some relatives and sort out some relations.

1.1 Notation. As we introduce axioms we give their names in round brackets, e.g. (A4) or Ax(A4) and to set of axioms, i.e. axiomatic frameworks as AxFr_2 or (AxFr_2). Later we write an axiom in square brackets to indicate in the case of a theorem that the axiom is needed to prove it and in the case of a definition that we only use the defined concept when the indicated axiom holds.

We may feel it reasonable to demand $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ (etc) are defined reasonably. Note however that by 3.16(2) (really by Chapter

I), under enough (but not many) assumptions, K and $(K, \leq_{\mathfrak{K}})$ (i.e. $\{(M,N) : N \leq_{\mathfrak{K}} M\}$) are $\mathrm{PC}_{\chi,2^\chi}$-classes.

1.2 Context: In all the frameworks, $\mathfrak{s}$ denotes a tuple consisting of classes and relations whose properties we axiomatize. E.g. $\mathfrak{s} = \langle K, \leq, \mathrm{NF}\rangle$ and $\mathfrak{K} = \mathfrak{K}_{\mathfrak{s}} = (K, \leq)$, so we may instead of $\leq$ write $\leq_{\mathfrak{K}}$ or $\leq_{\mathfrak{s}}$ and $\mathrm{NF}_{\mathfrak{s}}$ instead of NF. For our $\mathfrak{K}$'s, K will be a class of models of a fixed vocabulary $\tau(K) = \tau(\mathfrak{K}) = \tau_{\mathfrak{s}}, \leq_{\mathfrak{K}}$ a two-place relation on K (a generalization of being elementary submodel) and usually a four-place relation $\mathrm{NF} = \mathrm{NF}_{\mathfrak{s}}$ (where $\mathrm{NF}(M_0, M_1, M_2, M_3)$ means (i.e. intend to mean) that M_1, M_2 are in stable amalgamation over M_0 inside M_3. [In AxFr_4 we use $\mathrm{NF}^e = \mathrm{NF}^e_{\mathfrak{s}}$ (where $\mathrm{NF}^e(M_0, M_1, a, M_3)$ means that $\mathbf{tp}(a, M_1, M_3)$ does not fork over M_0 so $a \in M_3$)]. We may like to say in the former case that M_3 is generated by $M_1 \cup M_2$ ($M_3 = \langle M_1 \cup M_2\rangle^{\mathrm{gn}}_{M_3}$) or at least is prime over $M_1 \cup M_2$ (say $\mathrm{Pr}(M_0, M_1, M_2, M_3)$) or just any two possible M_3's are compatible. Also sometimes an increasing union is not by itself a member of K but we can close it or take over it a prime model or just any two possible bounds are compatible. Naturally we adopt:

1.3 Meta Axiom. K and all relations on it (here we shall have $\leq_{\mathfrak{K}} = \leq_{\mathfrak{s}}$, $\mathrm{NF} = \mathrm{NF}_{\mathfrak{s}}$, $\mathrm{NF}^e = \mathrm{NF}^e_{\mathfrak{s}}$ below and Pr, etc.), are closed under isomorphisms.

1.4 Group A. The following axioms always[1] will be assumed on $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$:

(A0) $M \leq_{\mathfrak{K}} M$ for $M \in K$

(A1) $M \leq_{\mathfrak{K}} N$ implies $M \subseteq N$ (M a submodel of N)

(A2) $\leq_{\mathfrak{K}}$ is transitive

(A3) if $M_0 \subseteq M_1 \subseteq N, M_0 \leq_{\mathfrak{K}} N$ and $M_1 \leq_{\mathfrak{K}} N$ <u>then</u> $M_0 \leq_{\mathfrak{K}} M_1$.

[1] In fact, Ax(A3) will be used considerably less than the others, but we shall not seriously investigate this; still we many times will mention which of the axioms we shall use.

1.5 Definition. We say $f : M \rightarrow N$ is a $\leq_{\mathfrak{K}}$-embedding if f is an isomorphism from M onto some $M' \leq_{\mathfrak{K}} N$.

1.6 Definition. The Main Framework: (AxFr_1):
Here $\mathfrak{s} = (K, \leq, \mathrm{NF}, <>^{\mathrm{gn}})$ or $(\mathfrak{K}, \mathrm{NF}, <>)$, where "gn" stands for "generated" and we may write $\leq_{\mathfrak{s}}$ or $\leq_{\mathfrak{K}}$ for $\leq, K_{\mathfrak{s}}$ for $K, \mathfrak{K}_{\mathfrak{s}}$ for $(K, \leq_{\mathfrak{K}}), \mathrm{NF}_{\mathfrak{s}}$ for NF, $\langle A \rangle^{\mathrm{gn},\mathfrak{s}}_M$ for $\langle A \rangle^{\mathrm{gn}}_M$ and we omit M when clear from the context (and we are assuming Ax(B3)); $K^{\mathfrak{s}}_\lambda = \{M \in K : \|M\| = \lambda\}, \mathfrak{K}^{\mathfrak{s}}_\lambda = \mathfrak{K}_{\mathfrak{s}} \restriction K^{\mathfrak{s}}_\lambda$ and $K^{\mathfrak{s}}_{<\lambda}, \mathfrak{K}^{\mathfrak{s}}_{<\lambda}$, etc., similarly.
AxFr_1 consists of (1.3, and (A0) - (A3) of 1.4 and) (A4),(B),$(\mathrm{C})_{\mathrm{gn}}$ where:

(A4) Existence of General Union
If $M_i (i < \delta)$ is $\leq_{\mathfrak{K}}$-increasing then

$$M := \bigcup_{j<\delta} M_j \in K \text{ and } M_j \leq_{\mathfrak{K}} M \text{ for } j < \delta.$$

The second group deals with the "algebraic closure".

Group B.

(B0) if $B = \langle A \rangle^{\mathrm{gn}}_M$ then $A \subseteq M \in K$ and $A \subseteq B \subseteq M$

(B1) If $B = \langle A \rangle^{\mathrm{gn}}_M$ then $\langle B \rangle^{\mathrm{gn}}_M = B$

(B2) If $A \subseteq B \subseteq M$ then $\langle A \rangle^{\mathrm{gn}}_M \subseteq \langle B \rangle^{\mathrm{gn}}_M$

(B3) If $A \subseteq M \leq_{\mathfrak{K}} N$ then $\langle A \rangle^{\mathrm{gn}}_M = \langle A \rangle^{\mathrm{gn}}_N$.

The third group of axioms deals with stable amalgamation.

Group C_{gn}.

(C0) NF is a four-place relation on K (we may say that M_1, M_2 are stably (or NF-stably) amalgamated over M_0 inside M instead $\mathrm{NF}(M_0, M_1, M_2, M))$; we usually shall not mention this axiom; we may say "in stable amalgamation" or say "M_0, M_1, M_2 are in stable amalgamation inside M".

(C1) If $\mathrm{NF}(M_0, M_1, M_2, M)$ then $M_0 \leq_{\mathfrak{K}} M_1 \leq_{\mathfrak{K}} M$, and $M_0 \leq_{\mathfrak{K}} M_2 \leq_{\mathfrak{K}} M$ (hence $M_0, M_1, M_2, M \in K$).

(C2) Existence for NF: If M_0, M_1 and M_2 such that $M_0 \leq_{\mathfrak{K}} M_1$ and $M_0 \leq_{\mathfrak{K}} M_2$, then there are M_1^*, M_2^*, M from K and f_1, f_2 such that: f_ℓ is an isomorphism from M_ℓ onto M_ℓ^* over M_0 for $\ell = 1, 2$ and $\mathrm{NF}(M_0, M_1^*, M_2^*, M)$.
Note that without loss of generality $M_1^* = M_1$ (so we can use $f_1 = \mathrm{id}_{M_1}$); also if $M_1 \cap M_2 = M_0$ then without loss of generality $M_1^* = M_1, M_2^* = M_2$ provided that we have disjoint amalgamation, i.e. $M_1^* \cap M_2^* = M_0$ above.

$(C2)^-$ Existence of amalgamation: if $M_0 \leq_{\mathfrak{K}} M_1$ and $M_0 \leq_{\mathfrak{K}} M_2$ then there are M_1^*, M_2^*, M from K and f_1, f_2 such that: f_ℓ is an isomorphism from M_ℓ onto M_ℓ^* for $\ell = 1, 2$ and $M_0 \leq_{\mathfrak{K}} M_1^* \leq_{\mathfrak{K}} M$ and $M_0 \leq_{\mathfrak{K}} M_2^* \leq_{\mathfrak{K}} M$
(i.e. amalgamation exists with no reference to NF).

(C3) Monotonicity:

- (a) $\mathrm{NF}(M_0, M_1, M_2, M)$ implies $\mathrm{NF}(M_0, M_1, M_2^*, M)$ when $M_0 \leq_{\mathfrak{K}} M_2^* \leq_{\mathfrak{K}} M_2$
- $(a)^d$ $\mathrm{NF}(M_0, M_1, M_2, M)$ implies $\mathrm{NF}(M_0, M_1^*, M_2, M)$ when $M_0 \leq_{\mathfrak{K}} M_1^* \leq_{\mathfrak{K}} M_1$
- (b) $\mathrm{NF}(M_0, M_1, M_2, M), M \leq_{\mathfrak{K}} M^*$ implies $\mathrm{NF}(M_0, M_1, M_2, M^*)$
- (c) $\mathrm{NF}(M_0, M_1, M_2, M)$, $M_1 \cup M_2 \subseteq M^* \leq_{\mathfrak{K}} M$ implies $\mathrm{NF}(M_0, M_1, M_2, M^*)$; [note that the superscript d stands for dual, actually (a)^d follows from (a) and symmetry].

(C4) Base enlargement: If $\mathrm{NF}(M_0, M_1, M_2, M)$ and $M_0 \leq_{\mathfrak{K}} M_0' \leq_{\mathfrak{K}} M_2$, then $\mathrm{NF}(M_0', \langle M_1 \cup M_0' \rangle_M^{\mathrm{gn}}, M_2, M)$ so in particular $\langle M_1 \cup M_0' \rangle_M^{\mathrm{gn}}$, that is M restricted to this set, belongs to K and is $\leq_{\mathfrak{K}} M$.

(C5) Uniqueness: If for $\ell = 1, 2$, $\mathrm{NF}(M_0^\ell, M_1^\ell, M_2^\ell, M^\ell)$ for $\ell = 1, 2$ and for $m = 0, 1, 2$ the mapping f_m is an isomorphism from M_m^1 onto M_m^2 and $f_0 \subseteq f_1, f_0 \subseteq f_2$ then for some $N \in K$ satisfying $M^2 \leq_{\mathfrak{K}} N$ there is a $\leq_{\mathfrak{K}}$-embedding h of M^1 into N, which extends $f_1 \cup f_2$.

(C6) Symmetry: $\mathrm{NF}(M_0, M_1, M_2, M)$ implies $\mathrm{NF}(M_0, M_2, M_1, M)$.

(C7) Finite Character: if $\langle M_{1,i} : i \leq \delta \rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous, $M_0 \leq_{\mathfrak{K}} M_{1,0}$ and $\mathrm{NF}(M_0, M_{1,\delta}, M_2, M)$ <u>then</u> $\langle M_{1,\delta} \cup M_2 \rangle^{\mathrm{gn}}_M = \bigcup\limits_{i<\delta} \langle M_{1,i} \cup M_2 \rangle^{\mathrm{gn}}_M$.

1.7 Remark. 1) Below when using $\langle A \rangle^{\mathrm{gn}}_N$, we may always assume the group (B) of axioms; in any case we always assume Ax(B0).
2) Note that Ax(C1),(C2),(C3)(a),(c) implies Ax(A3); but we are assuming (A).
3) If we use NF we may as well assume Ax(C0) and Ax(C1).
4) We shall not pay much attention to not using all the axiom group (B), as it will not be used.

1.8 Definition. 1) AxFr_1^+ is defined like AxFr_1 adding axiom (C8), see below.
2) We define two more axioms.

(C8) If $\langle M_{1,i} : i < \delta \rangle$ is $\leq_{\mathfrak{K}}$-increasing and $\mathrm{NF}(M_0, M_{1,i}, M_2, M)$ for each $i < \delta$ <u>then</u> for some $M_{1,\delta}$ and M' we have $M \leq_{\mathfrak{K}} M'$ and $(\forall i < \delta)(M_{1,i} \leq_{\mathfrak{K}} M_{1,\delta})$ and $\mathrm{NF}(M_0, M_{1,\delta}, M_2, M')$.

(C8)* Like (C8) adding that $M_{1,\delta} = \cup\{M_{1,i} : i < \delta\}$.

1.9 Definition. The Existential Framework: (AxFr_3)
Here $\mathfrak{K} = (K, \leq_{\mathfrak{K}}, \mathrm{NF})$.
We have 1.3 and Axioms (A0) - (A3) from 1.4 and $(\mathrm{A4})^-$ and $(\mathrm{C})_{\mathrm{ex}}$ where:

$(A4)^-$ Limit Existence: If $\langle M_i : i < \delta \rangle$ is $\leq_{\mathfrak{K}}$-increasing, <u>then</u> there is $M \in K$ satisfying $M_i \leq_{\mathfrak{K}} M$ for $i < \delta$.

Group C_{ex}. $\mathrm{Ax}(C0), (C1), (C2), (C3), (C5), (C6)$.

<u>1.10 Discussion</u>: In the first order case one defines prime models over arbitrary subsets of members of K. But this cannot be expected generally, and experience has shown that it suffices for many purposes to have prime models only in more specific cases: over unions of chains and over pairs of independent models. The following axioms describe the properties of such prime models.

There are (at least) three ways in which one could introduce prime models; first locally (i.e. within a specified model), second relatively or within compatibility and thirdly absolutely. (The compatibility class of N over $\bar{N}$ is $\{N' \in K$: for some $N^* \in K$ we have $N \leq_{\mathfrak{K}} N^*$ and $N' \leq_{\mathfrak{K}} N^*$ and $N_i \leq_{\mathfrak{K}} N'$ for each $i\}$; we may consider the closure of compatibility to an equivalence relation.) The axioms below are the compatibility version; we describe the absolute version in Definition 1.15; at present the relative version does not seem useful.

1.11 Definition. The Primal Framework (AxFr_2) is: to (AxFr_3) we add

<u>Group D: On prime models</u>
$(D1)$ If $\langle M_i : i \leq \delta \rangle$ is $\leq_{\mathfrak{K}}$-increasing <u>then</u> there is a model $N \leq_{\mathfrak{K}} M_\delta$ satisfying $(\forall i < \delta)[M_i \leq_{\mathfrak{K}} N]$ such that: if $(\forall i < \delta)[M_i \leq_{\mathfrak{K}} N' \leq_{\mathfrak{K}} N^*]$ and $N \leq_{\mathfrak{K}} N^*$ then there is a $\leq_{\mathfrak{K}}$-embedding f of N into N' over $\bigcup\limits_{i<\delta} M_i$.
We write in this case $\mathrm{rPr}(\langle M_i : i < \delta \rangle, N)$ and say that N is relatively prime over $\langle M_i : i < \delta \rangle$.

(D2) If $\mathrm{NF}(M_0, M_1, M_2, M_3)$ <u>then</u> there is N relatively prime over $M_1 \cup M_2$ inside M_3, i.e.:

- (i) $M_1 \cup M_2 \subseteq N \leq_{\mathfrak{K}} M_3$ and
- (ii) for every M, M_3^*, if $M_1 \leq_{\mathfrak{K}} M_3^*, M_2 \leq_{\mathfrak{K}} M_3^*$ and $M_3^* \leq_{\mathfrak{K}} M$ and $N \leq_{\mathfrak{K}} M$ <u>then</u> there is a $\leq_{\mathfrak{K}}$-embedding f of N into M_3^* over $M_1 \cup M_2$.

We write in this case $\mathrm{rPr}(M_0, M_1, M_2, N)$.

$\mathrm{Ax(C4)_{pr}}$ Base enlargement: If $\mathrm{NF}(M_0, M_1, M_2, M_3)$ and $M_0 \leq_{\mathfrak{s}} M_0' \leq_{\mathfrak{s}} M_2$ then:

(a) we can find M'_1, M'_3 such that $M_1 \cup M'_0 \subseteq M'_1 \leq_{\mathfrak{s}} M'_3, M_3 \leq_{\mathfrak{s}} M'_3$ and $\mathrm{NF}(M'_0, M'_1, M_2, M'_3)$;

(b) moreover the last assertion follows if $\mathrm{rPr}(M_0, M'_0, M_1, M'_1)$.

1.12 Definition. $(\mathrm{AxFr}_2)^+$, the primal framework with uniqueness means that to (AxFr_2) we add:
(D3) Uniqueness of the prime model over $\langle M_i : i < \delta\rangle$:
If $\mathrm{rPr}(\langle M_i : i < \delta\rangle, N^\ell)$ and $N^\ell \leq_{\mathfrak{K}} N$ for $\ell = 1, 2$ then N^1, N^2 are isomorphic over $\bigcup_{i<\delta} M_i$.

(D4) Uniqueness of the Prime Model over $M_1 \cup M_2$:
If $\mathrm{rPr}(M_0, M_1, M_2, N^\ell)$ and $N^\ell \leq_{\mathfrak{K}} N$ for $\ell = 1, 2$ then N^1, N^2 are isomorphic over $M_1 \cup M_2$. More exactly: if $\mathrm{rPr}(M^\ell_0, M^\ell_1, M^\ell_2, N^\ell)$ and $N^\ell \leq_{\mathfrak{K}} N$ for $\ell = 1, 2$ and f_i is an isomorphism from M^1_i onto M^2_i for $i = 0, 1, 2$ and $f_0 \subseteq f_1, f_0 \subseteq f_2$ then there is an isomorphism f from N^1 onto N^2 extending f_1 and f_2 (what is the difference? well we have to consider $M^\ell_1 \cap M^\ell_2 \backslash M^\ell_0$ for $\ell = 1, 2$).

1.13 Definition. The NF for elements framework (AxFr_4)
Here $\mathfrak{s} = (\mathfrak{K}, \leq_{\mathfrak{K}}, \mathrm{NF}^e)$.
We have here Ax(A0) - (A3),(A4)$^-$ and[2] the group (E) where

Group E:

(E1) $\mathrm{NF}^e(M_0, M_1, a, M_3)$ implies: $M_0 \leq_{\mathfrak{K}} M_1 \leq_{\mathfrak{K}} M_3$ and $a \in M_3$.

(E2) Existence: For every M_0, M_1, M_2, a such that $a \in M_2, M_0 \leq_{\mathfrak{K}} M_1, M_0 \leq_{\mathfrak{K}} M_2$ there are M and f, such that $M_1 \leq_{\mathfrak{K}} M, f$ is a $\leq_{\mathfrak{K}}$-embedding of M_2 into M over M_0, and $\mathrm{NF}^e(M_0, M_1, f(a), M)$.

(E3) Monotonicity:

(a) $\mathrm{NF}^e(M_0, M_1, a, M)$ and $M_0 \leq_{\mathfrak{K}} M^*_1 \leq_{\mathfrak{K}} M_1$ implies $\mathrm{NF}^e(M_0, M^*_1, a, M)$

[2] it is natural to demand Ax(C2)$^-$ or at least enough instances of it

(b) $\mathrm{NF}^e(M_0, M_1, a, M)$ and $M \leq_{\mathfrak{K}} M^*$ implies $\mathrm{NF}^e(M_0, M_1, a, M^*)$

(c) $\mathrm{NF}^e(M_0, M_1, a, M), M_1 \cup \{a\} \subseteq M^* \leq_{\mathfrak{K}} M$ implies $\mathrm{NF}^e(M_0, M_1, a, M^*)$

(E4) Base Enlargement: $\mathrm{NF}^e(M_0, M_1, a, M)$ and $M_0 \leq M_0^* \leq M_1$ implies $\mathrm{NF}^e(M_0^*, M_1, a, M)$.

(E5) Uniqueness: Suppose $M_0 \leq_{\mathfrak{K}} M_1 \leq_{\mathfrak{K}} M$, $\mathrm{NF}^e(M_0, M_1, a, M)$, $\mathrm{NF}^e(M_0, M_1, b, M)$ and $M_0 \cup \{a\} \subseteq N^a \leq_{\mathfrak{K}} M$, $M_0 \cup \{b\} \subseteq N^b \leq_{\mathfrak{K}} M$, and there is an isomorphism from N^a onto N^b over M_0 mapping a to b then there are N_a, N_b, M^* and f such that: $M \leq_{\mathfrak{K}} M^*, M_1 \cup \{a\} \subseteq N_a \leq_{\mathfrak{K}} M^*, M_1 \cup \{b\} \subseteq N_b \leq_{\mathfrak{K}} M^*$ and f is an isomorphism from N_a onto N_b over M_1 mapping a to b.

(E6) Continuity: If $\langle M_{1,i} : i < \delta \rangle$ is $\leq_{\mathfrak{K}}$-increasing, $\langle M_i : i < \delta \rangle$ is $\leq_{\mathfrak{K}}$-increasing and $\mathrm{NF}^e(M_0, M_{1,i}, a, M_i)$ for every $i < \delta$, then we can find $M_{1,\delta}$ and M_δ such that $M_{1,i} \leq_{\mathfrak{K}} M_{1,\delta}$ and $M_i \leq_{\mathfrak{K}} M_\delta$ (for $i < \delta$) and $\mathrm{NF}^e(M_0, M_{1,\delta}, a, M_\delta)$.

Here are some properties which do not obviously follow from the axioms we have given but are plausible additional axioms. As an example of their use note that the proof of V.E.1.3(1) is carried out without recourse to (F1), disjointness for NF; but assuming it would materially simplify the proof.

1.14 Definition. We define additional properties of frames $\mathfrak{s}$:
0) (C10),(Rigidity) If $\mathrm{NF}(M_0, M_1, M_2, M_3)$ and $M_3 = \langle M_1 \cup M_2 \rangle^{\mathrm{gn}}_{M_3}$ then the only automorphism of M_3 over $M_1 \cup M_2$ is the identity.
(1)(F1) Disjointness: $\mathrm{NF}(M_0, M_1, M_2, M_3)$ implies $M_1 \cap M_2 = M_0$.
(F2) Disjointness: if $\mathrm{NF}^e(M_0, M_1, a, M_3)$ and $a \notin M_0$ then $a \notin M_1$.
$(A5)^-$ Limit Uniqueness: If $\langle M_i : i < \delta \rangle$ is $\leq_{\mathfrak{K}}$-increasing and for $\ell = 1, 2$ and $[i < \delta \Rightarrow M_i \leq_{\mathfrak{K}} N^\ell]$ then there is $N, N^2 \leq_{\mathfrak{K}} N$ and a $\leq_{\mathfrak{K}}$-embedding f of N^1 into $N, f \restriction M_i = \mathrm{id}_{M_i}$ for $i < \delta$.
(2)(G1) Connecting NF^e to NF: if $\mathrm{NF}^e(M_0, M_1, a, M_3)$, then there are M_2', M_3' such that $M_0 \cup \{a\} \subseteq M_2' \leq_{\mathfrak{K}} M_3'$ and $\mathrm{NF}(M_0, M_1, M_2', M_3')$.

1.15 Definition. Let $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ be as in 1.4. Parts (1) and (2) of the following define the absolute notion of prime. (As hoped for, analogue of §1 of Chapter V.C would derive from (D1) a dichotomy between condition (1) and non-structure.)
(1) N is prime over $\langle M_i : i < \delta \rangle$, (where M_i is $\leq_{\mathfrak{K}}$-increasing) <u>if</u>:

(a) $M_i \leq_{\mathfrak{K}} N$ for $i < \delta$ and
(b) if $(\forall i < \delta) M_i \leq_{\mathfrak{K}} N^*$ then N can be $\leq_{\mathfrak{K}}$-embedded into N^* over $\bigcup\limits_{i<\delta} M_i$.

We write $\mathrm{Pr}(\langle M_i : i < \delta \rangle, N)$ for this.
(2) N is a prime stable amalgamation of M_1, M_2 over M_0 when:

(a) $\mathrm{NF}(M_0, M_1, M_2, N)$ and
(b) there is a $\leq_{\mathfrak{K}}$-embedding N into M^* extending $f_1 \cup f_2$ <u>when</u>:
 (α) $\mathrm{NF}(M_0, M_1^*, M_2^*, M^*)$,
 (β) f_1 an isomorphism from M_1 onto M_1^* over M_0
 (γ) f_2 an isomorphism from M_2 onto M_2^* over M_0.

We write $\mathrm{Pr}(M_0, M_1, M_2, N)$ for this.
(3) For $M \in K$ we define a relation $\mathscr{E}_M^{\mathrm{at}}$ between pairs $(\bar{a}, N), \bar{a} \in N, M \leq_{\mathfrak{K}} N$ as follows: $(\bar{a}_1, N_1)\mathscr{E}_M^{\mathrm{at}}(\bar{a}_2, N_2)$ <u>if and only if</u> there are $N_1^*, N_1^+, N_2^*, N_2^+, f$ such that:

(a) $M \leq_{\mathfrak{K}} N_1^* \leq_{\mathfrak{K}} N_1^+, N_1 \leq_{\mathfrak{K}} N_1^+$
(b) $M \leq_{\mathfrak{K}} N_2^* \leq_{\mathfrak{K}} N_2^+, N_2 \leq_{\mathfrak{K}} N_2^+$
(c) $\bar{a}_1 \in N_1^*$
(d) $\bar{a}_2 \in N_2^*$
(e) f is an isomorphism from N_1^* onto N_2^* over M mapping $\bar{a}_1$ to $\bar{a}_2$.

(4) $\mathscr{E}_M^{\mathrm{tp}}$ will be the closure of $\mathscr{E}_M^{\mathrm{at}}$ to an equivalence relation and $\mathbf{tp}(\bar{a}, M, N)$ is $(\bar{a}, N)/\mathscr{E}_M^{\mathrm{tp}}$ (note: if $\mathfrak{K}$ has amalgamation then $\mathscr{E}_M^{\mathrm{tp}} = \mathscr{E}_M^{\mathrm{at}}$).
(5) We say $\langle M_i : i < \alpha \rangle$ is $\leq_{\mathfrak{K}}$-increasing semi-continuous if it is

$\leq_{\mathfrak{K}}$-increasing and for each limit ordinal $\delta < \alpha, M_\delta$ is $\leq_{\mathfrak{K}}$-prime over $\langle M_i : i < \delta \rangle$.
(6) Of course, $\mathscr{E}^{\rm at}_M, \mathscr{E}^{\rm tp}_M, \mathbf{tp}$ depends on $\mathfrak{K}$ so we may write $\mathscr{E}^{\rm at}_{\mathfrak{K},M}, \mathscr{E}^{\rm tp}_{\mathfrak{K},M}$ and $\mathbf{tp}_{\mathfrak{K}}$, i.e., $\mathbf{tp}_{\mathfrak{K}}(\bar{a}, M, N)$ or use $\mathfrak{s}$ instead of $\mathfrak{K}_{\mathfrak{s}}$.

Now we note some interactions between the axioms and later define some related notions.

1.16 Definition. 1) $\mathfrak{K}$ has the λ-Löwenheim-Skolem property ($\lambda -$ LSP) if:

$$[A \subseteq M \text{ and } |A| \leq \lambda \leq \|M\|] \Rightarrow (\exists N \leq_{\mathfrak{K}} M)[A \subseteq N \text{ and } \|N\| = \lambda]$$

2) $\mathfrak{K}$ has the $(< \lambda)$-Lowenheim-Skolem property ($(< \lambda)$-LSP) means:

$$[A \subseteq M \text{ and } |A| < \lambda] \Rightarrow (\exists N \leq_{\mathfrak{K}} M)[A \subseteq N \text{ and } \|N\| < \lambda].$$

The $(\leq \lambda)$-LSP is defined similarly, it is quite closed to the λ-LSP.
3) LS($\mathfrak{K}$) is the minimal λ for which $\mathfrak{K}$ has λ-LSP. We also write $\chi_{\mathfrak{K}}$ for LS($\mathfrak{K}$) and LS($\mathfrak{s}$) $=$ LS($\mathfrak{K}_{\mathfrak{s}}$) $=$ LS$_{\mathfrak{s}}$.
4) Instead λ-LSP we also write LSP(λ). LSP(μ, λ) means that in (1), $|A| \leq \lambda$, $\|M\| \leq \mu$. We define LPS($< \mu, < \lambda$) etc., similarly.

1.17 Remark. The statement "$\lambda < \mu$ and the λ-Lowenheim-Skolem property $\Rightarrow$ μ-Lowenheim-Skolem property" will be considered.

1.18 Definition. 1) We say that $\mathfrak{K}$ (or $\mathfrak{s}$) has κ-smoothness when:

If $\langle M_i : i < \kappa \rangle$ is $\leq_{\mathfrak{K}}$-increasing, then there is N prime over $\langle M_i : i < \kappa \rangle$. (If Ax(A4) holds, e.g., in AxFr$_1$ this means: if each $M_i \leq_{\mathfrak{K}} M$ for $i < \kappa$ and $\langle M_i : i < \kappa \rangle$ is $\leq_{\mathfrak{K}}$-increasing, then $\bigcup\limits_{i<\kappa} M_i \leq_{\mathfrak{K}} M$).
1A) We may add "full" when the union is the prime.
2) The weak κ-smoothness means: if $\langle M_i : i < \kappa \rangle$ is $\leq_{\mathfrak{K}}$-increasing

semi-continuous and $i < \kappa \Rightarrow M_i \leq_{\mathfrak{K}} M$ <u>then</u> there is M_κ which is prime over $\langle M_i : i < \kappa \rangle$. [Semi-continuous is defined in 1.15(5). This condition is weaker than 1.18(1) since we have assumed the $\langle M_i : i < \kappa \rangle$ is semi-continuous].
3) Let (λ, κ)-smoothness be defined as in (1) but demanding $\|M_i\| \leq \lambda$, and $\|M\| \leq \lambda + \kappa$. Let $(\lambda, \kappa)^+$ -smoothness be defined as in (1) but demanding $\|M_i\| \leq \lambda$ for $i < \kappa$. Let (λ, μ, κ)-smoothness be defined in (1) but demanding $\|M_i\| \leq \lambda$ for $i < \kappa$ and $\|M\| \leq \mu$.
4) $(< \kappa)$-smoothness, etc. has the obvious meaning.
5) $\mathfrak{K}$ has smoothness (or is smooth) <u>if</u> it has κ-smoothness for every κ.
6) Let Ax(A4)$^-_\kappa$ mean that if $\langle M_i : i < \delta \rangle$ is $\leq_{\mathfrak{s}}$-increasing and $\mathrm{cf}(\delta) = \kappa$ <u>then</u> for some M we have $i < \delta \Rightarrow M_i \leq_{\mathfrak{s}} M$.
7) Let Ax(A4)$^-_{\lambda,\kappa}$ be defined similarly except that we assume $i < \delta \Rightarrow \|M_i\| < \lambda$ and we demand $\|M\| < \lambda$.
8) We define Ax(A4)$_\kappa$, Ax(A4)$_{\lambda,\kappa}$ similarly only $M = \cup\{M_i : i < \delta\}$. Let Ax(A6)$_\kappa$ mean: if $\langle M_\alpha : \alpha \leq \kappa \rangle$ is $\leq_{\mathfrak{s}}$-increasing then $i < \kappa \Rightarrow M_i \leq_{\mathfrak{s}} \bigcup\limits_{i<\kappa} M_i$, see V.F§2. Let (A6) mean (A6)$_\kappa$ for every regular κ, and (A6)$_\kappa$, (A6)$_{\lambda,\kappa}$, etc. similarly.
9) Let

$(A4)_*$ if $\langle M_\alpha : \alpha < \delta \rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous then $\alpha < \delta \Rightarrow M_\alpha \leq_{\mathfrak{K}} \cup\{M_\beta : \beta < \delta\}$.

10) Let (A4)$^*_\theta$ be like (A4)$_*$ but $\delta = \theta$, (A4)$^*_{<\theta}$, etc. are defined naturally.

<u>1.19 Exercise</u>: Check where we use (A4) rather than (A4)$_*$, see Definition 1.18(9).
[See more in V.C§1.]

1.20 Observation. 1) $\mathfrak{K}$ satisfies Ax(A4), and moreover smoothness when it satisfies

- (a) over every $\leq_{\mathfrak{K}}$-increasing sequence there is a prime and
- (b) if N is prime over $\langle M_i : i < \delta \rangle$ which is a $\leq_{\mathfrak{K}}$-increasing sequence then $N = \cup\{M_i : i < \delta\}$.

1A) If $\mathfrak{K}$ satisfies $(A4)^-$, see Definition 1.9, and is smooth then (A4) holds.
2) So smoothness and (A4) are together paralllel to the Tarski-Vaught theorem on elementary chains in this context and the union being prime.
3) $\mathfrak{K}$ has κ-smoothness if for every regular $\theta \le \kappa$, it has weak θ-smoothness.
4) [weak] κ-smoothness is equivalent to [weak] $\mathrm{cf}(\kappa)$-smoothness.
5) Our framework is $(< \kappa)$-smooth if and only if our framework in weakly $(< \kappa)$-smooth.

Remark. 1) But if we shall weaken Ax(A4) to, e.g., cases of cofinality $> \aleph_0$, then being prime is a new notion.
2) So with smoothness, in axiom (A4) it does not matter if we have or have not "$\langle M_i : i < \delta \rangle$ is not just $\le_{\mathfrak{K}}$-increasing but also continuous".
3) On 1.21(1) see more in V.C.1.2.

1.21 Lemma. *1) [*AxFr_1 *or just (B0),(C1),(C4)].*

If $\mathrm{NF}(M_0, M_1, M_2, M)$ *then* $M_3 := \langle M_1 \cup M_2 \rangle^{\mathrm{gn}}_M$ *(i.e. the restriction of M to this set is well defined), is a member of K and $M_1 \cup M_2 \subseteq M_3 \le_{\mathfrak{K}} M$ hence $M_1 \le_{\mathfrak{K}} M_3$ and $M_2 \le_{\mathfrak{K}} M_3$.*
2) [AxFr_1 *or just* $(B0), (B3), (C2)^-$*].*
Suppose that the conclusion of 1.21(1) holds, then Ax*(C5), uniqueness is equivalent to:*

- $(*)$ *if* $\mathrm{NF}(M^\ell_0, M^\ell_1, M^\ell_2, M^\ell)$ *for $\ell = 1, 2$ and for $m = 0, 1, 2$, f_m is an isomorphism from M^1_m onto M^2_m and $f_0 \subseteq f_1, f_0 \subseteq f_2$ then $f_1 \cup f_2$ can be extended to an isomorphism from $\langle M^1_1 \cup M^1_2 \rangle^{\mathrm{gn}}_{M^1}$ onto $\langle M^2_1 \cup M^2_2 \rangle^{\mathrm{gn}}_{M^2}$.*

3) AxFr_1^+ *implies* AxFr_1 *which implies* AxFr_3 *and* AxFr_1+ *smoothness implies* AxFr_2 *which implies* AxFr_3*.*
4) Ax*(C8) follows from (A4);(B) or just (B0),(B2),(B3);(C1),(C2), (C3), (C4),(C5) and smoothness (see Definition 1.18). Moreover we get $(C8)^*$, that is we can add $M_{1,\delta} = \cup\{M_{1,i} : i < \delta\}$.*
5) If Ax*(C1),(C3)(c),* Ax*(C5) and* $\mathrm{rPr}(M_0, M_1, M_2, M)$*, then M is a prime stable amalgamation of M_1, M_2 over M_0; recalling Definition 1.15 and* Ax*(D2) from Definition 1.11 which defines* $\mathrm{rPr}(\dots)$*.*

6) If (C3)(a),(C3)(c),(A4) and $\mathfrak{K}$ is smooth, then Ax*(C8) equivalent to* Ax*(C8)***, that is we can add* $M_{1,\delta} = \cup\{M_{1,i} : i < \delta\}$ *and* $M' = M_3$.
7) If (B0),(B3),(C1),(C2),(C3)(c),(C4),(C5) then prime stable amalgamation exists. In fact if $\mathrm{NF}(M_0, M_1, M_2, M_3)$ *and* $M_3 = (M_1 \cup M_2)^{\mathrm{gn}}_{M_3}$ *then* $\mathrm{Pr}(M_0, M_1, M_2, M_3)$.

Proof. 1) Apply Ax(C4) with $M'_0 := M_2$. [Note $M_0 \leq_{\mathfrak{K}} M'_0$ as $M_0 \leq_{\mathfrak{K}} M_2$ by Ax(C1) and $M'_0 \leq_{\mathfrak{K}} M_2$ by Ax(A0)]. So $\mathrm{NF}(M_2, \langle M_1 \cup M_2\rangle^{\mathrm{gn}}_M, M_2, M)$. Now by Ax(B0),(C1) this implies $M_1 \cup M_2 \subseteq \langle M_1 \cup M_2\rangle^{\mathrm{gn}}_M \leq_{\mathfrak{K}} M$.
2) First, assume that $(*)$ holds, and we shall prove Ax(C5); let f_3 be as guaranteed in $(*)$, i.e. f_3 is an isomorphism from $M^1_3 = \langle M^1_1 \cup M^1_2\rangle^{\mathrm{gn}}_{M^1}$ onto $M^2_3 = \langle M^2_1 \cup M^2_2\rangle^{\mathrm{gn}}_{M^2}$, and by part (1) of the claim $M^\ell_3 \leq_{\mathfrak{K}} M^\ell$ for $\ell = 1, 2$. Trivially, there is a pair (M_4, f_4) such that $\mathrm{M}^2_3 \leq_{\mathfrak{K}} M_4$ and f_4 is an isomorphism from M^1 onto M_4 extending f_3. By the existence axiom Ax(C2), in fact just Ax(C2)$^-$ there is a pair (M_5, f_5) such that $M^2 \leq_{\mathfrak{K}} M_5$ and f_5 is a $\leq_{\mathfrak{K}}$-embedding of M_4 into M_5 over M^2_3. So the pair $(f_5 \circ f_4, M_5)$ satisfies the demand on (h, N) in Ax(C5).

Second, assume that Ax(C5) holds and we should prove $(*)$ of 1.21(2); this is easy too, (or see the proof of V.C.1.2).
3) First, AxFr^+_1 implies AxFr_1 by Definition 1.8 as the former has all the axioms of the latter. Second, AxFr_1 implies AxFr_3 by Definition 1.9 as Ax(A4) $\Rightarrow$ Ax(A4)$^-$. Third, AxFr_2 implies AxFr_3 by Definition 1.11. Fourth, AxFr_1+ smoothness implies AxFr_2, for this we have to check just the axioms of AxFr_2 not from AxFr_3, i.e. (D1),(D2) from Definition 1.11, now (D1) holds by Definition 1.18(1),(5), and Ax(D2) holds by 1.21(2).
4) Let M_2, M_0, M and $\langle M_{1,i} : i < \delta\rangle$ be as in the assumption of Ax(C8), that is $M_{1,i} \leq_{\mathfrak{K}} M_{1,j}$ for $i < j < \delta$ and $\mathrm{NF}(M_0, M_{1,i}, M_2, M)$ for $i < \delta$ where, of course, δ is a limit ordinal. For $i \leq \delta$ let $M'_{1,i}$ be $\cup\{M_{1,j} : j < i\}$ when i is a limit ordinal and $M_{1,i}$ when i is a non-limit ordinal.

So by smoothness and Ax(A4) the sequence $\langle M'_{1,i} : i < \delta\rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous, and by Ax(A4) also the sequence $\langle M'_{1,i} : i \leq \delta\rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous. By Ax(C2) we can find a triple

$(f, M''_{1,\delta}, M')$ such that $\mathrm{NF}(M_0, M''_{1,\delta}, M_2, M')$ and f is an isomorphism from $M'_{1,\delta}$ onto $M''_{1,\delta}$ over M_0. So by part (1) and Ax(C3)(c) without loss of generality $M' = \langle M''_{1,\delta} \cup M_2\rangle^{\mathrm{gn}}_{M'}$. By Ax(C3)(a)^d for each $i < \delta$ we have $\mathrm{NF}(M_0, f(M'_{1,i}), M_2, M')$ hence as above there is $M'_{2,i} \le_{\mathfrak{K}} M'$ such that $M'_{2,i} = \langle f(M'_{1,i}) \cup M_2\rangle^{\mathrm{gn}}_{M'}$.

By Ax(B2) clearly $\langle M'_{2,i} : i \le \delta\rangle$ is $\subseteq$-increasing; as $M'_{2,i} \le_{\mathfrak{K}} M'$ it follows by Ax(A3) that $\langle M'_{2,i} : i \le \delta\rangle$ is $\le_{\mathfrak{K}}$-increasing. For non-limit $i \le \delta$ let $M''_{2,i} = M'_{2,i}$ so $M''_{2,i} \le_{\mathfrak{K}} M'$ and $M''_{2,i} = \langle f(M'_{1,i}) \cup M_2\rangle^{\mathrm{gn}}_{M'}$.

By smoothness and Ax(A4) of course, if $i \le \delta$ is a limit ordinal then $M''_{2,i} := \cup\{M'_{2,j} : j < i\}$ is $\le_{\mathfrak{K}} M'$ and includes $f(M'_{1,i}) \cup M_2$ hence $\mathrm{NF}(M_0, f_i(M'_{1,i}), M_2, M''_{2,i})$ by Ax(C3)(c). Now $\langle f(M'_{1,i}) \cup M\rangle^{\mathrm{gn}}_{M'} = \langle f(M'_{1,i}) \cup M\rangle^{\mathrm{gn}}_{M''_{2,i}} \subseteq M''_{2,i} = \cup\{M'_{2,j} : j < i\} \subseteq \langle \bigcup_{j<i} f(M'_{1,j}) \cup M_2\rangle^{\mathrm{gn}}_{M'} = \langle f(M'_{1,i}) \cup M_2\rangle^{\mathrm{gn}}_{M'}$ hence $M''_{2,i} = \langle f(M'_{1,i}) \cup M_2\rangle^{\mathrm{gn}}_{M'}$. It follows that $M''_{2,i} = \langle f_i(M'_{1,i}) \cup M_2)^{\mathrm{gn}}_{M'} \subseteq M'_{2,i}$ for $i \le \delta$, combining the statements for non-limit and limit $i \le \delta$. So $\langle M''_{2,j} : j \le \delta\rangle$ is not only $\le_{\mathfrak{s}}$-increasing but also continuous.

Let $f_i = f \restriction M'_{1,i}$. Now by induction on $i \le \delta$ we choose g_i, a $\le_{\mathfrak{K}}$-embedding of $M''_{2,i}$ into M such that $g_i \supseteq f_i^{-1} \cup \mathrm{id}_{M_2}$ is increasing and continuous (in i):

- For $i = 0$ this is trivial.
- For i limit take unions and use smoothness.
- For $i = j + 1$ we have $\mathrm{NF}(f_j(M'_{1,j}), f_i(M'_{1,i}), M_2, M''_{2,i})$ and also $\mathrm{NF}(M'_{1,j}, M'_{1,i}, M_2, M)$ by monotonicity, Ax(C3)(a)^d and base enlargement, Ax(C4). Hence by uniqueness Ax(C5), more exactly $(*)$ of part (2) we can choose g_i.

Having carried the induction, using g_δ we get $\mathrm{NF}(M_0, M'_{1,\delta}, M_2, g_\delta(M''_{2,\delta}))$ and $g_\delta(M''_{2,\delta}) \le_{\mathfrak{K}} M$ and $M'_{1,\delta} \cup M_2 \subseteq g_\delta(M''_{2,\delta})$. So by monotonicity Ax(C3)(b) we get $\mathrm{NF}(M_0, M'_{1,\delta}, M_2, M)$, as $M'_{1,\delta} = \cup\{M'_{1,i} : i < \delta\} = \cup\{M_{1,i} : i < \delta\}$ we are done.

5)-7) The other proofs are left to the readers. $\square_{1.21}$

There are more implications. Note that the claim above (i.e. 1.21(4)) says that in AxFr_1^++ smoothness, Ax(C8) is redundant so AxFr_1+ smoothness is equivalent to AxFr_1^++ smoothness; moreover we deduce the strong version of Ax(C8). Smoothness is necessary for this.

1.22 Example: A) $\mathfrak{K}$ is the class of metric graphs:

(a) $\tau_K = \{R, P_q : q$ a positive rational$\}$ where R, P_q are binary predicates

(b) A $\tau_{\mathfrak{K}}$-model M belongs to K when

(α) R^M is a symmetric irreflexive two-place relation

(β) P_q^M is symmetric

(γ) $\mathbf{d}^M$ is a metric on M when $\mathbf{d}^M(a,b) := \inf\{q : (a,b) \in P_q^M\}$

(δ) $P_q^M = \{(a,b) : \mathbf{d}^M(a,b) < q\}$

(c) $M_1 \le_{\mathfrak{K}} M_2$ iff $M_1 \subseteq M_2$ and under $\mathbf{d}^{M_1}$, M_1 is a closed subset of M_2.

B) We define $\mathfrak{s}$:

(a) $\mathfrak{K}$ as in (A)

(b) $\langle A \rangle_M^{\mathrm{gn}} = A$

(c) $\mathrm{NF}(M_0, M_1, M_2, M_3)$ iff:

(α) $M_0 \le_{\mathfrak{K}} M_\ell \le_{\mathfrak{K}} M_3$ for $\ell = 1, 2$

(β) if $a_1 \in M_1 \backslash M_0$ and $a_2 \in M_2 \backslash M_0$ then $(a_1, a_2) \notin R^{M_3}$

(γ) if $a_1 \in M_1 \backslash M_0$ and $a_2 \in M_2 \backslash M_0$ then $\mathbf{d}^M(a_1, a_2) = \inf\{\mathbf{d}^{M_1}(a_1, b) + \mathbf{d}^{M_2}(b, a_2) : b \in M_0\}$

(δ) $M_1 \cup M_2$ is a closed subset of M_3.

0) AxFr_2 holds for $\mathfrak{s}$.
1) AxFr_1 holds for $\mathfrak{s}$ and κ-smoothness holds iff $\mathrm{cf}(\kappa) > \aleph_0$.
2) The framework $\mathfrak{s}$ satisfies also Ax(D4),(F1). Also:
$\mathrm{rPr}(M_0, M_1, M_2, M_3)$ iff $\mathrm{Pr}(M_0, M_1, M_2, M_3)$ iff $M \cap N, M_0 \le_{\mathfrak{K}} M_\ell \le_{\mathfrak{K}} M_3$ for $\ell = 1, 2, M_1 \cap M_2 = M_0, |M_3| = |M_1| \cup |M_2|$ and $R^{M_3} = R^{M_1} \cup R^{M_2}$. But Ax(D1) fails, (use cases with $\mathrm{cf}(\delta) = \aleph_0$).
3) The framework $\mathfrak{s}$ fails $(\mathrm{AxFr}_2)^+$, as it fails Ax(C8).
[Hint: Note that if $\mathrm{cf}(\delta) = \aleph_0$ then the union $M_{1,\delta} := \cup\{M_{1,i} : i < \delta\}$ will satisfy $i < \delta \Rightarrow M_{1,i} \le_{\mathfrak{K}} M_{1,\delta} \in K$ but possibly $M_{1,\delta} \not\le_{\mathfrak{K}} M_\delta$.]

1.23 Example: Like the earlier but we omit R (from $\tau_{\mathfrak{K}}$ and so omit (A)(b)(α), (B)(c)(β).

Now all AxFr_1^+ is satisfied including Ax(C8) but not Ax(C8)* (because one has to take the closure of the union). Also $\aleph_0$-smoothness fails.

A canonical example of AxFr_2 is the following:
1.24 Example Let T be a first order complete strictly stable theory (i.e. stable not superstable).

Let K be the class of $|T|^+$-saturated models of T and $\mathfrak{K} = (K, \leq_{\mathfrak{K}}) = (K, \prec)$. Define NF by

$$\begin{aligned}\mathrm{NF}(M_0, M_1, M_2, M_3) \text{ iff } & M_0 \leq_{\mathfrak{K}} M_\ell \leq_{\mathfrak{K}} M_3 \\ & \text{and } \mathrm{tp}(\bar{a}, M_1, M_3) \text{ does not fork over } M_0 \\ & \text{for every } \bar{a} \in {}^{\omega>}(M_2).\end{aligned}$$

This is an example of AxFr_2 and we have primes, not just relatively prime.

1.25 Remark. In V.C§2 a central notion is "NF is κ-based" which means:

if $M \leq_{\mathfrak{K}} N, A \subseteq N, |A| \leq \kappa$ then there are M_0, M_1, such that $\mathrm{NF}(M_0, M, M_1, N)$, $\|M_1\| \leq \kappa$ and $A \subseteq M_1$.

1.26 Definition. 1) $\lambda_0(\mathfrak{s}) = \lambda_0(\mathfrak{K}) = \lambda_0(K)$ is the first λ such that K is a PC_λ-class, (equivalently the class of $\tau(K)$-reducts of models of some $\psi \in \mathbb{L}_{\lambda^+,\omega}$).
2) $\lambda_1(\mathfrak{s}) = \lambda_1(\mathfrak{K}) = \lambda(K, \leq_{\mathfrak{K}})$ is the first λ such that $\{(M, N) : M \in K, N \in K, N \leq_{\mathfrak{K}} M\}$ is a PC_λ-class.
3) $\lambda_2(\mathfrak{s}) = \lambda(\mathrm{NF}, \mathrm{gn})$ is the first cardinal λ such that $(M_0, M_1, M_2, M) : \mathrm{NF}(M_0, M_1, M_2, M)$ and $M = \langle M_1 \cup M_2\rangle^{\mathrm{gn}}_M\}$ is a PC_λ-class.
4) $\lambda_*(\mathfrak{s}) = \sum\limits_{\ell<3} \lambda_\ell(\mathfrak{s})$ and $\lambda_{\ell_1,\ell_2}(\mathfrak{s}) = \lambda_{\ell_1}(\mathfrak{s}) + \lambda_{\ell_2}(\mathfrak{s})$.
5) We define $\lambda_0^*(\mathfrak{s}) = \lambda_0^*(\mathfrak{K}) = \lambda_0^*(K)$, $\lambda_1^*(\mathfrak{s}) = \lambda_1^*(\mathfrak{K})$, $\lambda_2^*(\mathfrak{s}) =$

λ^*_2(NF,gn) and $\lambda^*(\mathfrak{s})$ similarly when we replace PC_λ by $PC_{\lambda,2^\lambda}$.
6) In all cases above, if there is no such λ then the result is ∞.

1.27 Definition. We say λ is $\mathfrak{s}$-inaccessible also written as $\le_{\mathfrak{s}}$-inaccessible [or $\mathfrak{K}$-inaccessible (also written as $\le_{\mathfrak{K}}$-inaccessible)] when:

- (A) if $M_0 \le_{\mathfrak{K}} M_1, M_0 \le_{\mathfrak{K}} M_2$ (all in K) each of cardinality $< \lambda$ such that we can $\le_{\mathfrak{K}}$-amalgamated M_1, M_2 over M_0 (which usually holds), then there is $M \in K, \|M\| < \lambda$, and for $\ell = 1,2$ $\le_{\mathfrak{K}}$-embeddings f_ℓ of M_ℓ into M over M_0 [and, for the $\mathfrak{s}$-version we also have $\mathrm{NF}_{\mathfrak{s}}(M_0, f_1(M_1), f_2(M_2), M)$]
- (B) if $\delta < \lambda, \|\bigcup\limits_{i<\delta} M_i\| < \lambda$ and $\langle M_i : i < \delta\rangle$ is $\le$-increasing, then for some $M \in K$ of cardinality $< \lambda$, we have $M_i \le_{\mathfrak{K}} M$ for $i < \lambda$.

The following definition of pseudo cardinality is an attempt to axiomatize the idea of a structure being generated by χ elements; we shall not use it.

1.28 Definition. [AxFr$_2$]
We define $\mathrm{pscard}^\chi_{\mathfrak{s}}(M)$ as follows:
(I) for $M \in K, \mathrm{pscard}^\chi_{\mathfrak{s}}(M) = \chi$ if $\|M\| \le \chi$
(II) for $M \in K, \lambda \ge \chi : \mathrm{pscard}^\chi_{\mathfrak{s}}(M) = \lambda$ iff:

- (i) for some $\le_{\mathfrak{s}}$-increasing sequence $\langle M_i : i < \delta\rangle$:
  - (a) $\delta \le \lambda$
  - (b) $\Pr(\langle M_i : i < \delta\rangle, M)$
  - (c) $\mathrm{pscard}^\chi_{\mathfrak{s}}(M_i) < \lambda$ for every $i < \delta$
- (ii) for no $\mu < \lambda, \mathrm{pscard}^\chi_{\mathfrak{s}}(M) = \mu$.

1.29 Remark. 1) Rather than defining pscard, we can use it as a basic function and put on it an axiom.
2) It may be more natural to say $\mathrm{pscard}^N_{\mathfrak{s}}(M)$ iff there is a directed

partial order I and $\bar{M} = \langle M_t : t \in I\rangle$ which is $\le_{\mathfrak{K}}$-increasing and "nice enough" each of cardinality $\le \chi$, M is prime over $\cup\{M_t : t \in I\}$ in a suitable sense. But too cumbersome for now.

§2 The Main Examples

We consider here three examples: first order, universal classes and (D, λ)-homogeneous models.

2.1 First Order Theories. *Let T be a stable first order theory. Assume that $T^{\rm eq}$ has elimination of quantifiers. Let*

(i) *$K = \{M : M$ is a submodel of some $N \models T^{\rm eq}$ and $|M| = ac\ell_N(M)\}$.*
(If you like - omit the unnecessary elements of N, note that $M \prec N$ is not demanded)

(ii) *$\le_{\mathfrak{K}}$ is being a submodel*

(iii) *let for some $N, M \subseteq N \models T^{\rm eq}$; then: $B = \langle A\rangle^{\rm gn}_M$ if and only if $A \subseteq M, B = ac\ell_N A$ (i.e., B is the algebraic closure of A inside N)*

(iv) *$A_\ell \subseteq N$ for $\ell < 4$, $N \models T^{\rm eq}$, <u>then</u> $\mathrm{NF}(A_0, A_1, A_2, A_3)$ holds if and only if:*
$A_\ell = ac\ell_N A_\ell$ for $\ell = 0, 1, 2, 3$, $A_0 \subseteq A_1 \subseteq A_3$ and
$A_0 \subseteq A_2 \subseteq A_3$ and $\mathrm{tp}_*(A_2, A_1)$ does not fork over A_0.

Remark. In this context "models" disappear. I.e. "model" in our context, is just an algebraically closed set. Later "λ-saturated model, $\lambda > |T|$" are defined. But "models of T" are not naturally defined in this context. As we prefer to have theorems which say something when specialized to this case, we will try to have non-structure theorems saying not only
"there are many $M \in K$" but
"there are many quite homogeneous ($\equiv$ quite saturated) models"
or at least
"there are many models in $K^{\rm us}_\mu$" (see Definition 3.20 below).

2.2 Fact. All axioms from §1 except Ax(C10) hold under those circumstances with the LS-number being $\le |T| + \aleph_0$.

Remark. So most of [Sh:c] can be done in this framework, and some of the proofs here are adaptations of proofs from [Sh:c] to our context under this translation.

* * *

2.3 Universal Classes.

2.4 Definition. A class K of $\tau(K)$-models is called universal <u>if</u> it is closed under submodels and under unions of increasing chains.

Remark. Recall that for $A \subseteq M, c\ell_M(A) = \{\sigma(\bar{a}) : \bar{a} \in A, \sigma(\bar{x})$ a τ_M-term, $\ell g(\bar{a}) = \ell g(\bar{x})\}$.

2.5 Claim. *The following are equivalent for a class K of $\tau(K)$-models:*

(i) *K is a universal class*

(ii) *a $\tau(K)$-model M belongs to K <u>iff</u> every finitely generated submodel of M belongs to K. (Of course, N is a finitely generated submodel of M when $N = M \restriction c\ell_M(A)$ for some finite $A \subseteq M$ where $c\ell_M(A)$ is the closure of A under the functions of M including the zero place functions, i.e. individual constants, and, of course, we assume that $A \neq \emptyset$ or at least $c\ell_M(A) \neq \emptyset$).*

Proof. Now (ii) $\Rightarrow$ (i) should be clear.
So assume (i). Let M be a $\tau(K)$-model. Now

(a) If $M \in K$ then every finitely generated submodel of M belongs to $\mathfrak{K}$.
[Why? It is true as "membership in K" is closed under being a submodel.]

(b) If every finitely generated submodel of M belongs to K then $M \in K$.

Why? We prove by induction on κ that if $N = N \restriction c\ell_M(A)$ where $A \subseteq M, |A| \le \kappa$ and every finitely generated submodel of N belongs to K, then $N \in K$.
For κ finite ($< \aleph_0$) it is trivial, i.e. holds by the assumptions.
For $\kappa \ge \aleph_0$ let $A = \{a_i : i < |A|\}$ and let

$$N_i = N \restriction c\ell_M\{a_j : j < 1 + i\}.$$

So N_i (for $i < |A|$) is $\subseteq$-increasing and $N = \bigcup_{i<\kappa} N_i$ (as the functions F^M have finite arity). Clearly every finitely generated submodel of N_i is a finitely generated submodel of M hence it belongs to K hence by the inductive hypothesis (as $1 \le |\{a_j : j < 1 + i\}| \le |i| < \kappa$) we have $N_i \in K$. But K, being universal, is closed under unions of increasing chains, hence

$$N = \bigcup_{i<\kappa} N_i \in K.$$

So we are done proving (b) hence 2.5. $\square_{2.5}$

Recall

2.6 Claim. *Let K be a universal class and $\Delta_{\rm qf}(\tau_K)$ be the set of first order quantifier free formulas in the vocabulary τ_K and we may write* qf *instead of $\Delta_{\rm qf}(\tau_K)$.*
0) K is a $\mathrm{PC}_{\kappa,2^\kappa}$ class, where $\kappa = |\tau_K| + \aleph_0$.
1) If K has the $(\mathrm{qf}, \beth_\alpha)$-order property, for every $\alpha < (2^{|\tau(K)|+\aleph_0})^+$ <u>then</u>

> $(*)$ *for some quantifier free $\mathbb{L}(\tau_K)$-formula $\varphi(\bar{x}, \bar{y}), K$ has the $(\varphi(\bar{x}; \bar{y}), \chi)$-order property for every χ.*

2) If K satisfies $(*)$, <u>then</u> for every $\lambda \ge \aleph_1 + |\tau_{\mathfrak{K}}|$, there are 2^λ non-isomorphic models in K_λ.

Proof. As this direction is not continued here we do not elaborate.
0) Let $\tau_1 = \tau_K$ and

$\Gamma = \{p(\bar{x})$: for some $n < \omega$ and τ_K-model N and sequence $\bar{a} \in {}^n N$
we have $p(\bar{x}) = \text{tp}_{\Delta_{\text{qf}}(\tau_K)}(\bar{a}, \emptyset, N)$ but there are no
$M \in K$ and $\bar{b} \in {}^n M$ such that $\bar{b}$ realizes $p(\bar{x})$ in $M\}$.

Clearly

$\boxdot_1$ K is the class of τ_K-models M omitting every $p(\bar{x}) \in \Gamma$ hence

$\boxdot_2$ K is a $\text{PC}_{\kappa,2^\kappa}$ class and even $\text{EC}_{\kappa,2^\kappa}$ one.

1) By part (0) we know that K is a $\text{PC}_{\kappa,2^\kappa}$ class. Now apply V.A.1.3(0).
2) By [Sh:e, III] or see [Sh 300, III], (in fact stronger non-structure theorems hold).

$\square_{2.6}$

This (in particular 2.6(1),(2)) is a reasonable justification of:

2.7 Hypothesis. K is a universal class and it has (qf, χ^+)-non-order, $\chi \geq |\tau(K)|$; the non-order means that for no $n < \omega$ and quantifier free (i.e. from $\Delta_{\text{qf}}(\tau_K)$) formula $\varphi = \varphi(\bar{x}, \bar{y}) = \varphi(\langle x_\ell : \ell < n\rangle, \langle y_\ell : \ell < n\rangle)$ do we have $M \in K$ and $\bar{a}_\alpha \in {}^n M$ for $\alpha < \chi^+$ such that $M \models \varphi[\bar{a}_\alpha, \bar{a}_\beta] \Leftrightarrow \alpha < \beta$.

2.8 Definition. We define $\mathfrak{s} = (K, \leq, \text{NF}, <>^{\text{gn}}) = (K_\mathfrak{s}, \leq_\mathfrak{s}, \text{NF}_\mathfrak{s}, <>^{\text{gn}}_\mathfrak{s})$ as follows, letting $\mu = 2^{2^\chi}$:

(*a*) $K_\mathfrak{s} = K$ so $\tau(\mathfrak{s}) = \tau(K)$

(*b*) $\leq_\mathfrak{s} = \leq^{\aleph_0}_{\text{qf},\mu,\chi}$, see V.A.4.1

(*c*) $\langle A\rangle^{\text{gn}}_N$ be $c\ell_N(A)$, i.e. the closure of A under the functions of N

(*d*) $\text{NF}(M_0, M_1, M_2, M_3)$ iff (they belong to K and)

  (α) $M_0 \leq_\mathfrak{s} M_2 \subseteq M_3$

  (β) $M_0 \leq_\mathfrak{s} M_1 \subseteq M_3$ (see 2.12)

(γ) if $\bar{c} \in {}^{\omega>}(M_1)$ and $\mathbf{J}$ is a (qf, χ^+)-convergent subset of ${}^{\ell g(\bar{c})}(M_0)$ inside M_0 of cardinality μ^+ such that $\mathrm{Av}_{\mathrm{qf}}(\mathbf{J}, M_0) = \mathrm{tp}_{\mathrm{qf}}(\bar{c}, M_0, M_1)$, then $\mathrm{Av}_{\mathrm{qf}}(\mathbf{J}, M_2) = \mathrm{tp}_{\mathrm{qf}}(\bar{c}, M_2, M_3)$

(δ) $\langle M_1 \cup M_2 \rangle^{\mathrm{gn}}_{M_3} \leq_{\mathfrak{s}} M_3$.

Remark. $\mathrm{NF}(M_0, M_1, M_2, M_3)$ is very close to "M_0, M_1, M_2 are in $(\mathrm{qf}, \mu, \chi, \aleph_0)$-stable amalgamation inside M_3 (see Definition V.A.4.9) and $\langle M_1 \cup M_2 \rangle^{\mathrm{gn}}_{M_3} \leq_{\mathfrak{s}} M_3$" and see V.A.4.10.

We shall below prove that $\mathrm{NF}(M_0, M_1, M_2, M_3)$ implies it, and inversely; note that in V.A.4.9, e.g. $M_\ell \leq_{\mathfrak{s}} M_3$ is not required and that M_3 here stands for M there. However, if $M_3 = \langle M_1 \cup M_2 \rangle^{\mathrm{gn}}_{M_3}$ then "M_0, M_1, M_2 are in $(\mathrm{qf}, \mu, \chi, \aleph_0)$-stable amalgamation" is equivalent to $\mathrm{NF}(M_0, M_1, M_2, M_3)$.

2.9 Lemma. *From the axioms from §1, $\mathfrak{s}$ satisfies* AxFr_1^+, *Ax(C8)*, Ax(C10) and* $\mathrm{LS}(\mathfrak{K}_{\mathfrak{s}}) \leq \mu^+$, *of course, $\mathfrak{s}$ is from Definition 2.8.*

Proof. Most are totally routine (using Lemma V.A.2.8, V.A.4.4). Note that we use types consisting of quantifier free formulas.

Now the Meta Axiom 1.3, preservation under isomorphism is obvious, (A0),(A1),(A3) hold by the definition (as we use quantifier free formulas) and (A2) holds by V.A.4.5. Lastly (A4) holds because the union $M_\delta = \cup\{M_i : i < \delta\}$ belongs to K because any finite sequence from the union of an increasing chain is a finite sequence from some of the models and $i < \delta \Rightarrow M_i \subseteq M_\delta$ hence $i < \delta \Rightarrow M_i \leq_{\mathrm{qf}} M_\delta$ and lastly $i < \delta \Rightarrow M_i \leq_{\mathfrak{K}} M_\delta$ by V.A.4.7, noting that $\aleph_0$ here stand for κ there.

The axioms (B0),(B1),(B2),(B3) hold trivially by our choice of $<>^{\mathrm{gn}}$. In the rest of the proof we shall rely on 2.10 - 2.13 below. Now Ax(C0) holds trivially. As for Ax(C1), assume $\mathrm{NF}_{\mathfrak{s}}(M_0, M_1, M_2, M_3)$ and let $M_3' = \langle M_1 \cup M_3 \rangle^{\mathrm{gn}}_{M_3}$; now $M_0 \leq_{\mathfrak{s}} M_1, M_0 \leq_{\mathfrak{s}} M_2$ holds by Definition 2.8, clauses $(d)(\alpha), (\beta)$ and $M_3' \leq_{\mathfrak{s}} M_3$ also holds by the definition of $\mathfrak{s}$, i.e., clause $(d)(\delta)$ of Definition 2.8. Now $M_2 \leq_{\mathfrak{s}} M_3$

by 2.12 below and $M_1 \leq_{\mathfrak{s}} M_3$ holds also by 2.12 below because $\mathrm{NF}_{\mathfrak{s}}(M_0, M_2, M_1, M_3)$ holds by symmetry, i.e. 2.11. So all the demands in Ax(C1) hold.

Ax(C2), existence holds by 2.10(1) below. As for monotonicity in M_1 and M_2, i.e. in Ax(C3)(a),(a)^d only clause (δ) of 2.8(d) is not obvious. For (C3)(a) we assume $\mathrm{NF}(M_0, M_1, M_2, M_3)$ and say $M_0 \leq_{\mathfrak{s}} M_2' \leq_{\mathfrak{s}} M_2$, by 2.13 letting $M_1' = \langle M_1 \cup M_2' \rangle^{\mathrm{gn}}_{M_3}$ we have $\mathrm{NF}(M_0', M_1', M_2', M_3)$ hence by 2.12 + symmetry we have $M_1' \leq_{\mathfrak{s}} M_3$, as required. Of course (C3)(a)^d follows by symmetry (and Ax(C3)(a)). As for Ax(C3)(b) upward monotonicity in M_3, it holds by Ax(A2), i.e. $\leq_{\mathfrak{s}}$ being transitive. Concerning Ax(C3)(c), downward monotonicity in M_3, only clause (δ) of 2.8(d) is not obvious and it holds by Ax(A3).

Now concerning Ax(C4), base enlargement, it holds by 2.13 and Ax(C5), uniqueness holds by 2.10(2) below and Ax(C6), symmetry, holds by 2.11 below and Ax(C7), finite character is trivial by the choice of $\langle - \rangle^{\mathrm{gn}}$. Lastly, Ax(C8) is easy; moreover, Ax(C8)* hold. Why? Let $M_{1,\delta} := \cup\{M_{1,i} : i < \delta\}$ and $\mathbf{J}$ be a (qf, χ^+)-convergent subset of ${}^{\ell g(\bar{c})}(M_0)$ of cardinality μ^+ and $\bar{c}$ be a finite sequence from $M_{1,\delta}$. Then for some $i < \delta$ the sequence $\bar{c}$ is from $M_{1,i}$ but $\mathrm{NF}(M_0, M_{1,i}, M_2, M)$ hence $\mathrm{Av}_{\mathrm{qf}}(\mathbf{J}, M_2) = \mathrm{tp}_{\mathrm{qf}}(\bar{c}, M_2, M)$ as required. Also Ax(C10), rigidity holds trivially, moreover $M \restriction c\ell_M(A)$ has no non-trivial automorphisms over A for any $A \subseteq M \in K$; all this because the closure is closing by the function, not something like algebraic closure.

Lastly, $\mathrm{LS}(\mathfrak{K}) \leq \mu^+$ where $\mu = 2^{2^\chi}$ by V.A.4.4. So modulo the sublemmas below we are done.

2.10 Sublemma. *1) Ax(C2), existence, holds.*
2) Ax(C5), uniqueness, holds.

Proof. 1) So suppose $M_\ell \in K$ for $\ell < 3$, $M_0 \leq_{\mathfrak{s}} M_1$ and $M_0 \leq_{\mathfrak{s}} M_2$.

We shall find $M, M_0 \leq_{\mathfrak{s}} M$ and $\leq_{\mathfrak{s}}$-embeddings $f_\ell : M_\ell \to M$ over M_0 for $\ell = 1, 2$ such that f_ℓ is an isomorphism from M_ℓ onto $M_\ell', M_\ell' \leq_{\mathfrak{s}} M$ and $f_1 \restriction M_0 = f_2 \restriction M_0$ such that $M = \langle f_1(M_1) \cup f_2(M_2) \rangle^{\mathrm{gn}}_M$ and $\mathrm{NF}(M_0, M_1', M_2', M)$. This clearly suffices.

For $\ell = 1,2$ we let $M_\ell = \{c^\ell_i : i < \|M_\ell\|\}$ and for finite $u \subseteq \|M_\ell\|$ let $\bar{c}^\ell_u = \langle c^\ell_i : i \in u\rangle, \bar{x}^\ell_u = \langle x^\ell_i : i \in u\rangle$. The universe of M will be the set $\{\sigma(\bar{c}^1_u, \bar{c}^2_v) : u \subseteq \|M_1\|, v \subseteq \|M_2\|$ are finite, σ a $\tau(\mathfrak{K})$-term$\}$ $(\ell = 1,2)$ divided by an equivalence relation $\mathscr{E}$ defined below; we may identify $c \in M_0$ with $c/\mathscr{E}$ then f_ℓ will be an isomorphism over M_0 for $\ell = 1,2$.
The operations will be defined in the obvious way and we still have to prove their being well defined.
Let Γ be the set of all $\varphi(\sigma_1(\bar{c}^1_{u_1}, \bar{c}^2_{v_1}), \ldots, \sigma_m(\bar{c}^1_{u_m}, \bar{c}^2_{v_m}))$ such that
* $m < \omega$
* $u_\ell \subseteq \|M_1\|$, $v_\ell \subseteq \|M_2\|$ finite for $\ell = 1, \ldots, m$
* $\varphi = \varphi(y_1, \ldots, y_m)$ is a (first order) quantifier free formula and
* for some finite $u \subseteq \|M_1\|$, $v \subseteq \|M_2\|$ such that $\ell \in \{1, \ldots, m\} \Rightarrow u_\ell \subseteq u \;\&\; v_\ell \subseteq v$ for $\ell \in \{1, \ldots, m\}$, and some (qf, χ^+)-convergent (in M_0) set $\mathbf{J} \subseteq {}^u(M_0)$ of cardinality μ^+ such that $\mathrm{Av}_{\mathrm{qf}}(\mathbf{J}, M_0) = \mathrm{tp}_{\mathrm{qf}}(\bar{c}^1_u, M_0, M_1)$ we have $\varphi(\sigma_1(\bar{x}^1_{u_1}, \bar{c}^2_{v_1}), \ldots, \sigma_m(\bar{x}^1_{u_m}, \bar{c}^2_{v_m})) \in \mathrm{Av}_{\mathrm{qf}}(\mathbf{J}, M_2)$.

The averages are well defined as $\mathbf{J}$ is convergent in M_0 hence by V.A.4.6(1) also in M_2 because $M_0 \leq_{\mathfrak{s}} M_2$. Note that in the definition of Γ, the satisfaction of the demand does not depend on the choice of $\mathbf{J}$ (and u, v) by part (2) of the claim V.A.4.6. So

$(*)_1$ Γ is complete ($\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$ for φ (f.o.) quantifier free with parameters from M_0) and

$(*)_2$ Γ is finitely satisfiable in M_0.

Also there are such $\mathbf{J}$'s with the convergence property because $M_0 \leq_{\mathfrak{s}} M_1$, see clause (β) of Definition 2.8(d) and Definition V.A.4.1.

Now $\mathscr{E}$, a two-place relation on the set of suitable terms, is defined by:

$$\sigma_1(\bar{c}^1_u, \bar{c}^2_v) \mathscr{E} \sigma_2(\bar{c}^1_{u'}, \bar{c}^2_{v'})$$

if and only if:

$$[\sigma_1(\bar{x}^1_u, \bar{c}^2_v) = \sigma_2(\bar{x}^1_{u'}, \bar{c}^2_{v'})] \in \Gamma.$$

As Γ is finitely satisfiable in M_0, $\mathscr{E}$ is a congruence relation (and of course an equivalence relation). So M is well defined, and f_1, f_2

are defined naturally by $f_\ell(c^\ell_\alpha) = c^\ell_\alpha/\mathscr{E}$ and they are embeddings (is clear or see the proof of $M \in K$ below).

Now, why is $M \in K$? By 2.5 it is enough to show that every finitely generated submodel of M belongs to K. Say such a submodel is generated by $\bar{c}^1_u, \bar{c}^2_v$ for some finite $u \subseteq \|M_1\|, v \subseteq \|M_2\|$, (pedantically we should replace c^ℓ_i by $c^\ell_i/\mathscr{E}$). As said above because $M_0 \leq_{\mathfrak{s}} M_1$ for some $\mathbf{J}$ of cardinality μ^+ we have $\mathrm{Av}_{\mathrm{qf}}(\mathbf{J}, M_0) = \mathrm{tp}_{\mathrm{qf}}(\bar{c}^1_u, M_0, M_1)$ and $\mathbf{J}$ is (qf, χ^+)-convergent in M_0, hence: for all but $\leq \chi$ of the sequences $\bar{d} \in \mathbf{J}$ the quantifier free type of $\bar{d}\char`^\bar{c}^2_v$ in $M_2$ is equal to the quantifier free type of $\bar{c}^1_u\char`^\bar{c}^2_v$ in M (recall $\chi \geq |\tau(K)|$ so there are $\leq \chi$ quantifier free formulas each may have $\leq \chi$ "exceptions" so together there are $\leq \chi$ exceptions). The models they generate are isomorphic but the first being a submodel of M_2 belongs to K, as K is universal, so also the second one belongs to K.

It is easy that f_ℓ embeds M_ℓ into M for $\ell = 1, 2$ and $f_1 \restriction M_0 = f_2 \restriction M_0$. Also $M = \langle f_1(M_1) \cup f_2(M_2) \rangle$, so easily $\mathrm{NF}(f_\ell(M_0), f_1(M_1), f_2(M_2), M)$ and by renaming $f_1 \restriction M_0 = f_0 \restriction M_2$ is the identity so we are done.

2) As we have proved Ax(C2) hence Ax(C2)$^-$ and the axiom group (B) we can use 1.21(2), so it is enough to prove $(*)$ there. So assume $\mathrm{NF}(M^\ell_0, M^\ell_1, M^\ell_2, M^\ell)$ for $\ell = 1, 2$ and for $m = 0, 1, 2$ the mapping f_m is an isomorphism from M^1_m onto M^2_m such that $f_0 \subseteq f_1, f_0 \subseteq f_2$. For $\ell = 1, 2$ let $M^\ell_3 = \langle M^\ell_1 \cup M^\ell_2 \rangle_{M^\ell}$, so by the definition of NF we know that $M^\ell_3 \leq_{\mathfrak{s}} M^\ell$. It is enough to find an isomorphism f from M^1_3 onto M^2_3 extending $f_1 \cup f_2$.

We try to define f by:

$(*)$ $f(\sigma^{M^1}(\bar{a}^1, \bar{a}^2)) = \sigma^{M^2}(f_1(\bar{a}^1), f_2(\bar{a}^2))$ whenever $\sigma(\bar{x}^1, \bar{x}^2)$ is a τ_K-term, (with $\bar{x}^1, \bar{x}^2$ finite), $\bar{a}^\ell \in {}^{\ell g(\bar{x}^\ell)}(M_\ell)$ for $\ell = 1, 2$.

We can prove that f is well defined (using the averages) and similarly it extends f_1, f_2, is onto M^2_3 and it is an isomorphism. $\square_{2.10}$

2.11 Sublemma. Ax(C6) *(symmetry) holds, that is:*
if $\mathrm{NF}(M_0, M_1, M_2, M_3)$ *<u>then</u>* $\mathrm{NF}(M_0, M_2, M_1, M_3)$.

Proof. Let $\boxtimes \quad M_3' := \langle M_1 \cup M_2 \rangle^{\rm gn}_{M_3}$.

As $\mathrm{NF}(M_0, M_1, M_2, M_3)$ we know that $M_3' \le_{\mathfrak{s}} M_3$, but clearly $M_3' = \langle M_2 \cup M_1 \rangle^{\rm gn}_{M_3}$, so clause (δ) of Definition 2.8(d) holds. Concerning $\mathrm{NF}(M_0, M_2, M_1, M_3)$ we know that for $\ell = 1, 2$ we have $M_0 \le^{\aleph_0}_{\mathrm{qf},\mu,\chi} M_\ell \subseteq M_3$ by clauses $(\alpha), (\beta)$ of Definition 2.8(d) hence clauses $(\beta), (\alpha)$ of Definition 2.8(d) holds concerning $\mathrm{NF}(M_0, M_2, M_1, M_3)$. Also as we use the set of quantifier free formulas, for every $\mathbf{I} \subseteq {}^m(M_0)$ and $\ell \in \{0, 1, 2\}$ we have $\mathrm{Av}_{\rm qf}(\mathbf{I}, M_\ell, M_\ell) = \mathrm{Av}_{\rm qf}(\mathbf{I}, M_\ell, M_3)$ even if $\mathbf{I}$ is (qf, χ^+)-convergent in M_ℓ but not in M_3. By V.A.4.6(1), for $\ell = 1, 2$ as $M_0 \le_{\mathfrak{s}} M_\ell$ clearly

- $(*)_\ell$ for $\mathbf{I} \subseteq {}^m(M_0)$ of cardinality μ^+, we have: $\mathbf{I}$ is (qf, χ^+)-convergent in M_ℓ iff it is (qf, χ^+)-convergent in M_0.

So, by the statements above, the problem is to prove clause $(d)(\gamma)$ of Definition 2.8 for verifying $\mathrm{NF}(M_0, M_2, M_1, M_3)$.

Let $\bar a \in {}^{\omega>}(M_2)$; as $M_0 \le_{\mathfrak{s}} M_2$ clearly there is $\mathbf{J} \subseteq M_0, |\mathbf{J}| = \mu^+$ such that $\mathbf{J}$ is (qf, χ^+)-convergent inside M_0 and $\mathrm{Av}_{\rm qf}(\mathbf{J}, M_0, M_0) = \mathrm{tp}_{\rm qf}(\bar a, M_0, M_2)$; hence by $(*)_1$ the type $q := \mathrm{Av}_{\rm qf}(\mathbf{J}, M_1, M_1)$ is well defined. We should show that it is equal to $\mathrm{tp}_{\rm qf}(\bar a, M_1, M_3)$. So assume $\bar b \in M_1, \varphi$ quantifier free, and $M_3 \models \varphi[\bar a, \bar b]$ and it is enough to show that $\varphi(\bar x, \bar b) \in q$.

Let $\mathbf{I} \subseteq M_0$, $|\mathbf{I}| = \mu^+$ be (qf, χ^+)-convergent such that $\mathrm{Av}(\mathbf{I}, M_0, M_0) = \mathrm{tp}_{\rm qf}(\bar b, M_0, M_1)$ (exists for similar reasons, as $M_0 \le_{\mathfrak{s}} M_1$).

<u>Picture</u>:

M_3

$\bar b \in M_1$ $\qquad\qquad$ $\bar a \in M_2$

M_0

$\mathbf{J}$

$\mathbf{I}$

Now as we assume $\mathrm{NF}(M_0, M_1, M_2, M_3)$ we have $\mathrm{Av}_{\mathrm{qf}}(\mathbf{I}, M_2, M_3) = \mathrm{tp}_{\mathrm{qf}}(\bar{b}, M_2, M_3)$, see $(d)(\delta)$ of Definition 2.8.

Hence (satisfaction of $\varphi(-,-)$ means in M_3)

$$\begin{aligned} M_3 \models \varphi[\bar{a}, \bar{b}] &\Rightarrow \varphi(\bar{a}, \bar{x}) \in \mathrm{tp}_{\mathrm{qf}}(\bar{b}, M_2, M_3) \\ &\Rightarrow \varphi(\bar{a}, \bar{x}) \in \mathrm{Av}_{\mathrm{qf}}(\mathbf{I}, M_2, M_3) \\ &\Rightarrow (\exists^{>\chi} \bar{b}' \in \mathbf{I})\varphi(\bar{a}, \bar{b}') \\ &\Rightarrow (\exists^{>\chi} \bar{b}' \in \mathbf{I})[\exists^{>\chi} \bar{a}' \in \mathbf{J}]\varphi(\bar{a}', \bar{b}'). \end{aligned}$$

[Why? The first implication by the definition of $\mathrm{tp}_{\mathrm{qf}}(\bar{b}, M_2, M_3)$, the second implicaiton by the previous sentence, the third implication by the definition of Av and the fourth implication by choice of $\mathbf{J}$.]

Hence we deduce

$$M_3 \models \varphi[\bar{a}, \bar{b}] \Rightarrow (\exists^{>\chi} \bar{a}' \in \mathbf{J})(\exists^{>\chi} \bar{b}' \in \mathbf{I})\varphi(\bar{a}', \bar{b}')$$

by the symmetry Lemma I, V.A.3.1. Now if $\bar{a}' \in \mathbf{J}$ then $\bar{a}' \subseteq M_0$ then by the choice of $\mathbf{I}$ we have $(\exists^{>\chi} \bar{b}' \in \mathbf{I})\varphi(\bar{a}', \bar{b}') \Rightarrow \varphi(\bar{a}', \bar{b})$. Hence $M_3 \models \varphi[\bar{a}, \bar{b}] \Rightarrow (\exists^{>\chi} \bar{a}' \in \mathbf{J})\varphi(\bar{a}', \bar{b})$. So $M_3 \models \varphi[\bar{a}, \bar{b}] \Rightarrow \varphi(\bar{x}, \bar{b}) \in \mathrm{Av}(\mathbf{J}, M_1)$; as this holds for every quantifier free φ and $\bar{b}$ from M_1, we get $\mathrm{tp}_{\mathrm{qf}}(\bar{a}, M_1, M_3) = \mathrm{Av}(\mathbf{J}, M_1, M_3)$. As $\bar{a}$ is any finite sequence from M_2 we have gotten the desired result. $\square_{2.11}$

2.12 Claim. *If* $\mathrm{NF}_{\mathfrak{s}}(M_0, M_1, M_2, M_3)$ *<u>then</u>* $M_2 \leq_{\mathfrak{s}} M_3$, *so by symmetry also* $M_1 \leq_{\mathfrak{s}} M_3$.

Proof. Note that we can use symmetry to get $M_1 \leq_{\mathfrak{s}} M_3$, as the proof of symmetry in 2.11 does not rely on this claim. Let $M_3' = \langle M_1 \cup M_2 \rangle^{\mathrm{gn}}_{M_3}$ so by clause $(d)(\delta)$ of Definition 2.8 we have $M_3' \leq_{\mathfrak{s}} M_3$. As $\leq_{\mathfrak{s}}$ is transitive (by V.A.4.5) it is enough to prove that $M_2 \leq_{\mathfrak{s}} M_3'$.

Let $\bar{c} \in {}^{\omega>}(M_3')$, so for some finite sequence of τ_K-terms $\bar{\sigma}(\bar{x}, \bar{y})$ and $\bar{c}_1 \in {}^{\ell g(\bar{x})}(M_1)$ and $\bar{c}_2 \in {}^{\ell g(\bar{y})}(M_2)$ we have $M_3' \models$ "$\bar{c} = \bar{\sigma}(\bar{c}_1, \bar{c}_2)$". As we are assuming $\mathrm{NF}_{\mathfrak{s}}(M_0, M_1, M_2, M_3)$ there is a (qf, χ^+)-convergent set $\mathbf{J} \subseteq {}^{\ell g(\bar{x})}(M_0)$ in M_0 of cardinality μ^+ such that $\mathrm{Av}_{\mathrm{qf}}(\mathbf{J}, M_2)$

$= \text{tp}_{\text{qf}}(\bar{c}_1, M_2, M_3) = \text{tp}_{\text{qf}}(\bar{c}_1, M_2, M'_3)$. Let $\mathbf{J}' = \{\bar{\sigma}(\bar{a}, \bar{c}_2) : \bar{a} \in \mathbf{J}\}$, so $\mathbf{J}' \subseteq {}^{(\ell g(\bar{c})}(M_2)$ and, as an indexed set, it has cardinality μ^+. By the convergence, for any $\bar{a}' \in \mathbf{J}$, consider the set $\{\bar{a} \in \mathbf{J} : \bar{\sigma}(\bar{a}, \bar{c}_2) = \sigma(\bar{a}', \bar{c}_2)\}$, if it has cardinality $> \chi$ we shall get $\bar{c} = \bar{\sigma}(\bar{c}_1, \bar{c}_2) \in M_2$ and the desired conclusion is trivial; so assume that this never happens hence $|\mathbf{J}'| = \mu^+$, in fact

$\odot$ if $\bar{c} \notin {}^{\ell g(\bar{c})}(M_2)$ then the mapping $\bar{a} \mapsto \bar{\sigma}(\bar{a}, \bar{c}_2)$ is $(\leq \chi)$-to-1 so without loss of generality it is one-to-one so $|\mathbf{J}'| = \mu^+$.

We have to prove that $\mathbf{J}'$ is (qf, χ^+)-convergent in M_2 and $\text{Av}_{\text{qf}}(\mathbf{J}', M_2)$ is well defined and equal to $\text{tp}(\bar{c}, M_2, M'_3)$ and $|\mathbf{J}'| = \mu^+$. All this follows by ($\odot$ and)

$\boxdot$ if $\varphi(\bar{z}_1, \bar{z}_2)$ is a quantifier formula in $\mathbb{L}(\tau_\kappa), \bar{d} \in {}^{\ell g(\bar{z}_2)}(M_2)$ and $M'_3 \models \varphi[\bar{c}, \bar{d}]$ <u>then</u> the following set has cardinality $\leq \chi$

$$\{\bar{c}' \in \mathbf{J} : M \models \neg\varphi[\bar{\sigma}(\bar{c}', \bar{c}_2), \bar{d}]\}.$$

This trivially holds. $\square_{2.12}$

2.13 Sublemma. Ax(C4) *(base enlargement) holds.*

Proof. The proof relies on 2.12.

It is enough to prove: if $\text{NF}(N_0, B, C, M)$ (in particular by 2.12 we have $N_0 \leq_{\mathfrak{s}} B \leq_{\mathfrak{s}} M, N_0 \leq_{\mathfrak{s}} C \leq_{\mathfrak{s}} M$) and $N_0 \subseteq C' \leq_{\mathfrak{s}} C$ then $\text{NF}(C', B', C, M)$ where $B' = \langle C' \cup B\rangle^{\text{gn}}_M$. As the proof of Ax(C3)(b),(c), monotonicity in M_3, does not rely on 2.13 we can use it (and (A0)-(A4)), so without loss of generality $M = \langle B \cup C\rangle^{\text{gn}}_M$. Also it is straightforward to check that $\text{NF}(N_0, B, C', \langle B \cup C'\rangle^{\text{gn}}_M)$ holds, hence by 2.12 we have $C' \leq_{\mathfrak{s}} \langle B \cup C'\rangle^{\text{gn}}_M$, i.e. $C' \leq_{\mathfrak{s}} B'$. So $C' \leq_{\mathfrak{s}} C \subseteq M, C' \leq_{\mathfrak{s}} B' \subseteq M$ and $\langle B' \cup C\rangle^{\text{gn}}_M = \langle B \cup C\rangle^{\text{gn}}_M = M \leq_{\mathfrak{s}} M$. So in checking Definition 2.8 for $\text{NF}(C', C, B', M)$, the only clause left is subclause (γ) of clause (d).

So let $\bar{b}' = \langle b'_\ell : \ell < n'\rangle$ be a finite sequence from B' and we will find a (qf, χ^+)-convergent sequence $\mathbf{I}'$ inside C' such that $\text{tp}_{\text{qf}}(\bar{b}', C, M) = \text{Av}_{\text{qf}}(\mathbf{I}', C)$. As $B' = \langle B \cup C'\rangle^{\text{gn}}_M = c\ell_M(B \cup C')$

there are finite sequences $\bar{a}, \bar{c}$ from B, C' respectively and $\tau_{\mathfrak{K}}$-term $\sigma_\ell(\ell < n')$ such that $b'_\ell = \sigma_\ell(\bar{a}, \bar{c})$ for $\ell < n'$. As $N_0 \leq_{\mathfrak{s}} B$ there is a (qf, χ)-convergent set $\mathbf{I} \subseteq {}^{\ell g(\bar{a})}(N_0)$ in N_0 (hence in M, B, C) such that $\mathrm{tp}_{\mathrm{qf}}(\bar{a}, N_0, M) = \mathrm{Av}_{\mathrm{qf}}(\mathbf{I}, N_0)$. As $\mathrm{NF}(N_0, B, C, M)$, we know that $\mathrm{tp}_{\mathrm{qf}}(\bar{a}, C, M) = \mathrm{Av}_{\mathrm{qf}}(\mathbf{I}, C)$. Let $\mathbf{I}' = \{\bar{a}' \hat{\ } \bar{c} : \bar{a}' \in \mathbf{I}\}$, so trivially also $\mathbf{I}'$ is (qf, χ^+)-convergent inside M and $\mathrm{tp}_{\mathrm{qf}}(\bar{a} \hat{\ } \bar{c}, C, M) = \mathrm{Av}_{\mathrm{qf}}(\mathbf{I}', C)$. Let $\mathbf{J} = \{\langle \sigma_\ell(\bar{a}', \bar{c}) : \ell < n' \rangle : \bar{a}' \in \mathbf{I}\}$, so $\mathbf{J}$ is a set of sequences from C' (as $\bar{a}', \bar{c}$ are from C), it is (qf, χ^+)-convergent in M, as in the proof of 2.12 without loss of generality $|\mathbf{J}| = \mu^+$ and $\mathrm{tp}_{\mathrm{qf}}(\bar{b}', C, M) = \mathrm{Av}_{\mathrm{qf}}(\mathbf{J}, C)$.

As $\bar{b}'$ was any finite sequence from B' we are done proving that $\mathrm{NF}(C', B', C, M)$.

$\square_{2.13}$

* * *

For the rest of this section we consider another example (which relies on [Sh 3]).

2.14 Sequence Homogenous Models.

<u>2.15 Context</u>: *Let τ be a vocabulary, Δ a set of $\mathbb{L}_{\omega,\omega}(\tau)$-formulas closed under subformulas, D a set of types, each a complete (Δ, n)-type for some n.*

For this example knowledge of, e.g. [Sh 3], [Sh 54] is assumed (in [Sh 3] we use $\Delta = \mathbb{L}_{\omega,\omega}(\tau)$, in [Sh 54], Δ is $\mathbb{L}_{\omega,\omega}(\tau)$ is the set quantifier free formulas, etc.; it does not matter).

2.16 Definition. 1) We say that D is μ-good <u>if</u> there is a (D, μ)-sequence homogeneous model of cardinality $\geq \mu$ for simplicity (see Definition I.2.3(5) or [Sh 3]). We say D is good if it is μ-good for every μ.
1A) Let K_D be the class of τ-models M such that $\mathrm{tp}_\Delta(\bar{a}, \emptyset, M) \in D$ for every $\bar{a} \in {}^{\omega>}M$.
2) Let $K = K^\mu_D$ be the set of τ-models M which are (D, μ)-sequence homogeneous; let $\mathfrak{K} = \mathfrak{K}^\mu_D = (K^\mu_D, \leq_{\mathfrak{K}^\mu_D})$ where $M \leq_{\mathfrak{K}} N$ iff $M \prec_\Delta N$.
3) Let $\kappa^-(D) = \aleph_0$ mean that if $M \prec_\Delta N \in K, \bar{a} \in {}^{\omega>}N$ then

$\text{tp}_\Delta(\bar{a}, M, N)$ does not split strongly over some finite subset of M (by [Sh 3], $\kappa^-(D) > \aleph_0$ with an additional assumption weaker than "D is good", implies non-structure). Sometimes we use the following variant.

4) Let $\kappa(D) = \aleph_0$ mean: if $A \subseteq N \in K, \bar{a} \in {}^{\omega>}N$ then $\text{tp}_\Delta(\bar{a}, A, N)$ does not split strongly over some finite subset of A (equivalent to $\kappa^-(D) = \aleph_0$ when D is good).

5) We let $\text{NF}(M_0, M_1, M_2, M_3)$ mean: $M_0 \prec_\Delta M_1 \prec_\Delta M_3, M_0 \prec_\Delta M_2 \prec_\Delta M_3$, and for $\bar{a} \in {}^{\omega>}(M_1)$, the type $\text{tp}_\Delta(\bar{a}, M_2, M_3)$ does not split strongly over some finite subset of M_0. Clearly $\text{NF}^e(M_0, M_1, a, M)$ is defined similarly.

6) Let $\lambda(D)$ be minimal λ such that D in λ-stable (see [Sh 3]).

7) We say M is (D, μ)-primary over A when ($M \in K^\mu_D$ and) we can find a sequence $\langle a_\alpha : \alpha < \alpha^* \rangle$ such that $|M| = A \cup \{a_\alpha : \alpha < \alpha^*\}$ and for every $\alpha < \alpha^*$ for some $B_\alpha \subseteq A_\alpha := A \cup \{a_\beta : \beta < \alpha\}$ of cardinality $< \mu$, the type $\text{tp}_\Delta(a_\alpha, A_\alpha)$ is the unique $p \in \mathbf{S}_D(A, M)$ which extends $\text{tp}_\Delta(a_\alpha, B_\alpha)$; see [Sh 3, §5] on it, this is called prime there, but we use the terminology of [Sh:c, IV].

Let us check when the axioms hold (recall D stable implies D is good, [Sh 3, 3.4]).

2.17 Lemma. *Assume D is good and $\kappa(D) = \aleph_0$ (also called "D is superstable") and $\mu \geq \lambda(D)^+$. Let $\mathfrak{s} = (K^\mu_D, \prec_\Delta, <>^{\text{gn}}, \text{NF})$ with trivial $<>^{\text{gn}}$, i.e., $\langle A \rangle^{\text{gn}}_M = A$.*

Then for this framework satisfies AxFr_2 *(and Ax(A4) holds).*

Proof: Note that by [Sh 3]

$(*)_1$ for every $M \in K_D$ there is N such that $M \prec_\Delta N \in K^\mu_D$.

[Why? As D is good.]

$(*)_2$ if $M \leq_{\mathfrak{s}} N$ and $\bar{a}$ is a finite sequence from N then there is an indiscernible (index) set, $\langle \bar{a}_\alpha : \alpha < \lambda(D)^+ \rangle$ of $\ell g(\bar{a})$-tuples from M with $\text{Av}_\Delta(\{\bar{a}_\alpha : \alpha < \mu^+\}, M) = \text{tp}_\Delta(\bar{a}, M, N)$ so if $\bar{a}$ is from M the set is trivial, $\bar{a}_\alpha = \bar{a}_0$, otherwise it is with no repetitions.

[Why? Let $A \subseteq M$ be finite such that $p = \text{tp}_\Delta(\bar{a}, M, N)$ does not strongly split over A, this holds as D is superstable. We can find $M_\alpha \leq_{\mathfrak{s}} M$ of cardinality $\lambda(D)$, including A, increasing continuous with α such that every $q \in \mathbf{S}_D^{<\omega}(M_\alpha)$ is realized in $M_{\alpha+1}$ and $\bar{a}_\alpha \in {}^{\ell g(\bar{a})}(M_{\alpha+1})$ realizes $p \restriction M_\alpha$. By [Sh 3] there is an unbounded $\mathscr{U} \subseteq \mu^+$ such that $\{\bar{a}_\alpha : \alpha \in \mathscr{U}\}$ is an indiscernible set and imitating the proof of V.A.4.4, it is as required.]

Ax(A0). ($M \leq_{\mathfrak{K}} M$ for $M \in K$).
Obvious.

Ax(A1). ($M \leq_{\mathfrak{K}} N$ implies $M \subseteq N$).
By the definition.

Ax(A2). ($\leq_{\mathfrak{K}}$ is transitive).
Proved as in 2.3, i.e. as in V.A.4.5.

Ax(A3). (if $M_0 \subseteq M_1 \leq_{\mathfrak{K}} N$ and $M_0 \leq_{\mathfrak{K}} N$ then $M_0 \leq_{\mathfrak{K}} M_1$).
Obvious.

Ax(A4). The problem is whether $M := \bigcup_{i<\delta} M_i$ is (D, μ)-homogeneous. For $\mu = \aleph_0$ this is trivial. Generally it still holds because $\kappa(D) = \aleph_0$ and D good by [Sh 54, Th1.15].

Smoothness: Holds trivially (by Ax(A4) and the relevant version of Tarski-Vaught Theorem).

Ax(A4)$_*$. Follows from Ax(A4)

Ax(A4)$^-$. Follows from Ax(A4)

Ax(C0),(C1). Obvious

Ax(C2)(Existence). As D is good, $\mu > \lambda(D)$ (an overkill), it is clear by [Sh 3]

[We prove this as an exercise. Without loss of generality Δ is the set of quantifier free formulas. So assume $M_0 \prec_\Delta M_\ell$ for $\ell = 1, 2$. Without loss of generality $M_2 \prec_\Delta \mathfrak{C}$, let $\langle c^1_\alpha : \alpha < \|M_1\| \rangle$ list the elements of M_1 and for finite $u \subseteq \|M_1\|$ let $\bar{c}^1_u = \langle c^1_\alpha : \alpha \in u \rangle$. Now by $(*)_2$ we can define $p_u \in \mathbf{S}_D(M_2)$ for finite $u \subseteq \|M_1\|$ such that for some $\mathbf{J}_u \subseteq {}^u(M_0)$ as in $(*)_2$ we have $p_u = \text{Av}_\Delta(\mathbf{J}_u, M_2)$. As in 2.3, i.e. 2.9's proof we can find a τ_K-model M_3 extending M_2 and $\langle c'_\alpha : \alpha \in u \rangle$ such that $\langle c'_\alpha : \alpha \in u \rangle$ realizes p_u in M_3 for every finite $u \subseteq \|M_1\|$ and $|M_3| = |M_2| \cup \{c'_\alpha : \alpha < \|M_1\|\}$. It is easy to see that $|M_2| \cup \{c'_\alpha : \alpha < \|M_1\|\}$ is a D-set but D is good hence there is a (μ, D)-sequence homogenous $M_4 \in K_D$ such that $M_3 \subseteq M_4$. In fact we can choose M_4 which is (D,μ)-primary over M_3 as D is stable in $\lambda(D)$ which is $< \mu$, actually $2^\mu > \lambda(D)$ suffice.]

Ax(C3). (Monotonicity)
Should be clear.

Ax(C4)$_{\rm pr}$. (Base enlargement, see Definition 1.11)
Assume $\text{NF}(M_0, M_1, M_2, M_3)$ and $M_0 \prec_\Delta M'_0 \prec_\Delta M_2$. We can find $M'_3 \prec_\Delta M_3$ which is (D,μ)-primary over $M_1 \cup M'_0$ (do it first for μ regular then use Ax(A4) for every regular $\mu' \in (\lambda(D), \mu]$. The rest should be clear, too.

Ax(C5). (Uniqueness)
Holds for good D, i.e. if $\text{NF}(M_0, M_1, M_2, M^\ell_3)$ for $\ell = 1, 2$ then first note $M_1 \cap M_2 = M_0$ (see Ax(F1) below), second show that $\bar{c} \in {}^{\omega>}(M_1 \cup M_2) \Rightarrow \text{tp}_\Delta(\bar{c}, \emptyset, M^1_3) = \text{tp}_\Delta(\bar{c}, \emptyset, M^2_3)$ and then we use the goodness of D (which implies D-sets can be amalgamated).

Ax(C6). (Symmetry)
Holds by "no order" as in V.A.3.5 or 2.11 as $\kappa(D) = \aleph_0$.

Ax(C7). (Finite character)
Holds as we deal with types of finite sequences

Ax(C8). Follows by Ax(C8)*.

Ax(C8)*. (Continuity of NF)
Holds by the definition of NF.

Ax(D1). Obvious as (A4) + smoothness holds

Ax(D2). This is how Ax(C2) was proved (as D is good, $\mu > \lambda(D)$), i.e. as said in the end there, M_4 is (D,μ)-primary over $|M_3|$ and (D,μ)-primary models over $|M_1| \cup |M_2|$ are primes over $M_1 \cup M_2$ (in K^{μ}_{D}, see [Sh 3, 5.2](1).

We can obviously generalize to "(D,μ)-sequence-homogeneous" the theorems on the uniqueness of prime models of [Sh:c, IV,§4] (in our case we can use induction on rank, D good, $\mu > \lambda(D)$).

Now we turn to NF^e.

Ax(E1). Should be clear.

Ax(E2). (Existence)

Holds as D is good, just easier than Ax(C2).

Ax(E3). (Monotonicity)

Should be obvious.

Ax(E4). (Base enlargement).

Trivial.

Ax(E5). (Uniqueness)

True as D is good.

Ax(E6). (Continuity)

Holds (take unions), as Ax(A4) holds.

Also

Ax(F1). (Disjointness for NF)

Holds as the indiscernible set in the definition is not trivial - the elements are distinct.

Ax(F2). (Disjointness for NF^e)

Should be clear.

Ax(G1). (from NF^e to NF, see Definition 1.14).

Just let $M'_3 = M_3$ and $M'_2 \prec_{\Delta} M_3$ be (D,μ)-primary over $M_0 \cup \{a\}$. $\square_{2.17}$

2.18 Problem. What if for D good, $\mu > \lambda(D)$ and we assume just $\kappa(D) < \infty$, and $K = \{M : M(D,\mu)\text{-homogeneous}\}$: we have many results, but not yet enough to prove the main gap.
On the superstable case see Hyttinen-Shelah [HySh 629], [HySh 676].

2.19 Exercise: 1) Assume D is good, stable and $\kappa(D) \le \kappa = \text{cf}(\kappa)$ and $\mu = \lambda(D)^+$.

Then $\mathfrak{s}$ defined as in 2.17 satisfies AxFr_2.
2) Similarly with $\mu = \lambda(D)$ but using $K = \{M : M$ is (D,μ)-homogeneous and if $M \le_\Delta N \in K_D$ and $\bar{a} \in {}^{\omega>}M$ then there is an indiscernible set $\mathbf{I} \subseteq {}^{\ell g(\bar{a})}M$ of cardinality μ with $\text{Av}_\Delta(\mathbf{I}, M) = \text{tp}_\Delta(\bar{a}, M, N)$.

§3 Existence/Uniqueness of Homogeneous quite Universal Models

3.1 Hypothesis. $\mathfrak{K}$ satisfies the axioms of group (A) and has the χ-LSP, see 1.16.

3.2 Remark. If we omit the χ-LSP but demand that $\chi_1 := \text{Min}\{\lambda : \lambda \ge \chi$ and $\mathfrak{K}$ has the λ-LSP$\}$ is well defined, we just sometimes have to use χ_1 instead of χ and replace $= \chi$ by "$\in [\chi, \chi_1)$".

3.3 Definition. 1) We define a two place relation $\mathscr{E}_{\mathfrak{K},\mu}$ on $K_{\ge\mu}$: the transitive closure of $\mathscr{E}^{\text{mat}}_{\mathfrak{K},\mu}$, defined in part (2).
2) $M \mathscr{E}^{\text{mat}}_{\mathfrak{K},\mu} N$ iff both are from $K_{\ge\mu}$ and they are isomorphic to $\le_{\mathfrak{K}}$-submodels of some common member of K.
3) If $\mu = \chi$ we may omit it (similarly in Definition 3.6).

It is straightforward to show:

3.4 Fact. 1) $\mathscr{E}_{\mathfrak{K},\chi}$ is an equivalence relation with $\le 2^{\text{LS}(\mathfrak{K})+|\tau(\mathfrak{K})|+\chi}$ equivalence classes, each having a member of cardinality χ.
2) $K_{\ge\chi} = \cup\{K_{\mathbb{D}} : \mathbb{D} \in \mathfrak{D}'_{\mathfrak{K},\chi}\}$, (see definition below; disjoint union).
3) Also $\mathfrak{K}_{\ge\chi}$ satisfies 3.1 and $\text{LS}(\mathfrak{K}_{\ge\chi}) = \chi$.

3.5 Exercise 1) If $N' \mathscr{E}_{\mathfrak{K},\chi} N''$ and $N', N'' \in K_\chi$ then there are $k < \omega$ and $N_0, \ldots, N_k \in K_\chi$ such that $N_0 \cong N', N_k \cong N'', N_{2\ell} \leq_{\mathfrak{K}} N_{2\ell+1}, N_{2\ell+2} \leq_{\mathfrak{K}} N_{2\ell+1}$ when $2\ell+1 \leq k, 2\ell+2 \leq k$, respectively.
2) If $N' \mathscr{E}_{\mathfrak{K},\mu} N''$ and $N', N'' \in K_{\leq \chi_1}$ and $\mu \leq \chi_1$ and $\mathfrak{K}$ has the χ_1-LSP, then there are $k < \omega$ and $N_0, \ldots, N_k \in K_{\leq \chi_1}$ as above.

Remark. To clarify our notation, note that $\mathfrak{D}$ is a set of $\mathbb{D}$'s, $\mathbb{D}$ a set of isomorphism types of models (and also a function with such values).

3.6 Definition. 1) For $M \in K, \|M\| \geq \chi$ let

$$\mathbb{D}_{M,\chi} = \mathbb{D}_\chi(M, \mathfrak{K}) = \{N/\cong : \|N\| = \chi \text{ and } N \leq_{\mathfrak{K}} M\}$$

2)

(a) $\mathfrak{D}''_{\mathfrak{K},\chi} = \{\mathbb{D}_\chi(M, \mathfrak{K}) : M \in K \text{ and } \|M\| \geq \chi\}$
(b) $\mathbb{D}^M_{\mathfrak{K},\chi} = \cup\{\mathbb{D}_\chi(N, \mathfrak{K}) : N \mathscr{E}_{\mathfrak{K},\chi} M\}$
(c) $\mathfrak{D}'_{\mathfrak{K},\chi} = \{\mathbb{D}^M_{\mathfrak{K},\chi} : M \in K \text{ and } \|M\| \geq \chi\}$
(d) $\mathbb{D}_{\mathfrak{K},\chi} = \cup\{\mathbb{D}_\chi(M) : M \in K_{\geq \chi}\}$
(e) $\mathbb{D}_{\mathfrak{K}}$ means $\mathbb{D}_{\mathfrak{K},\chi}$ where $\chi = \text{LS}(\mathfrak{K})$, if $\mathfrak{K} = \mathfrak{K}_{\mathfrak{s}}$ we let $\mathbb{D}_{\mathfrak{s}} = \mathbb{D}_{\mathfrak{K}}$.

3) For $\mathbb{D} \subseteq \mathbb{D}_{\mathfrak{K},\chi}$ non-empty, of course, let $K_{\mathbb{D}} = K_{\mathbb{D},\chi} = \{M \in K_{\geq \chi} : \mathbb{D}_{M,\chi} \subseteq \mathbb{D}\}$ and $\mathfrak{K}_{\mathbb{D}} = \mathfrak{K}_{\mathbb{D},\chi} = (K_{\mathbb{D},\chi}, \leq_{\mathfrak{K}} \restriction K_{\mathbb{D},\chi})$.
4) If $\mathfrak{s}$ is a framework with $\mathfrak{K}_{\mathfrak{s}} = \mathfrak{K}$ and $\mathbb{D} \subseteq \mathbb{D}_{\mathfrak{K},\chi}$ non-empty, of course, then let $\mathfrak{s}_{\mathbb{D}} = (\mathfrak{K}_{\mathbb{D},\chi}, \leq_{\mathfrak{K}_{\mathbb{D}}}, \text{NF}_{\mathfrak{s}} \restriction \mathfrak{K}_{\mathbb{D},\chi}, <>^{\text{gn}})$.

3.7 Discussion: 1) Translating the symbols into words we have:

(a) $\mathbb{D}_{M,\chi}$ is the collection of isomorphism types of models of cardinality χ, which are $\leq_{\mathfrak{K}}$-embeddable in M,
(b) $\mathbb{D}^M_{\mathfrak{K},\chi}$ is the collection of isomorphism types of models of cardinality χ which are "compatible" with M i.e. are $\mathscr{E}_{\mathfrak{K},\chi}$-equivalent to some $N \in \mathbb{D}_{M,\chi}$
(c) $\mathfrak{D}''_{\mathfrak{K},\chi}$ is the collection of $\mathbb{D}_{M,\chi}$ for $M \in K$ with $|M| \geq \chi$
(d) $\mathfrak{D}'_{\mathfrak{K},\chi}$ is the collection of $\mathbb{D}^M_{\mathfrak{K},\chi}$ for $M \in K$ with $|M| \geq \chi$

(*e*) $\mathbb{D}_{\mathfrak{K},\chi}$ is in fact the set of isomorphism types of members of K of cardinality χ. But in the sense (not denotation, see Frege) of our definition, $\mathbb{D}_{\mathfrak{K},\chi}$ is the union over all $M \in K$ of the collection $\mathbb{D}_{M,\chi}$ of isomorphism-types of models of power χ which can be embedded in M

(*f*) Thus $\mathfrak{D}''_{\mathfrak{K},\chi}, \mathfrak{D}'_{\mathfrak{K},\chi}$ are objects of one higher type than $\mathbb{D}_{\mathfrak{K},\chi}, \mathbb{D}_{M,\chi}$ and $\mathbb{D}^M_{\mathfrak{K},\chi}$

(*g*) Finally, if $\mathbb{D}$ is a collection of isomorphism types of models in K, each with cardinality $\chi, K_{\mathbb{D},\chi}$ is the collection of those $M \in K_{\geq\chi}$ such that each $\leq_{\mathfrak{K}}$-submodel of M of cardinality χ is isomorphic to a member of $\mathbb{D}$.

3.8 Claim. *1) Assume* AxFr_ℓ *is one of the frameworks from §1. If* $\mathbb{D} \subseteq \mathbb{D}_{\mathfrak{K},\chi}, \mathfrak{K}_{\mathbb{D},\chi} \neq \emptyset, \mathfrak{K}_{\mathfrak{s}} = \mathfrak{K}, \mathfrak{s}$ *satisfies* AxFr_ℓ *then* $\mathfrak{s}_{\mathbb{D}} = \langle K_{\mathbb{D},\chi}, \leq_{\mathfrak{K}_{\mathbb{D},\chi}}, \langle\rangle^{\mathrm{gn}}_{\mathfrak{s}}, \mathrm{NF}_{\mathfrak{s}}\rangle$ *satisfies* AxFr_ℓ *with* LS*-number* χ *provided that:* $\mathbb{D} \in \mathfrak{D}'_{\mathfrak{K},\chi}$.
2) If $\mathbb{D} \in \mathfrak{D}'_{\mathfrak{K},\chi}$ *then* $\mathfrak{K}' = \mathfrak{K}_{\mathbb{D},\chi}$ *satisfies 3.1 and* $\mathrm{LS}(\mathfrak{K}') = \chi$ *and* $\mathfrak{D}'_{\mathfrak{K}',\chi} = \{\mathbb{D}\}$.
3) $K_{\geq\chi}$ *is the disjoint union of* $\langle K_{\mathbb{D},\chi} : \mathbb{D} \in \mathfrak{D}'_{\mathfrak{K},\chi}\rangle$.

Proof. Easy. $\square_{3.8}$

In the following convention we are fixing a particular compatibility class (to guarantee joint embedding) and restricting our attention to it.

3.9 Convention. If $\mathfrak{D}'_{\mathfrak{K}}$ is a singleton $\mathrm{Ax(A4)}^-$ and $\mathrm{Ax(C2)}^-$, i.e. $\mathfrak{K}$ has amalgamation, then we can have a monster model $\mathfrak{C}$, i.e. one which is $(\mathbb{D}, <\infty)$-homogeneous (see below; really $(\mathbb{D}, \bar{\kappa})$-homogeneous) as in [Sh 3, §1] (but for uniqueness we have to assume smoothness). The existence of $\mathfrak{C}$ is shown in 3.14 below.

3.10 Definition. 1) $M \in K_{\mathbb{D}}$ or $(\mathbb{D}, \lambda)$-model homogeneous or $(\mathbb{D}, \lambda)$-homogeneous (where $\lambda \geq \chi^+, \mathbb{D} \in \mathfrak{D}'_{\mathfrak{K},\chi}$) if:

(a) if $N_0, N_1 \in K_{\mathbb{D}}$ satisfy $N_0 \leq_{\mathfrak{K}} M, N_0 \leq_{\mathfrak{K}} N_1$ and $\|N_1\| < \lambda$ then there is a $\leq_{\mathfrak{K}}$-embedding of N_1 into M over N_0

(b) every $N_1 \in K_{\mathbb{D}}$ of cardinality $< \lambda$ can be $\leq_{\mathfrak{K}}$-embedded into M.

2) $M \in K_{\mathbb{D}}$ is strongly $(\mathbb{D}, \lambda)$-model homogeneous or strongly $(\mathbb{D}, \lambda)$-homogeneous (where $\lambda \geq \chi_1^+$) if (b) above holds and

$(a)^+$ if N_0, N_1 are from $K_{\mathbb{D}}, N_0 \leq_{\mathfrak{K}} M, N_1 \leq_{\mathfrak{K}} M$, h an isomorphism from N_0 onto N_1 and $\|N_0\| < \lambda$ then h can be extended to an automorphism of M.

Remark. We may consider replacing "$\|N_1\| < \lambda$" by "$\|N_1\| \leq \lambda$" in Definition 3.10(1). If $\mathfrak{K}$ is smooth (see Definition 1.18(1),(5)), there is no difference.

3.11 Exercise: In Definition 3.10(1), clause (b) is redundant.
[Hint: Assume $\mathbb{D} \in \mathfrak{D}'_{\mathfrak{K},\chi}, \lambda \geq \chi^+$ and $M \in K_{\mathbb{D}}$ satisfies clause (a) of Definition 3.10. Assume further $N \in K_{\mathbb{D}}$ has cardinality $< \lambda$. We can find $N' \leq_{\mathfrak{K}} N$ of cardinality χ (as $\mathfrak{K}$ has the χ-LSP by 3.1) and $N'' \leq_{\mathfrak{K}} M$ of cardinality χ. Now $N' \mathscr{E}_{\mathfrak{K},\chi} N''$ so by 3.5(1) we can find k and a sequence $\langle N_\ell : \ell \leq k \rangle$ of members of K_χ such that $N_{2\ell} \leq_{\mathfrak{K}} N_{2\ell+1}, N_{2\ell+2} \leq_{\mathfrak{K}} N_{2\ell+1}$ when $2\ell + 1 \leq k, 2\ell + 2 \leq k$, respectively and $N_0 \cong N'', N_k \cong N'$. Now choose a $\leq_{\mathfrak{K}}$-embedding h_ℓ of N_ℓ into M: for $\ell = 0$ such h_0 exists as $N_0 \cong N''$. If $h_{2\ell}$ is defined and $2\ell < k$ then such $h_{2\ell+1}$ exists, in fact one extending $h_{2\ell}$ as M satisfies clause (a) of 3.10(1). If $h_{2\ell+1}$ is defined and $2\ell+2 \leq k$ let $h_{2\ell+2} = h_{2\ell+1} \restriction N_{2\ell+2}$.

Let g_0 be an isomorphism from N' onto N_k so $h_k \circ g_0$ is an isomorphism from N' onto $h_k(N_k) \leq_{\mathfrak{K}} M$ hence there is a pair (g_1, N^*) such that $h_k(N_k) \leq_{\mathfrak{K}} N^*$ and g_1 is an isomorphism from N onto N^* extending $h_k \circ g_0$. As we are assuming clause (a) of Definition 3.10(1) there is a $\leq_{\mathfrak{K}}$-embedding f of N^* into M (over $h_k(N_k)$) so $f \circ g_1$ is a $\leq_{\mathfrak{K}}$-embeding of N into M, as required.]

3.12 Exercise: If $\mathbb{D} \in \mathfrak{D}'_{\mathfrak{K},\chi}, \lambda \geq \chi^+$ and $M \in K_{\mathbb{D}}$ is $(\lambda, \mathbb{D})$-homogeneous then $(K_{\mathbb{D}})_{<\lambda}$ has the amalgamation property (and the joint embedding property).

3.13 Definition. $\mathfrak{K}$ is trivial if $[M \le_{\mathfrak{K}} N \Rightarrow M = N]$; hence in this case, if $\mathfrak{D}'_{\mathfrak{K},\chi}$ is a singleton then K has in cardinality χ, at most one member up to isomorphism (and $K_{>\chi} = \emptyset$ recalling $\mathfrak{K}$ has the χ-LSP).

3.14 Lemma. *Assume $\mathfrak{D}'_{\mathfrak{K},\chi} = \{\mathbb{D}\}$, see 3.8(2),(3) and $\mathfrak{K}$ satisfies Ax(A4),Ax(C2)$^-$.*
1) If λ is $\mathfrak{K}$-inaccessible (see Definition 1.27) and λ is regular, $\lambda = \lambda^{<\lambda} > \chi$ and $\mathfrak{K}_{<\lambda}$ has amalgamation and no $<_{\mathfrak{K}_{<\lambda}}$-maximal member then there is $M \in K$ of cardinality λ which is strongly $(\mathbb{D}, \lambda)$-homogeneous and the model M is smooth (which means that there is a sequence $\langle M_i : i < \lambda \rangle$ which $\le_{\mathfrak{K}}$-represent M, that is $M = \bigcup_{i<\lambda} M_i, \|M_i\| < \lambda, M_i$ is $\le_{\mathfrak{K}}$-increasing continuous, $M_i \le_{\mathfrak{K}} M$ for $i < \lambda$).
2) If λ is regular and the models M, N are $(\mathbb{D}_{\mathfrak{K}}, \lambda)$-homogeneous of cardinality λ and are smooth, then $M \cong N$.

Remark. We can weaken somewhat the λ-inaccessibility demands.

Proof. Left to the reader (and on this see I.2.17 and I.2.14,I.2.15). $\square_{3.14}$

3.15 Exercise: Assume $\mathfrak{D}'_{\mathfrak{K},\chi} = \{\mathbb{D}\}$ and in $\mathfrak{K}_{<\lambda}$ every $\le_{\mathfrak{K}}$-increasing chain, if the length is $< \lambda$ it has an $\le_{\mathfrak{K}}$-upper bound, and if the length of the chain is λ, the union is such $\le_{\mathfrak{K}}$-upper bound in $\mathfrak{K}_{\le\lambda}$. If $\mathfrak{K}_{<\lambda}$ has amalgamation and no $\le_{\mathfrak{K}}$-maximal member then there is a $(\mathbb{D}, \lambda)$-homogeneous model of cardinality λ.

3.16 Claim. *1) If $\mathfrak{K}$ satisfies (A4) and has smoothness, $\lambda > \mathrm{LS}(\mathfrak{K})$, then λ is $\mathfrak{K}$-inaccessible (and for $A \subseteq M \in K, \|A\| < \lambda \le \|M\|$ there is $N \le_{\mathfrak{K}} M, \|N\| = \lambda, A \subseteq N$).*
2) If $\mathrm{LS}(\mathfrak{K}) + |\tau(\mathfrak{K})| \le \chi$ and $\mathfrak{K}$ has smoothness and satisfies Ax(A4), then $\mathfrak{K}$ and $\{(M, N) : M \le_{\mathfrak{K}} N\}$ are $\mathrm{PC}_{\chi,(2^\chi)^+}$-class, hence $\mathfrak{K}_{\beth_\mu} \ne \emptyset \Rightarrow (\forall \lambda \ge \chi) K_\lambda \ne \emptyset$ for $\mu = (2^\chi)^+$.

Remark. Using NF, we can improve 3.16(2).

Proof. 1) Now $\mathfrak{K}$ satisfies the axioms (A0)-(A3), also (A4) and smoothness hence $\mathfrak{K}$ is an a.e.c. (see Definition I.1.2) with $\mathrm{LS}(\mathfrak{K}) \le \chi$, see 3.1. Now apply I.1.10.
2) The first phrase by I.1.7, I.1.9 and the "hence" by I.1.11 or V.A.1.3. $\square_{3.16}$

3.17 Conclusion. If $\mathfrak{K}$ satisfies Ax(A4) and has smoothness, λ is regular, $\|M\| = \lambda > \mathrm{LS}(\mathfrak{K})$, <u>then</u> every $M \in K$ of cardinality λ is smooth (see the statement of Lemma 3.14).

Proof. As in 3.16(1), obvious from I§1. $\square_{3.17}$

Remark. We can begin classification theory for a class satisfying Ax(A0)-(A4) + smoothness + amalgamation (= Ax(C2)$^-$) + $\chi = \mathrm{LS}(\mathfrak{K})$, using strong splitting. But we do not succeed to move the properties between cardinals. We can arrive, e.g., that for a class of suitable λ either union of $(\mathbb{D}_{\mathfrak{K}}, \lambda)$-homogeneous is $(\mathbb{D}_{\mathfrak{K}}, \lambda)$-homogeneous, or suitable non-structure results holds.

The following Lemma states that in order to verify that a model is homogeneous it is enough to check that types of singletons (in the appropriate sense) are realized. So it shows that $(\mathbb{D}_{\mathfrak{K}}, \mu)$-homogeneity is equivalent to $\mathbb{D}_{\mathfrak{K}}$-saturativity.

3.18 The Model-homogeneity = Saturativity Lemma. *Let $\mu > \mathrm{LS}(\mathfrak{K})$ and $\mathfrak{K}$ satisfies (A4) and smoothness, i.e. is an a.e.c. and for simplicity $\mathfrak{D}'_{\mathfrak{K}}$ is a singleton.*
1) M is $(\mathbb{D}_{\mathfrak{K}}, \mu)$-homogeneous <u>if and only if</u> for every $N_1 \le_{\mathfrak{K}} N_2 \in K, \|N_2\| < \mu, N_1 \le_{\mathfrak{K}} M$, and $a \in N_2 \backslash N_1$ there are models $N'_2, N_3 \in K$, such that $N_1 \le_{\mathfrak{K}} N'_2 \le_{\mathfrak{K}} N_3, N_2 \le_{\mathfrak{K}} N_3, a \in N'_2$ and there is $\le_{\mathfrak{K}}$-embedding f of N'_2 into M over N_1.
2) Assume $\mathfrak{K}$ has amalgamation and let $\mathfrak{C}$ be a monster model. $M \le_{\mathfrak{K}} \mathfrak{C}$ is $(\mathbb{D}_{\mathfrak{K}}, \mu)$-homogeneous <u>if and only if</u> for every $N \le_{\mathfrak{K}} M, \|N\| < \mu$

and $a \in \mathfrak{C}$, *there is* $a' \in M$ *realizing* $\mathbf{tp}(a, N, \mathfrak{C})$, *i.e., there is an automorphism* f *of* $\mathfrak{C}, f \restriction N = \mathrm{id}_N$ *and* $f(a) \in M$, *see Definition 1.15(4).*

Proof. 1) Clearly without loss of generality μ is regular. The "only if" direction is trivial. Let us prove the other direction.

So assume $N_1 \leq_{\mathfrak{K}} N_2 \in K_{<\mu}$ and $N_1 \leq_{\mathfrak{K}} M$ and we should find a $\leq_{\mathfrak{K}}$-embedding of N_2 into M over N_1. Let $|N_2| = \{a_i : i < \kappa\}$, and we know $\kappa < \mu$ as $\|N_2\| < \mu$. We choose by induction on $i \leq \kappa, N_1^i, N_2^i, f_i$ such that:

(a) $N_1^i \leq_{\mathfrak{K}} N_2^i, \|N_2^i\| < \mu$

(b) N_1^i is $\leq_{\mathfrak{K}}$-increasing continuous in i

(c) N_2^i is $\leq_{\mathfrak{K}}$-increasing continuous in i

(d) f_i is a $\leq_{\mathfrak{K}}$-embedding of N_1^i into M

(e) f_i is increasing with i

(f) $a_i \in N_1^{i+1}$

(g) $N_1^0 = N_1, N_2^0 = N_2, f_0 = \mathrm{id}_{N_1}$

For $i = 0$, clause (g) gives the definition. For i limit let
$N_1^i = \bigcup_{j<i} N_1^j, N_2^i = \bigcup_{j<i} N_2^j, f_i = \bigcup_{j<i} f_j$.
Now (a)-(f) continues to hold by continuity as μ is regular and $\mathfrak{K}$ is smooth.

For i successor if $a_{i-1} \in N_1^{i-1}$ then we let $(N_1^i, N_2^i, f_i) = (N_1^{i-1}, N_2^{i-1}, f_{i-1})$, so assume $a_{i-1} \notin N_1^{i-1}$; we use our assumption; more elaborately, let $M_1^{i-1} \leq_{\mathfrak{K}} M$ be $f_{i-1}(N_1^{i-1})$ and let M_2^{i-1}, g_{i-1} be such that g_{i-1} is an isomorphism from N_2^{i-1}
onto M_2^{i-1} extending f_{i-1}, so recalling $\leq_{\mathfrak{K}}$ is preserved under isomorphisms we have $M_1^{i-1} \leq_{\mathfrak{K}} M_2^{i-1}$, now apply the assumption with $M, M_1^{i-1}, M_2^{i-1}, g_{i-1}(a_{i-1})$ here standing for M, N_1, N_2, a there (note: $a_{i-1} \in N_2 = N_2^0 \subseteq N_2^{i-1}$); so there are $M_3^{i,*}, M_2^{i,*}, f_i^*$ such that:

(i) $M_1^{i-1} \leq_{\mathfrak{K}} M_2^{i,*} \leq_{\mathfrak{K}} M_3^{i,*}$,

(ii) $\|M_3^{i,*}\| < \mu$

(iii) $M_1^{i-1} \leq_{\mathfrak{K}} M_2^{i-1} \leq_{\mathfrak{K}} M_3^{i,*}$,

(iv) $g_{i-1}(a_{i-1}) \in M_2^{i,*}$

(v) f_i^* a $\leq_{\mathfrak{K}}$-embedding of $M_2^{i,*}$ into M,

(vi) $f_i^* \restriction M_1^{i-1} =$ id.

Let N_2^i, h_i be such that $N_2^{i-1} \leq_{\mathfrak{K}} N_2^i, h_i$ an isomorphism from N_2^i onto $M_3^{i,*}$ extending g_{i-1}. Let $N_1^i = h_i^{-1}(M_2^{i,*})$ and $f_i = f_i^* \circ (h_i \restriction N_1^i)]$.

We have carried the induction. Now f_κ is a $\leq_{\mathfrak{K}}$- embedding of N_1^κ into M over N_1, but $|N_2| = \{a_i : i < \kappa\} \subseteq N_1^\kappa$, so $f_\kappa \restriction N_2 : N_2 \to M$ is as required.

2) This follows by (1). $\square_{3.18}$

So far we have only spoken about mappings between models. The following fact says that also mappings from a set to a homogeneous model can be extended, but we need models in order to state it properly (as we do not know what a $\leq_{\mathfrak{K}}$-mapping from a set is), these are N_1 and M_2.

3.19 Fact. 1) If M is strongly $(\mathbb{D}_{\mathfrak{K}}, \mu)$-homogeneous and $\mu \geq \lambda >$ LS($\mathfrak{K}$) <u>then</u> M is $(\mathbb{D}, \lambda)$-homogeneous.

2) Assume $\mathfrak{K}$ has the LSP(λ). If M is strongly $(\mathbb{D}_{\mathfrak{K}}, \lambda^+)$-homogeneous, $A \subseteq N_1 \leq_{\mathfrak{K}} M, |A| \leq \lambda, N_1 \leq_{\mathfrak{K}} M_2, h \in$ AUT(M_2) or just h is a $\leq_{\mathfrak{K}}$-embedding of N_1 into M_2 and $h(A) \subseteq N_1$ <u>then</u> for some $g \in$ AUT(M) and $g \restriction A = h \restriction A$.

Proof. 1) Left to the reader.

2) First, we can find $N_0 \leq_{\mathfrak{K}} N_1$ such that $A \cup h(A) \subseteq N_0$ and $\|N_0\| \leq \lambda$ as $\mathfrak{K}$ has LSP(λ). Then we can find $M_1 \leq_{\mathfrak{K}} M_2$ of cardinality $\leq \lambda$ such that $N_0 \cup h(N_0) \subseteq M_1$ as $\mathfrak{K}$ has LSP(λ) hence by Ax(A3) we have $N_0 \leq_{\mathfrak{K}} M_1$ and $h(N_0) \leq_{\mathfrak{K}} M_1$. As M is $(\mathbb{D}_{\mathfrak{K}}, \lambda^+)$-homogeneous and $N_0 \leq_{\mathfrak{K}} M_1$ there is an $\leq_{\mathfrak{K}}$-embedding g_0 of M_1 into M satisfying $g_0 \restriction N_0 =$ id. Let $M_1' = g_0(M_1)$, so g_0 is an isomorphism from M_1 onto M_1' and $M_1' \leq_{\mathfrak{K}} M$. Let $h_1 = h \restriction N_0$ and let $g_1 = g_0 \circ h_1$, so g_1 is an isomorphism from N_0 onto $g_0 \circ h_1(N_0)$ both of which

has cardinality $\leq \lambda$ and are $\leq_{\mathfrak{K}} M$. So (as M is strongly $(\mathbb{D}_{\mathfrak{K}}, \lambda^+)$-homogeneous) g_1 can be extended to an automorphism g of M, but $a \in A \Rightarrow a \in N_0 \wedge h(a) \in N_0 \Rightarrow g_0(a) = a \wedge g_0(h(a)) = h(a) \Rightarrow g_1(a) = g_0(h(a)) = h(a)$ hence $g \supseteq h \restriction A$ so g is as required]. $\square_{3.19}$

3.20 Definition. 1) $K^{\mathrm{us}}_{\mu,\kappa} = \{M$: there is a $(< \kappa)$-directed partial order I and $(\mathbb{D}_{\mathfrak{K}}, \mu)$-homogeneous models $M_t \in K_\mu$ for $t \in I$ such that $M = \bigcup\limits_{t \in I} M_t\}$.

2) If $\kappa = \aleph_0$, we may omit it.

3.21 Remark. E.g. in 2.1 above, K^{us}_μ is included in the class of models of T of cardinality $\geq \mu$.

UNIVERSAL CLASSES: A FRAME IS NOT SMOOTH OR NOT χ-BASED SH300C

§0 Introduction

We deal in this chapter with two dividing lines: smoothness and being χ-based. Both are absent in the first order case (but the second is somewhat parallel to stability).

We do some positive theory without them, just enough to show that their negation has strong non-structure consequences. Once they are out of the way, much of the theory for stable first order complete theories in [Sh:c] can be redone.

Recall that we work in (AxFr_1) from II,§1 (in particular, limits exist but smoothness may fail: $\langle M_i : i \le \delta \rangle$ is $\le_{\mathfrak{s}}$-increasing does not imply $\bigcup\limits_{i<\delta} M_i \not\le_{\mathfrak{s}} M_\delta$ inspite of our having $\cup\{M_i : i < \delta\}$ not only belongs to K but also $\le_{\mathfrak{K}}$-extending M_i for every $i < \delta$, i.e. Ax(A4).

§1 Non-smooth Stability

In this section we try to prove that an $\subseteq$-increasing continuous sequence of models is $\le_{\mathfrak{K}}$-increasing, i.e., a case of smoothness holds under appropriate assumptions. We prove cases of NF holds in limit. A major point concerning those claims is that they do not rely on smoothness, in fact, their application is in §2, showing that non-smoothness implies non-structure. Naturally there is a price for avoiding to use smoothness: we heavily use the properties of $\langle \dots \rangle^{\text{gn}}$.

Note that we may tend to accept full smoothness "without saying", as it is a basic property for first order classes, hence we should

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be careful of proofs claiming not to use it. However, the phenomenon occurs also for first order T, if we look at

$$\{M : M \text{ a } |T|^+\text{-saturated model of } T\}$$

under a suitable order $<^*$. This is related to a non-smoothness property called didip (dimensional discontinuity property, see [Sh 132], [Sh:c, Ch.X]). But there we always have κ-smoothness for $\kappa \geq \kappa_r(T)$ so the "problematic" sequences are of length $< \kappa_r(T)$ and any two upper bounds are compatible, but in more complicated orders (even for elementary classes) even this may be missing. Also for the order $<^*$ from [Sh 48], not equal to the above, this may occur.

1.1 Context. Axiomatic Framework 1, AxFr_1 from V.B.1.6 for $\mathfrak{s}$ and $\mathfrak{K} = \mathfrak{K}_{\mathfrak{s}}$, $\text{NF} = \text{NF}_{\mathfrak{s}}$ and $\mathfrak{C}$ a monster model.

However, we shall pay special attention to the use of Ax(A4) (mainly in exercises).

The next several results are Lemmas for the proof of Theorem 2.6. Specifically Claim 1.11 carries out a major step in the construction; Claims 1.3 and 1.10 are used to prove Claim 1.11.

We shall use freely

1.2 Observation. If $\text{NF}(M_0^\ell, M_1^\ell, M_2^\ell, M_4^\ell)$ and $M_3^\ell = \langle M_1^\ell \cup M_2^\ell \rangle^{\text{gn}}_{M_4^\ell}$ for $\ell = 1, 2$ and f_i is an isomorphism from M_i^1 onto M_i^2 for $i = 0, 1, 2$ such that $f_0 \subseteq f_1, f_0 \subseteq f_2$ <u>then</u> $M_3^\ell \leq_{\mathfrak{s}} M_4^\ell$ for $\ell = 1, 2$ and $\text{NF}(M_0^\ell, M_1^\ell, M_2^\ell, M_3^\ell)$ and we can extend $f_1 \cup f_2$ to an isomorphism f_3 from M_3^1 onto M_3^2.

Proof. We have $M_3^\ell \leq_{\mathfrak{s}} M_4^\ell$ by Ax(C4) applied with $M_0' = M_2^\ell$, more exactly by V.B.1.21(1). Hence we get $\text{NF}_{\mathfrak{s}}(M_0^\ell, M_1^\ell, M_2^\ell, M_3^\ell)$ by Ax(B0),(C3)(c).

Lastly, by Ax(C5), uniqueness, we can find M, f such that $M_4^2 \leq_{\mathfrak{s}} M$ and f is an $\leq_{\mathfrak{s}}$-embedding of M_4^1 into M extending $f_1 \cup f_2$. Now M_3^2 and $f(M_3^1)$ are $\leq_{\mathfrak{s}} M$ and have universe $\langle M_1^2 \cup M_2^2 \rangle^{\text{gn}}_{M_4^2} = \langle M_1^2 \cup M_2^2 \rangle^{\text{gn}}_M$ and $f(\langle M_1^1 \cup M_2^1 \rangle^{\text{gn}}_{M_4^1}) = \langle f(M_1^1) \cup f(M_2^1) \rangle^{\text{gn}}_{f(M_4^1)} = \langle M_1^2 \cup M_2^2 \rangle^{\text{gn}}_M$ respectively, recalling Ax(B3). So $M_3^2, f(M_3^1)$ are both $\subseteq M$

and have the same universe, hence are equal, so $f_3 := f \restriction M_3^1$ is as required. $\square_{1.2}$

One of the basic tools of first order stability theory is the "transitivity of non-forking": let $A \subseteq B \subseteq C$, if $\mathrm{tp}(a, C)$ does not fork over B and $\mathrm{tp}(a, B)$ does not fork over A then $\mathrm{tp}(a, C)$ does not fork over A. Claim 1.3 is a slightly disguised version of this principle in framework AxFr_1. (Let M_1 play the role of a and M_0, M_2, M_4 play the role of A, B, C respectively; the second hypothesis of Claim 1.3 is then apparently stronger than a direct translation. However, replacing M_3 by the model generated by M_1 and M_2 yields by 1.2 the original situation as the assumptions are "$\mathbf{tp}_{\mathfrak{s}}(M_1, M_2, M_3)$ does not fork over M_0" and (as rephrased here) "$\mathbf{tp}_{\mathfrak{s}}(M_3, M_4, M_5)$ does not fork over M_2" and the conclusion is "$\mathbf{tp}_{\mathfrak{s}}(M_1, M_4, M_5)$ does not fork over M_0").

1.3 Claim. *If* $\mathrm{NF}(M_0, M_1, M_2, M_3)$ *and* $\mathrm{NF}(M_2, M_3, M_4, M_5)$ *then* $\mathrm{NF}(M_0, M_1, M_4, M_5)$.

1.4 Definition. We call this claim "transitivity of NF" and denote it Ax(C9).

Proof. Let $M_3' = \langle M_1 \cup M_2 \rangle^{\mathrm{gn}}_{M_3}$, so by 1.2 or Axiom (C4) (and Ax(C1), or use V.B.1.21(1)) we have $M_3' \le_{\mathfrak{s}} M_3$. So by Ax(C3)(c) (monotonicity) we have $\mathrm{NF}(M_0, M_1, M_2, M_3')$. So by Ax(C1), $M_2 \le_{\mathfrak{s}} M_3' \le_{\mathfrak{s}} M_3$ and by Axiom (C3)(a) + (C6) (symmetry), [alternatively, by (C3)(a)^d] we get $\mathrm{NF}(M_2, M_3', M_4, M_5)$. Similarly, letting $M_5' = \langle M_3' \cup M_4 \rangle^{\mathrm{gn}}_{M_5}$ we get $M_5' \le_{\mathfrak{s}} M_5$ and $\mathrm{NF}(M_2, M_3', M_4, M_5')$.

By Axiom (C2) (existence) there are M_4'', M_5'' and an isomorphism g from M_4 onto M_4'' over M_0, such that $\mathrm{NF}(M_0, M_1, M_4'', M_5'')$ and as above without loss of generality (by 1.2) we have $M_5'' = \langle M_1 \cup M_4'' \rangle^{\mathrm{gn}}_{M_5''}$. Let $M_2'' := g(M_2)$ so $M_0 \le_{\mathfrak{s}} M_2'' \le_{\mathfrak{s}} M_4''$.

Let $M_3'' = \langle M_1 \cup M_2'' \rangle^{\mathrm{gn}}_{M_5''}$. By the base enlargement axiom (C4) (and (C1)) as $\mathrm{NF}(M_0, M_1, M_4'', M_5'')$ we have $M_3'' \le_{\mathfrak{s}} M_5''$ so by Ax(C3), (first (a), then (c)) we have $\mathrm{NF}(M_0, M_1, M_2'', M_3'')$. By Ax(C4) we have $\mathrm{NF}(M_2'', M_3'', M_4'', M_5'')$, and clearly $M_5'' = \langle M_3'' \cup M_4'' \rangle^{\mathrm{gn}}_{M_5''}$ and recall $M_3'' = \langle M_1 \cup M_2'' \rangle^{\mathrm{gn}}_{M_5''}$. Applying twice the uniqueness (Axiom (C5) and in fact 1.2) we can extend g to an isomorphism g'' from M_5' onto M_5'' such that $g''(M_3') = M_3''$ and g'' is the identity on M_1. As everything is preserved by isomorphism, clearly $\mathrm{NF}(M_0, M_1, M_4, M_5')$. By Ax(C3)(b) we have $\mathrm{NF}(M_0, M_1, M_4, M_5)$ as required.

$\square_{1.3}$

* * *

The next two lemmas can also be understood as part of the proof of Lemma V.D.1.2. Specifically Lemma 1.6 is in the core of the proof of the μ-based implies μ'-based (for $\mu' > \mu$ when $\mathfrak{K}_\mathfrak{s}$ is $(\leq \mu, \leq \mu)$-smooth). Lemma 1.5 is used to prove Lemma 1.6 (and the proof of 1.6 is used in the proof of 1.10).

Lemma 1.5 is a case of smoothness: it asserts that if $\langle M_i : i \leq \delta \rangle$ is an $\leq_\mathfrak{s}$-increasing continuous sequence, $N_i = \langle M_i \cup N_0 \rangle^{\mathrm{gn}}_{N_i}$ is also $\leq_\mathfrak{s}$-increasing continuous and for $i < j < \delta$ we have $\mathrm{NF}(M_i, N_i, M_j, N_j)$ then $M_\delta \leq_\mathfrak{s} N_\delta$ and some further corollaries. If, in the non-forking condition, we could replace M_i by M_0, M_j by M_δ, and N_j by N_δ we would be in the situation of axiom (C7). Note this can be viewed as long transitivity. The proof proceeds by showing that we achieve this happy situation by replacing M_δ, N_δ by isomorphic copies which are independent from N_0 over M_0. After applying axiom (C7) we return to the original models by the invariance of non-forking under isomorphism.

1.5 Claim. *Assume $\langle M_i : i \leq \delta \rangle$, $\langle N_i : i \leq \delta \rangle$ are $\leq_\mathfrak{s}$-increasing continuous, (δ a limit ordinal) and for $i < j < \delta$, $\mathrm{NF}(M_i, N_i, M_j, N_j)$ and $N_i = \langle M_i \cup N_0 \rangle^{\mathrm{gn}}_{N_i}$. Then $M_\delta \leq_\mathfrak{s} N_\delta$ and for $i < \delta$ we have $\mathrm{NF}(M_i, N_i, M_\delta, N_\delta)$ and $N_\delta = \langle M_\delta \cup N_0 \rangle^{\mathrm{gn}}_{N_\delta}$.*

Proof. By Ax(C2), existence and 1.2, there are M_δ', N_δ' and g such that $\mathrm{NF}(M_0, N_0, M_\delta', N_\delta')$ and g is an isomorphism from M_δ onto M_δ' over M_0 and $N_\delta' = \langle M_\delta' \cup N_0 \rangle^{\mathrm{gn}}_{N_\delta'}$. Let $N_i' = \langle M_i' \cup N_0 \rangle^{\mathrm{gn}}_{N_\delta'}$ where $M_i' = g(M_i)$ so clearly $N_0' = N_0, M_0' = M_0$. By Axiom (C3),(C4) for $i < j < \delta$, $\mathrm{NF}(M_i', N_i', M_j', N_j')$ hence $\langle N_i' : i \leq \delta \rangle$ is $\leq_\mathfrak{s}$-increasing and by Ax (C7) this sequence is also continuous. So by Observation 1.2

(uniqueness) we can define by induction on $i < \delta$, g_i an isomorphism from N_i onto N'_i extending $(g \restriction M_i) \cup \mathrm{id}_{N_0}$ and every $g_j (j < i)$. Now $g_\delta = \bigcup\limits_{i<\delta} g_i$ is an isomorphism from $\bigcup\limits_{i<\delta} N_i = N_\delta$ onto $\bigcup\limits_{i<\delta} N'_i = N'_\delta$ (the first equality by an assumption, the second equality holds by Ax(C7)). Hence the mapping g_δ is an isomorphism from N_δ onto N'_δ mapping M_i, N_i, M_δ onto M'_i, N'_i, M'_δ respectively for each $i < \delta$ hence it follows that $\mathrm{NF}(M_i, N_i, M_\delta, N_\delta)$ (as $\mathrm{NF}(M'_i, N'_i, M'_\delta, N'_\delta)$ by Ax(C4)) and $N_\delta = \langle M_\delta \cup N_0 \rangle^{\mathrm{gn}}_{N_\delta}$ (as $N'_\delta = \langle M'_\delta \cup N'_0 \rangle^{\mathrm{gn}}_{N'_\delta}$). $\square_{1.5}$

1.6 Claim. *Suppose $\langle M_i : i \le \delta \rangle$, $\langle N_i : i \le \delta \rangle$ are $\subseteq$-increasing continuous, and for $i < j < \delta$, $\mathrm{NF}(M_i, N_i, M_j, N_j)$. Then $M_\delta \le_{\mathfrak{s}} N_\delta$ and for each $i < \delta$ we have $\mathrm{NF}(M_i, N_i, M_\delta, N_\delta)$.*

Proof. Clearly $i < j < \delta \Rightarrow \mathrm{NF}(M_i, N_i, M_j, N_j) \Rightarrow M_i \le_{\mathfrak{s}} M_j \le_{\mathfrak{s}} N_j$. As $\langle M_i : i < \delta \rangle$ is $\subseteq$-increasing continuous it follows that $\langle M_i : i < \delta \rangle$ is $\le_{\mathfrak{s}}$-increasing continuous.

By assumption $M_\delta = \cup \{ M_i : i < \delta \}$ hence by the previous sentence and Ax(A4) clearly $\langle M_i : i \le \delta \rangle$ is $\le_{\mathfrak{s}}$-increasing continuous. Similarly $\langle N_i : i \le \delta \rangle$ is $\le_{\mathfrak{s}}$-increasing continuous.

The proof will proceed by applying the following subclaim first to the given pair $\langle M_i : i \le \delta \rangle$, $\langle N_i : i \le \delta \rangle$ and then to a second pair of sequences of models. We shall use the following notation:

$\circledast$ (a) for $i \le j < \delta$ let $N_{i,j} = \langle M_j \cup N_i \rangle^{\mathrm{gn}}_{N_j}$

(b) for $i < \delta$ let $N_{i,\delta} = \bigcup\limits_{i \le j < \delta} N_{i,j}$.

1.7 Subclaim. *Assume that $\langle M_i : i \le \delta \rangle$, $\langle N_i : i \le \delta \rangle$ are $\subseteq$-increasing continuous and $i < j < \delta \Rightarrow \mathrm{NF}(M_i, N_i, M_j, N_j)$ and $N_{i,j}$ for $i \le j \le \delta$ are defined as in $\circledast$ above. Then $\langle M_i : i \le \delta \rangle, \langle N_i : i \le \delta \rangle$ are $\le_{\mathfrak{s}}$-increasing continuous and for each $i < \delta$:*

(a) $M_\delta \le_{\mathfrak{s}} N_{i,\delta}$
(b) $\mathrm{NF}(M_j, N_{i,j}, M_\delta, N_{i,\delta})$ *(when $i \le j < \delta$)*
(c) $N_{i,\delta} = \langle M_\delta \cup N_i \rangle^{\mathrm{gn}}_{N_{i,\delta}}$

(*d*) *For* $i \leq j_1 < j_2 < \delta$, $\mathrm{NF}(N_{i,j_1}, N_{j_1}, N_{i,j_2}, N_{j_2})$

(*e*) *for each* $i < \delta, \bar{N}^i = \langle N_{i,j} : i \leq j \leq \delta \rangle$ *is* $\leq_{\mathfrak{s}}$*-increasing continuous.*

Proof of 1.7. The first assertion (on $\leq_{\mathfrak{s}}$-increasing) was proved above. By Observation 1.2, $i \leq j < \delta \Rightarrow N_{i,j} \leq_{\mathfrak{s}} N_j$.
Let $i \leq j_1 < j_2 < \delta$ so by Axiom (C4) together with symmetry Axiom (C6) applied to $\mathrm{NF}(M_i, N_i, M_{j_2}, N_{j_2})$, recalling $M_i \leq_{\mathfrak{s}} M_{j_1} \leq_{\mathfrak{s}} M_{j_2}$ we get $\mathrm{NF}(M_{j_1}, N_{i,j_1}, M_{j_2}, N_{j_2})$. Hence by 1.2 we get $\mathrm{NF}(M_{j_1}, N_{i,j_1}, M_{j_2}, N_{i,j_2})$ and clearly $N_{i,j_2} = \langle M_{j_2} \cup N_{i,j_1} \rangle^{\mathrm{gn}}_{N_{i,j_2}}$.
[Why? As $N_{i,j_2} = \langle M_{j_2} \cup N_i \rangle^{\mathrm{gn}}_{N_{j_2}} = \langle M_{j_2} \cup N_i \rangle^{\mathrm{gn}}_{N_{i,j_2}}$ (by the Definition of N_{i,j_2} and Axiom (C4),(B3)) and as $N_i \subseteq N_{i,j_1} \subseteq N_{i,j_2}$ we have $\langle M_{j_2} \cup N_i \rangle^{\mathrm{gn}}_{N_{i,j_2}} = \langle M_{j_2} \cup N_{i,j_1} \rangle^{\mathrm{gn}}_{N_{i,j_2}}$ so we are done.]
By Ax (C7) for each $i < \delta$ the sequence $\langle N_{i,j} : i \leq j < \delta \rangle$ is not only $\leq_{\mathfrak{s}}$-increasing but also continuous [i.e., clause (e) almost holds]. Remember that we have defined $N_{i,\delta} = \bigcup_{i \leq j < \delta} N_{i,j}$ hence by Ax(A4) we have $j \in (i, \delta) \Rightarrow N_{i,j} \leq_{\mathfrak{s}} N_{i,\delta}$ [so together clause (e) holds].
Now by Claim 1.5 it follows that: for each $i < \delta$ we have $M_\delta \leq_{\mathfrak{s}} N_{i,\delta}$ [so clause (a) holds] and for $i \leq j < \delta$ we have $\mathrm{NF}(M_j, N_{i,j}, M_\delta, N_{i,\delta})$ [so clause (b) holds] and $N_{i,\delta} = \langle M_\delta \cup N_i \rangle^{\mathrm{gn}}_{N_{i,\delta}}$ [so clause (c) holds].
By Axiom (C4) if $i \leq j_1 < j_2 < \delta$ as (by an assumption of the subclaim) we have $\mathrm{NF}(M_{j_1}, N_{j_1}, M_{j_2}, N_{j_2})$ and $M_{j_1} \leq_{\mathfrak{s}} N_{i,j_1} \leq_{\mathfrak{s}} N_{j_1}$ clearly we get $\mathrm{NF}(N_{i,j_1}, N_{j_1}, N_{i,j_2}, N_{j_2})$ [so clause (d) holds] so all the conclusions of 1.7 holds. $\square_{1.7}$

Continuation of the Proof of 1.6. We return to the proof of 1.6. Applying the subclaim 1.7 to the original sequences $\langle M_i : i \leq \delta \rangle$ and $\langle N_i : i \leq \delta \rangle$ we see by clauses (e) and (d) of 1.7 that for each $i < \delta$ the sequences $\langle N_{i,j} : i \leq j \leq \delta \rangle$ and $\langle N_j : i \leq j \leq \delta \rangle$ satisfy the hypothesis of 1.7 [standing for $\langle M_j : j \leq \delta \rangle$ and $\langle N_j : j \leq \delta \rangle$ respectively], hence they satisfy the conclusion of Subclaim 1.7 (now indexed by j with fix i).

To clarify notation we expand, fixing $i < \delta$ for a while, for $j \in [i, \delta]$ let $M^*_j = N_{i,j}$ and $N^*_j = N_j$ and when $i \leq j_1 \leq j_2 < \delta$ let

$N^*_{j_1,j_2} = \langle M^*_{j_2} \cup N^*_{j_1} \rangle^{\rm gn}_{N^*_{j_2}} = \left\langle \langle M_{j_2} \cup N_i \rangle^{\rm gn}_{N_{j_2}} \cup N_{j_1} \right\rangle^{\rm gn}_{N^*_{j_2}} = \langle M_{j_2} \cup N_i \cup N_{j_1} \rangle^{\rm gn}_{N_{j_2}} = \langle M_{j_2} \cup N_{j_1} \rangle^{\rm gn}_{N_{j_2}} = N_{j_1,j_2}$ and $N^*_{j,\delta} = \cup\{N^*_{j,\varepsilon} : \varepsilon \in [j,\delta)\} = \cup\{N_{j,\varepsilon} : \varepsilon \in [j,\delta)\} = N_{j,\delta}$. Note that $\langle M^*_j : j \in [i,\delta] \rangle$ is $\subseteq$-increasing continuous and even $\leq_{\mathfrak{s}}$-increasing continuous (this holds by clause (e) of 1.7 when applied above). Also $\langle N^*_j : j \in [i,\delta] \rangle$ is $\subseteq$-increasing continuous and even $\leq_{\mathfrak{s}}$-increasing continuous, as this is the sequence $\langle N_j : j \in [i,\delta] \rangle$. Lastly, if $j_1 \leq j_2$ are from the interval $[i,\delta)$ then $\mathrm{NF}(M^*_{j_1}, N^*_{j_1}, M^*_{j_2}, N^*_{j_2})$ by clause (d) of 1.7 when applied above. So the pair $\langle M^*_j : j \in [i,\delta] \rangle, \langle N^*_j : j \in [i,\delta] \rangle$ of sequences satisfies the assumptions of subclaim 1.7.

Applying the subclaim 1.7 to these sequences we conclude by clause (a) there, for $j \in [i,\delta)$ that $M^*_\delta \leq_{\mathfrak{s}} N^*_{j,\delta}$ which means $N_{i,\delta} \leq_{\mathfrak{s}} N_{j,\delta}$. As $i < \delta$ was arbitrary, we have proved $j_1 < j_2 < \delta \Rightarrow N_{j_1,\delta} \leq_{\mathfrak{s}} N_{j_2,\delta}$.

Now note that $\bigcup\limits_{i<\delta} N_{i,\delta}$ includes each N_i (for $i < \delta$) hence includes $\bigcup\limits_{i<\delta} N_i$, but this is N_δ (as $\langle N_i : i \leq \delta \rangle$ is $\subseteq$-increasing continuous (by an assumption, in fact it was proved that it is $\leq_{\mathfrak{s}}$-increasing continuous)) so $N_\delta \subseteq \bigcup\limits_{i<\delta} N_{i,\delta}$. However, $N_{i,\delta} = \cup\{N_{i,j} : j \in [i,\delta)\} \subseteq \cup\{N_j : j \in [i,\delta)\} \subseteq \{N_\delta : j \in [i,\delta)\} = N_\delta$ so together $N_\delta = \bigcup\limits_{i<\delta} N_{i,\delta}$. As we have noted above that $\langle N_{i,\delta} : i < \delta \rangle$ is $\leq_{\mathfrak{s}}$-increasing, hence by Ax(A4), we can for $i < \delta$ deduce that $N_{i,\delta} \leq_{\mathfrak{s}} \bigcup\limits_{\zeta<\delta} N_{\zeta,\delta}$, in fact we can use a weaker version of Ax(A4) as we see in 1.8 below. By 1.5 and "$M^*_\delta \leq N^*_{0,\delta}$" noted above we know that $M_\delta \leq_{\mathfrak{s}} N_{0,\delta}$ and $N_\delta = \bigcup\limits_{\zeta<\delta} N_{\zeta,\delta}$, we get $M_\delta \leq_{\mathfrak{s}} N_\delta$, one of the desired conclusions of 1.6 and, of course, $i < \delta \Rightarrow N_{i,\delta} \leq_{\mathfrak{s}} N_\delta$.

The second desired conclusion of 1.6 is $\mathrm{NF}(M_i, N_i, M_\delta, N_\delta)$ when $i < \delta$. Now as $N_{i,\delta} \leq_{\mathfrak{s}} N_\delta$ was proved above, the (second) desired conclusion will follow from $\mathrm{NF}(M_i, N_i, M_\delta, N_{i,\delta})$. Now we shall apply 1.5 to the pair of sequences $\bar{M}^i = \langle M_j : i \leq j \leq \delta \rangle, \bar{N}^i = \langle N_{i,j} : i \leq j \leq \delta \rangle$, this is permissible as $\bar{M}^i$ is $\leq_{\mathfrak{s}}$-increasing continuous (proved in the beginning of the proof of 1.6) and $\bar{N}^i$ is $\leq_{\mathfrak{s}}$-increasing continuous (by clause (e) of 1.7) and $M_j \leq_{\mathfrak{s}} N_{i,j}$ (by clause (b) of 1.7 and Ax(C1)) and $i \leq j_1 \leq j_2 < \delta \Rightarrow \mathrm{NF}(M_{j_1}, N_{i,j_1}, M_{j_2}, N_{i,j_2})$ by clause (b) of 1.7 + monotonicity for NF and $N_{i,j_2} = \langle N_{i,j_1} \cup M_{j_2} \rangle^{\mathrm{gn}}_{N_{i,j_2}}$ easily. Hence the conclusion of 1.5 applied to $(\bar{M}^i, \bar{N}^i)$ holds so we get $j \in [i, \delta) \Rightarrow \ \mathrm{NF}(M_j, N_{i,j}, M_\delta, N_{i,\delta})$; or just use 1.7(6). But for $j = i$ this means $\mathrm{NF}(M_i, N_{i,i}, M_\delta, N_{i,\delta})$, so as $N_{i,i} = N_i$ because $N_{i,i} = \langle M_i \cup N_i \rangle^{\mathrm{gn}}_{N_i} = \langle N_i \rangle^{\mathrm{gn}}_{N_i} = N_i$ and $N_{i,\delta} \leq_{\mathfrak{s}} N_\delta$ by the end of the previous paragraph, by Ax(C3) we get $\mathrm{NF}(M_i, N_i, M_\delta, N_\delta)$ so we are done. $\square_{1.6}$

<u>1.8 Exercise</u>: 1) Prove that is the section so far (that is in 1.5 - 1.7) we can weaken the axiom (A4) to $(\mathrm{A4})_*$ and even just $(\mathrm{A4})^*_{\leq \mathrm{cf}(\delta)}$; (recall that it says that if $\langle M_i : i < \delta \rangle$ is $\leq_{\mathfrak{s}}$-increasing and continuous and $M_\delta = \cup\{M_i : i < \delta\}$ then $M_0 \leq_{\mathfrak{s}} M_\delta$).
2) In 1.7 we can add

(*f*) $\langle N_{i,\delta} : i \leq \delta \rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous.

3) The use of Ax(A4) in the proof of 1.9, near the end can be replaced by $\mathrm{Ax(A4)}^*_{\leq \mathrm{cf}(\delta)}$.
4) The use of Ax(A4) in the proof of 1.10 can be replaced by $(\mathrm{A4})^*_{\leq \mathrm{cf}(\delta)}$.

5) Check that there are no more uses in this section of Ax(A4) by $\mathrm{Ax(A4)}_*$.

[Hint: 1) Ax(A4) was used three times. The first (in the proof of 1.6 before 1.7), as well as the second (in the proof of 1.7) we leave to the reader. The third is in the proof of 1.6 after 1.7 (the fourth paragraph), on which we elaborate.

We prove by induction on $\beta \leq \delta$ that the sequence $\langle N_{i,\beta} : i < \beta \rangle$ is (not only $\leq_{\mathfrak{s}}$-increasing but also) continuous. Let $\alpha < \beta$ be a limit ordinal and we should show that $N_{\alpha,\beta} = \bigcup\limits_{i<\alpha} N_{i,\beta}$. For the $\supseteq$ inclusion recall $i < \alpha \Rightarrow N_i \leq_{\mathfrak{s}} N_\alpha$ hence if $i \leq \alpha \leq j < \delta$, then by Ax(B2) we have $N_{i,j} = \langle M_j \cup N_i \rangle^{\mathrm{gn}}_{N_j} \subseteq \langle M_j \cup N_\alpha \rangle^{\mathrm{gn}}_{N_j} = N_{\alpha,j}$, so (if $\beta = \delta$ taking unions) $N_{i,\beta} \subseteq N_{\alpha,\beta}$. Hence $\bigcup\limits_{i<\alpha} N_{i,\beta} \subseteq N_{\alpha,\beta}$.

For the other inclusion, we know that $N_{\alpha,\beta} = \bigcup_{\alpha\le\zeta<\beta} N_{\alpha,\zeta}$ (by 1.7(e) if $\beta < \delta$, and by definition of $N_{\alpha,\beta}$ otherwise) hence it suffices to prove, for a given ζ satisfying $\alpha \le \zeta < \beta$, that $N_{\alpha,\zeta+1} \subseteq \bigcup_{i<\alpha} N_{i,\beta}$; but as $N_{i,\zeta+1} \subseteq N_{i,\beta}$ it suffices to prove that $N_{\alpha,\zeta+1} \subseteq \bigcup_{i<\alpha} N_{i,\zeta+1}$; for this it suffices to know that $\langle N_{i,\zeta+1} : i \le \alpha\rangle$ is continuous, but as $\alpha \le \zeta < \beta$ this holds by the induction hypothesis on β.
2) The proof is included in the proof of part (1).
3),4),5) Left to the reader.]

From Claim 1.5 we can derive the "local character of dependence". Specifically

1.9 Lemma. *Axiom (C8)* holds, (it is from V.B.1.8) if smoothness holds. That is, assume (that $\mathfrak{s}$ has) $\mathrm{cf}(\delta)$-smoothness; if $\langle M_{1,i} : i \le_{\mathfrak{s}} \delta\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous and for each $i < \delta$, $\mathrm{NF}(M_0, M_{1,i}, M_2, M)$ holds, then $\mathrm{NF}(M_0, M_{1,\delta}, M_2, M)$.*

Remark. This reproves V.B.1.21(4).

Proof. Why the second sentence implies the first? Assuming the second we should prove that if $\langle M_{1,i} : i < \delta\rangle$ is $\le_{\mathfrak{s}}$-increasing and $\mathrm{NF}(M_0, M_{1,i}, M_2, M)$ holds for every $i < \delta$ then $\mathrm{NF}(M_0, \cup\{M_{1,i} : i < \delta\}, M_2, M)$ and we prove this by induction on δ. For each δ by the induction hypothesis and axiom (A4) the sequence $\langle M'_{1,i} : i \le \delta\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous and $i < \delta \Rightarrow \mathrm{NF}(M_0, M'_{1,i}, M_2, M)$ where $M'_{1,i} = \cup\{M_{1,j} : j < i\}$ for i limit and $M'_{1,i} = M_{1,i}$ otherwise. Applying the second sentence to $\langle M'_{1,i} : i \le \delta\rangle, M_0, M_2, M$ we get the desired statement. So we have to turn to the second sentence.

By the choice of the way Claim 1.5 was written we must first apply symmetry, i.e. Ax(C6), to rewrite the hypothesis as $\mathrm{NF}(M_0, M_2, M_{1,i}, M)$ and without loss of generaltiy $M_{1,0} = M_0$. Now for each $i < \delta$, let N_i denote $\langle M_{1,i} \cup M_2\rangle^{\mathrm{gn}}_M$ and let $N_\delta = \bigcup_{i<\delta} N_i$. By Ax(C4) (and monotonicity) we have $\mathrm{NF}(M_{1,i}, N_i, M_{1,j}, N_j)$ if $i < j < \delta$ hence $\langle N_i : i < \delta\rangle$ is $\le_{\mathfrak{s}}$-increasing and by Ax(C7) even continuous and by Ax(A4) even $\langle N_i : i \le \delta\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous. Now Claim 1.5 yields that $\mathrm{NF}(M_{1,i}, N_i, M_{1,\delta}, N_\delta)$ for $i < \delta$ but $M_{1,0} = M_0$ and easily $N_0 = M_2$ so we got $\mathrm{NF}(M_0, M_2, M_{1,\delta}, N_\delta)$ but $\mathrm{cf}(\delta)$-smoothness gives $N_\delta \le_{\mathfrak{s}} M$, so by monotonicity, i.e. Ax(C3)(b) this implies $\mathrm{NF}(M_0, M_2, M_{1,\delta}, M)$ as required. $\square_{1.9}$

1.10 Claim. *1) Suppose* $\langle M_i : i \le \delta + 1\rangle, \langle N^a_i : i \le \delta\rangle, \langle N^b_i : i \le \delta\rangle$ *are* $\le_{\mathfrak{s}}$*-increasing continuous sequences and* $\mathrm{NF}(M_i, N^a_i, M_{\delta+1}, N^b_i)$ *and* $N^b_i = \langle M_{\delta+1} \cup N^a_i\rangle^{\mathrm{gn}}_{N^b_i}$ *for* $i < \delta$. *Then* $\mathrm{NF}(M_\delta, N^a_\delta, M_{\delta+1}, N^b_\delta)$.
2) If $K_{\mathfrak{s}}$ *satisfies* $\mathrm{cf}(\delta)$*-smoothness, we can omit the assumption* "$N^b_i = \langle M_{\delta+1} \cup N^a_i\rangle^{\mathrm{gn}}_{N^b_i}$" *for* $i < \delta$.
3) If $\lambda_1 = \|N^b_\delta\|$, $\lambda_2 > \|N^b_i\| + \|M_{\delta+1}\|$ *for* $i < \delta$ *and* $\mathfrak{K}_{\mathfrak{s}}$ *has* $(\lambda_1, <\lambda_2, \mathrm{cf}(\delta))$*-smoothness then we can omit* $N^b_i = \langle M_{\delta+1} \cup N^a_i\rangle^{\mathrm{gn}}_{N^b_i}$.

Proof. We shall mention when we use smoothness and/or "$N^b_i = \langle M_{\delta+1} \cup N^a_i\rangle^{\mathrm{gn}}_{N^b_i}$".

Let $M = N^b_\delta$; as $i < \delta \Rightarrow \mathrm{NF}(M_i, N^a_i, M_{\delta+1}, N^b_i)$ we have $i < \delta \Rightarrow M_i \le_{\mathfrak{s}} N^a_i \le_{\mathfrak{s}} N^b_i \le_{\mathfrak{s}} N^b_\delta = M$ and $i \le j \le \delta \Rightarrow M_j \le_{\mathfrak{s}} M_{\delta+1} \le_{\mathfrak{s}} N^b_i \le_{\mathfrak{s}} N^b_\delta = M$. Hence clearly $i < \delta \Rightarrow N^b_i \le_{\mathfrak{s}} M$ and $M_i \le_{\mathfrak{s}} M, N^a_j \le_{\mathfrak{s}} M$ for $i \le \delta + 1, j \le \delta$ and by Ax(C3)(b) we have $\mathrm{NF}(M_i, N^a_i, M_{\delta+1}, M)$ for $i < \delta$ hence $\mathrm{NF}(M_i, N^a_i, M_j, M)$ so $\mathrm{NF}(M_i, N^a_i, M_j, N^a_j)$ when $i < \delta, j \in [i, \delta + 1)$. We use the proof of 1.6 with M_i $(i \le \delta)$, $N^a_i (i \le \delta)$ here corresponding to M_i $(i \le \delta)$ and N_i $(i \le \delta)$ there. Using its notation

$\circledast_1$ $\langle N_{i,\delta} : i \le \delta\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous.

[Why? Recall $N_{i,j} := \langle M_j \cup N^a_i\rangle^{\mathrm{gn}}_M$ for $j \in [i, \delta)$ and $N_{i,\delta} = \cup\{N_{i,j} : j \in [i, \delta)\}$, they are defined in $\circledast$ of the proof of 1.6, the sequence is $\le_{\mathfrak{s}}$-increasing continuous by 1.8(2), or toward the end of the proof of 1.6.]

And easily

$\circledast_2$ $N^a_\delta = N_{\delta,\delta} := \bigcup_{i<\delta} N_{i,\delta}$.

[Why? As $N^a_\delta = \cup\{N^a_i : i < \delta\} \subseteq \cup\{N_{i,\delta} : i < \delta\}$ but if $i < \delta$ then $N_{i,\delta} = \cup\{N_{i,j} : j \in [i,\delta)\} \subseteq \cup\{N^a_j : j \in [i,\delta)\} = N^a_\delta$ so $\circledast_2$ holds].

Clearly for $i \le j < \delta, N_{i,j} \le_{\mathfrak{s}} N^b_i$, hence $N_{i,\delta} \subseteq N^b_i$; and by 1.7(c) + Ax(B)(3) we have $N_{i,\delta} = \langle M_\delta \cup N^a_i\rangle_{N^a_i} = \langle M_\delta \cup N^a_i\rangle^{\rm gn}_{N^b_i} = \langle M_\delta \cup N^a_i\rangle^{\rm gn}_M$. By Ax(C4) for $i < \delta$ as $M_i \le_{\mathfrak{K}} M_\delta \le_{\mathfrak{K}} M_{\delta+1}$ and $\mathrm{NF}(M_i, N^a_i, M_{\delta+1}, M)$ clearly

$(*)_0$ $\mathrm{NF}(M_\delta, N_{i,\delta}, M_{\delta+1}, M)$ for $i < \delta$.

For $i < \delta$ let $N'_i := \langle M_{\delta+1} \cup N_{i,\delta}\rangle^{\rm gn}_M$ and let $N'_\delta := \bigcup\limits_{i<\delta} N'_i$; next

$(*)_1$ for $i < j < \delta$ we have $\mathrm{NF}(N_{i,\delta}, N'_i, N_{j,\delta}, N'_j)$.

[Why? By $(*)_0$ we have $\mathrm{NF}(M_\delta, N_{j,\delta}, M_{\delta+1}, M)$ so the symmetry axiom (C6) we have $\mathrm{NF}(M_\delta, M_{\delta+1}, N_{j,\delta}, M)$ and $M_\delta \le_{\mathfrak{K}} N_{i,\delta} \le_{\mathfrak{K}} N_{j,\delta}$ (by 1.7(a) and by 1.8(2)) so by Ax(C4) we have $\mathrm{NF}(N_{i,\delta}, \langle N_{i,\delta} \cup M_{\delta+1}\rangle^{\rm gn}_M, N_{j,\delta}, M)$ which by the definition of N'_i means $\mathrm{NF}(N_{i,\delta}, N'_i, N_{j,\delta}, M)$ but $\langle N'_i \cup N_{j,\delta}\rangle^{\rm gn}_M = \langle (M_{\delta+1} \cup N_{i,\delta}) \cup N_{j,\delta}\rangle^{\rm gn}_M = \langle M_{\delta+1} \cup (N_{i,\delta} \cup N_{j,\delta})\rangle^{\rm gn}_M = \langle M_{\delta+1} \cup N_{j,\delta}\rangle^{\rm gn}_M = N'_j$.
[Why the equalities? The first by the definition of N'_i and Ax(B2,3), second by the properties of union; third as trivially $N_{i,\delta} \subseteq N_{j,\delta}$, fourth by the definition of N'_j.]
As we have gotten $\mathrm{NF}(N_{i,\delta}, N'_i, N_{j,\delta}, M)$ and $\langle N'_i \cup N_{j,\delta}\rangle^{\rm gn}_M = N'_j$ by Ax(C3)(c) or 1.2 we conclude $\mathrm{NF}(N_{i,\delta}, N'_i, N_{j,\delta}, N'_j)$ as promised in $(*)_1$.]

Also

$(*)_2$ $\mathrm{NF}(M_\delta, M_{\delta+1}, N_{0,\delta}, N'_0)$.

[Why? We know by $(*)_0$ that $\mathrm{NF}(M_\delta, N_{i,\delta}, M_{\delta+1}, M)$ for $i < \delta$ hence in particular $\mathrm{NF}(M_\delta, N_{0,\delta}, M_{\delta+1}, M)$ and by 1.2 we have $N'_0 \le_{\mathfrak{K}} M$ and $\mathrm{NF}(M_\delta, N_{0,\delta}, M_{\delta+1}, N'_0)$ and by the symmetry axiom (C6) we finish proving $(*)_2$.]

We now apply 1.6 with $\langle N_{i,\delta} : i \le \delta\rangle, \langle N'_i : i \le \delta\rangle$ here standing for $\langle M_i : i \le \delta\rangle, \langle N_i : i \le \delta\rangle$ there; why its assumption holds?

First, $i < j < \delta \Rightarrow \mathrm{NF}(N_{i,\delta}, N'_i, N_{j,\delta}, N'_j)$ as said in $(*)_1$ above, hence $\langle N_{i,\delta} : i < \delta\rangle, \langle N'_i : i < \delta\rangle$ are $\le_{\mathfrak{K}}$-increasing. Second, above in $\circledast_1$ we prove (more than) that $\langle N_{i,\delta} : i < \delta\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous and $N_{\delta,\delta} = \cup\{N_{i,\delta} : i < \delta\}$ hence $\langle N_{i,\delta} : i \le \delta\rangle$ is $<_{\mathfrak{s}}$-increasing continuous by Ax(A4). Third, $N'_i = \langle M_{\delta+1} \cup N_{i,\delta}\rangle^{\rm gn}_M$ for $i < \delta$ is $\subseteq$-increasing by Ax(B1); and by the "first" above and Ax(C4) we have $\langle N'_i : i < \delta\rangle$ is $\le_{\mathfrak{s}}$-increasing and by Ax(A4) even $\langle N'_i : i \le \delta\rangle$ is $\le_{\mathfrak{s}}$-increasing recalling $N'_\delta = \cup\{N'_j : j < \delta\}$ by the definition of N'_δ. Hence to prove a $\langle N'_i : i \le \delta\rangle$ is continuous it is enough to show $\langle N'_i : i < \delta\rangle$ is (not only $\le_{\mathfrak{s}}$-increasing but also) continuous. For part (1) we have $N'_i = \langle M_{\delta+1} \cup N_{i,\delta}\rangle^{\rm gn}_M = \langle M_{\delta+1} \cup M_\delta \cup N^a_i\rangle^{\rm gn}_M = \langle M_{\delta+1} \cup N^a_i\rangle^{\rm gn}_M = N^b_i$ but by an assumption $\langle N^b_i : i < \delta\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous.

For parts (2),(3), by smoothness for $i < \delta$ limit, $N''_i := \cup\{N'_j : j < i\} \le_{\mathfrak{s}} N'_i$. But $M_{\delta+1} \le_{\mathfrak{s}} N'_0$ (as $N'_0 = \langle M_{\delta+1} \cup N^a_0\rangle^{\rm gn}_M$) and $N'_0 \le_{\mathfrak{s}} N''_i$ (as $N''_i = \cup\{N'_j : j < i\}, i$ limit) so $M_{\delta+1} \subseteq N''_i$. Also $N^a_i = \cup\{N^a_j : j < i\}$ (as $\langle N^a_j : j \le \delta\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous by an assumption of the claim) and $\cup\{N^a_j : j < i\} \subseteq \cup\{N'_j : j < i\}$ (by the definition of N'_j) and $\cup\{N'_j : j < i\} = N''_i$ (by the definition of N''_i) so together $N^a_i \subseteq N''_i$. By the last two sentences, $M_{\delta+1} \cup N^a_i \subseteq N''_i$ and $N''_i \le_{\mathfrak{s}} N'_i \le_{\mathfrak{s}} M$ by the one before, hence $N'_i = \langle M_{\delta+1} \cup N_{i,\delta}\rangle^{\rm gn}_M = \langle M_{\delta+1} \cup M_\delta \cup N^a_i\rangle^{\rm gn}_M = \langle M_{\delta+1} \cup N^a_i\rangle^{\rm gn}_{N''_i} \subseteq N''_i$, i.e. $N'_i \subseteq N''_i$, but we have shown above the other inclusion so $N'_i = N''_i$. This means $N'_i = \cup\{N'_j : j < i\}$, so as i was any limit ordinal $< \delta$ we have finished showing $\langle N'_j : j < \delta\rangle$ is continuous. But this finishes the proof of "$\langle N'_i : i \le \delta\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous", hence all the assumptions of 1.6 for $\langle N_{i,\delta} : i \le \delta\rangle, \langle N'_i : i \le \delta\rangle$ holds recalling $(*)_1$.

So the conclusions of 1.6 for $\langle N_{i,\delta} : i \le \delta\rangle, \langle N'_i : i \le \delta\rangle$ holds so

- (a) $N_{\delta,\delta} \le_{\mathfrak{s}} N'_\delta$ and
- (b) $\mathrm{NF}(N_{0,\delta}, N'_0, N_{\delta,\delta}, N'_\delta)$

but $N_{\delta,\delta}$ was defined as $\bigcup\limits_{i<\delta} N_{i,\delta}$ and as said in $\circledast_2$ above is equal to N^a_δ and N'_δ was defined as $\bigcup\limits_{j<\delta} N'_j$ so we get $\mathrm{NF}(N_{0,\delta}, N'_0, N^a_\delta, \bigcup\limits_{j<\delta} N'_j)$ and as, see $(*)_2$ above, $\mathrm{NF}(M_\delta, M_{\delta+1}, N_{0,\delta}, N'_0)$ we get (by 1.3, transitivity) that $\mathrm{NF}(M_\delta, M_{\delta+1}, N^a_\delta, \bigcup\limits_{j<\delta} N'_j)$,

i.e., $\mathrm{NF}(M_\delta, M_{\delta+1}, N^a_\delta, N'_\delta)$.

So by symmetry for NF, that is Ax(C6), in order to get the desired conclusion of 1.10, by Ax(C3)(b) it is enough to prove that $N'_\delta \le_{\mathfrak{s}} M$. If $\mathfrak{s}$, i.e., $\mathfrak{K}_{\mathfrak{s}}$ is cf(δ)-smooth, i.e., for part (2), this is obvious (as $N'_i \le_{\mathfrak{K}} M$ for $i < \delta$ by (Ax(C4) and $\langle N'_i : i \le \delta\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous). The proof for part (3) is similar. In the case left, i.e., part (1) we have

$$\begin{aligned} M = N^b_\delta &= \bigcup_{i<\delta} N^b_i = \bigcup_{i<\delta} \langle M_{\delta+1} \cup N^a_i\rangle^{\rm gn}_{N^b_i} \\ &= \bigcup_{i<\delta} \langle M_{\delta+1} \cup (M_\delta \cup N^a_i)\rangle^{\rm gn}_{N^b_i} \\ &= \bigcup_{i<\delta} \langle M_{\delta+1} \cup N_{i,\delta}\rangle^{\rm gn}_{N^b_i} = \bigcup_{i<\delta} \langle M_{\delta+1} \cup N_{i,\delta}\rangle^{\rm gn}_{M} \\ &= \bigcup_{i<\delta} N'_i = N'_\delta. \end{aligned}$$

$\square_{1.10}$

1.11 Claim. *Suppose $\langle M_i : i < \delta\rangle$, $\langle N_i : i < \delta\rangle$ are $\le_{\mathfrak{s}}$-increasing continuous, and for $i < j < \delta$, $\mathrm{NF}(M_i, N_i, M_j, N_j)$. If $M_i \le_{\mathfrak{K}} M$ for $i < \delta$, then we can find N such that $N_i \le_{\mathfrak{s}} N$ for $i < \delta$ and M can be $\le_{\mathfrak{s}}$-embedded into N over $\bigcup\limits_{i<\delta} M_i$.*

1.12 Remark. 1) This is a strengthened version of the existence of an amalgamation as possibly $\bigcup\limits_{i<\delta} M_i \not\le_{\mathfrak{s}} M$.
2) Note that for a successor ordinal instead of a limit δ, the proof is trivial — use Axiom (C2).

Proof. We define by induction on $i \le \delta$ models N^a_i, N^b_i and a function f_i such that:

⊙ (a) f_i is an isomorphism from N_i onto N^a_i over M_i;
  (b) $\langle N^a_j : j \le i\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous;

(c) $\langle N^b_j : j \leq i \rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous;

(d) f_i is increasing continuous in i;

(e) $\mathrm{NF}(M_i, N^a_i, M, N^b_i)$;

(f) $N^b_i = \langle M \cup N^a_i \rangle^{\mathrm{gn}}_{N^b_i}$.

For $i = 0$, we just have to define N^a_0, f_0, N^b_0 such that clauses (a), (e) and (f) holds. This is possible by Axiom (C2).

For $i = j+1$: let $N^y_i = \langle M_{j+1} \cup N^a_j \rangle^{\mathrm{gn}}_{N^b_j}$. By clause (e) (i.e. the induction hypothesis) we have $\mathrm{NF}(M_j, N^a_j, M, N^b_j)$, so by Axiom (C4) as $M_j \leq_{\mathfrak{s}} M_{j+1} \leq_{\mathfrak{s}} M$ we have $N^y_i \leq_{\mathfrak{K}} N^b_j$; and by Axiom (C3)(a), (b) we have $\mathrm{NF}(M_j, N^a_j, M_{j+1}, N^y_i)$. Let $N^x_i = \langle M_{j+1} \cup N_j \rangle^{\mathrm{gn}}_{N_{j+1}}$, so as $\mathrm{NF}(M_j, N_j, M_i, N_i)$ by 1.2 or Axiom (C4), $N^x_i \leq_{\mathfrak{s}} N_{j+1}$ and $\mathrm{NF}(M_j, N_j, M_i, N^x_i)$ by Ax(C3)(b); i.e., $\mathrm{NF}(M_j, N_j, M_{j+1}, N^x_i)$.

By Axiom (C5) (uniqueness) or more exactly 1.2 there is an isomorphism g_i from N^x_i onto N^y_i, extending $f_j \cup \mathrm{id}_{M_{j+1}}$. By Axiom (C2) (existence) there are N^a_i, N^b_i, f_i such that f_i is an isomorphism from N_i onto N^a_i extending g_i and $\mathrm{NF}(N^y_i, N^a_i, N^b_j, N^b_i)$ holds and (by Observation 1.2) without loss of generality $N^b_i = \langle N^b_j \cup N^a_i \rangle^{\mathrm{gn}}_{N^b_i}$ which is equal to $\left\langle \langle M \cup N^a_j \rangle^{\mathrm{gn}}_{N^b_j} \cup N^a_i \right\rangle^{\mathrm{gn}}_{N^b_i} = \langle (M \cup N^a_j) \cup N^a_i \rangle^{\mathrm{gn}}_{N^b_i} = \langle M \cup (N^a_j \cup N^a_i) \rangle^{\mathrm{gn}}_{N^b_i} = \langle M \cup N^a_i \rangle^{\mathrm{gn}}_{N^b_i}$. So by the previous sentence and symmetry $\mathrm{NF}(N^y_i, N^b_j, N^a_i, N^b_i)$. By the choice of N^y_i, as $\mathrm{NF}(M_j, M, N^a_j, N^b_j)$ by clause (e) of the induction hypothesis for j and as $M_j \leq_{\mathfrak{s}} M_i \leq_{\mathfrak{s}} M$ by the axiom (C4) we get $\mathrm{NF}(M_i, M, N^y_i, N^b_j)$ but by the last sentence $\mathrm{NF}(N^y_i, N^b_j, N^a_i, N^b_i)$, so together by transitivity, 1.3 we get $\mathrm{NF}(M_i, M, N^a_i, N^b_i)$ hence by the symmetry axiom (C6) we deduce $\mathrm{NF}(M_i, N^a_i, M, N^b_i)$. This is clause (e) for i, also the other clauses holds for i: we have dealt with clause (f) just after choosing N^b_i, clause (a) holds by the choice of f_i. Clause (b) holds by transitivity of $\leq_{\mathfrak{s}}$ means $N^a_j \leq_{\mathfrak{s}} N^a_i$ which holds as $N^a_j \leq_{\mathfrak{s}} N^y_i$ (as after the choice of N^y_i we note $\mathrm{NF}(M_j, N^a_j, M_{j+1}, N^y_i)$) and $N^y_i \leq_{\mathfrak{s}} N^a_i$ (as $\mathrm{NF}(N^y_i, N^b_j, N^a_i, N^b_i)$, see above) hence by transitivity of $\leq_{\mathfrak{s}}$ we get $N^a_j \leq_{\mathfrak{s}} N^a_i$ so indeed clause (b) holds.

As for clause (c), again by the transitivity of $\leq_{\mathfrak{s}}$ (and the induction hypothesis) it means $N^b_j \leq_{\mathfrak{s}} N^b_i$, which holds because $\mathrm{NF}(N^y_i, N^b_j, N^a_i,$

N^b_i) was deduced above.

Lastly, for clause (d) it suffices to have $f_j \subseteq f_i$ which holds as $f_j \subseteq g_i \subseteq f_i$ by the choices of g_i and f_i respectively. So we have carried the induction step for successor, $i = j + 1$.

For i limit $< \delta$: let $N^b_i = \bigcup_{j<i} N^b_j$, $f_i = \bigcup_{j<i} f_j$, $N^a_i = \bigcup_{j<i} N^a_j$. As $\langle N_j : j < i\rangle, \langle M_j : j < i\rangle$ are $\le_{\mathfrak{s}}$-increasing continuous by Ax(A4) clauses (b),(c) hold and clauses (a),(d) holds trivially.

As for clause (e) use 1.10(1) with $\langle M_j : j \le i\rangle\hat{}\langle M\rangle, \langle N^a_j : j \le i\rangle, \langle N^b_j : j \le i\rangle$ here standing for $\langle M_i : i \le \delta + 1\rangle, \langle N^a_i : i \le \delta\rangle, \langle N^b_i : i \le \delta\rangle$ there. Why the assumptions of 1.10(1) hold? The sequence $\langle M_j : j \le i\rangle\hat{}\langle M\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous by the assumptions of 1.11, and the two other sequences by clauses (b),(c) of $\odot$. Also for 1.10(1) we need $\mathrm{NF}(M_j, N^a_j, M, N^b_j)$ for $j < i$ which holds by clause (e) of $\odot$ and $N^b_j = \langle M \cup N^a_j\rangle^{\mathrm{gn}}_{N^b_j}$ for $j < i$ which holds by clause (f) of $\odot$. So the conclusion of 1.10(1) holds which means $\mathrm{NF}(M_i, N^a_i, M, N^b_i)$, so we have indeed proved clause (e).

Lastly for clause (f), for each $j < i$,

$$N^b_j = \langle M \cup N^a_j\rangle^{\mathrm{gn}}_{N^b_j} = \langle M \cup N^a_j\rangle^{\mathrm{gn}}_{N^b_i} \subseteq \langle M \cup N^a_i\rangle^{\mathrm{gn}}_{N^b_i},$$

hence $N^b_i = \bigcup_{j<i} N^b_j \subseteq \langle M \cup N^a_i\rangle^{\mathrm{gn}}_{N^b_i} \subseteq N^b_i$ so $N^b_i = \langle M \cup N^a_i\rangle^{\mathrm{gn}}_{N^b_i}$ as required.

So we can carry the induction. In the end using $f_\delta = \bigcup_{i<\delta} f_i$, $N^b_\delta = \bigcup_{i<\delta} N^b_i$, $N^a_\delta = \bigcup_{i<\delta} N^a_i$ and chasing arrows, we finish. $\square_{1.11}$

§2 Non-smoothness implies non-structure

We shall continue to assume

2.1 Hypothesis. $\mathfrak{s}$ satisfies AxFr_1 of V.B§1.

Our main aim in this section is told by its title. Remember that $\mathfrak{s}$ is smooth if: $\bigcup_{i<\delta} M_i \le_{\mathfrak{s}} M$ when $\langle M_i : i < \delta\rangle$ is $\le_{\mathfrak{s}}$-increasing, and for every $i < \delta$, $M_i \leq_{\mathfrak{s}} M$. The main theorem is 2.6: if λ is regular and $\mathfrak{s}$-inaccessible, and there is a counterexample to smoothness by $\langle M_i : i < \delta \rangle$, M, with $|\delta| + \|M\| + \sum_{i<\delta} \|M_i\| < \lambda$ then $\dot{I}(\lambda, K_{\mathfrak{s}}) = 2^\lambda$. (Usually there are 2^λ models no one $\leq_{\mathfrak{s}}$-embeddable into another).

Our main theorem 2.6 has some defects. First the requirement that λ is regular and $\leq_{\mathfrak{s}}$-inaccessible. By our "adopted rules of the game" (see in V.A§0(A)) this is not serious and we intend to look somewhere else whether when the singular case is covered by the theorems in [Sh:e]. Second, and apparently more troublesome is that we have no theorem showing that if κ-smoothness fails then $(< \kappa_{\rm sm}(\mathfrak{K}_{\mathfrak{s}}))$-smoothness fails for some reasonably small $\kappa_{\rm sm}(\mathfrak{K}_{\mathfrak{s}})$. The remedy we will have is to use V.D.1.2; by it $\mathrm{LSP}(\chi) + (\leq \chi)$-smoothness + "NF is χ-based" implies smoothness (in all cardinals).

So "if $\mathfrak{s}$ is not $(\leq \mathrm{LS}(\mathfrak{s}))$-smooth or NF is not $\mathrm{LS}(\mathfrak{s})$-based <u>then</u> $\dot{I}(\lambda, K_{\mathfrak{s}}) = 2^\lambda$ for every regular $\lambda = \lambda^{\mathrm{LS}(\mathfrak{s})}$ etc". See end of the section.

Here is a rough prescription for deducing the existence of many models of power λ from the failure of smoothness at some $\kappa < \lambda$ for models of cardinality $< \lambda$ (i.e., the existence of an $\leq_{\mathfrak{s}}$-increasing sequence $\langle M_i : i \leq \kappa \rangle$ with $\bigcup_{i<\kappa} M_i \nleq_{\mathfrak{s}} M_\kappa$). For each $\eta \in 2^\lambda$ build a sequence of models $\langle M_{\eta \restriction \alpha} : \alpha < \lambda \rangle$ such that $M_\eta = \cup\{M_{\eta\restriction\alpha} : \alpha < \lambda\}$ has cardinality λ and $\mathrm{Smth}(M_\eta) = \{\delta < \lambda : M_{\eta\restriction\delta} \leq_{\mathfrak{s}} M_\eta$ and $\mathrm{cf}(\delta) = \kappa\}/\mathscr{D}_\lambda$ is equal to $\eta^{-1}(\{0\}) \cap \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}/\mathscr{D}_\lambda$, ($\mathscr{D}_\lambda$ is the club filter on λ, Cf. Definition 2.9, Fact 2.10). Now 2^λ of the M_η's will be pairwise non-isomorphic since if $M_\eta \cong M_\nu$, then $\mathrm{Smth}(M_\eta) = \mathrm{Smth}(M_\nu)$. The failure of smoothness should allow us to decide for δ of cofinality κ whether $\bigcup_{\beta<\delta} M_{\eta\restriction\beta} \leq_{\mathfrak{s}} M_\eta$ or not depending on the value of $\eta(\delta)$.

But there is a fly in the ointment. If $\mathscr{T} \subseteq {}^{\kappa >}\lambda$, $|\mathscr{T}| = \lambda$, $\langle \mathscr{T}_i : i < \lambda \rangle$ a representation of $\mathscr{T}$ (i.e., $\mathscr{T} = \bigcup_{i<\delta} \mathscr{T}_i$, $\mathscr{T}_i$ increasing continuous, $|\mathscr{T}_i| < \lambda$), we may wonder whether (for suitable $\mathscr{T}$) for "many" $\delta < \lambda, \mathrm{cf}(\delta) = \kappa$ and there is $\eta_\delta \in {}^\kappa\lambda$ such that $\{\eta_\delta \restriction \zeta : \zeta < \kappa\} \subseteq \mathscr{T}_\delta$, but $(\forall \alpha < \delta)[\{\eta_\delta \restriction \zeta : \zeta < \kappa\} \nsubseteq \mathscr{T}_\alpha]$. Under mild cardinality restrictions we can circumvent this difficulty by working on a "good" stationary subset of λ (which are quite abundant). The required definition and background facts are laid out in 2.2 and 2.4.

2.2 Definition. 1) For a regular $\lambda > \aleph_0$, $S \subseteq \lambda$ is called good if we can find $\bar{u} = \langle u_i : i < \lambda \rangle$ where u_i is a subset of i and for some closed unbounded $E \subseteq \lambda$ for every limit $\delta \in E \cap S$, for some unbounded $u^*_\delta \subseteq \delta$ of order type $< \delta$ we have $(\forall \alpha < \delta)[u^*_\delta \cap \alpha \in \{u_i : i < \delta\}]$.
2) The set of good $S \subseteq \lambda$ is called $\check{I}[\lambda]$.
3) We say $(\bar{u}, E)$ witness $S \in \check{I}[\lambda]$ when they are as above and moreover as in 2.5 below.

2.3 Remark. 1) We can weaken the definition by replacing u_i by $< \lambda$ candidates, and modulo a club we get an equivalent definition. More exactly, let $S \subseteq \lambda$ be called $*$-good if there are $\langle \langle u_{i,\xi} : \xi < \xi(i) \rangle : i < \lambda \rangle, u_{i,\xi} \subseteq \lambda, \xi(i) < \lambda$ and for every limit $\delta \in S$, for some closed unbounded $u^*_\delta \subseteq \delta$ of order type $< \delta$ we have $(\forall \zeta < \delta)[u^*_\delta \cap \alpha \in \{u_{i,\xi} : i < \alpha, \xi < \xi(i)\}]$.
Now (for $S \subseteq \lambda$, λ regular), S is good if and only if S is $*$-good, see [Sh 420].
2) On 2.4, see [Sh 108], probably better to look at [Sh 88a], (last phrase in 2.4(2) — by [Sh 420, §1])):

Recall

2.4 Lemma. *Let $\lambda > \kappa$ be regular, $S = \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$.*
1) S is good if $(\forall \mu < \lambda)[\mu^{<\kappa} < \lambda]$ or $\lambda = \mu^+$ & $\mu = \mathrm{cf}(\mu) > \kappa$; (why? [Sh 351, §4]).
2) Some stationary $S' \subseteq S$ is good if: $\lambda = \lambda^{<\kappa}$ or $\lambda = \mu^+$ & $(\forall \chi < \mu)\chi^\kappa < \mu$, or just $\lambda > \kappa^+$. [Why? See [Sh 420].]
3) If $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ is a good stationary set and $\mu < \kappa$ is regular then there is a good stationary $S' \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \mu\}$ such that for a club of $\delta \in S$, for a club of $\alpha < \delta$, $\mathrm{cf}(\alpha) = \mu \Rightarrow \alpha \in S'$. [Why? An exercise.]

2.5 Observation. In Definition 2.2, without loss of generality, we can demand that for limit $\delta \in S, u^*_\delta = u_\delta$ has order type $\mathrm{cf}(\delta), \alpha \in u_\beta \Rightarrow$

$u_\alpha = u_\beta \cap \alpha$ so $u_\alpha \triangleleft u_\beta$, i.e. u_α is an initial segment of u_β and $[\alpha \in \lambda \backslash S \Rightarrow \text{otp}(u_\alpha) < \kappa]$ and u_α is a set of non-limit ordinals and $\alpha > 0 \Rightarrow 0 \in u_\alpha$.

Proof. See [Sh 420, §1].

2.6 Theorem. *1) Assume λ is regular and $\leq_{\mathfrak{s}}$-inaccessible, see Definition V.B.1.27 and $S \subseteq \{i < \lambda : \text{cf}(i) = \kappa\}$ is a good stationary subset of λ. Suppose $\mathfrak{K}_{\mathfrak{s}}$ is not $(<\lambda, \kappa)$-smooth (i.e. some $M_i (i \leq \kappa)$ are models from $K_{\mathfrak{s}}$ of cardinality $< \lambda$, $\langle M_i : i \leq \kappa \rangle$ is $\leq_{\mathfrak{s}}$-increasing, but $\bigcup_{i<\kappa} M_i \not\leq_{\mathfrak{s}} M_\kappa$). Then $\dot{I}(\lambda, K_{\mathfrak{s}}) = 2^\lambda$.*
2) Moreover, if $\lambda^{<\chi} + 2^{|\tau(\mathfrak{K})|} = \lambda$ and $\chi > \text{LS}(\mathfrak{s})$, then K has 2^λ models, $(\mathbb{D}_{\mathfrak{K}}, \chi)$-homogeneous pairwise non-isomorphic each of cardinality λ.

2.7 Remark. 1) Not only do we get 2^λ pairwise non-isomorphic models or $(\mathbb{D}_{\mathfrak{K}}, \chi)$-homogeneous models in K_λ, but the construction yields usually that one has no $\leq_{\mathfrak{s}}$-embedding into any other. (See Fact 2.11).
2) In the proof below, we may change κ as we argue that without loss of generality $\mathfrak{K}$ is $(<\lambda, \theta)$-smooth for every regular $\theta < \kappa$. We can retain the same κ, if we assume that for some stationary $S \subseteq \{i < \lambda : \text{cf}(i) = \kappa\}$ we have square (i.e., there is $S', S \subseteq S' \subseteq \{i : \text{cf}(i) \leq \kappa\}$ and u_δ a club of δ of order type $\leq \kappa$ for $\delta \in S'$ such that $[\delta_1 \in u_{\delta_2} \Rightarrow u_{\delta_1} = \delta_1 \cap u_{\delta_2}]$); see [Sh 351, §4] by which it holds for successor of regular $\lambda > \kappa^+$.
3) If we would like to use $(\text{AxA4})^*_{\mu,\lambda} + (\text{AxA4})_{>\mu}$ only, see Definition V.B.1.18(7),(8) (instead (AxA4)), we have to assume $\mu < \lambda$ and a square on $\{\delta < \lambda : \text{cf}(\delta) < \mu\}$ avoiding S. In fact, just good $S \subseteq S^\lambda_\theta$ not reflecting in $S^\lambda_{<\mu}$ for every regular $\theta < \mu$, see [Sh:E54].
4) What if we want in 2.6(1) or 2.6(2) to get at least 2^λ non-isomorphic models each of cardinality $\mu \geq \lambda$?
If whenever $\text{NF}(M_0, M_1, M_2, M_3) \& M_3 = \langle M_1 \cup M_2 \rangle^{\text{gn}}_{M_3}$, M_3 has a reasonable representation over $M_1 \cup M_2$, we can (more below, see [Sh:e, III]). This seems not so a restrictive demand.

Proof of 2.6. 1) Without loss of generality, for our λ, and under the assumptions on $\langle M_i : i \leq \kappa \rangle$, κ is minimal (see 2.4(3)). So without loss of generality, $\langle M_i : i < \kappa \rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous. Alternatively, just assume that $\mathfrak{K}$ fails the weak $(< \lambda, \theta)$-smoothness recalling that the failure of the $(< \lambda, \kappa)$-smoothness implies that for some regular $\theta \leq \kappa, \mathfrak{K}$ fails the weak $(< \lambda, \theta)$-smoothness; see V.B.1.20(3).

Without loss of generality let $\langle u_i : i < \lambda \rangle$ exemplify that $S \subseteq \lambda$ is good (see Definition 2.2) and without loss of generality the demands of 2.5 holds; this includes $[i \in \lambda \backslash S \Rightarrow |u_i| < \kappa]$. Let $u'_\delta := \{\alpha < \delta : \alpha = \sup(\alpha \cap u_\delta)\}$.

For $\alpha \leq \lambda$ let

$$\mathscr{T}_\alpha := \{\eta : \eta \text{ a function from } \alpha + 1 \text{ to } \{0,1\} \text{ and } [i \notin S \Rightarrow h(i) = 0]\}$$

Now we define by induction on $\alpha < \lambda$, for every $\eta \in \mathscr{T}_\alpha$ a model M_η and also a function f_η (when $\alpha \notin S$) such that:

$\boxtimes$ (a) $M_\eta \in K$ has as universe some ordinal $\alpha_\eta < \lambda$;

(b) for $\beta < \alpha, M_{\eta \restriction \beta} \leq_{\mathfrak{s}} M_\eta$;

(c) if α is a limit ordinal and $\alpha \notin S$ then $M_\eta = \bigcup_{\beta < \alpha} M_{\eta \restriction \beta}$;

(d) if $\alpha \in \lambda \backslash S$ then f_η is a $\leq_{\mathfrak{s}}$-embedding of $M_{\mathrm{otp}(u_{\ell g(\eta)})}$ into M_η;

(e) if $\alpha \in \lambda \backslash S$ and $u_\beta \triangleleft u_\alpha$ then $f_{\eta \restriction \beta} \subseteq f_\eta$;

(f) if $\alpha \in S$ and $\eta(\alpha) = 0$ then $M_\eta = \bigcup_{\beta < \alpha} M_{\eta \restriction \beta}$;

(g) if $\alpha \in S$ and $\eta(\alpha) = 1$ then $\bigcup_{\beta < \alpha} M_{\eta \restriction \beta} \not\leq_{\mathfrak{s}} M_\eta$;

(h)$_1$ if $\alpha \in \lambda \backslash S, \beta \in u_\alpha$ and $\eta \in \mathscr{T}_\alpha$, then $\mathrm{NF}(f_{\eta \restriction \beta} \left(M_{\mathrm{otp}(u_\beta)}\right), M_{\eta \restriction \beta}, f_\eta \left(M_{\mathrm{otp}(u_\alpha)}\right), M_\eta)$

(h)$_2$ if $\alpha \in \lambda \backslash S, \beta < \alpha, \beta \in u'_\alpha$ and $\eta \in \mathscr{T}_\alpha$ then $\mathrm{NF}(\bigcup_{\gamma \in \beta \cap u_\alpha} f_{\eta \restriction \gamma}(M_{\mathrm{otp}(u_\gamma)}), M_{\eta \restriction \beta}, f_\eta(M_{\mathrm{otp}(u_\alpha)}), M_\eta)$

(h)$_3$ if $\alpha_1 < \alpha_2$ belongs to $u_\beta \cup u'_\beta$ and $\eta \in \mathscr{T}_\beta$ then $\mathrm{NF}(\cup\{f_{\eta \restriction \gamma}(M_{\mathrm{otp}(u_\gamma)})\colon \gamma \in u_\beta, \gamma \leq \alpha_1\}, M_{\eta \restriction \alpha_1}, \cup\{f_{\eta \restriction \gamma}(M_{\mathrm{otp}(u_\gamma)}) : \gamma \in u_\beta, \gamma \leq \alpha_2\}, M_{\eta \restriction \alpha_2})$.

Note that it follows that in fact $(h)_3$ implies $(h)_1 + (h)_2$ so we have to take care of it only.

The construction is by cases:

Case 0. α is zero.
Easy.

Case 1. α is a limit ordinal and $[\alpha \in S \Rightarrow \eta(\alpha) = 0]$.
We let $M_\eta = \bigcup\limits_{\beta<\alpha} M_{\eta\restriction\beta}$ and when $\alpha \notin S, f_\eta = f_{<>}$. Note that $(h)_3$ holds by 1.6 (using monotonicity); note that $(h)_3$ deals also with $\beta > \alpha$ and that we do not use smoothness.

Case 2. $\alpha = \beta + 1$.
So, recalling $0 \in u_\alpha$, necessarily $u_\alpha \cup u'_\alpha$ has a last element, say $\zeta = \zeta_\alpha = \zeta(\alpha) < \alpha$ and if $\zeta \in u'_\alpha$ then $\mathfrak{s}$ has $(< \lambda, \mathrm{cf}(\zeta))$-smoothness. Let $M'_{\eta\restriction\zeta} = \cup\{f_{\eta\restriction\gamma}(M_{\mathrm{otp}(u_\alpha\cap\gamma)}) : \gamma \in u_\alpha, \gamma \leq \zeta\}$. Note that it may depend on $\eta \restriction \alpha$ not just on $\eta \restriction \zeta$. By $(h)_3$ we have $M'_{\eta\restriction\zeta} \leq_{\mathfrak{s}} M_{\eta\restriction\zeta} \leq_{\mathfrak{s}} M_{\eta\restriction\beta}$ (no use of smoothness). By Axiom (C2) there is an extension f_η of $f_{\eta\restriction\zeta}$ and models N_η, M_η such that f_η is an isomorphism from $M_{\mathrm{otp}(C_\alpha)}$ onto N_η satisfying $\mathrm{NF}(f_{\eta\restriction\zeta}(M_{\mathrm{otp}(u_\alpha\cap\zeta)}), M_{\eta\restriction\beta}, N_\eta, M_\eta)$. Without loss of generality the universe of M_η is an ordinal $< \lambda$ (we use "λ is $\leq_{\mathfrak{s}}$-inaccessible").

Case 3. $\alpha \in S$ and $\eta(\alpha) = 1$.
We apply Claim 1.11 twice. In each case the $\langle N_i : i < \delta\rangle$ from Claim 1.11 stands for $\langle M_{\eta\restriction\beta} : \beta \in u_\alpha \cup u'_\alpha\rangle$ here and the $\langle M_i : i < \delta\rangle$ there stands for $\langle \cup\{f_{\eta\restriction\gamma}(M_{\mathrm{otp}(u_\gamma)}) : \gamma \leq \beta \text{ and } \gamma \in u_\alpha\} : \beta \in u_\alpha \cup u'_\alpha\rangle$ here. The assumption holds by $(h)_3$. In the first application M is $\cup\{f_{\eta\restriction\beta}(M_{\mathrm{otp}(C_\beta)} : \beta \in u_\alpha\}$ and in the second application M is an M' such that there is an isomorphism g from M_κ onto M' extending $f_{\eta\restriction\beta}$ whenever $\beta \in u_\alpha$. We find models N^1, N^2 in K such that:

(*i*) $M_{\eta\restriction\beta} \leq_{\mathfrak{s}} N^\ell$ for $\beta \in u_\alpha \cup u'_\alpha$ and $\ell = 1, 2$

(ii) $\cup\{f_{\eta\restriction\beta} : \beta \in u_\alpha\}$ is a $\leq_{\mathfrak{K}}$-embedding of $\bigcup\limits_{i<\kappa} M_i$ into N^1; we call this embedding g^1

(iii) there is a $\leq_{\mathfrak{s}}$-embedding g^2 of M_κ into N^2 which extends $\cup\{f_{\eta\restriction\beta} : \beta \in u_\alpha\}$

moreover

$(i)^+$ $M_{\eta\restriction\beta} \leq_{\mathfrak{s}} N^\ell$ for $\beta < \alpha, \ell = 1, 2$.

Condition $(i)^+$ follows from (i) because $\leq_{\mathfrak{K}}$ is transitive and $\{M_{\eta\restriction\beta} : \beta \in u_\alpha\}$ is cofinal in $\{M_{\eta\restriction\beta} : \beta < \alpha\}$ as $\alpha \in S$. Now we will show $\bigcup\limits_{\beta<\alpha} M_{\eta\restriction\beta} \nleq_{\mathfrak{s}} N^\ell$ for $\ell = 1$ or for $\ell = 2$.

If $\bigcup\limits_{\beta<\alpha} M_{\eta\restriction\beta} \leq_{\mathfrak{s}} N^\ell$ for $\ell = 1, 2$, then by axiom $(C2)^-$ we can find $N \in K$ and $\leq_{\mathfrak{s}}$-embeddings f^ℓ of N^ℓ into N over $\bigcup\limits_{\beta<\alpha} M_{\eta\restriction\beta}$ for $\ell = 1, 2$. So $(f^1 \circ g^1)$ is a $\leq_{\mathfrak{s}}$-embedding of $\bigcup\limits_{\beta<\kappa} M_\beta$ into N so $(f^1 \circ g^1)(\bigcup\limits_{\beta<\kappa} M_\beta) \leq_{\mathfrak{s}} N$.

Also $f^2 \circ g^2$ is a $\leq_{\mathfrak{s}}$-embedding of M_κ into N so $(f^2 \circ g^2)(M_\kappa) \leq_{\mathfrak{s}} N$.

But $(f^1 \circ g^1)(\bigcup\limits_{\beta<\kappa} M_\beta) \subseteq (f^2 \circ g^2)(M_\kappa)$ hence (by Axiom (A3)) we have
$(f^1 \circ g^1)(\bigcup\limits_{\beta<\alpha} M_\beta) \leq_{\mathfrak{s}} (f^2 \circ g^2)(M_\kappa)$, hence (by invariance under isomorphisms) $\bigcup\limits_{\beta<\kappa} M_\beta \leq_{\mathfrak{s}} M_\kappa$, contradicting that $\langle M_i : i \leq \kappa\rangle$ is a counterexample to smoothness.

So for some ℓ, $\bigcup\limits_{\beta<\alpha} M_{\eta\restriction\beta} \nleq_{\mathfrak{s}} N^\ell$; in fact $\ell = 2$ is O.K. by Ax(A3); and (as λ is $\leq_{\mathfrak{s}}$-inaccessible) without loss of generality $\|N^\ell\| < \lambda$, so without loss of generality N^ℓ has universe an ordinal $< \lambda$ and let $M_\eta = N^\ell$.

We have carried the induction.

Now

⊛ let $M_\eta = \cup\{M_{\eta \restriction i} : i < \lambda\}$ for every $\eta \in \mathscr{T}_\lambda$.

Clearly by Ax(A4)

⊠ if $\eta \in \mathscr{T}_\lambda$ then $M_\eta \in K_\lambda$ and $i \in \lambda \backslash S \Rightarrow M_{\eta \restriction i} \le_{\mathfrak{s}} M_\eta$ and $i \in S \Rightarrow [\eta(i) = 0 \Leftrightarrow M_{\eta \restriction i} \le_{\mathfrak{s}} M_\eta]$

The proof of 2.6(1) is finished by Fact 2.8, Definition 2.9 and Fact 2.10 below.

2.8 Fact. If $\eta \in \mathscr{T}_\lambda$ (recall $\mathscr{T}_\lambda = \{\eta : \eta$ belongs to ${}^\lambda 2$ and $[i \in \lambda \backslash S \Rightarrow \eta(i) = 0]\})$ and the $\subseteq$-increasing sequence $\langle M^i : i < \lambda\rangle$ satisfies $M_\eta := \bigcup\limits_{i<\lambda} M^i, \|M^i\| < \lambda$, <u>then</u> $\text{Smth}(M_\eta) = \eta^{-1}(\{0\}) \mod \mathscr{D}_\lambda$ where

2.9 Definition. For $M \in K_\lambda, \lambda$ regular, $|M| = \bigcup\limits_{i<\lambda} A_i$, A_i increasing continuous, $|A_i| < \lambda$, $M_i := M \restriction A_i$, then
$\text{Smth}(M) = \text{Smth}_{\mathfrak{K}}(M) = \text{Smth}_{\mathfrak{s}}(M) = \{i : M_i \le_{\mathfrak{s}} M\}/\mathscr{D}_\lambda$ recalling $\mathscr{D}_\lambda$ is the club filter.

2.10 Fact. $\text{Smth}(M)$ does not depend on the choice of $\langle A_i : i < \lambda\rangle$.

End of the Proof of 2.6. 2) Now Theorem 2.6(2) is an easy variant: for α successor ordinal, by any reasonable bookkeeping, take care to make all the M_η ($\eta \in \mathscr{T}_\lambda$) to be $(\mathbb{D}_{\mathfrak{K}}, \chi)$-homogeneous. $\square_{2.6}$

2.11 Fact. 1) We can conclude in 2.6 that in K_λ there are 2^λ models, no one $\le_{\mathfrak{s}}$-embeddable into another (and when $\lambda = \lambda^{<\chi} + 2^{\chi(\mathfrak{K})+|\tau(\mathfrak{K})|}$, each $(\mathbb{D}_{\mathfrak{K}}, \chi)$-homogeneous) provided that

(∗) if $M, N \in K_\lambda$ and M is $\le_{\mathfrak{s}}$ -embeddable into N then $\text{Smth}(N) \subseteq \text{Smth}(M)$.

2) The statement $(*)$ above holds when:

$(*)_1$ $\leq_{\mathfrak{s}}$ (i.e. the class $\{(M,N) : M \leq_{\mathfrak{s}} N\}$) is a $\mathrm{PC}_{\mu,\omega}$-class, where $\mu < \lambda$
or just

$\circledast_{\lambda,\kappa}$ $\mathfrak{K}$ is a $\mathrm{PC}_{\mu,\theta}$-class where $\mu < \lambda$ and $\theta \leq 2^{\mu}$.

3) Assume that κ is minimal such that $\mathfrak{K}_{<\lambda}$ fails κ-smoothness, which is not a loss for 2.6. Then, letting $S = \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$, for the desired conclusion in part (1), we can weaken $(*)$ to $(*)_S$ for some stationary $S \subseteq \lambda$, see below, and $(*)_S$ is implied by $\circledast'_{\lambda,\kappa}$ where

$(*)_S$ if $M \leq_{\mathfrak{K}} N$ are from K_λ, then $\mathrm{Smth}(N) \cap S \subseteq \mathrm{Smth}(M) \cap S$

$\circledast'_{\lambda,\kappa}$ $(\forall \theta < \lambda)\theta^{<\kappa} < \lambda$ and there is a vocabulary $\tau' \supseteq \tau$ of cardinality $< \lambda$ and $\psi \in \mathbb{L}_{\lambda,\kappa}(\tau)$ such that $\leq_{\mathfrak{K}}$ is the class of τ-reducts of models of ψ.

4) Assume (in 2.6) that S, as a subset of λ, is not small see I.0.5(1) (or [DvSh 65], [Sh:b, Ch.XIV]). Let $\mu_{\mathrm{wd}}(\lambda)$ be as in I.0.5(2) (so it is "usually" 2^λ). Then in K_λ there is no universal member and we can find $M_i \in K_\lambda$ for $i < \mu_{\mathrm{wd}}(\lambda)$ such that:

(a) for $i > j$ then M_i cannot be $\leq_{\mathfrak{s}}$-embedded into M_j

(b) if $\lambda = \lambda^{<\chi} + 2^{\chi(\mathfrak{K}) + |\tau(\mathfrak{K})|}$ then each M_i is $(\mathbb{D}_{\mathfrak{K}}, \chi)$-homogeneous.

Proof. 1) Obvious we elaborate. (Let $S \in \check{I}[\lambda]$ be a stationary subset of λ, hence there is a sequence $\langle S_\varepsilon : \varepsilon < \lambda \rangle$ of pairwise disjoint subsets of S. Now for every $\mathscr{U} \subseteq \lambda$ let $\eta_{\mathscr{U}} \in {}^{\lambda}2$ be: $\eta_{\mathscr{U}}(\alpha) = 1$ if $\alpha \in S_{2\zeta+1}$ for some $\zeta \in \mathscr{U}$ and $\eta_{\mathscr{U}}(\alpha) = 0$ otherwise).
2) So suppose without loss of generality that $M \leq_{\mathfrak{s}} N$. Let $\langle M_i : i < \lambda \rangle, \langle N_i : i < \lambda \rangle$ be representations of M, N respectively. As $M \leq_{\mathfrak{s}} N$ by the assumption that "$\leq_{\mathfrak{K}}$ is a $\mathrm{PC}_{\mu,\omega}$-class, $\mu < \lambda$" the set $E = \{\delta < \lambda : N_\delta \cap M = M_\delta$ and $M_\delta \leq_{\mathfrak{s}} N_\delta\}$ contains a closed unbounded subset of λ.
3) Similar.
4) See Chapter VII or imitate the proof of [Sh 87b, Theorem 3.4];

only here the construction of M_η is as above and Case A disappears. $\square_{2.11}$

Remark. Some parts of 2.11 deal really just with $\mathfrak{K}$.

§3 Non χ-Based

3.1 Hypothesis. AxFr_1 (of course) and χ is such that $\mathfrak{K}$ has the χ-LSP except in claims in which χ does not appear or appear but a weaker relative of the χ-LSP is an assumption (e.g. LSP$(\leq \mu, \chi)$).

3.2 Remark. 0) Under a smoothness hypothesis we will show that χ-LSP this implies $\mathfrak{K}$ has the λ-LSP for all λ larger than χ.
1) We can through §3-5 replace χ^+ by a regular uncountable cardinal.
2) Some of the claims below really says something on any $\mathfrak{K}$ satisfying (A0)-(A4).
3) Note that as long as we do not assume smoothness we cannot really work only inside the monster $\mathfrak{C}$; because if we choose (or construct) $M_\alpha \leq_{\mathfrak{K}} \mathfrak{C}$ by induction on $\alpha \leq \alpha^*, \leq_{\mathfrak{K}}$-increasing, in limit steps $\delta < \alpha^*$ we may like to use $M_\delta := \cup\{M_\alpha : \alpha < \delta\}$, but possibly $M_\delta \not\leq_{\mathfrak{K}} \mathfrak{C}$. But, of course, we can then continue outside $\mathfrak{C}$ and in the end embed M_{α^*} into $\mathfrak{C}$ over say M_0.

3.3 Convention. $\mathfrak{C}$ is a monster model, i.e., large $(\mathbb{D}_{\mathfrak{K}}, \bar{\kappa})$-homogeneous model (see V.B.3.10, $\bar{\kappa}$-universality is problematic as long as we do not have smoothness).

* * *

We did not assume an axiom bounding the cardinality of $\langle A\rangle^{\mathrm{gn}}_M$ in terms of $|A|$. Thus even if $\mathfrak{K}$ has Lowenheim Skolem property down to κ that is LSP(κ), it may not have it down to $\lambda > \kappa$. This problem disappears in the presence of smoothness.

3.4 Claim. *1) For $\lambda \geq \chi$, $\mathrm{LSP}(\lambda)$ holds <u>if</u> $(\leq \lambda, \leq \lambda)^+$-smoothness holds (see Definition V.B.1.18(3),(4)).*
2) If $\mathfrak{K}$ is $(< \mu, < \mu)$-smooth and has $\mathrm{LSP}(\leq \mu, \chi)$ <u>then</u> for every λ satisfying $\chi \leq \lambda < \mu$, $\mathfrak{K}$ has $\mathrm{LSP}(\mu, \lambda)$. (See Definition V.B.1.16(4)).
3) In 2), if $A \subseteq M \in K_\mu, |A| < \mu$ <u>then</u> we can find $\langle M_i : i < \mu \rangle$ which is $\leq_{\mathfrak{s}}$-increasing continuous, $A \subseteq M_0$, $\|M_i\| < \mu$, $M = \bigcup\limits_{i<\mu} M_i$.

See proof below, but we shall need the following observation.

3.5 Claim. *1) Suppose $\langle M_t : t \in I \rangle$ is given where I is a directed partial order and $[t \in I \Rightarrow \|M_t\| \leq \lambda]$ and $|I| \leq \lambda^+$:*

- (a) *if $(\lambda, \leq |I|)$-smoothness holds and $[I \models t < s \Rightarrow M_t \leq_{\mathfrak{s}} M_s]$ <u>then</u> for $s \in I$,*
  $M_s \leq_{\mathfrak{s}} M := \bigcup\limits_{t\in I} M_t = \langle \bigcup\limits_{t\in I} M_t \rangle^{\mathrm{gn}}_M$,
- (b) *if $(\lambda, \leq |I|)^+$-smoothness holds and $[t \in I \Rightarrow M_t \leq_{\mathfrak{s}} M]$ and $[I \models t < s \Rightarrow M_t \subseteq M_s]$ <u>then</u> for every $s \in I$ we have* $M_s \leq_{\mathfrak{s}} \bigcup\limits_{t\in I} M_t = \langle \bigcup\limits_{t\in I} M_t \rangle^{\mathrm{gn}}_M \leq_{\mathfrak{s}} M$.

2) If $A \subseteq M \in \mathfrak{K}$ and $\mathrm{LSP}(|A|)$ (or just $\mathrm{LSP}(\|M\|, |A|)$ <u>then</u> we can find a directed I and $M_t \leq_{\mathfrak{s}} M$, $\|M_t\| = |A|$ for $t \in I$ such that $A \subseteq M_t \subseteq M_s$ for $t \leq_I s$ from I and $M = \bigcup\limits_{t\in I} M_t$.
3) In (1)(b) if $\mathrm{NF}(M^a, M_t, N^a, M)$ whenever $t \in I$ (so $M^a \leq_{\mathfrak{s}} M_t$ for every t) and we are assuming $(\leq \|\bigcup\limits_{t\in I} M_t\|, \leq |I|)^+$-smoothness <u>then</u> $\mathrm{NF}(M^a, \bigcup\limits_{t\in I} M_t, N^a, M)$.
4) In parts (1),(2) instead $(\lambda, \leq |I|)^+$-smoothness we can use $(\lambda, \leq |I|, \|M\|)$-smoothness for clause (b), see Definition V.B.1.18(3).

Proof of Claim 3.5. 1) By induction on $|I|$, (i.e., we prove (a) and (b) simultaneously by induction on $|I|$):

<u>Case (i)</u>: $|I|$ is finite.

The result is trivial, use the maximal member.

Case (ii): $|I| \geq \aleph_0$.

Let $I = \bigcup_{\alpha<|I|} I_\alpha$, I_α increasing, $|I_\alpha| < |I|$ and each I_α directed.

Let $M_\alpha = \bigcup_{t \in I_\alpha} M_t$. Clearly $\langle M_\alpha : \alpha \leq |I|\rangle$ is $\subseteq$-increasing continuous. Let $\alpha < |I|$. By clause (a) of the induction hypothesis, if $t \in I_\alpha$ then $M_t \leq_{\mathfrak{s}} M_\alpha$, in particular, for every $\beta < \alpha$ we have $t \in I_\beta \Rightarrow M_t \leq_{\mathfrak{s}} M_\alpha$. Applying clause (b) of the induction hypothesis with M_α, I_β here standing for M, I there we get $M_\beta = \cup\{M_t : t \in I_\beta\} \leq_{\mathfrak{s}} M_\alpha$. As this holds for any $\beta < \alpha < |I|$ we have proved that $\langle M_\alpha : \alpha < |I|\rangle$ is $\leq_{\mathfrak{s}}$-increasing hence by Ax(A4) we have $\alpha < |I| \Rightarrow M_\alpha \leq_{\mathfrak{s}} M := \cup\{M_\beta : \beta < |I|\} = \cup\{M_t : t \in I\}$. So if $t \in I$ then for some $\alpha < |I|, t \in I_\alpha$ so $M_t \leq_{\mathfrak{s}} M_\alpha \leq_{\mathfrak{s}} \cup\{M_t : t \in I\}$. Also $|M| \subseteq \cup\{|M_t| : t \in I\} \subseteq \cup\{\langle|M_t|\rangle^{\mathrm{gn}}_M : t \in I\} \subseteq \langle\bigcup_{t\in I} M_t\rangle^{\mathrm{gn}}_M \subseteq \langle M\rangle^{\mathrm{gn}}_M = M$. So we have proved clause (a).

For clause (b), by the induction hypothesis $M_\alpha \leq_{\mathfrak{s}} M$ for each α and clearly for $\beta < \alpha$ we have $M_\beta \subseteq M_\alpha$ hence by Ax(A3) we have $\beta < \alpha \Rightarrow M_\beta \leq_{\mathfrak{s}} M_\alpha$. So by the assumption on smoothness $\bigcup_{\alpha<|I|} M_\alpha \leq_{\mathfrak{s}} M$ but $\bigcup_{\alpha<|I|} M_\alpha = \bigcup_{t\in T} M_t$ so we are done.

2) By I.1.7, replacing $\mathrm{LS}(\mathfrak{K})$ by $|A|$, (e.g. replace $\mathfrak{K}$ by $\mathfrak{K}_{\geq |A|}$); or just like the[1] proof of 3.4(2) which appears below.

3) Like the proof of (1), using Lemma 1.9 in the induction step is O.K. as the relevant cases of smoothness holds by the assumptions of part (1).

4) Easy. $\square_{3.5}$

3.6 Remark. In some circumstances, e.g., Banach spaces or $|T|^+$-saturated models of T, where (full) smoothness fails, if we still have a prime model on (or closure of) the union of increasing chains, we can "save" $(\forall\mu \geq \chi)\mathrm{LSP}(\mu)$ by replacing the cardinality of a model M by, e.g., the density character, i.e. the minimal cardinality μ,

[1] no viscious circle, the order is 3.5(1), 3.4(1),(2), 3.5(2),(3), 3.4(3)

such that for some $A \subseteq M$, $|A| = \mu$, M the closure of A (for Banach models) or is $|T|^+$-primary over M (for $|T|^+$-saturated models) or by pscard(M) as in V.B.1.28.

Proof of Claim 3.4. 1) Let $A \subseteq M, |A| \leq \lambda$. Choose by induction on $n < \omega$ for every finite set $u \subseteq A$ of cardinality n, a model N_u such that: $u \subseteq N_u, N_u \leq_{\mathfrak{s}} M$, $\|N_u\| \leq \chi$ and $w \subset u \Rightarrow N_w \subseteq N_u$. There is no problem to do it, see I§1 $A \subseteq \bigcup\limits_u N_u \subseteq M$, $\|\bigcup\limits_u N_u\| \leq \lambda$ and $\bigcup\limits_u N_u \leq_{\mathfrak{s}} M$ by Claim 3.5(1)(b).

2) So assume $M \in K$, $\|M\| \leq \mu$ and $A \subseteq M$ with $|A| = \lambda$. For each finite set $u \subseteq |M|$ choose $N_u \leq_{\mathfrak{s}} M$ with $\|N_u\| \leq \chi$ such that $u \subseteq N_u$ and [$v \subset u$ implies $N_v \subseteq N_u$] (so they form a directed indexed set of models). Since $\mathfrak{K}$ is $(< \mu, < \mu)$-smooth, for each $B \subseteq M$ of cardinality $< \mu$ the model $N_B := \cup\{N_u : u \subseteq B \text{ is finite}\}$ is in K and $[u \in [B]^{<\aleph_0} \Rightarrow N_u \leq_{\mathfrak{s}} N_B]$ and $\|N_B\| \leq |B| + \chi$ (all by 3.5(1)(a)). It remains to show $N_A \leq_{\mathfrak{s}} M$.

Note again by $(< \mu, < \mu)$-smoothness:

$(*)$ $[C \subseteq B \subseteq M \wedge |B| < \lambda \Rightarrow N_C \leq_{\mathfrak{s}} N_B]$

(use 3.5(1)(b)). Write M as $\bigcup\limits_{i<\mu} A_i$ with $A = A_0$, the A_i increasing continuous and $|A_i| < \mu$. Then $M = \bigcup\limits_{i<\mu} N_{A_i}$, and by $(*)$ we have $\langle N_{A_i} : i < \mu \rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous. So by Ax(A4) for $j < \mu$, $N_{A_j} \leq_{\mathfrak{s}} \bigcup\limits_{i<\mu} N_{A_i}$; i.e., $N_{A_j} \leq_{\mathfrak{s}} M$; taking $j = 0$, we finish.

3) Included in the proof of 3.4(2).

The central definition of this section is "NF is κ-based", 3.7(1):

3.7 Definition. 1) NF is κ-based when: if $M \leq_{\mathfrak{s}} M^*$ and $A \subseteq M^*$ where $|A| \leq \kappa$ <u>then</u> for some N_0, N_1 we have $\|N_1\| \leq \kappa, N_0 = M \cap N_1, A \subseteq N_1$ and NF(N_0, M, N_1, M^*). We define "$(< \kappa)$-based" similarly. We may say $\mathfrak{s}$ is κ-based.

2) NF is (λ, κ)-based if (1) hold when $\|M\| = \lambda$ (similarly we define

"NF is $(\leq \lambda, \kappa)$-based",etc).
3) NF is χ-weakly κ-based if we weakened the conclusion to "N_0, M, N_1 are in χ-weakly stable amalgamation inside M^*", which mean that: $N_0 \leq_{\mathfrak{s}} M' \leq_{\mathfrak{s}} M \;\;\&\;\; M' \in K_{\chi+\kappa} \Rightarrow N_0, M', N_1$ are in stable amalgamation inside M^*. Similarly χ-weakly (λ, κ)-based.

The following lemma will lead via Section 4 to the conclusion in theorem 5.2, that if $\mathfrak{K}$ is not χ-based (but has some smoothness) then for suitable μ the class K_λ has 2^λ non-isomorphic $(\mathbb{D}_{\mathfrak{K}}, \mu)$-homogeneous models in many cardinals λ.

3.8 Lemma. *Assume $\lambda > \chi, \mathfrak{K}$ is $(\leq \lambda, \leq \lambda)$-smooth (see V.B.1.18(3)), $\mathfrak{K}$ has* LSP(λ, χ), NF *is not $(\leq \lambda, \chi)$-based as exemplified by M, A, M^* hence $\|M^*\| \leq \lambda$, $|A| \leq \chi$. Then there are $M_i, N_i (i < \chi^+)$ such that:*

$\odot$ (a) $\|M_i\|,\ \|N_i\| \leq \chi$;
(b) $A \subseteq N_0$;
(c) $M_i = M \cap N_i$;
(d) $M_i \leq_{\mathfrak{s}} N_i \leq_{\mathfrak{s}} M^*$ *and* $M_i \leq_{\mathfrak{s}} M \leq_{\mathfrak{s}} M^*$;
(e) *the triple M_i, M_{i+1}, N_i is not in stable amalgamation (inside M^*);*
(f) $\langle M_i : i < \chi^+ \rangle$ *is continuous increasing;*
(g) $\langle N_i : i < \chi^+ \rangle$ *is continuous increasing.*

Proof. We choose (M_i, N_i) by induction on i.

Case 1. $i = 0$. We choose by induction on $\zeta < \chi, A_\zeta,\ B_\zeta$ such that $|A_\zeta| + |B_\zeta| \leq \chi, A_\zeta \leq_{\mathfrak{s}} M, B_\zeta \leq_{\mathfrak{s}} M^*, B_\zeta \supseteq \bigcup_{\xi<\zeta} B_\xi \cup (A_\zeta \cap M)$, $A_\zeta \supseteq A \cup \bigcup_{\xi<\zeta} A_\xi \cup \bigcup_{\xi<\zeta} B_\xi$. Now $N_0 := \bigcup_{\zeta<\chi} A_\zeta$ is as required: $\bigcup_{\zeta<\chi} A_\zeta \leq_{\mathfrak{s}} M^*$, (by $(\leq \lambda, \leq \lambda)$-smoothness and the choice of A_ζ, B_ζ) and $(\bigcup_{\zeta<\chi} A_\zeta) \cap M = \bigcup_{\zeta<\chi} (A_\zeta \cap M) = \bigcup_{\zeta<\chi} B_\zeta \leq_{\mathfrak{s}} M$, (by smoothness).

Let $M_0 = M \cap N_0 = \bigcup_{\zeta<\chi} B_\zeta$. So clauses (a),(b),(c) hold and $N_0 \leq_{\mathfrak{s}} M^*$ and $M_0 \leq_{\mathfrak{s}} M$. Now by Ax(A3) we have $M_0 \leq_{\mathfrak{K}} N_0$ hence clause (d) holds, too.

Case 2. i limit: Take unions (use smoothness).

Case 3. $i = j+1$: Clearly "not NF(M_j, N_j, M, M^*)" (by the choice of A, M, M^*). We can represent M as a direct limit of a directed set of $\leq_{\mathfrak{s}}$-submodels including M_j of cardinality $\leq \chi$, $M = \bigcup_{t\in I} M_t$ (use LSP(λ,χ) and 3.5(2)). Necessarily for some t, M_j, M_t, N_j are not in stable amalgamation inside M^*. [Why? By 3.5(3),(4)]. Now define M_i, N_i as in the case $i = 0$ such that $M_t \subseteq M_i$, $N_j \subseteq N_i$ and (a), (c), (d) holds. Now by monotonicity of NF (i.e., Ax(C3)) we have: (e) holds. $\square_{3.8}$

3.9 Remark. 0) In the proof of 3.8 we use $(\leq\lambda, \leq\lambda)$-smoothness only when we quote 3.5(4) in case 3. So we can replace "$(\leq\lambda,\leq\lambda)$-smooth" by $(\leq\lambda,\leq\chi)$-smooth at the price of strengthening the assumption to NF is not χ-weakly $(\leq\lambda,\chi)$-based. Similarly below.
1) In Case 1 we can choose A_ζ, B_ζ only for $\zeta < \theta$ where θ is a regular cardinal $\leq\chi$. Then we use $(\lambda,\leq\chi,\theta)$-smoothness only. Still we should consider the use of smoothness in case 3.
2) Let $\theta = \text{cf}(\theta) \leq \chi$ and assume only $(\chi,\theta)^+$-smoothness that is (λ,χ,θ)-smoothness, see Definition V.B.1.18(3). Then as explained in 3.9(0),(1) above we can still prove the weaker version of lemma (assuming even just that $\mathfrak{s}$ is not χ-weakly $(\leq\lambda,\chi)$-based), but in clauses (f) and (g) of the conclusion we know that we get continuity only for $\delta < \chi^+$ of cofinality θ. This complicates the combinatorics in section 4.

3.10 Claim. *Suppose* LSP$(\leq\chi^+,\chi)$ *and* $\mathfrak{s}$ *is* $(\leq\chi^+,\leq\chi^+)$*-smooth.* <u>*Then*</u> *the existence of* M_i, N_i $(i<\chi^+)$ *as in 3.8 is equivalent to "*$\mathfrak{s}$ *is not* (χ^+,χ)*-based".*

Proof. The implication "if" holds by 3.8. For the implication "only if" let $\langle (M_i, N_i : i < \chi^+\rangle$ is as in 3.8, i.e. satisfies clauses (a)-(g) there.

Note that by clause (e) we have $M_i \neq M_{i+1}$, hence $M := \cup\{M_i : i < \chi^+\}$ belongs to K_{χ^+} and $i < \chi^+ \Rightarrow M_i \leq_{\mathfrak{s}} M$. Let $N = \cup\{N_i : i < \chi^+\}$ so $N \in K_{\chi^+}$ and $i < \chi^+ \Rightarrow N_i \leq_{\mathfrak{s}} N$. But by the $(\leq \chi^+, \leq \chi^+)$-smoothness we have $M \leq_{\mathfrak{s}} N$. Toward contradiction assume that $\mathfrak{s}$ is (χ^+, χ)-based then for each $\alpha < \chi^+$ applying the definition to $A = N_\alpha$ we can find M'_α, N'_α of cardinality $\leq \chi$ such that $N_\alpha \subseteq N'_\alpha \leq_{\mathfrak{s}} N, M'_\alpha = N'_\alpha \cap M$ and $\mathrm{NF}(M'_\alpha, N'_\alpha, M, N)$.

Let $\beta_\alpha = \min\{\beta < \chi^+ : N'_\alpha \subseteq N_\beta\}$. Now let $\gamma_0 = 0, \gamma_{n+1} = \beta_{\gamma_n}$ and $\gamma_\omega = \cup\{\gamma_n : n < \omega\}$. Also let $M^*_\omega = M_{\gamma_\omega} = \cup\{M'_{\gamma_n} : n < \omega\}, N^*_\omega = N_{\gamma_\omega} = \cup\{N'_{\gamma_n} : n < \omega\}, M^*_{\omega+1} = M_{\gamma_\omega+1}, N^*_{\omega+1} = N_{\gamma_\omega+1}$ and let $M^*_n = M'_{\gamma_n}, N^*_n = N'_{\gamma_n}$. Now $\neg\mathrm{NF}(M^*_\omega, N^*_\omega, M^*_{\omega+1}, N^*_{\omega+1})$ holds as it means $\neg\mathrm{NF}(M_\alpha, N_\alpha, M_{\alpha+1}, N_{\alpha+1})$ for $\alpha = \gamma_\omega$. But for each $n < \omega$ we have $\mathrm{NF}(M^*_n, N^*_n, M, N)$ by the choice of (M^*_n, N^*_n) hence by monotonicity we have $\mathrm{NF}(M^*_n, N^*_n, M^*_{\omega+1}, N^*_{\omega+1})$.

Apply 1.10(2) with $\langle M^*_i : i \leq \omega + 1\rangle, \langle N^*_i : i \leq \omega\rangle, \langle N^*_{\omega+1} : i \leq \omega\rangle$ here standing for $\langle M_i : i \leq \delta + 1\rangle, \langle N^a_i : i \leq \delta\rangle, \langle N^b_i : i \leq \delta\rangle$ there. This is O.K. as the assumption of 1.10 holds by a previous sentence, hence we get its conclusion, i.e. $\mathrm{NF}(M^*_\omega, N^*_\omega, M^*_{\omega+1}, N^*_{\omega+1})$, contradicting a previous sentence. The contradiction comes from assuming "$\mathfrak{s}$ is (χ^+, χ)-based", so we are done also with the "only if" direction. $\square_{3.10}$

3.11 Remark. In Definition 3.7 we may ask that N_0, N_1 exist not as submodels of M^* but of some M^{**}, where $M^* \leq_{\mathfrak{s}} M^{**}$. This is apparently a weaker definition. However, assuming, e.g., $(\leq \chi, \theta)^+$-smoothness for some $\theta \leq \chi$ is enough to get back the old definition (use 1.10(2)).

3.12 Claim. *Assume $\mathfrak{s}$ is κ-smooth, $\kappa \leq \chi$ and $\mathfrak{s}$ is χ-based. If $M^1_t \leq_{\mathfrak{s}} M^2_t$ and $A_t \subseteq M^2_t, |A_t| \leq \chi$ for every $t \in I$ and $|I| \leq \chi$ <u>then</u> for some set Y of cardinality χ, for every $t \in I$ the models $M^1_t \restriction Y, M^1_t, M^2_t \restriction Y$ are in stable amalgamation inside M^2_t and $A_t \subseteq M^2_T \restriction Y$.*

Proof. As without loss of generality $I \neq \emptyset$ we can let $I = \{t_\varepsilon : \varepsilon < \chi\}$ and we choose $M^1_{t,\varepsilon}, M^2_{t,\varepsilon}$ by induction on $\varepsilon < \kappa$ such that:

(a) $M^1_{t,\varepsilon} \le_{\mathfrak{s}} M^2_{t,\varepsilon} \le_{\mathfrak{s}} M^2_t$
(b) M^2_t include $A_t \cup \bigcup\{M^2_{s,\zeta} \cap M^2_t : s \in I, \zeta < \varepsilon\}$
(c) $\|M^2_{t,\varepsilon}\| \le \chi$
(d) $M^1_{t,\varepsilon}, M^1_t, M^2_{t,\varepsilon}$ are in stable amalgamation inside M^2_t.

This is possible as $\mathfrak{s}$ is χ-based. Let $Y = \cup\{M^2_{t,\varepsilon} : t \in I, \varepsilon < \kappa\}$ and let $M^\ell_{t,\kappa} := Y \cap M^\ell_t$ so we have

(α) $\langle M^\ell_{t,\varepsilon} : \varepsilon < \kappa\rangle$ is increasing with union $M^\ell_{t,\kappa}$ for $\ell = 1, 2$
(β) $M^1_{t,\varepsilon}, M^1_t, M^2_{t,\varepsilon}$ is in stable amalgamation inside M^2_t for every $\varepsilon < \kappa$.

We can deduce that $M^1_{t,\kappa}, M^1_t, M^2_{t,\kappa}$ is in stable amalgamation by 1.10(2).

Using Definition 4.8(1) below.

3.13 Claim. *If $\mathfrak{s}$ is $(\le \kappa)$-smooth and χ-based and $M_1 \le_{\mathfrak{s}} M_2$ <u>then</u> for the χ-majority of $Y \in [M]^{\le \chi}$ the triple $M_1 \restriction Y, M_1, M_2 \restriction Y$ is in stable amalgamation inside M_2.*

Proof. Using a directed system of submodels. $\square_{3.13}$

§4 Stable Constructions

The following definition generalizes the notion of a construction from Chapter IV of [Sh:c]. More precisely, since we are using independence rather than isolation, an $\mathbf{F}^f_\lambda$-construction recalling $\mathbf{F}^\ell_\lambda = \{(p, B)$: for some set $A \subseteq \mathfrak{C})$ and type $p \in \mathbf{S}^{<\omega}(A)$ we have $B \subseteq A$ and p does not fork over $A\}$. We shall return to this in V.D§5 and say there somewhat more, see in particular V.D.5.7(3) why this apply.

4.1 Context. AxFr_1, i.e. $\mathfrak{s}$ satisfies AxFr_1.

4.2 Definition. 1) $\mathscr{A} = \langle A, B_i, w_i : i < \alpha \rangle$ is a stable construction inside N (of length $\alpha = \ell g(\mathscr{A})$) when (letting for $u \subseteq \alpha$, $A_u = \langle A \cup \bigcup\limits_{j \in u} B_j \rangle_N^{\mathrm{gn}}$):

- (i) $A, B_i \le_{\mathfrak{s}} N$ and $A_j \le_{\mathfrak{s}} N$ (note $A_j = A_{\{\gamma:\gamma<j\}}$) for $i < \alpha$, $j \le \alpha$
- (ii) (a) $w_i \subseteq i$
  - (b) w_i is closed for $\mathscr{A} \restriction i$ [defined below in 4.2(2)]
- (iii) $B_i \cap A_i \subseteq A_{w_i}$
- (iv) $\mathrm{NF}(B_i \cap A_i, B_i, A_i, N)$
- (v) $B_i \cap A \le_{\mathfrak{s}} A$
- (vi) For each i one of the following occurs:
  - Case (a) : $B_i \subseteq A$, $w_i = \emptyset$
  - Case (b) : For some $\gamma_i < i, w_i = w_{\gamma_i} \cup \{\gamma_i\}$ and $B_i \cap A_i = B_{\gamma_i}$
  - Case (c) : $B_i = \langle \bigcup\limits_{j \in w_i} B_j \rangle_N^{\mathrm{gn}}$.

1A) Alternatively we define $\mathscr{A} = \langle A, A_i, B_j, w_j : i \le \alpha, j < \alpha \rangle$ by simultaneous induction on α such that $(i), (ii), (vi)$ and

- $(iii)'$ $A_i <_{\mathfrak{s}} \mathfrak{C}$ is $\le_{\mathfrak{s}}$-increasing continuous
- $(iv)'$ $B_i \cap A_i, B_i, A_i$ is in stable amalgamation inside A_{i+1}
- (v) $A_{i+1} = \langle A_i \cup B_i \rangle_{\mathfrak{C}}^{\mathrm{gn}}$.

2) For such $\mathscr{A}$, u is called closed or $\mathscr{A}$-closed if:

- (a) $u \subseteq \alpha$
- (b) $i \in u \Rightarrow w_i \subseteq u$.

3) $\mathscr{A}$ is a $(< \mu)$-stable construction inside N if $\mathscr{A}$ is a stable construction inside N and $|w_i| + |B_i| < \mu$ for $i < \ell g(\mathscr{A})$. In this case we say $A_{\ell g(\mathscr{A})}$ is $(< \mu)$-stably constructible over A.
4) Let $\|\mathscr{A}\| = |\alpha| + |A| + \Sigma\{|B_i| : i < \alpha\}$. For $u \subseteq \ell g(\mathscr{A})$ let $c\ell_{\mathscr{A}}(u)$ be the minimal $\mathscr{A}$-closed $v \subseteq \alpha$ such that $u \subseteq v$.

4.3 Notation. 1) If $\mathscr{A} = \langle A, B_i, w_i : i < \alpha\rangle$ then $\mathscr{A} \restriction \beta := \langle A, B_i, w_i : i < \alpha \cap \beta\rangle$.
2) In Definition 4.2 we let $A^{\mathscr{A}} = A, B_i^{\mathscr{A}} = B_i, A_i^{\mathscr{A}} = A_i, w_i^{\mathscr{A}} = w_i$.

4.4 Claim. *Assume $\mathscr{A}$ is a stable construction inside N.*
1) If $\beta \le \ell g(\mathscr{A})$ then $\mathscr{A} \restriction \beta$ is a stable construction inside N.
2) If $\gamma \le \ell g(\mathscr{A})$ then γ is closed for $\mathscr{A}$.
3) The intersection of any family of sets each closed for $\mathscr{A}$ is closed for $\mathscr{A}$.
4) The union of any family of subsets of $\ell g(\mathscr{A})$ closed for $\mathscr{A}$ is closed for $\mathscr{A}$.
5) If $u \subseteq \ell g(\mathscr{A})$ is closed for $\mathscr{A}$ where $\mathscr{A}$ is a stable construction inside N then $A_u^{\mathscr{A}} \le_{\mathfrak{s}} N$.
6) If $\mathscr{A}$ is a $(< \mu)$-stable construction, μ regular, $u_1 \subseteq \ell g(\mathscr{A})$ is $\mathscr{A}$-closed and $a \in A_{u_1}$ then for some $\mathscr{A}$-closed $u_2 \subseteq u_1$ of cardinality $< \mu$ we have $a \in A_{u_2}^{\mathscr{A}}$.
7) If $u_1 \subseteq u_2$ are $\mathscr{A}$-closed then $A_{u_1} \subseteq A_{u_2}$ hence (by (5)), $A_{u_1} \le_{\mathfrak{s}} A_{u_2}$.

Proof. 1) - 4). Easy.
5),6),7) Are proved in 4.5; more specifically parts (5),(7) are proved in 4.5(1) and part (6) is proved in 4.5(2). $\square_{4.4}$

4.5 Claim. *0) The two definitions 4.2(1), 4.2(1A) of a stable construction are compatible.*
1) If $\mathscr{A}$ is a stable construction inside N, for $\ell = 0,1,2$, $u_\ell \subseteq \alpha = \ell g(\mathscr{A})$ is closed, and $u_0 = u_1 \cap u_2$ then $A_{u_\ell} \le_{\mathfrak{s}} N$ and $\mathrm{NF}(A_{u_0}, A_{u_1}, A_{u_2}, N)$ and $A_{u_1 \cup u_2} = \langle A_{u_1} \cup A_{u_2}\rangle_N^{\mathrm{gn}}$.
2) 4.4(6) holds for $a \in A_\alpha$, in fact if $\mathscr{A}$ is a stable construction, μ regular and $\bar a \in {}^{\mu>}(A^{\mathscr{A}}_{\ell g(\mathscr{A})})$ then for some $u \subseteq \ell g(\mathscr{A})$ of cardinality $< \mu$ we have $\bar a \in {}^{\mu>}(A_u^{\mathscr{A}})$. If $\mathscr{A}$ is a $(< \mu)$-stable construction then without loss of generality u is $\mathscr{A}$-closed, in fact any $v \subseteq \ell g(\mathscr{A})$ of cardinality $< \mu$ is included in some $\mathscr{A}$-closed $u \subseteq \ell g(\mathscr{A})$ of cardinality $< \mu$.

Proof. 0) First, clearly if $\mathscr{A}$ is as in definition 4.2(1), then $A_i := \langle A \cup \{B_j : j \in i\}\rangle^{\rm gn}_N = \langle A \cup \{B_j : j < i\}\rangle^{\rm gn}_N$ is well defined (by Ax(B3)) and easily $\langle A, A_i, B_j, w_j : i \le \alpha, j < \alpha\rangle$ is as required in Definition 4.2(1A).
Second, assume $\mathscr{A}$ is as in Definition 4.2(1A). Now we prove by induction on $\alpha \le \ell g(\mathscr{A})$ that

$\circledast$ (a) if $u \subseteq \alpha$ is $\mathscr{A}$-closed <u>then</u> $A_u <_{\mathfrak{s}} \mathfrak{C}$

(b) $\mathscr{A}_\alpha := \langle A, B_i, w_i : i < \alpha\rangle$ satisfies Definition 4.2(1)

(c) $\mathscr{A}_\alpha$ satisfies Claim 4.5(1).

This is enough.
1) Straightforward by induction on $\alpha \le \ell g(\mathscr{A})$, (for successor remember to use 1.3, for limit use 1.10(1)).
2) We prove by induction on $\alpha \le \ell g(\mathscr{A})$ that if $\bar a \in {}^{\mu>}(A^{\mathscr{A}}_\alpha)$ then for some $u \subseteq \alpha$ we have $|u| < \mu$ and $\bar a \in {}^{\mu>}(A^{\mathscr{A}}_u)$. This is straight. Also the additional assertions are. $\square_{4.5}$

4.6 Claim. *1) If $\mathscr{A} = \langle A, B_i, w_i : i < \alpha\rangle$ is a stable construction inside N, h a one-to-one function from α onto β satisfying $[j \in w_i \Rightarrow h(j) < h(i)]$ and let $w^*_{h(i)} = \{h(j) : j \in w_i\}$ and $B^*_{h(i)} = B_i$ <u>then</u> $\mathscr{A}^* = \langle A, B^*_i, w^*_i : i < \beta\rangle$ is a stable construction inside N.*
2) [Smoothness + Ax(C8)] If $\mathscr{U}$ is an unbounded subset of δ, a limit ordinal, $\mathscr{A} = \langle A, B_i, w_i : i < \delta\rangle$ and for every $\alpha \in \mathscr{U}$ the sequence $\mathscr{A}_\alpha = \langle A, B_i, w_i : i < \alpha\rangle$ is a stable construction inside N <u>then</u> $\mathscr{A}$ is a stable construction inside N.
3) [Smoothness + Ax(C8)] If $\mathscr{U}$ is an unbounded subset of δ, a limit ordinal, $\mathscr{A} = \langle A, B_i, w_i : i < \delta\rangle$ and for every $\alpha \in \mathscr{U}$ the sequence $\mathscr{A}_\alpha = \langle A, B_i, w_i : i < \alpha\rangle$ is a stable construction inside N_α satisfying $N_\alpha = \langle \cup\{B_i : i < \alpha\} \cup A\rangle^{\rm gn}_{N_i}$ and for $\alpha, \beta \in \mathscr{U}$ we have $\alpha < \beta \Rightarrow N_\alpha \le_{\mathfrak{s}} N_\beta$ <u>then</u> $\mathscr{A}$ is a stable construction inside $N_\delta := \cup\{N_\alpha : \alpha < \delta\}$.

Remark. As it is clear how to add one step to a stable construction, this enables us to "construct".

Proof. 1) Easy by 4.5.
2),3) Easy, too. $\square_{4.6}$

4.7 Claim. *1) If $\lambda^{<\chi} + 2^{|\tau(\mathfrak{K})|} = \lambda$, $\mathrm{LSP}(<\chi), M \in K_{\mathfrak{s}}$ and $\|M\| \leq \lambda$ and λ^+ is $\leq_{\mathfrak{s}}$-inaccessible <u>then</u> there is a stable construction $\mathscr{A} = \langle A, B_i, w_i : i < \delta\rangle$ inside some $N \in K_{\mathfrak{s}}$ such that $A = |M|, A_\delta = |N|$, $\|N\| \leq \lambda$ and N is $(\mathbb{D}_{\mathfrak{s}}, \chi)$-homogeneous.*

Proof. Straightforward. $\square_{4.7}$

4.8 Definition. 1) Let $\mathscr{P} \subseteq [B]^\lambda$; we say that for the λ-majority of $A \subseteq B$, we have $A \in \mathscr{P}$ <u>when</u>: for some algebra $\mathscr{B}$ with universe B and with vocabulary with $\leq \lambda$ functions, if $\mathscr{B}'$ is a subalgebra of $\mathscr{B}$ of cardinality λ, then $|\mathscr{B}'|$ belongs to $\mathscr{P}$. We can replace "$A \in \mathscr{P}$" by $\varphi(A), \varphi(-)$ a property, well if $|B| \leq \lambda$, "for the λ-majority of $A \subseteq B \ldots$" means "for $A = B, \ldots$".
2) We say "for the λ-majority of $M \subseteq N$, M satisfies φ" instead "for the λ-majority of $A \subseteq |N|$ the model $N \restriction A$ satisfies φ"; see 4.9 below.
3) We say that for the $(<\lambda)$-majority of $A \subseteq B$ we have φ <u>when</u>: for some algebra $\mathscr{B}$ with universe $B \cup \lambda$ and $(<\lambda)$ many functions for every $\mathscr{B}'$, a subalgebra of $\mathscr{B}, \mathscr{B}' \cap \lambda$ an ordinal and the set $\mathscr{B}' \cap B$ satisfies φ.

We can conclude from 4.6:

4.9 Observation. 1) If M is a model, τ_M has $\leq \lambda$ function symbols (including individual constants) <u>then</u> for the λ-majority of $A \subseteq |M|, M \restriction A$ is a submodel, i.e. is $\subseteq M$.
2) If $\mathfrak{s}$ has $\mathrm{LSP}(\lambda)$ and has $(\leq \lambda, \leq \lambda)^+$-smoothness and $M \in K_{\mathfrak{s}}$ <u>then</u> for the λ-majority of $A \subseteq |M|$ we have $M \restriction A \leq_{\mathfrak{s}} M$.
3) If $\mathfrak{s}$ has smoothness, $\lambda = \mathrm{LS}(\mathfrak{s})$ and M is a $\tau_{\mathfrak{s}}$-model <u>then</u> $M \in K_{\mathfrak{s}}$ iff for the λ-majority $A \subseteq M$ we have $M \restriction A \in K_{\mathfrak{s}}$.
4) If $\mathfrak{s}$ has smoothness, $\lambda = \mathrm{LS}(\mathfrak{s})$ and $M \subseteq N$ are $\tau_{\mathfrak{s}}$-models <u>then</u> $M \leq_{\mathfrak{s}} N$ iff for the λ-majority of $A \subseteq M \cup N$ we have $(M \restriction A) \leq_{\mathfrak{s}} (N \restriction A)$.
5) In parts (3),(4) we can replace $\mathfrak{K}_{\mathfrak{s}}$ by an a.e.c. $\mathfrak{K}$ with $\lambda = \mathrm{LS}(\mathfrak{K})$.

Proof. 1) Obvious.
2) Easy.
3),4) By (5).
5) By Chapter I. $\square_{4.9}$

4.10 Claim. *1) Suppose $\mathscr{A} = \langle A, B_i, w_i : i < \alpha\rangle$ is a $(< \lambda^+)$-stable construction, $|A| \le \lambda$ and λ^+ is $\le_{\mathfrak{s}}$-inaccessible. Then for the λ-majority of $X \subseteq A_\alpha \cup \alpha$, $A_\alpha \restriction (X \cap A_\alpha) \le_{\mathfrak{s}} A_\alpha$ and $\langle A \cap X, B_i, w_i : i \in \alpha \cap X\rangle$ is a $(< \lambda^+)$-stable construction (of $A_\alpha \restriction (X \cap A_\alpha)$).*
2) If $\mathscr{A}$ is a $(< \lambda^+)$-stable construction, $|A| < \lambda^+$, λ^+ is $\le_{\mathfrak{s}}$-inaccessible and $\ell g(\mathscr{A}) < \lambda^+$ then $A^{\mathscr{A}}_{\ell g(\mathscr{A})}$ has cardinality $\le \lambda$.
3) We can replace λ^+ by any regular uncountable cardinal.

Proof. Easily, by induction on $\ell g(\mathscr{A})$. $\square_{4.10}$

4.11 Claim. *Suppose $\mathscr{A} = \langle A, B_i, w_i : i < \alpha\rangle$ is a $(< \theta)$-stable construction, θ is a $\le_{\mathfrak{s}}$-inaccessible regular uncountable cardinal.*
1) If for the $(< \theta)$-majority of $X \subseteq A$, we have $A \restriction X \le_{\mathfrak{s}} A$, then for the $(< \theta)$-majority of $X \subseteq A_\alpha \cup \alpha$ we have $A \restriction (A \cap X) \le_{\mathfrak{s}} A$, $A_\alpha \restriction (A_\alpha \cap X) \le_{\mathfrak{s}} A_\alpha$ and $\langle A \cap X, B_i, w_i : i \in \alpha \cap X\rangle$ is $(< \theta)$-stable construction.
2) If A is $(< \theta)$-smooth then A_α is $(< \theta)$-smooth, see Definition 4.12 below.

Remark. If $|A| < \theta$ then it follows that A is $(< \theta)$-smooth.

Proof. Easy by 1.10(1). $\square_{4.11}$

4.12 Definition. We say the model A is $(< \theta)$-smooth when we can find $\langle A_t : t \in I\rangle$ such that $A = \bigcup\limits_{t \in I} A_t, A_t \le_{\mathfrak{s}} A, |A_t| < \theta, I$ directed, $[s \le_I t \Rightarrow A_s \le_{\mathfrak{s}} A_t]$ and $[J \subseteq I$ directed $\& |J| < \theta \Rightarrow \bigcup\limits_{t \in J} A_t \le_{\mathfrak{s}} A]$.

4.13 Theorem. *1) In 2.6 suppose in addition that λ is $\leq_{\mathfrak{s}}$-inaccessible.* *Then for every $\mu > \lambda$, $\dot{I}\dot{E}(\mu, \mathfrak{K}) \geq 2^\lambda$.*
2) If in addition $\mu = \mu^{<\theta} + 2^{|\tau(\mathfrak{K})|}, \theta \leq \lambda$ we can have 2^λ pairwise non-isomorphic models, each $(\mathbb{D}_{\mathfrak{K}}, \theta)$-homogeneous.

Proof. Left to the reader.

Remark. We can generalize [Sh:c, IV,§3], (presenting a uniqueness theorem) to the present context.

4.14 Exercise: Assume that

(a) $\bar{w} = \langle w_\alpha : \alpha < \alpha(*)\rangle, w_\alpha \subseteq \alpha$ and $0 \in w_{1+\alpha}$
(b) $\alpha \in w_\beta \Rightarrow w_\alpha \subseteq w_\beta$
(c) $\bar{C} = \langle C_\alpha : \alpha < \alpha(*)\rangle$
(d) $C_\alpha <_{\mathfrak{s}} \mathfrak{C}$
(e) if $\beta \in w_\alpha$ then $w_\beta = w_\alpha \cap \beta$ and $\langle C_\alpha : \alpha \in w_\beta \cup \{\beta\}\rangle$ is $<_{\mathfrak{s}}$-increasing and $\beta \in w_\alpha \Rightarrow \cup\{C_\gamma : \gamma \in w_\beta\} <_{\mathfrak{s}} C_\alpha$
or just
$(e)^-$ if $\beta < \alpha(*)$ and $\langle \alpha^\beta_\varepsilon : \varepsilon < \zeta_\alpha\rangle$ list w_β in increasing order then there is a stable construction $\mathscr{A}_\alpha$ such that
  (α) $\ell g(\mathscr{A}_\alpha) = \zeta_\alpha$
  (β) $A^{\mathscr{A}_\alpha}_0 = C^1_0$
  (γ) $B^{\mathscr{A}_\alpha}_\varepsilon = C_{\alpha^\beta_\varepsilon}$.

Then we can find $\mathscr{A}, \bar{f}$ such that:

(a) $\bar{f} = \langle f_\alpha : \alpha < \alpha(*)\rangle$
(b) $\mathscr{A}$ is a stable construction inside $\mathfrak{C}$
(c) $\alpha(*) = \ell g(\mathscr{A})$
(d) f_α is an isomorphism from C_α onto $B^{\mathscr{A}_\alpha}_\alpha$
(e) if $\alpha \in w_\beta$ then $f_\alpha \subseteq f_\beta$.

[Hint: Straight using Definition 4.2(1A).]

§5 Non-structure from "NF is not χ-Based"

We are trying to get non-structure from "NF is not χ-based" for suitable regular χ.
Remember the definition of "λ is $\leq_{\mathfrak{s}}$-inaccessible" (see V.B.1.27).

5.1 Context. AxFr_1, i.e. $\mathfrak{s}$ satisfies AxFr_1.

5.2 Theorem. *Assume $\chi^+ \geq \mu > \mathrm{LS}(\mathfrak{K})$ and $(\leq \chi^+, \leq \chi^+)$-smoothness holds*[2] *but* NF *is not χ-based with a counterexample as in 3.8* *Then:*
0) If λ^+ is $\leq_{\mathfrak{s}}$-inaccessible and $\lambda = \lambda^{\mu^+}$ then there are 2^λ pairwise non-isomorphic $(\mathbb{D}_{\mathfrak{s}}, \mu)$-homogeneous models of cardinality λ.
1) For every $\lambda = \lambda^{<\mu} + 2^\chi$ which is regular and $\leq_{\mathfrak{s}}$-inaccessible such that some $S^* \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \chi^+\}$ is good and stationary[3] *there are 2^λ pairwise non-isomorphic $(\mathbb{D}_{\mathfrak{s}}, \mu)$-homogeneous models of cardinality λ; moreover with different reasonable definable invariants.*

5.3 Discussion. We give, in essence, three proofs of (variants of) Theorem 5.2. Items 5.8 through 5.10 reduce the proof of the general case (arbitrary λ) to results in [Sh 300, III], [Sh:e, IV] using black boxes so by 5.11 we deduce 5.2(0), which is enough for the later parts. Items 5.13 through 5.17 (using the construction of 5.8) prove Theorem 5.2(1) as stated. We then comment on models of cardinality $> \lambda$.

5.4 Idea of Proof.
In 3.8 from a counterexample we get a canonical counterexample with $\langle M_i : i \leq \chi^+\rangle, \langle N_i : i \leq \chi^+\rangle$ (as in the picture). We now copy $\langle M_i : i < \chi^+\rangle$ along the tree ${}^{\chi^+>}\lambda$: i.e., choose to define $M_\eta <_{\mathfrak{s}} \mathfrak{C} (\eta \in {}^{\chi^+ \geq}\lambda)$ and $f_\eta : M_{\ell g(\eta)} \underset{\text{isomorphism}}{\overset{\text{onto}}{\rightarrow}} M_\eta, f_\eta$ increasing, amalgamating them freely (i.e. by $\mathrm{NF}_{\mathfrak{s}}$) say inside $\mathfrak{C}$. For $\eta \in {}^{(\chi^+)}\lambda$

[2] hence the cardinals χ^+, μ are $\leq_{\mathfrak{s}}$-inaccessible
[3] if $\lambda > \chi^{++}$ then there is such S

we can choose N_η and $g_\eta \supseteq \bigcup_{\alpha<\chi^+} f_{\eta\restriction\alpha}$ such that $g_\eta : N \to N_\eta$, (isomorphism onto), again amalgamating them freely inside $\mathfrak{C}$. For $S \subseteq {}^{(\chi^+)}\lambda$ let $N_S = \langle N_\eta, M_\nu : \eta \in S, \nu \in {}^{(\chi^+)>}\lambda\rangle^{\rm gn}_{\mathfrak{C}}$. Now in N_S over M_η there is a copy of N_η if and only if $\eta \in S$ (i.e. we shall prove this).

So we have coded S, see [Sh 300, III,§5] or [Sh:e, IV,§2] for why this implies non-structure. We shall give the proof of 5.2 after some further discussion.

5.5 Discussion. NF is not χ-based generalizes (roughly) the first order notion "$\chi \geq |T|$, T unstable" (in Chapter V.A we have considered another one) for the stable first order class $\kappa(T) \leq |T|^+$, the case however does not not appear when $\chi < |T|$, as $|acl(\emptyset)| = |T|$ by the definition of $\mathfrak{C}^{\rm eq}$. But it would appear if we varied the first order notions slightly (perhaps to deal more precisely with algebra), and instead of using the cardinality of a set A in the definitions used the cardinality of a minimal set of generators for A. The following example explores this possibility.

5.6 Example. $T = T^{\rm eq}$ is (first order complete) stable, not superstable. Now if $A, B \subseteq \mathfrak{C}$ are algebraically closed, $B = acl(\bar b), \ell g(\bar b) < \kappa, \kappa$ regular <u>then</u> we can find $\bar a \in A, \ell g(\bar a) < \kappa$ such that $acl(\bar a)$, $acl(\bar a \cup \bar b)$, A are in stable amalgamation <u>if and only if</u> $\kappa \geq \kappa_r(T)$. We can just use $\|A\|_{\rm gen}$ in the definition of χ-based. There are two reasonable ways to define $\|A\|_{\rm gen}$:

$\|A\|_{\rm gen} = \ {\rm Min}\{|B| : B \subseteq A \subseteq acl(B)\}$.
$\|A\|'_{\rm gen} = \ {\rm Min}\{|B| : A \subseteq acl(B)\}$.

The second is less natural but it satisfies $A_1 \subseteq A_2 \Rightarrow \|A_1\|'_{\rm gen} \leq \|A_2\|'_{\rm gen}$ (i.e., monotonicity holds). So "NF κ-based" is a generalization of $\kappa \geq \kappa_r(T)$.

5.7 Discussion Continued. Later, in Chapter V.D, we shall have another notion, capturing the parallel of $\kappa(T)$ and so in particular "superstability". But remember that "stable" was captured in some sense in Chapter V.A and axiomatized in Chapter V.B. Looking carefully at universal classes (see V.B.2.3) we see that for this case (i.e., $\leq_{\mathfrak{s}}$ is $\leq^{\aleph_0}_{\mathrm{qf},\mu^+,\chi^+}$ - see V.B.2.8, K a universal class without the (χ^+,qf)-order property, $\mu = 2^{2^\chi}$) the statement "$\mathfrak{s}$ is $\chi_{\mathfrak{s}}$-based" follows. However, this is seemingly not true for the general $\mathfrak{s}$ we are dealing with. Also note that if, e.g., K is the class of submodels of models of T, T first order, stable not superstable with elimination of quantifiers, so K is a universal class, then in V.B.2.3 we get $(\mathfrak{K}, \leq_{\mathfrak{K}}, \mathrm{NF}, \langle\rangle^{\mathrm{gn}})$ satisfying AxFr_1. After the following theorem and assumption we shall be able to generalize some definitions and facts on stable theories to our context, e.g., $|T|^+$-primary model, parallelism. In other words, only assuming smoothness and $\mathfrak{s}$ is χ-based we can really generalize stability theory.

5.8 Proof of Theorem 5.2. By our assumption (see Lemma 3.8), there are sequences $\langle M_i : i \leq \chi^+\rangle, \langle N_i : i \leq \chi^+\rangle$ such that:

$(*)_1$ (*i*) both are $\leq_{\mathfrak{s}}$-increasing continuous

(*ii*) $i < \chi^+ \Rightarrow \|M_i\| + \|N_i\| \leq \chi$

(*iii*) $\neg\, \mathrm{NF}(M_i, N_i, M_{i+1}, N_{i+1})$ for $i < \chi^+$

(*iv*) $M_i \leq_{\mathfrak{s}} N_i \leq_{\mathfrak{s}} N_{\chi^+}$ for $i \leq \chi^+$.

Concerning clause (iv) note that (for $i = \chi^+$ we use (χ^+,χ^+)-smoothness) to show that $M_{\chi^+} \leq_{\mathfrak{s}} N_{\chi^+}$. Let $N := N_{\chi^+}$ and $M := M_{\chi^+}$.

Let $\{\eta_i : i < i^*(0)\}$ be a list of ${}^{(\chi^+)>}\lambda$ such that $[i \leq j \Rightarrow \ell g(\eta_i) \leq \ell g(\eta_j)]$. By induction on $i < i^*(0)$ we choose f_{η_i}, M_{η_i}, C_i such that:

$(*)_2$ (a) f_{η_i} is an isomorphism from $M_{\ell g(\eta_i)}$ onto M_{η_i}

(b) $\eta_j = \eta_i \restriction \alpha \Rightarrow f_{\eta_j} \subseteq f_{\eta_i}$ (hence $M_{\eta_j} \leq_{\mathfrak{s}} M_{\eta_i}$)

(c) $M_{\eta_j} \leq_{\mathfrak{s}} C_i$ for $j < i$

(d) C_i is $\leq_{\mathfrak{s}}$-increasing continuous

(e) if $\ell g(\eta_i) = \gamma+1$, let $\eta_j = \eta_i\restriction\gamma$ and $\mathrm{NF}(M_{\eta_j}, C_i, M_{\eta_i}, C_{i+1})$, $C_{i+1} = \langle M_{\eta_i} \cup C_i\rangle^{\mathrm{gn}}_{C_{i+1}}$

(f) $M_{<>} = M_{\eta_0} = C_0$

There is no problem.
Now for $\mathscr{T} \subseteq {}^{(\chi^+)>}\lambda$, let

$(*)_3$ $C^{\mathscr{T}} = \langle \bigcup_{\eta \in \mathscr{T}} M_\eta \rangle^{\text{gn}}_{C_{i^*(0)}}$.

5.9 Claim. *1) There a $(< \chi^+)$-stable construction $\mathscr{A}$ inside $C_{i^*(0)}$ with $A^{\mathscr{A}} = \emptyset, B^{\mathscr{A}}_i = M_{\eta_j}$ if $\ell g(\eta_i) = \ell g(\eta_j) + 1$ and $\eta_j \triangleleft \eta_i, w_i = \{j < i : \eta_j \triangleleft \eta_i\}$ and even $A^{\mathscr{A}}_i = C_i$ for $i \le i^*(0)$.*
2) If $\mathscr{T}_0, \mathscr{T}_1, \mathscr{T}_2 \subseteq {}^{(\chi^+)>}\lambda$ are closed under initial segments, $\mathscr{T}_0 = \mathscr{T}_1 \cap \mathscr{T}_2$ then $\text{NF}\big(C^{\mathscr{T}_0}, C^{\mathscr{T}_1}, C^{\mathscr{T}_2}, C_{i^*(0)}\big)$.

Proof of 5.9. 1) Should be clear by comparing the construction with Definition 4.2(3) recalling claim 4.5(0).
2) It is immediate by 4.5(1). $\square_{5.9}$

Remark. That is, is does not matter in which order we carry out the definition.

Continuation of the Proof of 5.2. 0) We have built a tree of the $\{M_\eta : \eta \in {}^{\chi^+>}\lambda\}$. Since the original sequence $\langle M_i : i \le \chi^+\rangle$ was continuous any model containing this tree will contain all the $M_\eta := \bigcup_{i<\chi^+} M_{\eta \restriction i}$ for η from ${}^{\chi^+}\lambda$ such that $\ell g(\eta) = \chi^+$. Note: by 5.9(2) we know that $M_\eta \le_{\mathfrak{s}} C_{i^*(0)}$. Now we paste independent copies of $N = N_{\chi^+}$ on the top of the tree. We will see that we can realize or omit a particular N_η (for $\eta \in {}^{\chi^+}\lambda$) at will.

Formally let $\{\nu_\alpha : \alpha < \lambda^{\chi^+}\}$ list ${}^{(\chi^+)}\lambda$ and we can easily define g_{ν_α}, N_{ν_α}, $g_{\nu_\alpha,\zeta}$, $N_{\nu_\alpha,\zeta}$, C^+_α and $C^\oplus$ for $\alpha < \lambda^{\chi^+}, \zeta < \chi^+$ such that:

$(*)_4$ (α) $g_{\nu_\alpha} : N_{\chi^+} \rightarrow N_{\nu_\alpha}$ is an isomorphism onto extending $f_{\nu_\alpha} := \bigcup_{\xi<\chi^+} f_{\nu_\alpha \restriction \xi}$

(β) $g_{\nu_\alpha,\zeta} = g_{\nu_\alpha} \restriction N_\zeta$ and $N_{\nu_\alpha,\zeta} = g_{\nu_\alpha,\zeta}(N_\zeta)$ for $\zeta < \chi^+$

(γ) C^+_α is $\le_{\mathfrak{s}}$-increasing continuous

(δ) $C^+_0 = C_{i^*(0)}$

(ε) $C^\oplus = C^+_{(\lambda\chi^+)}$

(ζ) $\mathrm{NF}(M_{\nu_\alpha}, C^+_\alpha, N_{\nu_\alpha}, C^+_{\alpha+1})$ and $C^+_{\alpha+1} = \langle C^+_\alpha \cup N_{\nu_\alpha}\rangle^{\mathrm{gn}}_{C^+_{\alpha+1}}$.

For $\mathscr{T} \subseteq {}^{\chi^+ \ge}\lambda$ let (this is compatible with $(*)_3$):

$(*)_5$ $C^{\mathscr{T}} = \langle \cup\{M_\eta : \eta \in \mathscr{T} \cap {}^{\chi^+>}\lambda\} \cup \{N_\nu : \nu \in \mathscr{T} \cap {}^{\chi^+}\lambda\}\rangle^{\mathrm{gn}}_{C^\oplus}$.

If $\lambda = \lambda^{<\mu} + 2^\chi$ let $C^{\mathscr{T}}_*$ be $(\mathbb{D}_{\mathfrak{s}}, \mu)$ -homogeneous and $(< \mu)$-stably constructible over $L^{\mathscr{T}}$ and let $\mathscr{A}^{\mathscr{T}} = \langle C^{\mathscr{T}}, B^{\mathscr{T}}_i, w^{\mathscr{T}}_i : i < i^{\mathscr{T}}\rangle$ be such a construction and without loss of generality $i^{\mathscr{T}} = \lambda$, see 4.7 (if we would like to deal with $(\mathbb{D}_{\mathfrak{s}}, \mu)$-homogeneous models of cardinality $\lambda_1 = \lambda_1 < \mu + 2^\lambda \ge \lambda$, we shall use larger $i^{\mathscr{T}}$). For other λ or when proving the version without "$(\mathbb{D}_\kappa, \mu)$-homogeneous" let $C^{\mathscr{T}}_* := C^{\mathscr{T}}$.

Easily by 4.4, 4.5, 4.6, 4.7

$(*)_6$ $C^{\mathscr{T}}_*$ is $(< \chi^{++})$-stably constructible over $C_{i^*(0)}$

$(*)_7$ $C_{i^*(0)}$ is $(< \chi^+)$-stably constructible over $\emptyset$

$(*)_8$ if $\mathscr{T}^1 \subseteq {}^{\chi^+\ge}\lambda$ is downward closed and $\mathscr{T}^0 = \mathscr{T}^1 \cap {}^{\chi^+>}\lambda$ then

(a) $C^{\mathscr{T}^0}$ is $(< \chi^+)$-stably constructible over $\emptyset$

(b) $C^{\mathscr{T}^1}$ is $(< \chi^{++})$-stably constructible over $C^{\mathscr{T}_0}$ (and over $\emptyset$)

(c) $C^{\mathscr{T}^1}_*$ (see below) is $(< \mu)$-stably constructible over $C^{\mathscr{T}_0}$ (and over $\emptyset$)

(d) $\|C^{\mathscr{T}^1}_*\| = \lambda$ when $|\mathscr{T}^1| \le \lambda$.

Recall $N = N_{\chi^+}$ (see the beginning of the proof after $(*)_1$).

5.10 Claim. *If $\mathscr{T} \subseteq {}^{\chi^+\ge}\lambda$ and $\nu \in {}^{\chi^+}\lambda$ are such that $\{\nu \restriction \alpha : \alpha < \chi^+\} \subseteq \mathscr{T}$ but $\nu \notin \mathscr{T}$, <u>then</u>:*

1) $f_\nu = \bigcup\limits_{\xi<\chi^+} f_{\nu\restriction\xi}$ cannot be extended to a $\le_{\mathfrak{s}}$-embedding of N into

$C^{\mathscr{T}}$.
2) Similarly for $C_*^{\mathscr{T}}$.

Proof. 1) Toward contradiction assume that $g : N \to C^{\mathscr{T}}$ is an $\le_{\mathfrak{s}}$-embedding, extending f_ν. Without loss of generality $\mathscr{T}$ is closed under initial segments. For $\xi < \chi^+$, let

$$\mathscr{T}_\xi = \{\rho \in \mathscr{T} : \neg((\nu \restriction \xi) \trianglelefteq \rho)\}.$$

Clearly (see 4.4, 4.5, 4.6):

$\odot$ (*i*) $C^{\mathscr{T}} = \bigcup\limits_{\xi<\chi^+} C^{\mathscr{T}_\xi}$ [why? as $\nu \notin \mathscr{T}$]

(*ii*) $C^{\mathscr{T}_\xi}$ is increasing continuous in ξ
[why? because if ξ is a limit ordinal, then note that $\nu \restriction \xi = \bigcup\limits_{\zeta<\xi} \nu \restriction \zeta$]

(*iii*) $\mathrm{NF}(M_{\nu\restriction\zeta}, C^{\mathscr{T}_\zeta}, M_\nu, C^{\mathscr{T}})$ remembering $M_\nu = \bigcup\limits_{\xi<\chi^+} M_{\nu\restriction\xi}$
[why? by 5.9(2)].

Now by $\odot(i)$ for every $\zeta < \chi^+$ the set $g''(N_\zeta)$ is $\subseteq \bigcup\limits_{\zeta<\chi^+} C^{\mathscr{T}_\zeta}$. But $C^{\mathscr{T}_\zeta}$ increasing with ζ by $\odot(ii)$ and $|g''(N_\zeta)| \le \chi$; <u>hence</u> for some $\xi(\zeta) < \chi^+$ we have

$$g''(N_\zeta) \subseteq C^{\mathscr{T}_{\xi(\zeta)}}.$$

Hence

$$E := \{\alpha < \chi^+ : (\forall\zeta < \alpha)\xi(\zeta) < \alpha \text{ and } \alpha \text{ is a limit ordinal}\}$$

is a closed unbounded subset of χ^+. Fix ζ from E. Then $g''(N_\zeta) = \bigcup\limits_{\epsilon<\zeta} g''(N_\epsilon) \subseteq \bigcup\limits_{\xi<\zeta} C^{\mathscr{T}_{\xi(\epsilon)}} = C^{\mathscr{T}_\zeta}$; so noting that

$$g''(N_\zeta) \leq_{\mathfrak{s}} C^{\mathscr{T}}, C^{\mathscr{T}_\zeta} \leq_{\mathfrak{s}} C^{\mathscr{T}}$$

clearly by Ax(A3) we have $g''(N_\zeta) \leq_{\mathfrak{s}} C^{\mathscr{T}_\zeta}$.

Remember $\mathrm{NF}\big(M_{\nu\restriction\zeta}, C^{\mathscr{T}_\zeta}, M_\nu, C^{\mathscr{T}}\big)$ by $\odot(iii)$, hence by monotonicity

$$\mathrm{NF}\left(M_{\nu\restriction\zeta}, g''(N_\zeta), M_\nu, C^{\mathscr{T}}\right).$$

Again by monotonicity

$$\mathrm{NF}\left(M_{\nu\restriction\zeta}, g''(N_\zeta), M_{\nu\restriction(\zeta+1)}, C^{\mathscr{T}}\right)$$

but

$$g''(N_\zeta) \cup M_{\nu\restriction(\zeta+1)} \subseteq g''(N_{\zeta+1}) \leq_{\mathfrak{s}} C^{\mathscr{T}}$$

hence

$$\mathrm{NF}\left(M_{\nu\restriction\zeta}, g''(N_\zeta), M_{\nu\restriction(\zeta+1)}, g''(N_{\zeta+1})\right)$$

which contradicts the hypothesis on $\langle M_i, N_i : i \leq \chi^+\rangle$ (and g being an $\leq_{\mathfrak{K}}$-embedding).

2) If μ is singular then $\lambda = \lambda^{<\mu} \Rightarrow \lambda = \lambda^\mu = \lambda^{<\mu^+}$, so without loss of generality μ is regular and $C^{\mathscr{T}}_* \neq C^{\mathscr{T}}$. Suppose $g : N \to C^{\mathscr{T}}_*$ is a $\leq_{\mathfrak{s}}$-embedding extending f_ν. So $|\mathrm{Rang}(g)| \leq \chi^+$, and by 4.4 for every $a \in C^{\mathscr{T}}_*$ for some closed $w_a \subseteq C^{\mathscr{T}}, |w_a| < \mu$ and $a \in A^{\mathscr{A}^{\mathscr{T}}}_{w_a}$. So by 4.4 for some closed $w \subseteq \ell g(\mathscr{A}^{\mathscr{T}})$, $|w| \leq \chi^+$ and $\mathrm{Rang}(g) \subseteq A^{\mathscr{A}^{\mathscr{T}}}_w$. Recalling $\mu \leq \chi^+$ by the assumption of 5.2, by 4.6 without loss of generality w is a subset of χ^+. Now define $h : \chi^+ \to \chi^+$ by: $h(\xi)$ is the first ordinal such that (using the notation from the proof of part (1)): for every $\zeta < \xi, B^{\mathscr{T}}_\zeta \cap C^{\mathscr{T}} \subseteq C^{\mathscr{T}_\xi}$ and clearly $E_0 = \{\xi : \zeta < \xi \Rightarrow h(\zeta) < \xi$ and ξ is a limit ordinal$\}$ is a club of λ. For $\zeta \in E_0$ let $A^\zeta_* = \langle C^{\mathscr{T}_\zeta} \cup \bigcup\limits_{\xi<\zeta} B^{\mathscr{T}}_\xi\rangle^{\mathrm{gn}}_{C^{\mathscr{T}}_*}$. By 4.5 we know that $\langle A^\zeta_* : \zeta \in E_0\rangle$

is $\leq_{\mathfrak{s}}$-increasing continuous and $\bigcup\limits_{\zeta \in E_0} A^\zeta_* \leq_{\mathfrak{s}} C^{\mathscr{T}}_*$; and for $\zeta < \xi$ from E_0 we have $\mathrm{NF}(N_{\nu \restriction \zeta}, A^\zeta_*, N_{\nu \restriction \xi}, A^\xi_*)$. The rest is as in part (1). $\square_{5.10}$

5.11 Continuation of the proof of 5.2. Part (0).
By [Sh:e, IV,§2] or see [Sh 300, III,§5].

Remark. Note that the rest of the section is not essential for our program.

5.12 Continuation of the proof of 5.2, Part (1).: The assumptions on λ (in particular its being $<_{\mathfrak{K}}$-inaccessible) imply that there are $\mathscr{T} \subseteq {}^{\chi^+ >}\lambda$ and $\langle \mathscr{T}_\alpha : \alpha < \lambda \rangle$ such that $\mathscr{T} = \bigcup\limits_{\alpha < \lambda} \mathscr{T}_\alpha, \mathscr{T}_\alpha$ is $\subseteq$-increasing continuous, $\mathscr{T}_\alpha$ is closed under initial segments, $|\mathscr{T}_\alpha| < \lambda$, and for $\delta \in S^*$ we have $\eta_\delta \in {}^{(\chi^+)}\lambda$ such that $\{\eta_\delta \restriction \zeta : \zeta < \chi^+\} \subseteq \mathscr{T}_\delta$ and for no $\alpha < \delta$ do we have $\{\eta_\delta \restriction \zeta : \zeta < \chi^+\} \subseteq \mathscr{T}_\alpha$ (i.e., as S^* is good — see statement of Theorem). For $S \subseteq S^*$, let $\mathscr{T}[S] = \mathscr{T}_S := \mathscr{T} \cup \{\eta_\delta : \delta \in S\}$ and let $C_{[S]} = C^{\mathscr{T}[S]}_* = C^{\mathscr{T} \cup \{\eta_\delta : \delta \in S\}}_*$. Clearly $C_{[S]}$ is a model of cardinality λ which is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-homogeneous when demanded. Decompose $C_{[S]}$ as $\bigcup\limits_{\alpha < \lambda} C_{[S],\alpha}$, the sequence $\bar{C}_{[S]} = \langle C_{[S],\alpha} : \alpha < \lambda \rangle$ is $\subseteq$-increasing continuous, $\|C_{[S],\alpha}\| < \lambda$. We would like to reconstruct $S/\mathscr{D}_\lambda$ from $C_{[S]}/\cong$, this is clearly beneficial for our purpose. For $t \subseteq \mathscr{T}_S$ let $M_t = M^S_t = C_{[S]} \restriction \langle \cup \{M_\eta, N_\nu : \eta \in t \cap \mathscr{T}, \nu \in t \cap \mathscr{T}_S \backslash \mathscr{T}\}\rangle^{\mathrm{gn}}_{C_{[S]}}$.

5.13 Definition. 1) For any $M \in K_\lambda, \lambda$ regular $>$ LSP($\mathfrak{K}$) and representation $\langle M_i : i < \lambda \rangle$ of M (i.e., it is $\subseteq$-increasing continuous, $M = \bigcup\limits_{i < \lambda} M_i$ and $i < \lambda \Rightarrow \|M_i\| < \lambda$), we let:

$$\begin{aligned}\mathrm{Bs}_\chi(\langle M_i : i < \lambda \rangle) := \{\delta < \lambda : & \mathrm{cf}(\delta) = \chi^+ \text{ and for every } A \subseteq M,\\ & |A| \leq \chi^+ \ \underline{\text{some pair}}\ (\bar{N}_0, \bar{N}_1)\\ & \text{is a non-base } \chi\text{-witnesses for } A\\ & \text{inside } (M_\delta, M)\}.\end{aligned}$$

where
2) We say that the pair $(\bar{N}_0, \bar{N}_1)$ is a non-based χ-witnesses for A inside (M_0, M_1) when:

- (a) $M_0 \leq_{\mathfrak{K}} M_1$
- (b) $A \subseteq M_1, |A| \leq \chi^+$
- (c) $\bar{N}_\ell = \langle N^\ell_\alpha : \alpha < \chi^+ \rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous for $\ell = 1, 2$
- (d) $N^\ell_\alpha \in K_\chi$ for $\alpha < \chi^+$
- (e) $N^1_\alpha \leq_{\mathfrak{s}} N^2_\alpha$ for every $\alpha \leq \chi^+$
- (f) $N^\ell_\alpha \leq_{\mathfrak{s}} M_\ell$ for $\alpha < \chi^+, \ell = 1, 2$
- (g) $A \subseteq \cup \{N^2_\alpha : \alpha < \chi^+\}$
- (h) $\mathrm{NF}(N^1_\alpha, N^2_\alpha, N^1_\beta, N^2_\beta)$ fail for $\alpha < \beta < \chi^+$.

3) For $M \in K_\lambda$, let $\mathrm{Bs}_\chi(M) :=\ \mathrm{Bs}_\chi(\bar{M})/\mathscr{D}_{\chi^+}$ whenever $\bar{M} = \langle M_i : i < \lambda \rangle$ is a $\leq_{\mathfrak{K}}$-representation of M, i.e. is as in part (1), recalling $\mathscr{D}_{\chi^+}$ is the club filter on χ^+.

Remark. We can replace χ^+ by a regular uncountable cardinal.

5.14 Observation. $\mathrm{Bs}_\chi(\bar{M})$ is an $\mathscr{D}_\lambda$-invariant of M, i.e. if $\bar{M}^1, \bar{M}^2$ are $\leq_{\mathfrak{K}}$-representations of $M \in K_\lambda$ then $\mathrm{Bs}_\chi(\bar{M}^1) =\ \mathrm{Bs}_\chi(\bar{M}^2)$ mod $\mathscr{D}_\lambda$.

Now we shall prove the result without the homogeneity condition; using our proof of 5.10(1), but first:

5.15 Claim. *A sufficient condition for* $\mathrm{NF}(M^S_{t_0}, M^S_{t_1}, M^S_{t_2}, C_{[S]})$ *is: if* $t_1, t_2, \bar{N}^1, \bar{N}^2$ *satisfies* $\boxdot$ *below then* $(\alpha) \Leftrightarrow (\beta)$ *where*

- (α) *for some club E of χ^+, for every $\alpha < \beta$ from E we have* $\mathrm{NF}(N^1_\alpha, N^2_\alpha, N^1_\beta, N^2_\beta)$
- (β) *if* $\nu \in t_2 \backslash \mathscr{T}$ *and* $\zeta < \chi^+ \Rightarrow \nu \restriction \zeta \in t_1$ *<u>then</u>* $\nu \in t_1$
  and

$\boxdot$ (a) *$t_1 \subseteq t_2$ are subsets of $\mathscr{T}_S$ of cardinality χ^+*
  (b) *t_ℓ is closed under initial segments*

(c) $\bar{N}^\ell = \langle N^\ell_\alpha : \alpha < \chi^+ \rangle$ *is a* $\leq_{\mathfrak{K}}$*-representation of* $\langle \cup\{M_\eta : \eta \in t_\ell \cap \mathscr{T}\} \cup \{N_\nu : \nu \in t_\ell \cap {}^{\chi^+}\lambda\}\rangle^{\rm gn}_{C^\oplus}$.

Proof. Let $\{\eta^\ell_\varepsilon : \varepsilon < \chi^+\}$ list $t_\ell \cap \mathscr{T}$ and $\langle \nu^\ell_\varepsilon : \varepsilon < \chi^+\rangle$ list $t_\ell \cap {}^{\chi^+}\lambda$; we are assuming for notational simplicity that $t_\ell \nsubseteq \mathscr{T}$ and also assume $\varepsilon < \ell g(\eta^\ell_\zeta) \Rightarrow \eta^\ell_\zeta \restriction \varepsilon \in \{\eta^\ell_\xi : \xi < \zeta\}$.

For $\zeta < \chi^+$ let

$$N^{\ell,*}_\zeta = \langle \cup\{M_{\eta^\ell_\varepsilon} : \varepsilon < \zeta\} \cup \{N_{\nu^\ell_{\varepsilon,\zeta}} : \varepsilon < \zeta\}\rangle^{\rm gn}_{C^\oplus}.$$

Now for some club E_1 of χ^+ we have

$\odot$ (i) if $\ell \in \{1,2\}$ and $\zeta \in E_1$ then $N^\ell_\zeta = N^{\ell,*}_\zeta$ and

(ii) $\{\eta^2_\varepsilon, \nu^2_\varepsilon : \varepsilon < \zeta\} \cap t_1 = \{\eta^1_\varepsilon, \nu^1_\varepsilon : \varepsilon < \zeta\}$.

Now for the $N^{\ell,*}_\zeta$'s the result should be clear. $\square_{5.15}$

5.16 Fact. For part (1) of 5.2, for any stationary $S \subseteq S^*, S^* \cap (\lambda \setminus \mathrm{Bs}_\chi(\bar{C}_{[S]})) = S \bmod \mathscr{D}_\lambda$.

Proof. We shall show that a club E of λ as required in Claim 5.15 is

$$E = \left\{\alpha : C_{[S],\alpha} = \langle \{M_\eta : \eta \in \mathscr{T}_\alpha\} \cup \{N_{\nu_\delta} : \delta < \alpha \cap S\}\rangle^{\rm gn}_{C^\oplus}\right\}.$$

It is easy to see E is a club of λ since $C_{[S]}$ is generated by $\{M_\eta : \eta \in \mathscr{T}\} \cup \{N_{\nu_\delta} : \delta \in S\}$.

The result follows by 5.15. $\square_{5.16}$

5.17 End of the Proof of 5.2 part(1). First, without homogeneity Theorem 5.2 easily follows from 5.16. For, if $C_{[S]} \cong C_{[S']}$ (with $S, S' \subseteq S^*$), Fact 5.16 implies that S and S' agree on a club. But there are 2^λ stationary subsets of S^* which are pairwise not equal mod $\mathscr{D}_\lambda$.

Second, with homogeneity the proof is similar replacing the role of 5.15 by 5.18.

$\square_{5.2}$

Similarly to 5.15

5.18 Claim. *If $t_1, t_2, u_1, u_2, \bar{N}^1, \bar{N}^2$ satisfies $\boxdot$ below then $(\alpha) \Leftrightarrow (\beta)$ when*

(α) for some club E of χ^+, for every $\alpha < \beta$ from E we have $NF(N^1_\alpha, N^2_\alpha, N^1_\beta, N^2_\beta)$

(β) if $\nu \in t_2 \backslash \mathscr{T}$ and $\zeta < \chi^+ \Rightarrow \nu \restriction \zeta \in t_1$ <u>then</u> $\nu \in t_1$

$\boxdot$ *(a) $t_1 \subseteq t_2$ are subsets of $\mathscr{T}_S$ of cardinality χ^+*

(b) t_ℓ is closed under initial segments

(c) $u_\ell \subseteq i^{\mathscr{T}}$ has cardinality $\leq \chi^+$ is $\mathscr{A}^{\mathscr{T}}$-closed and is such that for every $i \in u_\ell$ we have $B^{\mathscr{T}}_i \cap C^{\mathscr{T}} \subseteq \langle \{M_\eta : \eta \in t_\ell \cap {}^{\chi^+}\lambda\} \cup \{N_\nu : \nu \in t_\ell \cap {}^{\chi^+}\lambda\}\rangle^{\text{gn}}_{C^{\mathscr{T}}}$ for $\ell = 1, 2$

(d) $u_1 \subseteq u_2$

(e) $\bar{N}^\ell = \langle N^\ell_\alpha : \alpha < \chi^+ \rangle$ is a $\leq_{\mathfrak{K}}$-representation of $N_\ell = \langle \{M_\eta : \eta \in t_\ell \cap {}^{\chi^+ >}\lambda\} \cup \{N_\nu : \nu \in t_\ell \cap {}^{\chi^+}\lambda\} \cup \{B^{\mathscr{T}}_i : i \in u_\ell\}\rangle_{\mathfrak{C}^\oplus}$.

Proof. Similar to 5.15. $\square_{5.18}$

* * *

5.19 Remark. 1) So it was enough for 5.9 (so really 5.2) that

$$\{i < \chi^+ : \neg\, \text{NF}(M_i, N_i, M_{i+1}, N_{i+1})\}$$

is stationary.
2) By [Sh:e, VI,2.1] we can get other variants of 5.2 as we have the right representation.

5.20 Fact. We can use the proof of 5.2 to get 2^λ models in $\lambda_1 \geq \lambda$. Using models which have a stable construction $\langle C^{\mathscr{T}}, B^{\mathscr{T}}_\alpha, w^{\mathscr{T}}_\alpha : \alpha < \alpha(T)\rangle$, $\|B^{\mathscr{T}}_\alpha\| \leq \chi$ (so we get something for singular λ_1).
3) We can in 5.2 omit the "$(\mathbb{D}_{\mathfrak{s}}, \mu)$-homogeneous" demand gaining the omission of "$\lambda = \lambda^{<\mu}$". If we demand only $\lambda \geq 2^\mu$ we have the models in $\mathfrak{K}^{\rm us}_{\mu,\chi^+}$ (see Definition V.B.3.20).

Proof. We have to use an extension of the definition of Bs as defined in 5.21 below.

5.21 Definition. Let $M \in K_{\geq\lambda}$. We say $\mathrm{Bs}^\lambda_\chi(M) = S/\mathscr{D}_\lambda$ iff for the λ-majority of $A \subseteq |M|, |A| = \lambda \Rightarrow M \restriction A \leq_{\mathfrak{s}} M \;\;\&\;\; \mathrm{Bs}_\chi(M \restriction A) = S/D$.
To carry the proof we just need 4.11.

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§0 Introduction

Lemma 1.2 is a necessary crucial step. It says that if μ is such that $\mathfrak{s}$ satisfies AxFr_1, is $(\le \mu, \le \mu)$-smooth, $(\le \mu^+, \mu)$-based and satisfies LSP(μ), then those conditions hold for all $\mu' \ge \mu$. From Section 2 we fix the least such μ as $\chi_{\mathfrak{s}}$ and from then on assume $\chi_{\mathfrak{s}} < \infty$.

Note that in Chapter V.C we have gotten non-structure results from the failure of such properties so this assumption is justified.

After assuming the existence of $\chi_{\mathfrak{s}}$, this chapter is quite parallel to Ch.III,IV of [Sh:c], and many theorems are parallel.

In Section 2 we define a $(\mathbb{D}_{\mathfrak{s}}, \mu)$-prime models and prove their existence (for $\mu > \chi_{\mathfrak{s}}$). In Section 3 we begin the development of non-forking for types. The first step is to extend the NF relation from a relation between models to a relation between conjugacy classes. The "orbital", "algebraic" version of $\mathrm{tp}(M,N)$ is $\mathbf{tp}(M,N) := \{F(M) : F \in \mathrm{AUT}_N(\mathfrak{C})\}$ where $\mathfrak{C}$ is "the" monster in this context.

In Sections 3 and 4 we develop this notion. But in this chapter we concentrate almost exclusively on the case when the domain of the type is a model and, moreover, a realization of a type is an enumeration of a model.

In Sections 5 and 6 we develop further properties of $(\mathbb{D}_{\mathfrak{s}}, \mu)$-prime models.

Note that our models here correspond to algebraically closed sets in [Sh:c], $(\mathbb{D}_{\mathfrak{s}}, \chi_{\mathfrak{s}}^+)$-homogeneous models here correspond to $\mathbf{F}^a_{\kappa_r(T)}$-saturated models there, so some theorems in [Sh:a] on models have no corresponding theorems here (like [Sh:c, IV,5.6]).

§1 Being smooth and based progagate up

This section contains two results. In Lemma 1.2 we show that if $\mathfrak{s}$ is $(\leq \chi, \leq \chi^+)$-smooth and (χ^+, χ)-based and LSP(χ) <u>then</u> $\mathfrak{s}$ is μ-smooth and μ-based and has the μ-LSP for $\mu \geq \chi$. The proof of this result requires the refining of the notion of smooth to keep track of the size of the models in the union. (Cf. Definition V.B.1.18).

Secondly, (Fact 1.12) we show that if the framework $\mathfrak{s}$ contains a single non-trivial amalgamation (e.g. NF(N_0, N_1, N_2, N_3) with $N_1 \neq N_0$ isomorphic to N_2 over N_0 (by a non-identity isomorphism) then $\mathfrak{s}$ has arbitrarily large models.

1.1 Context. (AxFr$_1$) if not said otherwise.

1.2 Lemma. *If $\mathfrak{s}$ is $(\leq \chi, \leq \chi^+)$-smooth (really $\mathfrak{K}_\mathfrak{s}$ is),* NF *is (χ^+, χ)-based,* LSP(χ) *holds and $\mu \geq \chi$ <u>then</u> $\mathfrak{s}$ is $(\leq \mu, \leq \mu)$-smooth,* NF *is $(\leq \mu, \leq \mu)$-based and* LSP(μ, λ) *holds when $\mu \geq \lambda \geq \chi$.*

Before we prove 1.2, we do 1.4.

1.3 Explanation. 1) Note that if $\mathfrak{s}$ is $(\leq \chi, \leq \chi, \leq \chi)$-smooth but not $(\leq \chi, \chi^+, \chi^+)$-smooth then there is a $\leq_\mathfrak{s}$-increasing continuous sequence $\langle M_\alpha : \alpha < \chi^+ \rangle$ of members of K_χ and $N \in K_{\chi^+}$ such that $\alpha < \chi^+ \Rightarrow M_\alpha \leq_\mathfrak{s} N$ but $\cup\{M_\alpha : \alpha < \chi^+\} \nleq_\mathfrak{s} N$. Recall that in "$(\lambda, \chi, \kappa)$-smoothness", the λ is the size of each member of the $\leq_\mathfrak{s}$-increasing sequence say $\langle M_\alpha : \alpha < \delta \rangle, \kappa$ is the length of the sequence (so $\delta = \kappa$) and χ is the size of the model M in which they are (so $M_\alpha \leq_\mathfrak{s} M$ for $\alpha < \kappa$ and the conclusion is $\cup\{M_\alpha : \alpha < \delta\} \leq_\mathfrak{s} M$) and we omit χ when $\chi = \lambda$.
2) Remember that $\mathfrak{s}$ is κ-based means: for every $M \leq_\mathfrak{s} M^*$ and $A \subseteq M^*$ with $|A| \leq \kappa$ there is an $N \leq_\mathfrak{s} M^*$ with $|N| \leq \kappa, A \subseteq N$ such that M is independent from N over $M \cap N$ and $\mathfrak{s}$ is (λ, κ)-based when we restrict ourselves to $M \in K_{\leq \lambda}$. The next result shows that: if A is contained in M_δ with $\bigcup\limits_{i<\delta} M_i \subseteq M_\delta$ then we can find an N which satisfies this condition simultaneously for a subsequence containing less than μ of the M_i.

More precisely,

1.4 Fact. 1) Suppose $\mathrm{LSP}(\mu,\lambda), \mu > \lambda \geq \kappa$ and $\mathfrak{s}$ has $(\leq \lambda, \kappa)$-smoothness. If $\langle M_i : i \leq \kappa\rangle$ is $\leq_{\mathfrak{s}}$-increasing, $\|M_\kappa\| = \mu$ and $\sum\limits_{i<\kappa} \|M_i\| \leq \lambda$ <u>then</u> $\bigcup\limits_{i<\kappa} M_i \leq_{\mathfrak{s}} M_\kappa$.
2) Suppose $\mathfrak{K}_{\mathfrak{s}}$ has $(< \mu, < \mu)$-smoothness and satisfies $\mathrm{LSP}(\leq \mu, \lambda)$, $\lambda < \mu$. If $|I| \leq \lambda$, $\{M_t : t \in I\} \subseteq K^{\mathfrak{s}}_{\leq \mu}$ and $A \subseteq \bigcup\limits_{t\in I} M_t$, $|A| \leq \lambda$ <u>then</u> we can find $\{N_t : t \in I\}$ such that $A \subseteq \bigcup\limits_{t\in I} N_t$ and for each $t \in I$ we have $N_t \leq_{\mathfrak{s}} M_t, \|N_t\| \leq \lambda$ and $|N_t| = |M_t| \cap \left(\bigcup\limits_{s\in I} N_s\right)$.
3) In (2) we can assume only $(\leq \lambda, \kappa)$-smoothness, for every $\kappa \leq \lambda$ (or just the conclusion of 1.4(1), (i.e., $(\leq \lambda, \leq \mu, \kappa)$-smoothness).

Remark. In 1.4(1) we need to assume $\mathrm{LSP}(\mu,\lambda)$ since $\|M_\kappa\|$ may be greater than λ. But using $\mathrm{LSP}(\mu,\lambda)$ we interpolate a model of size λ.

Proof. 1) By $\mathrm{LSP}(\mu,\lambda)$ applied to $M_\kappa, \bigcup\limits_{i<\kappa} M_i$ as M, A respectively, there is N, $N \leq_{\mathfrak{s}} M_\kappa, \|N\| \leq \lambda$ such that $\bigcup\limits_{i<\kappa} M_i \subseteq N$. By Ax(A3), $M_i \leq_{\mathfrak{s}} N$ for $i < \kappa$. So by $(\leq \lambda, \kappa)$-smoothness $\bigcup\limits_{i<\kappa} M_i \leq_{\mathfrak{s}} N$ hence by transitivity of $\leq_{\mathfrak{s}}$ we have $\bigcup\limits_{i<\kappa} M_i \leq_{\mathfrak{s}} M_\kappa$, as required.
2) We choose $\langle N^n_t : t \in I\rangle$ by induction on $n < \omega$ such that:

- (a) $\|N^n_t\| \leq |A| + \lambda + |I| = \lambda$
- (b) $N^n_t \leq_{\mathfrak{s}} M_t$
- (c) $M_t \cap (A \cup \bigcup\{N^\ell_s : \ell < n, s \in I\}$ is included in N^n_t
- (d) If $\|M_t\| \leq \lambda$ then $M_t \subseteq N^0_t$.

For this we use just LSP($\leq \mu, \lambda$). Let $N_t = \bigcup_{n<\omega} N_t^n$; the only non-trivial point is "$N_t \leq_{\mathfrak{s}} M_t$" which follows by 1.4(1) (see V.C.3.4(2)).
3) Same proof but we define $\langle N_t^\varepsilon : t \in I\rangle$ by induction on $\varepsilon < \kappa$. $\square_{1.4}$

Proof of 1.2. We prove by induction on $\mu \geq \chi$ that $\mathfrak{K}_{\mathfrak{s}}$ is $(\leq \mu, \leq \mu)$-smooth, LSP($\leq \mu, \lambda$) when $\mu \geq \lambda \geq \chi$ and NF is $(\leq \mu, \leq \mu)$ - based. For $\mu = \chi$ this is given. So suppose $\mu > \chi$ and for every μ', $\chi \leq \mu' < \mu$ the induction hypothesis holds. We shall prove it for μ by the following series of subfacts.

1.5 Subfact. If $M \in K_\mu^{\mathfrak{s}}$ and $A \subseteq |M|$, <u>then</u> there is $N \leq_{\mathfrak{s}} M$, such that $A \subseteq |N|, \|N\| \leq \chi + |A|$ (i.e., LSP(μ, λ) for $\mu \geq \lambda \geq \chi$, i.e. the last conclusion of 1.2 holds for μ).

Proof. If $|A| < \mu$ <u>then</u> by Claim V.C.3.4(2); why? as the induction hypothesis on μ holds we have $(< \mu, \mu)$-smoothness and recalling we are assuming LSP(χ) hence in particular LSP(μ, χ)]. Lastly, if $|A| = \mu$ choose $N = M$. $\square_{1.5}$

1.6 Subfact. If $M \in K_\mu^{\mathfrak{s}}$, the sequence $\langle M_i : i < \delta\rangle$ is $\leq_{\mathfrak{s}}$-increasing, for each $i < \delta, M_i \leq_{\mathfrak{s}} M$ and $\sum_{i<\delta} \|M_i\| < \mu$ <u>then</u> $\bigcup_{i<\delta} M_i \leq_{\mathfrak{s}} M$.

Proof. Without loss of generality $\langle M_i : i < \delta\rangle$ is not eventually constant, and without loss of generality $\delta =$ cf(δ), and (see our assumptions) $\lambda := \sum_{i<\delta} \|M_i\| < \mu$ and clearly $\delta < \mu$. Now we apply 1.4(1) with $\langle M_i : i < \delta\rangle\hat{\ }\langle M\rangle, \mu, \lambda, \delta$ here standing for $\langle M_i : i \leq \kappa\rangle, \mu, \lambda, \kappa$ there. The assumption "$\mathfrak{s}$ has $(\leq \lambda, \kappa)$-smoothness" demanded there holding by the induction hypothesis and the assumption "LSP(μ, λ)" holds by subfact 1.5. So the conclusion of 1.4 holds and it says that $\bigcup_{i<\delta} M_i \leq_{\mathfrak{s}} M$ as required. $\square_{1.6}$

1.7 Subfact. 1) If $M \leq_{\mathfrak{s}} M^*, A \subseteq M^*, \|M^*\| = \mu$ and $|A| < \mu$ <u>then</u> for some $N \leq_{\mathfrak{s}} M^*$ we have:

> $\odot$ $A \subseteq N$, $\|N\| \leq \chi + |A|$
> and $N \cap M, N, M'$ is in stable amalgamation inside M^*
> whenever $M \cap N \subseteq M' \leq_{\mathfrak{s}} M, \|M'\| < \mu$.

2) If $\mu > \chi^+ + |A|$ then in part (1) we can replace the assumption "$M \leq_{\mathfrak{s}} M^*$" by:

> $\boxplus$ $M \subseteq M^*$ and if $N \subseteq M, \|N\| < \mu$ then $N \leq_{\mathfrak{s}} M \Leftrightarrow N \leq_{\mathfrak{s}} M$.

Proof. If $\mu = \chi^+$ then for part (1) this is said in the assumptions of 1.2 and for part (2) this does not occur so without loss of generality $\mu > \chi^+$; hence $|A| + \chi^+ < \mu$. We prove the statement by induction on $|A|$. First assume $|A| \leq \chi$.

Note that LSP(μ, λ) holds for $\lambda \in [\chi, \mu]$ by 1.5; imitating the proof of V.C.3.8, i.e. by V.C.3.9(0), we can find $N \leq_{\mathfrak{s}} M^*$ of cardinality $\leq \chi$ such that $M \cap N \leq_{\mathfrak{s}} N, A \subseteq N, N \subseteq M^*$ and $[M \cap N \leq_{\mathfrak{s}} N' \leq M \ \ \& \ \ \|N'\| \leq \chi \Rightarrow \text{NF}(N \cap M, N, N', M^*)]$. By $\boxplus$ we have $M \cap N \leq_{\mathfrak{s}} M^*$ hence by Ax(C3) we have $M \cap N \leq_{\mathfrak{s}} N$. We shall prove that N is as required. Suppose $N \cap M \subseteq M' \leq_{\mathfrak{s}} M$ and $\|M'\| < \mu$; by 1.5 there is M^*_1 such that $M' \cup N \subseteq M^*_1 \leq_{\mathfrak{s}} M^*$ and $\|M^*_1\| < \mu$ hence by Ax(A3) $N \leq_{\mathfrak{s}} M^*_1$ and by $\boxplus$ we have $M' \leq_{\mathfrak{s}} M^*_1$. As we are inside the induction on μ, letting $\mu_1 = \|M^*_1\|$ we know that NF is $(\leq \mu_1, \leq \mu_1)$-based and we apply it to the pair of models $M' \leq_{\mathfrak{s}} M^*_1$ and the set $|N|$. So recalling we are assuming $|A| \leq \|N\| \leq \chi$ there is N_1, $N \subseteq N_1 \leq_{\mathfrak{s}} M^*_1$ such that $\|N_1\| \leq \chi$ and the triple $N_1 \cap M'$, N_1, M' is in stable amalgamation inside M^*_1, i.e. $\text{NF}(N_1 \cap M', N_1, M', M^*_1)$ hence by monotonicity Ax(C3) also $\text{NF}(N_1 \cap M', N_1, M', M^*)$. Note that $N_1 \cap M' \leq_{\mathfrak{s}} M$ by Ax(A3) because $N_1 \cap M' \leq_{\mathfrak{s}} M^*$ and $N_1 \cap M' \subseteq M' \subseteq M \leq_{\mathfrak{s}} M^*$ and, for part (2) the cardinality of $N_1 \cap M'$ is $\leq \chi$. By the choice of N, we have $N \cap M$, $N_1 \cap M'$, N is in stable amalgamation inside M^* hence (by monotonicity Ax(C3)) inside N_1 so $\text{NF}(N \cap M, N_1 \cap M', N, N_1)$ hence by symmetry, Ax(C6) we have $\text{NF}(N \cap M, N, N_1 \cap M', N_1)$. By transitivity of NF (i.e., Ax(C9) see V.C.1.3,V.C.1.4) we have $\text{NF}(N \cap M, N, M', M^*)$ as required.

So we have proved for $|A| \leq \chi$; let $|A| > \chi$.
Now let $A = \{a_i : i < |A|\}$. We choose N_i by induction on $i \leq |A|$ such that:

$\circledast$ (i) $\|N_i\| \leq \chi + |i|$

(ii) $N_i \leq_{\mathfrak{s}} M^*$

(iii) $a_i \in N_{i+1}$

(iv) N_i is $\leq_{\mathfrak{s}}$-increasing continuous

(v) $N_i \cap M, N_i, M'$ is in stable amalgamation (inside M^*) whenever

$$N_i \cap M \subseteq M' \leq_{\mathfrak{s}} M, \|M'\| < \mu.$$

For $i = 0$ and for i successor — choose N_i satisfying (i)-(v) by the induction hypothesis on $|A|$.
For i limit: by 1.6 we know that

$$N_i := \bigcup_{j<i} N_j \leq_{\mathfrak{s}} M^*$$

and

$$N_i \cap M = \bigcup_{j<i} (N_j \cap M) \leq_{\mathfrak{s}} M^*.$$

Clearly clauses (i)-(iv) of $\circledast$ holds. For clause (v) let $N_1 \cap M \subseteq M' \leq_{\mathfrak{s}} M, \|M'\| < \mu$; by $\boxplus$ we have $M' \leq_{\mathfrak{s}} M$. By the induction hypothesis $j < i \Rightarrow$ NF$(N_i \cap M, M', N_i', M^*)$.
By Claim V.C.1.10(3) the models $M \cap N_i, N_i, M'$ are in stable amalgamation inside M^*. Now $N_{|A|}$ is as required. $\square_{1.7}$

1.8 Subfact. NF is $(\leq \mu, \leq \mu)$-based (i.e., the second conclusion of 1.2 holds).

Proof. So we should prove

$(*)$ if $M \leq_{\mathfrak{s}} M^*, A \subseteq M^*, \|M^*\| \leq \mu$, $|A| \leq \mu$ <u>then</u> for some $N \subseteq M^*$, we have: $A \subseteq N$, $\|N\| \leq \chi + |A|$ and $N \cap M, N, M$ is in stable amalgamation inside M^*

If $|A| = \mu$ this is trivial (let $N = M^*$), so assume $|A| < \mu$. Let $|M^*| = \{a_i : i < \mu\}$ where $A = \{a_i : i < |A|\}$.

We now choose N_i by induction on $i < \mu$ exactly as in $\circledast$ in the proof of 1.7. Now we cannot be stuck in $i = 0$ or $i = j + 1$ as 1.7 says so and we can prove we are not stuck for i limit as in the proof of 1.7. So we have $\langle N_i : i < \mu\rangle$ satisfying (i) - (v) of $\circledast$ there. Clearly $\langle N_i : i < \mu\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous (by (iv)) with union M^* (by (ii),(iii) and the choice of $\langle a_i : i < \mu\rangle$ above). Now $\langle N_i \cap M : i < \mu\rangle$ is an $\subseteq$-increasing continuous sequence of members of $K_{\mathfrak{s}}$ with union M. Next $N_i \cap M \le_{\mathfrak{s}} M$ by (v) and note that $M \le_{\mathfrak{s}} M^*$. By (v) and monotonicity + Ax(C3)(c) we have $i < j < \mu \Rightarrow \mathrm{NF}(N_i \cap M, N_i, N_j \cap M, N_j)$ hence by V.C.1.10(3) we have $\mathrm{NF}(N_i \cap M, N_i, M, M^*)$; so as $A = \{a_i : i < |A|\}$ clearly for $i = |A|, N_i$ is as required. $\square_{1.8}$

<u>1.9 Subfact</u>: The framework $\mathfrak{s}$ has $(\le \mu, \le \mu)$-smoothness (i.e. the first conclusion of 1.2 holds for μ).

Proof. By the assumption (of 1.2) without loss of generality $\mu > \chi$. Toward contradiction suppose $M_i \in K$ (for $i \le \delta$) is $\le_{\mathfrak{s}}$-increasing, $\|M_\delta\| \le \mu$ but $M^* := \bigcup\limits_{i<\delta} M_i \not\le_{\mathfrak{s}} M_\delta$. Without loss of generality $\langle M_i : i < \delta\rangle$ is not eventually constant and $\delta = \mathrm{cf}(\delta)$ hence $\delta \le \mu, \|M_\delta\| = \mu$ and $\langle M_i : i < \delta\rangle$ is continuous.

[Why? As we can prove by induction on δ; there is an increasing continuous sequence $\langle \alpha_\varepsilon : \varepsilon < \mathrm{cf}(\delta)\rangle$ of ordinals $< \delta$ with limit δ; if $\mathrm{cf}(\delta) < \delta$ then the subfact holds for $\langle M_{\alpha_\varepsilon} : \varepsilon < \mathrm{cf}(\delta)\rangle\hat{\ }\langle M_\delta\rangle$ and this implies the desired conclusion, so without loss of generality $\delta = \mathrm{cf}(\delta)$.

Also we let N'_i be N_i if $i = \delta$ or $i < \delta$ is a non-limit ordinal and $N'_i = \cup\{N_j : j < i\}$ otherwise. Now $N'_i \le_{\mathfrak{s}} M_i$ for $i < \delta$, because if i is a limit ordinal by the minimality of δ and trivially otherwise.]

Now we first prove:

1.10 SubFact:. If (δ and $\langle M_i : i \le \delta\rangle$ are as above and) $A \subseteq M_\delta, |A| < \mu, i \le \delta, i < \mu$ <u>then</u> there is N satisfying $\|N\| = \chi + |A| + |i|$, $N \le_{\mathfrak{s}} M_\delta$

such that for every $j \in \{\gamma : \gamma \le i\} \backslash \{\delta\}$ we have $N \cap M_j \le_{\mathfrak{s}} M_j$ and $\mathrm{NF}(N \cap M_j, M_j, N, M_\delta)$.

Proof. We choose by induction on $n < \omega$ for every $\alpha \in \{j : j \le i\} \backslash \{\delta\}$ models M^n_α, N^n_α such that:

(*i*) $M^n_\alpha \le_{\mathfrak{s}} M_\alpha, N^n_\alpha \le_{\mathfrak{s}} M_\delta, N^n_\alpha \cap M_\alpha = M^n_\alpha$

(*ii*) $\|M^n_\alpha\| \le \chi + |A| + |i|$

(*iii*) $\|N^n_\alpha\| = \chi + |A| + |i|$

(*iv*) N^n_α includes $A \cup \cup \{N^k_\beta : k < n, \beta \in \{j : j \le i\} \backslash \{\delta\}\}$

(*v*) $\mathrm{NF}(M^n_\alpha, M_\alpha, N^n_\alpha, M_\delta)$ holds for $\alpha \le i$.

This is easily done as we are assuming $i < \delta \le \mu$ and NF is $(\le \mu, \le \mu)$-based by 1.8. Let $N = \bigcup_{n<\omega} N^n_0$; by clause (iv) clearly $N = \bigcup_{n<\omega} N^n_\alpha$ for each $\alpha \in \{j : j \le i\} \backslash \{\delta\}$. Now $\|N\| \le \chi + |A| + |i|$, hence (by (iii)) equality holds. Now $N \le_{\mathfrak{s}} M_\delta$ by 1.6 above.
Clearly for $j \in \{\gamma : \gamma \le i\} \backslash \{\delta\}$ we have:

$$\begin{aligned}\bigcup_{n<\omega} M^n_j = \left(\bigcup_{n<\omega} M^n_j\right) \cap M_j &\subseteq (\bigcup_{n<\omega} N^n_0) \cap M_j \\ &\subseteq N \cap M_j \subseteq \bigcup_{n<\omega} (N^n_0 \cap M_j) \\ &\subseteq \bigcup_{n<\omega} (M^{n+1}_j \cap M_j) = \bigcup_{n<\omega} M^{n+1}_j = \bigcup_{n<\omega} M^n_j\end{aligned}$$

hence $N \cap M_j = \bigcup_{n<\omega} M^n_j$ and by 1.6 we have $\bigcup_n M^n_j \le_{\mathfrak{s}} M_j$. So $N \cap M_j = \bigcup_{n<\omega} M^n_j \le_{\mathfrak{s}} M_j$ for each $j \in \{\gamma : \gamma \le i\} \backslash \{\delta\}$ as required, but also for $j = \delta$ because it means $N \le_{\mathfrak{s}} M_\delta$ which was proved. Lastly for $j \in \{\gamma : \gamma \le i\} \backslash \{\delta\}$, by clause (v) we have $\mathrm{NF}(M^n_j, M_j, N^n_j, M_\delta)$ for $n < \omega$ hence $\mathrm{NF}\left(\bigcup_{n<\omega} M^n_j, M_j, \bigcup_{n<\omega} N^n_j, M_\delta\right)$ by V.C.1.10(3) recalling subfact 1.6. But this means that $\mathrm{NF}(N \cap M_j, M_j, N, M_\delta)$, i.e., N is as required recalling $\|N\| = \chi + |A| + |i|$. $\square_{1.10}$

1.11 Continuation of the proof of 1.9.:

The proof splits to two cases.

Case α**.** $\delta < \mu$.
Let $M_\delta = \{a_\gamma : \gamma < \mu\}$ and by induction on $\gamma < \mu$ choose N_γ such that:

- (i) $N_\gamma \leq_{\mathfrak{s}} M_\delta$
- (ii) $\|N_\gamma\| \leq |\gamma| + \chi + |\delta|$
- (iii) $a_\gamma \in N_{\gamma+1}$
- (iv) for every $j < \delta$, $N_\gamma \cap M_j \leq_{\mathfrak{s}} M_j$ and $\mathrm{NF}(N_\gamma \cap M_j, M_j, N_\gamma, M_\delta)$

  (note: $\delta < \mu$)
- (v) $\langle N_\gamma : \gamma < \mu \rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous hence
- (vi) $M_\delta = \cup\{N_\gamma : \gamma < \mu\}$.

Successor stages and $\gamma = 0$ are done by Subfact 1.10 above. For limit stages γ let $N_\gamma = \bigcup\limits_{\beta<\gamma} N_\beta$, then (i) holds by $(< \mu, < \mu)$-smoothness and 1.6; clause (ii) is trivial, clause (iii) is irrelevant; for clause (iv) we have $N_\gamma \cap M_j \leq_{\mathfrak{s}} M_j$ again by 1.6 (and the induction hypothesis) and $\mathrm{NF}(N_\gamma \cap M_j, M_j, N_\gamma, M_\delta)$ holds by V.C.1.10(3) and the induction hypothesis; lastly, clause (v) holds trivially so we can carry the induction and there are such N_γ's.

Suppose $\gamma(1) < \gamma(2) < \mu$; for $j < \delta$ by clause (iv) we have $\mathrm{NF}(N_{\gamma(1)} \cap M_j, M_j, N_{\gamma(1)}, M_\delta)$ but $N_{\gamma(1)} \cap M_j \leq_{\mathfrak{s}} N_{\gamma(2)} \cap M_j \leq_{\mathfrak{s}} M_j$ (by Ax(A3) as $N_{\gamma(\ell)} \cap M_j \leq_{\mathfrak{s}} M_j$ for $\ell = 1, 2$ and $N_{\gamma(1)} \cap M_j \subseteq N_{\gamma(2)} \cap M_j$).
Also $(N_{\gamma(2)} \cap M_j) \cup N_{\gamma(1)} \subseteq N_{\gamma(2)} \leq_{\mathfrak{s}} M_\delta$ hence by monotonicity of NF we have $\mathrm{NF}\big(N_{\gamma(1)} \cap M_j, N_{\gamma(2)} \cap M_j, N_{\gamma(1)}, N_{\gamma(2)}\big)$. So for each $\gamma < \mu$ the sequence $\langle N_\gamma \cap M_j : j < \delta \rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous, and $j < \delta \Rightarrow N_\gamma \cap M_j \leq_{\mathfrak{s}} N_\gamma \in K_{<\mu}$ hence by the induction hypothesis on μ the sequence $\langle N_\gamma \cap M_j : j < \delta \rangle \hat{} \langle (\bigcup\limits_{j<\delta} M_j) \cap N_\gamma, N_\gamma \rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous for each $\gamma < \mu$. Hence by V.C.1.10(2)

$$\mathrm{NF}(N_{\gamma(1)} \cap (\bigcup_{j<\delta} M_j), N_{\gamma(2)} \cap (\bigcup_{j<\delta} M_j), N_{\gamma(1)}, N_{\gamma(2)}).$$

Now applying V.C.1.10(3) + monotonicity (i.e. Ax(C3)(b)) with M_i there standing for $N_i \cap \bigcup_{j<\delta} M_j$ here and N_i there standing for N_i here (and recalling $\bigcup_{i<\delta} N_i = M_\delta$) we finish the case $\delta < \mu$.

Case β**.** $\delta = \mu$ (so μ is regular).
Note that if N is a submodel of $\bigcup_{i<\delta} M_i$ of cardinality $< \mu$, then $N \leq_{\mathfrak{s}} \bigcup_{i<\delta} M_i$ iff $N \leq_{\mathfrak{s}} M_\delta$ [because each such N is a submodel of M_i for some $i < \delta$ and then $N \leq_{\mathfrak{s}} M_\delta \Leftrightarrow N \leq_{\mathfrak{s}} M_i \Leftrightarrow N \leq_{\mathfrak{s}} \bigcup_{j<\delta} M_j$].

Clearly there is $\langle N_\alpha : \alpha < \delta \rangle$ which is $\subseteq$-increasing continuous with union $\cup\{M_\alpha : \alpha < \delta\}$ such that $\|N_\alpha\| < \mu$ for $\alpha < \delta \, (= \mu)$ and $N_\alpha \leq_{\mathfrak{s}} M_\alpha$ for α non-limit hence $N_\alpha \subseteq M_\alpha$ for every $\alpha < \mu$. So if $\alpha \leq \beta < \delta$ and α is a non-limit ordinal then $N_\alpha \leq_{\mathfrak{s}} M_\alpha \leq_{\mathfrak{s}} M_\beta$ so $N_\alpha \leq_{\mathfrak{s}} M_\beta \leq_{\mathfrak{s}} \cup\{M_\gamma : \gamma < \delta\}$. By subfact 1.6 we deduce that $N_\alpha \leq_{\mathfrak{s}} \cup\{M_\beta : \beta < \delta\}$ for every $\alpha < \mu$; so for $\alpha < \delta$ we have $N_\alpha \leq_{\mathfrak{s}} M_\alpha \leq_{\mathfrak{s}} M_\delta$ hence $N_\alpha \leq_{\mathfrak{s}} M_\delta$. We can replace $\langle M_\alpha : \alpha \leq \delta \rangle$ by $\langle N_\alpha : \alpha < \delta \rangle\hat{\ }\langle M_\delta \rangle$ so without loss of generality $\alpha < \delta \Rightarrow \|M_\alpha\| < \mu$.

If $\mu = \chi^+$, then we can use the assumption "$\mathfrak{s}$ is $(\leq \chi, \chi^+)$-smooth (of 1.2), so assume $\mu > \chi^+$. Under those circumstances we can use 1.7(2) (with $\bigcup_{i<\delta} M_i, M_\delta$ here corresponding to M, M^* there).

So we get

(∗) if $A \subseteq M_\delta$ and $|A| < \mu$ then for some $N_1 \leq_{\mathfrak{s}} N_2$ we have $N_2 \leq_{\mathfrak{s}} M_\delta, N_1 = N_2 \cap \bigcup_{i<\delta} M_i, \|N_2\| \leq |A| + \chi$ and: if $N_1 \leq_{\mathfrak{s}} M' \in K_{<\mu}$ and $M' \leq_{\mathfrak{s}} \cup\{M_i : i < \delta\}$ then $\mathrm{NF}(N_1, N_2, M', M_\delta)$ so $N_2 \cap M' = N_1 = N_2 \cap \cup\{M_i : i < \delta\}$.

Let $M_\delta = \{a_i : i < \mu\}$. Now we can choose $(N_{1,i}, N_{2,i})$ by induction on $i < \mu$ such that

$\circledast$ (a) $N_{1,i} \leq_{\mathfrak{s}} N_{2,i} \leq_{\mathfrak{s}} M_\delta$

(b) $\|N_{2,i}\| = \chi + |i|$

(c) $N_{1,i} \leq_{\mathfrak{s}} \cup\{M_\alpha : \alpha < \delta\}$

(d) $a_i \in N_{2,i+1}$ and $a_i \in \bigcup_{\alpha<\delta} M_\alpha \Rightarrow a_i \in N_{1,i+1}$

(e) $\langle N_{\ell,j} : j \leq i\rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous for $\ell = 1, 2$

(f) if $N_{1,i} \leq_{\mathfrak{s}} M' \leq_{\mathfrak{s}} \cup\{M_\alpha : \alpha < \delta\}$ and $M' \in K_{<\mu}$ then $\mathrm{NF}(N_{1,i}, N_{2,i}, M', M_\delta)$.

For $i = 0$ and i successor use $(*)$, for i limit use V.C.1.10(3). Lastly, having carried the induction clearly $M_\delta = \cup\{N_{2,i} : i < \mu\}$ and $\cup\{M_\alpha : \alpha < \delta\} = \cup\{N_{1,i} : i < \mu\}$ and by V.C.1.6 we get $\cup\{M_\alpha : \alpha < \delta\} \leq_{\mathfrak{s}} M_\delta$ as required. $\square_{1.9}, \square_{1.2}$

1.12 Fact.* $[\mathrm{AxFr}_3 + \mathrm{LSP}(\chi) + \chi \geq |\tau_{\mathfrak{K}}|]$.
1) If $N_0 \leq_{\mathfrak{s}} N_1 \leq_{\mathfrak{s}} N_3$ and $N_0 \leq_{\mathfrak{s}} N_2 \leq_{\mathfrak{s}} N_3$ and f is an isomorphism from N_1 onto N_2 over N_0, (so $\mathrm{id}_{N_0} \subseteq f$) such that $f \neq \mathrm{id}_{N_1}$ then $K_{\geq\lambda} \neq \emptyset$ for every λ.
2) There are such $N_\ell (\ell < 4)$ if $K_{\geq(2^\chi)^+} \neq \emptyset$.
3) If $K_{\geq\lambda} = \emptyset$ for some λ, then K has a member M^* of cardinality $\leq 2^\chi$ such that: M^* is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-homogeneous for every μ (even $> 2^\chi!$); also if $N \leq_{\mathfrak{s}} M^*$ then M^* has no non-trivial automorphism over N.

1.13 Remark. 0) On AxFr_3, see Definition V.B.1.9 and it follows from AxFr_1.
1) We are using only $\chi \geq \mathrm{LS}(\mathfrak{s}) + |\tau(\mathfrak{s})|$ and axioms (A0),(A1),(A2), (A4),(C1),
(C2)(existence),(C3)(monotonicity),(C5)(uniqueness).
2) Instead of (C5) we can assume Ax(F1).
3) Instead (A4) we can use just $(\mathrm{A4})^-$, but then we have to omit clause (d) in the proof of part (1).

Proof. We shall prove 1) later.
2) If $\|M\| > 2^\chi$, $M \in K$, as $\mathrm{LSP}(\chi)$, we can find $N_0 \leq_{\mathfrak{s}} M$, $\|N_0\| \leq \chi$. For each $c \in M$ let N_c be such that:

$$N_0 \leq_{\mathfrak{s}} N_c \leq_{\mathfrak{s}} M$$

$$\|N_c\| \leq \chi$$

$$c \in N_c$$

Now the number of isomorphism types of $(N_c, c, d)_{d \in N_0}$ is $\leq 2^\chi$ hence there are $c_1 \neq c_2$ from M such that

$$(N_{c_1}, c_1, d)_{d \in N_0} \cong (N_{c_2}, c_2, d)_{d \in N_0}.$$

So there is an isomorphism f from N_{c_1} onto N_{c_2} such that

$$f \restriction N_0 = \text{id}_{N_0}, f(c_1) = c_2.$$

So $N_0, N_{c_1}, N_{c_2}, M, f$ are as required on N_0, N_1, N_2, N_3, f in part (1).
1) Let $c \in N_1$ be such that $f(c) \neq c$. We choose by induction on $\alpha < \lambda$, f^α, N^α, M_α such that:

- (a) f^α is an isomorphism from N_3 onto N^α over N_0,
- (b) $M_0 = N_3$
- (c) $\text{NF}(N_0, N^\alpha, M_\alpha, M_{\alpha+1})$
- (d) for α limit $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$
- (e) M_α is $\leq_{\mathfrak{s}}$-increasing.

Clearly $\langle M_\alpha : \alpha < \lambda \rangle$ is $\leq_{\mathfrak{s}}$-increasing and $N_\ell, N^\beta \leq_{\mathfrak{s}} M_\alpha$ for $\beta < \alpha, \ell < 4$. How do we define? For α limit let $M_\alpha := \cup\{M_\beta : \beta < \alpha\}$, so by Ax(A4) we deduce $M_\alpha \in K$ and $\beta < \alpha \Rightarrow M_\beta \leq_{\mathfrak{s}} M_\alpha$ (clearly $N^\beta \leq_{\mathfrak{s}} M_\alpha$ by transitivity of $\leq_{\mathfrak{s}}$, that is Ax(A2)). For $\alpha + 1$ use Ax(C2) to choose $M_{\alpha+1}, N^\alpha, f^\alpha$ and by Ax(C1) we have $M_\alpha \leq_{\mathfrak{s}} M_{\alpha+1}$, $N^\alpha \leq_{\mathfrak{s}} M_{\alpha+1}$. Let $M_\lambda = \cup\{M_\alpha : \alpha < \lambda\}$. Of course, $N_\ell \leq_{\mathfrak{s}} M_\alpha$ as $N_\ell \leq_{\mathfrak{s}} M_0 \leq_{\mathfrak{s}} M_\alpha \leq_{\mathfrak{s}} M_\beta$ for $\ell < 3, \alpha \leq \beta \leq \lambda$. It is enough to show

- $(*)$ if $\alpha < \lambda$ then $f^\alpha(c) \notin M_\alpha$.

Suppose not, by clause (c) and monotonicity (i.e., Ax(C3)):

$$\mathrm{NF}\,(N_0, f^\alpha(N_1), M_\alpha, M_{\alpha+1})$$

and

$$\mathrm{NF}\,(N_0, f^\alpha(N_2), M_\alpha, M_{\alpha+1})\,.$$

Let $h_\alpha = (f^\alpha) \circ f \circ (f^\alpha \restriction N_1)^{-1}$, it is an isomorphism from $f^\alpha(N_1)$ onto $f^\alpha(N_2)$ over $f^\alpha(N_0)$. So $\mathrm{id}_{f^\alpha(N_0)}, h_\alpha, \mathrm{id}_{M_\alpha}$ is an isomorphism from $f^\alpha(N_0), f^\alpha(N_1), M_\alpha$ onto $f^\alpha(N_0), f^\alpha(N_2), M_\alpha$, respectively and $\mathrm{id}_{f^\alpha(N_0)} \subseteq h_\alpha$, id_{M_α} and $h_\alpha(f^\alpha(c)) \neq f^\alpha(c)$. By the uniqueness Ax(C5) there is a pair $(M^+_{\alpha+1}, h^+_\alpha)$ such that: $M_{\alpha+1} \leq_{\mathfrak{s}} M^+_{\alpha+1}$ and h^+_α is a $\leq_{\mathfrak{s}}$-embedding of $M_{\alpha+1}$ into $M^+_{\alpha+1}$ extending $h_\alpha \cup \mathrm{id}_{M_\alpha}$. So $h^+_\alpha(f^\alpha(c)) = h_\alpha(f^\alpha(c)) \neq f^\alpha(c)$, however if indeed $f^\alpha(c) \in M_\alpha$ then $h^+_\alpha(f^\alpha(c)) = \mathrm{id}_{M_\alpha}(f^\alpha(c)) = f^\alpha(c)$, contradiction. So $\|M_\lambda\| \geq \|\{f^\alpha(c) : \alpha < \lambda\}\|$ is as required.
3) Left to the reader. $\square_{1.12}$

1.14 Remark. 1) Thus we have described models "generated" by long sequences of indiscernibles. They are analogous to free algebras or Ehrenfeucht Mostowksi models.
2) Of course, in 1.12(1) we can find $M \in K_{\geq\lambda}$ which is $(\mathbb{D}_{\mathfrak{K}}, \lambda)$-homogeneous.

1.15 Theorem. *Suppose* LSP(χ) *but* $\mathfrak{s}$ *does not have* $(\leq \chi^+, \leq \chi^+)$-*smoothness or* $\mathfrak{s}$ *is not* $(\leq \chi^+, \chi)$-*based.* *Then:*

(1) *if* λ *is regular* $\leq_{\mathfrak{s}}$-*inaccessible* $> \chi^{++}$ *then* $\dot{I}(\lambda, K_{\mathfrak{s}}) = 2^\lambda$
(2) *if* $\lambda = \lambda^{<\mu}$, $\mu \leq \chi^+$, λ $\mathfrak{K}_{\mathfrak{s}}$-*inaccessible* $> \chi$ *then there are* 2^λ *non-isomorphic* $(\mathbb{D}_{\mathfrak{s}}, \mu)$-*homogeneous models of cardinality* λ.

Proof. 1) By V.C.2.4 we know that λ has stationary good subsets S_θ of $\{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$ for each $\theta = \mathrm{cf}(\theta) \leq \chi^+$. If $\mathfrak{s}$ is not $(\leq \chi^+, \leq$

χ^+)-smooth then for some $\theta = \text{cf}(\theta) \le \chi, \mathfrak{s}$ is not $(\le \chi^+, \theta)$-smooth now use V.C.2.6. So we can assume that $\mathfrak{s}$ is $(\le \chi^+, \le \chi)$-smooth, now by 1.17, the conclusion of V.C.3.8 holds so V.C.5.2(0) applies.
2) Similar using V.C.5.2(1). $\square_{1.15}$

1.16 Exercise:

(a) no $\bar{M}, \bar{N}, M^*$ satisfies $\odot$ of V.C.3.8 for χ
(b) $\langle M_\alpha : \alpha < \chi^+\rangle, \langle N_\alpha : \alpha < \chi^+\rangle$ are $\le_{\mathfrak{s}}$-increasing continuous sequences in K_χ.

Then for some club E of χ^+ we have:
if $\alpha < \beta$ from E then $\text{NF}(M_\alpha, N_\alpha, M_\beta, N_\beta)$.

[Hint: For $\alpha < \chi^+$ let $S_\alpha = \{\beta < \chi^+ : \beta > \alpha \text{ and } \text{NF}(M_\alpha, N_\alpha, M_\beta, N_\beta)\}$ and $S = \{\alpha < \chi^+ : S_\alpha \text{ is a stationary subset of } \chi^+\}$.

$(*)_1$ if $\alpha < \beta < \gamma$ and $\gamma \in S_\alpha$ then $\beta \in S_\alpha$.

[Why? By monotonicity of NF.]

$(*)_2$ if $\alpha \in S$ then $S_\alpha = (\alpha, \chi^+)$.

[Why? By $(*)_1$ and the definition of S.]

$(*)_3$ if $\delta < \chi^+$ and $\delta = \sup(\delta \cap S)$ then $\delta \in S$.

[Why? We prove this by induction on δ. So arriving to $\delta, S \cap \delta$ is a closed subset of δ, let its order type be ζ and a limit ordinal and let $\langle \gamma_\varepsilon : \varepsilon < \zeta\rangle$ list $S \cap \delta$ in increasing order. Let $\gamma_\zeta = \delta, \gamma_{\zeta+1} = \beta$. Clearly $\varepsilon < \zeta \Rightarrow \text{NF}(M_{\gamma_\varepsilon}, N_{\gamma_\varepsilon}, M_{\gamma_{\zeta+1}}, N_{\gamma_{\zeta+1}})$ hence by V.C.1.10 we deduce $\text{NF}(M_{\gamma_\zeta}, N_{\gamma_\zeta}, M_{\gamma_{\zeta+1}}, N_{\gamma_{\zeta+1}})$ which means $\text{NF}(M_\delta, N_\delta, M_\beta, N_\beta)$. As β was any ordinal $\in (\delta, \chi^+)$ we conclude $\delta \in S$ as required.]

$(*)_4$ S is unbounded in χ^+.

[Why? Otherwise we get a contradiction to clause (a) of the assumption.]

$(*)_5$ S is a closed unbounded subset of χ^+.

[Why? By $(*)_3 + (*)_4$ it is closed unbounded.]

Now by $(*)_2 + (*)_5$ we are done.]

1.17 Exercise If $\mathfrak{s}$ is not $(\chi^+, \leq \chi)$-smooth but is $(\leq \chi, \leq \chi)$-smooth and $\mathrm{LSP}(\chi^+, \chi)$ then the conclusion of V.C.3.8 holds.

[Hint: Toward contradiction assume that the conclusion fails. By assumption "$\mathfrak{s}$ is not (χ^+, χ)-smooth", there are $\delta < \chi^+$, a limit ordinal and an $\subseteq$-increasing continuous sequence $\langle M_\alpha : \alpha \leq \delta + 1\rangle, M_\alpha \in K_{\chi^+}$ and $\alpha < \beta \leq \delta + 1 \wedge (\alpha, \beta) \neq (\delta, \delta + 1) \Rightarrow M_\alpha \leq_{\mathfrak{s}} M_\beta$ but $M_\delta \nleq_{\mathfrak{s}} M_{\delta+1}$ (we use Ax(A4)). Without loss of generality $\delta = \mathrm{cf}(\delta)$ and for each $\alpha \leq \delta + 1$ we can find an $\leq_{\mathfrak{s}}$-increasing continuous sequence $\langle M_{\alpha,i} : i < \chi^+\rangle$, a sequence of models of cardinality $\leq \chi$ such that $M_\alpha = \cup\{M_{\alpha,i} : i < \chi^+\}$. By $\mathrm{LSP}(\chi^+, \chi)$ without loss of generality $M_{\alpha,i} \leq_{\mathfrak{s}} M_\alpha$ for non-limit i. Now the assumptions of 1.6 holds with $\mu := \chi^+$, hence also for limit $i < \chi^+$ we have $M_{\alpha,i} \leq_{\mathfrak{s}} M_\alpha$. As we are assuming that the conclusion of V.C.3.8 fail, by 1.16 if $\alpha < \beta \leq \delta + 1, (\alpha, \beta) \neq (\delta, \delta + 1)$ then for some club $E_{\alpha,\beta}$ of χ^+ we have: if $i < j < \chi^+$ are from $E_{\alpha,\beta}$ then $\mathrm{NF}(M_{\alpha,i}, M_{\beta,i}, M_{\alpha,j}, M_{\beta,j})$. Let $E_{\delta,\delta+1}$ be a club of χ^+ such that for $i < j$ from $E_{\delta,\delta+1}, M_{\delta,j} \cap M_{\delta+1,i} = M_{\delta,i}$ and $M_{\delta,j} \cup M_{\delta+1,i} \subseteq M_{\delta+1,j}$.

Recall that $\delta < \chi^+$.

Let $E := \cap\{E_{\alpha,\beta} : \alpha < \beta \leq \delta + 1\}$, clearly it is a club of χ^+ so by renaming without loss of generality $E = \chi^+$. For $i < \chi^+$ we have: $\langle M_{\alpha,i} : \alpha < \delta\rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous, each is $\leq_{\mathfrak{s}} M_{\delta+1}$ and $\subseteq M_{\delta+1,i} \leq_{\mathfrak{s}} M_{\delta+1}$ so by Ax(A3) we have $\alpha < \delta \Rightarrow M_{\alpha,i} \leq_{\mathfrak{s}} M_{\delta+1,i}$ hence (by $\mathfrak{s}$ being $(\leq \chi, \leq \chi)$-smooth) we have $\cup\{M_{\alpha,i} : \alpha < \delta\} \leq_{\mathfrak{s}} M_{\delta+1,i}$ which means $M_{\delta,i} \leq_{\mathfrak{s}} M_{\delta+1,i}$. So $\langle M_{\alpha,i} : \alpha \leq \delta + 1\rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous for each $i < \chi^+$.

Let $i < j < \chi^+$, for $\alpha < \beta \leq \delta + 1, (\alpha, \beta) \neq (\delta, \delta + 1)$ we have $\mathrm{NF}(M_{\alpha,i}, M_{\beta,i}, M_{\alpha,j}, M_{\beta,j})$ hence by symmetry Ax(C5) also $\mathrm{NF}(M_{\alpha,i}, M_{\alpha,j}, M_{\beta,i}, M_{\beta,j})$. This means that we can apply V.C.1.6 and get $\mathrm{NF}(M_{\delta,i}, M_{\delta,j}, M_{\delta+1,i}, M_{\delta+1,j})$.

By symmetry Ax(C5) also $\mathrm{NF}(M_{\delta,i}, M_{\delta+1,i}, M_{\delta,j}, M_{\delta+1,j})$ holds for $i < j < \chi^+$. As $\langle M_{\delta,i} : i < \chi^+\rangle$ is $\leq_{\mathfrak{s}}$-increasing with union M_δ and $\langle M_{\delta+1,i} : i < \chi^+\rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous with union $M_{\delta+1}$ by V.C.1.6. By V.C.1.6 we get $M_\delta \leq_{\mathfrak{s}} M_{\delta+1}$ contradiction.]

The following is used in 4.5(6).

1.18 Claim. *1) Assume χ is as in 1.2 and $M_0 \le_{\mathfrak{s}} M_\ell \le_{\mathfrak{s}} M_3$ for $\ell = 1,2$. Then the folowing conditions are equivalent:*

$\circledast_1$ (a) $\mathrm{NF}(M_0, M_1, M_2, M_3)$

(b) *for the χ-majority of $Y \in [M_3]^{\le \chi}$ we have* $\mathrm{NF}(M_0 \restriction Y, M_1 \restriction Y, M_2 \restriction Y, M_3 \restriction Y)$

(c) *for the χ-majority of $Y \in [M_1]^{\le \chi}$ we have* $\mathrm{NF}(M_0 \restriction Y, M_1 \restriction Y, M_2, M_3)$.

2) If in addition $M_3' \subseteq M_3$ <u>then</u> the following conditions are equivalent:

$\circledast_2$ (a) $\mathrm{NF}(M_0, M_1, M_2, M_3)$ *and* $M_3' = \langle M_1 \cup M_2 \rangle^{\mathrm{gn}}_{M_3}$ *(hence $M_3' \le_{\mathfrak{s}} M_3$)*

(b) *for the χ-majority of $Y \in [M_3]^{\le \chi}$ we have* $\mathrm{NF}(M_0 \restriction Y, M_1 \restriction Y_1, M_2 \restriction Y, M_3 \restriction Y)$ *and* $M_3' \restriction Y = \langle (M_1 \restriction Y) \cup (M_2 \restriction Y) \rangle^{\mathrm{gn}}_{M_3 \restriction Y}$.

3) Assume that χ is as in 1.2 and $M_0 \le_{\mathfrak{s}} M_1 \le_{\mathfrak{s}} M_3$ and $a \in M_3$. <u>Then</u> the following conditions are equivalent:

$\circledast_3$ (a) $\mathbf{tp}_{\mathfrak{s}}(a, M_1, M_3)$ *does not fork over* M_0

(b) *for the χ-majority of $Y \subseteq M_3$ we have* $\mathbf{tp}_{\mathfrak{s}}(a, M_1 \restriction Y, M_3 \restriction Y)$ *does not fork over* $M_0 \restriction Y$.

Proof. 1) We prove this by induction on $\mu := \|M_3\|$ for every $\chi < \mu$.
Now the case $\mu \le \chi$ is obvious, so assume $\mu > \chi$.

<u>(a) ⇒ (c)</u>:
By V.C.3.13 and NF being transitivity.

<u>(c) ⇒ (b)</u>:
By monotonicity of NF.

<u>(b) ⇒ (a)</u>:

By V.C.3.13 the majority of $Y \in [M_3]^{\leq \chi}$ belongs to $\mathscr{Y} = \{Y \in [M_1]^{\leq \chi}$: $\mathrm{NF}(M_0 \restriction Y, M_1 \restriction Y, M_0, M_1)\}$. Now fix for awhile $Y_* \in \mathscr{Y}$. Now if $M_0 \restriction Y_* \leq_{\mathfrak{s}} M_2' \leq_{\mathfrak{s}} M_2, M_2' \in K_{<\mu}$, let $\mu_1 = \|M_1'\| + \chi$ then by the induction hypothesis (as the majorities form a filter) we can find $Y \in [M_3]^{\mu_1}$ such that: $Y_* \subseteq Y, M_2' \subseteq Y, M_\ell \restriction Y \leq_{\mathfrak{s}} M_\ell$ for $\ell \leq 3$ and $\mathrm{NF}(M_0 \restriction Y, M_1 \restriction Y, M_2 \restriction Y, M_3 \restriction Y)$. But we have $\mathrm{NF}(M_0 \restriction Y_*, M_1 \restriction Y_*, M_0, M_1)$ so by montonocity $\mathrm{NF}(M_0 \restriction Y_*, M_1 \restriction Y_*, M_0 \restriction Y, M_1 \restriction Y)$ hence by transitivity of NF we have $\mathrm{NF}(M_0 \restriction Y_*, M_1 \restriction Y_*, M_2 \restriction Y, M_3 \restriction Y)$. By monotonicity of NF (and Ax(A3)) we can deduce $\mathrm{NF}(M_0 \restriction Y_*, M_1 \restriction Y_*, M_2', M_3)$ and recall that the only requirements on M_2' were $M_2' \in K_{<\mu}, M_0 \restriction Y_* \leq_{\mathfrak{s}} M_2' \leq_{\mathfrak{s}} M_2$. Hence by V.C.1.10(2) we can deduce $\mathrm{NF}(M_0 \restriction Y_*, M_1 \restriction Y_*, M_2, M_3)$.

But Y_* was any member of $\mathscr{Y}$ and we can replace χ by any $\chi' \in (\chi, \mu)$, hence we can find a $\leq_{\mathfrak{s}}$-increasing continuous sequence $\langle M^1_\alpha : \alpha < \mu \rangle$ such that $M^1_\alpha \in K_{<\mu}$, $M_1 = \cup\{M^1_\alpha : \alpha < \mu\}$, $M^0_\alpha := M^1_\alpha \cap M_0 \leq_{\mathfrak{s}} M_0$ and $\mathrm{NF}(M^0_\alpha, M^1_\alpha, M_2, M_3)$ for $\alpha < \mu$. So again by V.C.1.10(2) we get $\mathrm{NF}(\bigcup\limits_{\alpha<\mu} M^0_\alpha, \bigcup\limits_{\alpha<\mu} M^1_\alpha, M_2, M_3)$, i.e. $\mathrm{NF}(M_0, M_1, M_2, M_3)$ as required.

2),3) Similar. $\square_{1.18}$

<u>1.19 Exercise</u>: If $\mathfrak{s}$ is not $(\chi^+, \leq \chi^+)$-smooth but is $(\leq \chi, \leq \chi)$-smooth and $\mathrm{LSP}(\chi^+, \chi)$ <u>then</u> the conclusion of V.C.3.8 holds.

[<u>Hint</u>: Assume toward contradiction that the conclusion fail. Let $\langle M_\alpha : \alpha \leq \delta + 1 \rangle$ be a counterexample to "$(\chi^+, \leq \chi^+)$-smoothness" as in the proof of 1.17; without loss of generality $\delta = \mathrm{cf}(\delta)$. By 1.17 without loss of generality $\delta = \chi^+$, let $\langle M_{\alpha,i} : i < \chi^+, \alpha \leq \delta + 1 \rangle$ and $\langle E_{\alpha,\beta} : \alpha < \beta \leq \delta + 1$ and $(\alpha, \beta) \neq (\delta, \delta + 1) \rangle$ be as in the proof of 1.17.

Let $E := \{j < \chi^+ : j$ is a limit ordinal and if $\alpha_1 < \alpha_2 < j$ then $j \in E_{\alpha_1,\alpha_2} \cap E_{\alpha_1,\delta} \cap E_{\alpha_2,\delta+1} \cap E_{\delta,\delta+1}\}$. Clearly E is a club of χ^+.

For $\alpha \in E$ let $M'_\alpha := \cup\{M_{\beta,\beta} : \beta < \alpha\}$. Easily if $(\beta_\ell, \gamma_\ell) \in \alpha \times \alpha$ for $\ell = 1, 2$ and $\alpha \in E$ then for some $(\beta, \gamma) \in \alpha \times \alpha$ we have $M_{\beta_\ell,\gamma_\ell} \leq_{\mathfrak{s}} M_{\beta,\gamma}$ for $\ell = 1, 2$.

So $M'_\alpha = \cup\{M_{\beta,\gamma} : \beta, \gamma < \alpha\}$ and $\langle M'_\alpha : \alpha \in E \rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous, each M'_α is $\subseteq M_\alpha$ (and $\leq_{\mathfrak{s}} M_\alpha$, applying 1.6 as above).

Also $\cup\{M'_\alpha : \alpha < \chi^+\}$ is $\subseteq M_\delta$ and for each $\beta < \chi^+$ it includes $\cup\{M_{\beta,i} : i < \chi^+\}$, i.e., include M_β. As $M_\delta = \cup\{M_\beta : \beta < \delta\}$ we conclude that $M_\delta = \cup\{M'_i : i < \chi^+\}$.

Now

$(*)_1$ if $i \in E$ then $\langle M_{\alpha,i} : \alpha < i\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous with union M'_i and $M'_i \le_{\mathfrak{s}} M_{\delta+1,i}$.

[Why? We use the definition of E so clearly $\langle M_{\alpha,i} : \alpha < i\rangle$ is increasing continuous, moreover, union is included in $M_{\delta+1,i}$, also $M'_i = \cup\{M_\alpha : \alpha < i\} \subseteq \{M_{\alpha,i} : \alpha < i\} \subseteq \cup\{M_{\alpha,j} : \alpha < i, j < i\} \subseteq \cup\{M_{\max\{\alpha,j\}} : \alpha, j < i\} = M'_i$, so the union of $\langle M_{\alpha,i} : \alpha < i\rangle$ is equal to M'_i and again by applying 1.6, it is $\le_{\mathfrak{s}} M_{\delta+1,i}$.]

$(*)_2$ if $i < j$ belongs to E then $\mathrm{NF}(M'_i, M_{\delta+1,i}, M'_j, M_{\delta+1,j})$.

[Why? As above we have: if $\alpha < \beta < i$ then $\mathrm{NF}(M_{\alpha,i}, M_{\beta,i}, M_{\alpha,j}, M_{\beta,j})$ hence by symmetry, Ax(C5) also $\mathrm{NF}(M_{\alpha,i}, M_{\alpha,j}, M_{\beta,i}, M_{\beta,j})$. As in Case α we can deduce that $\mathrm{NF}(M'_i, \bigcup\limits_{\alpha<i} M_{\alpha,j}, M_{\delta+1,i}, M_{\delta+1,j})$ which means that $\mathrm{NF}(M'_i, M_{i,j}, M_{\delta+1,i}, M_{\delta+1,j})$.
Now by Ax(A3),(C3),(C5) we get $(*)_2$, so we finish as in the proof of 1.17.]

1.20 Remark. 1) Note that 1.17 says that we can weaken the assumptions of 1.2.
2) Can we weaken the assumption "$\mathfrak{s}$ is $(\le \chi, \le \chi^+)$-smooth" in 1.2 to "$\mathfrak{s}$ is $(\le \chi, \le \chi)$-smooth? This is a motivation of 1.17, 1.19, but it uses a stronger version of (χ^+, χ)-based.

§2 Primeness

In this section we introduce (for our context) the notions prime and primary and isolation for $(\mathbb{D}_{\mathfrak{s}}, \mu)$ where $\mu > \chi_{\mathfrak{s}}$ and give their obvious properties. The main lemma is 2.9: existence of isolated types; the main point is to use "$\mathfrak{s}$ is μ-based for $\mu \ge \chi_{\mathfrak{s}}$"; this fact is not needed in the first order case. The main result is 2.11, the existence $(\mathbb{D}_{\mathfrak{s}}, \mu)$-primary models.

For the rest of this chapter we assume

2.1 Hypothesis. 1) $\mathfrak{s}$ satisfies $\mathrm{AxFr}_1, \mathfrak{C}$ a monster model (so $\langle A\rangle^{\mathrm{gn}} = \langle A\rangle^{\mathrm{gn}}_{\mathfrak{C}}$).
2) We assume $\chi_{\mathfrak{s}} < \infty$ where $\chi_{\mathfrak{s}} = \chi(\mathfrak{s})$ is defined as

$$\mathrm{Min}\Big\{\chi : \mathfrak{s} \text{ is } (\leq \chi^+, \chi)\text{-based}, \mathrm{LSP}(\chi) \text{ and } \mathfrak{s} \text{ is } (\leq \chi, \leq \chi^+)\text{-smooth} \quad \text{(see V.C.3.7,V.B.1.16,V.B.1.27 respectively)}\Big\};$$

[Hence, by 1.2 for $\chi_1 \geq \chi$, smoothness holds and $\mathfrak{s}$ is χ_1-based, $\mathrm{LSP}(\chi_1)$ holds and by V.B.3.9, the monster model $\mathfrak{C}$, is well defined].

2.2 Observation. If $M_0 \leq_{\mathfrak{s}} M_\ell \leq_{\mathfrak{s}} M_3$ for $\ell = 1,2$ and $\neg\mathrm{NF}(M_0, M_1, M_2, M_3)$ <u>then</u> there is M_2' such that $M_0 \leq_{\mathfrak{s}} M_2' \leq_{\mathfrak{s}} M_2; \|M_2'\| \leq \|M_0\| + \chi_{\mathfrak{s}}$ and $\neg\mathrm{NF}(M_0, M_1, M_2', M_3)$.

Proof. By transitivity of NF and $\mathfrak{s}$ being $(\|M_0\| + \chi)$-based (really used in a proof in §1). $\square_{2.2}$

2.3 Definition. 1) M is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-prime over A <u>if</u>:

- (i) $A \subseteq M \leq_{\mathfrak{s}} \mathfrak{C}$
- (ii) M is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-homogeneous[1], see V.B.3.10
- (iii) if M' satisfies (i) and (ii) then M can be $\leq_{\mathfrak{s}}$-embedded into M' over A.

2) M is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-primitive over A <u>if</u>:

- (i) $A \subseteq M \leq_{\mathfrak{s}} \mathfrak{C}$
- (ii) if $A \subseteq M' \leq_{\mathfrak{s}} \mathfrak{C}$ and M' is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-homogeneous <u>then</u> M can be $\leq_{\mathfrak{s}}$-embedded into M' over A.

[1] this means model homogeneous as we use $\mathbb{D}_{\mathfrak{s}}$

2.4 Remark. So [M is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-prime over A] is equivalent to [M is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-homogeneous and M is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-primitive over A].

2.5 Definition. 1) We say N is isolated over (M, M_0) if:

- (i) M_0, M, N are in stable amalgamation (i.e., inside $\mathfrak{C}$)
- (ii) if N' is isomorphic to N over M_0 (and $N' \leq_{\mathfrak{s}} \mathfrak{C}$) then M_0, M, N' are in stable amalgamation.

2) We say N is μ-isolated over (M, M_0) if:

- (i) M_0, M, N are in stable amalgamation and
- (ii) for some M_1 we have $M_0 \leq_{\mathfrak{s}} M_1 \leq_{\mathfrak{s}} M, \|M_1\| < \mu$ and $\langle M_1 \cup N \rangle^{\mathrm{gn}}$ is isolated over (M, M_1).

3) In parts (1) and (2) we may write M instead of (M, M_0) when $M_0 = M \cap N$.

2.6 Fact. 1) If $N \leq_{\mathfrak{s}} M$ then N is isolated over M.
2) If N' is isolated over M, (see 2.5(3)),

$$M' = \langle M \cup N' \rangle^{\mathrm{gn}}$$

$$\chi(\mathfrak{s}) + \|N'\| < \mu$$

then M' is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-primitive over M.
3) If $M_i (i \leq \alpha)$ is $\leq_{\mathfrak{s}}$-increasing continuous, $M_i \leq_{\mathfrak{s}} \mathfrak{C}$ and M_{i+1} is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-primitive over M_i for $i < \alpha$ then M_α is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-primitive over M_0.
4) If N is isolated over (M, M_0) and $\mu \geq \|N\|^+ + \chi_{\mathfrak{s}}^+$ then N is μ-isolated over (M, M_0).
5) If N is isolated over $(M, M_0), M_0 \leq_{\mathfrak{s}} M_1 \leq_{\mathfrak{s}} M, N_1 = \langle M_1 \cup N \rangle^{\mathrm{gn}}$ then N_1 is isolated over (M, M_1).
6) If N is isolated over (M, M_0), and $M_0 \leq_{\mathfrak{s}} N_0 \leq_{\mathfrak{s}} N_1 \leq_{\mathfrak{s}} N$ and $M_0 \leq_{\mathfrak{s}} M_1 \leq_{\mathfrak{s}} M, M_1' = \langle M_1 \cup N_0 \rangle^{\mathrm{gn}}$ and $M_0' = N_0$ then N_1 is isolated over (M_1', M_0').

7) If $A \subseteq M_1 \leq_{\mathfrak{s}} M_2$ and M_2 is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-primitive over A and M_1 is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-homogeneous <u>then</u> M_1 is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-prime over A.

Proof. 1) Easy.
2) Easy.
3) Suppose M is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-homogeneous, $M_0 \leq_{\mathfrak{s}} M \leq_{\mathfrak{s}} \mathfrak{C}$. We define by induction on $i \leq \alpha$ a $\leq_{\mathfrak{s}}$-embedding f_i of M_i into M, increasing with i.
4),5),6) and 7), too, are easy. $\square_{2.6}$

2.7 Definition. 1) M is primarily $(\mathbb{D}_{\mathfrak{s}}, \mu)$-constructible over M_0 <u>if</u> $M_0 \leq_{\mathfrak{s}} M \leq_{\mathfrak{s}} \mathfrak{C}$ and there are an ordinal α and models $M_i (i \leq \alpha), N_i (i < \alpha)$ such that:

(*i*) $M = M_\alpha$
(*ii*) M_0 is the given model M_0
(*iii*) $\langle M_i : i \leq \alpha \rangle$ is $\leq_{\mathfrak{s}}$ -increasing continuous
(*iv*) $M_{i+1} = \langle M_i \cup N_i \rangle^{\mathrm{gn}}_{M_{i+1}}$
(*v*) $\|N_i\| < \mu$
(*vi*) N_i is μ-isolated over M_i.

1A) We say $\mathscr{A} = \langle M_i, N_j, N'_j : i \leq \alpha, j < \alpha \rangle$ is a primarily $(\mathbb{D}_{\mathfrak{s}}, \mu)$-construction over M_0 when clauses (i)-(vi) hold and

(*vii*) $N_i \cap M_i \leq_{\mathfrak{s}} N'_i \leq_{\mathfrak{s}} M_i, \|N'_i\| < \mu$
(*viii*) $\langle N_i \cup N'_i \rangle^{\mathrm{gn}}$ is isolated over M_i.

1B) If in (1A) we omit N'_j we mean $N'_j = N_j \cap M_j$.
2) M is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-primary over M_0 <u>when</u>:

(*i*) M is primarily $(\mathbb{D}_{\mathfrak{s}}, \mu)$-constructible over M_0 and
(*ii*) M is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-homogeneous.

3) We say B is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-atomic over A <u>if</u> $A \leq_{\mathfrak{s}} B <_{\mathfrak{s}} \mathfrak{C}$, and for every $B_1 \subseteq B$ of power $< \mu$ for some B_2, $B_1 \subseteq B_2 \leq_{\mathfrak{s}} B, |B_2| < \mu$ and $\mathrm{TP}_*(B_2, A) \in \mathscr{S}_c^{<\mu}(A)$ is μ-isolated.
4) We may omit the "primarily" above.

Remark. See Definition 5.3.

2.8 Claim. *1) If M is primarily $(\mathbb{D}_{\mathfrak{s}}, \mu)$-constructible over M_0 then M is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-primitive over M_0.*
2) If M is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-primary over M_0 then M is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-prime over M_0.

Proof. Put together parts (1),(2),(3) of Fact 2.6 recalling 2.4.

We still do not know if $(\mathbb{D}_{\mathfrak{s}}, \mu)$-prime models exist in non-trivial cases. But now we remedy that situation.

2.9 Lemma - The Existence of Isolated Models. *If $M_0 \le_{\mathfrak{s}} M <_{\mathfrak{s}} \mathfrak{C}, M_0 \le_{\mathfrak{s}} N <_{\mathfrak{s}} \mathfrak{C}$ and $\theta = \|N\| + \chi(\mathfrak{s})$ then there is $N' <_{\mathfrak{s}} \mathfrak{C}$ isolated over M, (i.e., over $(M, N' \cap M)$) with $\|N'\| \le \theta$ such that N can be $\le_{\mathfrak{s}}$-embedded into N' over M_0.*

2.10 Remark. In the first order case, extending a type to an isolated one does not require addition of variables (as really the elements of N serve as variables; elements of M_0 as parameters). In the conclusion of 2.9, variables are members of N' (or $N' \backslash M_0$); parameters are members of $N' \cap M$.

Proof. Suppose this is impossible. We choose by induction on the ordinal $\alpha \le \theta^+$, models M_α, N_α and functions $f_{\beta,\alpha} (\beta < \alpha)$ such that:

$\circledast$ (*a*) $M_\alpha \le_{\mathfrak{s}} M, \|M_\alpha\| \le \theta + |\alpha|$
(*b*) $N_\alpha <_{\mathfrak{s}} \mathfrak{C}, \|N_\alpha\| \le \theta + |\alpha|$
(*c*) M_α is $\le_{\mathfrak{s}}$-increasing continuous with α
(*d*) $M_\alpha \le_{\mathfrak{s}} N_\alpha$
(*e*) for $\beta < \alpha, f_{\beta,\alpha}$ is $\le_{\mathfrak{s}}$-embedding of N_β into N_α such that:
(*f*) $\gamma < \beta < \alpha \Rightarrow f_{\gamma,\alpha} = f_{\beta,\alpha} \circ f_{\gamma,\beta}$

(g) α limit $\Rightarrow N_\alpha = \bigcup_{\beta<\alpha} f_{\beta,\alpha}(N_\beta)$

(h) $f_{\beta,\alpha} \restriction M_\beta = \mathrm{id}_{M_\beta}$

(i) $\neg\mathrm{NF}(M_\alpha, M_{\alpha+1}, f_{\alpha,\alpha+1}(N_\alpha), N_{\alpha+1})$

(j) $N_0 = N$, M_0 as in the lemma

(k) $N_\alpha \cap M = M_\alpha$ for $\alpha > 0$.

The construction follows.

Case 1. $\alpha = 0$.
Let $N_\alpha = N$, $M_\alpha = M_0$.

Case 2. $\alpha = \beta + 1$ (hence $\alpha < \theta$, $\|N_\beta\| \le \theta$).
We look at M_β, N_β as candidates for being $N' \cap M, N'$ in the conclusion of 2.9 with $f_{0,\beta}$ being the embedding. As $M_\beta \le_{\mathfrak{s}} N_\beta, M_\beta \le_{\mathfrak{s}} M, f_{0,\beta}(N) \le_{\mathfrak{s}} N_\beta, M_0 \le_{\mathfrak{s}} M_\beta, \|N_\beta\| \le \theta$, necessarily N_β is not isolated over (M, M_β), <u>hence</u> there is a model N'_β isomorphic to N_β over M_β, say by $h : N_\beta \to N'_\beta$ such that M_β, N'_β, M are not in stable amalgamation (in $\mathfrak{C}$). Now NF is θ-based (by 1.2 and the Definition of $\chi(\mathfrak{s})$) hence by 2.2 there are $M_{\beta+1}, N_{\beta+1}$ such that $M_\beta \le_{\mathfrak{s}} M_{\beta+1} \le_{\mathfrak{s}} M, N'_\beta \le_{\mathfrak{s}} N_{\beta+1}, \|N_{\beta+1}\| + \|M_{\beta+1}\| \le \theta$, $N_{\beta+1} \cap M = M_{\beta+1}$ and $M_{\beta+1}, N_{\beta+1}, M$ is in stable amalgamation. Now by transitivity of NF, that is V.C.1.3 the triple $M_\beta, N'_\beta, M_{\beta+1}$ is not in stable amalgamation. For $\gamma < \alpha$ let $f_{\gamma,\alpha} = h \circ f_{\gamma,\beta}$ stipulating $f_{\beta,\beta} = \mathrm{id}_{N_\beta}$.

Case 3. α a limit ordinal.
Let $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ and N_α, $f_{\beta,\alpha}(\beta < \alpha)$ be a limit of the $\le_{\mathfrak{s}}$-directed system $\langle N_\beta, f_{\beta,\gamma} : \beta < \gamma < \alpha\rangle$ (making $f_{\beta,\alpha} \restriction M_\beta = \mathrm{id}_{M_\beta}$).

Note that $\bigwedge_{\beta<\alpha} M_\beta \le_{\mathfrak{s}} M_\alpha$ by Ax(A4), $M_\alpha \le_{\mathfrak{s}} M$ by smoothness, N_α, $f_{\beta,\alpha}$ exists by Ax(A4) (and chasing arrows), $M_\alpha \le_{\mathfrak{s}} N_\alpha$ by smoothness.

So we finish the induction; let $N^*_\alpha = f_{\alpha,(\theta^+)}(N_\alpha)$. Now $\langle M_\alpha : \alpha \le \theta^+\rangle$ is increasing continuous; also $\langle N^*_\alpha : \alpha \le \theta^+\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous, $M_\alpha \le_{\mathfrak{s}} N^*_\alpha$ for $\alpha \le \theta^+$ and $\neg$ NF$(M_\alpha, M_{\alpha+1}, N^*_\alpha, N^*_{\alpha+1})$ for $\alpha < \theta^+$. So NF is not (θ^+, θ)-based (see V.C.3.10) contradicting 1.2. $\square_{2.10}$

2.11 Conclusion. 1) For $\mu > \chi_{\mathfrak{s}}, M_0 <_{\mathfrak{s}} \mathfrak{C}$ there is a $(\mathbb{D}_{\mathfrak{s}}, \mu)$-primary M over M_0 and $\|M\| \le \|M_0\|^{<\mu}$.
2) Moreover it is witnessed by a primarily $(\mathbb{D}_{\mathfrak{s}}, \mu)$-construction $\mathscr{A}$ such that $|N_i^{\mathscr{A}}| \le \chi_{\mathfrak{s}}$ for $i < \ell g(\mathscr{A})$.

Proof. 1) It is enough to find M of cardinality $\le \|M_0\| < \mu$ which is primarily $(\mathbb{D}_{\mathfrak{s}}, \mu)$-constructible (hence primitive) over M_0 and $(\mathbb{D}_{\mathfrak{s}}, \mu)$-homogeneous (by 2.7(2), 2.8(2)). By facts 2.6(1),(2),(3) (and standard bookkeeping) it suffices to show:

- (A) If $M \in K$, $N_1 \in K$, $N_0 \le_{\mathfrak{s}} M, N_0 \le N_1$ and $\|N_1\| < \mu$ then N_1 can be $\le_{\mathfrak{s}}$-embedded into some M' which is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-primitive over N_0.

Now (A) holds by Lemma 2.9, 2.6(2).
2) The same proof using V.B.3.18 and adapting 2.9 or use Chapter V.E. $\square_{2.11}$

Concerning "simultaneous isolation":

2.12 Fact. Suppose $\langle M_i : i \le i(*)\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous, $N^a \le_{\mathfrak{s}} M_0, N^a \le_{\mathfrak{s}} N^b$. <u>Then</u> we can find a closed $w \subseteq i(*) + 1$ of cardinality $\le \|N^b\| + \chi_{\mathfrak{s}}$ satisfying $0 \in w, i(*) \in w$ and $\langle N_i^a, N_i^b : i \in w\rangle$ such that $N_i^a \le_{\mathfrak{s}} M_i, N^a \le_{\mathfrak{s}} N_0^a, \langle N_i^a : i \in w\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous, $\langle N_i^b : i \in w\rangle$ is increasing continuous, $N_i^a \le_{\mathfrak{s}} N_i^b$, there is a $\le_{\mathfrak{s}}$-embedding of N^b into $N^b_{i(*)}$ over N^a and $\|N_i^a\| \le \|N_i^b\| \le \|N^b\| + \chi_{\mathfrak{s}}$ and for $i \le i(*), N_i^b$ is isolated over (M_j, N_i^a) where $j = \text{Min}\{j \in w : j \ge i\}$.

Proof. We repeat the proof of 2.9 letting $M = M_{i(*)}$ but we add to $\circledast$

- (i) $i_\alpha \le i(*)$ is increasing continuous with α

(j) $M_\alpha \leq_{\mathfrak{s}} M_{i_\alpha}$

(k) if $\alpha = \beta + 1$, then i_α is minimal, (for all possible choices in stage α) so if $i_\beta < i_\alpha$ then N_β is isolated over. $\square_{2.11}$

Remark. What is 2.12 about? It is intended for generalizing 2.10, 2.11 as in [Sh:c, IV,§3]. So it is not really used.

2.13 Conclusion. If $\langle M_i : i \leq \alpha \rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous and $\mu > \chi_{\mathfrak{s}}$ <u>then</u> we can find a $\leq_{\mathfrak{s}}$-increasing sequence $\langle N_i : i \leq \alpha \rangle$ such that for each $i, M_i \leq_{\mathfrak{s}} N_i$, N_i is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-primary over M_i and $\beta \leq \alpha \wedge \text{ cf}(\beta) \geq \mu \Rightarrow M_\beta = \cup\{M_i : i < \beta\}$.

§3 Theory of Types of Models

Previously we have studied the properties of a triple of models, say, N_0, N_1, N_2 which are in stable amalgamation. In this section we consider the collection of all of N'_2 which are conjugate to N_2 by an automorphism of $\mathfrak{C}$ which fixes N_1, this is called the type of N_2 over N_1. We begin by fixing some notation for describing the conjugacy class of a sequence of elements $\bar{a}$. Until Definition 3.15 such a sequence will always enumerate a member of $K_{\mathfrak{s}}$. Throughout this chapter the domain of a type will be always a member of $K_{\mathfrak{s}}$.

In this section we define in our context types, non-forking, stationarization, independence and give their basic properties. This leads to the definition of dimension and Lemma 3.20, which says that $(\mathbb{D}_{\mathfrak{s}}, \mu)$-homogeneity of M is equivalent to "every $p \in \mathscr{S}_c^{\chi(\mathfrak{s})}(N)$ with $N \leq_{\mathfrak{s}} M$ and $\|N\| \leq \chi(\mathfrak{s})$ has dimension $\geq \mu$ in M" (provided that the model M is $(\mathbb{D}_{\mathfrak{s}}, \chi^+_{\mathfrak{s}})$-homogeneous).

3.1 Definition. 1) For sequences $\bar{a}, \bar{b}$ and a set A (from $\mathfrak{C}$) we[2] define $\bar{a}\mathscr{E}_A\bar{b}$ if there is an automorphism f of $\mathfrak{C}$ over A such that $f(\bar{a}) = \bar{b}$.
2) Sometimes we use sets instead of sequences abusing notation in an understandable way.

[2]without smoothness (and amalgamation) this does not make sense)

3.2 Fact. $\mathscr{E}_A$ is an equivalence relation.

3.3 Definition. 1) Let $\mathrm{TP}(\bar{a}, A) = \bar{a}/\mathscr{E}_A$. For a set B, $\mathrm{TP}_*(B, A) = \mathrm{TP}(\langle a_b : b \in B \rangle, A)$ where $a_b = b$. (To conform with old notation, $p_1 \subseteq p_2$ has the inverse meaning, see 3.7(1)). When no confusion arises we may omit the "$*$" from $\mathrm{TP}_*(B, A)$.
2) If $A \subseteq B$ and $p = \mathrm{TP}(\bar{a}, B)$ <u>then</u> $p \restriction A := \mathrm{TP}(\bar{a}, A)$, easily well defined; and $p \subseteq q$ means that for some $A \subseteq B$ and $\bar{a}$ we have $p = \mathrm{TP}(\bar{a}, A), q = \mathrm{TP}(\bar{a}, B)$.

3.4 Remark. If you concentrated on the case of universal classes, you might be misled to use quantifier free-types which seem to lead to problems. If we define stable amalgamation in this way, we may lose uniqueness of NF. Maybe $\mathrm{TP}(N_2, N_1)$ is definable over N_0 (as in I) but $\langle N_1 \cup N_2 \rangle^{\mathrm{gn}} \nleq_{\mathfrak{s}} \mathfrak{C}$). Avoiding this trap is one of the pluses of the axiomatic approach.

We now define the collection of "complete" types over M which are used in this section. Essentially $\mathrm{TP}(\bar{a}, M)$ is a complete type over M if $\bar{a}$ if an enumeration of a model N and $(M \cap N, M, N)$ are in stable amalgamation.

3.5 Definition. 1) $\mathscr{S}^\alpha(A) = \{\mathrm{TP}(\bar{a}, A) : \bar{a} \in {}^\alpha\mathfrak{C}\}$ for $A \subseteq M$.
2) For $M <_{\mathfrak{s}} \mathfrak{C}$,

$$\mathscr{S}^\alpha_c(M) = \Big\{\mathrm{TP}(\bar{a}, M) : \bar{a} \in {}^\alpha\mathfrak{C} \text{ and } \mathrm{Rang}(\bar{a}) \text{ is (the universe of) a } \leq_{\mathfrak{s}}\text{-submodel } N \text{ of } \mathfrak{C} \text{ such that the triple } M \cap N, N, M \text{ is in stable amalgamation}\Big\}.$$

3.6 Remark. 1) Of course, $\mathscr{S}^\mu(A), \mathscr{S}^\mu_c(M)$ depends on $\mathfrak{s}$ and even $\mathfrak{C}$ (which can be avoided).
2) We shall really use just

$$\mathscr{S}^\mu_c(M) \text{ and } \mathscr{S}^m(M)\,(m < \omega).$$

The first is used for the parallel of theorems on stable theories. The latter is used (in Chapter V.E and later) to develop superstability theory and will be more different from the first order case; you have to use more a model N with $M \cup \bar a \subseteq N \leq_{\mathfrak{s}} \mathfrak{C}$ in order to analyze $\mathrm{TP}(\bar a, M)$. The set $\mathscr{S}^\mu_c(M)$ is more "closed in itself".

In conformity with our earlier definitions we define here the notion of a stationarization p_2 of a type p_1 (in better English a stationary extension). However, since we deal only with types over models it suffices, as will be pointed out in Claim 3.8, to deal only with the notion of a non-forking extension.

3.7 Definition. 1) If $p_\ell = \mathrm{TP}_*(N_\ell, M_\ell) \in \mathscr{S}^{<\infty}_c(M_\ell)$, for $\ell = 1,2$ and $M_1 \subseteq M_2$ (equivalently $M_1 \leq_{\mathfrak{s}} M_2$) we say $p_1 \subseteq p_2$ <u>if</u> some N' realizes both. In this case we say $p_1 = p_2 \restriction M_1$ (this is well defined). Similarly for $p_\ell \equiv \mathrm{TP}(\bar a, A_\ell)$ for $\ell = 1,2$.
2) Let $p_\ell \in \mathscr{S}^\mu_c(M_\ell)$ for $\ell = 1,2$ <u>then</u> p_2 <u>is the stationarization</u> of p_1 <u>over</u> M_2 <u>if</u>: there is N realizing p_2 and p_1 and $N \cap M_1, M_2, N$ in stable amalgamation and $M_1 \subseteq M_2$ (so $p_2 \supseteq p_1$ and $M_1 \leq_{\mathfrak{s}} M_2$).
3) $p_\ell \in \mathscr{S}^\mu_c(M_\ell)$, $\ell = 1,2$ are <u>parallel</u> if they have a common stationarization.
4) If $p_1 \subseteq p_2$ (see above) $M_1 \leq_{\mathfrak{s}} M_2$ and $p_\ell = \mathrm{TP}_*(N, M_\ell)$ for $\ell = 1,2$ we say p_2 forks over M_1 <u>if</u> p_2 is not the stationarization of p_1.

The following two claims are obvious from the basic properties of NF in Chapter V.B.

3.8 Claim. *1) If $A_1 \subseteq A_2 \subseteq A_3$ and $p \in \mathscr{S}^{<\infty}(A_3)$ <u>then</u> $p \restriction A_2$ is well defined, $p \restriction A_2 \subseteq p$ and $(p \restriction A_2) \restriction A_1 = p \restriction A_1$ and "$p_1 \subseteq p_2$" is a partial oder.*
2) If $M_1 \leq_{\mathfrak{s}} M_2$ $(\leq \mathfrak{C})$, $p \in \mathscr{S}^\mu_c(M_1)$ <u>then</u> p has one and only one stationarization over M_2, i.e. in $\mathscr{S}^\mu_c(M_2)$.

Proof. Trivial. $\square_{3.8}$

3.9 Claim. *1) Parallelism is an equivalence relation.*
2) If $p, q \in \mathscr{S}_c^{<\infty}(M)$ are parallel <u>then</u> *they are equal.*

Proof. Obvious. $\square_{3.9}$

3.10 Claim. *If N_1, N_2, N_3 are in stable amalgamation,* <u>then</u> $\mathrm{TP}_*(N_2, N_3)$ *is the stationarization of* $\mathrm{TP}(N_2, N_1)$ *over* N_3.

Proof. Check (just use the uniqueness axiom). $\square_{3.10}$

3.11 Claim. *Suppose $A \subseteq \mathfrak{C}$ and $\chi_{\mathfrak{s}} + |A| \le \mu$.* <u>Then</u> *for every M, for some N*

$$A \subseteq N \subseteq \mathfrak{C}, \|N\| \le \mu \ \underline{\text{and}} \ \mathrm{TP}_*(N, M) \in \mathscr{S}_c^{\mu}(M).$$

Proof. Follows from "NF is $(|A|+\chi_{\mathfrak{s}})$-based". See 1.2 (and Definition of $\chi_{\mathfrak{s}}$). $\square_{3.11}$

3.12 Claim. *1) [Symmetry]: If $\mathrm{TP}_*(N_\ell, M) \in \mathscr{S}_c^{<\infty}(M)$ for $\ell = 1, 2$* <u>then</u>*:*
$\mathrm{TP}_*(N_1, \langle M \cup N_2\rangle^{\mathrm{gn}})$ *forks over M if and only if* $\mathrm{TP}_*(N_2, \langle M \cup N_1\rangle^{\mathrm{gn}})$ *forks over M.*
2) If $M_0 \le_{\mathfrak{s}} M_2 \le_{\mathfrak{s}} M_3, M_0 \le_{\mathfrak{s}} M_1$, $\mathrm{TP}_*(M_2, M_1)$ does not fork over M_0, and $\mathrm{TP}_*(M_3, \langle M_2 \cup M_1\rangle^{\mathrm{gn}})$ does not fork over M_2, <u>then</u> $\mathrm{TP}_*(M_3, M_1)$ *does not fork over M_0.*

Proof. 1) Let $N'_\ell = \langle M \cup N_\ell\rangle^{\mathrm{gn}}$ for $\ell = 1, 2$; now by symmetry for NF it is easy to see that $\mathrm{TP}_*(N'_1, N'_2)$ does not fork over M if and only if M, N'_1, N'_2 is in stable amalgamation if and only if $\mathrm{TP}_*(N'_2, N'_1)$ does not fork over M. Now by the symmetry of the situation, it suffices to prove:

(∗) $\mathrm{TP}_*(N'_1, N'_2)$ does not fork over M <u>if and only if</u> $\mathrm{TP}_*(N_1, N'_2)$ does not fork over M.

Proof of $(*)$. only if direction, " $\Rightarrow$ ".

So we are assuming that $\mathrm{TP}_*(N'_1, N'_2)$ does not fork over M. Then M, N'_1, N'_2 are in stable amalgamation but $M \cap N_1, M, N_1$ are also in stable amalgamation; together by transitivity of NF, i.e. V.C.1.3 we get $M \cap N_1, N_1, N'_2$ are in stable amalgamation. So $\mathrm{TP}_*(N_1, N'_2)$ does not fork over $N_1 \cap M$ so inspecting the definitions, by Ax(C4) the type $\mathrm{TP}_*(N_1, N'_2)$ does not fork over M.

if direction, " $\Leftarrow$ ".

We assume $\mathrm{TP}_*(N_1, N'_2)$ does not fork over M; i.e. $\mathrm{TP}_*(N_1, N'_2)$ is the stationarization of $\mathrm{TP}_*(N_1, M)$. So $N_1 \cap M, N_1, N'_2$ is in stable amalgamation; by axiom (C4) the triple M, $\langle N_1 \cup M\rangle^{\mathrm{gn}}, N'_2$ is in stable amalgamation; i.e., M, N'_1, N'_2 is in stable amalgamation so $\mathrm{TP}_*(N'_1, N'_2)$ does not fork over M.

2) Combine 3.12(1) with 3.13(1) below. $\square_{3.12}$

3.13 Claim. *1) Transitivity: If* $M_1 \leq_{\mathfrak{s}} M_2 \leq_{\mathfrak{s}} M_3 \leq_{\mathfrak{s}} \mathfrak{C}$,

$$\mathrm{TP}_*(N, M_3) \in \mathscr{S}_c^{<\infty}(M_3) \textit{ does not fork over } M_2 \textit{ and}$$
$$\mathrm{TP}_*(N, M_2) \in \mathscr{S}_c^{<\infty}(M_2) \textit{ does not fork over } M_1$$

then

$$\mathrm{TP}_*(N, M_3) \in \mathscr{S}_c^{<\infty}(M_3) \textit{ does not fork over } M_1.$$

2) Continuity a) If N_i $(i < \delta)$ *is* $\leq_{\mathfrak{s}}$*-increasing continuous and* $\mathrm{TP}_*(N_i, M) \in \mathscr{S}_c^{<\infty}(M)$ *does not fork over* $M_0 \leq_{\mathfrak{s}} M$ *for each* $i < \delta$ *then* $\mathrm{TP}_*(\bigcup_{i<\delta} N_i, M) \in \mathscr{S}_c^{<\infty}(M)$ *does not fork over* $M_0 \leq_{\mathfrak{s}} M$.

b) If M_i $(i < \delta)$ *is* $\leq_{\mathfrak{s}}$*-increasing continuous,* $\mathrm{TP}_*(N, M_i) \in \mathscr{S}_c^{<\infty}(M_i)$ *does not fork over* M_0 *for* $i < \delta$ *then* $\mathrm{TP}_*(N, \bigcup_{i<\delta} M_i)$ *does not fork over* M_0.

3) Monotonicity a) If $\mathrm{TP}_*(N, M) \in \mathscr{S}_c^{<\infty}(M)$ *does not fork over* $M_0 \leq_{\mathfrak{s}} M$ *and* $M_0 \leq_{\mathfrak{s}} M_1 \leq_{\mathfrak{s}} M_2 \leq_{\mathfrak{s}} M$ *then* $\mathrm{TP}_*(N, M_2) \in \mathscr{S}_c^{<\infty}(M_2)$ *does not fork over* M_1.

b) If $\mathrm{TP}_*(N, M) \in \mathscr{S}_c^{<\infty}(M)$ *does not fork over* $M_0 \leq_{\mathfrak{s}} M$ *and* $M_0 \cap N \leq_{\mathfrak{s}} N_1 \leq_{\mathfrak{s}} N$ *then* $\mathrm{TP}_*(N_1, M) \in \mathscr{S}_c^{<\infty}(M)$ *does not fork over* M_0.

c) If $\mathrm{TP}_*(N, M) \in \mathscr{S}_c^{<\infty}(M)$ *does not fork over* $M_0 \leq_{\mathfrak{s}} M$ *and* $N \cap M_0 \leq_{\mathfrak{s}} N_1 \leq_{\mathfrak{s}} N$ *and* $N \cap M_0 \leq_{\mathfrak{s}} M_{-1}$ *and* $N' = \langle N \cup M_{-1}\rangle^{\mathrm{gn}}$ *and* $M'_0 = \langle N_1 \cup M_0\rangle^{\mathrm{gn}}, M' = \langle N_1 \cup M\rangle^{\mathrm{gn}}$ *then* $\mathrm{TP}(N', M')$ *does not fork over* M'_0*.*

Proof. 1) By transitivity for NF, i.e. V.C.1.3 (and monotonicity, i.e. Ax(C3)(c)); see the proof of $\Rightarrow$ in $(*)$ in the proof of 3.12(1) above.
2) By V.C.1.10 (and symmetry for NF).
3) By monotonicity for NF and base enlargement axiom. $\square_{3.13}$

3.14 Claim. *If* $M = \bigcup_{t \in I} M_t$ *and* $\mu \geq \chi_{\mathfrak{s}}, p \in \mathscr{S}_c^{\mu}(M)$ *and* $[t \leq_I s \Rightarrow M_t \leq_{\mathfrak{s}} M_s <_{\mathfrak{s}} \mathfrak{C}]$ *and* I *is* μ^+*-directed* *then* *for some* t*,* p *does not fork over* M_t*.*

Proof. Easy (let N realize p and $N_0 = N \cap M$, so $N_0 \leq_{\mathfrak{s}} M$ has cardinality $\leq \mu$ hence for some $t \in I$ we have $N_0 \subseteq M_t$ and this t is as required). $\square_{3.14}$

3.15 Definition. 1) We say $\{\bar{a}_\alpha : \alpha < \alpha^*\}$ is independent over M where $\mathrm{TP}_*(\bar{a}_\alpha, M) \in \mathscr{S}_c^{<\infty}(M)$ if $\mathrm{TP}_*(\bar{a}_\alpha, \langle M \cup \bigcup_{\beta \in \alpha^* \backslash \{\alpha\}} \bar{a}_\beta\rangle^{\mathrm{gn}})$ does not fork over M for every $\alpha < \alpha^*$.
2) We say $\{\bar{a}_\alpha : \alpha < \alpha^*\}$ is independent over (M_1, M_0) if $M_0 \leq_{\mathfrak{s}} M_1$ and $\mathrm{TP}(\bar{a}_\alpha, M_1) \in \mathscr{S}_c^{<\infty}(M_1)$ and $\mathrm{TP}_*(\bar{a}_\alpha, \langle M_1 \cup \cup\{\bar{a}_\beta : \beta \in \alpha^* \backslash \{\alpha\}\rangle^{\mathrm{gn}})$ does not fork over M_0 for every $\alpha < \alpha^*$.

Remark. Note that this relation does not satisfy enough axioms to guarantee the classical definition of dimension (just as in [Sh:a, III]).

3.16 Claim. *1) If* $\mathrm{TP}(\bar{a}_\alpha, M) \in \mathscr{S}_c^{<\infty}(M)$ *and* $\mathrm{TP}(\bar{a}_\alpha, \langle M \cup \bigcup_{\beta < \alpha} \bar{a}_\beta\rangle^{\mathrm{gn}})$ *does not fork over* M *for each* $\alpha < \alpha^*$ *then* $\{\bar{a}_\alpha : \alpha < \alpha^*\}$ *is independent over* M*.*

2) If in addition $\alpha < \alpha^*, I$ is directed and $[t \leq_I s \Rightarrow N \leq_{\mathfrak{s}} N_t \leq_{\mathfrak{s}} N_s \leq_{\mathfrak{s}} M \leq_{\mathfrak{s}} \mathfrak{C}], M = \bigcup_t N_t$ and $\mathrm{TP}(\bar{a}_\alpha, N_t) \in \mathscr{S}_c^{<\infty}(N_t)$ does not fork over N for every $t \in I$ <u>then</u> $\mathrm{TP}(\bar{a}_\alpha, M) \in \mathscr{S}_c^{<\infty}(M)$ does not fork over N.
3) If $\{\bar{a}_\alpha : \alpha < \alpha^*\}$ is independent over M or over $(M, M_0), w_1 \subseteq w_2 \subseteq \alpha^*$ <u>then</u> $\{\bar{a}_\alpha : \alpha \in w_2 \backslash w_1\}$ is independent over $\langle M \cup \bigcup_{\alpha \in w_1} \bar{a}_\alpha\rangle^{\mathrm{gn}}$ or over $(\langle M \cup \bigcup\{\bar{a}_\alpha : \alpha \in w_1\}\rangle, M_0)$.
4) If $w_i (i < i(*))$ is $\subseteq$-increasing with i, $\{\bar{a}_\alpha : \alpha \in w_i\}$ is independent over M <u>then</u> $\{\bar{a}_\alpha : \alpha \in \bigcup_i w_i\}$ is independent over M. Hence $\{\bar{a}_\alpha : \alpha < \alpha^*\}$ is independent over M <u>iff</u> $\{a_\alpha : \alpha \in w\}$ is independent over M for every finite $w \subseteq \alpha^*$.
5) If $\mathbf{J}$ is a subset of $\{\bar{b} : \mathrm{TP}(\bar{b}, M) \in \mathscr{S}_c^{<\infty}(M)\}$, $\mathbf{J}_0 \subseteq \mathbf{J}$ is independent over M (possibly $\mathbf{J}_0 = \emptyset$) <u>then</u> among $\{\mathbf{J} : \mathbf{J}_0 \subseteq \mathbf{J} \subseteq \mathbf{J}, \mathbf{J}$ independent over $M\}$ there is a maximal one.
6) If $\{\bar{a}_s : s \in I\}$ is independent over M and $\langle I_t : t \in J\rangle$ is a partition of I (i.e. I is the disjoint union of $\langle I_t : t \in J\rangle$) <u>then</u> $\{\langle \bigcup_{s \in I_t} \bar{a}_s \cup M\rangle^{\mathrm{gn}} : t \in J\}$ is independent over M.

Proof. 1) For each α we prove by induction on $\gamma \leq \alpha^*$, that

$$\mathrm{TP}_*(\bar{a}_\alpha, \langle M \cup \bigcup_{\substack{\beta<\gamma \\ \beta\neq\alpha}} \bar{a}_\beta\rangle^{\mathrm{gn}})$$

does not fork over M. For $\gamma \leq \alpha$ - by monotonicity and assumption. For $\gamma > \alpha$ successor - by symmetry (i.e., 3.12). For γ limit by 3.13(2).
2) Easy by 3.13(2).
3) Immediate by properties of $\langle - \rangle^{\mathrm{gn}}$.
4) Just use part (1) and monotonicity.
5) Immediate from (4).
6) Left to the reader. $\square_{3.16}$

3.17 Claim. *If* $\langle \bar{a}_\alpha : \alpha < \alpha^* \rangle$ *is independent over* M *and* $\mathrm{TP}(\bar{b}, M) \in \mathscr{S}_c^{<\infty}(M)$ *<u>then</u> for some* $w \subseteq \alpha^*$*:*

- (*i*) $|w| \le \chi := |\ell g(\bar{b})| + \chi_{\mathfrak{s}}$
- (*ii*) $\{\bar{a}_\alpha : \alpha < \alpha^*, \alpha \notin w\} \cup \{\bar{b}\}$ *is independent over* M *(in fact for some* N, $\|N\| \le |\bar{b}| + \chi_{\mathfrak{s}}$, $\bigcup\limits_{\alpha \in w} \bar{a}_\alpha \cup \bar{b} \subseteq N$, $\mathrm{TP}_*(N, M) \in \mathscr{S}_c^{<\infty}(M)$ *and* $\{N\} \cup \{\bar{a}_\alpha : \alpha < \alpha^*, \alpha \notin w\}$ *is independent over* M*).*

Proof. Use "NF is χ-based" and 3.16(1). $\square_{3.17}$

3.18 Definition. For $p \in \mathscr{S}_c^{<\infty}(M), M \le_{\mathfrak{s}} N <_{\mathfrak{s}} \mathfrak{C}$ we let

$$\dim(p, N) = \mathrm{Min}\Big\{|\mathbf{J}| : \mathbf{J} \text{ is a maximal family of sequences from } N \text{ realizing } p \text{ which is independent over } M\Big\}.$$

This is well defined by 3.16(5).

3.19 Conclusion. 1) If $M \le_{\mathfrak{s}} N \le_{\mathfrak{s}} \mathfrak{C}, p \in \mathscr{S}_c^{\mu}(M), \mathbf{J}$ is a maximal subset of $p(N)$ independent over M, <u>then</u> $\chi_{\mathfrak{s}} + \mu + \dim(p, M) \ge |\mathbf{J}| \ge \dim(p, N)$ recalling $p(N) = \{\bar{b} \in {}^{\mu}N : \bar{b} \text{ realizes } p\}$.
2) Above for the second inequality the maximality of $\mathbf{J}$ is not used. Hence $N \le_{\mathfrak{s}} N_1 \Rightarrow \dim(p, N) \le \dim(p, N_1) + \chi_{\mathfrak{s}} + \mu$.

Proof. First $|\mathbf{J}| \ge \dim(p, N)$ because $\mathbf{J}$ exemplifies this in the definition of $\dim(p, N)$. Second, assume toward contradiction that $|\mathbf{J}| > \chi_{\mathfrak{s}} + \mu + \dim(p, M)$ and let $\mathbf{J}_1$ exemplify the definition of $\dim(p, N)$, so $|\mathbf{J}_1| = \dim(p, N)$. Let $\mathbf{J} = \{\bar{a}_\alpha : \alpha < \alpha^*\}$. For any $\bar{b}$ which is the concatanation of finitely many members of $\mathbf{J}_1$ let $w_{\bar{b}}$ be as guaranteed by 3.17. So there is $\alpha \in \alpha^* \backslash (\cup \{w_{\bar{b}} : \bar{b} \text{ as above}\})$. Now obviously $\bar{a}_\alpha \notin \mathbf{J}_1$ and $\mathbf{J}_1 \cup \{\bar{a}_\alpha\}$ is independent over M (by 3.16(4)) so $\bar{a}_\alpha$ contradicts the choice of $\mathbf{J}_1$. $\square_{3.19}$

Note also:

3.20 Lemma. *If M is $(\mathbb{D}_{\mathfrak{s}}, \chi_{\mathfrak{s}}^+)$-homogeneous and $\mu > \chi_{\mathfrak{s}}$, <u>then</u> M is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-homogeneous if and only if: for every $N \leq_{\mathfrak{s}} M$, if $\|N\| \leq \chi = \chi_{\mathfrak{s}}$ and $p \in \mathscr{S}_c^{\chi(\mathfrak{s})}(N)$ then $\dim(p, M) \geq \mu$.*

Proof. The direction " $\Rightarrow$ " is easy.

For the other direction " $\Leftarrow$ ", we use V.B.3.18. So let $M_0 \subseteq M, \|M_0\| < \mu, M_0 \leq_{\mathfrak{s}} N_0 <_{\mathfrak{s}} \mathfrak{C}, c \in N_0$. As $\mathfrak{s}$ is $\chi_{\mathfrak{s}}$-based there are $M_1 \leq_{\mathfrak{s}} M_0$ and $N_1 \leq_{\mathfrak{s}} N_0$ such that $c \in N_1, \|M_1\| \leq \|N_1\| \leq \chi_{\mathfrak{s}}$ and M_1, N_1, M_0 is in stable amalgamation. By the assumption applied to $p = \text{TP}_*(N_1, M_1)$, for $\alpha < \mu$ there are $\leq_{\mathfrak{s}}$-embeddings $f_\alpha : N_1 \to N$ over M_1 (i.e. $f_\alpha \restriction M_1 = \text{id}_{M_1}$, with no repetitions, of course) such that $\{f_\alpha(N_1) : \alpha < \mu\}$ is independent over M_1. By 3.17 for some $\alpha, \text{TP}_*(f_\alpha(N_1), M_0)$ does not fork over M_1 so $f_\alpha(N_1)$ realizes $\text{TP}_*(N_1, M_0)$ hence $f_\alpha(c)$ is as required (in the criterion for M being $(\mathbb{D}_{\mathfrak{s}}, \mu)$-homogeneous in V.B.3.18). $\square_{3.20}$

3.21 Conclusion. If $\langle M_i : i < \delta \rangle$ is a $\leq_{\mathfrak{s}}$-increasing chain of $(\mathbb{D}_{\mathfrak{s}}, \mu)$-homogeneous models, and $\text{cf}(\delta) > \chi_{\mathfrak{s}}$ <u>then</u> $M = \bigcup_{i<\delta} M_i$ is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-homogeneous.

Proof. We, of course, use the criterion of 3.20; this is allowed as M is $(\mathbb{D}_{\mathfrak{s}}, \chi_{\mathfrak{s}}^+)$-homogeneous because $\text{cf}(\delta) > \chi_{\mathfrak{s}}$. Let $N \leq_{\mathfrak{s}} M, \|N\| \leq_{\mathfrak{s}} \chi_{\mathfrak{s}}$ and $p \in \mathscr{S}_c^{\chi_{\mathfrak{s}}}(N)$. Now $N \leq_{\mathfrak{s}} M, \|N\| \leq_{\mathfrak{s}} \chi_{\mathfrak{s}}$ implies that for some $i < \delta$, $N \subseteq M_i$ hence $N \leq_{\mathfrak{s}} M_i$. Now $\dim(p, M) \geq \dim(p, M_i) \geq \mu$ and we finish. $\square_{3.21}$

<u>3.22 Exercise</u> Assume

- (a) $\{M_t : t \in I\}$ is independent over M
- (b) π is a permutation of I
- (c) f_t is an isomorphism from M_t onto $M_{\pi(t)}$ for $t \in I$.

<u>Then</u> there is an autormorphism f of $\langle \cup\{M_t : t \in I\}\rangle^{\text{gn}}$ extending $\cup\{f_t : t \in I\}$.
[Why? Prove by induction on $|I|$, using uniqueness and continuity of NF.]

§4 Orthogonality

We introduce here weak orthogonality, orthogonality of a type to a model, and almost orthogonality in our context and give their basic properties.

4.1 Definition. $p_\ell \in \mathscr{S}_c^{<\infty}(M), \ell = 1,2$ are weakly orthogonal ($p_1 \underset{\text{wk}}{\perp} p_2$) <u>when</u>: if $p_\ell = \text{TP}_*(N_\ell, M)$ for $\ell = 1,2$ <u>then</u> M, $\langle N_1 \cup M\rangle^{\text{gn}}, \langle N_2 \cup M\rangle^{\text{gn}}$ are in stable amalgamation.

The next claim is an easy consequence of transitivity.

4.2 Claim. *If $p_\ell \in \mathscr{S}_c^{<\infty}(M)$ for $\ell = 1,2, p_1, p_2$ are not weakly orthogonal and $M \leq_{\mathfrak{s}} M'$ <u>then</u> the stationarizations of p_1, p_2 over M' are not weakly orthogonal.*

4.3 Observation. Assume that $\bar{a}_2 \in {}^{\infty>}\mathfrak{C}$ and $\bar{a}_1 = \bar{a}_2 \restriction w$ list N_1. Then $\mathbf{tp}(\bar{a}_1, M) \in \mathscr{S}_c^{<\infty}(M)$ is a reduct of $\mathbf{tp}(\bar{a}_2, M) \in \mathscr{S}_c^{<\infty}(M)$ <u>iff</u> for some $N_\ell <_{\mathfrak{s}} \mathfrak{C}$ we have $\bar{a}_\ell$ list $N_\ell \cap M, M, N_\ell$ is in stable amalgamation for $\ell = 1,2$ and $N \models (N_1 \cap M, N_1, N_2 \cap M, N_2)$.

Proof. Read the definition.

Remark. We are "better off" than in the first order case as every p is stationary.

Proof of 4.2. By the assumption there are N_ℓ realizing p_ℓ (for $\ell = 1,2$) such that M, $\langle N_1 \cup M\rangle^{\text{gn}}$, $\langle N_2 \cup M\rangle^{\text{gn}}$ are not in stable amalgamation. Choose $N \supseteq M \cup N_1 \cup N_2$ with $N <_{\mathfrak{s}} \mathfrak{C}$. Without loss of generality M, M', N are in stable amalgamation. Clearly $\text{TP}_*(N_\ell, M')$ does not fork over M, hence realizes the stationarization of p_ℓ for $\ell = 1,2$ (by transitivity of non-forking). Assume $M', \langle N_1 \cup M'\rangle^{\text{gn}}$, $\langle N_2 \cup M'\rangle^{\text{gn}}$ are in stable amalgamation, and we shall get a contradiction, this clearly suffices. This implies that $\mathrm{TP}_*(N_1, \langle N_2 \cup M'\rangle^{\mathrm{gn}})$ does not fork over M'. Remember $\mathrm{TP}_*(N_1, M')$ does not fork over M so by 3.13(1), transitivity, $\mathrm{TP}_*(N_1, \langle N_2 \cup M'\rangle^{\mathrm{gn}})$ does not fork over M. By monotonicity $\mathrm{TP}_*(N_1, \langle N_2 \cup M\rangle^{\mathrm{gn}})$ does not fork over M, contradiction. $\square_{4.2}$

4.4 Definition. 1) If $p_\ell \in \mathscr{S}_c^{<\infty}(M)$ for $\ell = 1, 2$ we say p_1 is a <u>reduct</u> of p_2 if there exist $\bar{a}_1, \bar{a}_2$ realizing p_1, p_2 respectively such that $\bar{a}_1$ is a subsequence of $\bar{a}_2$. So if $p_2 \in \mathscr{S}_c^\alpha(M), w \subseteq \alpha$, then $p_2 \restriction w \in \mathscr{S}_c^\alpha(M)$ is naturally (and uniquely) defined (but not always).
2) Let $p_\ell \in \mathscr{S}_c^{<\infty}(M_\ell), \ell = 1, 2$. We say p_1 and p_2 are orthogonal ($p_1 \perp p_2$) <u>if</u> for every M, any $p'_1, p'_2 \in \mathscr{S}_c^{<\infty}(M)$ parallel to p_1, p_2 respectively are weakly orthogonal.

4.5 Claim. *0) If $p_1 = p_2 \restriction w$ (both in $\mathscr{S}_c^{<\infty}(M)$) and $p'_2 \in \mathscr{S}_c^{<\infty}(M')$ is parallel to p_2 then $p'_1 = p'_2 \restriction w$ is parallel to p_1 and also $\restriction w_1, \restriction w_2$ commute.*
1) If p'_ℓ is a reduct of p_ℓ, (for $\ell = 1, 2$) <u>then</u> $p_1 \underset{\mathrm{wk}}{\perp} p_2 \Rightarrow p'_1 \underset{\mathrm{wk}}{\perp} p'_2$ and so $p_1 \perp p_2 \Rightarrow p'_1 \perp p'_2$.
2) Orthogonality of p_1, p_2 depends just on the parallelism type of the p_1, p_2.
3) If $p_1, p_2 \in \mathscr{S}_c^{<\infty}(M)$ are orthogonal <u>then</u> they are weakly orthogonal.

4) If M is $(\mathbb{D}_{\mathfrak{s}}, \chi_{\mathfrak{s}}^+)$-homogeneous $p_1, p_2 \in \mathscr{S}_c^{<\infty}(M)$ <u>then</u>

$$p_1 \underset{\mathrm{wk}}{\perp} p_2 \Leftrightarrow p_1 \perp p_2$$

5) If $\mathrm{TP}_*(N_i, M) \in \mathscr{S}_c^{<\infty}(M), N_i$ is $\leq_{\mathfrak{s}}$-increasing for $i < \delta$ and $p \in \mathscr{S}_c^{<\infty}(M)$ <u>then</u>

$$p \underset{(\mathrm{wk})}{\perp} \mathrm{TP}_*(\bigcup_i N_i, M) \Leftrightarrow \bigwedge_{i<\delta} [p \underset{(\mathrm{wk})}{\perp} \mathrm{TP}(N_i, M)]$$

6) Suppose $p_1, p_2 \in \mathscr{S}_c^{<\infty}(M)$; then $p_1 \underset{(\mathrm{wk})}{\perp} p_2$ <u>if and only if</u> for every pair of reducts p'_1, p'_2 of p_1, p_2 respectively with $\leq \chi_{\mathfrak{s}}$ places each we have $p'_1 \underset{(\mathrm{wk})}{\perp} p'_2$.

7) Suppose $M_0 \leq_{\mathfrak{s}} M, p \in \mathscr{S}_c^{<\mu}(M)$ *does not fork over* M_0, μ *is regular* $> \chi_{\mathfrak{s}}$, *and* $\dim(p \restriction M_0, M) \geq \mu$. *If* $q \in \mathscr{S}_c^{<\mu}(M)$ *is* μ*-isolated* *then* $q \underset{\text{wk}}{\perp} p$.

Proof. 0) Easy.
1) By (0) it suffices to deal with the case of weak orthogonality. So we assume $p_1 \underset{\text{wk}}{\perp} p_2$. So there are $M <_{\mathfrak{s}} \mathfrak{C}$ and $N_1, N_2 <_{\mathfrak{s}} \mathfrak{C}$ such that $p_\ell = \text{TP}_*(N_\ell, M)$. As p'_ℓ is a reduct of p_ℓ there are $N'_\ell \leq_{\mathfrak{s}} N_\ell$ such that $p'_\ell = \text{TP}_*(N'_\ell, M)$ for $\ell = 1, 2$, and p_1, p_2, p'_1, p'_2 all belong to $\mathscr{S}_c^{<\infty}(M)$. So $N_\ell \cap M$, N_ℓ, M is in stable amalgamation as well as $N'_\ell \cap M$, N'_ℓ, M (for $\ell = 1, 2$). Let $M_\ell = \langle N_\ell \cup M\rangle^{\text{gn}}$, $M'_\ell = \langle N'_\ell \cup M\rangle^{\text{gn}}$, so $M \leq_{\mathfrak{s}} M'_\ell \leq_{\mathfrak{s}} M_\ell$ for $\ell = 1, 2$..

Suppose p'_1, p'_2 are not weakly orthogonal. Then there is a $\leq_{\mathfrak{s}}$-embedding g (into $\mathfrak{C}$), $\text{Dom}(g) = M'_2, g \restriction M = \text{id}_M$ and $M, M'_1, g(M'_2)$ are not in stable amalgamation. As $\mathfrak{C}$ is homogeneous there is a $\leq_{\mathfrak{s}}$-embedding h (into $\mathfrak{C}$) extending g, $\text{Dom}(h) = M_2$. As $p_1 \underset{\text{wk}}{\perp} p_2$, we know that the triple M, M_1, $h(M_2)$ is in stable amalgamation, and by monotonicity we get a contradiction.
2) Easy.
3) By the definition and 3.9(2).
4) The direction " $\Leftarrow$ " holds by part (3). So let us prove " $\Rightarrow$ ", so we are assuming $p_1 \underset{\text{wk}}{\perp} p_2$ and $p_1 \pm p_2$ and we shall get a contradiction. So there are $M' \leq_{\mathfrak{s}} \mathfrak{C}$ and $p'_1, p'_2 \in \mathscr{S}_c^{<\infty}(M')$ parallel to p_1, p_2 respectively, such that p'_1, p'_2 are not weakly orthogonal. By 4.2 without loss of generality $M \leq_{\mathfrak{s}} M'$. For $\ell = 1, 2$ let N_ℓ be such that $p'_\ell = \text{TP}_*(N_\ell, M')$. By part (6), (i.e., (5) and (1)) of 4.5 without loss of generality $\|N_\ell\| \leq \chi_{\mathfrak{s}}$ recalling 1.7(1). We can find $N \leq_{\mathfrak{s}} \mathfrak{C}$ such that $N_1 \cup N_2 \subseteq N$, and $N \cap M', N, M'$ are in stable amalgamation and $\|N\| \leq \chi_{\mathfrak{s}}$. By 1.10(2) without loss of generality also $N \cap M, N, M$ are in stable amalgamation. Easily by 2.2 transitivity also $N \cap M'$, $\langle N_1 \cup (N \cap M')\rangle^{\text{gn}}$, $\langle N_2 \cup (N \cap M')\rangle^{\text{gn}}$ are not in stable amalgamation. Now by the assumption on M there is a $\leq_{\mathfrak{s}}$-embedding h_0 of $N \cap M'$ into M over $N \cap M$. We can extend h_0 to an automorphism g_0 of $\mathfrak{C}$ such that the triple $h_0(N \cap M'), g_0(N), M$ is in stable amalgamation. Easily $\text{TP}_*(g_0(N_\ell), M) = p'_\ell$ for $\ell = 1, 2$ and the rest should also be

clear, contradicting $p_1 \underset{\text{wk}}{\perp} p_2$.
5) For $\underset{\text{wk}}{\perp}$ by 3.13(2). For $\perp$ it follows.
6) First we deal with the $\underset{\text{wk}}{\perp}$ version. The implication $\Rightarrow$ is by part (1) and the implication $\Leftarrow$ by applying 1.18. We then deduce the equivalence for the $\perp$ version.
7) Easy too. $\square_{4.5}$

4.6 Definition. 1) Assume $p \in \mathscr{S}_c^{<\infty}(N)$. We say $p \perp M$ (p orthogonal to M) <u>if</u> p is orthogonal to every q which $\in \mathscr{S}_c^{<\infty}(N')$ for some N' which does not fork over M.
2) $p \underset{\text{a}}{\perp} M$ (p is almost orthogonal to M) <u>if</u> $p \in \mathscr{S}_c^{<\infty}(N)$ where $M \le_{\mathfrak{s}} N$ and p is weakly orthogonal to every $q \in \mathscr{S}_c^{<\infty}(N)$ which does not fork over M.
3) Assume $\mathscr{A} = \langle A, N_i, w_i : i < \alpha \rangle$ and $\kappa = \text{ cf}(\kappa) > \chi_{\mathfrak{s}} + \sup\{\|N_i\| : i < \alpha\}$ and $\mu > |A| + \chi_{\mathfrak{s}} + \sum\limits_{i<\alpha} \|N_i\|$. Then $\mathscr{A}$ is a $(< \mu)$-stable construction iff $\mathscr{A}$ is a stable $(\mathbb{D}_{\mathfrak{s}}, \mu, \kappa)$-construction.

Remark. 1) Why in 5.7(2) we cannot add $B_j^{\mathscr{A}^*} = N'_j$?

This is because in Definition 2.7(1A) we do not have the parallel to "w_j is $\mathscr{A}$-closed" in V.C.4.2. Of course, we could have made other choices in those definitions with no noticable difference in the results.
2) By 5.7(2),(3), results on stable $(\mathbb{D}_{\mathfrak{s}}, \mu, \kappa)$ constructions can be translated to results on $(< \mu)$-stable constructions and on primarily $(\mathbb{D}_{\mathfrak{s}}, \mu)$-constructions.

Again the relevant information from [Sh:c, V,X] generalizes, e.g.

4.7 Claim. *1) Monotonicity. If $p_1 \perp M_1$, p_0 is a reduct of p_1 and $M_0 \le_{\mathfrak{s}} M_1$, <u>then</u> $p_0 \perp M_0$. Also the parallel of 4.5(6) holds.*
2) Similarly for $\underset{\text{a}}{\perp}$.
2A) If $M_1 \le_{\mathfrak{s}} M_2 \le_{\mathfrak{s}} N_2, M_1 \le_{\mathfrak{s}} N_1 \le_{\mathfrak{s}} N_2$ and $p \in \mathscr{S}_c^{<\infty}(N_2)$ is almost orthogonal to M_2 <u>then</u> $p \restriction N_1$ is almost orthogonal to M_1.
3) If M is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-saturated, for $\alpha < \alpha(*)$ we have $\text{TP}(\bar{a}_\alpha, M) \in$

$\mathscr{S}_c^{<\infty}(M), \chi_{\mathfrak{s}} < \mu = \mathrm{cf}(\mu)$ and N_α is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-constructible (or $(\mathbb{D}_{\mathfrak{s}}, \mu)$-primary or $(\mathbb{D}, \mu)$-prime) over $M \cup \bar{a}_\alpha$ and $\{\bar{a}_\alpha : \alpha < \alpha^*\}$ is independent over M then:

(a) *if $q \in \mathscr{S}_c^{<\infty}(\langle M \cup \bar{a}_\alpha\rangle^{\mathrm{gn}})$ is realized in N_α then $q \underset{\mathrm{a}}{\perp} M$*

(b) *$\{N_\alpha : \alpha < \alpha^*\}$ is independent over M.*

Proof. Easy (e.g. for (2A), deal first with the case $M_1 = M_2$ and then with the case $N_1 = N_2$). $\square_{4.7}$

§5 Uniqueness of $(\mathbb{D}_{\mathfrak{s}}, \mu)$-Primary Models

We first prove in 5.1 that a restriction of an $(\mathbb{D}, \mu)$-isolated type is still isolated (if it does not fork) and similarly a reduct of a restriction, 5.4. The rest of this section is parallel to [Sh:c, IV,§3], defining and giving the basic properties of $(\mathbb{D}_{\mathfrak{s}}, \mu)$-constructions and primary models.

5.1 Lemma. *Suppose $\mathrm{TP}_*(N, M)$ does not fork over M_0 and N is μ-isolated over (M, M_0), see Definition 2.5(2), $\mu > \chi_{\mathfrak{s}}$ and, of course, $\|N\| < \mu$. If $M_0 \leq_{\mathfrak{s}} M^* \leq_{\mathfrak{s}} M$ then N is μ-isolated over (M^*, M_0).*

5.2 Remark. This generalizes [Sh:c, Ch.IV,4.2,4.3,pg.183,184]; so it is natural.

5.3 Definition. We say $p \in \mathscr{S}_c^{<\infty}(M)$ is μ-isolated if $p \in \mathscr{S}_c^{<\mu}(M)$, and for any ($\equiv$ some) N realizing it, N is μ-isolated over $(M, M \cap N)$.

Proof of 5.1. By the definition of N being μ-isolated over (M, M_0) from 2.5(2) there is M_1 satisfying $M_0 \leq_{\mathfrak{s}} M_1 \leq_{\mathfrak{s}} M$ such that $\langle N \cup M_1\rangle^{\mathrm{gn}}$ is isolated over (M, M_1) and $\mu > \|\langle N \cup M_1\rangle^{\mathrm{gn}}\|$. By $\mathfrak{s}$ being μ-based we can find $M_2, M_1 \leq_{\mathfrak{s}} M_2 \leq_{\mathfrak{s}} M, \|M_2\| = \|M_1\| + \chi_{\mathfrak{s}} < \mu$ such that $\mathrm{TP}_*(M_2, M^*) \in \mathscr{S}_c^{<\infty}(M^*)$ does not fork over $M_2 \cap M^*$.

By monotonicity and based enlargement (Axiom (C4)), we know that $\langle N \cup M_2 \rangle^{\rm gn}$ is isolated over (M, M_2). We shall show that $\langle N \cup (M_2 \cap M^*)\rangle^{\rm gn}$ is isolated over $(M^*, M^* \cap M_2)$ thus finishing as $\mathrm{TP}_*(N, M^*)$ does not fork over $M_0, M^* \cap M_2 \leq_{\mathfrak{s}} M^*$, by 3.13(3).

Suppose this fails so there is a $\leq_{\mathfrak{s}}$-embedding $g : \langle N \cup (M_2 \cap M^*)\rangle^{\rm gn} \rightarrow \mathfrak{C}$ such that $g \restriction (M_2 \cap M^*) =$ id and $\mathrm{TP}_*(\mathrm{Rang}(g), M^*)$ forks over $M^* \cap M_2$. Without loss of generality for some N^* we have $M^* \cup \mathrm{Rang}(g) \subseteq N^* \leq_{\mathfrak{s}} \mathfrak{C}$ and $\mathrm{TP}_*(N^*, M)$ does not fork over M^*. By 3.13(3)(a), i.e. monotonicity, the type $\mathrm{TP}_*(N^*, \langle M^* \cup M_2\rangle^{\rm gn})$ does not fork over M^*. As also $\mathrm{TP}_*(M^*, M_2)$ does not fork over $M_2 \cap M^*$ [as $\mathrm{TP}_*(M_2, M^*) \in \mathscr{S}_c^{<\infty}(M^*)$ does not fork over $M_2 \cap M^*$ and symmetry], and $M_2 \cap M^* \leq_{\mathfrak{s}} M^* \leq_{\mathfrak{s}} N^*$ clearly by 3.12(2) the type $\mathrm{TP}(N^*, M_2)$ does not fork over $M_2 \cap M^*$. By monotonicity it follows that $\mathrm{TP}_*(g(N), M_2)$ does not fork over $M_2 \cap M^*$, hence it is equal to $\mathrm{TP}_*(N, M_2)$. As $M_1 \leq_{\mathfrak{s}} M_2$ clearly $\mathrm{TP}_*(g(N), M_1) = \mathrm{TP}_*(N, M_1)$, hence by the choice of M_1 we know that $\mathrm{TP}_*(g(N), M)$ does not fork over M_0, hence by monotonicity $\mathrm{TP}_*(g(N), M^*)$ does not fork over M_0 hence over $M^* \cap M_2$, contradicting the choice of g. $\square_{5.1}$

5.4 Claim. *Suppose* $M_0 \leq_{\mathfrak{s}} M_1 \leq_{\mathfrak{s}} M, N_0 \leq_{\mathfrak{s}} N_1$, $\mathrm{TP}_*(N_\ell, M)$ *does not fork over* M_ℓ *for* $\ell = 0, 1$ *and* $\langle N_1 \cup M_1\rangle^{\rm gn}$ *is isolated over* (M, M_1) <u>*then*</u> $\langle N_0 \cup M_1\rangle^{\rm gn}$ *is isolated over* (M, M_1).

Proof. By the proof of 4.5(1). $\square_{5.4}$

5.5 Fact. Assume $N_0 \leq_{\mathfrak{s}} N_1, N_2$ and N_0, N_1, N_2 is in stable amalgamation. <u>Then</u> $\mathrm{TP}(N_1, N_2)$ is isolated over (N_2, N_0) <u>if and only if</u> $\mathrm{TP}(N_1, N_0) \underset{\rm wk}{\perp} \mathrm{TP}(N_2, N_0)$.

Proof. Easy (by the definitions). $\square_{5.5}$

We refine the notion of a stable construction (V.C.4.2). This is an elaboration of Definition 2.7, we need it for proving uniqueness, etc. The reader can restrict himself to the case $\kappa = \mu$ hence κ is regular.

5.6 Definition. 1) We say $\mathscr{A} = \langle A, N_i, w_i : i < \alpha\rangle$ is a stable $(\mathbb{D}_{\mathfrak{s}}, \mu, \kappa)$-construction, when (always $\mu \geq \kappa > \chi_{\mathfrak{s}}$ and for simplicity even $\mathrm{cf}(\mu) > \kappa = \mathrm{cf}(\kappa) > \chi_{\mathfrak{s}}$):

- (a) $\mathscr{A}$ is a stable construction (inside $\mathfrak{C}$) (see V.C.4.2, so we can use the notation there)
- (b) $\|N_i\| < \kappa, |w_i| < \kappa, A <_{\mathfrak{s}} \mathfrak{C}$
- (c) $\mathrm{TP}(N_i, A^{\mathscr{A}}_{\{j:j<i\}}) \in \mathscr{S}^{<\infty}_c(A^{\mathscr{A}}_{\{j:j<i\}})$ is μ-isolated.

2) We say N is stably $(\mathbb{D}_{\mathfrak{s}}, \mu, \kappa)$-constructible over M <u>if</u> for some stable $(\mathbb{D}_{\mathfrak{s}}, \mu, \kappa)$-construction $\mathscr{A}$ we have $M = A^{\mathscr{A}}, N = A^{\mathscr{A}}_{\ell g(\mathscr{A})}$.
3) We say B is $(\mathbb{D}_{\mathfrak{s}}, \mu, \kappa)$-atomic over A <u>if</u> $A \leq_{\mathfrak{s}} B <_{\mathfrak{s}} \mathfrak{C}$, and for every $B_1 \subseteq B$ of power $< \kappa$ for some $B_2, B_1 \subseteq B_2 \leq_{\mathfrak{s}} B, |B_2| < \kappa$ and $\mathrm{TP}_*(B_2, A) \in \mathscr{S}^{<\mu}_c(A)$ is μ-isolated.

5.7 Claim. *1) The parallel of V.C.4.6 holds; i.e., we can change the order of the construction.*
2) If $\mathscr{A} = \langle M_i, N_j, N'_j : i \leq \alpha, j < \alpha\rangle$ is a primarily $(\mathbb{D}_{\mathfrak{s}}, \mu)$-construction, see Definition 2.7(1A), μ is regular $> \chi_{\mathfrak{s}}$ <u>then</u> we can find a stable $(\mathbb{D}_{\mathfrak{s}}, \mu, \mu)$-construction $\mathscr{A}'$ such that $\ell g(\mathscr{A}') = \alpha$ and $A^{\mathscr{A}'}_i = M_i$ for $i \leq \alpha, N'_j \leq_{\mathfrak{s}} B^{\mathscr{A}'}_j$ for $j < \alpha$ for every $\alpha \leq \ell g(\mathscr{A})$.
3) Assume $\mathscr{A} = langleA, N_i, w_i : i < \alpha\rangle$ and $\kappa = cf(\kappa) > \chi_{\mathfrak{s}} + \sup\{\|N_i\| : i < \alpha\}$ and $\mu > |A| + \chi_{\mathfrak{s}} + \sum\limits_{i<\alpha} \|N_i\|$. Then $\mathscr{A}$ is a $(< \mu)$-stable construction iff $\mathscr{A}$ is a stable $(\mathbb{D}_{\mathfrak{s}}, \mu, \kappa)$-construction.

Proof. 1) Use the weak orthogonality from §4.
2),3) Easy. $\square_{5.7}$

Remark. 1) Why in 5.7(2) we cannot add $B^{\mathscr{A}'}_j = N'_j$? This is because in Definition 2.7(1A) we do not have the parallel to "w_j is $\mathscr{A}$-closed" in V.C.4.2. Of course, we could have made other choices in those definitions with no noticable difference in the results.
2) By 5.7(2),(3), results on stable $(\mathbb{D}_{\mathfrak{s}}, \mu, \kappa)$ constructions can be translated to results on $(< \mu)$-stable constructions and on primarily $(\mathbb{D}_{\mathfrak{s}}, \mu)$-constructions.

5.8 Remark. On existence see 2.11(2).

5.9 Fact. 1) If $\beta \le \alpha$ and $\mathscr{A} = \langle A_0, N_i, w_i : i < \alpha \rangle$ and $j \in w_i \Rightarrow w_j \subseteq w_i$, then: $\mathscr{A}$ is a stable $(\mathbb{D}_{\mathfrak{s}}, \mu, \kappa)$-construction, <u>iff</u> $\mathscr{A} \restriction \beta$ is a $(\mathbb{D}_{\mathfrak{s}}, \mu, \kappa)$-construction and $\mathscr{A}' = \langle A^{\mathscr{A}}_{\beta}, N^{\mathscr{A}}_{\beta+i}, w'_i : i < \alpha - \beta \rangle$ is a stable $(\mathbb{D}_{\mathfrak{s}}, \mu, \kappa)$-construction where $w'_i = \{\gamma : \beta + \gamma \in w_{\beta+i}\}$.
2) Let $A \subseteq B \subseteq C$; if C is $(\mathbb{D}_{\mathfrak{s}}, \mu, \kappa)$-constructible over B and B is $(\mathbb{D}_{\mathfrak{s}}, \mu, \kappa)$-constructible over A <u>then</u> C is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-constructible over A. Similarly for atomic.
3) If $\mathrm{NF}(N \cap A, N, A, \mathfrak{C})$, $\|N\| < \mu$, $\mathrm{TP}_*(N, A)$ is isolated over $N \cap A$ and $B = \langle A \cup N \rangle^{\mathrm{gn}}$ <u>then</u> B is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-constructible over A.
4) If $\mathscr{A}$ is a stable $(\mathbb{D}_{\mathfrak{s}}, \mu_1, \kappa_1)$-construction and $\mu_2 \ge \mu_1, \kappa_1 \ge \kappa_2 = \mathrm{cf}(\kappa_2) > \chi_{\mathfrak{s}}$ and $\mathrm{cf}(\mu_2) \ge \mu_2$ <u>then</u> $\mathscr{A}$ is a stable $(\mathbb{D}_{\mathfrak{s}}, \mu_2, \kappa_2)$-construction.

Proof. Should be clear, (on part (2) see [Sh:c, 3.2](4). $\square_{5.9}$

5.10 Fact. If M is primarily $(\mathbb{D}_{\mathfrak{s}}, \mu)$-constructible (see 2.7) over M_0, as exemplified by $\langle M_i, N_j, N'_j : i \le \alpha, j < \alpha \rangle, \mu > \chi_{\mathfrak{s}}$ and $\kappa = \sup(\{\|N_i\|^+ : i < \alpha\} \cup \{\chi^+_{\mathfrak{s}}\})$, <u>then</u> for some stable $(\mathbb{D}_{\mathfrak{s}}, \mu, \kappa)$-construction $\mathscr{A} = \langle M_0, N_i, w_i : i < \alpha \rangle$ we have $M = A^{\mathscr{A}}_{\alpha}$ (i.e., $\langle M \cup \bigcup_{i<\alpha} N_i \rangle^{\mathrm{gn}}$) (see V.C.4.2).

Proof. By $\mathfrak{s}$ being χ-based for $\chi \ge \chi_{\mathfrak{s}}$. $\square_{5.10}$

5.11 Claim. *If M is stably $(\mathbb{D}_{\mathfrak{s}}, \mu, \kappa)$-constructible over M_0, <u>then</u> M is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-atomic over M_0.*

Remark. Here we use $\mathrm{cf}(\mu) \ge \kappa = \mathrm{cf}(\kappa)$.

Proof. Let $A \subseteq M, |A| < \kappa$. By the definition and 5.10 there is a stable $(\mathbb{D}_{\mathfrak{s}}, \mu, \kappa)$-construction $\mathscr{A} = \langle M_0, N_i, w_i : i < \alpha \rangle$, $A^{\mathscr{A}}_{\alpha} = M$

and so $A^{\mathscr{A}} = M_0$. Clearly $\{A^{\mathscr{A}}_u : u \subseteq \alpha$ is closed for $\mathscr{A}$, $|u| < \kappa\}$ is a κ-directed family of $\leq_{\mathfrak{s}}$-submodels of M with union M, so for some $\mathscr{A}$-closed $u \subseteq \alpha$ of cardinality $< \kappa$ we have $A \subseteq A^{\mathscr{A}}_u (\leq_{\mathfrak{s}} M)$, see Definition V.C.4.2. By 1.2 there is $N \leq_{\mathfrak{s}} M, \|N\| \leq \chi_{\mathfrak{s}} + |\cup\{N_i : i \in u\}| < \mu, \bigcup\limits_{i \in u} N_i \subseteq N$, $\mathrm{TP}_*(N, M_0)$ does not fork over $N \cap M_0$. We can now prove by induction on $\beta \leq \alpha$ that $\langle N \cap M_0 \cup \cup\{N_i : i \in u \cap \beta\}\rangle^{\mathrm{gn}}$ is isolated over $(M_0, N \cap M_0)$. $\square_{5.11}$

5.12 Fact. Let $\mathrm{cf}(\mu) \geq \kappa = \mathrm{cf}(\kappa) > \chi_{\mathfrak{s}}$ be regular.
1) If $\mathscr{A}$ is a stable $(\mathbb{D}_{\mathfrak{s}}, \mu, \kappa)$-construction, $u \subseteq \alpha := \ell g(\mathscr{A})$ is $\mathscr{A}$-closed, $A \subseteq M := A^{\mathscr{A}}_\alpha$, $|A| < \mu$, then for some $N \leq_{\mathfrak{s}} M, \|N\| < \mu$, $A \subseteq N$ and N is isolated over $A^{\mathscr{A}}_u$.
2) If M is stably $(\mathbb{D}_{\mathfrak{s}}, \mu, \kappa)$-constructible over $M_0, N \leq_{\mathfrak{s}} M, \|N\| < \kappa$ and $\mathrm{TP}_*(N, M) \in \mathscr{S}^{<\mu}_c(M)$ then M is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-constructible over $\langle M_0 \cup N\rangle^{\mathrm{gn}}$.
3) If $M_0 \leq_{\mathfrak{s}} M_1 \leq_{\mathfrak{s}} M_2$ and $M_{\ell+1}$ is $(\mathbb{D}_{\mathfrak{s}}, \mu, \kappa)$-atomic over M_ℓ for $\ell = 0, 1$ then M_2 is $(\mathbb{D}_{\mathfrak{s}}, \mu, \kappa)$-atomic over M_0.

Proof. 1) By 5.7(1) without loss of generality u is an initial segment of α, by 5.9(1) without loss of generality $u = \emptyset$, and then apply 5.11.
2) As in the proof of 5.11 there is an $\mathscr{A}$-closed $u \subseteq \ell g(\mathscr{A})$ of cardinality $< \mu$ such that $N \subseteq A^{\mathscr{A}}_u$. By 5.7(1) without loss of generality u is an initial segment $\{i : i < \beta\}$ of $\ell g(\mathscr{A})$. There is $M^* \leq_{\mathfrak{s}} M_0, |M^*| < \kappa$ such that $\bigwedge\limits_{i<\beta} N_i \cap M_0 \subseteq M^*$, and $N \subseteq \langle M^* \cup \bigcup\limits_{i<\beta} N_i\rangle^{\mathrm{gn}}$. Define by induction on $i \leq \beta$, M^*_i, by: for $i = 0$ let $M^*_i = M^*$, for limit i let $M^*_i = \bigcup\limits_{j<i} M^*_j$ and if $i = j+1$, $M^*_i = \langle M^*_j \cup N_j\rangle^{\mathrm{gn}}$. Clearly $\|M^*_i\| < \kappa$ for $i \leq \beta$. By 5.11 the type $\mathrm{TP}(M^*_\beta, M_0)$ is μ-isolated. As $\|M^*_\beta\| < \kappa$, and the assumptions on N, by 2.6(6) the type $\mathrm{TP}_*(M^*_\beta, \langle M_0 \cup N\rangle^{\mathrm{gn}})$ is μ-isolated. In other words $A^{\mathscr{A}}_\beta$ is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-constructible over $\langle M \cup N\rangle^{\mathrm{gn}}$ by a construction of length 1 and by 5.9(1) + 5.7 the model $A^{\mathscr{A}}_{\ell g(\mathscr{A})}$ is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-constructible over $A^{\mathscr{A}}_\beta$. So by 5.9(2) we finish.
3) Left to the reader.

5.13 Lemma. *If* $\mathrm{cf}(\mu) \geq \kappa = \mathrm{cf}(\kappa) > \chi_{\mathfrak{s}}$ *is regular and* $M_0 <_{\mathfrak{s}} \mathfrak{C}$, *then any two* $(\mathbb{D}_{\mathfrak{s}}, \mu, \kappa)$*-primary models over* M_0 *are isomorphic.*

Proof. As in [Sh:c, IV,3.8,3.9].

* * *

5.14 Discussion. 1) A uniqueness of $(\mathbb{D}_{\mathfrak{s}}, \mu)$-prime models (as in [Sh:c, Ch.IV,§4]) would be a better result. See Section 6. The uniqueness without characterization (as in [Sh:c, Ch.IV,§5]) we had not looked at it as χ is quite large anyhow; and the main point there was doing it for models rather than quite saturated models.
2) For universal classes "prime among models" is not such a good notion: if $M \not\leq_{\mathfrak{s}} \mathfrak{C}$ it is not a good object and we know too much if $M \leq_{\mathfrak{s}} \mathfrak{C}$ then M is prime over itself.

§6 Uniqueness of $(\mathbb{D}_{\mathfrak{s}}, \mu)$-Prime Models

So now we can deal with prime models; we do not try to generalize the theorem of the uniqueness of $(\mathbb{D}_{\mathfrak{s}}, \mu)$-prime models when $\mu \leq \chi_{\mathfrak{s}}$. (The case of $\mathrm{cf}(\mu) > \chi_{\mathfrak{s}}$ is proved just like the case μ is regular because of 5.1).

We, of course, imitating [Sh:c, IV,§4].

6.1 Fact. If $\mu > \chi_{\mathfrak{s}}$ is regular, N is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-prime over M, $N_0 \leq_{\mathfrak{s}} N$, $\mathrm{TP}(N_0, M) \in \mathscr{S}_c^{<\mu}(M)$ then N is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-prime and $(\mathbb{D}_{\mathfrak{s}}, \mu)$-atomic over $\langle M \cup N_0 \rangle^{\mathrm{gn}}$.

Proof. We can find N_1 which is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-primary over M by 2.11; without loss of generality N_1 satisfies $N \leq_{\mathfrak{s}} N_1$. [Why? As by 2.8(2) primary $\Rightarrow$ prime and the definition of prime.] So by 5.12(2) the model N_1 is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-constructible over $\langle M \cup N_0 \rangle^{\mathrm{gn}}$, hence the model N_1 is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-prime over $\langle M \cup N_0 \rangle^{\mathrm{gn}}$ hence by 2.6(7) the model N is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-prime over $\langle M \cup N_0 \rangle^{\mathrm{gn}}$ as required. $\square_{6.1}$

6.2 Lemma. *Assume*

(*i*) N *is* $(\mathbb{D}_{\mathfrak{s}}, \mu)$*-prime over* M

(*ii*) $\mu > \chi_{\mathfrak{s}}$ *(*μ *regular for simplicity),*

(*iii*) $(M' \cap M, M', M)$ *is in stable amalgamation*

(*iv*) $M' \leq_{\mathfrak{s}} N$, $\|M'\| < \mu$,

(*v*) $p \in \mathscr{S}_c^{<\mu}(\langle M \cup M' \rangle_N^{\rm gn})$.

1) $\dim(p, N) \leq \mu$.
2) Moreover, there is $M'' \leq_{\mathfrak{s}} N$ *such that:*

(*a*) $\|M''\| \leq \mu$, $M' \subseteq M$"

(*b*) $M'' \cap M, M'', M$ *is in stable amalgamation*

and

(*c*) *the stationarization of* p *over* $\langle M \cup M'' \rangle^{\rm gn}$ *has a unique extension over* N.

Proof. It suffices to deal with N which is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-primary over M (for part (2) use 5.1). By the definition, N is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-constructible over M, say the $(\mathbb{D}_{\mathfrak{s}}, \mu)$-construction $\mathscr{A}$ witness this. Clearly $M' \subseteq A_u^{\mathscr{A}}$ for some $\mathscr{A}$-closed u of cardinality $< \mu$, by 5.7 without loss of generality $u = \beta < \mu$.

Assume towards a contradiction that $\dim(p, N) > \mu$ (or just $\dim(p, N) \geq \mu$, which may occur). Now by 5.7 without loss of generality there is $S \subseteq \mu$, $|S| = \mu$ and $\langle M_\alpha : \alpha \in S \rangle$ is $\leq_{\mathfrak{s}}$-increasing, $M_\alpha \leq_{\mathfrak{s}} A_\alpha^{\mathscr{A}}$, $\|M_\alpha\| < \mu$, p has a stationarization p_α over M_α which N_α realizes and $\bigcup\limits_{\beta < \alpha} N_\beta \subseteq M_\alpha$, $\mathrm{TP}(M_\alpha, M)$ does not fork over $M_\alpha \cap M$. Now we prove that: if $q \in \mathscr{S}_c^{<\mu}(A_\mu^{\mathscr{A}})$ is parallel to p then $q \underset{\rm wk}{\perp} \mathrm{TP}(A_\gamma^{\mathscr{A}}, A_\mu^{\mathscr{A}})$ for $\gamma \geq \mu$.

We prove this by induction on γ: use 4.5(7) for successor γ, 4.5(5) for limit γ. The rest should be clear (or see [Sh:c, IV,4.9]). $\square_{6.2}$

6.3 Theorem. *For $\mu > \chi_{\mathfrak{s}}$ regular, over any M:*
1) All $(\mathbb{D}_{\mathfrak{s}}, \mu)$-prime models over M are isomorphic over M.
2) If N satisfies the conclusion of the last lemma and $(\forall A)(A \subseteq M)(|A| \leq \chi(\mathfrak{s}) \Rightarrow \exists M'(\mathrm{TP}(M', M) \in \mathscr{S}_c^{\chi(\mathfrak{s})}(M)$ is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-isolated, $A \subseteq M'))$ then N is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-prime model over M.

Remark. By V.B.3.18 we can restrict ourselves to types of singletons (see ChV). More easily to types in $\leq \chi(\mathfrak{s})$-variables.

Proof. Similar to [Sh:a, IV,§4].

6.4 Exercise: Generalize 5.11 - 5.13 to the case $\mu >$ cf$(\mu) > \chi_{\mathfrak{s}}$.
[Hint: Read [Sh:c, IV,§4] and recalling §5.]

6.5 Exercise: We can generalize $\mathfrak{C}^{\mathrm{eq}}$, canonical basis from [Sh:c, III].

[Hint: See [Sh:E54].]

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§0 Introduction

In this chapter we continue the hypotheses first laid out in Chapter V.D. Namely, we assume AxFr_1 and $\chi_{\mathfrak{s}} < \infty$. Recall that $\chi_{\mathfrak{s}}$ is the minimal χ such that $\mathfrak{s}$ is (χ^+, χ)-based, $(\leq \chi, \leq \chi^+)$-smooth and satisfies $\mathrm{LSP}(\chi)$.

We have dealt in Chapter V.D with "good" types (i.e., $M <_{\mathfrak{s}} \mathfrak{C}$ and $p \in \mathscr{S}_c^{<\infty}(M)$). Now we shall deal with other types, particularly of finite sequences. As earlier, we could weaken our axiomatic framework to AxFr_5.
Note the following:
1) Even for $p \in \mathscr{S}_c^{<\infty}(M)$, we do not know whether there are such types which do not fork over $\emptyset$ (but we can remedy this by adding an individual constant $c_a, a \in M^*$ for a fixed $M^* <_{\mathfrak{s}} \mathfrak{C}$).
2) For $p \in \mathscr{S}^1(A)$, we don't know much for arbitrary A, we use "good sets" like models.
3) Dependence does not, in general, have finite character.

Note that not much is lost if considering "$p \in \mathscr{S}^{<\infty}(B)$ does not fork over A" we restrict ourselves to the case "p does not fork over some $M \subseteq A$".

§1 Forking over Models of Types of Sequences

In this section we define non-forking and stationarization and parallelism for $p \in \mathscr{S}^{<\infty}(M)$ (not necessarily in $\mathscr{S}_c^{<\infty}(M)$, but still over $M <_{\mathfrak{s}} \mathfrak{C}$). We have to prove that stationarization of p over

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M is unique (1.4(1)) and that parallelism is an equivalence relation (1.4(2)). We also prove, for such types, that if $\mathbf{tp}(\bar{c}, B)$ does not fork over $M, A \subseteq B <_{\mathfrak{s}} \mathfrak{C}$ and M is $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous ($\lambda$ big enough relative to $|A|$) then $\mathbf{tp}(\bar{c}, A)$ is realized in M. We also have the basic properties of non-forking (1.7, 1.5) and can get back the formulation with models (1.8).

As in V.D.2.1 we assume from now on

1.1 Hypothesis.: The framework $\mathfrak{s}$ satisfies AxFr_1 and $\chi = \chi_{\mathfrak{s}}$ is well defined, so $\mathfrak{s}$ is (χ^+, χ)-based and $(\leq \chi, \leq \chi^+)$-smooth and $\mathrm{LSP}(\chi)$ holds, $\mathfrak{C}$ a monster model; so M, N vary on (small) $<_{\mathfrak{s}}$-submodels of $\mathfrak{C}$ and A, B, C vary on (small) subsets of $\mathfrak{C}$ but we may use $A <_{\mathfrak{s}} \mathfrak{C}$.

1.2 Definition. 1) Let $M \leq_{\mathfrak{s}} N <_{\mathfrak{s}} \mathfrak{C}$; we say that $\mathbf{tp}(\bar{c}, N)$ does not fork over M if for some $N', \bar{c}'$ we have: $\bar{c}'$ realizes $\mathbf{tp}(\bar{c}, N)$, $\bar{c}' \in N'$ and the triple M, N, N' is in stable amalgamation (of course, without loss of generality $\bar{c}' = \bar{c}$ - use the definition of $\mathbf{tp}$ and the choice of $\mathfrak{C}$).
2) $\mathbf{tp}(\bar{c}, N)$ is the stationarization of $\mathbf{tp}(\bar{c}', N')$ over N if $N' \leq_{\mathfrak{s}} N, \mathbf{tp}(\bar{c}, N') = \mathbf{tp}(\bar{c}', N')$ and $\mathbf{tp}(\bar{c}, N)$ does not fork over N' (on the uniqueness, which is implicitly said here, see 1.4 below).
3) p^1, p^2 are parallel if they have a common stationarization.

1.3 Claim. *1) These definitions are compatible with the previous ones (from V.D.3.7).*
2) If $M_0 \leq_{\mathfrak{s}} M_1 \leq_{\mathfrak{s}} M_2 \leq_{\mathfrak{s}} M_3, p_\ell \in \mathscr{S}^{<\infty}(M_\ell)$ for $\ell \leq 3$, p_3 a stationarization of p_0, $p_2 = p_3 \restriction M_2$ and $p_1 = p_3 \restriction M_1$ then p_2 is a stationarization of p_1.
3) If $p \in \mathscr{S}^{<\infty}(M)$ then p is a stationarization of itself.

Proof. Easy.

1.4 Lemma. *1) If $M \leq_{\mathfrak{s}} N$ then every $p \in \mathscr{S}^{<\infty}(M)$ has one and only one stationarization in $\mathscr{S}^{<\infty}(N)$.*
2) Parallelism is an equivalence relation and for each M an equivalence class has at most one member in $\mathscr{S}^{<\infty}(M)$.

Remark. In our context the first statement (1.4(1)) is not totally trivial.

Proof. 1) Existence: easy (by Ax(C2), existence for NF).

Uniqueness:. Suppose $\mathbf{tp}(\bar{c}^\ell, N)$, for $\ell = 1, 2$ are different stationarizations of $\mathbf{tp}(\bar{c}, M)$ over N (so $M \leq_{\mathfrak{s}} N$). For $\ell = 1, 2$, as $\mathbf{tp}(\bar{c}, M) \subseteq \mathbf{tp}(\bar{c}^\ell, N)$ there is an automorphism g_ℓ of $\mathfrak{C}$ over M satisfying $g_\ell(\bar{c}) = \bar{c}^\ell$. Let $g = g_2 \circ g_1^{-1}$, so

(a) $g \restriction M = \mathrm{id}_M$ and $g(\bar{c}^1) = \bar{c}^2$.

Again for $\ell = 1, 2$ as $\mathbf{tp}(\bar{c}^\ell, N)$ is a stationarization of $\mathbf{tp}(\bar{c}, M)$, there is a model M_ℓ such that:

(b) $M \leq_{\mathfrak{s}} M_\ell \leq_{\mathfrak{s}} \mathfrak{C}, \bar{c}^\ell \in M_\ell$ and M, N, M_ℓ are in stable amalgamation.

However, maybe for every M', $[M_1 \cup M_2 \subseteq M' \Rightarrow \mathbf{tp}(M', N)$ forks over $M]$. Choose a model M^* such that:

(c) $M^* <_{\mathfrak{s}} \mathfrak{C}, N \cup M_1 \cup M_2 \subseteq M^*$ and M^* is closed under g, g^{-1}.

By Ax(C2) there is an automorphism h of $\mathfrak{C}$ over M, such that $\{M^*, h(M^*)\}$ is independent over M. For $\ell = 1, 2$, as $\{N, M_\ell\}$ is independent over M, $N \cup M_\ell \subseteq M^*$, by V.D.3.16(1) and monotonicity of non-forking (= V.B.3.19(3)), we have $\{N, M_\ell, h(M^*)\}$ is independent over M (for $\ell = 1, 2$). Hence (by Ax(C5)), $(h \restriction M_\ell) \cup \mathrm{id}_N$ can be extended to an automorphism of $\mathfrak{C}$ which we call h_ℓ. So

$$h_\ell \restriction N = \mathrm{id}_N,\ h_\ell(\bar{c}^\ell) = h(\bar{c}^\ell),$$

hence

$(*)_\ell$ $\mathbf{tp}(\bar{c}^\ell, N) = \mathbf{tp}(h(\bar{c}^\ell), N)$.

Obviously, hgh^{-1} is an automorphism of $h(M^*)$ mapping $h(\bar{c}^1)$ to $h(\bar{c}^2)$ (remember that g is an automorphism of M^* mapping $\bar{c}^1$ to $\bar{c}^2$) and (hgh^{-1}) is the identity on M (as $h \restriction M = g \restriction M = \mathrm{id}_M$). As $\{N, h(M^*)\}$ is independent over M there is an automorphism h^* of $\mathfrak{C}$, extending $\mathrm{id}_N \cup ((hgh^{-1}) \restriction h(M^*))$. So $h^* \restriction N = \mathrm{id}_N, h^*(h(\bar{c}^1)) = h(\bar{c}^2)$. Hence

$(**)$ $\mathbf{tp}(h(\bar{c}^1), N) = \mathbf{tp}(h(\bar{c}^2), N)$.

Together by $(*)_1, (*)_2$ and $(**)$ we get the desired conclusion.
2) Why is parallelism an equivalence relation? Symmetry holds by the definition. Reflexivity holds by 1.3(3).

We are left with transitivity. It will follows from 1.3 and 1.4(1) by a straightforward computation. So suppose $p_\ell \in \mathscr{S}^{<\infty}(M_\ell)$ for $\ell < 3$; for $\ell = 0, 1$ let q_ℓ be a common stationarization of p_ℓ, $p_{\ell+1}$. So $q_\ell \in \mathscr{S}^{<\infty}(N_\ell)$ where $M_\ell \leq_{\mathfrak{s}} N_\ell$ and $M_{\ell+1} \leq_{\mathfrak{s}} N_\ell <_{\mathfrak{s}} \mathfrak{C}$. Choose N such that $N_0 \cup N_1 \subseteq N <_{\mathfrak{s}} \mathfrak{C}$ and for $\ell = 0, 1, 2$ let $r_\ell \in \mathscr{S}^{<\infty}(N)$ be a stationarization of p_ℓ. By 1.3(2) for $\ell = 0, 1$ the type $r_\ell \restriction N_0$ is a stationarization of p_ℓ hence by 1.4(1), $r_\ell \restriction N_0 = q_1$. Also by 1.3(2) for $\ell = 0, 1$ the type r_ℓ is a stationarization of $r_\ell \restriction N_0$, hence by 1.4(1), $r_0 = r_1$.

Similarly, for $\ell = 1, 2$ by 1.3(2) the type $r_\ell \restriction N_1$ is a stationarization of p_ℓ hence by 1.4(1), $r_\ell \restriction N_1 = q_1$. Also by 1.3(2) for $\ell = 1, 2$, r_ℓ is a stationarization of $r_\ell \restriction N_1$ hence by 1.4(1), $r_1 = r_2$.

Thus, p_0 and p_2 have a common stationarization: $r_0 = r_1 = r_2$. So parallelism satisfies transitivity.

So parallelism is an equivalence relation. The second phrase of 1.4(2) (every equivalence class has in $\mathscr{S}^{<\infty}(M)$ at most one member) follows by the above using 1.4(1). $\square_{1.4}$

1.5 Lemma. *(Transitivity). If $M_0 \leq_{\mathfrak{s}} M_1 \leq_{\mathfrak{s}} M_2 <_{\mathfrak{s}} \mathfrak{C}$, $\mathbf{tp}(\bar{c}, M_2)$ does not fork over M_1 and $\mathbf{tp}(\bar{c}, M_1)$ does not fork over M_0 then $\mathbf{tp}(\bar{c}, M_2)$ does not fork over M_0.*

Proof. We can deduce this from 1.4(2). $\square_{1.5}$

Remark. Alternatively, as $\mathbf{tp}(\bar{c}, M_2)$ does not fork over M_1, there is N_1 such that $M_1 \leq_{\mathfrak{s}} N_1 <_{\mathfrak{s}} \mathfrak{C}$, $\bar{c} \in N_1$, and M_1, M_2, N_1 is in stable amalgamation. Similarly, there is N_0 such that $M_0 <_{\mathfrak{s}} N_0 <_{\mathfrak{s}} \mathfrak{C}$,$\bar{c} \in N_0$ and M_0, M_1, N_0 is in stable amalgamation. Let $\lambda > \lambda_{\mathfrak{s}} + \|M_2\| + \|N_0\| + \|N_1\|$. There is N_1^a such that $N_1 \leq_{\mathfrak{s}} N_1^a <_{\mathfrak{s}} \mathfrak{C}, N_1^a$

is strongly $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous and M_1, N_1^a, M_2 in stable amalgamation. So there is a $\leq_{\mathfrak{s}}$-embedding h of $\langle M_1, N_0\rangle^{\rm gn}$ into N_1^a over M_1. So $\mathbf{tp}(\bar{c}, M_1) = \mathbf{tp}(h(\bar{c}), M_1)$ hence there is an automorphism f of N_1^a, $f \restriction M_1 = \mathrm{id}_{M_1}$, $f(h(\bar{c})) = \bar{c}$. Let $N_0^a = f(h(N_0))$, so clearly M_0, N_0^a, M_1 is in stable amalgamation, and $\bar{c} = f(h(\bar{c})) \in N_0^a$.

Now as M_0, N_0^a, M_1 and M_1, N_1^a, M_2 are in stable amalgamation, $M_0 \leq_{\mathfrak{s}} M_1 \leq_{\mathfrak{s}} M_2 \leq_{\mathfrak{s}} \mathfrak{C}, N_0^a \leq_{\mathfrak{s}} N_1^a$ by V.D.3.12(2), M_0, N_0^a, M_2 is in stable amalgamation, and as $\bar{c} \in N_0^a$ we finish.

1.6 Claim. *If $M \subseteq B$, $\mathbf{tp}(\bar{c}, B)$ does not fork over M, $A \leq_{\mathfrak{s}} B <_{\mathfrak{s}} \mathfrak{C}$ and M is $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous and $\lambda = (\chi_{\mathfrak{s}} + \ell g(\bar{c}) + |A|)^+$, then $\mathbf{tp}(\bar{c}, A)$ is realized by some $\bar{c}' \in M$.*

Proof. There are $N_0, N_1 <_{\mathfrak{s}} \mathfrak{C}$ such that N_0, N_1, M is in stable amalgamation, $A \cup \bar{c} \subseteq N_1$, $N_1 \cap M = N_0$, $N_1 \cap B \leq_{\mathfrak{s}} B$ and $\|N_1\| < \lambda$ (as by V.D.1.2, $\mathfrak{s}$ is $(\chi_{\mathfrak{s}} + \ell g(\bar{c}) + |A|)^+$-based, more exactly by V.C.3.12). We can find a $\leq_{\mathfrak{s}}$-embedding f of N_1 into M over N_0 (as M is $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous). By monotonicity, $N_0, N_1, f(N_1)$ is in stable amalgamation hence $\{N_1, f(N_1)\}$ is independent over N_0. By Definition 1.2 letting $\bar{d} := f(\bar{c})$, the type $\mathbf{tp}(\bar{d}, N_1)$ is parallel to $\mathbf{tp}(\bar{d}, N_0)$ which is equal to $\mathbf{tp}(\bar{c}, N_0)$ which is parallel to $\mathbf{tp}(\bar{c}, M)$. Now $N_0 \leq_{\mathfrak{s}} N_1 \cap B \leq_{\mathfrak{s}} N_1$ hence by monotonicity $\mathbf{tp}(\bar{d}, N_1 \cap B)$ is parallel to $\mathrm{tp}(\bar{c}, N_0)$. As $\mathbf{tp}(\bar{c}, B)$, $\mathbf{tp}(\bar{c}, M)$ do not fork over M, N_0 respectively, by transitivity (see 1.5) the type $\mathbf{tp}(\bar{c}, B)$ does not fork over N_0, hence by monotonicity (see 1.3(2)) the types $\mathbf{tp}(\bar{c}, N_1 \cap B)$ does not fork over N_0. So, by "parallelism is an equivalence relation" (see 1.4(2)) the types $\mathbf{tp}(\bar{d}, N_1 \cap B)$, $\mathbf{tp}(\bar{c}, N_1 \cap B)$ are parallel hence (by 1.3) equal. Hence $\bar{c}' := \bar{d}$ is as required. $\square_{1.6}$

1.7 Lemma. *For every $M <_{\mathfrak{s}} \mathfrak{C}$ and $\bar{c} \in {}^{\infty>}\mathfrak{C}$ there is an $N \leq_{\mathfrak{s}} M$ such that $\|N\| \leq |\ell g(\bar{c})| + \chi_{\mathfrak{s}}$, and $\mathbf{tp}(\bar{c}, M)$ does not fork over N.*

Proof. By NF being $(\ell g(\bar{c}) + \chi_{\mathfrak{s}})$-based. $\square_{1.7}$

1.8 Claim. *If $M \leq_{\mathfrak{s}} N <_{\mathfrak{s}} \mathfrak{C}$, $\mathbf{tp}(\bar{c}, N)$ does not fork over M and $M \cup \bar{c} \subseteq M_1 <_{\mathfrak{s}} \mathfrak{C}$, then for some automorphism h of $\mathfrak{C}$, $h \restriction (M \cup \bar{c}) =$ id and $M, N, h(M_1)$ is in stable amalgamation.*

Proof. As $\mathbf{tp}(\bar{c}, N)$ does not fork over M there is M_2, $M \cup \bar{c} \subseteq M_2$, such that M, N, M_2 is in stable amalgamation. Let $N \cup M_2 \subseteq M_2^+ <_{\mathfrak{s}} \mathfrak{C}$ and let $\lambda = \|N\| + \|M_2\| + \|M_1\| + \chi_{\mathfrak{s}}$, so there is a strongly $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous model M_3 satisfying $M_2 \leq_{\mathfrak{s}} M_3 <_{\mathfrak{s}} \mathfrak{C}$. Without loss of generality M_2, M_2^+, M_3 is in stable amalgamation hence by transitivity of NF also M, N, M_3 is in stable amalgamation. Clearly there is an automorphism h_0 of $\mathfrak{C}$ satisfying $h_0 \restriction M =$ id and $h_0(M_1) \subseteq M_3$. Clearly $\mathbf{tp}(\bar{c}, M) = \mathbf{tp}(h_0(\bar{c}), M)$, hence there is an automorphism h_1 of $\mathfrak{C}$ satisfying $h_1 \restriction M =$ id, $h_1(h_0(\bar{c})) = \bar{c}$. By the choice of λ and M_3 and as $\bar{c} \in M_2 \subseteq M_3$, $h_0(\bar{c}) \in M_3$, clearly without loss of generality $h_1 \restriction M_3$ is an automorphism of M_3. So $(h_1 \circ h_0)(M_1) \subseteq M_3$, $h_1 \circ h_0(\bar{c}) = \bar{c}$, $(h_1 \circ h_0) \restriction M =$ id and $M, N, (h_1 \circ h_0)(M_1)$ is in stable amalgamation (by monotonicity as M, N, M_3 is). So we finish. $\square_{1.8}$

Remark. On existence for Definition 1.2, see 2.11(2) below.

§2 Forking Over Sets

In this section we deal with types over $A \subseteq \mathfrak{C}$ without requiring $A <_{\mathfrak{s}} \mathfrak{C}$ so A, B, C denote such sets but $M, N <_{\mathfrak{s}} \mathfrak{C}$. We redo non-forking for this, 2.1 - 2.5 and also 2.10 (symmetry) and we define strong splitting (definition 2.6).

We also deal with convergent sequences, (2.7 (Definition), 2.9,2.11 (existence) independence (2.12 (Definition), 2.13) and parallelism (2.14 (Definition), 2.15). In 1.2 we have defined "$\mathbf{tp}(c, M)$ does not fork over $N, N \leq_{\mathfrak{s}} M <_{\mathfrak{s}} \mathfrak{C}$". In (1) of 2.1 we shall drop the requirement $N <_{\mathfrak{s}} \mathfrak{C}$, and in (2) of 2.1 we also drop the requirement $M <_{\mathfrak{s}} \mathfrak{C}$.

2.1 Definition. 1) Suppose $A \subseteq M <_{\mathfrak{s}} \mathfrak{C}$ we say that $p = \mathbf{tp}(\bar{c}, M)$ does not fork over A <u>if</u> for every N, $M \leq_{\mathfrak{s}} N <_{\mathfrak{s}} \mathfrak{C}$, and automorphism h of N over A, h maps the stationarization of p over N to itself.
2) Suppose $A \subseteq B \subset \mathfrak{C}$, <u>then</u> $\mathbf{tp}(\bar{c}, B)$ does not fork over A <u>if</u> for every model $N <_{\mathfrak{s}} \mathfrak{C}$ satisfying $B \subseteq N$, for some sequence $\bar{c}'$ realizing $\mathbf{tp}(\bar{c}, B)$, $\mathbf{tp}(\bar{c}', N)$ does not fork over A (according to part (1)).

Remark. Unfortunately, Definition 2.1(2) does not specialize for first order T to the usual definition (but to a variant, see 2.5(8)).

2.2 Claim. *1) Definitions 2.1(1), 2.1(2) and 1.2(1) are compatible.*
2) Assume $A \subseteq B \subset \mathfrak{C}$. The type $p = \mathbf{tp}(\bar{c}, B)$ does not fork over A <u>iff</u>: $\circledast_1$ iff $\circledast_2$ where

- $\circledast_1$ *for every N satisfying $B \subseteq N <_{\mathfrak{s}} \mathfrak{C}$, there is an extension $q \in \mathscr{S}^{<\infty}(M)$ of p such that for every automorphism h of N over A, h maps p to itself*
- $\circledast_2$ *like $\circledast_1$ for some N which is strongly $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous, where $\lambda = (\chi_{\mathfrak{s}} + |A| + \ell g(\bar{c}))^+$.*

Proof. 1) Obviously, by 1.4(1) <u>if</u> $p = \mathbf{tp}(\bar{c}, B)$ does not fork over A by 1.2(1) (so A, $B <_{\mathfrak{s}} \mathfrak{C}$) then p does not fork over A by 2.1(1).

Also if $\mathbf{tp}(\bar{c}, B)$ does not fork over A by 2.1(1) (so $B <_{\mathfrak{s}} \mathfrak{C}$) then p does not fork over A by 2.1(2). [Why? Assume $A \subseteq B = M <_{\mathfrak{s}} \mathfrak{C}$, $p = \mathbf{tp}(\bar{c}, B) = \mathbf{tp}(\bar{c}, M)$ and 2.1(1) holds, now we should prove 2.1(2). So let $B \subseteq N <_{\mathfrak{s}} \mathfrak{C}$, and we should find an extension $q = \mathbf{tp}(\bar{c}', N)$ of p such that q does not fork over A by 2.1(1). Let q be the stationarization of p over N (see 1.2(1), 1.4(1)), we should prove 2.1(1) holds; i.e., let $N_1 \leq_{\mathfrak{s}} \mathfrak{C}, N \leq_{\mathfrak{s}} N_1$, r be the stationarization of q over N_1, and h be an automorphism of N_1 over A, we should prove $h(r) = r$. By 1.5 the type r does not fork over M, hence r is the stationarization of p over N_1, so according to 2.1(1), $h(r) = r$ as required.]

Next suppose $\mathbf{tp}(\bar{c}, M)$ does not fork over $A(\subseteq M)$ in 2.1(1)'s sense, and Definition 1.2(1) is applicable, so $A = |M_0|, A \subseteq M <_{\mathfrak{s}} \mathfrak{C}$.

In the proof of the present this implication "does not fork" mean in 1.2(1)'s sense which is compatible with Chapter V.D's sense. We should show that $\mathbf{tp}(\bar{c}, M)$ does not fork over M_0. Let $\lambda = \chi_{\mathfrak{s}} + \|M\|$, and $N <_{\mathfrak{s}} \mathfrak{C}$ be a strongly $(\mathbb{D}_{\mathfrak{s}}, \lambda^+)$-homogeneous model such that $M \subseteq N$. So by Definition 2.1(1) there is $\bar{c}^1$ realizing $\mathbf{tp}(\bar{c}, M)$, such that any automorphism of N over M_0 maps $\mathbf{tp}(\bar{c}^1, N)$ to itself. By (Hypothesis 1.1 and) Claim V.D.1.2 clearly $\mathfrak{s}$ is λ-based so there are M_1, N_1 such that $M \leq_{\mathfrak{s}} M_1 \leq_{\mathfrak{s}} N, M_1 \leq_{\mathfrak{s}} N_1 <_{\mathfrak{s}} \mathfrak{C}$, $\|M_1\| \leq \|N_1\| \leq \lambda$, $\bar{c}^1 \in N_1$ and $\{N_1, N\}$ is independent over M_1. There is an automorphism h of $\mathfrak{C}$ such that $h \restriction M_0 = \text{id}_{M_0}$ and $\{M_1, h(M_1)\}$ is independent over M_0; as N is strongly $(\mathbb{D}_{\mathfrak{s}}, \lambda^+)$-homogeneous, without loss of generality h maps N onto itself. By the choice of $\bar{c}^1$ without loss of generality $h(\bar{c}^1) = \bar{c}^1$. As $\{N_1, N\}$ is independent over M_1, $\mathbf{tp}(\bar{c}^1, N)$ does not fork over M_1, hence $\mathbf{tp}(\bar{c}^1, N) = \mathbf{tp}(h(\bar{c}^1), h(N))$ does not fork over $h(M_1)$.

Now we shall use "does not fork" in Chapter V.D's sense in the next two sentences. But $\mathbf{tp}(N_1, N)$ does not fork over M_1, hence, by symmetry, $\mathbf{tp}(N, N_1)$ does not fork over M_1, hence as $h(M_1) \leq_{\mathfrak{s}} N$ also $\mathbf{tp}(h(M_1), N_1)$ does not fork over M_1. Now recall that $\mathbf{tp}(h(M_1), M_1)$ does not fork over M_0, so by transitivity (V.D.3.13(1)) the type $\mathbf{tp}(h(M_1), N_1)$ does not fork over M_0; hence (by symmetry (V.D.3.12(1)), the type $\mathbf{tp}(N_1, h(M_1))$ does not fork over M_0.

So in Definition 1.2 sense the type $\mathbf{tp}(\bar{c}^1, h(M_1))$ does not fork over M_0, hence by an implication already proved (ie. from 1.2 to 2.1(1)) we know $\mathbf{tp}(\bar{c}^1, h(M_1))$ does not fork over M_0 (from now on, in 2.1(1) sense). But by few sentences above we get $\mathbf{tp}(\bar{c}^1, N)$ does not fork over $h(M_1)$ hence (by transitivity for non-forking, 1.5) the type $\mathbf{tp}(\bar{c}^1, N)$ does not fork over M_0, hence by monotonicity (1.3(2)) the type $\mathbf{tp}(\bar{c}^1, M)$ does not fork over M_0. But $\mathbf{tp}(\bar{c}^1, M) = \mathbf{tp}(\bar{c}, M)$ so we have finished this implication.

To finish the proof that definitions 2.1(1), 2.1(2), 1.2(1) are compatible, suppose $\mathbf{tp}(\bar{c}, B)$ does not fork over A in Definition 2.1(2)'s sense, and Definition 2.1(1) applies; i.e., $A \subseteq B = |M|, (M <_{\mathfrak{s}} \mathfrak{C})$, and we shall show that $\mathbf{tp}(\bar{c}, N)$ does not fork over A according to Definition 2.1(1). This is easy, just apply the definitions (choosing N is 2.1(2) as M).

2) By part (1) we should prove just the equivalence of 2.1(2) with $2.2\circledast_1$, $2.2\circledast_2$.
Recall $p = \mathbf{tp}(\bar c, B)$, $A \subseteq B$.

First, assuming 2.1(2) holds proving $2.2\circledast_1$ is trivial (using $N = M$ in Definition 2.1(1)).

Second, if $2.2\circledast_1$ holds, clearly $2.2\circledast_2$ holds.

Third, and lastly, assume that $2.2\circledast_2$ holds, let it be exemplified by N, q; it suffices to show that q does not fork over A according to Definition 2.1.
[Why? Given N' such that $B \subseteq N' <_{\mathfrak{s}} \mathfrak{C}$, we let $\mu = \|N'\| + \|N\|$ and $N'' <_{\mathfrak{s}} \mathfrak{C}$ be strongly $(\mathbb{D}_{\mathfrak{s}}, \mu^+)$-homogeneous containing $N' \cup N''$.

By the present assumption on the pair (N, q) there is $\bar c'$ realizing q such that $\mathbf{tp}(\bar c', N'')$ does not fork over N. By transitivity of non-forking in the sense of 1.2 and part (1), clearly q' does not fork over A in the sense of 1.2 hence by part (1) in the sense of 2.1(1). Now this implies that every automorphism h of N'' over A maps $\mathbf{tp}(\bar c', N'')$ to itself. But $\bar c'$ realizes also $\mathbf{tp}(\bar c, B) = \mathbf{tp}(\bar c, N) \restriction B$ and every automorphism of N' over A can be extended to an automorphism of N'' over A, so we are done.]

Toward contradiction assume q forks over A according to 2.1(1). So there are N_1 and $\bar e \subseteq \mathfrak{C}$ such that $N \le_{\mathfrak{s}} N_1 <_{\mathfrak{s}} \mathfrak{C}$, an automorphism h of $N_1, r = \mathbf{tp}(\bar e, N_1)$ a stationarization of q such that $h \restriction A = \mathrm{id}_A$ and h maps r to $r_1 = h(r) \neq r$. We can choose an automorphism h^+ of $\mathfrak{C}$ extending h so $r_1 = \mathbf{tp}(h^+(\bar e), N_1)$.

As $\mathfrak{s}$ is $(\chi_{\mathfrak{s}} + |A|)$-based, by V.C.3.12 we can find N^a and N^b such that $N^a \le_{\mathfrak{s}} N_1$, $A \subseteq N^a, N^a \le_{\mathfrak{s}} N^b, \bar e\hat{\ } h^+(\bar e) \subseteq N^b$, $h(N^a) = N^a$, $\mathbf{tp}(N^b, N_1)$ does not fork over N^a, $\|N^b\| < \lambda, N^b \cap N = N^a \cap N \le_{\mathfrak{s}} N$ and $\mathbf{tp}(N^b, N)$ does not fork over $N^b \cap N = N^a \cap N$ and $h^+(N^b) = N^b$.

As $\mathbf{tp}(N^b, N)$ does not fork over $N^b \cap N$, there is an automorphism f of $\mathfrak{C}$ such that $f \restriction (N^a \cap N) =$ id, and $f(N^a) \subseteq N$ and $f(N^b) \underset{f(N^a)}{\bigcup} N_1$. Let $\bar e_0 = \bar e, \bar e_1 = f(\bar e)$. Let $N_0 = N$

- $(*)_1$ $\mathbf{tp}(\bar e, N_\ell)$ does not fork over $N^a \cap N_\ell$ for $\ell = 0, 1$
  [Why? As $\bar e \subseteq N^b$ and $\mathbf{tp}(N^b, N_\ell)$ does not fork over $N^b \cap N_\ell = N^a \cap N_\ell$.]
- $(*)_2$ $\mathbf{tp}(h^+(\bar e), N_\ell)$ does not fork over $N^b \cap N_\ell = N^a \cap N_\ell$ for

$\ell = 0, 1$
[Why? As $h^+(\bar{e}) \subseteq N^b$ and $\mathbf{tp}(N^b, N)$ does not fork over $N^b \cap N = N^a \cap N$.]

$(*)_3$ $\mathbf{tp}(\bar{e}, N^a) \neq \mathbf{tp}(h^+(\bar{e}), N^a)$
[Why? If this fails, as $\mathbf{tp}(\bar{e}, N_1) = r, r_1 = \mathbf{tp}(h^+(\bar{e}), N_1)$ and $(*)_1$ and $(*)_2$ we get $r = r_1$ contradicting the choice of h and $\bar{e}$.]

$(*)_4$ $\mathbf{tp}(f(\bar{e}), f(N^a)) \neq \mathbf{tp}(f(h^+(\bar{e})), f(N^a))$
[Why? By $(*)_3$ as f is an automorphism of $\mathfrak{C}$.]

$(*)_5$ $\mathbf{tp}(f(\bar{e}), N)$ does not fork over $f(N^a)$
[Why? By the choice of f, $\mathbf{tp}(f(N^b), N_1)$ does not fork over $f(N^a)$. Now $f(N^a) \leq_{\mathfrak{s}} N \leq_{\mathfrak{s}} N_1$, and use monotonicity of non-forking (of course, in Chapter V.D's sense) to get $\mathbf{tp}(f(N^b), N)$ does not fork over $f(N^a)$. But $\bar{e} \subseteq N^b$, so we are done.]

$(*)_6$ $\mathbf{tp}(f(h^+(\bar{e})), N)$ does not fork over $f(N^a)$
[Why? As in $(*)_5$ because $h^+(\bar{e}) \subseteq N^b$ as h^+ maps N^b onto itself.]

$(*)_7$ $\mathbf{tp}(f(\bar{e}), N) \neq \mathbf{tp}(f(h^+(\bar{e})), N)$
[Why? By $(*)_4$ as $f(N^a) \subseteq N$.]

$(*)_8$ $\mathbf{tp}(\bar{e}, N_1)$ does not fork over $N^a \cap N$
[Why? Recall that $\mathbf{tp}(\bar{e}, N_1)$ does not fork over N by the choice of $\bar{e}$ such that $r = \mathbf{tp}(\bar{e}, N_1)$ is the stationarization of $q \in \mathscr{S}(N)$. Also $\mathbf{tp}(\bar{e}, N)$ does not fork over $N^a \cap N$ by $(*)_1$ hence by transitivity (see 1.5) we know that $\mathbf{tp}(\bar{e}, N_1)$ does not fork over $N^a \cap N$.]

$(*)_9$ $\mathrm{tp}(\bar{e}, N^a)$ does not fork over $N^a \cap N$
[Why? By $(*)_8$ and monotonicity, i.e. 1.3(2) as $N^a \leq_{\mathfrak{s}} N_1$.]

$(*)_{10}$ $\mathbf{tp}(f(\bar{e}), N)$ does not fork over $N^a \cap N$
[Why? By $(*)_5$ the type $\mathbf{tp}(f(\bar{e}), N)$ does not fork over $f(N^a)$. By $(*)_9$ it follows that $\mathbf{tp}(f(\bar{e}), f(N^a))$ does not fork over $f(N^a \cap N)$ but $f \supseteq \mathrm{id}_{N^a \cap N}$ hence $\mathbf{tp}(f(\bar{e}), f(N^a))$ does not fork over $N^a \cap N$. Together by transitivity (1.5) we get $\mathbf{tp}(f(\bar{e}), N)$ does not fork over $N^a \cap N$ as required.]

$(*)_{11}$ there is an automorphism h^* of N over A such that
$h^*(\mathbf{tp}(f(\bar{e}), N)) = \mathbf{tp}(f(h^+(\bar{e})), N)$

[Why? As $h(N^a) = N^a$ by the choice of N^a, clearly $h' := (f \circ h \circ f^{-1}) \restriction f(N^a)$ is an automorphism of $f(N^a)$, (chasing arrows) and clearly $h' \restriction A = \mathrm{id}_A$ as $f \restriction A = \mathrm{id}_A$ and $h \restriction A = \mathrm{id}_A$ and $h'(\mathbf{tp}(f(\bar{e}), f(N^a))) = \mathbf{tp}(f(h^+(\bar{e})), f(N^a))$. As $f(N^a) \le_{\mathfrak{s}} N$ and $\|f(N^a)\| \le \|N^b\| < \lambda$ there is an automorphism h^* of N which extends h'.
Now $h^*(\mathbf{tp}(f(\bar{e}), N)) = \mathbf{tp}(f(h^+(\bar{e})), N)$ as both types does not fork over $f(N^a)$ by $(*)_5$ and $(*)_6$ respectively and the parallel statement holds for $h' = h^* \restriction f(N^a), f(N^a)$ by the first sentence. So we are done.]

$(*)_{12}$ $\mathbf{tp}(f(\bar{e}), N) = q$
[Why? As $\mathbf{tp}(\bar{e}, N)$ is q by the choice of $r, \bar{e}$, it is enough to prove $\mathbf{tp}(f(\bar{e}), N) = \mathbf{tp}(\bar{e}, N)$. Now $\mathbf{tp}(\bar{e}, N)$ does not fork over $N^a \cap N$ (by $(*)_1$) and by $(*)_8$ also $\mathbf{tp}(f(\bar{e}), N)$ does not fork over $N^a \cap N$, hence it is enough to prove $\mathbf{tp}(f(\bar{e}), N^a \cap N) = \mathbf{tp}(\bar{e}, N^a \cap N)$. But f is the identity on $N^a \cap N$ by its choice so we are done.]

By $(*)_{11} + (*)_{12} + (*)_7$ we deduce: h^* is an automorphism of N over A such that $h^*(q) \neq q$ contradiction, so we have finished proving that: if $\circledast_2$ then $\mathbf{tp}(\bar{c}, B)$ does not fork over A. $\square_{2.2}$

At the beginning of this chapter we have defined the notion of the type of an element $\bar{c}$ over a model N, being stationary over $N_0 \subseteq N$ (the stationarization of $\mathbf{tp}(\bar{c}, N_0)$). Now we extend this notation by allowing sets as domains.

2.3 Definition. 1) The type $p = \mathbf{tp}(\bar{c}, B)$ is stationary over A, $A \subseteq B$ if for every $C, B \subseteq C \subset \mathfrak{C}$, the type p has one and only one extension in $\mathscr{S}^{\ell g(\bar{c})}(C)$ which does not fork over A; see 2.4(4) second sentence.
2) $p = \mathbf{tp}(\bar{c}, B)$ is explicitly stationary over A if for some $M \subseteq A, M <_{\mathfrak{s}} \mathfrak{C}$ and p does not fork over M.
3) If $A = B$ we omit "over A" (in both cases).

2.4 Claim. *1) If $\mathbf{tp}(\bar{c}, M)$ does not fork over $A \subseteq M$ then every automorphism of M over B maps $\mathbf{tp}(\bar{c}, M)$ to itself; i.e.: if $f \in AUT(M)$, $f \restriction B = \mathrm{id}_B$ then there is $g \in \mathrm{AUT}(\mathfrak{C})$ with $f \subseteq g$,*

$g(\bar{c}) = \bar{c}$.
2) If $A \subseteq M$, $\lambda = |A| + \chi_{\mathfrak{s}} + |\ell g(\bar{c})|$, M is strongly $(\mathbb{D}_{\mathfrak{s}}, \lambda^+)$-homogeneous, then: $\mathbf{tp}(\bar{c}, M)$ does not fork over A if and only if every automorphism of M over A maps $\mathbf{tp}(\bar{c}, M)$ to itself iff $\mathbf{tp}(\bar{c}, M)$ is stationary over A.
3) If $p = \mathbf{tp}(\bar{c}, B), B <_{\mathfrak{s}} \mathfrak{C}, \lambda = \chi_{\mathfrak{s}} + |\ell g(\bar{c})| + |B|$, then p has character $\leq \lambda$ or is λ-local which means: p is the unique extension in $\mathscr{S}^{\ell g(\bar{c})}(B)$ of all members of $\{\mathbf{tp}(\bar{c}, C) : C \subseteq B, |C| \leq \lambda\}$.
4) If $p = \mathbf{tp}(\bar{a}, B)$ is explicitly stationary over A then it it stationary over A. If $\mathbf{tp}(\bar{a}, B)$ is stationary over A then it does not fork over A.
5) If $A \subseteq B, \bar{c}'$ is a permutation of $\bar{c}$ and $\mathbf{tp}(\bar{c}, B)$ is [explicitly] stationary over A then so does $\mathbf{tp}(\bar{c}', B)$.
6) If $p = \mathbf{tp}(\bar{a}, B)$ does not fork over A, $A \subseteq B \subseteq C$ then there is $q \in \mathscr{S}^{\ell g(\bar{a})}(C)$ extending p which does not fork over A.

Proof. Check for part (1) + (2). Note that we are using Definition 2.1 and not the version of [Sh:c]. $\square_{2.4}$

In the following lemma we consider a number of basic facts of first order stability theory in the present context. Note that 2.5(8) fails in the first order case and that we do not assert $\mathbf{tp}(a, A)$ does not fork over A. This dilutes the power of Claim 2.4(6). Moreover, it changes the emphasis of the definition of stationary types from the first order case. Now here the assertion that a stationary type has a non-forking extension is an essential component of the definition. The situation is similar to [Sh 87a], [Sh 87b], i.e. not every type has a stationarization.

2.5 Lemma. *1) (monotonicity). Assume $A \subseteq B \subseteq C$. If $\mathbf{tp}(\bar{c}, C)$ does not fork over A, then $\mathbf{tp}(\bar{c}, C)$ does not fork over B and $\mathbf{tp}(\bar{c}, B)$ does not fork over A.*
2) If $M <_{\mathfrak{s}} \mathfrak{C}$, then $\mathbf{tp}(\bar{c}, M)$ is explicitly stationary, hence is stationary (hence does not fork over M).
3) If $A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \subseteq C$, $\mathbf{tp}(\bar{c}, A_3)$ does not fork over A_0 and $\mathbf{tp}(\bar{c}, A_2)$ is stationary over A_1 then $\mathbf{tp}(\bar{c}, A_3)$ is stationary over A_0.
4) (transitivity) If $A \subseteq B_1 \subseteq B_2 \subseteq C$, $\mathbf{tp}(\bar{c}, B_2)$ does not fork over

A, and $\mathbf{tp}(\bar{c}, B_2)$ *is stationary over* B_1 *and* $\mathbf{tp}(\bar{c}, C)$ *does not fork over* B_1 *then* $\mathbf{tp}(\bar{c}, C)$ *does not fork over A and is stationary over A.*
5) If $A \subseteq B$, $\mathbf{tp}(\bar{c}_1, B)$ *does not fork over A and* $\mathbf{tp}(\bar{c}_2, B \cup \bar{c}_1)$ *does not fork over* $A \cup \bar{c}_1$ *then* $\mathbf{tp}(\bar{c}_1{}^\frown\bar{c}_2, B)$ *does not fork over A.*
6) Suppose $\mathrm{Rang}(\bar{c}_1) \subseteq \mathrm{Rang}(\bar{c}_2)$. *If* $\mathbf{tp}(\bar{c}_2, B)$ *does not fork over A, then* $\mathbf{tp}(\bar{c}_1, B)$ *does not fork over A.*
7) If $\mathrm{Rang}(\bar{c}_1) = \mathrm{Rang}(\bar{c}_2)$, $\mathbf{tp}(\bar{c}_1, B)$ *is stationary over A if and only if* $\mathbf{tp}(\bar{c}_2, B)$ *is stationary over A and* $\mathbf{tp}(\bar{c}_1, B)$ *does not fork over A if and only if* $\mathbf{tp}(\bar{c}_2, B)$ *does not fork over A.*
8) If $\mathbf{tp}(\bar{c}, B)$ *does not fork over A, (so $A \subseteq B$),* $\bar{b}_1 \in B$, $\bar{b}_2 \in B$ *and* $\mathbf{tp}(\bar{b}_1, A) = \mathbf{tp}(\bar{b}_2, A)$ *then* $\mathbf{tp}(\bar{c}{}^\frown\bar{b}_1, A) = \mathbf{tp}(\bar{c}{}^\frown\bar{b}_2, A)$.
9) If $A \subseteq B$, $A \subseteq C$, $\mathbf{tp}(B, C)$ does not fork over A and $B_0, B_1 \subseteq B$ and $\mathbf{tp}(B_1, A \cup B_0)$ does not fork over A then $\mathbf{tp}(B_1, C \cup B_0)$ does not fork over A.

Proof. 1) Check by reading Definition of non-forking (2.1(1),(2)), or use 2.2(2).
2) By 2.2 + 1.4(1) (and the two consequences by 2.4(4)).
3) Toward contradiction suppose $\mathbf{tp}(\bar{c}, A_3)$ is not stationary over A_0. Let $\lambda = |A_3| + \chi_{\mathfrak{s}} + |\ell g(\bar{c})|$ and let N be a strongly $(\mathbb{D}_{\mathfrak{s}}, \lambda^+)$-homogeneous model satisfying $A_3 \subseteq N <_{\mathfrak{s}} \mathfrak{C}$. As $\mathbf{tp}(\bar{c}, A_3)$ does not fork over A_0 but is not stationary over A_0, by 2.2 (using $\circledast_2$ there) there are $\bar{c}^1, \bar{c}^2$ realizing $\mathbf{tp}(\bar{c}, A_3), B <_{\mathfrak{s}} \mathfrak{C}$, $\mathbf{tp}(\bar{c}^\ell, N)$ does not fork over A_0 for $\ell = 1, 2$, but $\mathbf{tp}(\bar{c}^1, N) \neq \mathbf{tp}(\bar{c}^2, N)$. So (by 2.5(1)) also the type $\mathbf{tp}(\bar{c}^\ell, N)$ does not fork over A_1, and extends $\mathbf{tp}(\bar{c}, A_2)$. So $\mathbf{tp}(\bar{c}, A_2)$ is not stationary over A_1, contradiction.
4) As $\mathbf{tp}(\bar{c}, B_2)$ does not fork over A, by 2.4(6) there is $\bar{c}'$ realizing $\mathbf{tp}(\bar{c}, B_2)$ such that $\mathbf{tp}(\bar{c}', C)$ does not fork over A. By 2.5(1), $\mathbf{tp}(\bar{c}', C)$ does not fork over B_1 and over B_2; as also $\mathbf{tp}(\bar{c}, C)$ does not fork over B and both extend $\mathbf{tp}(\bar{c}, B_2)$ which is stationary, clearly $\mathbf{tp}(\bar{c}', C) = \mathbf{tp}(\bar{c}, C)$. By the choice of $\bar{c}'$, this type does not fork over A so $\mathbf{tp}(\bar{c}, C)$ does not fork over A as required. Concerning "stationary" it follows by part (3) with (A, B_1, B_2, C) here standing for (A_0, A_1, A_2, A_3) there.
5) Let N be such that $B \subseteq N$ and we should find an extension p of $\mathbf{tp}(\bar{c}_1{}^\frown\bar{c}_2, B)$ in $\mathscr{S}^{\ell g(\bar{c}_1)+\ell g(\bar{c}_2)}(N)$ such that every automorphism of N over A maps p itself.

As $\mathbf{tp}(\bar{c}_1, B)$ does not fork over A, there is $\bar{c}_1^*$ realizing $\mathbf{tp}(\bar{c}_1, B)$ such that every automorphism of N over A maps $\mathbf{tp}(\bar{c}_1^*, N)$ to itself. Let $\lambda = \|N\| + \chi_{\mathfrak{s}} + |\ell g(\bar{c}_1 \hat{\ } \bar{c}_2)|$ and $M <_{\mathfrak{s}} \mathfrak{C}$ be a strongly $(\mathbb{D}_{\mathfrak{s}}, \lambda^+)$-homogeneous model extending N such that $\bar{c}_1^* \subseteq M$. We can find $\bar{c}_2^*$ such that $\mathbf{tp}(\bar{c}_1^* \hat{\ } \bar{c}_2^*, B) = \mathbf{tp}(\bar{c}_1 \hat{\ } \bar{c}_2, B)$. So $\mathbf{tp}(\bar{c}_2^*, B \cup \bar{c}_1^*)$ does not fork over $A \cup \bar{c}_1^*$, hence there is $\bar{c}_2^{**}$ realizing $\mathbf{tp}(\bar{c}_2^*, B \cup \bar{c}_1^*)$ such that every automorphism of M over $A \cup \bar{c}_1^*$ maps $\mathbf{tp}(\bar{c}_2^{**}, M)$ to itself.

Now $\bar{c}_1^* \hat{\ } \bar{c}_2^{**}$ is as required: clearly, it realizes $\mathbf{tp}(\bar{c}_1 \hat{\ } \bar{c}_2, B)$; let $f \in \mathrm{AUT}(N)$, $f \restriction A = \mathrm{id}_A$, then by the choice of $\bar{c}_1^*$, there is $g \in \mathrm{AUT}(\mathfrak{C})$ satisfying $f \subseteq g$ and $g(\bar{c}_1^*) = \bar{c}_1^*$; now by the choice of M without loss of generality $g \restriction M \in \mathrm{AUT}(N)$, so by the choice of $\bar{c}_2^{**}$ without loss of generality $g(\bar{c}_2^{**}) = \bar{c}_2^{**}$, so we finish.
6) - 7) Easy.
8) Let $f \in \mathrm{AUT}(\mathfrak{C})$ be the identity on A and maps $\bar{b}_1$ to $\bar{b}_2$. Let $M <_{\mathfrak{s}} \mathfrak{C}$ be a model such that $B \subseteq M$ and $f(M) = M$. Let $\bar{c}^*$ realizing $\mathbf{tp}(\bar{c}, B)$ be such that every automorphism of M maps $\mathbf{tp}(\bar{c}^*, M)$ to itself, it exists by Definition 2.1 but $f \restriction M$ is a counterexample. Note again that we are using Definition 2.1 and not the one of [Sh:c].
9) Let $\lambda = \chi_{\mathfrak{s}} + |B \cup C|$ and let M be a strongly $(\mathbb{D}_{\mathfrak{s}}, \lambda^+)$-homogeneous model which contains $A \cup C$. As $\mathbf{tp}(B, C)$ does not fork over A we can find $f \in \mathrm{AUT}(\mathfrak{C})$ mapping M onto itself such that $f \restriction C = \mathrm{id}_C$ and $\mathbf{tp}(f(B), M)$ does not fork over A, so:

- $(*)_1$ every automorphism h_0 of M over A can be extended to an automorphism h^+ of $\mathfrak{C}$ which is the identity over B.

Let $\mu = \chi_{\mathfrak{s}} + \|M \cup B\|$ and let N be a strongly $(\mathbb{D}_{\mathfrak{s}}, \mu^+)$-homogeneous models which contains $M \cup B$. As $\mathbf{tp}(B_1, A \cup B_0)$ does not fork over A there is an automorphism $g \in \mathfrak{C}$ over $A \cup f(B_0)$ such that $\mathbf{tp}(g(f(B_1)), N))$ does not fork over A; clearly

- $(*)_2$ every automorphism h of N over A can be extended to an automorphism h^+ of $\mathfrak{C}$ which is the identity on $g(f(B_1))$.

Now by the choice of f if h_0 is an automorphism of M over A then $h_0(\mathbf{tp}(f(B), M)) = \mathbf{tp}(f(B), M)$ but $B_0 \subseteq B \wedge g \restriction f(B_0) = \mathrm{id}_{f(B_0)}$ hence $h_0(\mathbf{tp}(f(B_0), M)) = \mathbf{tp}(f(B_0), M)$. As $B_0 \subseteq B$, by $(*)_1$ we know that $h_0 \cup \mathrm{id}_{f(B_0)}$ can be extended to an automorphism of $\mathfrak{C}$. Hence it can be extended to an automorphism h_1 of N hence by $(*)_2$

we know that $h_1 \cup \mathrm{id}_{(g(f(B_1)))}$ can be extended to an automorphism of $\mathfrak{C}$ so it extends $h_0 \cup \mathrm{id}_{f(B_0) \cup g(f(B_1))}$.

The previous paragraph implies that $\mathbf{tp}(gf(B_1), M)$ does not fork over A and $\mathbf{tp}(gf(B_1) \cup f(B_0), M)$ does not fork over A, so as $C \subseteq M$ we get that $f_* := ((gf) \restriction B_1) \cup (f \restriction B_0) \cup \mathrm{id}_C$ can be extended to an automorphism of $\mathfrak{C}$. Also by the construction $\mathbf{tp}(gf(B_1), N)$ does not fork over $A \cup f(B_0)$, hence by monotonicity, i.e. part (1) the type $\mathbf{tp}(fg(B_1), f(B_0) \cup C)$ does not fork over $f(B_0) \cup A$. Applying $(gf)^{-1} = f_*^{-1}$ we get the desired result. $\square_{2.5}$

2.6 Definition. Let $A \subseteq B$, we say $\mathbf{tp}(\bar{c}, B)$ does μ-strongly splits over A <u>if</u> there are $\bar{b}_i$ satisfying $\ell g(\bar{b}_i) < \mu$, $\langle \bar{b}_i : i < ((\chi_{\mathfrak{s}} + |A|)^\mu)^+ \rangle$ is 2-indiscernible over A (i.e., $\mathbf{tp}(\bar{b}_i {}^\frown \bar{b}_j, A)$ is the same for $i < j < ((\chi_{\mathfrak{s}} + |A|)^\mu)^+$) and $\bar{b}_0, \bar{b}_1 \in B, \mathbf{tp}(\bar{c} {}^\frown \bar{b}_1, A) \neq \mathbf{tp}(\bar{c} {}^\frown \bar{b}_2, A)$ (if we omit μ we mean $\mu = \bar{\kappa}$).

2.7 Definition. 1) We say $\mathbf{J} = \langle \bar{b}_\alpha : \alpha < \alpha^* \rangle$ is (μ, κ)-convergent over A <u>if</u> for every $\bar{c}$ of length $< \mu$, for some $w \subseteq \alpha^*$ satisfying $|w| < \kappa$, for every $i \in \alpha^* \backslash w$ the type $\mathbf{tp}(\bar{c} {}^\frown \bar{b}_i, A)$ is the same.
2) For $C \subseteq \mathfrak{C}$ we let $\mathrm{Av}^s(\mathbf{J}, C) = \{\mathbf{tp}(\bar{b}_\alpha, A \cup \bar{c}) : \bar{c} \in C$, and for all but $< |\alpha^*|$ ordinals $\gamma < \alpha^*$ we have $\mathbf{tp}(\bar{b}_\alpha, A \cup \bar{c}) = \mathbf{tp}(\bar{b}_\gamma, A \cup \bar{c})\}$.
3) The superscript s in part (1) signifies that we have found a set of averages as we vary $\bar{c} \in C$. If there is a unique $p \in \mathscr{S}^{\ell g(\bar{b}_0)}(C)$ extending all those types we denote it by $\mathrm{Av}(\mathbf{J}, C)$.
4) We say $\langle \bar{b}_\alpha : \alpha < \alpha^* \rangle$ is based on A <u>if</u> for every C, extending A, $\mathrm{Av}(\mathbf{J}, C)$ is well defined and does not fork over A.

2.8 Observation. If $\mathbf{J} = \{\bar{a}_\alpha : \alpha < \alpha^*\}$, <u>then</u> there is at most one $p \in \mathscr{S}^{\ell g(\bar{a}_0)}(N)$ which extends every $q \in \mathrm{Av}^s(\mathbf{J}, N)$ when $\mathrm{cf}(|\alpha^*|) > \chi_{\mathfrak{s}} + |\ell g(\bar{a}_\alpha)|$, so the "unique" in Definition 2.7(4) is redundant.

Proof. By 2.4(3) there is at most one p. $\square_{2.8}$

2.9 Lemma. *1) If* $\bar{b}_\alpha = \langle b^\alpha_i : i < i_0\rangle$, $\bar{c}_\alpha = \langle b^\alpha_{h(i)} : i < i_1\rangle$ *where* h *is a function from* i_1 *into* i_0, *and* $\langle \bar{b}_\alpha : \alpha < \alpha^*\rangle$ *is* (μ,κ)*-convergent over* A *then* $\langle \bar{c}_\alpha : \alpha < \alpha^*\rangle$ *is* (μ,κ)*-convergent over* A*. Similarly for being based on* A.
2) If $M \le_{\mathfrak{s}} M_\alpha <_{\mathfrak{s}} \mathfrak{C}$, $(\mathrm{Rang}(\bar{b}_\alpha)) = M_\alpha <_{\mathfrak{s}} \mathfrak{C}, \bar{b}_\alpha = \langle b^\alpha_i : i < i_0\rangle$ *for* $\alpha < \alpha^*, b^0_i \mapsto b^\alpha_i$ *is an isomorphism from* M_0 *onto* M_α *extending* id_M, *and* $\{M_\alpha : \alpha < \alpha^*\}$ *is independent over* M *then* $\{\bar{b}_\alpha : \alpha < \alpha^*\}$ *is* $(\chi_{\mathfrak{s}} + |\ell g(\bar{b}_0)|)^+$*-convergent and is based on* M.

Proof. Check.

2.10 Lemma. *(Symmetry). If* $\mathbf{tp}(\bar{d}, B \cup \bar{c})$ *does not fork over* A_2 *and* $A_1 \subseteq B$, $A_2 \subseteq B$ *and* $\mathbf{tp}(\bar{c}, B)$ *does not fork over* A_1, *then* $\mathbf{tp}(\bar{c}, B \cup \bar{d})$ *does not fork over* A_1.

Proof. Let $\lambda = (|B| + 2)^{\chi(\mathfrak{s})}$ and assume that the conclusion fails. We choose by induction on $\alpha < \lambda^+$, $\bar{c}_\alpha, \bar{d}_\alpha$ such that:

(*i*) $\bar{c}_0 = \bar{c}$, $\bar{d}_0 = \bar{d}$

(*ii*) $\mathbf{tp}(\bar{c}_\alpha, (\bigcup_{\beta<\alpha} \bar{c}_\beta\hat{\ } d_\beta) \cup B)$ does not fork over A_1 and extend $\mathbf{tp}(\bar{c}_\alpha, B)$

(*iii*) $\mathbf{tp}(\bar{d}_\alpha, (\bigcup_{\beta<\alpha} \bar{c}_\beta\hat{\ } b_\beta) \cup B \cup \bar{c}_\alpha)$ does not fork over A_2 and extend $\mathbf{tp}(\bar{d}, B \cup \bar{c})$.

This is possible by 2.4(6) [and for $\alpha = 0$, (ii), (iii) holds by assumptions].

Now if $\alpha \le \beta < \lambda^+$, by (iii) and 2.5(8) as $\bar{c}_0 = \bar{c}$ and $\bar{c}_\alpha$ realizes $\mathbf{tp}(\bar{c}, B)$ then $\mathbf{tp}(\bar{d}_\beta\hat{\ } c_\alpha, B) = \mathbf{tp}(\bar{d}_\beta\hat{\ }\bar{c}, B)$. By the second phrase of (iii), $\mathbf{tp}(\bar{d}_\beta, B \cup \bar{c}) = \mathbf{tp}(\bar{d}, B \cup \bar{c})$. So $[\alpha \le \beta \Rightarrow \mathbf{tp}(\bar{d}_\beta\hat{\ }\bar{c}_\alpha, B) = \mathbf{tp}(\bar{d}\hat{\ }\bar{c}, B)]$.

On the other hand, by (ii) and 2.5(1), for $\beta < \alpha$, $\mathbf{tp}(\bar{c}_\alpha, B \cup \bar{d}_\beta)$ does not fork over A_1, hence necessarily $\mathbf{tp}(\bar{d}_\beta\hat{\ }\bar{c}_\alpha, B) \ne \mathbf{tp}(\bar{d}\hat{\ }\bar{c}, B)$. So we get an order on $\{\bar{c}_\alpha\hat{\ }\bar{d}_\alpha : \alpha < \lambda^+\}$ contradiction as we shall prove in 2.11(3) below. $\square_{2.10}$

2.11 Lemma. *1) Suppose $A \subseteq M_0, M_\alpha(\alpha \le \gamma)$ is $\le_{\mathfrak{s}}$-increasing, $\bar{a}_\alpha \in M_{\alpha+1}, \mathbf{tp}(\bar{a}_\alpha, M_\alpha)$ does not fork over A and increase with α, $\chi \ge \chi_{\mathfrak{s}}$ and γ is infinite. Then $\{\bar{a}_\alpha : \alpha < \gamma\}$ is indiscernible over M_0, is (χ^+, χ^+)-convergent on A, and if $\gamma \ge \chi^+$, based on A.*
2) If $p \in \mathscr{S}^\gamma(N), N <_{\mathfrak{s}} \mathfrak{C}$ and $\chi = \chi_{\mathfrak{s}} + |\gamma|$, then for some $M \le_{\mathfrak{s}} N$ of cardinality $\le \chi$ the type p does not fork over M.
3) If $\chi = \chi_{\mathfrak{s}} + |\gamma|$ and $\bar{a}_\alpha \in {}^\gamma\mathfrak{C}$ for $\alpha < \mu$ and $(\forall \alpha < \mu)(|\alpha|^\chi < \mu = \mathrm{cf}(\mu))$ then

- (a) *for some unbounded $\mathscr{U} \subseteq \mu$ the (index) set $\{\bar{a}_\alpha : \alpha \in \mathscr{U}\}$ is indiscernible and (χ^+, χ^+)-convergent*
- (b) *there is a function h on $S = \{\delta < \mu : \mathrm{cf}(\delta) > \chi\}$ which is regressive (i.e. $h(\delta) < \delta$ for $\delta \in S$) satisfying: if $\mathscr{U} \subseteq S \cap E$ and $h \restriction \mathscr{U}$ is constant then $\{\bar{a}_\alpha : \alpha \in \mathscr{U}\}$ is indiscernible and (χ^+, χ^+)-convergent.*

Proof. 1) The indiscernibility follows by 2.5(8). That is, we can prove by induction on n that if $\beta \le \beta_{\ell,0} < \beta_{\ell,1} < \ldots < \beta_{\ell,n-1} < \gamma$ for $\ell = 1, 2$ then $\bar{a}_{\beta_{1,0}} \char`^ \ldots \char`^ \bar{a}_{\beta_{1,n-1}}$ and $\bar{a}_{\beta_{2,0}} \char`^ \ldots \char`^ \bar{a}_{\beta_{2,n-1}}$ realizes the same type over M_β; the induction step is by 2.5(8) as $\mathbf{tp}(\bar{a}_\alpha, M_\alpha)$ increasing with α, so this is exactly as in V.A.2.8 and even better V.A.3.2.

Let us prove convergence: without loss of generality $\langle M_\alpha : \alpha \le \gamma\rangle$ is not only $\le_{\mathfrak{s}}$-increasing but also continuous and recall $M_\alpha <_{\mathfrak{s}} \mathfrak{C}$. For any $\bar{c} \in \mathfrak{C}$, $\ell g(\bar{c}) < \chi^+$, there are $w, N, N', \bar{c}'$ such that $N \le_{\mathfrak{s}} N' <_{\mathfrak{s}} \mathfrak{C}, \|N'\| \le \chi, \bar{c} \in N', N \le_{\mathfrak{s}} M_\gamma, \{N', M_\gamma\}$ independent over $N, 0 \in w \subseteq \gamma, |w| \le \chi$, $N = \bigcup\limits_{\alpha \in w} (N' \cap M_{\alpha+1} \backslash M_\alpha) \cup (N' \cap M_0)$, and for $\beta \in w \cup \{\gamma\}$, $\{N', M_\beta\}$ independent over $N \cap M_\beta$. [This holds by the proof of V.C.3.12].

Now for $\beta \in \gamma \backslash w$, let $\alpha = \mathrm{Min}(w \cup \{\gamma\} \backslash \beta)$, (so $\alpha > \beta$) so $\{N', M_\alpha\}$ is independent over $N \cap M_\alpha$; but $N \cap M_\alpha = N \cap M_\beta$, hence $\{N', M_{\beta+1}\}$ is independent over $N \cap M_\beta$, hence $\{\langle N', M_\beta\rangle^{\mathrm{gn}}, M_{\beta+1}\}$ is independent over M_β, so $\mathbf{tp}(\bar{a}_\beta, \langle N', M_\beta\rangle^{\mathrm{gn}})$ does not fork over M_β, hence by 2.5(4) does not fork over A. So now as in V.A.2.8, using the symmetry from 2.10, (recall that its proof will be completed below) we finish.

Also "based on A" is easy.
2) Let $\bar{c} \in {}^{\gamma}\mathfrak{C}$ realize p. As $\mathfrak{s}$ is χ-based we can find $M \leq_{\mathfrak{s}} M_1$ such that $\|M_1\| \leq \chi, \bar{c} \subseteq M_1$ and M, N, M_1 is in stable amalgamation. By Definition 1.2(1) we can deduce that $\mathbf{tp}(\bar{c}, N)$ does not fork over M, but $p = \mathbf{tp}(\bar{c}, N)$ so M is as required.
3) For completing the proof of 2.10, using its notation let $\mu := \lambda^+$ and $\bar{a}_\alpha := \bar{c}_\alpha \hat{} \bar{d}_\alpha$ for $\alpha < \lambda^+ = \mu$ and choose $M_\alpha <_{\mathfrak{s}} \mathfrak{C}$ by induction on $\alpha < \mu$ such that M_α is $\leq_{\mathfrak{s}}$-increasing continuous and $\|M_\alpha\| \leq \lambda, A \subseteq M_0$ and $\alpha = \beta + 1 \Rightarrow \bar{a}_\beta \subseteq M_\alpha$. Now (for completing 2.10's proof), it suffices to prove that $\mathbf{tp}(\bar{a}_\alpha \hat{} \bar{a}_\beta, M_0) = \mathbf{tp}(\bar{a}_\beta \hat{} \bar{a}_\alpha, M_0)$ for $\alpha < \beta < \mu$, as done below.

For proving 2.11(3), first we choose $M_\alpha <_{\mathfrak{s}} \mathfrak{C}$ of cardinality $< \lambda$, $\leq_{\mathfrak{s}}$-increasing continuous, $\bar{a}_\alpha \subseteq M_{\alpha+1}$. Now for each $\alpha < \lambda$ we can find $N_\alpha \leq_{\mathfrak{s}} M_{\alpha+1}$ of cardinality $\leq \chi$ such that $\bar{a}_\alpha \subseteq N_\alpha$ and $N'_\alpha = N_\alpha \cap M_\alpha, N'_\alpha, M_\alpha$ are in stable amalgamation. Let $\bar{b}_\alpha \in {}^{\gamma+\chi}(N_\alpha)$ list N_α such that $\bar{a}_\alpha \triangleleft \bar{b}_\alpha$. We define a two-place relation $\mathscr{E}$ on λ: $\alpha \mathscr{E}_\beta$ iff the function $h_{\alpha,\beta}$ mapping $\bar{b}_\alpha$ to $\bar{b}_\beta$ is an isomorphism from N_α onto N_β, mapping (necessarily $\bar{a}_\alpha$ to $\bar{a}_\beta$ and) $N'_\alpha = N_\alpha \cap M_\alpha$ onto $N'_\beta = N_\beta \cap N_\beta$, moreover it is the identity on $N_\alpha \cap M_\alpha$. Easily $\mathscr{E}$ is an equivalence relation and for every $\alpha < \mu, \{\beta/\mathscr{E} : N'_\beta \subseteq M_\alpha\}$ has $\leq \|M_\alpha\|^\chi < \mu$ members.

By Fodor lemma there are $\mathscr{E}$-equivalent $\alpha \neq \beta \in S$. Now by symmetry for NF (for models) there is an automorphism h of $\mathfrak{C}$ such that $h \restriction N'_\alpha = \text{ id}, h(\bar{b}_\alpha) = \bar{b}_\beta \wedge h(\bar{b}_\beta) = \bar{b}_\alpha$. This completes the proof of 2.10. Also for 2.11(3) we are done.

$\square_{2.11}$

2.12 Definition. We say $\{\bar{a}_\alpha : \alpha \in u\}$ is independent over A <u>if</u> $\mathbf{tp}(\bar{a}_\alpha, A \cup \{\bar{a}_\beta : \beta \in u, \beta \neq u\})$ does not fork over A for every $\alpha \in u$.

2.13 Claim. *If $n < \omega$, $\mathbf{tp}(\bar{a}_m, A \cup \bigcup_{\ell<m} \bar{a}_\ell)$ does not fork over A for $m < n$ <u>then</u> $\{\bar{a}_m : m < n\}$ is independent over A.*

Proof. By induction on n and use of symmetry (2.10). $\square_{2.13}$

2.14 Definition. 1) The types $\mathbf{tp}(\bar a_1, A_1), \mathbf{tp}(\bar a_2, A_2)$ are parallel <u>if</u> both are stationary and they have a common stationarization (see below) over some M satisfying $A_1 \cup A_2 \subseteq M$.
2) Suppose $p = \mathbf{tp}(\bar b, B)$ does not fork over A and $A \subseteq B \subseteq C$, <u>then</u> $\mathbf{tp}(\bar c, C)$ is a stationarization of (p, A) <u>if</u> it extends p and does not fork over A. If $A = \text{Dom}(p)$ we may write p instead of (p, A).
3) Assume p_ℓ is stationary over A_ℓ (for $\ell = 1, 2$) then (p_1, A_1), (p_2, A_2) are parallel <u>if</u> they have a common stationarization over some M satisfying $\text{Dom}(p_1) \cup \text{Dom}(p_2) \subseteq M$.

2.15 Claim. *1) If $\mathbf{tp}(\bar a_1, A_1)$, $\mathbf{tp}(\bar a_2, A_1)$ are parallel (so each is stationary) <u>then</u> for every $B \subseteq \mathfrak{C}$ containing $A_1 \cup A_2$, the stationarization of $\mathbf{tp}(a_1, A_1)$, $\mathbf{tp}(a_2, A_2)$ over B are equal.*
2) "parallel" is an equivalence relation.

Proof. 1) Suppose $\mathbf{tp}(\bar a_1, A_1)$, $\mathbf{tp}(\bar a_2, A_2)$ are parallel <u>then</u> for some M, $A_1 \cup A_2 \subseteq M \leq_{\mathfrak{s}} \mathfrak{C}$ they have a common stationarization $\mathbf{tp}(\bar a, M)$. Let $A_1 \cup A_2 \subseteq B \subset \mathfrak{C}$. Choose $N <_{\mathfrak{s}} \mathfrak{C}$, $B \cup M \subseteq N$, and without loss of generality $\mathbf{tp}(\bar a, N)$ is the stationarization of $\mathbf{tp}(\bar a, M)$ over N. By 2.5(4), the type $\mathbf{tp}(\bar a, N)$ does not fork over A_ℓ (for $\ell = 1, 2$) hence is a stationarization of $\mathbf{tp}(\bar a_\ell, A_\ell)$ (as clearly it extends $\mathbf{tp}(\bar a_\ell, A_\ell)$). By 2.5(1), the type $\mathbf{tp}(\bar a, B)$ does not fork over A_1 and over A_2, and clearly it extends $\mathbf{tp}(\bar a_\ell, A_\ell)$, $(\ell = 1, 2)$. So $\mathbf{tp}(\bar a, B)$ exemplifies $\mathbf{tp}(\bar a_1, A_1)$, $\mathbf{tp}(\bar a_2, A_2)$ has a common stationarization over B; being stationary we finish.
2) Easy. $\square_{2.15}$

<u>2.16 Exercise</u>: Assume $\text{Rang}(\bar a_1) \subseteq \text{Rang}(\bar a_2), A_1 \subseteq A_2 \subseteq B$. If $\mathbf{tp}(\bar a_2, B)$ does not fork over A_1 and $\bar a_2'$ realizes $\mathbf{tp}(\bar a_2, B)$ and $\mathbf{tp}(\bar a_2', A_2)$ does not fork over A_1, <u>then</u>

- (a) there is a unique $\bar a_1'$ such that $\bar a_1' {}^\frown \bar a_2'$ realizes $\mathbf{tp}(\bar a_1 {}^\frown \bar a_2, A_1)$
- (b) it follow that $\mathbf{tp}(\bar a_1', B)$ does not fork over A_1
- (c) if $\mathbf{tp}(\bar a_1, A_2)$ is stationary over A_1 then $\mathbf{tp}(\bar a_1', B)$ is the stationarization of $(\mathbf{tp}(\bar a_1, A_2), A_1)$.

§3 Defining Superstability and $\kappa(\mathfrak{s})$

We here define the parallel to $\kappa(T)$: a set $\kappa(\mathfrak{s})$ of regular cardinals in Definition (3.1); we say $\mathfrak{s}$ is superstable if and only if $\kappa(\mathfrak{s}) = \emptyset$; we show that $\kappa(\mathfrak{s}) \neq \emptyset$ implies a strong non-structure (3.5) and prove $\sup \kappa(\mathfrak{s}) \leq \chi_{\mathfrak{s}}$ (3.7). We further show that a $(< \omega)$-type over a directed union of models does not fork over one of them for $\mathfrak{s}$ superstable; also the union is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-homogeneous if each model is (and a weaker Lemma when $\kappa(\mathfrak{s}) \neq \emptyset$) (in 3.8, 3.9, 3.10, 3.11, 3.13). We then define stability in λ and derive some basic facts.

We also give some explanation for the "deviations" from the first order case.

3.1 Definition. $\kappa(\mathfrak{s})$ is the class of regular κ such that for some $M_i <_{\mathfrak{s}} \mathfrak{C}$ $(i \leq \kappa)$, $\leq_{\mathfrak{s}}$-increasing and for some finite $\bar{c} \in M_\kappa$ the type $\mathbf{tp}(\bar{c}, \bigcup\limits_{i<\kappa} M_i)$ forks over M_j for every $j < \kappa$.

Because we have not shown that forking has finite character we can't show that $\kappa(\mathfrak{s})$ is an initial segment of the cardinals. Thus, we must use the set $\kappa(\mathfrak{s})$ instead of its supremum which we denoted $\kappa(T)$ in the first order case. (Note: we did not say that $\mathbf{tp}(\bar{c}, M_{j+1})$ forks over M_j).

3.2 Remark. 1) "Finite $\bar{c}$" is essential, otherwise, for any $\mathfrak{C}$ even with no relations, just equality, let M_i $(i < \kappa)$ be strictly increasing, $\bar{c} = \langle c_i : i < \kappa \rangle$, $c_i \in M_{i+1} \backslash M_i$.
2) What about demanding just "$\kappa > |\ell g(\bar{c})|$"? There is no reason of opposing this, but it just complicates our non-structure theorem, with no gain visible now. But see 3.9.
3) Why not "$\mathbf{tp}(\bar{c}, M_{j+1})$ forks over M_j for every $j < \kappa$"? Definition 3.1 is more natural in our context and the witness for forking is no longer a finite formula. Definition 3.1 is seemingly a different and better definition, not harming much the non-structure proofs, while helping with structure.

3.3 Definition. We say $\mathfrak{s}$ is superstable if $\kappa(\mathfrak{s}) = \emptyset$.

3.4 Remark. 1) So the finiteness in the definition of superstable disappears; neither "a formula catching the rank" nor "does not fork over a finite set" appear on the surface.
2) We can define $\mathfrak{C}^{\mathrm{eq}}$ (naturally, delayed to [Sh:E54]) and in it for every parallelism class of $(<\omega)$-type p, an element $p/\|$ such that $p/\| \in M^{\mathrm{eq}} \Leftrightarrow (\exists q)[p\|q \ \& \ q \in \mathscr{S}^{<\infty}(M)]$ and p does not fork over $p/\|$.
3) Anyhow, types of models (i.e., $\mathscr{S}_c(M)$) play a greater role here than in first order case, but less than in Chapter V.D, as still we need elements for the $\bar{c}$. For superstability we shall see examples.
4) Why do we look at $\kappa(\mathfrak{s})$ and not, *e.g.* $\mathrm{Min}(\kappa(\mathfrak{s}))$? If we would like, e.g., to characterize $\{\lambda : \mathfrak{s}$ has a $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous model of cardinality $\lambda\}$ we shall need the set.
5) Even though we have established in V.D.3.16(4) the local character of dependence for models, this does not extend automatically to elements. That is, it seems that may be in general there is a $\leq_{\mathfrak{s}}$-increasing sequence $\langle M_i : i < \omega\rangle$ such that $\mathbf{tp}(c, M_i)$ does not fork over M_0 for each i but $\mathbf{tp}(\bar{c}, \bigcup\limits_{i<\omega} M_i)$ forks over M_0. In the presence of superstability this is impossible since for some i we have $\mathbf{tp}(c, M_\delta)$ does not fork over M_i and we can apply transitivity. This observation is the key to the central Lemma 5.3.
Essentially the theorems on superstable first order theories generalize.

3.5 Theorem. *[The non-structure theorem for unsuperstability] If κ is a cardinal in $\kappa(\mathfrak{s})$ <u>then</u> we have strong non-structure as follows.*

If $\lambda = \lambda^\kappa + \chi^{\chi(\mathfrak{s})}, \chi \geq \kappa + \chi_{\mathfrak{s}}$, <u>then</u> there are 2^λ non-isomorphic models in $\mathfrak{s}_\lambda$ and even in

$$\mathfrak{K}^{\mathrm{us}}_{\lambda,\chi,\kappa} = \mathfrak{K}^{\mathrm{us}}_{\chi,\kappa} \cap K_\lambda = \{M \in K_\lambda : M \textit{ is a } (<\kappa)\textit{-directed union of } (\mathbb{D}_{\mathfrak{s}}, \chi^+)\textit{-homogeneous models}\}.$$

Remark. 1) We can replace $\chi^{\chi(\mathfrak{s})}$ by Min$\{\|N\| : N$ is a $(\mathbb{D}_{\mathfrak{s}}, \chi^+)$-homogeneous model$\}$; we can replace χ^+ by suitable regular χ'.
2) We can get strong homogeneity, too.

Proof. Let M_i $(i \le \kappa)$, $\bar{c}$ witness $\kappa \in \kappa(\mathfrak{s})$. By Lowenheim-Skolem and the fact that NF, i.e. $\mathfrak{s}$ is $\chi_{\mathfrak{s}}$-based without loss of generality (by V.D.1.18 and Definition 1.2):

$$\|M_i\| \le \kappa + \chi_{\mathfrak{s}}.$$

We shall use below V.C§4.
Without loss of generality the sequence of M_i's is $\le_{\mathfrak{s}}$-increasing and continuous (except for $i = \kappa$) (if we waive smoothness, we can restore the continuity demand using $\kappa =$ Min$[\kappa(\mathfrak{s}) \cup \{\kappa : \kappa$ smoothness fail$\}]$ or redefining $\kappa(\mathfrak{s})$). Now we can find M_η, f_η for $\eta \in {}^{\kappa \ge}\lambda$ such that:

- (a) $f_\eta : M_{\ell g(\eta)} \xrightarrow{\text{onto}} M_\eta <_{\mathfrak{s}} \mathfrak{C}$
- (b) for $\alpha < \ell g(\eta)$, $\mathbf{tp}(M_\eta, M_{\eta \restriction \alpha} \cup \bigcup\{M_\nu : \nu \in {}^{\kappa >}\lambda, \nu \restriction (\alpha+1) \ne \eta \restriction (\alpha+1)\})$ does not fork over $M_{\eta \restriction \alpha}$
- (c) $f_{\eta \restriction \alpha} \subseteq f_\eta$.

Why? Let $\langle \eta_\alpha : \alpha < \alpha(*) \rangle$ list ${}^{\kappa \ge}\lambda$ such that $\eta_\alpha \triangleleft \eta_\beta \Rightarrow \alpha < \beta$ (we can list only $\bigcup\{{}^{\zeta}\lambda : \zeta < \kappa$ successor or $\zeta = \kappa\}$).

Now let $w^*_\alpha = \{\beta : \eta_\beta \triangleleft \eta_\alpha\}$ and $C_\alpha = M_{\ell g(\eta)}$. Now apply V.C.4.14 and get a stable construction $\mathscr{A}$ with $\ell g(\mathscr{A}) = \alpha(*)$ and $w^{\mathscr{A}}_\alpha = w^*$ for $\alpha < \ell g(\mathscr{A})$. Now by V.C.4.5(1) we have

- $(*)$ if $u_1, u_2 \subseteq \alpha(*)$ are $\mathscr{A}$-closed then $A^{\mathscr{A}}_{u_1 \cap u_2}, A^{\mathscr{A}}_{u_1}, A^{\mathscr{A}}_{u_2}$, are in stable amalgamation in $\mathfrak{C}$.

Translating this to our terms (for $S \subseteq {}^{\kappa \ge}\lambda$ let $c\ell(S) = \{\eta \restriction i : \eta \in S, i \le \ell g(\eta)\}$ and say S is $\triangleleft$-downward closed if $S = c\ell(S)$, this is compatible with V.C§4), note

$(*)$
- (a) if $S \subseteq {}^{\kappa \ge}\lambda$ <u>then</u> $\langle \bigcup\{M_\eta : \eta \in S \cup \{<>\}\} \rangle^{\text{gn}}_{\mathfrak{C}} = \langle \bigcup\{M_\eta : \eta \in c\ell(S)\} \rangle^{\text{gn}}_{\mathfrak{C}}$
- (b) if $S_1, S_2 \subseteq {}^{\kappa \ge}\lambda$ are $\triangleleft$-downward closed and non-empty then $M_{S_1 \cap S_2}, M_{S_1}, M_{S_2}$ is in stable amalgamation inside $\mathfrak{C}$.

Now for $\eta \in {}^{\kappa}\lambda$, let $\bar{c}_\eta := f_\eta(\bar{c})$.

3.6 Fact. Assume ${}^{\kappa>}\lambda \subseteq S \subseteq {}^{\kappa\geq}\lambda$ and recall $M_S = \langle \bigcup\limits_{\eta \in S} M_\eta \rangle^{\mathrm{gn}}$.

Then $\mathbf{tp}(\bar{c}_\eta, \bigcup\limits_{\alpha<\kappa} M_{\eta\restriction\alpha})$ is realized in M_S if and only if $\eta \in S$.

[Proof of Fact: For every $\bar{c}^* \in M_S$ (of length $\ell g(\bar{c})$) there is a finite $w \subseteq S$ such that $\bar{c} \subseteq \langle \bigcup\limits_{\nu \in w} M_\nu \rangle^{\mathrm{gn}}$ (by the "finite character", explicitly V.C.4.5(2) for $\mu = \aleph_0$). We can prove by induction on $|w|$, that $\mathbf{tp}(\bigcup\limits_{\nu\in w} M_\nu, \bigcup\limits_{\alpha<\kappa} M_{\eta\restriction\alpha})$ does not fork over $M_{\eta\restriction\alpha(w)}$ for some $\alpha(w) < \kappa$, and finish easily].

By Chapter III of [Sh 300] or [Sh:e, IV,§2,§3] this is enough for proving $\dot{I}(\lambda, \mathfrak{K})$ is large. But we would like to show that even $\dot{I}(\lambda, \mathfrak{K}^{\mathrm{us}}_{\lambda,\chi,\kappa})$ is large.

We let $\{u_i : i < \lambda\}$ list a cofinal subset of $\{u : u \subseteq S, |u| < \kappa\}$ such that $[u_i \subseteq u_j \Rightarrow i \leq j]$ and let $\langle w_i : i < \lambda\rangle$ be such that:

- (a) $w_i \subseteq i, |w_i| <_{\mathfrak{s}} \kappa$,
- (b) $[j \in w_i \Rightarrow w_j \subseteq w_i]$
- (c) for each i and j $(< \lambda)$ there is ζ satisfying $\max\{i, j\} < \zeta < \lambda$ such that $w_\zeta = w_i \cup \{j\} \cup w_j$
- (d) if $w \subseteq \mu$ & $|w| < \kappa$ & $\bigwedge\limits_{i \in w} w \cap i = w_i$ then for some j, $w = w_j$.

Let $u_i^+ = \bigcup\limits_{j \in w_i} u_j$. We now define by induction on $i \leq \lambda$, M_S^i, N_S^j (for $j < \lambda$), and g_{j_1,j_2} (for $j_1, j_2 \leq i$) such that:

- (a) $M_S^0 = M_S <_{\mathfrak{s}} \mathfrak{C}$
- (b) $M_S^\delta = \bigcup\limits_{i<\delta} M_S^i$ for limit $\delta \leq \lambda$, moreover $\langle M_S^i : i \leq \lambda\rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous
- (c) N_S^i is $(\mathbb{D}_{\mathfrak{s}}, \chi^+)$-homogeneous of cardinality $\chi^{\chi(\mathfrak{s})}$ (and $N_S^i <_{\mathfrak{s}} \mathfrak{C}$)

(d) $j \in w_i \Rightarrow N^j_S \leq_{\mathfrak{s}} N^i_S$

(e) $\eta \in u^+_i \Rightarrow M_\eta \leq_{\mathfrak{s}} N^i_S$

(f) $\mathbf{tp}(N^i_S, M_S)$ does not fork over $\langle \cup\{M_\eta : \eta \in u_i\}\rangle^{\rm gn}$

(g) $\langle M_S, N^i_S, w_i : i < \lambda\rangle$ is a stable construction,

(h) if $i(0), i(1) < \lambda$, and there are functions h_1, h_2 satisfying (α)–(ε) below, <u>then</u> $g_{i(0),i(1)}$ is well defined, is an isomorphism from $N^{i(0)}_S$ onto $N^{i(1)}_S$ extending $g_{j,h_2(j)}$ for $j \in w_{i(0)}$ and extending $f_{h_1(\eta)} \circ f^{-1}_\eta$ for $\eta \in u^+_{i(0)}$ where

(α) $h_1 : u_{i(0)} \to u_{i(1)}$ one to one onto,

(β) $h_2 : w_{i(0)} \to w_{i(1)}$ one to one order preserving onto,

(γ) $(\forall \eta \in u_{i(0)})[\ell g(\eta) = \ell g(h_1(\eta))]$

(δ) $(\forall \eta, \nu \in u_{i(0)})[\eta \triangleleft \nu \Leftrightarrow h_1(\eta) \triangleleft h_1(\nu)]$

(ε) $(\forall \eta \in u^+_{i(0)})(\forall j \in w_{i(0)})[\eta \in u^+_j \Leftrightarrow h_1(\eta) \in u^+_{h_2(j)}]$.

The rest should be clear. $\square_{3.5}$

3.6 Remark.

0) We may elaborate in [Sh:e].

1) Suppose we would like to apply 3.5 to the first order case, i.e., to prove for a first order unsuperstable T that $\dot{I}(\lambda, T) = 2^\lambda$. From the first conclusion in 3.5 we get only "many algebraically closed sets". In order to get "many models", we use the second conclusion in 3.5: $\dot{I}(\lambda, \mathfrak{K}^{\rm us}_{\lambda,\chi,\kappa}) = 2^\lambda$.

2) Here is the first order analog of the situation in Lemma 3.5. Let T be a countable stable but not superstable theory. Then one cannot prove (there are counterexamples) that for large κ, T has 2^κ pairwise non-isomorphic $\aleph_1$-saturated models of cardinality κ; but there are 2^κ models which are direct limits of $\aleph_1$-saturated models.

3.7 Lemma. *1)* $\kappa \in \kappa(\mathfrak{s}) \Rightarrow \kappa \leq \chi_{\mathfrak{s}}$.
2) If $\kappa \in \kappa(\mathfrak{s})$ *<u>then</u> there is a* $\leq_{\mathfrak{s}}$*-increasing continuous sequence* $\langle M_i : i \leq \kappa \rangle$ *of models from* $K^{\mathfrak{s}}_{\chi_{\mathfrak{s}}}$ *and* $p \in \mathscr{S}^{<\omega}(M_\kappa)$ *which forks over* M_i *for every* $i < \kappa$.

Proof. 1) Let M_i $(i \leq \kappa)$ and $\bar{c}$ witnesses $\kappa \in \kappa(\mathfrak{s})$; suppose for contradiction that $\kappa > \chi_{\mathfrak{s}}$ and note that by smoothness $\bigcup\limits_{i<\kappa} M_i \leq_{\mathfrak{s}} M_\kappa$.
As NF is $\chi_{\mathfrak{s}}$-based there are N_1, N_2 satisfying

$$N_1 \leq_{\mathfrak{s}} \bigcup_{i<\kappa} M_i, N_2 \leq M_\kappa, \bar{c} \in N_2$$

such that the triple $N_1, N_2, \bigcup\limits_{i<\kappa} M_i$ is in stable amalgamation and $\|N_\ell\| \leq \chi_{\mathfrak{s}}$ for $\ell = 1, 2$. As $\|N_1\| \leq \chi_{\mathfrak{s}}$ and κ is regular by definition of $\kappa(\mathfrak{s})$ (i.e. the choice of $\langle M_i : i \leq \kappa \rangle$ and $\bar{c}$) there is $\alpha < \kappa$ such that $N_1 \subseteq M_\alpha$. Now $\mathbf{tp}(N_2, \bigcup\limits_{i<\kappa} M_i)$ does not fork over N_1 hence by monotonicity it does not fork over M_α, hence $\mathbf{tp}(\bar{c}, \bigcup\limits_{i<\kappa} M_i)$ does not fork over M_α, contradiction.
2) Easy (or see the proof of 3.5). $\square_{3.7}$

3.8 Lemma. *1) If* I *is a* μ*-directed partial order,* $(\forall \kappa)[\kappa \in \kappa(\mathfrak{s}) \rightarrow \kappa < \mu]$,

$$M_t \leq_{\mathfrak{s}} M \subseteq N \textit{ for } t \in I,$$

$[t < s \Rightarrow M_t \subseteq M_s]$, *and* $M = \bigcup\limits_{t \in I} M_t$ *<u>then</u> for every* $\bar{c} \in {}^{\omega>}N$ *there is* $t \in I$ *such that* $\mathbf{tp}(\bar{c}, M)$ *does not fork over* M_t.

Proof. Easy.

3.9 Remark. 1) Using $\bar{c}$ of greater length, say ω, we can get the right conclusion, provided that we redefine $\kappa(\mathfrak{s})$. Let

$$\begin{aligned}\kappa_\theta(\mathfrak{s}) = \{\kappa :& \mathrm{cf}(\kappa) = \kappa > \theta \text{ and there are } M_i <_{\mathfrak{s}} \mathfrak{C},\\ & \leq_{\mathfrak{s}} \text{-increasing continuous for } i \leq \kappa + 1,\\ & \text{and } \bar{c} \in M_{\kappa+1}, \ell g(\bar{c}) = \theta \text{ such that}\\ & \text{for each } i < \kappa,\ \mathbf{tp}(\bar{c}, M_\kappa) \text{ forks over } M_i\}.\end{aligned}$$

In Lemma 3.8 we demand: $\ell g(\bar{c}) < \mu$ and $[\kappa \in \kappa_{\ell g(\bar{c})}(\mathfrak{s}) \Rightarrow \kappa < \mu]$. Note that, e.g., if $\bar{c} = \langle c_\ell : \ell < \omega\rangle$ <u>then</u>

$$\begin{aligned}[\bigwedge_{n<\omega} \mathbf{tp}(\bar{c} \restriction n, N) \text{ does not fork over } M \Rightarrow \\ \Rightarrow \mathbf{tp}(\bar{c}, N) \text{ does not fork over } M]\end{aligned}$$

does not seem true in general.
2) So if, e.g., $\kappa(\mathfrak{s}) = \{\aleph_0\}$ we can still be interested but we are not now. If we look at the class of something like $\aleph_1$-saturated (or $\aleph_1$-compact) then $\kappa_{\aleph_0}(\mathfrak{s}) = \emptyset$ is a good dividing line but see 3.12.

3.10 Theorem. *1) If M_i $(i < \delta)$ is $\leq_{\mathfrak{s}}$-increasing, each M_i is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-homogeneous, $\mu > \chi_{\mathfrak{s}}$, and $\mathrm{cf}(\delta) \notin \kappa(\mathfrak{s})$, <u>then</u> $\bigcup_{i<\delta} M_i$ is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-homogeneous.*
2) If $\kappa = \mathrm{cf}(\delta) \in \kappa(\mathfrak{s}), \lambda^{<\mu} = \lambda, \mu > \chi_{\mathfrak{s}} + |\tau(\mathfrak{s})|$ <u>then</u> we can find a $(\mathbb{D}_{\mathfrak{s}}, \mu)$-homogeneous model of cardinality λ, morever models $M_i (i < \delta)$ $\leq_{\mathfrak{s}}$-increasing with i, each of cardinality λ such that $\bigcup_{i<\delta} M_i$ is not a $(\mathbb{D}_{\mathfrak{s}}, \mu)$-homogeneous model.

3.11 Remark. 1) A priori it looks as though the old proof fails to generalize, as there we have dealt with the type of an element, but the model-homogeneity=saturativity Lemma V.B.3.18 saves us (it shows

homogeneity can be achieved by realizing only types of elements).
2) Without loss of generality μ is regular (and even is a successor cardinal).

Proof. 1) By V.B.3.18 it is enough to prove that for every $p \in \mathscr{S}^1(\bigcup_{i<\delta} M_i)$ and $A \subseteq \bigcup_{i<\delta} M_i$ of cardinality $< \mu$ the type $p \restriction A$ is realized in $\bigcup_{i<\delta} M_i$. As $\mathrm{cf}(\delta) \notin \kappa(\mathfrak{s})$ for some $i < \delta$, p does not fork over M_i. Now M_i is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-homogeneous, and we can apply 1.6.
2) Let $M_i (i \le \kappa+1)$ be increasing continuous, $\bar{c} \in {}^{\omega>}|M_\kappa|$, $\mathbf{tp}(\bar{c}, M_\kappa)$ forks over M_α for each $\alpha < \kappa$. By V.D.1.18 without loss of generality $\|M_i\| \le \chi_{\mathfrak{s}}$ for $i \le \kappa$. We now choose by induction on $i \le \kappa$, model N_i such that:

- (a) $j < i \Rightarrow N_j \le_{\mathfrak{s}} N_i$,
- (b) if i is a limit ordinal then $N_i = \cup\{N_j : j < i\}$
- (c) if i is non-limit then N_i is $(\mathbb{D}_{\mathfrak{s}}, \mu)$-homogeneous
- (d) $M_i \subseteq N_i$ and $M_i, N_i, M_{\kappa+1}$ is in stable amalgamation,
- (e) $\|N_i\| = \lambda$.

This is easily done, e.g. in limit stage i use V.C.1.10. In particular the triple $M_\kappa = \bigcup_{i<\kappa} M_i, N_\kappa = \bigcup_{i<\kappa} N_i, M_{\kappa+1}$ is in stable amalgamation. If $\mathbf{tp}(\bar{c}, \bigcup_{i<\kappa} M_i)$ is realized in $\bigcup_{i<\kappa} N_i$, let $\bar{c}' \in {}^{\omega>}(\bigcup_{i<\kappa} N_i)$ realize it, so for some $\alpha < \kappa$, $\bar{c}' \in {}^{\omega>}|N_\alpha|$, <u>then</u> $\mathbf{tp}(\bar{c}', \bigcup_{i<\kappa} M_i)$ does not fork over M_α by clause (e), contradiction. So $\bigcup_{i<\kappa} N_i$ is not $(\mathbb{D}_{\mathfrak{s}}, \chi_{\mathfrak{s}}^+)$-homogeneous, and we finish. $\square_{3.10}$

3.12 Claim. *1) We can use $\kappa_\theta(\mathfrak{s})$ in proving 3.10(2) when* $\mathrm{cf}(\delta) > \theta$ *instead of $\kappa(\mathfrak{s})$, this holds for 3.10(1) too, trivially.*
2) We can conclude that $\kappa_\theta(\mathfrak{s}) = \kappa(\mathfrak{s}) \backslash \theta^+$.

Proof. 1) Similar proof.
2) For $\kappa = \mathrm{cf}(\kappa) > \theta$, Claim 3.10(1),(2) give a necessary and sufficient condition for $\kappa \in \kappa(\mathfrak{s})$ and part (1) gives the same necessarily and sufficient condition for $\kappa \in \kappa_\theta(\mathfrak{s})$. $\square_{3.12}$

3.13 Conclusion. If $\mu^* > \chi_{\mathfrak{s}}, I$ is a μ_1-directed partial order, and $(\forall \mu)[\mu \in \kappa(\mathfrak{s}) \Rightarrow \mu < \mu_1 \vee \mu > |I|]$ and $M_t <_{\mathfrak{s}} \mathfrak{C}$ for $t \in I$, $[I \models s \leq t \Rightarrow M_t \leq_{\mathfrak{s}} M_s]$ and each M_t is $(\mathbb{D}_{\mathfrak{s}}, \mu^*)$-homogeneous. Then $\bigcup_{t \in I} M_t$ is $(\mathbb{D}_{\mathfrak{s}}, \mu^*)$-homogeneous.

Proof. Easy by now.

3.14 Definition. We say $\mathfrak{s}$ (actually $\mathfrak{K}_{\mathfrak{s}}$) is stable in λ when: for every $M <_{\mathfrak{s}} \mathfrak{C}$ of cardinality λ we have $|\mathscr{S}^{<\omega}(M)| \leq \lambda$ (remember $\mathscr{S}^m(M) = \{\mathbf{tp}(\bar{c}, M) : \bar{c} \in {}^m|\mathfrak{C}|\}$), and (if not said otherwise) $\lambda \geq \chi_{\mathfrak{s}}$.

Remark. We could have restricted ourselves in Definition 3.14 to $m = 1$, i.e. to $\mathscr{S}^1(M)$ see 3.16.

3.15 Lemma. *1) If $\mathfrak{s}$ is stable in λ and $\kappa = \mathrm{Min}\{\kappa : \lambda^\kappa > \lambda\}$ then $\kappa \notin \kappa(\mathfrak{s})$ and κ is regular, of course.*
2) If $\mathfrak{s}$ is stable in λ and $\lambda > \chi_{\mathfrak{s}}$, then $\mathfrak{s}$ has a $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous model of cardinality λ.
3) If $\kappa(\mathfrak{s}) = \emptyset$ and $\lambda \geq 2^{\chi(\mathfrak{s}) + |\tau(\mathfrak{s})|}$ then $\mathfrak{s}$ is stable in λ.
4) If $\mathfrak{s}$ is stable in λ and $\lambda < \lambda^{<\kappa>_{\mathrm{tr}}}$ (i.e., there is $\mathscr{T} \subseteq {}^{\kappa \geq}\lambda$ closed under initial segments such that $|\mathscr{T} \cap {}^{\kappa >}\lambda| \leq \lambda < |\mathscr{T}|$) then $\kappa \notin \kappa(\mathfrak{s})$.

Proof. 1) Easy. If not let $M_\eta (\eta \in {}^{\kappa \geq}\lambda)$, $\bar{c}_\nu (\nu \in {}^\kappa\lambda)$ be as in the proof of 3.5. Choose M such that $\cup\{M_\eta : \eta \in {}^{\kappa >}\lambda\} \subseteq M <_{\mathfrak{s}} \mathfrak{C}$ and $\|M\| = \lambda$ (clearly possible) so

$$|\mathscr{S}^{<\omega}(M)| \geq |\{\mathbf{tp}(\bar{c}_\nu, M) : \nu \in {}^\kappa\lambda\}| = \lambda^\kappa > \lambda$$

2) Let κ be as in part (1), $\delta = \lambda \times \kappa$ (ordinal multiplication). We choose by induction on $\alpha \le \delta$, M_α such that:

(i) $M_\alpha <_{\mathfrak{s}} \mathfrak{C}, \|M_\alpha\| = \lambda$

(ii) every $p \in \mathscr{S}^{<\omega}(M_\alpha)$ is realized in $M_{\alpha+1}$

(iii) $M_\beta \le_{\mathfrak{s}} M_\alpha$ for $\beta < \alpha$.

There are no problems and $M = \bigcup\limits_{\alpha<\delta} M_\alpha$ is as required.
[Why? By 3.17 below, $M_{\lambda\times(i+1)}$ is $\le_{\mathfrak{s}}$-universal over $M_{\lambda\times i}$ (in $\mathfrak{K}^{\mathfrak{s}}_\lambda$). Now M_κ is $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous is proved as in 3.10(1), reproving 1.6 under the present assumptions (or see II§4).]
3) Let $M \in \mathfrak{K}$, $\|M\| = \lambda$. We shall define by induction on n, $S_n \subseteq {}^n\lambda$ and models $M_\eta (\eta \in S_n)$. Let $S_0 = \{<>\}$, $M_{<>} = M$. If M_η is defined, $\|M_\eta\| > \chi_{\mathfrak{s}}$ let $M_\eta = \cup\{M_{\eta\hat{}<i>} : i < \|M_\eta\|\}$, $\|M_{\eta\hat{}<i>}\| < \|M_\eta\|$, $M_{\eta\hat{}<i>} \le_{\mathfrak{s}} M_\eta$ and $\langle M_{\eta\hat{}<i>} : i < \|M_\eta\|\rangle$ is $\le_{\mathfrak{s}}$-increasing. So for every $p \in \mathscr{S}^m(M)$ we try to choose by induction on n, $\eta_n \in S_n$, $\eta_{n+1} \restriction n = \eta_n$ such that p does not fork over M_{η_n}. For $n = 0$ this certainly holds by 1.3(3). If η_n is defined and $\|M_{\eta_n}\| > \chi_{\mathfrak{s}}$ then $\langle M_{\eta_n\hat{}<\alpha>} : \alpha < \|M_{\eta_n}\|\rangle$ is well defined and as above, and for some $\alpha, p \restriction M_{\eta_n}$ does not fork over $M_{\eta\hat{}<\alpha>}$ as $\mathrm{cf}(\|M_{\eta_n}\|) \notin \kappa(\mathfrak{s})$; so let $\alpha_n = \min\{\alpha < \|M_{\eta_n}\| : p \restriction M_{\eta_n}$ does not fork over $M_{\eta_n\hat{}<\alpha>}\}$ and let $\eta_{n+1} = \eta_n\hat{}\langle\alpha_n\rangle$. As $\langle\|M_{\eta_n}\| : n < \omega\rangle$ is decreasing, for some $n = n(p), \eta_n$ is well defined and $\|M_{\eta_n}\| \le \chi_{\mathfrak{s}}$. So for some $\eta \in \cup\{S_n : n < \omega\}, p$ does not fork over M_η, $\|M_\eta\| \le \chi_{\mathfrak{s}}$. Now if p, $q \in \mathscr{S}^m(M)$ both do not fork over M_η and $p \restriction M_\eta = q \restriction M_\eta$ then $p = q$ (by 1.5), so $|\mathscr{S}^m(M)| \le \sum\{|\mathscr{S}^m(M_\eta)| : \eta \in \bigcup\limits_n S_n, \|M_\eta\| \le \chi_{\mathfrak{s}}\}$, now $|\bigcup\limits_n S_n| \le \sum\limits_n \lambda^n = \lambda$, and $M \in \mathfrak{K}_{\chi(\mathfrak{s})} \Rightarrow |\mathscr{S}^m(M)| \le 2^{\chi(\mathfrak{s})+|\tau(\mathfrak{s})|}$ (count isomorphism types over M of extensions of M of cardinality $\le \chi(\mathfrak{s}) + |\tau(\mathfrak{s})|$ with expanded by a distinguished element).
4) As in the proof of part (1). $\square_{3.15}$

3.16 Conclusion. 1) $\mathfrak{K}$ is superstable (i.e. $\kappa(\mathfrak{s}) = \emptyset$) <u>if and only if</u> $\mathfrak{K}$ is stable in every $\lambda \ge 2^{\chi(\mathfrak{s})+|\tau(\mathfrak{s})|}$, <u>iff</u> for every $\theta = \mathrm{cf}(\theta) \le \chi_{\mathfrak{s}}, \mathfrak{s}$ is stable in some $\lambda < \lambda^{<\theta>_{\mathrm{tr}}}$ (which is $\ge \chi_{\mathfrak{s}}$).

2) For $\lambda > \chi_{\mathfrak{s}}, \mathfrak{K}$ is stable in λ <u>if and only if</u> there is a $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous model of cardinality λ <u>if and only if</u> $|\mathscr{S}^1(M)| \leq \lambda$ whenever $M \in K_\lambda$.

Proof. 1) Easy.
2) The last statement implies the second as in the proof of 3.15(2). Let us prove that the second statement implies the first. Suppose M is $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous of cardinality λ. Let $\langle M_i : i < \mathrm{cf}(\lambda)\rangle$ be $\leq_{\mathfrak{s}}$-increasing continuous, $\|M_i\| < \lambda$, $M = \bigcup\limits_{i<\mathrm{cf}\lambda} M_i$. As M is $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous for every $i < \mathrm{cf}(\lambda)$, every $p \in \mathscr{S}^{<\omega}(M_i)$ is realized in M, hence $|\mathscr{S}^{<\omega}(M_i)| \leq \lambda$, hence $\mathscr{S} = \{p \in \mathscr{S}^{<\omega}(M) : p$ does not fork over M_i, for some $i < \mathrm{cf}(\lambda)\}$ has cardinality $\leq \lambda + \mathrm{cf}(\lambda) = \lambda$. If $\mathscr{S} = \mathscr{S}^{<\omega}(M)$, then (as every $M' \in K_\lambda$ can be $\leq_{\mathfrak{s}}$-embedded into M) $\mathfrak{K}_{\mathfrak{s}}$ is stable in λ and we finish. So toward contradiction we assume $\mathscr{S} \neq \mathscr{S}^{<\omega}(M)$ hence we can choose $p \in \mathscr{S}^{<\omega}(M)\backslash\mathscr{S}$. Clearly $p, \langle M_i : i < \mathrm{cf}(\lambda)\rangle$ exemplify $\kappa := \mathrm{cf}(\lambda) \in \kappa_{\mathfrak{s}}$. As $\lambda > \chi_{\mathfrak{s}}$ by 3.7 we have $\mathrm{cf}(\lambda) \leq \chi_{\mathfrak{s}} < \lambda$ and there is a $\leq_{\mathfrak{s}}$-increasing continuous sequence $\langle N_\varepsilon : \varepsilon \leq \kappa\rangle$ of members of $K^{\mathfrak{s}}_{\leq\chi(\mathfrak{s})}$ and $p \in \mathscr{S}^m(N_\kappa)$ which forks over M_ε for every $\varepsilon < \kappa$ and let $\bar{c} \in {}^m\mathfrak{C}$ realize p.

If $\lambda^{<\kappa>_{\mathrm{tr}}} > \lambda$, see 3.15(4), we may try to imitate the proof of 3.15(1), but we do not know to deduce $\lambda^{<\kappa>_{\mathrm{tr}}} > \lambda$, so we shall use the choice of M and $\langle M_i : i < \mathrm{cf}(\lambda)\rangle$, in fact imitate in the proof of 3.10(2).

We choose $(f_\varepsilon, M_\varepsilon, M^*_\varepsilon)$ by induction on $\varepsilon \leq \kappa$ such that:

$\circledast$ (a) $M_\varepsilon \leq_{\mathfrak{s}} M^*_\varepsilon \leq_{\mathfrak{s}} M$
(b) $\|M^*_\varepsilon\| \leq \|M_\varepsilon\| + \chi_{\mathfrak{s}} < \lambda$
(c) M^*_ε is $\leq_{\mathfrak{s}}$-increasing continuous with ε
(d) f_ε is a $\leq_{\mathfrak{s}}$-embedding of N_ε into M^*_ε
(e) f_ε is $\subseteq$-increasing continuous with ε
(f) if $\zeta < \varepsilon$ then the triple $f_\zeta(N_\zeta), f_\varepsilon(N_\varepsilon), M^*_\zeta$ is in stable amalgamation inside M.

There is no problem to carry the definition recalling M is $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous, $M_\varepsilon \leq_{\mathfrak{s}} M, \|M_\varepsilon\| < \lambda$. Now clearly $M^*_\kappa = M$ (as

$M^*_\kappa = \cup\{M^*_\varepsilon : \varepsilon < \kappa\} \subseteq M$ and $M = \cup\{M_\varepsilon : \varepsilon < \kappa\} \subseteq \cup\{M^*_\varepsilon : \varepsilon < \kappa\} = M^*_\kappa$).

Also if $\bar c \in {}^m M$ then for some $\varepsilon < \kappa, \bar c \in {}^m(M^*_\varepsilon)$ but $f_\varepsilon(N_\varepsilon), f_\kappa(N_\kappa)$, M^*_ε is in stable amalgamation hence $\mathbf{tp}(\bar c, f_\kappa(N_\kappa))$ does not fork over $f_\varepsilon(N_\varepsilon)$ hence $\bar c$ does not realize the type $f_\kappa(p)$. But $f_\kappa(p) \in \mathscr{S}^m(f_\kappa(N_\kappa))$, $\|f_\kappa(N_\kappa)\| = \|N_\kappa\| \le \chi_{\mathfrak{s}} < \lambda$, so this contradicts the choice of M. Hence $\mathscr{S} = \mathscr{S}^m(M)$ so we are done proving "the second statement" in 3.16(2) implies the first.

Lastly, the third statement follows from the first (see Definition 3.14) so we are done. $\square_{3.16}$

We may feel the lack of a $(\mathbb{D}_{\mathfrak{s}}, \chi_{\mathfrak{s}})$-homogeneous model in cardinality $\chi_{\mathfrak{s}}$, when $\mathfrak{K}_{\mathfrak{s}}$ is stable in $\chi_{\mathfrak{s}}$ (except that $\chi_{\mathfrak{s}}^{<\chi_{\mathfrak{s}}} > \chi_{\mathfrak{s}}$); the following claim is a redemption of this.

3.17 Claim. *If $\|M^1\| = \|M^2\| = \lambda \ge \chi_{\mathfrak{K}}$, and for $\ell = 1, 2, M^\ell = \cup\{M^\ell_i : i < \lambda\}$, $\langle M^\ell_i : i < \lambda\rangle$ is $\le_{\mathfrak{s}}$-increasing, and every $p \in \mathscr{S}^1(M^\ell_i)$ is realized in M^ℓ_{i+1} then $M^1 \cong M^2$.*

Proof. Like V.B.3.18 (or see II§1). $\square_{3.17}$

Similarly (or see II§1).

3.18 Subfact. If $\langle M_i : i \le \lambda\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous, $\|M_i\| = \lambda$, $\lambda \ge \chi_{\mathfrak{s}}$ every $p \in \mathscr{S}^1(M_i)$ is realized in M_{i+1} and $M_0 \le_{\mathfrak{s}} N <_{\mathfrak{s}} \mathfrak{C}$, $\|N\| = \lambda$ then N can be $\le_{\mathfrak{s}}$-embedded into M_λ over M_0.

Remark. Really we need on $\mathfrak{K}$ only LSP(λ) and amalgamation for $\mathfrak{K}_\lambda$.

§4 Orthogonality

In this section we redefine orthogonality and weak orthogonality, then give their basic properties (4.1 - 4.6).

4.1 Definition. 1) We say $\mathrm{tp}(\bar a, A) \underset{\mathrm{wk}}{\perp} \mathbf{tp}(\bar b, A)$ (they are weakly orthogonal) <u>if</u> for every $\bar a', \bar b'$ realizing $\mathbf{tp}(\bar a, A)$, $\mathbf{tp}(\bar b, A)$ respectively,

$$\mathbf{tp}(\bar a\char`^\bar b, A) = \mathbf{tp}(\bar a'\char`^\bar b', A).$$

2) We say $\mathbf{tp}(\bar a, A) \perp \mathbf{tp}(\bar c, B)$ (they are orthogonal) <u>if</u> they both are stationary and for every N satisfying $A \cup B \subseteq N <_{\mathfrak{s}} \mathfrak{C}$, and stationarizations $\mathbf{tp}(\bar a', N)$, $\mathbf{tp}(\bar b', N)$ of $\mathbf{tp}(\bar a, A)$, $\mathbf{tp}(\bar b, B)$ respectively we have

$$\mathbf{tp}(\bar a', N) \underset{\mathrm{wk}}{\perp} \mathbf{tp}(\bar b', N).$$

3) We say $\mathbf{tp}(\bar a, B) \underset{a}{\perp} A$ (where $A \subseteq B$) <u>if</u> $\mathbf{tp}(\bar a, B)$ is weakly orthogonal to $\mathbf{tp}(\bar b, B)$ for every $\bar b$ such that $\mathbf{tp}(\bar b, B)$ does not fork over A (a - for almost). We say for this $\mathbf{tp}(\bar a, A)$ is almost orthogonal to A.
4) We say $\mathbf{tp}(\bar a, A) \perp B$ when: $\mathbf{tp}(\bar a, A)$ is stationary and if $A \cup B \subseteq M$, $\mathbf{tp}(\bar a', M)$ is the stationarization of $\mathbf{tp}(\bar a, A)$ and $\mathbf{tp}(\bar b, M)$ does not fork over B <u>then</u> $\mathbf{tp}(\bar a', M) \perp \mathbf{tp}(\bar b, M)$.

4.2 Observation. Assume $\mathrm{Rang}(\bar a_1) \subseteq \mathrm{Rang}(\bar a_2)$.
1) If $\mathbf{tp}(\bar a_2, A) \underset{\mathrm{wk}}{\perp} \mathbf{tp}(\bar b, A)$ then $\mathbf{tp}(\bar a_1, A) \underset{\mathrm{wk}}{\perp} \mathbf{tp}(\bar b, A)$.
2) Assume $\mathbf{tp}(\bar a_\ell, A)$ is stationary for $\ell = 1, 2$. <u>Then</u> for any B and $\bar b$

- (a) $\mathbf{tp}(\bar a_2, A) \perp \mathbf{tp}(\bar b, B)$ implies $\mathbf{tp}(\bar a_1, A) \perp \mathbf{tp}(\bar b, B)$ and
- (b) $\mathbf{tp}(\bar a_2, A) \underset{a}{\perp} B$ implies $\mathbf{tp}(\bar a_1, A) \underset{a}{\perp} B$ when $B \subseteq A$
- (c) $\mathbf{tp}(\bar a_2, A) \perp B$ implies $\mathbf{tp}(\bar a_1, A) \perp B$.

3) If $M \subseteq B$ and $\bar a \in \mathfrak{C}$ then the following are equivalent

- (a) $\mathbf{tp}(\bar a, A) \underset{a}{\perp} M$
- (b) $\mathbf{tp}(\bar a, A) \underset{\mathrm{wk}}{\perp} \mathbf{tp}(\bar b, N)$ for every sequence $\bar b$ listing the elements of some N satisfying $M \leq_{\mathfrak{s}} N <_{\mathfrak{s}} \mathfrak{C}$ and $\mathbf{tp}(N, A)$ does not fork over A (equivalently $\mathbf{tp}(\bar b, A)$ does not fork over A)
- (c) like (b) but $N = \langle M, N_1 \rangle$ where $\|N_1\| \leq |\ell g(a)| + \chi_{\mathfrak{s}}$ and the triple $M \cap N_1, N_1, M$ is in stable amalgamation.

Remark. Note that $p \underset{a}{\perp} A$ has problematic cases if there are types in $\mathscr{S}(A)$ with no extension in $\mathscr{S}^{<\omega}(B)$ which does not fork over A. However, the main case is $M \leq_{\mathfrak{s}} N, \bar{c} \in {}^{\omega>}N$ and $\mathbf{tp}(N, M + \bar{c}) \underset{a}{\perp} M$.

Proof of 4.2. Easy.

(For (2) recall 2.16 hence if $B \supseteq A$ then there are $\bar{a}'_1, \bar{a}'_2$ such that $\mathbf{tp}(\bar{a}'_\ell, B)$ is a stationarization of $\mathbf{tp}(\bar{a}_\ell, A)$ for $\ell = 1, 2$ and $\bar{a}'_1 {}^\frown \bar{a}'_2$ realize $\mathbf{tp}(\bar{a}_1 {}^\frown \bar{a}_2, A)$. $\square_{4.2}$

4.3 Claim. *Definition 4.1 is compatible with Definitions V.D.4.1, V.D.4.4(2), V.D.4.6(1),(2).*

Proof. Easy. $\square_{4.3}$

Remark. We do not have: $\mathbf{tp}(\bar{a}, A) \underset{\text{wk}}{\perp} \mathbf{tp}(\bar{b}, A)$ implies $\mathbf{tp}(\bar{a}, A \cup \bar{b})$ does not fork over A. (Moreover, there may not be a non-forking extension of $\mathbf{tp}(\bar{a}, A)$ to a member of $\mathscr{S}^{\ell g(\bar{a})}(A \cup \bar{b})$). The following claim provides an acceptable substitute (cf. proof of Lemma 5.3).

4.4 Claim. *Suppose* $\mathbf{tp}(\bar{c}, M)$ *does not fork over* $M^* \leq_{\mathfrak{s}} M$ *and* $M^* \cup \bar{c} \subseteq N^*$. *The following are equivalent:*

(a) $\mathbf{tp}(N^*, M^* \cup \bar{c}) \underset{\text{wk}}{\perp} \mathbf{tp}(M, M^* \cup \bar{c})$

(b) *for any* N' *realizing* $\mathbf{tp}(N^*, M \cup \bar{c})$, $\mathbf{tp}(N', M)$ *does not fork over* M^*.

Proof. Suppose (b) fails; this gives an N' realizing $\mathbf{tp}(N^*, M^* \cup \bar{c})$ which depends on M over M^* (i.e., $\neg\, \mathrm{NF}(M^*, M, N', \mathfrak{C})$). But $\mathbf{tp}(\bar{c}, M)$ does not fork over M^* implies, by Claim 1.8, that there is an N'' isomorphic to N' over $M \cup \bar{c}$ such that $\mathrm{NF}(M^*, M, N'', C)$. This contradicts (a).

Since types over models [or which do not fork over models] are stationary (Claim 2.4(4)), clearly (b) implies (a) is immediate. $\square_{4.4}$

4.5 Claim. *If* $\mathbf{tp}(\bar a, M)$, $\mathbf{tp}(\bar b, M)$ *are not weakly orthogonal,* $M \subseteq A$, $\mathbf{tp}(\bar a', A)$, $\mathbf{tp}(\bar b', A)$ *are the stationarization of* $\mathbf{tp}(\bar a, M)$ *and* $\mathbf{tp}(\bar b, M)$ *respectively, then* $\mathbf{tp}(\bar a', A)$, $\mathbf{tp}(\bar b', A)$ *are not weakly orthogonal.*

Proof. Like the proof of V.D.4.2. $\square_{4.5}$

4.6 Claim. *1) If* $\mathbf{tp}(\bar b, N)$, $\mathbf{tp}(\bar c, N)$ *are orthogonal, then they are weakly orthogonal.*
2) If N *is* $(\mathbb{D}_{\mathfrak{s}}, \lambda^+)$*-homogeneous and* $\lambda \geq \ell g(\bar b_1) + \ell g(\bar b_2) + \chi_{\mathfrak{s}}$, *then:*

$$\mathbf{tp}(\bar b_1, N) \perp \mathbf{tp}(\bar b_2, N) \textit{ if and only if } \mathbf{tp}(\bar b_1, N) \underset{\text{wk}}{\perp} \mathbf{tp}(\bar b_2, N).$$

Proof. 1) Obvious.
2) As in V.D.4.5(4), or deduce by V.D.2.9, V.D.4.5 it and 4.4, (or read 4.8 first).

$\square_{4.6}$

4.7 Claim. *If* $A \subseteq B, N \leq_{\mathfrak{s}} M, M$ *is* $(\mathbb{D}_{\mathfrak{s}}, \lambda^+)$*-homogeneous,* $\lambda \geq |B| + \|N\| + \chi_{\mathfrak{s}}$, $\mathbf{tp}(B, M)$ *does not fork over* N, *and* $\mathbf{tp}(B, N \cup A)$, $\mathbf{tp}(M, N \cup A)$ *are weakly orthogonal, then* $\mathbf{tp}(B, N \cup A)$ *and* $\mathbf{tp}(B, M \cup A)$ *are almost orthogonal to* N.

Proof. Easy. $\square_{4.7}$

4.8 Lemma. *1) Suppose* $\mathbf{tp}(N, M \cup C) \underset{a}{\perp} M, M \leq_{\mathfrak{s}} N$ *and* $C \subseteq N$. *If* $M \leq_{\mathfrak{s}} M_1$ *and* $\mathbf{tp}(C, M_1)$ *does not fork over* M *then* $\{N, M_1\}$ *is independent over* M *and*

$$\mathbf{tp}(\langle N \cup M_1 \rangle^{\text{gn}}, M_1 \cup C) \underset{a}{\perp} M_1$$

2) If $M_i (i < \delta)$ *is* $\leq_{\mathfrak{s}}$*-increasing,* $N_i (i < \delta)$ *is* $\leq_{\mathfrak{s}}$*-increasing,* $C \subseteq N_0$, $\mathbf{tp}(C, M_i)$ *does not fork over* $M_0, M_i \leq_{\mathfrak{s}} N_i$ *and* $\mathbf{tp}(N_i, M_i \cup C) \underset{a}{\perp} M_i$

for $i < \delta$ *then*
$\mathbf{tp}(\bigcup_{i<\delta} N_i, \bigcup_{i<\delta} M_i \cup C) \underset{a}{\perp} \bigcup_{i<\delta} M_i$.
3) If $\bar{c} \in {}^\alpha\mathfrak{C}, M \le_{\mathfrak{s}} M^+, \mathbf{tp}(\bar{c}, M^+)$ *does not fork over* M*,* $M \le_{\mathfrak{s}} N, \bar{c} \subseteq M^+, N \le_{\mathfrak{s}} N^+, M^+ \le_{\mathfrak{s}} N^+, \{N, M^+\}$ *is independent over* M *and* $\mathbf{tp}(N^+, N \cup \bar{c}) \underset{a}{\perp} N$ *then* $\mathbf{tp}(M^+, M \cup \bar{c}) \underset{a}{\perp} M$.
4) Suppose $\mathbf{tp}(N, M \cup C) \underset{a}{\perp} M$*,* M *is* $(\mathbb{D}_{\mathfrak{s}}, \lambda^+)$*-homogeneous,* $M \le_{\mathfrak{s}} N, N_1 \le_{\mathfrak{s}} N <_{\mathfrak{s}} \mathfrak{C}, N_1 \le_{\mathfrak{s}} N_2, \|N_2\| + |C| + \chi_{\mathfrak{s}} \le \lambda, \{N_2, N\}$ *independent over* N_1 *and* $\mathbf{tp}(N_2, N)$ *isolated over* N_1 *then* $\mathbf{tp}(\langle N \cup N_2\rangle^{\mathrm{gn}}, M \cup C) \underset{a}{\perp} M$.

Proof. 1),2) Easy (details on part (2), see [Sh:E54]).
3) If not then there is M_1 satisfying $M \le_{\mathfrak{s}} M_1$ and $\mathbf{tp}(\bar{c}, M_1)$ does not fork over M, but $\mathbf{tp}(M^+, M_1 \cup \bar{c})$ forks over M hence $\mathbf{tp}(M^+, M_1)$ forks over M. Choose $M_1^+ <_{\mathfrak{s}} \mathfrak{C}$ such that $M_1 \cup M^+ \subseteq M_1^+$ and we can find an automorphism g of $\mathfrak{C}$ satisfying $g \restriction M^+ = \mathrm{id}_{M^+}$ and the triple M^+, M_1^+, $g(N^+)$ is in stable amalgamation, note that in particular $g \restriction \mathrm{Rang}(\bar{c}) = \mathrm{id}_{\mathrm{Rang}(\bar{c})}$.

But we know M, N, M^+ is in stable amalgamation, now each is $\le_{\mathfrak{s}} N^+$ hence the triple $M = g(M), g(N), M^+ = g(M^+)$ is in stable amalgamation inside $g(N^+)$, so by transitivity of NF (and the previous paragraph) M, M_1^+, $g(N)$ is in stable amalgamation — hence (by monotonicity) M, M_1, $g(N)$ is. Let $N_1 = \langle g(N), M_1\rangle^{\mathrm{gn}}$, so $N_1 \le_{\mathfrak{s}} \mathfrak{C}$ and (by Ax(C4)) as $M \le_{\mathfrak{s}} M_1 \le_{\mathfrak{s}} M_1^+$ and $M, M_1^+, g(N)$ is in stable amalgamation clearly also M_1, M_1^+, N_1 is in stable amalgamation. Now $\mathrm{Rang}(\bar{c}) \subseteq M^+ \subseteq M_1^+$ hence $\mathbf{tp}(\bar{c}, N_1)$ does not fork over M_1.

As (by the choice of M_1) also $\mathbf{tp}(\bar{c}, M_1)$ does not fork over M we get (by transitivity) $\mathbf{tp}(\bar{c}, N_1)$ does not fork over M; hence (by monotonicity) $\mathbf{tp}(\bar{c}, N_1)$ does not fork over $g(N)$ but $g(\bar{c}) = \bar{c}$ hence $\mathbf{tp}(g(\bar{c}), N_1)$ does not fork over $g(N)$. So as $\mathbf{tp}(N^+, N \cup \bar{c})$ is almost orthogonal to N, we know that $\mathbf{tp}(g(N^+), g(N) \cup g(\bar{c})) \underset{a}{\perp} g(N)$, hence (as $g(N) \le_{\mathfrak{s}} N_1$ and the previous sentence) $g(N)$, $g(N^+)$, N_1 is in stable amalgamation; so $\mathbf{tp}(N_1, g(N^+))$ does not fork over $g(N)$. By the previous paragraph, M, M_1^+, $g(N)$ is in stable amalgamation, so as $M \le_{\mathfrak{s}} M_1 \le_{\mathfrak{s}} M_1^+$ we have $\mathbf{tp}(M_1, g(N))$ does not fork over

M; remember $M \leq_{\mathfrak{s}} g(N) \leq_{\mathfrak{s}} N_1, M \leq_{\mathfrak{s}} M_1 \leq_{\mathfrak{s}} N_1$. By the last two sentences (by transitivity of NF) the type $\mathbf{tp}(M_1, g(N^+))$ does not fork over M. Hence (by monotonicity) as $M \leq_{\mathfrak{s}} M^+ \leq_{\mathfrak{s}} N^+, g \restriction M^+ = \text{id}_{M^+}$, we have $\mathbf{tp}(M_1, M^+)$ does not fork over M, contradicting (by symmetry) the choice of M_1.
4) For (4) use 1.6. $\square_{4.8}$

Note that we have showed

4.9 Lemma. *Suppose M is $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous, $\lambda > \chi_{\mathfrak{s}}, M_0 \leq_{\mathfrak{s}} M, M_0 \leq_{\mathfrak{s}} N_0, C \subseteq N_0$, $\mathbf{tp}(C, M)$ does not fork over $M_0, \{N_0, M\}$ is independent over M_0, $\mathbf{tp}(N_0, M_0 \cup C) \underset{a}{\perp} M_0$ and $\|N_0\| < \lambda$. If N is $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-prime model over $\langle N_0 \cup M \rangle^{\text{gn}}$, <u>then</u> $\mathbf{tp}(N, M \cup C) \underset{a}{\perp} M$.*

Proof. By V.D§4 (and the definitions) without loss of generality N is $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-primary over $\langle N_0 \cup M \rangle^{\text{gn}}$. Prove by induction on length of construction using 4.6(2) for limit stages, 4.6(1),(4) for successor stages. $\square_{4.9}$

4.10 Lemma. *If $M \leq_{\mathfrak{s}} N$, $\mathbf{tp}(\bar{a}, N \cup \bar{c}) \underset{a}{\perp} N$, $\mathbf{tp}(\bar{a}\hat{\ }\bar{c}, N)$ does not fork over M, $\lambda = \chi_{\mathfrak{s}} + |\ell g(\bar{a}\hat{\ }\bar{c})| + \|M\|$ and N is $(\mathbb{D}_{\mathfrak{s}}, \lambda^+)$-homogeneous <u>then</u> $\mathbf{tp}(\bar{a}, M \cup \bar{c}) \underset{a}{\perp} M$.*

Proof. Easy. $\square_{4.10}$

<u>4.11 Exercise</u>: 1) If $\lambda \geq |\ell g(\bar{b})| + |\ell g(\bar{c})| + \chi_{\mathfrak{s}}$ and $\mathbf{tp}(\bar{b}, M) \underset{\text{wk}}{\pm} \mathbf{tp}(\bar{c}, M)$ and $A \subseteq M, |A| \leq \lambda$ <u>then</u> for some $M_0 \leq_{\mathfrak{s}} M$ we have $\|M_0\| \leq \lambda, A \subseteq M_0$ and $\mathbf{tp}(\bar{\mathbf{b}}, M_0) \underset{\text{wk}}{\pm} \mathbf{tp}(\bar{c}, M_0)$ (see 2.4(3)).
2) Similarly for $\pm$.

<u>4.12 Exercise</u>: 1) If $M \leq_{\mathfrak{s}} N, \lambda = \chi_{\mathfrak{s}} + |\ell g(\bar{c})|$
<u>then</u> we have $\mathbf{tp}(\bar{c}, M) \underset{\text{wk}}{\perp} \mathbf{tp}(N, M)$ iff $\mathbf{tp}(\bar{c}, M) \underset{\text{wk}}{\perp} \mathbf{tp}(\bar{a}, M)$ for every $\bar{a} \in {}^{\lambda \geq} M$.

2) Similarly for $\perp$.
3) If M is $(\mathbb{D}_{\mathfrak{s}}, \lambda^+)$-homogeneous and $\mathbf{tp}(\bar{c}, M) \underset{\text{wk}}{\perp} \mathbf{tp}(N, M)$ then $\mathbf{tp}(\bar{c}, M) \perp \mathbf{tp}(N, M)$.

§5 Niceness of Types

Note that not only do we have "problems" with types over sets, but even with types (of sequences) over models.

When we want to generalize more facts on independence, we seemingly need to translate those questions to questions on models. Nice and/or prenice types are the ones for which we succeed to carry out this intention. In §6 we shall prove that for superstable $\mathfrak{s}$, every $p \in \mathscr{S}^{<\omega}(M)$ is prenice.

In 5.1, 5.2 we define, 5.3 sum some facts, 5.4(1) says that being prenice has a "Lowenheim-Skolem property"; then we say how homogeneity of models simplify the matter. In 5.4 - 5.10 we show that for sequences realizing prenice types, the theory of dependence is similar to the one for $\mathfrak{C}$ a model of a first order stable T: has local character by 5.9; everything can be translated to models (5.6, 5.8) and indiscernible set of M has to lose little to become indiscernible over $A, (M \subseteq A)$.

5.1 Definition. We call $\mathbf{tp}(\bar{c}, M)$ nice if there is N satisfying $M \cup \bar{c} \subseteq N <_{\mathfrak{s}} \mathfrak{C}$ such that $\mathbf{tp}(N, M \cup \bar{c})$ is almost orthogonal to M. We call N a witness to $\mathbf{tp}(\bar{c}, M)$ being nice.

5.2 Definition. 1) We call $\mathbf{tp}(\bar{c}, M)$ prenice if for some M', $M \leq_{\mathfrak{s}} M' <_{\mathfrak{s}} \mathfrak{C}$, $\mathbf{tp}(\bar{c}, M')$ does not fork over M and $\mathbf{tp}(\bar{c}, M')$ is nice.
2) We call $\mathbf{tp}(\bar{c}, A)$ prenice when for some M, p we have $p \in \mathscr{S}^{\ell g(\bar{c})}(M)$ is prenice and is the stationarization of $\mathbf{tp}(\bar{c}, A)$.
3) Let $\kappa_{\text{nice}}(\mathfrak{s}) =$ Min$\{\kappa$: if $M <_{\mathfrak{s}} \mathfrak{C}$ and $\bar{c} \in {}^{\kappa >}\mathfrak{C}$ then $\text{tp}(\bar{c}, M)$ is prenice$\}$.

5.3 Fact. 1) If $M_1 \leq_{\mathfrak{s}} M_2$ and $\mathbf{tp}(\bar{c}, M_2)$ does not fork over M_1 and $\mathbf{tp}(\bar{c}, M_1)$ is nice then $\mathbf{tp}(\bar{c}, M_2)$ is nice.

2) If $M \leq_{\mathfrak{s}} N, N \backslash M \subseteq$ Range$(\bar{c}) \subseteq N$ then $\mathbf{tp}(\bar{c}, M)$ is nice.
3) Every nice type $p \in \mathscr{S}^{<\infty}(M)$ is prenice.
4) If $M \leq_{\mathfrak{s}} N$ and $\mathbf{tp}(\bar{c}, N)$ does not fork over M <u>then</u>: $\mathbf{tp}(\bar{c}, M)$ is prenice iff $\mathbf{tp}(\bar{c}, N)$ is prenice. Hence of $p_\ell \in \mathscr{S}(M_\ell)$ for $\ell = 1, 2$ and p_1, p_2 are parallel <u>then</u> p_1 is prenice iff p_2 is prenice.
5) If $\langle N_i : i < \delta \rangle$ is $\leq_{\mathfrak{s}}$-increasing, $\mathbf{tp}(\bar{c}, N_0)$ is prenice, and $\mathbf{tp}(\bar{c}, N_0) \underset{\text{wk}}{\perp} \mathbf{tp}(N_i, N_0)$ for $i < \delta$, <u>then</u> $\mathbf{tp}(\bar{c}, N_0) \underset{\text{wk}}{\perp} \mathbf{tp}(\bigcup_{i<\delta} N_i, N_0)$.
6) If $\langle N_\alpha : \alpha \leq \delta \rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous, $\mathbf{tp}(\bar{c}, N_0)$ is prenice and $\mathbf{tp}(\bar{c}, N_\alpha)$ does not fork over N_0 for $\alpha < \delta$ <u>then</u> $\mathbf{tp}(\bar{c}, N_\delta)$ does not fork over N_0.
7) If $\mathbf{tp}(\bar{c}, N)$ is prenice <u>then</u> for some $M \leq_{\mathfrak{s}} N, \|N\| \leq \chi_{\mathfrak{s}} + |\ell g(\bar{c})|, \mathbf{tp}(\bar{c}, N)$ does not fork over M and $\mathbf{tp}(\bar{c}, M)$ is prenice.

Proof. 1) By 4.8(1).
2) Easy.
3) Trivial.
4) By part (1) and the definition and transitivity of non-forking.
5) We rely on part (6). Let $\bar{c}'$ realizes $\mathbf{tp}(\bar{c}, N_0)$ and we shall prove that $\bar{c}'$ realizes $\mathbf{tp}(\bar{c}, \cup\{N_i : i < \delta\})$, this suffices. For $i < \delta$, as $\mathbf{tp}(\bar{c}, N_0) \underset{\text{wk}}{\perp} \mathbf{tp}(N_i, N_0)$ clearly $\mathbf{tp}(\bar{c}', N_i) = \mathbf{tp}(\bar{c}, N_i)$ and they do not fork over N_0. By part (6) we know that $\mathbf{tp}(\bar{c}', \cup\{N_i : i < \delta\})$ and $\mathbf{tp}(\bar{c}, \cup\{N_i : i < \delta\}$ both do not fork over N_0. We can conclude that they are equal, so we are done.
6) Let $N_\delta := \cup\{N_i : i < \delta\}$ and let $N_{\delta+1}$ be such that $N_\delta \cup \bar{c} \subseteq N_{\delta+1} <_{\mathfrak{s}} \mathfrak{C}$ and $N_{\delta+1}$ is $(\mathbb{D}_{\mathfrak{s}}, \|N_\delta\| + \chi_{\mathfrak{s}})$-homogeneous. We can find M such that $N_0 \leq_{\mathfrak{s}} M$ and $\mathbf{tp}(\bar{c}, M)$ does not fork over N_0 and is nice. There is an $\leq_{\mathfrak{s}}$-embedding f of M into $\mathfrak{C}$ over N_0 such that the triple $N_0, f(M), N_{\delta+1}$ is in stable amalgamation. As $\bar{c} \in N_{\delta+1}$, this implies that $\mathbf{tp}(\bar{c}, f(M))$ does not fork over N_0. As also $\mathbf{tp}(\bar{c}, M)$ does not fork over N_0, we can extend $f \cup \text{id}_{\bar{c}}$ to an automorphism of $\mathfrak{C}$. So without loss of generality $f = \text{id}_M$. As $\mathbf{tp}(\bar{c}, M)$ is nice, there is M^+ satisfying $M \cup \bar{c} \subseteq M^+ <_{\mathfrak{s}} \mathfrak{C}$ such that $\mathbf{tp}(M^+, M \cup \bar{c})$ is almost orthogonal to M. Fix for awhile $i < \delta$, recall that the triple $N_0, M, N_{\delta+1}$ is in stable amalgamation inside $\mathfrak{C}$. By an assumption $\mathbf{tp}(\bar{c}, N_i)$ does not fork over N_0 hence there is N_i^+ such that $\bar{c} \in N_i^+, N_0 \leq_{\mathfrak{s}} N_i^+$ and N_0, N_i^+, N_i is in stable amalgamation inside

$\mathfrak{C}$. By the choice of $N_{\delta+1}$ without loss of generality $N_i^+ \leq_{\mathfrak{s}} N_{\delta+1}$. Recall that $\{M, N_{\delta+1}\}$ is independent over N_0 and $\{N_i^+, N_i\}$ is independent over N_0 inside $N_{\delta+1}$ hence easily $\{M, N_i^+, N_i\}$ is independent over N_0 (see V.D§3) hence $\mathbf{tp}(\bar{c}, \langle N_i \cup M\rangle^{\text{gn}})$ does not fork over N_0. By the choice of M^+, clearly $\mathbf{tp}(M^+, \langle N_i \cup M\rangle^{\text{gn}})$ does not fork over N_0 hence $\mathbf{tp}(M^+, N_i)$ does not fork over N_0. So $\langle N_i : i \leq \delta\rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous, the triple N_0, M^+, N_i is in stable amalgamation for every $i < \delta$ hence by V.D.3.13(2) the triple N_0, M^+, N_δ is in stable amalgamation (recalling Definition V.D.3.5,V.D.3.7).

So $\mathbf{tp}(\bar{c}, \cup\{N_i : i < \delta\})$ does not fork over N_0 as required.

7) By "$\mathfrak{s}$ is $(\chi_{\mathfrak{s}} + |\ell g(\bar{c})|)$-based" and part (4). $\square_{5.3}$

5.4 Fact. 1) If $\mathbf{tp}(\bar{c}, N)$ is nice <u>then</u> for some $M \leq_{\mathfrak{s}} N$ satisfying $\|M\| \leq \chi_{\mathfrak{s}} + |\ell g(\bar{c})|$, the type $\mathbf{tp}(\bar{c}, N)$ does not fork over M and $\mathbf{tp}(\bar{c}, M)$ is nice.

2) If $M \leq_{\mathfrak{s}} N$, $\mathbf{tp}(\bar{c}, N)$ does not fork over M, and $\mathbf{tp}(\bar{c}, M)$ is prenice, $\lambda > \chi_{\mathfrak{s}} + |\ell g(\bar{c})|$ and N is $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous, <u>then</u> $\mathbf{tp}(\bar{c}, N)$ is nice.

3) If $N \leq_{\mathfrak{s}} M$, $N \leq_{\mathfrak{s}} N^+, \bar{c} \subseteq N^+$, $\mathbf{tp}(N^+, N \cup \bar{c}) \underset{\text{wk}}{\perp} \mathbf{tp}(M, N \cup \bar{c})$, $\lambda = |(\ell g(\bar{c})| + \|N^+\| + \chi_{\mathfrak{s}})$ and M is $(\mathbb{D}_{\mathfrak{s}}, \lambda^+)$-homogeneous, $\mathbf{tp}(\bar{c}, M)$ does not fork over N <u>then</u> $\mathbf{tp}(\bar{c}, N)$ is nice.

4) If $\lambda = (|\ell g(\bar{c})| + \chi_{\mathfrak{s}})$, the type $\mathbf{tp}(\bar{c}, M)$ is prenice and M is $(\mathbb{D}_{\mathfrak{s}}, \lambda^+)$-homogeneous <u>then</u> $\mathbf{tp}(\bar{c}, M)$ is nice.

Proof. 1) There is N^+, $N \cup \bar{c} \subseteq N^+ \leq_{\mathfrak{s}} \mathfrak{C}$, such that $\mathbf{tp}(N^+, N \cup \bar{c})$ is almost orthogonal to N. As $\mathfrak{s}$ is χ-based for $\chi \geq \chi_{\mathfrak{s}}$ there are M, M^+ such that:

$\circledast$ (*a*) $\bar{c} \subseteq M^+ \leq_{\mathfrak{s}} N^+$
  (*b*) $\|N^+\| \leq \chi_{\mathfrak{s}} + |\ell g(\bar{c})|$
  (*c*) $M = M^+ \cap N$
  (*d*) $\text{NF}(M, M^+, N, N^+)$.

Clearly $\mathbf{tp}(\bar{c}, N)$ does not fork over M. It suffices to prove: $\mathbf{tp}(M^+, M \cup \bar{c})$ is almost orthogonal to M, thus finishing. But this holds by 4.8(3).

2) By "$\mathfrak{s}$ is $(\chi_{\mathfrak{s}} + |\ell g(\bar{c})|)$-based" there is $M_0 \leq_{\mathfrak{s}} M$ of cardinality $\leq (\chi_{\mathfrak{s}} + |\ell g(\bar{c})|) < \lambda$ such that $\mathbf{tp}(\bar{c}, M)$ does not fork over M_0, hence $\mathbf{tp}(\bar{c}, N)$ does not fork over M_0. By 5.3(4) also $\mathbf{tp}(\bar{c}, M_0)$ is prenice. By the definition of "$\mathbf{tp}(\bar{c}, M)$ is prenice", there are M_1, M_2 such that $M \leq_{\mathfrak{s}} M_1 \leq_{\mathfrak{s}} M_2, \bar{c} \in M_2$ and $\mathbf{tp}(M_2, M_1 \cup \bar{c})$ is almost orthogonal to M_1. By part (1) and "$\mathfrak{s}$ is $(\chi_{\mathfrak{s}} + |\ell g(\bar{c})|)$-based" without loss of generality $\|M_1\|, \|M_2\| < \lambda$, so there is a $\leq_{\mathfrak{s}}$-embedding f_1 of M_1 into M over M_0. Now $\mathbf{tp}(\bar{c}, N)$ does not fork over M hence also does not fork over $f_1(M_1)$ by monotonicity (see 1.3(2)). Also $\mathbf{tp}(\bar{c}, M_1)$ does not fork over M hence $f(\mathbf{tp}(\bar{c}, M_1))$ does not fork over M and extend $\mathbf{tp}(\bar{c}, M)$, so $\bar{c}$ realizes $f(\mathbf{tp}(\bar{c}, M_1))$.

Hence $f_1 \cup \mathrm{id}_{\bar{c}}$ can be extended to an automorphism f_2 of $\mathfrak{C}$. Trivially $\mathbf{tp}(\bar{c}, f_1(M_1))$ is nice, $M \leq_{\mathfrak{s}} f(M_1) \leq_{\mathfrak{s}} N$ and $\mathbf{tp}(\bar{c}, N)$ does not fork over $f_1(M_1)$, so by 5.3(1) we are done.

3) By 5.4(1), (2).

4) Should be clear. $\square_{5.4}$

5.5 Claim. *Assume $\lambda > \chi_{\mathfrak{s}}, M$ is $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous, $\mathbf{tp}(\bar{a}, M)$ is nice and $\ell g(\bar{a}) < \lambda$. Then over $M \cup \bar{a}$ there is a $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-prime model.*

Remark. 1) See Definition V.D.2.3(1), it naturally means: if N' is $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous, f is an automorphism of $\mathfrak{C}$ mapping $M + \bar{c}$ into N' then $f \restriction (M + \bar{c})$ can be extended to a $\leq_{\mathfrak{s}}$-embedding of N into M; without loss of generality $f \restriction M = \mathrm{id}_M$ so $M \leq_{\mathfrak{s}} N'$.

Proof. By 5.4(1) there is $M_0 \leq_{\mathfrak{s}} M$ of cardinality $\chi_{\mathfrak{s}} + |\ell g(\bar{c})|$ such that $\mathbf{tp}(\bar{c}, M_0)$ is nice and $\mathbf{tp}(\bar{c}, M)$ does not fork over M_0. Hence there is $N_0 \prec_{\mathfrak{s}} \mathfrak{C}$ such that $M_0 \cup \bar{c} \subseteq N_0$ and $\mathbf{tp}(N_0, M + \bar{c})$ is almost orthogonal to M_0, and by 5.4(5) without loss of generality $\|N_0\| \leq \chi_{\mathfrak{s}} + |\ell g(\bar{c})|$ hence M_0, N_0, M are in stable amalgamation and so $N := \langle N_0 \cup M \rangle^{\mathrm{gn}}$ is $<_{\mathfrak{s}} \mathfrak{C}$, hence by V.D.2.11,V.D.2.8(2) there is a $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-prime model N^+ over N_0. We shall show that N is as required. So let N', f as above be given, so $M \leq_{\mathfrak{s}} N'$ and N' is $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous and $f(\bar{c}') \in {}^{\ell g(\bar{c})}(N')$ realizes $\mathbf{tp}(\bar{c}, M)$ hence by "N' is $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous" we can find a $\leq_{\mathfrak{s}}$-embedding g of N_0 into N' such that $f \restriction M_0 = \mathrm{id}_{M_0}$ and $f(\bar{c}) = \bar{c}'$.

Now $g \cup \mathrm{id}_M$ can be extended to an automorphism g^+ of $\mathfrak{C}$ by the almost orghogonality. So g^+ necessarily maps N onto $\langle M \cup f(N_0)\rangle^{\mathrm{gn}}$.

As N^+ is $(\mathbb{D}, \lambda)$-prime over $\langle M \cup N_0\rangle^{\mathrm{gn}}$ it follows by the definition (V.D.2.3(1)) that we can extend $g^+ \restriction \langle M \cup f(N_0)\rangle$ to a $\leq_{\mathfrak{s}}$-embedding of N^+ into N', so we are done. $\square_{5.5}$

5.6 Claim. *If $\alpha(*)$ is an ordinal, for each $\alpha < \alpha(*)$, $\mathbf{tp}(\bar{a}_\alpha, M)$ is prenice and for every finite $w \subseteq \alpha$, $\mathbf{tp}(\bar{a}_\alpha, M \cup \bigcup_{\beta \in w} \bar{a}_\beta)$ does not fork over M <u>then</u> we can find $M_\alpha(\alpha < \alpha(*))$ such that:*

- *(i) $\bar{a}_\alpha \subseteq M_\alpha <_{\mathfrak{s}} \mathfrak{C}$ and $M \subseteq M_\alpha$*
- *(ii) $\mathbf{tp}(M_\alpha, \bigcup_{\beta < \alpha} M_\beta)$ does not fork over M.*

5.7 Remark. 1) This will help us to deal with independence of sets of finite sequences (if $\mathfrak{s}$ superstable) — realizing prenice types by translating to problems on sets of models.
2) The inverse of 5.6 is easy by 2.5(6).

Proof. We can find M^1, M^2 such that:

- (a) M^1, M^2 are $(\mathbb{D}_{\mathfrak{s}}, \mu^+)$-homogeneous, where $\mu \geq \chi_{\mathfrak{s}} + \|M \cup \bigcup_\alpha \bar{a}_\alpha\|$
- (b) M, M^1, M^2 are in stable amalgamation.
- (c) $\bigcup_{\alpha < \alpha(*)} \bar{a}_\alpha \subseteq M^1$.

So $\mathbf{tp}(\bar{a}_\alpha, M^2)$ does not fork over M, hence we can find (by Definition 5.1 and 5.4(1),(2)) models M_α, N_α such that: $M \leq_{\mathfrak{s}} M_\alpha <_{\mathfrak{s}} M^2$, $M_\alpha \cup \bar{a}_\alpha \subseteq N_\alpha$, $\|N_\alpha\| \leq \mu$ and $\mathbf{tp}(N_\alpha, M_\alpha \cup \bar{a}_\alpha) \underset{a}{\perp} M_\alpha$. Clearly if $M'_\alpha \leq_{\mathfrak{s}} M^2$ is isomorphic to M_α over M, we can find N'_α such that (M'_α, N'_α) satisfies the demand on (M_α, N_α). So without loss of generality $\{M_\alpha : \alpha < \alpha(*)\}$ is independent over M, see V.D.3.15-V.D.3.19. By 2.13, 2.5(6), we can prove by induction on $m + n <$

ω, that if $\alpha_0, \ldots, \alpha_{n-1}$, $\beta_0, \ldots, \beta_{m-1}$ are distinct ordinals on $< \alpha(*)$ then $\{N_{\alpha_0}, \ldots, N_{\alpha_{n-1}}, \bar{a}_{\beta_0}, \ldots, \bar{a}_{\beta_{m-1}}\}$ is independent over M^2. Then we use it for $m = 0$ to show that any finite subfamily over M and then use the finite character of independence of models (by V.D.3.16(4)). $\square_{5.6}$

5.8 Fact. If $\{\bar{a}_\alpha : \alpha < \alpha(*)\}$ is independent over M (see Definition 2.12), $M \cup \bar{a}_\alpha \subseteq N_\alpha$ and $\mathbf{tp}(N_\alpha, M \cup \bar{a}_\alpha) \underset{a}{\perp} M$ <u>then</u> $\{N_\alpha : \alpha < \alpha(*)\}$ is independent over M.

Proof. Easy, as in the proof of 5.6. $\square_{5.8}$

5.9 Conclusion. 1) If $\mathbf{tp}(\bar{a}_\alpha, M)$ is prenice for every $\alpha < \alpha(*)$, <u>then</u>: $\{\bar{a}_\alpha : \alpha < \alpha(*)\}$ is independent over M <u>if and only if</u> every finite subfamily is independent.
2) If $\{\bar{a}_\alpha : \alpha < \alpha(*)\}$ is independent over M, each $\mathbf{tp}(\bar{a}_\alpha, M)$ is prenice and does not depend on α, $\alpha(*) > \chi_{\mathfrak{s}}$ <u>then</u> $\{\bar{a}_\alpha : \alpha < \alpha(*)\}$ is indiscernible over M, [in fact for every permutation h of $\alpha(*)$ there is an automorphism f_h of $\mathfrak{C}$, $f_h \restriction M = \mathrm{id}_M, f_h(\bar{a}_\alpha) = \bar{a}_{h(\alpha)}$].

Proof. By 5.6 (and V.D.3.22). $\square_{5.9}$

5.10 Claim. *Suppose $\alpha(*) > \chi(\mathfrak{s})$, $\{\bar{a}_\alpha : \alpha < \alpha(*)\}$ independent over M_0 and each $\mathbf{tp}(\bar{a}_\alpha, M_0)$ is prenice. <u>Then</u>:*
1) For every sequence $\bar{a}$, for some $w \subseteq \alpha(*)$, w has cardinality $\leq |\ell g(\bar{a})| + \chi_{\mathfrak{s}}$ and $\{\bar{a}_\alpha : \alpha \in (\alpha(*) \backslash w)\} \cup \{\bar{a}\}$ is independent over $(M_0 \cup \bigcup\limits_{\alpha \in w} \bar{a}_\alpha, M_0)$.
2) Suppose that for every α we have $\mathbf{tp}(\bar{a}_\alpha, M_0) = p$. For every $N <_{\mathfrak{s}} \mathfrak{C}$ for some $w \subseteq \alpha(*)$, $|w| \leq \|N\| + \chi(\mathfrak{s})$, for all $\alpha \in (\alpha(*) \backslash w)$, $\mathbf{tp}(\bar{a}_\alpha, N)$ is the same; (i.e. $\{\bar{a}_\alpha : \alpha < \alpha(*)\}$ is convergent, see 2.7).

Proof. Easy by 5.6 and the parallel result for the case $\mathbf{tp}(\bar{a}_\alpha, M_0) \in \mathscr{S}_c^{<\infty}(M_0)$, see V.D.3.17. $\square_{5.10}$

5.11 Fact. Suppose $p \in \mathscr{S}^{<\lambda}(N)$ is nice, $\|N\| + \chi_{\mathfrak{s}} < \lambda, N \le_{\mathfrak{s}} M$, M is $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous, $\mathbf{I} \subseteq p(M) := \{\bar{c} \in M : \bar{c} \text{ realizes } p\}$ has cardinality $\ge \lambda, \mathbf{I}$ is independent over N, for $\bar{a} \in \mathbf{I}$, $\bar{a} \in N_{\bar{a}} \le_{\mathfrak{s}} M$, $N \le_{\mathfrak{s}} N_{\bar{a}}, \mathbf{tp}(N_{\bar{a}}, N \cup \bar{a}) \underset{a}{\perp} N$ and M is $(\mathbb{D}_{\mathfrak{s}}, \lambda^+)$-primary over $\bigcup\limits_{\bar{a} \in \mathscr{I}} N_{\bar{a}}$ (or just $(\mathbb{D}_{\mathfrak{s}}, \lambda^+)$-atomic). If $\mathbf{tp}(\bar{c}, M)$ is the stationarization of p then $\mathbf{tp}(\bar{c}, N \cup \cup \mathbf{I}) \vdash \mathbf{tp}(\bar{c}, M)$.

Proof. Easy. $\square_{5.11}$

§6 Superstable frames

Ranks are less important here as p "every type has of rank $< \infty$" is seemingly not equivalent to superstability.

6.1 Definition. 1) For finite $\bar{c} \in {}^{\omega>}\mathfrak{C}, N <_{\mathfrak{s}} \mathfrak{C}$ and ordinal α, the truth value of $\text{rk}[\mathbf{tp}(\bar{c}, N)] \ge \alpha$ is defined as follows: $\text{rk}[\mathbf{tp}(\bar{c}, N)] \ge \alpha$ if and only if for every $\beta < \alpha$ there are M such that $N \le_{\mathfrak{s}} M$ and $\bar{c}_1, \bar{c}_2 \in {}^{\ell g(\bar{c})}\mathfrak{C}$ realizing $\mathbf{tp}(\bar{c}, N)$ such that $\text{rk}[\mathbf{tp}(\bar{c}_\ell, M)] \ge \beta$ for $\ell = 1, 2$ and $\mathbf{tp}(\bar{c}_1, M) \ne \mathbf{tp}(\bar{c}_2, M)$ (check that the definition depends on $\mathbf{tp}(\bar{c}, N)$, in particular in N but not on $\bar{c}$).
2) Now $\text{rk}[\mathbf{tp}(\bar{c}, M)] = \alpha$ if and only if it is $\ge \alpha$ but $\not\ge \alpha + 1$.
If $\text{rk}[\mathbf{tp}(\bar{c}, M)]$ is $\ge \alpha$ for every α, we say it is not defined and also write $\mathbf{tp}(\bar{c}, M) = \infty$ (stipulating $\alpha < \infty$ for any ordinal α).
3) Let $\text{rk}[\bar{c}, M] = \text{rk}[\mathbf{tp}(\bar{c}, M)]$.

6.2 Lemma. *1) Assume $M <_{\mathfrak{s}} \mathfrak{C}$ and $\bar{c}$ finite, α, β ordinals. If: $\alpha \le \beta$ and $\text{rk}[\mathbf{tp}(\bar{c}, N)] \ge \beta$ then $\text{rk}[\mathbf{tp}(\bar{c}, N)] \ge \alpha$.*
2) For $M <_{\mathfrak{s}} \mathfrak{C}$ and $\bar{c} \in \mathfrak{C}$ (finite) we have: $\text{rk}[\mathbf{tp}(\bar{c}, M)]$ is equal to a unique object, an ordinal or ∞, which is $\ge \alpha$, if and only if "$\text{rk}[\mathbf{tp}(\bar{c}, M)] \ge \alpha$".
3) If $M \le_{\mathfrak{s}} N <_{\mathfrak{s}} \mathfrak{C}$ and $\bar{c}$ is finite, then

$$\text{rk}[\mathbf{tp}(\bar{c}, N)] \le \text{rk}[\mathbf{tp}(\bar{c}, M)].$$

4) If for no M' we have: $(M \le_{\mathfrak{s}} M' \;\; \& \;\; \text{rk}(\bar{c}, M') = \alpha)$ and $\text{rk}(\bar{c}, M) \ge \alpha$ then $\text{rk}(\bar{c}, M) = \infty$.

5) If $M \leq_{\mathfrak{s}} N <_{\mathfrak{s}} \mathfrak{C}$, $\bar{c}$ finite, $\mathbf{tp}(\bar{c}, N)$ does not fork over M then $\text{rk}[\mathbf{tp}(\bar{c}, N)] = \text{rk}[\mathbf{tp}(\bar{c}, M)]$.
6) There is $\alpha < \left(2^{\chi(\mathfrak{s})+|\tau(\mathfrak{s})|}\right)^+$ such that (for finite $\bar{c}$):

$$\text{rk}[\mathbf{tp}(\bar{c}, M)] \geq \alpha \Rightarrow \text{rk}[\mathbf{tp}(\bar{c}, M)] = \infty.$$

7) If for some $\bar{c} \in M$ we have $\text{rk}[\mathbf{tp}(\bar{c}, M)] = \infty$ *then* $\aleph_0 \in \kappa(\mathfrak{s})$.
8) If $M \leq_{\mathfrak{s}} N <_{\mathfrak{s}} \mathfrak{C}$ the type $\mathbf{tp}(\bar{c}, N)$ forks over M and $\text{rk}[\mathbf{tp}(\bar{c}, N)] < \infty$ *then* $\text{rk}[\mathbf{tp}(\bar{c}, N)] < \text{rk}[\mathbf{tp}(\bar{c}, M)]$. *Also if* $\infty > \text{rk}[\mathbf{tp}(\bar{c}, M)] > \beta$ *then for some N we have $M \leq_{\mathfrak{s}} N <_{\mathfrak{s}} \mathfrak{C}, \mathbf{tp}(\bar{c}, N)$ forks over M and* $\text{rk}[\mathbf{tp}(\bar{c}, N)] \geq \beta$ *and even* $= \beta$.

Proof. 1) Check Definition 6.1(1).
2) By (1) and Definition 6.1(2).
3) Trivial by the definition.
4) We prove by induction on $\beta \geq \alpha$ that: if $N <_{\mathfrak{s}} \mathfrak{C}$ and $M <_{\mathfrak{s}} N$ and $\text{rk}(\mathbf{tp}(\bar{c}, N)) \geq \alpha$ then $\text{rk}[\mathbf{tp}(\bar{c}, N)] \geq \beta$.
5) By 6.2(3) it suffices to prove: $\text{rk}[\mathbf{tp}(\bar{c}, M)] \leq \text{rk}[\mathbf{tp}(\bar{c}, N)]$; hence it suffices to prove by induction on α:
$\text{rk}[\mathbf{tp}(\bar{c}, M)] \geq \alpha \Rightarrow \text{rk}[\mathbf{tp}(\bar{c}, N)] \geq \alpha$.
For $\alpha = 0$, α limit: trivial.

For $\alpha = \beta + 1$, as $\text{rk}[\mathbf{tp}(\bar{c}, M)] \geq \alpha = \beta + 1$, there is M' such that $M \leq_{\mathfrak{s}} M'$ and $\bar{c}_1, \bar{c}_2 \in {}^{\ell g(\bar{c})}\mathfrak{C}$ realizing $\mathbf{tp}(\bar{c}, M)$ and realizing different types in $\mathscr{S}^{\ell g(\bar{c})}(M')$ such that $\text{rk}[\mathbf{tp}(\bar{c}_\ell, M')] \geq \beta$ for $\ell = 1, 2$. Choose M'' such that $M' \cup \bar{c}_1 \cup \bar{c}_2 \subseteq M''$.

For some $f \in \text{AUT}(\mathfrak{C})$, $f \restriction M = \text{id}_M$ and $\{N, f(M'')\}$ is independent over M. For $\ell = 1, 2$ as $\bar{c}_\ell \subseteq M''$, clearly $\mathbf{tp}(f(\bar{c}_\ell), N)$ does not fork over M and as $f \restriction M = \text{id}_M$ clearly $\mathbf{tp}(\bar{c}_\ell, M) = \mathbf{tp}(f(\bar{c}_\ell), M) = \mathbf{tp}(\bar{c}, M)$ and recall $\mathbf{tp}(\bar{c}, N)$ does not fork over M so together $\mathbf{tp}(\bar{c}, N) = \mathbf{tp}(f(\bar{c}_\ell), N)$. As we can replace $(M', M'', \bar{c}_1, \bar{c}_2)$ by $(f(M'), f(M''), f(\bar{c}_1), f(\bar{c}_2))$, without loss of generality f is the identity. As $\mathbf{tp}(\bar{c}_\ell, \langle M' \cup N\rangle^{gn})$ does not fork over M', [because $\{f(M''), N\}$ is independent over M and $M \leq_{\mathfrak{s}} M' \leq_{\mathfrak{s}} M''$ by the base enlargement axiom], by the induction hypothesis $\text{rk}[\mathbf{tp}(\bar{c}_\ell, \langle M' \cup N\rangle^{gn})] \geq \beta$. Also $\mathbf{tp}(\bar{c}_1, \langle M' \cup N\rangle^{\text{gn}}) \neq \mathbf{tp}(\bar{c}_2, \langle M' \cup N\rangle^{\text{gn}})$ because

$\mathbf{tp}(\bar{c}_1, M') \neq \mathbf{tp}(\bar{c}_2, M')$. So $\langle M'' \cup N \rangle^{\rm gn}, \bar{c}_1, \bar{c}_2$ satisfies the requirement in Definition 6.1. So clearly we are done.
6) By 6.2(5)

$$\{\mathrm{rk}[\mathbf{tp}(\bar{c}, M)] : \bar{c} \in {}^{\omega >}\mathfrak{C}, M <_{\mathfrak{s}} \mathfrak{C}\} =$$

$$\{\mathrm{rk}[\mathbf{tp}(\bar{c}, M)] : \bar{c} \in {}^{\omega >}\mathfrak{C}, M <_{\mathfrak{s}} \mathfrak{C}, \|M\| \leq \chi_{\mathfrak{s}}\}.$$

By 6.2(4) this set is an ordinal (plus maybe ∞). Clearly for $h \in$ AUT$(\mathfrak{C})$

$$\mathrm{rk}[\mathbf{tp}(\bar{c}, M)] = \mathrm{rk}[\mathbf{tp}(h(\bar{c}), h(M))],$$

so the second set above has cardinality $\leq 2^{\chi(\mathfrak{s}) + |\tau(\mathfrak{s})|}$. Together we get the result.
7) Let $\alpha(*) = (2^{\chi(\mathfrak{s}) + |\tau(\mathfrak{s})|})^+$. We choose M_n by induction on n such that $M_0 = M$, $M_n \leq_{\mathfrak{s}} M_{n+1} <_{\mathfrak{s}} \mathfrak{C}$ and $\mathrm{rk}[\mathbf{tp}(\bar{c}, M_n)] = \infty$.
For $n = 0$ this is assumed. For $n = m + 1$, as $\mathrm{rk}[\mathbf{tp}(\bar{c}, M_m)] = \infty > \alpha(*)$ we can find $M', \bar{c}_1, \bar{c}_2$ such that $M_1 \leq_{\mathfrak{s}} M'$, $\mathrm{rk}[\mathbf{tp}(\bar{c}_\ell, M')] \geq \alpha(*)$, $\mathbf{tp}(\bar{c}_\ell, M_m) = \mathbf{tp}(\bar{c}, M_m)$ for $\ell = 1, 2$, and $\mathbf{tp}(\bar{c}_1, M') \neq \mathbf{tp}(\bar{c}_2, M')$. So for some $\ell \in \{1, 2\}$ the type $\mathbf{tp}(\bar{c}_\ell, M')$ is not the stationarization of $\mathbf{tp}(\bar{c}, M_m)$ hence it forks over M_m. By using an automorphism of $\mathfrak{C}$ over M_m without loss of generality $\bar{c}_\ell = \bar{c}$ and let $M_m = M'$. Having carried the induction, $\bar{c}, \langle M_n : n < \omega \rangle$ more than exemplifies $\aleph_0 \in \kappa(\mathfrak{s})$.
8) Should be clear. $\square_{6.2}$

6.3 The Existence of Nice Types Lemma: [$\mathfrak{s}$ is superstable].
1) Suppose $M <_{\mathfrak{s}} \mathfrak{C}, \bar{c}$ a finite sequence, $\lambda = \chi_{\mathfrak{s}}$. Then there are $M^* \leq_{\mathfrak{s}} M$ and $N^* <_{\mathfrak{s}} \mathfrak{C}$ such that $M^* \leq_{\mathfrak{s}} N^*$, $\bar{c} \in N^*$, $\|N^*\| \leq \lambda$, $\mathbf{tp}(\bar{c}, M)$ does not fork over M^* and the types $\mathbf{tp}(N^*, M^* \cup \bar{c})$, $\mathbf{tp}(M, M^* \cup \bar{c})$ are weakly orthogonal.
2) So in (1) if M is $(\mathbb{D}_{\mathfrak{s}}, \lambda^+)$-homogeneous, then $\mathbf{tp}(\bar{c}, M^*)$ is nice, hence $\mathbf{tp}(\bar{c}, M)$ is nice.

Proof. 1) We assume that such M^*, N^* does not exist and will eventually derive a contradiction. We choose M_i, $N_i (i < \lambda^+)$, $f_{i,j} (j < i < \lambda^+)$ by induction on i such that:

$\boxdot$ (a) $M_i \leq_{\mathfrak{s}} M$ is $\leq_{\mathfrak{s}}$-increasing, $\mathbf{tp}(\bar{c}, M)$ does not fork over M_0

(b) $\bar{c} \in N_i, M_i \leq_{\mathfrak{s}} N_i, \|N_i\| \leq \lambda$

(c) $f_{i,j}$ is a $\leq_{\mathfrak{s}}$-embedding of N_j into N_i over M_j and $f_{i,j}(\bar{c}) = \bar{c}$

(d) $j_1 < j_2 < j_3 \Rightarrow f_{j_3,j_1} = f_{j_3,j_2} \circ f_{j_2,j_1}$

(e) $\mathbf{tp}(f_{i+1,i}(N_i), M_{i+1})$ forks over M_i

(f) for i limit, $M_i = \bigcup_{j<i} M_j$, $N_i = \bigcup_{j<i} f_{i,j}(N_j)$.

Construction:.

Case 1:. $i = 0$

Choose (as $\mathfrak{s}$ is $\chi_{\mathfrak{s}}$-based), $N_0 <_{\mathfrak{s}} \mathfrak{C}$ such that $\bar{c} \subseteq N_0$ and $N_0 \cap M$, N_0, M is in stable amalgamation and $\|N_0\| \leq \lambda$. Let $M_0 = N_0 \cap M$. Clearly clause (b) holds as well as "$M_0 \leq_{\mathfrak{s}} M$" from clause (a) and the other conditions are inapplicable.

Case 2:. $i = j + 1$.

So N_j, M_j are defined (and are as required). Consider N_j, M_j as candidates for N^*, M^* in the conclusion of 6.3(1), so they should fail some demand. As $\|M_j\| \leq \|N_j\| \leq \lambda$, $M_j \leq_{\mathfrak{s}} M$, $M_j \leq_{\mathfrak{s}} N_j <_{\mathfrak{s}} \mathfrak{C}$ and $\bar{c} \in N_j$ necessarily $\mathbf{tp}(N_j, M_j \cup \bar{c})$ is not weakly orthogonal to $\mathbf{tp}(M, M_j \cup \bar{c})$. So there is $N'_j <_{\mathfrak{s}} \mathfrak{C}$ isomorphic to N_j over $M_j \cup \bar{c}$, say by the isomorphism h_j, such that:

$$\mathbf{tp}(N'_j, M) \text{ forks over } M_j.$$

Then we can find $N_i <_{\mathfrak{s}} \mathfrak{C}$, $\|N_i\| \leq \lambda$ such that $N'_j \subseteq N_i$ and $N_i \cap M$, N_i, M are in stable amalgamation (exists as $\mathfrak{s}$ is λ-based). We let $M_i := M \cap N_i$ and $f_{i,j} := h_j$ and for $\zeta < j$ we have $f_{i,\zeta} = f_{i,j} \circ f_{j,\zeta}$. As for checking the conditions, the main point is clause (e), now remember $\mathbf{tp}(N_i, M)$ does not fork over M_i, so if $\mathbf{tp}(f_{i,j}(N_j), M_i)$ does not fork over M_j then (by transitivity, 1.5) the type $\mathbf{tp}(f_{i,j}(N_j), M)$ does not fork over M_j, but the former is just N'_j, so this contradicts the choice of N'_j. So (e) holds.

Case 3. $i = \delta$ is a limit ordinal.

Let $M_\delta = \bigcup_{\beta<\delta} M_\beta$.

Now $\langle N_\alpha, f_{\gamma,\beta} : \alpha < \delta, \beta < \gamma\rangle$ is a directed system hence it has a limit: N^*_δ and $f^*_{\delta,\beta}$ for $\beta < \delta$ which means that $f^*_{\delta,\beta}$ (for $\beta < \delta$) is a $\le_{\mathfrak{s}}$-embedding of N_β into N^*_δ, $[\beta < \gamma < \delta \Rightarrow f^*_{\delta,\beta} = f^*_{\delta,\gamma} \circ f_{\gamma,\beta}]$ and $N^*_\delta = \bigcup_{\beta<\delta} f^*_{\delta,\beta}(N_\beta)$. Clearly $\langle f^*_{\beta,\delta} \restriction M_\beta : \beta < \delta\rangle$ is increasing and $f^*_{\delta,\beta} \restriction M_\beta$ is an $\le_{\mathfrak{s}}$-embedding of M_β into N^*_δ, so $\bigcup_{\beta<\delta} f^*_{\delta,\beta}(M_\beta) \le_{\mathfrak{s}} N^*_\delta$, so without loss of generality $\bigcup_{\beta<\delta} f^*_{\delta,\beta} \restriction M_\beta$ is the identity on $M_\delta := \bigcup_{\beta<\delta} M_\beta$. Now we come to the main point: we would like also to have $f_{\delta,j}(\bar{c}) = \bar{c}$. In order to get this we need $\mathbf{tp}(\bar{c}, \bigcup_{\beta<\delta} M_\beta) = \mathbf{tp}(f_{\delta,0}(\bar{c}), \bigcup_{\beta<\delta} M_\beta)$. We know that for each $\beta < \delta$, $\mathbf{tp}(\bar{c}, M_\beta) = \mathbf{tp}(f_{\delta,0}(\bar{c}), M_\beta)$, but we need continuity for this property. By our choice of M_0, $\mathbf{tp}(\bar{c}, M)$ does not fork over M_0, hence for $\beta \le \delta$, $\mathbf{tp}(\bar{c}, M_\beta)$ does not fork over M_0. By the superstability for some $\beta(*) < \delta, \mathbf{tp}(f_{\delta,0}(\bar{c}), \bigcup_{\beta<\delta} M_\beta)$ does not fork over $M_{\beta(*)}$; now as $\mathbf{tp}(f_{\delta,0}(\bar{c}), M_{\beta(*)}) = \mathbf{tp}(\bar{c}, M_{\beta(*)})$ by 1.4(1) we get the desired equality $\mathbf{tp}(f_{\delta,0}(\bar{c}), M_\delta) = \mathbf{tp}(\bar{c}, M_\delta)$, so without loss of generality $f_{\delta,0}(\bar{c}) = \bar{c}$ hence $f_{\delta,\beta}(\bar{c}) = \bar{c}$ for $\beta < \delta$. So we finish the case $i = \delta$ limit.

So we have finished the construction, we can choose M_{λ^+}, N_{λ^+}, $\langle f_{\lambda^+,i} : i < \lambda^+\rangle$ such that the relevant demands in $\boxdot(a) - (f)$ hold. But then $\langle M_i, f_{\lambda^+,i}(N_i) : i < \lambda^+\rangle$ contradict "$\mathfrak{s}$ is $\chi_{\mathfrak{s}}$-based" (see V.C.3.11).

2) Left to the reader (use 5.4(4)). $\square_{6.3}$

6.4 Remark. 1) See more in [Sh:E54].

2) If $\bar{c} \subseteq N$ and $|\ell g(\bar{c})| = \lambda$, then $\mathbf{tp}(N, M \cup \bar{c})$ has character (= localness) $\leq \lambda + \chi_{\mathfrak{s}}$ as $\mathfrak{s}$ is $(\lambda + \chi_{\mathfrak{s}})$-based.

6.5 Conclusion. [$\mathfrak{s}$ superstable]. Every $p \in \mathscr{S}^m(N)$, (such that $N <_{\mathfrak{s}} \mathfrak{C}, m < \omega$) is prenice; in other words $\kappa_{\text{nice}}(\mathfrak{s}) \geq \aleph_0$.

Proof. Let $M <_{\mathfrak{s}} \mathfrak{C}$ be a $(\mathbb{D}_{\mathfrak{s}}, \chi^+_{\mathfrak{s}})$-homomorphic $\leq_{\mathfrak{s}}$-extension of N and $q \in \mathscr{S}^m(M)$ is a non-forking extension of p. By 6.3(2) the type q is nice hence p is prenice. $\square_{6.5}$

6.6 Lemma. [$\mathfrak{s}$ *superstable*]. *If* $\{\bar{a}_\alpha : \alpha < \alpha(*)\}$ *is independent over* M*, each* $\bar{a}_\alpha$ *finite, and* $\bar{b}$ *a finite sequence* <u>*then*</u> *for some finite* $w \subseteq \alpha(*)$, $\{\bar{a}_\alpha : \alpha \in (\alpha(*)\backslash w)\} \cup \{\bar{b}\}$ *is independent over* M *(and also over some* $N, M \cup \bigcup_{\beta \in w} \bar{a}_\beta \subseteq N <_{\mathfrak{s}} \mathfrak{C}$*, such that* $\{\bar{a}_\alpha : \alpha \in (\alpha(*)\backslash w)\}$ *is independent over* (N, M)*).*

Proof. As in the proof of 5.5 without loss of generality $\mathbf{tp}(\bar{a}_\alpha, M)$ is nice as well as $\mathbf{tp}(\bar{b}, M)$. Let N_a, $N_{\bar{a}}$ witness it. Choose by induction on n finite $w_n \subseteq \alpha(*)$ such that $\mathbf{tp}(\bar{b}, \langle \cup\{N_\alpha : \alpha \in \bigcup_{\ell \leq n} w_\ell\}\rangle^{gn}$ forks over $\langle \cup\{N_\alpha : \alpha \in \bigcup_{\ell < n} w_\ell\}\rangle$. By 6.2(7) eventually we stop, then use symmetry. $\square_{6.6}$

6.7 Exercise:. In 6.6, replace the model M by a set.

6.8 Definition. 1) We say $\mathfrak{s}$ has DOP (the dimensional order property) when for some $\lambda \geq \chi_{\mathfrak{s}}$ and $(\mathbb{D}_{\mathfrak{s}}, \lambda^+)$-homogeneous models M_0, M_1, M_2 which are in stable amalgamation, choosing M_3 as a $(\mathbb{D}_{\mathfrak{s}}, \lambda^+)$-prime models over $M_1 \cup M_2$ we have: there is $M'_3 <_{\mathfrak{s}} M_3$ which is $(\mathbb{D}_{\mathfrak{s}}, \lambda^+)$-homogeneous and includes $M_1 \cup M_2$.
2) We say a prenice $p \in \mathscr{S}(M)$ is of depth zero when: if $M \leq_{\mathfrak{s}} M_1 \leq_{\mathfrak{s}} M_2$ and M_1, M_2 are $(\mathbb{D}_{\mathfrak{s}}, \chi_{\mathfrak{s}})$-homogeneous, $a \in M_2, \mathbf{tp}(a, M_1)$ is a

non-forking extension of p, M_2 is $\chi^+_{\mathfrak{s}}$-prime over $M_1 \cup \{a\}$ then M_2 is minimal over $M_1 \cup \{a\}$ among the $(\mathbb{D}_{\mathfrak{s}}, \chi^+_{\mathfrak{s}})$-homogeneous model.

6.9 Exercise: Imitate [Sh:c, Ch.X].

§7 Regular Types and Weight

We deal mainly with regular types and state a definition and a theorem on weight ([Sh:c, V,§4]). Also our knowledge on the orders $\leq_w, \leq_{st}$ between stationary types as well (see [Sh:c, V,§2]) can be generalized to the present context.

7.1 Hypothesis. $\mathfrak{s}$ is superstable.

Actually we shall mention when we use superstability (however 7.9 which assumes it is our main interest).

7.2 Definition. We say $\mathbf{tp}(\bar{a}, M)$ is regular provided that it is prenice and for every N: if $M \leq_{\mathfrak{s}} N <_{\mathfrak{s}} \mathfrak{C}$ and $\mathbf{tp}(\bar{a}, N)$ forks over M then $\mathbf{tp}(\bar{a}, N) \perp \mathbf{tp}(\bar{a}, M)$.

7.3 Claim. *1) If $\mathbf{tp}(\bar{a}_0, M_0)$, $\mathbf{tp}(\bar{a}_1, M_1)$ are parallel and prenice then: $\mathbf{tp}(\bar{a}_0, M_0)$ is regular if and only if $\mathbf{tp}(\bar{a}_1, M_1)$ is regular.*
2) Being regular is a property of $\mathbf{tp}(\bar{a}, M)$ not of $\bar{a}$; moreover of the parallelism equivalence class of it.

Proof. Part (2) is obvious, using part (1) (and moreover by 5.3(4)). For part (1), let M be such that $M_\ell \leq_{\mathfrak{s}} M <_{\mathfrak{s}} \mathfrak{C}$, for $\ell = 0, 1$.

For $\ell = 1, 2$ let $\alpha = \ell g(\bar{a}_\ell)$ and let $p \in \mathscr{S}^\alpha(M)$ be a non-forking extension of p_ℓ hence it is prenice, clearly α, p does not depend on ℓ. By symmetry it suffices to show [p_1 is regular $\Leftrightarrow$ p is regular]. Without loss of generality $\mathbf{tp}(\bar{a}_1, M)$ does not fork over M_1 so $\mathbf{tp}(\bar{a}, M) = p$.

If p_1 is not regular, then is $N_1 <_{\mathfrak{s}} \mathfrak{C}$ satisfying $M_1 \leq_{\mathfrak{s}} N_1$ such that $\mathbf{tp}(\bar{a}_1, N_1)$ forks over M_1 and is not orthogonal to $\mathbf{tp}(\bar{a}_1, M_1)$. Let $N_1 \cup \bar{a}_1 \subseteq N_1^+ <_{\mathfrak{s}} \mathfrak{C}$, and using automorphisms of $\mathfrak{C}$ over $M \cup \bar{a}_1$ as $\mathbf{tp}(\bar{a}, M)$ does not fork over M_1 without loss of generality the triple

M_1, N_1^+, M is in stable amalgamation, hence also M_1, N_1, M is. Let $N = \langle N_1 \cup M\rangle^{\rm gn}$, easily N witnesses that $\mathbf{tp}(\bar a_1, M) = p$ is not regular.

The other direction (p not regular $\Rightarrow \mathbf{tp}(\bar a_1, M_1)$ not regular) is even easier. $\square_{7.3}$

7.4 Lemma. *Suppose $p \in \mathscr{S}^{<\infty}(M)$ is regular. If*

- *(i)* $\mathbf{I} \cup \mathbf{J} \cup \{\bar c\} \subseteq \{\bar a \in \mathfrak{C} : \bar a$ *realizes* $p\}$
- *(ii)* *every* $\bar b \in \mathbf{J}$ *depends on* $\mathbf{I}$ *over* M *(i.e.* $\mathbf{tp}(\bar b, M \cup \cup \mathbf{I})$ *forks over* M*)*
- *(iii)* $\bar c$ *depends on* $\mathbf{I} \cup \mathbf{J}$ *over* M

<u>then</u> $\bar c$ *depends on* $\mathbf{I}$ *over* M.

7.5 Remark. We can use A instead of M and also a set $B \subseteq \mathfrak{C}$ instead $\mathbf{I}$ if we say $\mathbf{tp}(\bar a, A)$ is regular and is defined similarly.

Proof. Assume the conclusion fails; i.e. $\mathbf{tp}(\bar c, M \cup \mathbf{I})$ does not fork over M. Let $\lambda = (\chi_{\mathfrak{s}} + \|M\| + |\mathbf{I}| + |\mathbf{J}|)^+$ and N a $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous model satisfying $M \cup \mathbf{I} \subseteq N <_{\mathfrak{s}} \mathfrak{C}$ such that $\mu = \mu^{\chi(\mathfrak{s})} = \|N\|$. As $\mathbf{tp}(\bar c, M \cup \mathbf{I})$ does not fork over M there is $\bar c'$ realizing $\mathbf{tp}(\bar c, M \cup \mathbf{I})$ such that $\mathbf{tp}(\bar c', N)$ does not fork over M. Without loss of generality $\bar c = \bar c'$. We now try to choose N_i by induction on $i \le \mu^+$ such that:

- (a) $N_0 = N$
- (b) for i limit $N_i = \bigcup_{j<i} N_j$
- (c) for $i = j+1$, if N_j is $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous <u>then</u> for some $\bar b_j \in \mathbf{J}, \bar b_j \notin N_j$ and $N_j^*, \|N_j^*\| \le \chi(\mathfrak{s}) < \lambda$, $N_j^{**} = N_j^* \cap N_j$, $\bar b_j \in N_j^*$ and $\mathbf{tp}(N_j^*, N_j^{**} \cup \bar b_j) \underset{a}{\perp} N_j^{**}$ and $N_i = \langle N_j \cup N_j^*\rangle^{\rm gn}$
- (d) for $i = j+1$, if N_j not $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous, <u>then</u> $N_i = \langle N_j \cup N_j^*\rangle^{\rm gn}, \|N_j^*\| \le \chi(\mathfrak{s}) < \lambda$, $N_j^{**} = N_j^* \cap N_j$ and N_j^* is isolated
(i.e. $\mathbf{tp}(N_j^*, N_j^{**}) \underset{\rm wk}{\perp} \mathbf{tp}(N_j, N_j^{**})$).

So for some $i(*) \le \mu^+$ we stop. Note that by monotonicity of forking (and regularity of p)

(*e*) if $j < i(*)$ and N_j is $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous, <u>then</u> $\mathbf{tp}(\bar{b}_j, N_j)$ is orthogonal to p.
[Why? As $\mathbf{tp}(\bar{b}_j, N_j)$ extend $\mathbf{tp}(\bar{b}_j, M) = p$ and forks over M (as $\mathbf{tp}(\bar{b}_j, M \cup \mathbf{I})$ forks over M and $\mathbf{I} \subseteq N \subseteq N_j$) so the definition of "$p$ regular" gives the desired conclusions.]

(*f*) $i(*) < \mu^+$.
[Why? Otherwise for some club E of λ^+ we have $[\delta \in E$ & $\mathrm{cf}(\delta) > \chi_{\mathfrak{s}} \Rightarrow N_\delta$ is $(\lambda, \mathbb{D}_{\mathfrak{s}})$-homogeneous$]$, hence for every such δ clause (c) applies, but for each $\bar{b} \in \mathbf{J}, \bar{b} = b_\delta$ holds for at most one δ whereas $|\mathbf{J}| \le \lambda$, contradiction.]
Hence

(*g*) $\mathbf{I} \subseteq N_{i(*)}$.
[Why? As $\mathbf{I} \subseteq N = N_0 \subseteq N_{i(*)}$.]

(*h*) $N_{i(*)}$ is $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous.
[Why? By the density of isolated types from V.D.2.9.]

(*i*) $\mathbf{J} \subseteq N_{i(*)}$.
[Why? Similarly.]

Before we finish we prove

7.6 Subfact. $\mathbf{tp}(\bar{c}, N_i)$ does not fork over M for $i \le i(*)$.
Why? Just prove by induction.

Case A:. For $i = 0$: as $\mathbf{tp}(\bar{c}, N_i) = \mathbf{tp}(\bar{c}, N)$ and the choice of N.

Case B:. For i limit: use 5.3(6) (which is applicable as in Definition 7.2 we have demanded "prenice" (or justify it by 6.5 + superstability)).

Case C:. For $i = j + 1$, N_j is $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous:
Now

($C1$) $\mathbf{tp}(\bar{c}, N_j \cup \bar{b}_j)$ does not fork over N_j.
[Why? By clause (e).]

(C2) $\mathbf{tp}(N_i, N_j \cup \bar{b}_j) \underset{a}{\perp} N_j$.
[Why? Because N_j^{**}, N_i, N_j^* is in stable amalgamation and $\mathbf{tp}(N_j^*, N_j^{**} \cup \bar{b}_j) \underset{a}{\perp} N_j^{**}$ by clause (c).]

(C3) $\mathbf{tp}(\bar{c}, N_i)$ does not fork over N_j
[by (C2) and (C1) and definition of $\underset{a}{\perp}$].

As we know by the induction hypothesis that $\mathbf{tp}(\bar{c}, N_j)$ does not fork over M, by 2.5(4) + (C3) we know $\mathbf{tp}(\bar{c}, N_i) = \mathbf{tp}(\bar{c}, N_{j+1})$ does not fork over M.

Case D:. For $i = j + 1$, N_j not $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous.
Similar (remember 1.6 to get failure of the isolation in clause (d) from failure of the non-forking).

So we have proved Subfact 7.6. So $\mathbf{tp}(\bar{c}, N_{i(*)})$ does not fork over M, hence by monotonicity (2.5(1)) the type $\mathbf{tp}(\bar{c}, M \cup \mathbf{I} \cup \mathbf{J})$ does not fork over M, contradiction.

$\square_{7.4}$

7.7 Conclusion. 1) If $p \in \mathscr{S}^m(M)$ is regular, on $p(\mathfrak{C}) := \{\bar{c} \in {}^m\mathfrak{C} : \bar{c} \text{ realizes } p\}$ dependency satisfies the axioms of linear dependence; i.e.

(a) finite character: if $\bar{c}$ depends on $\mathbf{I}$ over M it depends on some finite subset (here we use $M <_{\mathfrak{s}} \mathfrak{C}$)

(b) transitivity: if $\bar{c}$ depends on $\mathbf{J}$, each $\bar{b} \in \mathbf{J}$ depends on $\mathbf{I}$ then $\bar{c}$ depends on $\mathbf{J}$

(c) exchange principle: if $\bar{c}_\ell$ does not depend on $\{\bar{c}_j : j < \ell\}$ for $\ell < n$ then $\{\bar{c}_\ell : \ell < n\}$ is independent.

2) So if $\mathbf{J}_1$, $\mathbf{J}_2$ are maximal independent subsets of $\mathbf{I} \subseteq p({}^m\mathfrak{C})$ then $|\mathbf{J}_1| = |\mathbf{J}_2|$.
3) If $p \in \mathscr{S}(M)$ is regular then on $\{\bar{c} : \text{tp}(\bar{c}, M) \text{ is regular } \pm p\}$ the parallel to part (1) holds.
4) On $\{\bar{c} : \bar{c} \text{ realize a regular type in } \mathscr{S}^{<\omega}(M)\}$ dependency over M satisfies enough properties to define dimensions (see [Sh:a, AP] or [Sh:c, V], i.e. in (A), $\mathbf{I}, \mathbf{J}$ are independent).

Proof. 1) Clause (a) by 5.9(1) + 6.5; clause (b) by 7.4 and clause (c) by 2.13 and clause (a).
2) Follows.
3) Similar (but use 7.9,7.10 below).
4) Also easy, as in the citations. $\square_{7.4}$

7.8 Claim. *Suppose $p \in \mathscr{S}^m(M)$ is prenice, M is $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous, p does not fork over $N_0 <_{\mathfrak{s}} M$ and $\|N_0\| + \chi_{\mathfrak{s}} < \lambda$. Then p is regular if and only if p is weakly orthogonal to every $q \in \mathscr{S}^m(M)$ extending $p \restriction N_0$ and forking over N_0 (equivalently $q \perp p$).*

Proof. Easy recalling 4.6(2). $\square_{7.8}$

7.9 Claim. *1) [$\mathfrak{s}$ superstable] If $\lambda > \chi(\mathfrak{s}), M <_{\mathfrak{s}} N$ are $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous, $p_\ell \in \mathscr{S}^{m(\ell)}(M)$, $(m(\ell) < \omega)$ for $\ell = 1, 2$, p_2 is regular, $p_1 \pm p_2$ and p_1 is realized in N then p_2 is realized in N.*
2) If $M \leq_{\mathfrak{s}} N <_{\mathfrak{s}} \mathfrak{C}$ are $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous, $\bar{c}_1 \in N$, $\bar{c}_2 \in \mathfrak{C}$ (not necessarily finite) $\mathbf{tp}(\bar{c}_\ell, M)$ does not fork over $M_\ell \leq_{\mathfrak{s}} M, \|M_\ell\| + |\ell g(\bar{c}_\ell)| + \chi_{\mathfrak{s}} < \lambda$ for $\ell = 1, 2$, and $\mathbf{tp}(\bar{c}_1, M)$, $\mathbf{tp}(\bar{c}_2, M)$ are not orthogonal then there is $\bar{c}_2' \in N$ realizing $\mathbf{tp}(\bar{c}_2, M_2)$ such that $\bar{c}_2' \notin M$.
3) [$\mathfrak{s}$ superstable] If $p \in \mathscr{S}^m(M)$ is regular, $M \leq_{\mathfrak{s}} N <_{\mathfrak{s}} C$ and M, N are $(\mathbb{D}_{\mathfrak{s}}, \chi_{\mathfrak{s}}^+)$-homogeneous, p is not realized in N then $p \underset{\text{wk}}{\perp} \mathbf{tp}(N, M)$ hence $p \perp \mathbf{tp}(N, M)$.

Proof. 1) Follows by (3).
2) As $\mathbf{tp}(\bar{c}_1, M), \mathbf{tp}(\bar{c}_2, M)$ are not orthogonal and M is $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous, there is M' satisfying $M_1 \cup M_2 \subseteq M' <_{\mathfrak{s}} M$ and $\|M'\| \leq \chi_{\mathfrak{s}} + \|M_1\| + \|M_2\| + |\ell g(\bar{c}_1)| + |\ell g(\bar{c}_2)| < \lambda$ such that $\mathbf{tp}(\bar{c}_1, M')$, $\mathbf{tp}(\bar{c}_2, M')$ are not weakly orthogonal; e.g. see 4.11. So for some $\bar{c}_1', \bar{c}_2'$ realizing $\mathbf{tp}(\bar{c}_1, M')$, $\mathbf{tp}(\bar{c}_2, M')$ respectively, $\mathbf{tp}(\bar{c}_1', M' \cup \bar{c}_2')$ forks over M'. Without loss of generality $\bar{c}_1' = \bar{c}_1$; and as N is $(\mathbb{D}_{\mathfrak{s}}, \lambda)$-homogeneous, $\lambda > |M' \cup \bar{c}_1|$, without loss of generality $\bar{c}_2' \in N$. Now $\bar{c}_2'$ is as required, it realizes $\mathbf{tp}(\bar{c}_2, M')$ hence $\mathbf{tp}(\bar{c}_2, M_2)$ (remember $M_2 \subseteq M'$). Why $\bar{c}_2' \notin M$? Because if $\bar{c}_2' \in M$, then by

monotonicity of non-forking $\mathbf{tp}(\bar{c}_1, M' \cup \bar{c}'_2)$ does not fork over M', contradiction.
3) By Definition 7.2, we know p is prenice, so by 5.4(3) it is nice. Moreover by 5.4(4) we can choose M_0 such that

$\circledast_1$ p does not fork over $M_0 \leq_{\mathfrak{s}} M$ and $\|M_0\| \leq \chi_{\mathfrak{s}}$.

Let $\langle M^i, N^j, \ell_j : i \leq \beta, j < \beta \rangle$ be maximal such that:

$\circledast_2$ (a) $\langle M^i : i \leq \beta \rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous

(b) $M^0 = M$, $M^i \leq_{\mathfrak{s}} N$

(c) $\|N^j\| \leq \chi_{\mathfrak{s}}$ and $N_j \nsubseteq M_j$

(d) if $\ell_j = 0$ then $\mathbf{tp}(N^j, M^j)$ is isolated (equivalently, $N^j \cap M^j \leq_{\mathfrak{s}} M^j$
and $\mathbf{tp}(N^j, M^j \cap N^j) \underset{\text{wk}}{\perp} \mathbf{tp}(M^j, M^j \cap N^j)$)

(e) if $\ell_j = 1$, then $N^j \cap M^j, N^j, M^j$ is in stable amalgamation and there
is $\bar{c}_j \in N^j$ such that $\mathbf{tp}(N^j, (M^j \cap N^j) \cup \bar{c}_j) \underset{a}{\perp} (M^j \cap N^j)$ and
$\mathbf{tp}(\bar{c}_j, M^j \cap N^j) \perp p$

(f) $\ell_j \in \{0, 1\}$.

Clearly there is such a sequence and $M \leq_{\mathfrak{s}} M^\beta \leq_{\mathfrak{s}} N$. Now:

$\circledast_3$ (A) $p \underset{\text{wk}}{\perp} \mathbf{tp}(M^i, M)$ for every $i \leq \beta$
[prove by induction on i, for limit i use 5.3(5), for $i = j + 1$, if $\ell_j = 0$ as in Case D of the proof of 7.6, if $\ell_j = 1$ as in case C of the proof of 7.6]

(B) M^β is $(\mathbb{D}_{\mathfrak{s}}, \chi^+_{\mathfrak{s}})$-homogeneous.
[Why? If not, then there is $M' \leq_{\mathfrak{s}} M^\beta$ of cardinality $< \lambda$ and $q \in \mathscr{S}^1(M')$ omitted by M^β and now use V.D.2.9.]

(C) there is no $\bar{d} \in {}^{\omega >}N$ satisfying $\bar{d} \notin {}^{\omega >}M$ and $\mathbf{tp}(\bar{d}, M) \perp p$.
[Why? As then choose $\bar{c}_\beta = \bar{d}, \ell_\beta = 1, N^\beta$ as required in (e) exists by 6.3 (here we use superstability) and act as in Case C of the proof of 7.6; contradicting $\beta's$ maximality]

(D) if $\bar{c} \in {}^m N$ and $\bar{c}$ realizes $p \restriction M_0$ (recalling $\circledast_1$) then $\bar{c} \in M^\beta$
[otherwise as p is not realized in N (by an assumption of 7.9(3) which we are proving) clearly $\mathbf{tp}(\bar{c}, M)$ is not $= p$ hence it forks over M_0 hence $\mathbf{tp}(\bar{c}, M^\beta)$ forks over M_0 hence (by $p's$ regularity) $p \perp \mathbf{tp}(\bar{c}, M^\beta)$ contradicting clause (C)]

(E) if $\bar{c} \in N$ but $\bar{c} \notin M^\beta$ then $\mathbf{tp}(\bar{c}, M^\beta) \perp p$
[otherwise by (B) using 7.9(2) there is $\bar{c}' \in {}^m M$, $\bar{c}' \notin {}^m(M^\beta)$, $\bar{c}'$ realizes $p \restriction M_0$, but this contradicts (D)]

(F) $N = M^\beta$
[otherwise choose $c \in N \backslash M^\beta$, by (E) we have $\mathbf{tp}(\bar{c}, M^\beta) \perp p$, but this contradicts (C)]

(G) $\mathbf{tp}(N, M) \underset{\text{wk}}{\perp} p$
[combine (F) and (A) for $i = \beta$].

The "hence $p \perp \mathbf{tp}(N, M)$" follows by 4.12 and actually not used. $\square_{7.9}$

7.10 Conclusion. [$\mathfrak{s}$ is superstable] Among regular types, non-orthogonality is an equivalence relation.

Proof. By 7.9. $\square_{7.10}$

7.11 Claim. *[$\mathfrak{s}$ superstable] Assume $m < \omega, p \in \mathscr{S}^m(N)$ is prenice, $N \le_{\mathfrak{s}} M, \lambda \ge \|N\| + \chi_{\mathfrak{s}}, M$ is $(\mathbb{D}_{\mathfrak{s}}, \lambda^+)$-homogeneous and $\bar{c} \in \mathfrak{C}$ realizes p.*

If in addition, $\mathbf{tp}(\bar{c}, M)$ does not fork over N hence it extends p, p regular and $\mathbf{I} \subseteq p(M)$ is a maximal subset independent over N <u>then</u> $\mathbf{tp}(\bar{c}, N \cup \cup \mathbf{I}) \vdash \mathbf{tp}(\bar{c}, M)$.

Proof. Assume that the conclusion fails, so there is $\bar{c}'$ such that

$\circledast_1$ (a) $\bar{c}'$ realizes $\mathbf{tp}(\bar{c}, N \cup \bigcup \mathbf{I})$
(b) $\bar{c}'$ does not realize $\mathbf{tp}(\bar{c}, M)$.

But $\circledast_1(b)$ and the uniqueness for non-forking extensions

$\circledast_2$ $\mathbf{tp}(\bar{c}', M)$ forks over N.

Of course

$\circledast_3$ $\mathbf{tp}(\bar{c}', M)$ extends $p \in \mathscr{S}(N)$.

But p is regular (by an assumption) hence

$\circledast_4$ $\mathbf{tp}(\bar{c}', M)$ is orthogonal to p.

By 6.3(2) there is $N_1 \leq_{\mathfrak{s}} M$ of cardinality $\leq \lambda$ such that:

$\circledast_5$ (a) $N \subseteq N_1$
(b) $\mathbf{tp}(\bar{c}', M)$ does not fork over N_1
(c) $\mathbf{tp}(\bar{c}', N_1)$ is nice.

By "majority" considerations there is N_2 such that

$\circledast_6$ (a) $N_2 \leq_{\mathfrak{s}} M$ and $\|N_2\| \leq \lambda$
(b) $N_1 \subseteq N_2$
(c) if $\bar{a} \in \mathbf{I}\backslash^m(N_2)$ then $\bar{a}$ realizes $\mathbf{tp}(\bar{c}, N_2)$, (a non-forking extension of p)
(d) $\mathbf{I}\backslash^m(N_2)$ is independent over N_2.

Now by 5.3(1)

$\circledast_7$ $\mathbf{tp}(\bar{c}', N_2)$ is nice.

By the definition of "nice" there is N_3 such that

$\circledast_8$ (a) $N_3 <_{\mathfrak{s}} \mathfrak{C}$ and $\|N_3\| \leq \lambda$
(b) $N_2 + \bar{c}' \subseteq N_2$
(c) $\mathbf{tp}(N_3, N_2 + \bar{c}') \underset{\mathrm{a}}{\perp} N_1$.

As M is $(\mathbb{D}_{\mathfrak{s}}, \lambda^+)$-homogeneous there is f such that:

$\circledast_9$ f is a $\leq_{\mathfrak{s}}$-embedding of N_3 into M over N_2.

Now as $f \restriction N = \text{ id }_N$ and $\bar{c}'$ realizes $\text{tp}(\bar{c}', N) = p$ clearly

$\odot_1$ $f(\bar{c}') \in {}^m M$ realizes p

moreover as $\mathbf{tp}(\bar{c}', N \cup \mathbf{I})$ does not fork over N by monotonicity:

$\odot_2$ $\mathbf{tp}(\bar{c}', N \cup (\mathbf{I} \cap {}^m(N_2))$ does not fork over N

hence

$\odot_3$ $\mathbf{tp}(f(\bar{c}'), N \cup (\mathbf{I} \cap {}^m(N_2)))$ does not fork over N.

As $\mathbf{tp}(\bar{c}', N_2) = \mathbf{tp}(f(\bar{c}'), N_2)$ is $\perp p$ (by $\circledast_4 + \circledast_5(b) + \circledast_6$) and $\mathbf{tp}(N_3, N_2 + \bar{c}')\underset{a}{\perp} N_2$ (by $\circledast_8(c)$) and $\circledast_6$ easily

$\odot_4$ for any $n < \omega$ and distinct $\bar{a}_1, \dots, \bar{a}_n \in \mathbf{I} \backslash {}^m(N_2)$ we have
- (α) $\mathbf{tp}(\bar{a}_m, \bar{a}_1 + \dots + \bar{a}_{m-1} + N_2)$ does not fork over N for $m = 1, \dots, n$
- (β) $\mathbf{tp}(\bar{a}_m, \bar{a}_1 + \dots + \bar{a}_{m-1} + f(N_2))$ does not fork over N for $m = 1, \dots, n$
- (γ) $\mathbf{tp}(\bar{a}_1 \hat{}\ \dots \hat{}\ \bar{a}_n, f(N_3))$ does not fork over N

hence by $\circledast_6(d)$

$\odot_5$ if $m < n, \bar{b}_0, \dots, \bar{b}_{m-1} \in \mathbf{I} \cap {}^n(N_2)$ are distinct, $\bar{b}_m = f(\bar{c}')$ and $\bar{a}_{m+1}, \dots, \bar{a}_{n-1} \in \mathbf{I} \backslash {}^m(N_2)$ are pairwise distinct <u>then</u> $\ell < n \Rightarrow \mathbf{tp}(\bar{b}_\ell, N + \bar{b}_0 + \dots + \bar{b}_{\ell-1})$ does not fork over N and extend p.

[Why? The sequence $\bar{b}_\ell$ realizes p for $\ell \neq m$ by the assumption on $\mathbf{I}$ and for $\ell = m$ by $\odot_1$. The non-forking for ℓ holds: if $\ell < m$ by the assumption on $\mathbf{I}$, for $\ell = m$ by $\odot_2$ for $\ell \in (m, n)$ by $\odot_4$.]

Now by Claim 5.6, i.e. the local character of being independent, $\mathbf{I} \cup \{f(\bar{c}')\}$ is a subset of $p(M)$ which is independent. This contradicts the maximality of $\mathbf{I}$. $\square_{7.11}$

7.12 Claim: [$\mathfrak{s}$ superstable]. *1) If $M \leq_{\mathfrak{s}} N <_{\mathfrak{s}} \mathfrak{C}$ are $(\mathbb{D}_{\mathfrak{s}}, \chi^+_{\mathfrak{s}})$-homogeneous models, $M \neq N$ <u>then</u> for some $c \in N \backslash M$ the type $\mathbf{tp}(c, M)$ is regular.*
2) If $M \leq_{\mathfrak{s}} N, p \in \mathscr{S}^{<\omega}(N)$ then: $p \perp M$ (see Definition 4.1(4)) <u>iff</u> whenever $M \leq_{\mathfrak{s}} M_1$, $N \underset{M}{\bigcup} M_1$ and $q \in \mathscr{S}^1(M_1)$ does not fork over

M we have $p \perp q$.
3) Assume that $p \in \mathscr{S}^{<\omega}(N_0)$, $\langle N_i : i \le i(*)\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous, $\lambda \ge \chi_{\mathfrak{s}}^+$, N_0 is $(\mathbb{D}_{\mathfrak{s}}, \lambda^+)$-homogeneous and for each $i < i(*)$:

- (a) $N_i \bigcup\limits_{N_i^{**}} N_i^*$, $\|N_i^*\| \le \chi_{\mathfrak{s}}$, $N_{i+1} = \langle N_i, N_i^*\rangle^{\rm gn}$ *and* $\|N_i^*\| \le \lambda$ *where* $N_i^{**} \le_{\mathfrak{s}} N_i$, $N_i^* \le_{\mathfrak{s}} N_{i+1}$
- (b) *<u>either</u> (i) or (ii) where*
  - (i) *there is $\bar b \in N_i^*$ finite, $N_i^*/(N_i^{**} \cup \bar b) \underset{a}{\perp} N_i^{**}$*
  - (ii) *N_i^*/N_i isolated over N_i^{**}.*

<u>Then</u> $p \underset{\rm wk}{\perp} N_{i(*)}/N_0$.

Proof. 1) Choose $c \in N\backslash M$ with rk[$\mathbf{tp}(c,M)$] minimal, see §6 in particular 6.2(7). Let $M_1 \le_{\mathfrak{s}} M$ be such that $\|M_1\| \le \chi_{\mathfrak{s}}$ and $\mathbf{tp}(c,M)$ does not fork over M_1. If $\mathbf{tp}(c,M)$ is not regular then by 7.3 also $\mathbf{tp}(c,M_1)$ is not regular, so there are M_1', $M_1 \le_{\mathfrak{s}} M_1'$ and $q \in \mathscr{S}^1(M_1')$ extending $\mathbf{tp}(c,M_1)$ such that q forks over M_1 and q not orthogonal to $\mathbf{tp}(c,M_1)$. Without loss of generality $\|M_1'\| \le \chi_{\mathfrak{s}}$ and $M_1' \le_{\mathfrak{s}} M$. By 7.9(2) some $d \in N\backslash M$ realizes q hence $\mathbf{tp}(d,M)$ extends $\mathbf{tp}(c,M_1)$ and forks over M_1 (as q does); so recalling 6.2(8) we have $\text{rk}(d,M) \le \text{rk}(d,M_1) = \text{rk}(q) < \text{rk}(c,M_1) = \text{rk}(c,M)$ contradicting the choice of c.
2), 3) Left to the reader. $\square_{7.12}$

Concerning weight (see more on this in Chapter N).

7.13 Definition. We define for $p \in \mathscr{S}^{<\omega}(M)$ (or stationary $p \in \mathscr{S}^{<\omega}(A)$) a number $w(p)$ (the weight of p) (or $w(\bar a, A)$ if $p = \mathbf{tp}(\bar a, A)$).

It is n <u>if and only if</u> there is a $(\mathbb{D}_{\mathfrak{s}}, \chi_{\mathfrak{s}}^+)$-homogeneous model N, $\bar c$, $\bar c_\ell (\ell < n)$ and N^* such that:

- (i) $\mathbf{tp}(\bar c, N)$ is the stationarization of p over N
- (ii) N^* is $(\mathbb{D}_{\mathfrak{s}}, \chi_{\mathfrak{s}}^+)$-prime over $N \cup \{\bar c\}$ (see 5.5)
- (iii) N^* is $(\mathbb{D}_{\mathfrak{s}}, \chi_{\mathfrak{s}}^+)$-prime over $N \cup \{\bar c_\ell : \ell < n\}$

(*iv*) $\mathbf{tp}(\bar{c}_\ell, M)$ is regular

(*v*) $\{\bar{c}_\ell : \ell < n\}$ is independent over N.

7.14 Theorem. *1) For every stationary p, $w(p)$ has at most one value.*
2) [$\mathfrak{s}$ superstable] For every stationary p, $w(p)$ has a value.
3) If $\mathbf{I} = \{\bar{a}_\alpha : \alpha < \alpha(*)\}$ is independent over M, $\bar{c}$ finite, then for some $w \subseteq \alpha(*)$ of cardinality $\leq w(\bar{c}, M)$,

$$\{\bar{a}_\alpha : \alpha \in (\alpha(*) \backslash w)\} \cup \{\bar{c}\} \text{ is independent over } M.$$

Proof. Similar to [Sh:c, V]. $\square_{7.13}$

§8 Trivial Regular Types

We generalize here [Sh:c, V,§7]:

8.1 Definition. We call $p \in \mathscr{S}^{<m}(M)$ trivial provided that p is regular, and if $\bar{a}_i (i \leq n)$ realizes p, $\bar{a}_n$ depends on $\{\bar{a}_i : i < n\}$ over M then for some $i < n$, $\bar{a}_n$ depends on $\{\bar{a}_i\}$ over M (we may say $\langle a_i : i \leq n \rangle$ witness the non-triviality).

8.2 Claim. *If $p_\ell \in \mathscr{S}^{<m}(M_\ell)$, $(\ell = 1, 2)$, p_1, p_2 parallel and nice, then p_1 is trivial if and only if p_2 is trivial.*

Proof. There are p, M satisfying $M_1 \cup M_2 \subseteq M$ and $p \in \mathscr{S}^m(M)$ is parallel to p_1 and to p_2 such that M is $(\mathbb{D}_\mathfrak{s}, \lambda^+)$-homogeneous, $\lambda := \|M_1\| + \|M_2\| + \chi(\mathfrak{s})$, so it is enough to prove that p is trivial iff p_ℓ is trivial for $\ell = 1, 2$. So by renaming without loss of generality $p_1 \subseteq p_2$, $M_1 \subseteq M_2$ and M_2 is $(\mathbb{D}_\mathfrak{s}, \|M_1\|^+ + \chi(\mathfrak{s}))$-homogeneous.

First assume p_2 is (regular and) trivial. Then p_1 is regular by 7.3. Suppose p_1 is not trivial and suppose $\bar{a}_0, \ldots, \bar{a}_{n-1}, \bar{a}_n$ exemplifies it; i.e. they realize p, $\bar{a}_n$ depends on $\{\bar{a}_0, \ldots, \bar{a}_{n-1}\}$ over M_1 but not on $\{\bar{a}_i\}$ over M for any $i < n$. Let $N_1 <_\mathfrak{s} \mathfrak{C}$ be such that $M_1 \cup \bar{a}_0 \cup \cdots \bar{a}_n \subseteq$

N_1; so there is an automorphism f of $\mathfrak{C}$ over M_1 such that M_1, M_2, $f(N_1)$ are in stable amalgamation. As $\bar{a}_i$ realize p_1, clearly $f(\bar{a}_i)$ realize p_2. As $\mathbf{tp}(\bar{a}_n, M_1 \cup \bar{a}_i)$ does not fork over M, by 2.5(9) the type $\mathbf{tp}(f(\bar{a}_n), M_2 \cup f(\bar{a}_i))$ does not fork over M_1 hence over M_2. Lastly as $\mathbf{tp}(f(\bar{a}_n), M_1 \cup \bigcup_{i<n} f(\bar{a}_i))$ forks over M_1, by 2.5(1) the type $\mathbf{tp}(f(\bar{a}_n), M_2 \cup \bigcup_{i<n} f(\bar{a}_i))$ forks over M_1 but $\mathbf{tp}(f(\bar{a}_n), M_2)$ does not fork over M_1 hence $\mathbf{tp}(f(\bar{a}_n), M_2 \cup \bigcup\{f(\bar{a}_i) : i < n\})$ forks over M_2. So $f(\bar{a}_0), \ldots, f(\bar{a}_n)$ exemplifies p_2 is not trivial - contradiction, so p_1 is trivial.

Secondly assume p_1 is trivial. By 7.3 the type p_2 is regular. Suppose p_2 is not trivial and we shall derive a contradiction. So let $n < \omega$, $\{\bar{a}_0, \ldots, \bar{a}_n\}$ exemplify this. Let $\mathbf{J}$ be a maximal subset of $p_1(M_2)$ independent over M_1. By their choice $\{\bar{a}_i : i < n\}$ is independent over M_2 hence easily $\mathbf{J} \cup \{a_\ell : \ell < n\}$ is independent over M_1. Let $\bar{a}'_n$ be such that $\mathbf{tp}(\bar{a}'_n, M_2 \cup \{a_\ell : \ell < n\})$ is the stationarization of p hence easily $\mathbf{tp}(\bar{a}_0 \hat{\ } \bar{a}_1 \hat{\ } \ldots \hat{\ } \bar{a}_{n-1} \hat{\ } \bar{a}'_n, M_2)$ is prenice, hence nice by 5.4(2). Now by 7.11

$$(*)_1 \quad \mathbf{tp}(\bar{a}_0 \hat{\ } \bar{a}_1 \hat{\ } \ldots \hat{\ } \bar{a}_{n-1} \hat{\ } \bar{a}'_n, M_1 \cup \mathbf{J}) \vdash \mathbf{tp}(\bar{a}_0 \hat{\ } \bar{a}_1 \hat{\ } \ldots \hat{\ } \bar{a}_{n-1} \hat{\ } \bar{a}'_n, M_2).$$

Now $\bar{a}_0 \hat{\ } \bar{a}_1 \hat{\ } \ldots \hat{\ } \bar{a}_{n-1} \hat{\ } \bar{a}_n$, does not realize the second type in $(*)_1$ by their choice hence

- $(*)_2$ $\bar{a}_n$ does not realize the stationarization of p over $M_1 \cup \mathbf{J} \cup \bigcup\{\bar{a}_\ell : \ell < n\}$.

By 6.6 for some finite $\mathbf{J}' \subseteq \mathbf{J}$

- $(*)_3$ $\bar{a}_n$ does not realize the stationarization of p over $M_1 \cup \mathbf{J}' \cup \{a_\ell : \ell < n\}$.

Now $\mathbf{J}' \cup \{\bar{a}_\ell : \ell < n\} \subseteq p(\mathfrak{C})$ is independent over M_1 (as $\mathbf{J} \cup \{\bar{a}_\ell : \ell < n\} \subseteq p_1(\mathfrak{C})$ is) but

- $(*)_4$ $\ell < n \Rightarrow \mathbf{tp}(\bar{a}_n, M_1 \cup \bar{a}_\ell)$ does not fork over M_2.
  [Why? as $\mathbf{tp}(\bar{a}_n, M_2 \cup \bar{a}_\ell)$ is a non-forking extension of p_2 hence of $p_2 \restriction M_2 = p_1$]
  and

$(*)_5$ $\bar{a} \in \mathbf{J}' \Rightarrow \mathbf{tp}(\bar{a}_n, M_1 \cup \bar{a})$ does not fork over M_1.
[Why? As $\mathbf{tp}(\bar{a}_n, M_2)$ does not fork over M_1.]

So we are done. $\square_{8.2}$

8.3 Claim. *If p_1, p_2 are regular not orthogonal <u>then</u>: p_1 is trivial if and only if p_2 is trivial.*

Proof. Let M be $(\mathbb{D}_{\mathfrak{s}}, \chi(\mathfrak{s})^+)$ -homogeneous, and without loss of generality $p_1 \in \mathscr{S}^{m(1)}(M)$, $p_2 \in \mathscr{S}^{m(2)}(M)$ and they are regular. Suppose p_1 is not trivial and let $n < \omega$, $\bar{a}_0, \dots, \bar{a}_n \in {}^{m(1)}(\mathfrak{C})$ exemplify this. Let M_ℓ be $\chi(\mathfrak{s})^+$-primary over $M \cup \bar{a}_\ell$ (see 5.5). By 7.9(1) there is $\bar{b}_\ell \in M_\ell$ realizing p_2. So (see 7.12(3)) the type $\mathbf{tp}(\bar{b}_\ell, M \cup \bar{a}_\ell)$ forks over M, so by symmetry (2.10) also $\mathbf{tp}(\bar{a}_\ell, M \cup \bar{b}_\ell)$ forks over M. Similarly by 7.12(3) for $\ell < n$, the type $\mathbf{tp}(\bar{b}_n, M_\ell)$ does not fork over M as $\mathbf{tp}(\bar{b}, M + \bar{b}_\ell)$ does not fork over M hence $\mathbf{tp}(\bar{b}_n, M \cup \bar{b}_\ell)$ does not fork over M. Also $\{M_\ell : \ell < n\}$ is independent over M hence $\{b_\ell : \ell < n\}$ is independent over M. On the other hand, by applying 7.7(3) twice (to $\mathbf{I} = \{\bar{b}_\ell : \ell < n\}, \mathbf{J} = \{\bar{a}_\ell : \ell < n\}$ and $\bar{a}_n$, then $\mathbf{I} = \{\bar{b}_\ell : \ell < n\}$, $\mathbf{J} = \{\bar{a}_n\}$ and $\bar{b}_n$) we get $\mathbf{tp}(\bar{b}_n, M \cup \{\bar{b}_\ell : \ell < n\})$ forks over M. So $b_0, \dots, \bar{b}_n$ exemplify "p_2 is not trivial". $\square_{8.3}$

8.4 Claim. *If $p_\ell = \mathbf{tp}(\bar{a}_\ell, M) \in \mathscr{S}^{<\omega}(M)$ is not orthogonal to a trivial q for $\ell = 1, 2$, <u>then</u> $p_1 \underset{\text{wk}}{\pm} p_2$.*

Proof. We can find N such that $M \cup \text{Dom}(q) \subseteq N <_{\mathfrak{s}} \mathfrak{C}, N$ is $(\mathbb{D}_{\mathfrak{s}}, \lambda^+)$-homogeneous where $\lambda = \|M\| + \chi(\mathfrak{s})$. Without loss of generality $q \in \mathscr{S}^{<\omega}(N)$. Let $q_\ell \in \mathscr{S}^{<\omega}(N)$ be the stationarization of p_ℓ over N for $\ell = 1, 2$. We continue as in [Sh:c, X,7.3,pg.552]. $\square_{8.4}$

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Introduction

The idea of this chapter is to start with a given (superstable) framework $\mathfrak{s}$, to strengthen the notion of submodel, and arrives to a similar $\mathfrak{s}^+$ with the same class of models, NF relation except restricting to the cases $M_0 \le_{\mathfrak{s}(+)} M_\ell \le_{\mathfrak{s}(+)} M_3$ for $\ell = 1, 2$ and the same closure operation $\langle - \rangle^{\rm gn}$) discarding along the way various non-structure cases. So we may ask — why should we trouble so much to return to our starting point? However, if we repeat this ω times, the limit is more similar to the first order case; we try to start to deal with this approach in Chapter V.G.

The first section revisits the problem of existence of indiscernibles dealt with in I,§5, throwing more light even in the first order case. A point is that we consider indiscernible sequences with other "index models" (in addition to well orders and sets), the main case being $[\lambda]^2$. The main point is that we get independent $\langle M_u : u \in [S]^{\le 2} \rangle$.

In §2 we define the order properties for some variants of Σ_1-formulas (for $\mathbb{L}_{\chi^+,\chi^+}[\mathfrak{K}_\mathfrak{s}]$). We would like to use it to get non-structure, but we have a problem in building from it many models using non-well ordered linear orders because we lose track of the satisfaction of those infinary formulas. Hence we use the indiscernibility existence from §1 and partial well orders to get non-structure.

This leads in §3, assuming the suitable non-order property, to the introduction of a successor $\mathfrak{s}^+$ of $\mathfrak{s}$, with a stronger notion of submodels from §2. We prove that it satisfies AxFr_1^-, an approximation to our main framework here, and more, relative of being based and

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weaker substitutes of existence of union and smoothness. The main lack is the existence of union.

So in §4 we show that the failure of "the union of an increasing ω-sequence does $\le_{\mathfrak{s}(+)}$-extend each of them" implies a non-structure result. However, it gives only $\dot{I}(\mu, K_{\mathfrak{s}}) \ge \mu^+$ for many μ's. In §5 we get a similar result about from failure of "the union of an $\le_{\mathfrak{s}(+)}$-increasing continuous sequence of length θ.

§1 More on Indiscernibility

1.1 Context. $\mathfrak{s}$ is as in V.B§1, satisfying AxFr_1, i.e. Ax(A1)-(A4),(B0)-(B3), (C1)-(C7) and (C8), smoothness and let $\mathfrak{C}$ be a monster.

1.2 Definition. 1) We say $\langle M_u : u \in W\rangle$ is independent inside M where $W \subseteq [I]^{<\aleph_0}$ is closed under subsets, <u>when</u>:

- (a) $u \subseteq w \in W \Rightarrow M_u \le_{\mathfrak{s}} M_w \le_{\mathfrak{s}} M$
- (b) $M_{W(1)} \le_{\mathfrak{s}} M$ for $W(1) \subseteq W$ where $M_{W(1)} = \langle \cup\{M_u : u \in W(1)\}\rangle^{\text{gn}}$ so without loss of generality $W(1)$ is downward closed
- (c) if $W(\ell) \subseteq W$ for $\ell = 1, 2$ are downward closed and $W(0) = W(1) \cap W(2)$ <u>then</u> $M_{W(0)}, M_{W(1)}, M_{W(2)}$ is in stable amalgamation.

1.3 Remark. 1) For a linear order I we do not strictly distinguish e.g. $[I]^{<n}$ and $\{(s_0, ..., s_{m-1}) : m < n$ and $I \models s_0 < s_1 < \ldots < s_{m-1}\}$.
2) Recall $\mathscr{P}^-(w) = \{u \subseteq w : u \ne w\}$.

1.4 Lemma. *1) For given $n_0 \le \omega$ and $\bar{M} = \langle M_u : u \in W\rangle$ where $W \subseteq [I]^{<n_0}$ is downward closed, the following are equivalent:*

- (a) *$\bar{M}$ is independent*
- (b) *there is $\{u_i : i < i(*)\}$, a list of W, such that:*
  - (α) *$u_i \subseteq u_j \Rightarrow i \le j$;*
  - (β) *for each $i, M_{\mathscr{P}^-(u_i)}, M_{u_i}, M_{\{u_j : j<i\}}$ is in stable amalgamation.*

2) Independence has finite character, that is, for any downward closed family W of finite sets, $\langle M_u : u \in W\rangle$ is independent iff $\langle M_u : u \in V\rangle$ is independent for any finite downward closed $V \subseteq W$.

Proof. 1) Now $(a) \Rightarrow (b)$ is trivial as any list $\langle u_i : i < i(*)\rangle$ of W satisfying clause $(b)(\alpha)$ satisfies clause $(b)(\beta)$ as we are assuming clause (a).

For the other direction let $A_i := M_{\{u_j : j<i\}} <_{\mathfrak{s}} \mathfrak{C}$ and $B_i := M_{\mathscr{P}^-(u_i)}, w_i = \{j : u_j \subset u_i\}$, so $\langle \emptyset, A_i, B_j, w_j : i \le i(*), j < i(*)\rangle$ is a stable construction (so Definition V.C.4.2(2) and use V.C.4.5(1).
2) Let $\langle u_i : i < i(*)\rangle$ be a list of W such that $u_i \subseteq u_j \Rightarrow i \le j$, i.e. as in $(b)(\alpha)$ of 1.4. We prove "$\langle M_u : u \in W\rangle$ is independent" by induction on $i(*)$. This is similr to the above except that for limit $i(*), \cup\{M_{\{u_j:j<i\}} : i < i(*)\}$ is $<_{\mathfrak{s}} \mathfrak{C}$ by Ax(C8), smoothness. $\square_{1.4}$

Remark. 1) In the proof of 1.4(1) we did not use Ax(C8).
2) Claim 1.4(2) can be proved without Ax(C8) under the additional assumption "$\langle M_u : u \in W\rangle$ is independent inside $\cup\{M_v : v \subseteq W\}$ and W is finite downward closed"?

1.5 Lemma. *[$\mathfrak{s}$ is χ-based and $\chi \ge |\tau_{\mathfrak{s}}| + \mathrm{LS}(\chi)$].*

If λ is regular, $(\forall \alpha < \lambda)[|\alpha|^\chi < \lambda]$ and $\bar a_\alpha \in {}^{\chi^+}\mathfrak{C}$ for $\alpha < \lambda$ <u>then</u> there are $S \subseteq S^\lambda_{>\chi} := \{\delta < \lambda : \mathrm{cf}(\delta) \ge \chi^+\}$ stationary, $N_{<>} \le_{\mathfrak{s}} N_{<\alpha>} <_{\mathfrak{s}} \mathfrak{C}$ for $\alpha \in S, \|N_\alpha\| \le \chi, \bar a_\alpha \subseteq N_\alpha, \{N_{<\alpha>} : \alpha \in S\}$ independent over $N_{<>}$ (see Definition V.D.3.15) and for $\alpha, \beta \in S$ there is an isomorphism $h_{<\alpha>,<\beta>}$ from $N_{<\alpha>}$ onto $N_{<\beta>}$ over $N_{<>}$ mapping $\bar a_\alpha$ to $\bar a_\beta$.

Proof. We choose $M_\alpha <_{\mathfrak{s}} \mathfrak{C}$ by induction on α, increasing continuous such that $\bar a_\alpha \subseteq M_{\alpha+1}$ and $\|M_\alpha\| = \chi + |\alpha|$. For each α find $N_\alpha \le_{\mathfrak{s}} M_{\alpha+1}$ such that $\bar a_\alpha \subseteq N_\alpha$ and $N_\alpha \cap M_\alpha, N_\alpha, M_\alpha$ is in stable amalgamation, exists by Definition V.C.3.7. For $\delta \in S^\lambda_{>\chi}$ let $\beta_\delta = \min\{\beta < \delta : N_\delta \cap M_\alpha \subseteq M_\beta\}$, it is $< \delta$ as $\|N_\delta\| \le$

$\chi <$ cf(δ). So by Fodor lemma for some stationary $S_1 \subseteq S^\lambda_{<\chi}$, we have $\delta \in S_1 \Rightarrow \beta_\delta = \beta_*$. As $\|M_{\beta_*}\|^\chi < \lambda$, for some stationary $S \subseteq S_1, N_\alpha \cap M_\alpha = N_{<>}$ for $\alpha \in S$, and $(N_\alpha, c)_{c \in N_{<>}}$ has the same isomorphism type. Now $\langle N_\alpha : \alpha \in S\rangle, N_{<>}$ are as required. $\square_{1.5}$

1.6 The Pair Index Theorem. *[$\chi_\mathfrak{s}$ is well defined so $\mathfrak{s}$ is μ-based for $\mu \geq \chi_\mathfrak{s}$].*

Suppose $\bar{a}_t \in M \in \mathfrak{K}_\mathfrak{s}$ for $t \in [\lambda]^{\leq 2}, \lambda = (2^\mu)^+, \ell g(\bar{a}_t) < \kappa, \mu = \mu^{<\kappa}, \kappa = \text{cf}(\kappa) > \chi_\mathfrak{s}$.

Suppose further g is a symmetric two place function from λ to μ. Then we can find $I \subseteq \lambda$ and N_t (for $t \in [I]^{\leq 2}$) and $h_{s,t}$ (for $s, t \in [I]^{\leq 2}, |s| = |t|$) such that:

- (a) *I has order type μ^+*
- (b) *$\bar{a}_t \subseteq N_t \leq_\mathfrak{s} M$ and $\|N_t\| < \kappa$*
- (c) *$\langle N_t : t \in [I]^{\leq 2}\rangle$ is independent (see Definition 1.2)*
- (d) *if $t, s \in I^{[\leq 2]}$; i.e. t, s are increasing sequences of length ≤ 2 from I having the same length, then $h_{s,t}$ is an isomorphism from N_t onto N_s mapping $\bar{a}_t$ to $\bar{a}_s$*
- (e) *if $\alpha < \beta$ and $\alpha_1 < \beta_1$ are from I, then*
  $h_{<>,<>} \subseteq h_{<\alpha>,<\alpha_1>} \subseteq h_{<\alpha,\beta>,<\alpha_1,\beta_1>}$,
  $h_{<>,<>} \subseteq h_{<\beta>,<\beta_1>} \subseteq h_{<\alpha,\beta>,<\alpha_1,\beta_1>}$
- (f) *if $t_1, t_2, t_3 \in I^{[\leq 2]}$ have the same length then $h_{t_3,t_1} = h_{t_3,t_2} \circ h_{t_2,t_1}$.*

Remark. 0) Recall $h_{<>,<>}$ is $h_{\emptyset,\emptyset}$ and $h_{<\alpha>,<\alpha_1>}$ is $h_{\{\alpha\},\{\alpha_1\}}$ and $h_{<\alpha,\alpha_1>,<\beta,\beta_1>}$ is $h_{\{\alpha,\alpha_1\},\{\beta,\beta_1\}}$.
1) This is similar to V.A§5, but we have here $a_{\{\alpha\}}, \bar{a}_{\{\alpha,\beta\}}$ rather than $\bar{a}_{\{\alpha\}}$ alone.
2) This is new even for first order stable T.
3) We can replace λ by a linear order of this cardinality, only then I may have order type the inverse of μ^+.

Proof. Let χ be large enough regular cardinal such that $\chi = \chi^\lambda$ and in particular,

$M, \langle \bar a_t : t \in \lambda^{[\le 2]}\rangle$ belongs to $\mathscr{H}(\chi)$.
Let $\mathfrak{B}$ be an $\mathbb{L}_{\mu^+,\mu^+}$-elementary submodel of $\mathfrak{B}^* := (\mathscr{H}(\chi^+), \in, \mathfrak{s}, M, \langle \bar a_\eta : \eta \in \lambda^{[\le 2]}\rangle)$ of cardinality 2^μ, such that $\{\alpha : \alpha \le 2^\mu\} \subseteq \mathfrak{B}$, so clearly $\mathfrak{B} \in \mathscr{H}(\chi^+)$ and $\mathfrak{B} \cap \lambda$ is an initial segment of λ recalling $\lambda = (2^\mu)^+$. Choose $\alpha = \text{Min}(\lambda \backslash \mathfrak{B})$; the $\mathfrak{s}$ in $\mathfrak{B}$ means that $\{M : M \in K_\mathfrak{s} \cap \mathscr{H}(\chi)\}, \{(M,N) : M \le_\mathfrak{s} N \in \mathscr{H}(\chi)\}$ are relations of $\mathfrak{B}$.

We say $\mathfrak{A}$ is a candidate if

$\circledast_0$ $\mathfrak{A} \prec \mathfrak{B}^*, \|\mathfrak{A}\| \le \mu, \mathfrak{B} \in \mathfrak{A}$ and $\mu + 1 \subseteq \mathfrak{A}$ hence $M \cap \mathfrak{B} \cap \mathfrak{A}, M \cap \mathfrak{B}, M \cap \mathfrak{A}$ is in stable[1] amalgamation by V.C.3.13, also $\mathfrak{B}, \lambda \in \mathfrak{A}$ hence $\alpha \in \mathfrak{A} \cap \lambda$.

We say $\mathfrak{x} = \big(N_{<>}, \bar b_{<>}, N_{<0>}, \bar b_{<0>}, N_{<1>}, b_{<1>}, N_{<0,1>}, b_{<0,1>}, h, i\big)$ is a possible <u>witness</u> for the candidate $\mathfrak{A}$ <u>when</u>:

$\circledast_1$ (*i*) $N_{<>} \le_\mathfrak{s} N_{<\ell>} \le_\mathfrak{s} N_{<0,1>} \le_\mathfrak{s} M$ for $\ell = 0, 1$

(*ii*) $N_{<>}, N_{<0>}, N_{<1>}$ is in stable amalgamation inside M,

(*iii*) $\|N_{<0,1>}\| < \kappa$,

(*iv*) $\bar b_\eta \subseteq N_\eta$ for $\eta \in \{<>, <0>, <1>, <0,1>\}$, satisfying $\bar b_{<>} = a_{<>}$ and $\bar b_{<1>} = \bar a_{<\alpha>}$

(*v*) $N_{<>} \subseteq |\mathfrak{A}| \cap |\mathfrak{B}| \subseteq M$ and $N_{<1>} \subseteq |\mathfrak{A}| \cap M$

(*vi*) h is an isomorphism from $N_{<0>}$ onto $N_{<1>}$ over $N_{<>}$ satisfying $h(\bar b_{<0>}) = \bar b_{<1>}$

(*vii*) $i < \mu$.

We say $\mathfrak{x}$ is a witness for the candidate $\mathfrak{A}$ if in addition:

$\circledast_2$ <u>for any</u> candidate $\mathfrak{A}_1$ such that $\mathfrak{A} \prec \mathfrak{A}_1$ and formula $\varphi(\bar x) \in \mathbb{L}_{\mu^+,\mu^+}$ with $\bar x = \langle x_c : c \in \mathfrak{A}_1\rangle$ such that $\mathfrak{B}^* \models \varphi(..., c, ...)_{c \in \mathfrak{A}_1}$ <u>there is</u> a one to one function f from $|\mathfrak{A}_1|$ into $|\mathfrak{B}|$ over $|\mathfrak{B}| \cap |\mathfrak{A}_1|$, and functions h_η for $\eta \in \{<>, <0>, <1>, <0,1>\}$ such that:

(a) $\mathfrak{B}^* \models \varphi(\dots, f(c), \dots)_{c \in \mathfrak{A}_1}$ and f maps $\mathfrak{A}_1 \cap M$ into M and $f \restriction (\mathfrak{A}_1 \cap M)$ is a $\le_\mathfrak{s}$-embedding of $M \restriction (\mathfrak{A}_1 \cap M)$ into M

[1] pedantically we should write "$M \restriction (M \cap \mathfrak{B} \cap \mathfrak{A}), M \restriction (M \cap \mathfrak{B}), M \restriction (M \cap \mathfrak{A})$

(b) $g(f(\alpha),\alpha) = i = i^{\mathfrak{x}}$ (remember: g was a two place function from λ to μ, a "coloring" and $f(\alpha) < \alpha$ as $\mathfrak{B} \cap \lambda = \alpha < \lambda$)

(c) $h_{<0,1>}$ is $a \leq_{\mathfrak{s}}$-embedding of $N_{<0,1>}$ into M

(d) $h_\eta = h_{<0,1>} \restriction N_\eta$ for $\eta = <>, <0>, <1>$

(e) $h_{<1>} = \mathrm{id}_{N_{<1>}}$

(f) $h_{<0>} = f \circ h$ and $h_{<0>}(\bar{b}_{<0>}) = \bar{a}_{f(\alpha)}$ which is necessarily equal to $f(\bar{a}_\alpha)$

(g) $h_{<0,1>}(\bar{b}_{<0,1>}) = \bar{a}_{<f(\alpha),\alpha>}$

(h) the triple $\langle h_{<0>}(N_{<0>}) \cup h_{<1>}(N_{<1>})\rangle^{\mathrm{gn}}, h_{<0,1>}(N_{<0,1>})$,

$\langle (\mathfrak{A}_1 \cap M) \cup \big(f(\mathfrak{A}_1 \cap M)\big)\rangle^{\mathrm{gn}}$ is in stable amalgamation inside M.

If $\mathfrak{A}_1, \varphi$ form a counterexample to $\circledast_2$, we say that the pair $(\mathfrak{A}_1, \varphi)$ exemplifies the failure of the possible witness $\mathfrak{x} = \big(N_{<>}, \bar{b}_{<>}, N_{<0>}, \bar{b}_{<0>}, N_{<1>}, \bar{b}_{<1>}, N_{<0,1>}, \bar{b}_{<0,1>}, h, i\big)$.

If $\mathfrak{x}, \mathfrak{y}$ are possible witnesses for a candidate $\mathfrak{A}$, then we say that $\mathfrak{x}, \mathfrak{y}$ are isomorphic when there is an isomorphism g from $\mathfrak{x}$ onto $\mathfrak{y}$ which means:

$\circledast_3$ (a) g is an isomorphism from $N^{\mathfrak{x}}_{<0,1>}$ onto $N^{\mathfrak{y}}_{<0,1>}$

(b) g maps $N^{\mathfrak{x}}_\eta$ to $N^{\mathfrak{y}}_\eta$ for $\eta = \langle\rangle, \langle 0\rangle, \langle 1\rangle$

(c) g maps $\bar{b}^{\mathfrak{x}}_\eta$ to $b^{\mathfrak{y}}_\eta$ for $\eta = \langle 0\rangle, \langle 1\rangle, \langle 1,2\rangle$ and also $\eta = \langle\rangle$ (which is automatic)

(d) $h^{\mathfrak{y}} g = g h^{\mathfrak{x}}$

(e) $i^{\mathfrak{y}} = i^{\mathfrak{x}}$.

1.7 Fact. For some candidate $\mathfrak{A}$ there is a witness.

Proof. Suppose not.

By induction on $\zeta < \mu$ we choose $\mathfrak{A}_\zeta, \mathfrak{A}'_\zeta, \varphi_\zeta$ and $\mathfrak{x}^\zeta = \big(N^\zeta_{<>}, b^\zeta_{<>}, N^\zeta_{<0>}, \bar{b}^\zeta_{<0>}, N^\zeta_{<1>}, b^\zeta_{<1>}, N^\zeta_{<0,1>}, \bar{b}^\zeta_{<0,1>}, h^\zeta, i^\zeta\big)$ such that:

$\circledast_4$ (α) $\mathfrak{A}_\zeta$ is a candidate (i.e., $\mathfrak{A}_\zeta \prec \mathfrak{B}, \|\mathfrak{A}_\zeta\| \le \mu, \mu + 1 \subseteq \mathfrak{A}_\zeta, \mathfrak{B} \in \mathfrak{A}_\zeta$,
(hence $\alpha \in \mathfrak{A}_\zeta$),

(β) $\mathfrak{A}_\zeta$ is increasing (by $\prec$)

(γ) $\mathfrak{x}^\zeta$ is a possible witness for the candidate $\mathfrak{A}_\zeta$

(δ) $\big(\mathfrak{A}_{\zeta+1}, \varphi_\zeta\big)$ exemplifies $\mathfrak{x}^\zeta$ is not a witness for the candidate $\mathfrak{A}_\zeta$

(ϵ) every possible witness for $\bigcup_{\zeta<\mu} \mathfrak{A}_\zeta$ is isomorphic to $\mathfrak{x}^\xi$ over $\mathfrak{A}_\xi$ for some $\xi < \mu$

(ζ) $\mathfrak{A}_{\zeta+1} \prec_{\mathbb{L}_{\kappa,\kappa}} \mathfrak{B}$.

For this, just note:

$\circledast_5$ if $\mathfrak{A} \prec \mathfrak{A}'$ are candidates, any possible witness for $\mathfrak{A}$ is a possible witness for $\mathfrak{A}'$

$\circledast_6$ if $\langle \mathfrak{A}_\zeta : \zeta \le \delta \rangle$ is an increasing continuous sequence of candidates and $\mathrm{cf}(\delta) \ge \kappa$ and $\mathfrak{x}$ is a possible witness for $\mathfrak{A}_\delta$, <u>then</u> $\mathfrak{x}$ is a possible witness for $\mathfrak{A}_\zeta$ for some $\zeta < \delta$

$\circledast_7$ if $\mathfrak{A}$ is a candidate <u>then</u> the number of possible witnesses over $\mathfrak{A}$ up to isomorphism is $\le \mu^{<\kappa} = \mu$.

Now there is no problem to carry the definition.
Let $\mathfrak{A} = \bigcup_{\zeta<\mu} \mathfrak{A}_\zeta$, so $\mathfrak{A}$ is a candidate and $\mathfrak{A} \prec_{\mathbb{L}_{\kappa,\kappa}} \mathfrak{B}^*$. For $\zeta < \mu$ we can find a formula $\psi_\zeta(\dots, x_\ell, \dots)_{c \in \mathfrak{A}_\zeta \cap M}$ such that for any function $f \in \mathscr{H}(\chi)$ with domain $\mathfrak{A}_\zeta \cap M$, we have $\mathfrak{B} \models \psi_\zeta(\dots, f(c), \dots)_{c \in \mathfrak{A}_\zeta \cap M}$ iff f is a $\le_{\mathfrak{s}}$-embedding of $M \restriction (\mathfrak{A}_\zeta \cap M)$ into M.

Clearly $\bigwedge_{\zeta<\mu} (\varphi_\zeta \wedge \psi_\zeta)$ is still an $\mathbb{L}_{\mu^+,\mu^+}$-formula and $\mathfrak{B} \models \bigwedge_{\zeta<\mu} (\varphi_\zeta(\dots, c, \dots)_{c \in \mathfrak{A}_\zeta} \& \psi_\zeta(\dots, f(c), \dots)_{c \in \mathfrak{A}_\zeta \cap M})$. Hence there is a one to one function f from $\mathfrak{A}$ into $\mathfrak{B}$ such that:

$(a')\qquad \mathfrak{B} \models \bigwedge_{\zeta<\mu} (\varphi_\zeta(\dots, f(c), \dots)_{c\in\mathfrak{A}} \ \&\ \psi_\zeta(\dots, f(c), \dots)_{c\in\mathfrak{A}_\zeta\cap M}).$

By $\circledast_0$ clearly $\mathrm{NF}(M\cap\mathfrak{B}\cap\mathfrak{A}, M\cap\mathfrak{A}, M\cap\mathfrak{B}, M)$ hence $(M\cap\mathfrak{B}\cap\mathfrak{A}) \le_{\mathfrak{s}} M\cap\mathfrak{B}$. Obviously $M\cap\mathfrak{B}\cap\mathfrak{A} \subseteq M\cap f(\mathfrak{A}) \subseteq M\cap\mathfrak{B}$.

Also $M\cap f(\mathfrak{A}) \le_{\mathfrak{s}} M$ by the choice of ψ_ζ, hence by Ax(A3), see V.B.1.4 we have $M\cap\mathfrak{B}\cap\mathfrak{A} \le_{\mathfrak{s}} M\cap f(\mathfrak{A}) \le_{\mathfrak{s}} M\cap\mathfrak{B}$. By the last two sentences it follows that $\mathrm{NF}((M\cap\mathfrak{B}\cap\mathfrak{A}, M\cap\mathfrak{A}, M\cap f(\mathfrak{A}), M)$ (by monotonicity as $\mathfrak{A}$ is a candidate). Hence $N^* := \langle M\cap\mathfrak{A}, M\cap f(\mathfrak{A})\rangle^{\mathrm{gn}} \le_{\mathfrak{s}} M$.

As $\chi_{\mathfrak{s}} + |\ell g(\bar{a}_{\langle f(\alpha),\alpha\rangle})| < \kappa$, by the assumption of 1.6 the framework $\mathfrak{s}$ is $(\chi_{\mathfrak{s}} + |\ell g(\bar{a}_{<f(\alpha),\alpha>})|)$-based we can find $N \le_{\mathfrak{s}} M$ of cardinality $< \kappa$ such that $\bar{a}_{<>} \cup \bar{a}_{<\alpha>} \cup \bar{a}_{<f(\alpha)>} \cup a_{<f(\alpha),\alpha>} \subseteq N$, $\|N\| < \kappa$, $\mathrm{NF}(N\cap N^*, N, N^*, M)$, and $\mathrm{NF}(N\cap\mathfrak{A}, N, M\cap\mathfrak{A}, M)$, $\mathrm{NF}(N\cap f(\mathfrak{A}), N, M\cap f(\mathfrak{A}), M)$, and
$\mathrm{NF}(N\cap\mathfrak{A}\cap\mathfrak{B}, N, M\cap\mathfrak{A}\cap\mathfrak{B}, M)$. Why having those four NF conditions is possible? By V.C.4.8 using $N^* \le_{\mathfrak{s}} M$ (see the previous paragraph), $M\cap\mathfrak{A} \le_{\mathfrak{s}} M$ (as $N\cap(M\cap\mathfrak{A}) = N\cap\mathfrak{A}$), $M\cap f(\mathfrak{A}) \le_{\mathfrak{s}} M$ and $M\cap\mathfrak{A}\cap\mathfrak{B} \le_{\mathfrak{s}} M$ respectively.

By $(a) \Rightarrow (b)$ in Claim V.D.1.18 the triple $N\cap\mathfrak{A}\cap\mathfrak{B}, N\cap\mathfrak{A}, N\cap f(\mathfrak{A})$ is in stable amalgamation inside N because $M\cap\mathfrak{A}\cap\mathfrak{B}, M\cap\mathfrak{A}, M\cap f(\mathfrak{A})$ is in stable amalgamation inside N^*.

Now we let

$\circledast_7$ (α) $N_{<0,1>} = N\cap N^*$
(β) $N_{<1>} = N\cap\mathfrak{A}$
(γ) $N_{<0>} = f(N_1) = f(N)$
(δ) $N_{<>} = N\cap\mathfrak{A}\cap\mathfrak{B}$
(ε) $i = g(f(\alpha), \alpha)$
(ζ) $\bar{b}_{<0,1>} = f(\bar{a}_{<f(\alpha),\alpha>})$
(η) $\bar{b}_{<1>} = \bar{a}_{<\alpha>}$
(θ) $\bar{b}_{<0>} = \bar{a}_{<f(\alpha)>}$ which is $f(\bar{a}_\alpha)$
(ι) $\bar{b}_{<>} = \bar{a}_{<>}$
(κ) $h = (f \restriction N_{<1>})^{-1}$.

Now $\mathfrak{x} := \big(N_{<>}, \bar{b}_{<>}, N_{<0>}, \bar{b}_{<0>}, N_{<1>}, \bar{b}_{<1>}, N_{<0,1>}, \bar{b}_{<0,1>}, h, i\big)$ is a possible witness for $\mathfrak{A}$, hence by $\circledast_5$ (as $\mathrm{cf}(\mu) \geq \kappa$) it is a possible witness for $\mathfrak{A}_\xi$ for some $\xi < \mu$ hence is isomorphic to $\mathfrak{x}^\zeta$ for some $\zeta > \mu$ so there is an isomorphism h from $\mathfrak{x}$ onto $\mathfrak{x}^\zeta$. But then "$(\varphi_\zeta, \mathfrak{A}_{\zeta+1})$ exemplifies the failure of $\mathfrak{x}^\zeta$ as a witness for $\mathfrak{A}_\zeta$". However, $f \restriction \mathfrak{A}_{\zeta+1}, h_\eta := h^{-1} \restriction N^{\mathfrak{x}^\zeta}_\eta$ contradicts this.

So the fact holds. $\square_{1.7}$

* * *

Continuing Proof of 1.6. Now it is quite easy to use the existence of a witness to produce the desired $\langle N_t : t \in [I]^{\leq 2}\rangle$ choosing $\alpha_\varepsilon, N_{\{\alpha_\varepsilon\}}, N_{\{\alpha_\xi,\alpha_\varepsilon\}}$ for $\xi < \varepsilon$ by induction on ε (and the appropriate maps, of course). $\square_{1.6}$

On partition theorems on trees see Rubin Shelah [RuSh 117], more in [Sh:f].

<u>1.8 Exercise</u>: Suppose $\kappa = \mathrm{cf}(\kappa) > \chi_{\mathfrak{s}}, \lambda$ is regular and $(\forall \alpha < \lambda)(|\alpha|^{<\kappa} < \lambda)$ and $\mathbb{I}_\eta$ is a normal ideal on λ for $\eta \in {}^{\omega>}\lambda$. If $\bar{a}_\eta \in {}^{\kappa>}\mathfrak{C}$ for $\eta \in {}^{\omega>}\lambda$ <u>then</u> we can find $\langle N_\eta : \eta \in \mathscr{T}\rangle$ such that

- (a) $\mathscr{T}$ is a non-empty set of finite increasing sequences of ordinals $< \lambda$
- (b) $\mathscr{T}$ is closed under initial segments
- (c) $\bar{a}_\eta \subseteq N_\eta <_{\mathfrak{s}} \mathfrak{C}$ and $\|N_\eta\| < \kappa$
- (d) $\eta \in \mathscr{T} \Rightarrow \{\alpha < \lambda : \eta\hat{\ }\langle\alpha\rangle \in \mathscr{T}\} \neq \emptyset \bmod \mathbb{I}_\eta$
- (e) if $\nu = \eta\hat{\ }\langle\alpha\rangle \in \mathscr{T}$ then $N_\eta, N_\nu, \langle\cup\{N_\rho : \rho \in \mathscr{T}$ but $\neg(\mu \trianglelefteq \rho)\rangle^{\mathrm{gn}}$ is in stable amalgamation.

[Hint: Combines the proofs or see [Sh:E54]. We give some details.

Suppose λ is regular, $\kappa > \aleph_0, (\mathscr{T}, \bar{\mathbb{I}})$ is a λ-tagged tree, each $\mathbb{I}_\eta$ is a normal ideal on λ and for notational simplicity the sets $\{\mathrm{Suc}_{\mathscr{T}}(\eta) : \eta \in \mathscr{T}\}$ are pairwise disjoint.

Suppose further that Pr_n is a 2n-place relation with the even places for sets $s \in [\lambda]^{<\kappa}$, odd places for ordinals $\delta < \lambda$, and $\mathscr{Y} = \{s_i : i < \lambda\}$ and $\mathscr{Y}$ is equal to $[\lambda]^{<\kappa}$ or just is a subset of $[\lambda]^{<\kappa}$ and:

(∗) if $\delta_0 < \delta_1 < \ldots < \delta_n < \ldots$ for $n < \omega$, $\langle \delta_0, \ldots, \delta_{n-1} \rangle \in \mathscr{T}$ for each n <u>then</u> for some $a \subseteq \bigcup_{n<\omega} \delta_n$ we have:

- (i) $a \cap \delta_n \in \{s_i : i < \delta_{n+1}\}$ for each $n < \omega$
- (ii) $Pr_n\big(a \cap \delta_0, \delta_0, a \cap \delta_1, \delta_1, \ldots, a \cap \delta_{n-1}, \delta_{n-1}\big)$ holds for each n.

Or just

(∗) for every closed unbounded $E \subseteq \lambda$ for some sequence $\langle \delta_\ell : \ell < \omega \rangle$ of member of E, the demands in (∗) above holds.

<u>Then</u> there are $\mathscr{T}' \subseteq \mathscr{T}$ and a_η (for $\eta \in \mathscr{T}'$) such that:

- (a) $(\mathscr{T}', \bar{\mathbb{I}})$ is a λ-tagged tree of increasing sequences
- (b) if $\nu = \eta\hat{}<\alpha> \in \mathscr{T}'$, then $a_\nu \cap \alpha = a_\eta \in \{s_i : i < \alpha\}$
- (c) if $\eta = \langle \alpha_0, \ldots, \alpha_{n-1} \rangle \in \mathscr{T}'$ then $Pr_n\big(a_{<>}, \alpha_0, a_{<\alpha_1>}, \alpha_1, \ldots, \alpha_{n-2}, a_{\langle \alpha_0, \ldots, \alpha_{n-2} \rangle}, \alpha_{n-1}\big)$.

Remark. Instead the $Pr_n(n < \omega)$ we can use any Borel subset of the tree of possible $\langle a \cap \delta_n, \delta_n : n < \omega \rangle$.]

Remark. Another such theorem is VII, 2.10 — it deals with $ds(\alpha) = \{\eta : \eta$ a strictly decreasing sequence of ordinals $< \alpha\}$ (so we get an isomorphic copy of $ds(\beta)$ for β not much smaller than α (in the $\beth_i$'s-sequence).

<u>1.9 Exercise</u>: Generalize the Erdös-Rado theorem $\beth_n(\lambda)^+ \to (\lambda^+)^{n+1}_2$ as done for the case $n = 2$ in 1.6.

§2 Order Properties Considered Again

2.1 Definition. 1) For a set $A \subseteq \mathfrak{C}$ let $\bar{x}_A = \langle x_a : a \in A \rangle, \bar{y}_A = \langle y_a : a \in A \rangle$, etc. and $\bar{a}_A = \langle a : a \in A \rangle$; for $N <_{\mathfrak{s}} \mathfrak{C}$, let $\varphi_N = \varphi_N(\bar{x}_N)$

be the formula saying: $a \mapsto x_a$ is a $\leq_{\mathfrak{s}}$-embedding of N into $\mathfrak{C}$; for $A \subseteq N$ let

$$\varphi_{N,A}(\bar{x}_A) = (\exists \bar{y}_N)[\varphi(\bar{y}_N) \ \ \& \ \bigwedge_{a \in A} y_a = x_a].$$

So $\varphi_{N,N}(\bar{x}_N)$ is equivalent to $\varphi_N(\bar{x}_N)$ and $\varphi_{N,A}$ means $\varphi_{N,A}(\bar{x}_A)$.
2) For $\chi \geq \chi_{\mathfrak{s}}$ let

$$\Lambda_{\mathfrak{s},\chi} = \Lambda^{\mathfrak{s}}_\chi = \{\varphi_{N,A}(\bar{x}_A) : A \subseteq N \in \mathfrak{K}_{\leq \chi}\}$$

$$\Lambda_{\mathfrak{s},<\chi} = \bigcup_{\mu < \chi} \Lambda_{\mathfrak{K},\mu} (\text{when } \chi > \chi_{\mathfrak{K}}).$$

3) If $\chi = \chi_{\mathfrak{s}}$ then we may omit it, and, of course, we may omit $\mathfrak{s}$ when clear from the context.

2.2 Remark. 1) Assume $\chi \geq$ LS$(\mathfrak{s})$. Remember that: $M^* \leq_{\Lambda_{\mathfrak{s},\chi}} N^*$ iff $M^* \leq_{\mathfrak{s}} N^*$ and for every $A \subseteq M^*$ of cardinality $\leq \chi$ satisfying $A \subseteq N <_{\mathfrak{s}} \mathfrak{C}$ and $\|N\| \leq \chi$ we have $M^* \models \varphi_{N,A}[\bar{a}_A] \Leftrightarrow N^* \models \varphi_{N,A}[\bar{a}_A]$.
2) In this section we deal with the expected "vice": an order property. So we consider $\chi \geq \chi_{\mathfrak{s}}$.

Now till 2.16 we assume

2.3 Hypothesis. The framework $\mathfrak{s}$ has the $(\Lambda_\chi, (\beth_2(\chi)^+)$-order property where $\chi \geq \chi_{\mathfrak{s}}$, i.e.:
there is a formula $\varphi^* = \varphi^*(\bar{x}, \bar{y}) \in \pm\Lambda_\chi$ which has the $(2^{2^\chi})^+$-order property, i.e. there are $\bar{a}_i, \bar{b}_i$ (for $i < (\beth_2(\chi))^+$) in a model M such that:

$$M \models \varphi^*[\bar{a}_i, \bar{b}_j] \text{ iff } i < j.$$

So $\varphi^*(\bar{x}, \bar{y})$ has the form $\pm(\exists \bar{z})\varphi_{N^*}(\bar{z}, \bar{x}, \bar{y})$ where $\ell g(\bar{z}) = \|N^*\| \leq \chi$ and $\ell g(\bar{x}) \leq \chi, \ell g(\bar{y}) \leq \chi$.

2.4 Claim. *In 2.3 without loss of generality* $\varphi^* \in \Lambda_\chi$.

Proof. Suppose $\varphi^*(\bar{x}, \bar{y}) = \neg(\exists \bar{z})\varphi_{N^*}(\bar{z}, \bar{x}, \bar{y})$.
Let $\psi^*(\bar{x}, \bar{y}) = (\exists \bar{z})\varphi_{N^*}(\bar{z}, \bar{y}, \bar{x})$ and interchange the $\bar{a}_i$'s and $\bar{b}_j$'s. (See V.A.1.18 and V.A.1.23). $\square_{2.4}$

2.5 Discussion. We would like to have a non-structure theorem assuming appropriate order property. For this we shall like to have indiscernible sequences, however the formulas in Λ_χ are not finitary, and so even getting an indiscernible sequence (and using stable amalgamation) we cannot vary the order at will. We do not know whether we can use linear orders which are not well orders, $\langle \bar{a}_t\hat{\ }\bar{b}_t : t \in I \rangle$ as if $t_{n+1} <_I t_n$ for $n < \omega$, maybe $\bar{a}_{t_0}\hat{\ }\bar{b}_{t_0}\hat{\ } \ldots \hat{\ }\bar{a}_{t_n}\hat{\ }\bar{b}_n\hat{\ } \ldots$ gives witness to $(\exists \bar{z})\varphi(\bar{z}, \bar{a}_s, \bar{b}_t)$ for some s, t not as in 2.3. Our solution is to remember we are in the context of universal classes and use §1 as follows.

2.6 Claim. *There are* $\bar{a}_i, \bar{b}_i (i < \chi^+), \bar{c}_{\{i,j\}}\ (i < j < \chi^+)$ *of length* $\ell g(\bar{x}), \ell g(\bar{y}), \ell g(\bar{z})$ *respectively and models* $M, M_u (u \in [\chi^+]^{\le 2})$ *and* $f_{u,w}$ *(for* $u, w \in [\chi^+]^{\le 2}$ *satisfying* $|u| = |w|$*) such that (so* φ^*, N^* *are from 2.3):*

(a) $\bar{a}_i, \bar{b}_i \subseteq M_{\{i\}}$
(b) $M = \langle \cup \{M_u : u \in [\chi]^{\le 2}\} \rangle^{\mathrm{gn}}_{\mathfrak{C}}$
(c) $M \models \varphi^*[\bar{a}_i, \bar{b}_j]$ *iff* $i < j$;
(d) *if* $i < j < \chi^+$ *then* $\bar{c}_{\{i,j\}} \subseteq M_{\{i,j\}}$ *and* $M_{\{i,j\}} \models \varphi_{N^*}[\bar{c}_{\{i,j\}}, \bar{a}_i, \bar{b}_j]$
(e) $u \subseteq w \in [\chi]^{\le 2}$ *implies* $M_u \subseteq M_w$
(f) $\langle M_u : u \in [\chi]^{\le 2} \rangle$ *is independent (see Definition 1.2),* $\|M_u\| \le \chi$*, and of course,* $M_u <_{\mathfrak{s}} \mathfrak{C}$
(g) $f_{u,w}$ *is an isomorphism from* M_w *onto* M_u*, when* $w, u \in [\chi^+]^{\le 2}$ *and* $|w| = |u|$ *and* $f_{w,u} = f^{-1}_{u,w}$
(h) $f_{u(1),u(2)} \circ f_{u(2),u(3)} = f_{u(1),u(3)}$ *when* $u(1), u(2), u(3) \in [\chi^+]^\iota$ *for* $\iota \le 2$
(i) $f_{\{i\},\{j\}}(\bar{a}_j\hat{\ }\bar{b}_j) = \bar{a}_i\hat{\ }\bar{b}_i$

(j) $f_{\{i_1,i_2\},\{j_1,j_2\}}(\bar{c}_{\{j_1,j_2\}}) = \bar{c}_{\{i_1,i_2\}}$ *when* $i_1 < i_2 < \chi, j_1 < j_2 < \chi$

(k) $f_{\{i_1,i_2\},\{j_1,j_2\}}$ *extend* $f_{\{i_1\},\{j_1\}}$ *and* $f_{\{i_2\},\{j_2\}}$ *when* $i_1 < i_2 < \chi^+, j_1 < j_2 < \chi^+$ *and* $f_{\{i\},\{j\}}$ *extend* $f_{\emptyset,\emptyset}$.

Proof. We shall apply the pair index theorem 1.6 with $\beth_2(\chi)^+, 2^\chi, \chi^+$ here standing for λ, μ, κ there. So the result follows.

By 1.6, 2.3 and 2.4 except clause (b), however easily letting $M' = \langle \cup\{M_u : u \in [\chi^+]^{\leq 2}\}\rangle^{\text{gn}}_{\mathfrak{C}}$ we have $M' \leq_{\mathfrak{s}} M$. But we can replace M by M' as clause (c) continues to hold even for M'. Why? For $i < j$ by clause (a) and φ_{N^*} being a "quantifier free" formula, and for $i \geq j$ as $M \models \neg\varphi^*[\bar{a}_i, \bar{b}_j] \Rightarrow M' \models \neg\varphi[\bar{a}_i, \bar{b}_j]$.

$\square_{2.6}$

2.7 Claim. *For any ordinal $\alpha(*)$ we can choose $M_u = M_u^{\alpha(*)}$, $f_{u(1),u(2)}$, $\bar{a}_i$, $\bar{b}_j$, $\bar{c}_{i_1,j_1}$ for u, $u(1)$, $u(2)$, i, j, i_1, j_1 satisfying $u \in [\alpha(*)]^{\leq 2}$, $u(1), u(2) \in [\alpha(*)]^{\leq 2}$, $|u(1)| = |u(2)|$, $i < \alpha(*)$, $j < \alpha(*)$, $i_1 < j_1$ so that clauses (a) and (d)-(k) of 2.6 holds.*

Remark. We may add a superscript $\alpha(*)$ but we can assume that for $i, j < \chi^+$ those objects are from 2.6.

Proof. Immediate using stable constructions (see V.C.4.2-V.C.4.6(2)). $\square_{2.7}$

2.8 Notation. For $\alpha(*)$ an ordinal and R a two place relation on $\alpha(*)$ which satisfies $[iRj \Rightarrow i < j]$, we let $M_R^{\alpha(*)} = \langle \cup\{M_u^{\alpha(*)} : u \in [\alpha(*)]^{\leq 1}$ or $u = \{i, j\}$ and $iRj\}\rangle^{\text{gn}}$ and similarly $M^w_{R\restriction w}$ for $w \subseteq \alpha(*)$.

2.9 Observation. For R as in 2.8 we have $M_R^{\alpha(*)} <_{\mathfrak{s}} \mathfrak{C}$.

Proof. By V.C.4.2 and V.C.4.6(2). $\square_{2.9}$

2.10 Main Claim.
$M_R^{\alpha(*)} \models \varphi^*[\bar{a}_i, \bar{b}_j]$ if iRj and $M_R^{\alpha(*)} \models \neg\varphi^*[\bar{a}_i, \bar{b}_j]$ if $j \le i$.

Remark. By 2.4 we are assuming $\varphi^* \in \Lambda_\chi$. We could do it similarly for $\varphi^* \in \neg\Lambda_\chi$, but there is no need.

Proof. If iRj, then clearly $\mathfrak{C} \models \varphi_{N^*}(\bar{c}_{\{i,j\}}, \bar{a}_i, \bar{b}_j)$ hence $M_R^{\alpha(*)} \models$
$\varphi_{N^*}[\bar{c}_{i,j}, \bar{a}_i, \bar{b}_j]$
hence $M_R^{\alpha(*)} \models \varphi^*[\bar{a}_i, \bar{b}_j]$. If $j \le i$, and $M_R^{\alpha(*)} \models \varphi^*[\bar{a}_i, \bar{b}_j]$ then for some
$\bar{c} \subseteq M_R^{\alpha(*)}$ we have $M_R^{\alpha(*)} \models \varphi_{N^*}[\bar{c}, \bar{a}_i, \bar{b}_j]$. So for some $w \subseteq \alpha(*)$ we have $|w| \le \chi, i, j \in w$ and $\bar{c} \subseteq M^w_{R \restriction w}$.

By an easy Lowenheim Skolem argument (or see 2.11 below) without loss of generality $i, j < \chi^+$ and $w \subseteq \chi^+$. So $M^{\chi^+} \models \varphi_{N^*}[\bar{c}, \bar{a}_i, \bar{b}_j]$, hence $M \models \varphi_{N^*}[\bar{c}, \bar{a}_i, \bar{b}_j]$ hence $M \models \varphi^*[a_i, b_j]$ contradicting (c) of 2.6. $\square_{2.10}$

2.11 Claim. *1) For any subsets w_1, w_2 of $\alpha(*)$, and two place relations R_1, R_2 on w_1, w_2 respectively, as in 2.8 so $[iR_\ell j \Rightarrow i < j]$, and isomorphism h from (w_1, R_1) onto (w_2, R_2), there is a isomorphism $H_h = H_{h,(w_1,R_1),(w_2,R_2)}$ from $M^{w_1}_{R_1}$ onto $M^{w_2}_{R_2}$, extending $f_{\{h(i)\},\{i\}}$ for $i \in w_1$ and extending $f_{\{h(i_1),h(i_2)\},\{i_1,i_2\}}$ for $\langle i_1, i_2\rangle \in R_1$.*
2) If $w^*_1 \subseteq w_1, w^*_2 = h''(w^*_1), R^*_\ell \subseteq R_\ell, h''(R^*_1) = R^*_2$, then
$H_{h\restriction w^*_1,(w^*_1,R^*_1),(w^*_2,R^*_2)} \subseteq H_{h,(w_1,R_1),(w_2,R_2)}$.

Proof. Immediate. $\square_{2.11}$

2.12 Claim. *If R is a partial order of $\alpha(*)$ included in $<$ (i.e. $[iRj \Rightarrow i < j]$ and $[i_1Ri_2 \ \& \ i_2Ri_3 \Rightarrow i_1Ri_3]$) then for $i, j < \alpha(*)$ we have*

$$M_R^{\alpha(*)} \models \varphi^*[\bar{a}_i, \bar{a}_j] \text{ iff } iRj.$$

Proof. By 2.10 it suffices to deal with the case $i < j$ & $\neg(iRj)$, i.e. to prove for such i, j that $M_R^{\alpha(*)} \models \neg\varphi^*[\bar{a}_i, \bar{b}_j]$. By 2.11 it suffices to find a $\beta(*), R_2$ (so $\beta(*)$ an ordinal, R_2 a two-place relation on $\beta(*)$ as in 2.8) and an isomorphism h from $(\alpha(*), R)$ onto $(\beta(*), R_2)$ such that $h(j) < h(i)$.

Let $I_1 \subseteq \alpha(*)$ be such that: $i \in I_1, j \in I_1, I_1$ is an R-antichain (i.e. a set of pairwise R-incomparable elements) and I_1 maximal under those conditions. Let

$$I_0 = \{\alpha < \alpha(*) : (\exists \gamma \in I_1)\alpha R \gamma\},$$

$$I_2 = \{\alpha < \alpha(*) : (\exists \gamma \in I_1)\gamma R \alpha\}.$$

Clearly

$(*)_1$ I_0, I_1, I_2 form a partition of $\alpha(*)$.

Let $\beta(0) = \text{otp}(I_0), \beta(1) = \text{otp}(I_1 \backslash \{i, j\}), \beta(2) = \text{otp}(I_2)$. Now let $\beta(*) = \beta(0) + 2 + \beta(1) + \beta(2)$, and we define the function h (with domain $\alpha(*)$) as follows:

⊛ (*a*) for $\alpha \in I_0, h(\alpha) = \text{otp}(I_0 \cap \alpha)$
(*b*) for $\alpha = j, h(\alpha) = \beta(0)$
(*c*) for $\alpha = i, h(\alpha) = \beta(0) + 1$
(*d*) for $\alpha \in I_1 \backslash \{i, j\}, h(\alpha) = \beta(0) + 2 + \text{otp}(\alpha \cap I_1 \backslash \{i, j\})$
(*e*) for $\alpha \in I_2, h(\alpha) = \beta(0) + 2 + \beta(1) + \text{otp}(\alpha \cap I_2)$.

Clearly h is one to one from $\alpha(*)$ onto $\beta(*)$ and $h(j) < h(i)$ and $[\alpha_1 R \alpha_2 \Rightarrow h(\alpha_1) < h(\alpha_2)]$.

So we can find R_2 such that h is an isomorphism from $(\alpha(*), R)$ onto $(\beta(*), R_2)$ and $[i_1 R j_1 \Rightarrow i_1 < j_1]$, and as said above this suffices. $\square_{2.12}$

2.13 Conclusion [Recall we are assuming Hypothesis 2.3].
1) For $\lambda = \lambda^\chi + \chi^{++}$ we have $\dot{I}(\lambda, \mathfrak{K}_\mathfrak{s}) = 2^\lambda$.
2) Moreover just $\lambda \geq \chi^{++}$ suffice.

Proof. See [Sh:e, Ch.IV,3.1] or [Sh 300, III,§3], (in fact, using the terms there, there is a representation, so life is easier). The "moreover" is by applying [Sh:e, III,§3] using 2.14 below. Note that we can use 2.15 below.

$\square_{2.13}$

In fact we can quote [Sh:e, III,§3] or [Sh 300, IV,§3] as

2.14 Claim. *For any linear order I and two-place relation R on I of cardinality $\lambda \geq \chi$ we can find M and $\langle (\bar{a}_t, \bar{b}_t) : t \in I \rangle$ such that*

(a) $M \in K^{\mathfrak{s}}_\lambda$

(b) $M \models \varphi^*[\bar{a}_s, \bar{b}_t]$ *iff* sRt *(for* $s, t \in I$*, so* $\ell g(\bar{a}_s) = \ell g(\bar{x}), \ell g(\bar{b}_s) = \ell g(\bar{y})$*)*

(c) *if* $\bar{c} \in {}^{\ell g(\bar{x}}(M)$ *then for some* $J \subseteq I$ *of cardinality* $\leq \chi$ *we have: if* $s, t \in I \backslash J$ *induce the same cut of* J *then* $M \models$ "$\varphi^*[\bar{c}, \bar{b}_s] \equiv \varphi^*[\bar{c}, \bar{b}_t]$"

(d) *if* $\bar{c} \in {}^{\ell g(\bar{y})}M$ *then for some* $J \subseteq I$ *of cardinality* $\leq \chi$ *we have: if* $s, t \in I \backslash J$ *induce the same cut of* J *then* $M \models$ "$\varphi[\bar{a}_s, \bar{c}] \equiv \varphi[\bar{a}_t, \bar{d}]$"

(e) $M = \langle \cup\{M_{\{s,t\}} : sRt\} \cup \bigcup\{M_{\{s\}} : s \in I\} \cup M_\emptyset \rangle^{\rm gn}_{\mathfrak{C}}$ *as in 2.6.*

Proof. Similar.

Let $\{t_\zeta : \zeta < \zeta(*)\}$ list I with no repetitions and let $\alpha(*) = \zeta(*) + \zeta(*)$ and let $R = \{(\zeta_1, \zeta(*) + \zeta_2) : t_{\zeta_1} <_I t_{\zeta_2}\}$, so $\alpha(*), R$ are as in 2.8, so $M := M^{\alpha(*)}_R$ is well defined. Now without loss of generality $\alpha(*) \cap I = \emptyset$ and let $a_{t_\zeta} = \bar{a}_\zeta, \bar{b}_{t_\zeta} = \bar{b}_{\zeta(*)+\zeta}$; now check. $\square_{2.14}$

* * *

So now we stop assuming Hypothesis 2.3.

2.15 Claim. *Suppose that* $\lambda \geq \chi \geq \chi_{\mathfrak{s}}$ *and we have* $M, M_\emptyset, M^1_i, M^2_j, M^3_{i,j}, h_{i_1,i_2}, h^2_{i_1,i_2}, h_{i_1,j_1,i_2,j_2}$ *for* $i, j, i_1, j_1, i_2, j_2 < \chi$ *such that:*

- (a) $M_\emptyset \leq_{\mathfrak{s}} M^1_i \leq_{\mathfrak{s}} M^3_{i,j}$ *for* $i, j < \chi$, *of course*
- (b) $M_\emptyset \leq_{\mathfrak{s}} M^2_j \leq_{\mathfrak{s}} M^3_{i,j}$
- (c) $h^\ell_{i_1,j_1}$ *is an isomorphism from* $M^\ell_{i_1}$ *onto* $M^\ell_{j_1}$ *over* $M_\emptyset$ *for* $\ell = 1, 2$
- (d) h_{i_1,j_1,i_2,j_2} *is an isomorphism from* $M^3_{i_1,j_1}$ *onto* $M^3_{i_2,j_2}$ *extending* $h^1_{i_1,i_2} \cup h^2_{j_1,j_2}$
- (e) $\langle N_u : u \in W\rangle$ *where* $W = \{u$: *if* $|u| = 2$ *then* $u = \{\langle i, 0\rangle, \langle j, 1\rangle\}$ *for some* $i, j < \lambda$ *and* $u \in [\lambda \times 2]^{\leq 2}\}$ *is an independent sequence of models, each of cardinality* $\leq \chi$ *where* $N_\emptyset = M_\emptyset$, $N_{\{<i,\ell>\}} = M^\ell_i$, *and*

$$N_{\{<i,1>,<j,2>\}} = M^3_{i,j}$$

- (f) $M = \langle M_\emptyset \cup \bigcup_{\ell,i} M^\ell_i \cup \bigcup\{M^3_{i,j} : i, j < \chi \text{ and } <i, j> \neq <0, 0>\}\rangle^{\text{gn}}_{\mathfrak{C}}$
- (g) $M^3_{0,0}$ *has no* $\leq_{\mathfrak{s}}$*-embedding into* M *over* $M^1_0 \cup M^2_0$
- (h) $\chi = \chi_{\mathfrak{s}} + \sup_{i,j} \|M^3_{i,j}\|$.

<u>Then</u>

- (α) $\mathfrak{s}$ *has the* (Λ_χ, λ)*-order property.*

Remark. 1) The (Λ_χ, λ)-order property is defined in V.A.1.1(1).
2) Used in 3.15.

Proof. We use freely stable constructions (see V.C§4).

We let $N'_\emptyset = M_\emptyset = N_\emptyset$. By induction on $i < \lambda$ we can choose $g^1_{\{i\}}, g^2_{\{\lambda+i\}}, N'_{\{i\}}, N'_{\{\lambda+i\}}$ such that

$\circledast_1$ (a) $N'_{\{i\}} <_{\mathfrak{s}} \mathfrak{C}$ for $i < \lambda + \lambda$

(b) $g^\ell_{\lambda(\ell-1)+i}$ is an isomorphism from M^ℓ_0 onto $N'_{\{\lambda(\ell-1)+i\}}$ over $M_\emptyset$ for $i < \lambda, \ell = 1, 2$

(c) $\langle N'_{\{i\}} : i < \lambda + \lambda\}$ is independent over $M_\emptyset$.

Next for $i, j < \lambda$ we choose $g_{\{i,\lambda+j\}}, N'_{\{i,\lambda+j\}}$ such that

$\circledast_2$ (a) $N'_{\{i,\lambda+j\}} <_{\mathfrak{s}} \mathfrak{C}$

(b) $g_{\{i,\lambda+j\}}$ is an isomorphism from $M^3_{i,j}$ onto $N'_{\{i,\lambda+j\}}$ extending $g_{\{i\}}, g_{\{\lambda+j\}}$

(c) $\langle N'_u : u = \emptyset$ or $u = \{i\}, i < \lambda+\lambda$ or $u = \{i, \lambda+j\}, i, j < \lambda\rangle^{\text{gn}}_{\mathfrak{C}}$ is independent (see Definition 1.2).

Now for every $R \subseteq \lambda \times \lambda$ let $N'_R = \langle \cup\{N'_{\{i\}} : i < \lambda+\lambda\} \cup \{N'_{\{i,\lambda+j\}} : (i, j) \in R\}\rangle^{\text{gn}}_{\mathfrak{C}}$.

To finish we need

$\odot$ for $i, j < \lambda$, there is a $\leq_{\mathfrak{s}}$-embedding f of $N'_{\{i,\lambda+j\}}$ into N'_R extending $g_{\{i\}} \cup g_{\{\lambda+i\}}$ <u>iff</u> $(i, j) \in R$.

Why does $\odot$ hold? The "if" direction is trivial; $g_{\{i,\lambda+j\}}$ is such an $\leq_{\mathfrak{s}}$-embedding.

For the other direction, assume toward contradiction that f is such an embedding but $(i, j) \notin R$. We can find sets $u, v \subseteq \lambda$ of cardinality $\leq \chi$ such that $\text{Rang}(f) \subseteq M_{u,v,R}$ where $M_{u,v,R}$ is defined as

$$M_{u,v,R} := \langle \bigcup\{N'_{\{i_1\}} : i_1 \in u\} \cup \bigcup\{N'_{\{\lambda+i_2\}} : i_2 \in v\} \\ \cup \bigcup\{N'_{\{i_1,\lambda+i_2\}} : i_1 \in u, i_2 \in v \text{ and } (i_1, i_2) \in R\}\rangle^{\text{gn}}_{\mathfrak{C}}.$$

Now we can find a one-to-one function π_1 from u into χ such that $\pi_1(i) = 0$ because $|u| \leq \chi$. Also we can find a one-to-one function π_2 from v into χ such that $\pi_2(j) = 0$. Now there is a mapping f^* such that, recalling clauses (c),(d) of the assumption

$\circledast_3$ (a) $\text{Dom}(f) = \bigcup\{N'_{\{i_1\}} : i_1 \in u\} \cup \bigcup\{N'_{\{\lambda+i_2\}} : i_2 \in v\} \cup \bigcup\{N'_{\{i_1,\lambda+i_2\}}, i_1 \in u, i_2 \in v \text{ and } (i_1, i_2) \in R\}$

(b) $f^* \restriction N'_{\{i_1\}} = h^1_{0,i_1} \circ g^{-1}_{\{i_1\}}$ for $i_1 \in v$

(c) $f^* \restriction N'_{\{\lambda+i_2\}} = h^2_{0,i_2} \circ g^{-1}_{\{\lambda+i_2\}}$ for $i_2 \in v$

(d) $f^* \restriction N'_{\{i_1,\lambda+i_2\}} = h_{0,i_1,0,i_2} \circ g^{-2}_{\{i_1,\lambda+i_2\}}$.

Clearly $f^* \restriction N'_{\{i_1\}}$ is an isomorphism from $N'_{\{i_1\}}$ onto $M^1_{i_1}$ and $f^* \restriction N'_{\{\lambda+i_2\}}$ is an isomorphism from $N'_{\{\lambda+i_2\}}$ onto $M^2_{i_2}$ and $f^* \restriction N'_{\{i_1,\lambda+i_2\}}$ is an isomorphism from $N'_{\{i_1,\lambda+i_2\}}$ onto $M^3_{i_1,i_2}$. Also $\langle f_*(N'_w) : w = \emptyset$ or $w = \{i_1\}, i_1 \in u$ or $w = \{\lambda + i_2\}, i_2 \in v$ or $w = \{i_1, \lambda + i_2\}, i_1 \in u_j, i_2 \in v$ and $(i_1, i_2) \in R\}\rangle$ is independent. As $(i_1, i_2) \in R \Rightarrow (i_1, i_2) \neq (i, j)$, necessarily we can extend f^* to a $\le_{\mathfrak{s}}$-embedding f^+ of $M_{u,v,R}$ into M. But by clause (g) of the assumption we get a contradiction.

$\square_{2.15}$

Now, recalling again that we are not assuming Hypothesis 2.3

2.16 Claim. *1) If $M_\emptyset \le_{\mathfrak{s}} M^1_0 \le_{\mathfrak{s}} M^3_{0,0}, M_\emptyset \le_{\mathfrak{s}} M^2_0 \le_{\mathfrak{s}} M^3_{0,0} <_{\mathfrak{s}} \mathfrak{C}, \|M^3_{0,0}\| \le \chi$ and $\{M^1_0, M^2_0\}$ is independent over $M_\emptyset$ then we can choose M^ℓ_i (for $\ell = 1,2$ and $i < \lambda$), $M^3_{i,j} (i,j < \chi), h^\ell_{i,j}, h_{i_1,j_1,i,j}$ satisfying (a) - (f) + (h) of 2.15.*
2) For $\chi \ge \chi_{\mathfrak{s}}$, the framework $\mathfrak{s}$ has the (Λ_χ, λ)-order property for some $\lambda \ge (\beth_2(\chi))^+$ iff $\mathfrak{s}$ satisfies the conclusion of 2.6 iff $\mathfrak{s}$ satisfies the assumption of 2.15 iff $\mathfrak{s}$ has the (Λ_χ, λ)-order property for every $\lambda \ge \chi$.

Proof. 1) Easy. By V.C§4.
2) Easy by now. $\square_{2.16}$

2.17 Exercise: In Hypothesis 2.3 we can replace "the $(\Lambda_\chi, (2^{2^\chi})^+)$-order property" by "the $(\Lambda^{\text{eb}}_\chi, (2^{2^\chi})^+)$-order property".
[Hint: See Definition V.A.1.18(1)(g). The only difference is that when we apply 1.6, we use g. That is, if e.g. $\psi^*(\bar{x}, \bar{y}_1\hat{\ }\bar{y}_2) = [\varphi^*(x, \bar{y}_1) \equiv \varphi^*(\bar{x}, \bar{y}_2)]$ and $M \models \varphi[\bar{a}_\alpha, \bar{b}_\beta, \bar{c}_\beta]$ iff $(\alpha < \beta) \equiv \mathbf{t}$ we choose a function g from $[(2^{2^\chi})^+]^2$ to ω such that $g\{\alpha, \beta\}$ code the truth values $M \models \varphi^*[\bar{a}_\alpha, \bar{b}_\beta], M \models \varphi^*[\bar{a}_\alpha, \bar{c}_\beta], M \models \varphi^*[\bar{a}_\beta, \bar{b}_\alpha], M \models \varphi^*[\bar{a}_\beta, \bar{c}_\alpha]$.]

2.18 Exercise: In 2.16(2) we can add: "iff $\mathfrak{s}$ is not (Λ_χ, λ)-stable for every $\lambda \ge \chi$ iff $\mathfrak{s}$ is not (Λ_χ, λ)-stable for some $\lambda = \lambda^\chi + \beth_4(\chi)$.

[Hint: See 3.2 and 2.14.]

§3 Strengthening the order $\leq_{\mathfrak{s}}$

We assume for this section

3.1 Hypothesis. The framework $\mathfrak{s}$ is as in Chapter V.E, $\theta^* > \chi_{\mathfrak{s}}$ and $\mathfrak{s}$ does not have the $(\Lambda_\chi, (2^{2^\chi})^+)$-order property whenever $\chi_{\mathfrak{s}} \leq \chi < \theta^*$. If not said otherwise, χ will denote such a cardinal (in this section).

It is O.K. to use $\theta^* = \beth_\omega(\chi_{\mathfrak{s}})$, this saves us complications compared to trying to show that one $\chi \geq \chi_{\mathfrak{s}}$ suffice.

We shall use mainly $x = i$ and to help it also $x = j$; note that 3.12 are not essential.

3.2 Conclusion. $\mathfrak{s}$ is (Λ_χ, λ)-stable when $\chi \in [\chi_{\mathfrak{s}}, \theta^*), \lambda = \lambda^\chi + \beth_4(\chi)$; i.e. for every $M \in \mathfrak{K}_{\mathfrak{s}}, A \subseteq M, |A| \leq \chi$ we have $|\{\text{tp}_{\Lambda_\chi}(\bar{c}, A, M) : \bar{c} \in {}^\chi|M|\}| \leq \lambda$.

Proof. By 2.16(2) and V.A.1.19. $\square_{3.2}$

The main notion in this section is $\leq^i_{\lambda,\chi}$ where

3.3 Definition. 1) For $x = i, \text{nc}$ and $\lambda \geq \chi \geq \text{LS}(\mathfrak{s})$ we say $M \leq^x_{\lambda,\chi} N$ <u>when</u>: ($M \leq_{\mathfrak{s}} N <_{\mathfrak{s}} \mathfrak{C}$ and) for every $\bar{c} \in {}^{\chi \geq}(N)$, and[2] $p \subseteq \text{tp}_{\Lambda^x_\chi}(\bar{c}, M, N)$ of cardinality $\leq \lambda$ there is $\bar{c}' \in {}^{\chi\geq}|M|$ realizing p, where recalling V.A.1.18(1) that $\Lambda^i_\chi = \Lambda_\chi, \Lambda^{\text{nc}}_\chi = \pm\Lambda_\chi = \{\varphi, \neg\varphi : \varphi \in \Lambda_\chi\}$. If $\lambda = \lambda^\chi$, this is equivalent to: for every $A \subseteq |M|, |A| \leq \lambda$ there is $\bar{c}' \in {}^{\chi\geq}|M|$ realizing $\text{tp}_{\Lambda^x_\chi}(\bar{c}, A, N)$; so for $x = \text{nc}$ this is $\text{tp}_{\pm\Lambda_\chi}(\bar{c}, A, N)$, for $x = i$ this is $\text{tp}_{\Lambda_\chi}(\bar{c}, A, N)$; recall V.A.0.8(2),(4)(c), so possibly, for some $\bar{c}, \bar{c}'$ we have $\text{tp}_{\Lambda_\chi}(\bar{c}, A, N)$ is a proper subset of $\text{tp}_{\Lambda_\chi}(\bar{c}', A, N)$.
2) Here $x = j$ means the same as $x = i$ and $\Lambda^j_\chi = \Lambda^i_\chi$. For $x \in \{i, j, \text{nc}\}, M \leq^x_{<\lambda,<\chi} N$ means $M \leq^x_{\lambda_1,\chi_1} N$ whenever $\lambda_1 < \lambda, \chi_1 < \chi$ when $\lambda, \chi > \chi_{\mathfrak{s}}$.

[2]so for $x = i$ the type p consists only of instances of formulas from Λ_χ (not their negations)

3.4 Remark. Recall, as Λ is closed under permuting the variables, (and we shall use this freely)

- (a) the (Λ_χ, λ)-order property is equivalent to the $(\Lambda^{\rm nc}_\chi, \lambda)$-order property, see V.A.1.23(3)
- (b) the $(\Lambda^{\rm eb}_\chi, (2^\lambda)^+)$-order property implies the $(\Lambda_\chi, \lambda^+)$-order property, see V.A.1.15(2) recalling the Erdős-Rado theorem: $(2^\lambda)^+ \rightarrow (\lambda^+)^2_2$.

3.5 Observation. Let $\chi \geq$ LS$(\mathfrak{s})$.
1) $M_1 \leq^i_{\chi,\chi} M_2$ <u>iff</u> $M_1 \leq_{\mathfrak{s}} M_2$ and for every $N_1 \leq_{\mathfrak{s}} N_2$ from $K^{\mathfrak{s}}_\chi$ such that $N_\ell \leq_{\mathfrak{s}} M_\ell$ for $\ell = 1,2$ (and can add $\mathrm{NF}_{\mathfrak{s}}(N_1, M_1, N_2, M_2)$) <u>there</u> is a $\leq_{\mathfrak{s}}$-embedding of N_2 into M_1 over N_1.
2) $M_1 \leq^{\rm nc}_{\chi,\chi} M_2$ <u>iff</u> $M_1 \leq_{\mathfrak{s}} M_2$ and

- $\circledast$ if $N_1 \leq_{\mathfrak{s}} N_2$ are from $K^{\mathfrak{s}}_\chi$ such that $N_\ell \leq_{\mathfrak{s}} M_\ell$ for $\ell = 1,2$ and $A_{2,i} \subseteq N_2$ for $i < \chi$ and $N_{2,i} \in K^{\mathfrak{s}}_\chi$ satisfying $A_{2,i} \subseteq N_{2,i} <_{\mathfrak{s}} \mathfrak{C}$ such that there is no $\leq_{\mathfrak{s}}$-embedding of $N_{2,i}$ into M_2 over $A_{2,i}$ <u>then</u> there is a $\leq_{\mathfrak{s}}$-embedding f of N_2 into M_0 over N_1 such that for $i < \chi$ there is no $\leq_{\mathfrak{s}}$-embedding g of $N_{2,i}$ into M_1 extending $f \restriction A_{2,i}$.

3) $\mathfrak{s}$ is $(\Lambda^x_\chi, \lambda)$-stable iff $\mathfrak{s}$ is $(\Lambda^y_\chi, \lambda)$-stable when $x, y \in \{i, \text{nc,eb}\}$.
4) The statements "$\langle \bar{a}_t : t \in I \rangle$ is (Δ^x_χ, λ)-convergent" for $x = i, j, \text{nc}$ are equivalent.

Proof. Trivial. $\square_{3.5}$

3.6 Claim. *: If $M \leq_{\mathfrak{s}} N, \lambda = \mathrm{cf}(\lambda) > \beth_4(\chi), \chi \in [\chi_{\mathfrak{s}}, \theta^*)$ and $(\forall \mu < \lambda)\mu^\chi < \lambda$ and $x = i\, [x = \mathrm{nc}]$ <u>then</u> the following are equivalent:*

- (a) *$M \leq^{x,*}_{\lambda,\chi} N$ which means:*
  for every $\bar{c} \in {}^{\chi\geq}N$ there is (Λ_χ, χ^+)-convergent set
  $\mathbf{I} = \{\bar{c}_i : i < \lambda^+\} \subseteq {}^{\chi\geq}|M|$ with $\mathrm{Av}_{\Lambda^x}(\mathbf{I}, M) \supseteq \mathrm{tp}_{\Lambda^x_\chi}(\bar{c}, M, N)$
- (b) *$M \leq^x_{\lambda,\chi} N$.*

Remark. 1) The exact choice "$\lambda > \beth_4(\chi)$" is immaterial here (as we shall use $\theta^* = \beth_\omega(\chi_{\mathfrak{s}})$). The $\beth_4(\chi)$ rather than $\beth_2(\chi)$ occurs as we use "the $(\Lambda_\chi, (2^{2^\chi})^+)$-non-order property". In 3.2 we use $(\Lambda_\chi, (2^{2^\chi})^+)$-non-order rather than $(\Lambda_\chi, (2^\chi)^+)$-non-order so that §1 applies. A use of $\Delta^{\rm eb}$ rather than Δ^i, see Definition V.A.1.18, cause no problem by Exercise 2.17. May see more [Sh:E54].
2) In the end of clause (a) of 3.6, if $x =$ nc the "$\supseteq$" is equality.

Proof. Use V.A.4.4 and see V.A.4.1, V.A.2.8. $\square_{3.6}$

3.7 Observation. For $x = i, \mathrm{nc}$.
1) If $\lambda_1 \le \lambda, \chi_1 \le \chi$ and $\lambda \ge \chi \ge \mathrm{LS}(\mathfrak{s}), \lambda_1 \ge \chi_1 \ge \mathrm{LS}(\mathfrak{s})$ <u>then</u> $M \le^x_{\lambda,\chi} N$ implies $M \le^x_{\lambda_1,\chi_1} N$.
2) $\le^x_{\lambda,\chi}$ is transitive.
3) Suppose $\chi \in [\chi_{\mathfrak{s}}, \theta^*)$. If $A \subseteq N \in \mathfrak{K}_{\mathfrak{s}}$ and $\lambda = \lambda^\chi + \beth_4(\chi)$ <u>then</u> there is M such that $A \subseteq M \le^x_{\lambda,\chi} N$ and $\|M\| \le |A|^\chi + \lambda$. (Use V.A.1.20, note that even without the assumption on λ, demanding only $\|M\| \le |A|^{\lambda+\chi}$ is easy).
4) If $M_\ell \le^x_{\lambda,\chi} N$ for $\ell = 1, 2$ and $M_0 \subseteq M_1$ <u>then</u> $M_0 \le^x_{\lambda,\chi} M_1$.
5) In the definition of $M \le^i_{\chi,\chi} N$ without loss of generality
$p = \{\varphi(\bar{x}, \bar{b})\}, \varphi = (\exists \bar{z})\varphi_N(\bar{z}, \bar{x}, \bar{y})$ and $\ell g(\bar{z}{}^\frown\bar{x}{}^\frown\bar{y}) < \chi^+$.
6) If $M_0 \le_{\mathfrak{s}} M_1 \le_{\mathfrak{s}} M_2$ and $M_0 \le^i_{\lambda,\chi} M_2$ then $M_0 \le^i_{\lambda,\chi} M_1$.

Proof. Easy. $\square_{3.7}$

3.8 Claim. *1) Suppose $\lambda \ge \mu > \chi \ge \chi_{\mathfrak{s}}, M \le^{\rm nc}_{\lambda,\chi} N, \bar{a}_i \in M, \ell g(\bar{a}_i) \le \chi$ for $i < \lambda, \lambda > \chi^+$. Then $\mathbf{I} = \{\bar{a}_i : i < \lambda\}$ is (Λ_χ, μ)-convergent inside M iff $\mathbf{I}$ is (Λ_χ, μ)-convergent inside N.*
2) Assume $\mathbf{I}^\ell = \{\bar{a}^\ell_i : i < \lambda\}, \lambda, \chi, M, N$ are as in 3.8(1) for $\ell = 1, 2$ and $\ell g(\bar{a}^1_i) = \ell g(\bar{a}^2_i)$ for $i < \lambda$. Then $\mathrm{Av}_{\Lambda_\chi}(\mathbf{I}^1, M) = \mathrm{Av}_{\Lambda_\chi}(\mathbf{I}^2, M)$ <u>iff</u> $\mathrm{Av}_{\pm\Lambda_\chi}(\mathbf{I}^1, N) = \mathrm{Av}_{\pm\Lambda_\chi}(\mathbf{I}^2, N)$.

Proof. Check. $\square_{3.8}$

3.9 Definition. 1) For $x = i, \mathrm{nc}, j$ and $\lambda \geq \chi \geq \chi_{\mathfrak{s}}$ let $\mathfrak{s}^x_{\lambda,\chi}$ be defined as $(K, \leq^x_{\lambda,\chi}, \mathrm{NF}^x_{\lambda,\chi}, \langle - \rangle^{\mathrm{gn}})$ where:

- (a) K is our $K_{\mathfrak{s}}$
- (b) $\langle - \rangle^{\mathrm{gn}}$ is as in $\mathfrak{s}$
- (c) $\leq^x_{\lambda,\chi}$ is from Definition 3.3.
- $(d)_1$ when $x = j$ or $x =$ nc we let $\mathrm{NF}^x_{\lambda,\chi}$ be the class of quadruple (M_0, M_1, M_2, M_3) satisfying:
  - (α) $M_\ell \in K$ for $\ell < 4$
  - (β) $M_0 \leq^x_{\lambda,\chi} M_\ell \leq^x_{\lambda,\chi} M_3$ for $\ell = 1, 2$
  - (γ) $\mathrm{NF}_{\mathfrak{s}}(M_0, M_1, M_2, M_3)$
  - (δ) if $\bar{c} \in {}^{\chi \geq}(M_2)$ and $p \subseteq \mathrm{tp}_{\Lambda^x_\chi}(\bar{c}, M_1, M_3)$ is of cardinality $\leq \chi$ <u>then</u> p is realized by some $\bar{c}' \in {}^{\ell g(\bar{c})}(M_0)$
- $(d)_2$ $\mathrm{NF}^i_{\lambda,\chi} = \{(M_0, M_1, M_2, M_3) \in \mathrm{NF}^j_{\lambda,\chi} : (M_0, M_2, M_1, M_3) \in \mathrm{NF}^j_{\lambda,\chi}\}$.

2) $\mathfrak{s}^x_{<\lambda,<\chi}$ is defined similarly when $\lambda \geq \chi > \chi_{\mathfrak{s}}$.

3.10 Observation. Let $x \in \{i, j, \mathrm{nc}\}$ and $\lambda \geq \chi \in [\chi_{\mathfrak{s}}, \theta^*)$.
1) $\mathrm{NF}^x_{\lambda,\chi}(M_0, M_1, M_2, M_3)$ <u>if</u> $M_0 \leq^x_{\lambda,\chi} M_\ell \leq^x_{\lambda,\chi} M_3$ for $\ell = 1, 2$ and $M_1 = M_0$ or $M_2 = M_0$.
2) If $\mathrm{NF}^x_{\lambda,\chi}(M_0, M_1, M_2, M_3)$ and $M_0 \leq^x_{\lambda,\chi} M'_\ell \leq^x_{\lambda,\chi} M_\ell$ for $\ell = 1, 2$ <u>then</u> $\mathrm{NF}^x_{\lambda,\chi}(M_0, M'_1, M'_2, M_3)$.
3) If $M_1 \cup M_2 \subseteq M'_3 \leq^x_{\lambda,\chi} M''_3$ <u>then</u> $\mathrm{NF}^x_{\lambda,\chi}(M_0, M_1, M_2, M'_3)$ iff $\mathrm{NF}^x_{\lambda,\chi}(M_0, M_1, M_2, M''_3)$.
4) Assume $x =$ nc, $\gamma < \chi^+$ and $\lambda \geq \beth_4(\chi)$. If $\mathrm{NF}^x_{\lambda,\chi}(M_0, M_1, M_2, M_3)$ and $\bar{c} \in {}^\gamma|M_2|$ and $\mathbf{I} = \{\bar{c}_\alpha : \alpha < \lambda^+\} \subseteq {}^\gamma(M_0)$ is $(\Lambda^x_\chi, \lambda)$-convergent with $\mathrm{Av}_{\Lambda^x_\chi}(\mathbf{I}, M_0) \supseteq \mathrm{tp}_{\Lambda^x_\chi}(\bar{c}, M_0, M_2)$ <u>then</u> $\mathrm{Av}(\mathbf{I}, M_1) \supseteq \mathrm{tp}_{\Lambda^x_\chi}(\bar{c}, M_1, M_3)$.

Proof. Easy, e.g. for part (3) use 3.7(4) and 3.7(2). For part (4), toward contradiction assume $\bar{b} \in {}^{\chi^+ >}|M|$ and $M_3 \models \varphi[\bar{b}, \bar{c}]$ and $\varphi \in \Lambda^x_\chi$ but for some $u \in [\lambda^+]^\lambda$ we have $\alpha \in u \Rightarrow M \models \neg\varphi[\bar{a}, \bar{c}_\alpha]$ then by the definition of $\mathrm{NF}^x_{\lambda,\chi}$ we can find such $\bar b' \in {}^{\ell g(\bar b)}(M)$ and get contradiction to an assumption.

$\square_{3.10}$

3.11 Observation. Let $x \in \{i, j, \mathrm{nc}\}$. Assume $\lambda \geq \chi \geq \chi_{\mathfrak{s}}$ and $\mathrm{NF}_{\mathfrak{s}}(M_0, M_1, M_2, M_3)$ and $M_0 \leq^x_{\lambda,\chi} M_\ell \leq^x_{\lambda,\chi} M_3$ for $\ell = 1, 2$.
1) If $\neg\mathrm{NF}^j_{\chi,\chi}(M_0, M_1, M_2, M_3)$ <u>then</u> there is $\bar N$ such that

$\circledast_{\bar N,\bar M}$ (a) $\bar N$ is the quadruple $\bar N = \langle N_\ell : \ell \leq 3\rangle$
(b) $\mathrm{NF}_{\mathfrak{s}}(N_0, N_1, N_2, N_3)$
(c) $N_\ell \in K^{\mathfrak{s}}_\chi$ and $N_\ell \leq_{\mathfrak{s}} M_\ell$ for $\ell \leq 3$
(d) $\mathrm{NF}_{\mathfrak{s}}(N_0, M_0, N_\ell, M_\ell)$ for $\ell = 1, 2$
(e) $N_3 \cap \langle M_1 \cup M_2\rangle^{\mathrm{gn}}_{M_3} = \langle N_1 \cup N_2\rangle^{\mathrm{gn}}_{M_3}$
$\odot$ there is no $\leq_{\mathfrak{s}}$-embedding f of N_3 into M_1 over N_1 such that $f(N_2) \subseteq M_0$.

2) If $A \subseteq M_1 \cup M_2, |A| \leq \chi$ then there is $\bar N$ satisfying $\circledast_{\bar N,\bar M}$ and $A \subseteq N_1 \cup N_2$.
3) If $\mathrm{NF}^j_{\chi,\chi}(M_0, M_1, M_2, M_3)$ and $\circledast_{\bar N,\bar M}$ holds <u>then</u> there is a $\leq_{\mathfrak{s}}$-embedding f of N_3 into M_1 such that $f(N_2) \subseteq M_0$.

Proof. Easy. $\square_{3.11}$

3.12 Claim. *[Weak Symmetry] Suppose* $x \in \{i, \mathrm{nc}\}$.
<u>*Then*</u> $\mathrm{NF}^x_{\lambda,\chi}(M_0, M_2, M_1, M_3)$ <u>*when*</u>*:*

(a) $\mathrm{NF}^x_{\theta,\chi}(M_0, M_1, M_2, M_3)$
(b) $\lambda \geq \beth_2(\chi) \wedge \theta = \chi$ *if* $x = nc$
(c) $\lambda \geq \chi \wedge \theta = \lambda$ *if* $x = i$
(d) $\chi \in [\chi_{\mathfrak{s}}, \theta^*)$.

Proof. Toward contradiction we suppose $\mathrm{NF}^x_{\theta,\chi}(M_0, M_2, M_1, M_3)$ fails. So there is $\bar c \in {}^{\chi\geq}|M_1|$ and $p \subseteq \mathrm{tp}_{\Lambda^x_\chi}(\bar c, M_2, M_3)$ of cardinality $\leq \theta$ which is not realized by any $\bar c' \in {}^{\chi\geq}|M_0|$. Let $B =$

$\mathrm{Dom}(p)$ and let $\bar{b}$ list B so p is over $B, B = \mathrm{Rang}(\bar{b}), \bar{b} \in {}^{\theta\geq}|M_2|$, and $p = \{\varphi_i(\bar{x}_{\bar{c}}, \bar{b}) : i < i(*) \leq \theta\}$, (pedantically we should write $\varphi_i(\bar{x}_i \restriction u_i, \bar{b} \restriction w_i)$ where $u_i \subseteq \ell g(\bar{x}_i), w_i \subseteq \ell g(\bar{b})$).

Case 1: $x =$ nc, so $\theta = \chi$.

We now choose by induction on $\alpha < \lambda^+, \bar{b}_\alpha, \bar{c}_\alpha \in {}^{\chi\geq}|M_0|$ such that:

(a) $\ell g(\bar{b}_\alpha) = \ell g(\bar{b})$ and $\ell g(\bar{c}_\alpha) = \ell g(\bar{c})$,

(b) $\bar{b}_\alpha$ realizes

$$p_\alpha(\bar{x}_b) := \{\varphi_i(\bar{c}, \bar{x}_{\bar{b}}) : i < i(*) \text{ and } M_3 \models \varphi_i[\bar{c}, \bar{b}]\} \cup \cup \{\neg\varphi_i(\bar{c}_\beta, \bar{x}_{\bar{b}}) : i < i(*), \beta < \alpha \text{ and } M_3 \models \neg\varphi_i[\bar{c}_\beta, \bar{b}]\}$$

(c) $\bar{c}_\alpha$ realizes

$$q_\alpha(\bar{x}_{\bar{c}}) := \{\varphi_i(\bar{x}_{\bar{c}}, \bar{b}_\beta) : i < i(*), \beta \leq \alpha \text{ and } M_3 \models \varphi_i[\bar{c}, \bar{b}_\beta]\}.$$

In stage α we can first choose $\bar{b}_\alpha \in {}^{\ell g(\bar{b})}|M_0|$ by the definition of $\mathrm{NF}^x_{\lambda,\chi}(M_0, M_1, M_2, M_3)$, as $p_\alpha(\bar{x}_{\bar{b}})$ is a set of $\leq \lambda$ formulas of the right kind, with parameters from M_1 realized by the sequence $\bar{b}$ from M_2.

Second we can choose $\bar{c}_\alpha$ by the definition of $M_0 \leq^x_{\lambda,\chi} M_1$ as $q_\alpha(\bar{x}_{\bar{c}})$ is a set of $\leq \lambda$ formulas of the right kind with parameters from M_0, realized by $\bar{c}$.

As $\bar{c}$ realizes $p = \{\varphi_i(\bar{x}_{\bar{c}}, \bar{b}) : i < i(*)\}$, clearly $M_3 \models \varphi_i[\bar{c}, \bar{b}]$ for $i < i(*)$, hence (by clause (b), the first set in the union), clearly $M_3 \models \varphi_i[\bar{c}, \bar{b}_\alpha]$ for $i < i(*), \alpha < \lambda^+$, hence (by clause (c)) clearly $M_3 \models \varphi_i[\bar{c}_\beta, \bar{b}_\alpha]$ for $i < i(*)$ and $\alpha \leq \beta < \lambda^+$.

On the other hand by the choice of p for $\alpha < \lambda^+, \bar{c}_\alpha$ does not realize p, hence for some $j(\alpha) < i(*)$ we have $M_3 \models \neg\varphi_{j(\alpha)}[\bar{c}_\alpha, \bar{b}]$ hence (by clause (b), second set in the union), clearly $M_3 \models \neg\varphi_{j(\alpha)}[\bar{c}_\alpha, \bar{b}_\beta]$ when β satisfies $\alpha < \beta < \lambda^+$. As $i(*) \leq \chi$ for some j and $W \subseteq \lambda^+$ of cardinality $\lambda^+, j(\alpha) = j$ for every $\alpha \in W$.
Together this contradicts Hypothesis 3.1.

Case 2: $x = i$ by the symmetry in clause $(d)_2$ of Definition 3.9(1). $\square_{3.12}$

For the rest of the section we assume:

3.13 Hypothesis. $\mathfrak{s}$ satisfies the rigidity axiom Ax(C10).

3.14 Remark. 1) Recall: we say that $\mathfrak{s}$ satisfies Ax(C10), rigidity, when: if $\mathrm{NF}(M_0, M_1, M_2, M_3)$ and $M_3 = \langle M_1 \cup M_2 \rangle^{\mathrm{gn}}_{M_3}$ then there is no automorphism of M_3 over $M_1 \cup M_2$ except the identity; and it holds for the $\mathfrak{s}$ derived from a universal class with the quantifier-order property in V.B.2.3.
2) It is not used in 3.10, 3.12 and 3.19-3.25, in fact, it is used only in 3.15, 3.19, 3.26. We intend to return to this in [Sh:E54].

3.15 Claim. *Suppose $x = i$ and $\chi \in [\chi_{\mathfrak{s}}, \theta^*)$.*
If $\mathrm{NF}^x_{\chi,\chi}(M_0, M_1, M_2, M_3)$ *then* $\langle M_1 \cup M_2 \rangle^{\mathrm{gn}}_{\mathfrak{C}} \leq^x_{\chi,\chi} M_3$.

3.16 Remark. 1) We can also use copies of M_1, M_2, M_3 to get a large order.
2) If we let $\lambda = \beth_4(\chi)$ and prove the version of 3.15 with the strengthened assumption $\mathrm{NF}^x_{\lambda,\lambda}(M_0, M_1, M_2, M_3)$, then this does not affect the results as we are using the case $\theta^* = \beth_\omega(\chi_{\mathfrak{s}})$.
3) What about $x = j$ in 3.15? See [Sh:E54].

Proof. Let $\lambda = \chi$.

Suppose not. Let $M_3^* := \langle M_1 \cup M_2 \rangle^{\mathrm{gn}}_{M_3}$, so $M_1 \cup M_2 \subseteq M_3^* \leq_{\mathfrak{s}} M_3$ but $\neg(M_3^* \leq^x_{\lambda,\chi} M_2)$. Then (by 3.7(5)) for some $\varphi(\bar{x}, \bar{y}) \in \Lambda^x_\chi, \bar{d} \in {}^{\chi \geq}(M_3^*)$ and $\bar{c} \in {}^{\chi \geq}|M_3|$, we have $M_3 \models \varphi[\bar{c}, \bar{d}]$ but there is no $\bar{c}' \in \langle M_1 \cup M_2 \rangle^{\mathrm{gn}}$ satisfying $\ell g(\bar{c}) = \ell g(\bar{c}')$ and realizing $\{\varphi(\bar{x}, \bar{d})\}$.

Using smoothness and $\mathrm{LSP}(\chi)$, (and $\mathfrak{s}$ being χ-based by V.C.3.12) there is $N_3 \leq_{\mathfrak{s}} M_3$ such that for $\ell \leq 3$ we have $N_3 \cap M_\ell \leq_{\mathfrak{s}} M_\ell, \|N_3\| \leq \chi, \bar{d} \subseteq N_3, N \models \varphi[\bar{c}, \bar{d}]$ and $\mathbf{tp}_{\mathfrak{s}}(N_3, M_\ell)$ does not fork over $N_3 \cap M_\ell$ for $\ell = 0, 1, 2, 3$. Hence $N_3 \cap M_0 \leq_{\mathfrak{s}} N_3 \cap M_1 \leq_{\mathfrak{s}} N_3$ and by monotonicity, $\mathbf{tp}_{\mathfrak{s}}(N_3 \cap M_1, M_0)$ does not fork over $N_3 \cap M_0$. As $\mathbf{tp}_{\mathfrak{s}}(M_1, M_2, M_3)$ does not fork over M_0 we can deduce $\mathbf{tp}_{\mathfrak{s}}(N_3 \cap M_1, M_2)$ does not fork over $N_3 \cap M_0$ and similarly $\mathbf{tp}_{\mathfrak{s}}(N_3 \cap M_2, M_1)$ does not fork over $N_3 \cap M_0$. Let $N_\ell = M_\ell \cap N_3$ and let $\bar{a}, \bar{b}, \bar{c}$ list

N_1, N_2, N_3 respectively, so by the rigidity axioms without loss of generality $\varphi = \varphi(\bar{x}, \bar{a}, b)$.

For $\alpha(*), \beta(*) \leq \lambda^+$, we say $\mathbf{x} = \langle f_\alpha, g_\beta, h_{\alpha,\beta} : \alpha \in \alpha(*) \cup \{\lambda^+\}, \beta \in \beta(*) \cup \{\lambda^+\}\rangle$ is a $(\alpha(*), \beta(*))$-approximation if:

- (a) $f_\alpha, g_\beta, h_{\alpha,\beta}$ are $\leq_{\mathfrak{s}}$-embeddings
- (b) for $\alpha < \alpha(*), f_\alpha$ is a $\leq_{\mathfrak{s}}$-embedding $N_3 \cap M_1$ into M_0
- (c) for $\beta < \beta(*), g_\beta$ is a $\leq_{\mathfrak{s}}$-embedding of $N_3 \cap M_2$ into M_0
- (d) f_{λ^+} is the identity on $N_3 \cap M_1$
- (e) g_{λ^+} is the identity on $N_3 \cap M_2$
- (f) (α) Dom $(h_{\alpha,\beta}) = N_3$
  - (β) $h_{\lambda^+,\lambda^+} = \text{id}_N$
  - (γ) $f_\alpha \cup g_\beta \subseteq h_{\alpha,\beta}$
  - (δ) $\alpha < \alpha(*) \Rightarrow \text{Rang}(h_{\alpha,\lambda^+}) \subseteq M_1$
  - (ε) $\beta < \beta(*) \Rightarrow \text{Rang}(h_{\lambda^+,\beta}) \subseteq M_2$
  - (ζ) $\alpha, \beta < \lambda^+ \Rightarrow \text{Rang}(h_{\alpha,\beta}) \subseteq M_0$
- (g) $\langle N_t : t \in I_{\alpha(*),\beta(*)}\rangle$ is independent where $I_{\alpha(*),\beta(*)} = \{\emptyset\} \cup \{\{\alpha\}, \{\lambda^+ + \beta\} : \alpha < \alpha(*) \vee \alpha = \lambda^+ \text{ and } \beta < \beta(*) \vee \beta = \lambda^+\} \cup \{\{\alpha, \lambda^+ + \beta\} : \alpha, \beta < \lambda^+\}$ and for $\alpha \in \alpha(*) \cup \{\lambda^+\}, \beta \in \beta(*) \cup \{\lambda^+\}$ we have
  - (α) $N_\emptyset = N \cap M_0$
  - (β) $N_{\{\alpha\}} = \text{Rang}(f_\alpha)$
  - (γ) $N_{\{\lambda^+ + \beta\}} = \text{Rang}(g_\beta)$
  - (δ) $N_{\{\alpha,\lambda^+ + \beta\}} = \text{Rang}(h_{\alpha,\beta})$.

[We could have used 3.11] The following two subfacts easily finish the proof.

<u>3.17 Subfact</u> There is a (λ^+, λ^+)-approximation.
[Proof: We choose by induction on $\max\{\alpha(*), \beta(*)\}$ then on $\alpha(*)$ and then on $\beta(*)$ we choose f_α, g_β and $h_{\alpha,\beta}$ for $\alpha \in \alpha(*) \cup \{\lambda^+\}$, and $\beta \in \beta(*) \cup \{\lambda^+\}$ so sometimes increasing $\alpha(*) < \lambda^+$, sometimes increasing $\beta(*) < \lambda^+$.

For $\alpha(*) = 0 = \beta(*)$ this is obvious. In limit stages we have no problem — independence has finite character here, see 1.4. By the symmetry let us increase $\beta(*) < \lambda^+$ to $\beta(*) + 1$.

Choose A such that

(α) $A \leq_{\mathfrak{s}} M_3$

(β) $|A| \leq \lambda$

(γ) the following set B_α is included in A and $A \cap M_1 \leq_{\mathfrak{s}} M_1$ where

$$B_\alpha := N_3 \cup \bigcup \big\{ \mathrm{Rang}(h_{\alpha,\beta}) : \alpha \in \alpha(*) \cup \{\lambda^+\}, \beta \in \beta(*) \cup \{\lambda^+\} \big\}.$$

It is enough to find a function h such that

$\circledast_2$ (α) $\mathrm{Dom}(h) = A$

(β) $\mathrm{Rang}(h) \subseteq M_1$

(γ) $h \restriction (A \cap M_1) = \mathrm{id}$,

(δ) $h(N_3 \cap M_2) \subseteq M_0$ (for independence use 1.4) i.e. $g_{\beta(*)} = h \restriction (N \cap M_2)$,

(ε) $h_{\alpha,\beta(*)} = h \restriction \big(\mathrm{Dom}(h_{\alpha,\lambda^+})\big)$.

For this it is enough to find a $\leq_{\mathfrak{s}}$-embedding h', $\mathrm{Dom}(h') = N_3 \cap M_2$, $\mathrm{Rang}(h') \subseteq M_0, h'(N_3 \cap M_2)$ realizes $\mathrm{tp}_{\Lambda^x_\chi}(N_3 \cap M_2, A \cap M_1)$. As $M_0 \leq^x_{\lambda,\chi} M_1$ for this it suffices that $h'(N_3 \cap M_2)$ realizes $\mathrm{tp}_{\Lambda^x_\chi}(N \cap M_2, B)$ where $B \subseteq M_0$ has cardinality $\leq \lambda$. But this follows from $\mathrm{NF}^i_{\lambda,\chi}(M_0, M_1, M_2, M_3)$]. $\square_{3.17}$

<u>3.18 Subfact</u> If $\langle f_\alpha, g_\beta, h_{\alpha,\beta} : \alpha \leq \lambda^+, \beta \leq \lambda^+ \rangle$ is a (χ^+, χ^+)-approximation, <u>then</u> $\mathfrak{s}$ has $(\Lambda_\chi, (2^{2^\chi})^+)$-order property (contradicting 3.1).

Proof. By 2.15 with $\langle M_1 \cup M_2 \rangle^{\mathrm{gn}}$ here standing for M there, it is immediate.

$\square_{3.18,3.18}$

3.19 Claim. *Suppose* $x = i$, $\chi \in [\chi_{\mathfrak{s}}, \theta^*)$, $\lambda = \chi$, *and* $\mathrm{NF}^x_{\lambda,\chi}(M_0, M_1, M_2, M_3)$ *and* $M_0 \leq^x_{\lambda,\chi} M^*_0 \leq^x_{\lambda,\chi} M_1$, $M^*_2 = \langle M^*_0 \cup M_2 \rangle^{\mathrm{gn}}$.
<u>*Then*</u> $\mathrm{NF}^x_{\lambda,\chi}(M^*_0, M_1, M^*_2, M_3)$.

Proof. By observation 3.10(2) we have $\mathrm{NF}^x_{\lambda,\chi}(M_0, M^*_0, M_2, M_3)$ hence by 3.15 we have $M^*_2 = \langle M^*_0 \cup M_2 \rangle^{\mathrm{gn}} \leq^x_{\lambda,\chi} M_3$. By that and 3.7(4) clearly $M^*_0 \leq^x_{\lambda,\chi} M^*_2 \leq^x_{\lambda,\chi} M_3$ and, of course, $M^*_0 \leq^x_{\lambda,\chi} M_1 \leq^x_{\lambda,\chi} M_3$. Also by the Definition 3.9(d) of $\mathrm{NF}^x_{\lambda,\chi}$ we have $\mathrm{NF}_{\mathfrak{s}}(M_0, M_1, M_2, M_3)$ hence $\mathrm{NF}_{\mathfrak{s}}(M^*_0, M_1, M^*_2, M_3)$ as $\mathfrak{s}$ satisfied Ax(C4). So only the two versions of one condition in Definition 3.9(d) is left to be confirmed: clause (δ). The first version is for (M_0, M_1, M_2, M_3); the second version is for (M_0, M_2, M_1, M_3).

For the first version, let $\bar{c}^* \in {}^{\chi\geq}(M^*_2)$ and $p \subseteq \mathrm{tp}_{\Lambda^x_\chi}(\bar{c}^*, M_1)$ be such that $|p| \leq \lambda$. Clearly there are $\bar{c}^0 \in {}^{\chi\geq}|M_0|, \bar{c}^1 \in {}^{\chi\geq}|M^*_0|, \bar{c}_2 \in {}^{\chi\geq}|M_2|$ such that for $\ell = 1, 2$ we have $\mathrm{Rang}(\bar{c}_\ell) \leq_{\mathfrak{s}} M_\ell$, $\mathrm{Rang}(\bar{c}^0) \subseteq \mathrm{Rang}(\bar{c}^\ell)$ and $\mathbf{tp}(\bar{c}^\ell, M_0)$ does not fork over $\mathrm{Rang}(\bar{c}^0)$ and $\bar{c}^* \subseteq \langle \bar{c}^1 \cup \bar{c}^2 \rangle^{\mathrm{gn}}_{M_3}$. We have to show that p is realized in ${}^{\chi\geq}|M^*_0|$.

We can find $N_\ell\, (\ell \leq 3), N^*_0, N^*_2$ such that:

- (a) $\|N_\ell\| \leq \chi$ and $\|N^*_0\|, \|N^*_2\| \leq \chi$
- (b) $\bar{c}_0 \subseteq N_0, \bar{c}_2 \subseteq N_2, \bar{c}_1 \subseteq N^*_0$, $\mathrm{Dom}(p) \subseteq N^*_2$
- (c) $\circledast_{<N_\ell:\ell\leq 3>,<M_\ell:\ell\leq 3>}$ and $\circledast_{<N_0,N^*_0,N_2,N^*_2>,<M_0,M^*_0,M_2,M^*_2>}$ and

  $\circledast_{<N^*_0,N_1,N^*_2,N_3>,<M^*_0,M_1,M^*_2,M_3>}$ from 3.11(1) holds.

So easily it suffices to prove the following

- $\boxdot_1$ there is a $\leq_{\mathfrak{s}}$-embedding f_2 of N_3 into M_1 over N_1 such that $f_2(N^*_2) \subseteq M^*_0$.

Why is $\boxdot_1$ true? As $\mathrm{NF}_{\mathfrak{s}}(M_0, M_1, M_2, M_3)$ by clause $(d)_1(\delta)$ of Definition 3.9(1) for $\ell = 1$ there is an $\leq_{\mathfrak{s}}$-embedding f_2 of N_3 into M_1 over N_1 such that $f_2(N_2) \subseteq M_0$. Now as $\mathrm{NF}_{\mathfrak{s}}(N_0, N_1, N_2, N_3)$ and $f_2(N^*_2) = f_2(\langle N^*_0 \cup M_2 \rangle)^{\mathrm{gn}} = \langle f(N^*_0) \cup f(N_2) \rangle^{\mathrm{gn}} = \langle N^*_0 \cup f(N_2) \rangle^{\mathrm{gn}} \subseteq M_0$ we are done.

For the second version we have $\bar{c}^* \in {}^{\chi\geq}(M_1)$ and $p \subseteq \text{tp}_{\Lambda^*_\chi}(\bar{c}^*, M^*_2)$ has cardinality $\leq \chi$ and we have to find $\bar{c}' \in {}^{\ell g(\bar{c}^*)}(M^*_0)$ realizing p in M^*_2. The proof is similar. $\square_{3.19}$

3.20 Claim. *If $x = i, \lambda \geq \chi \geq \chi_{\mathfrak{s}}$ and $M_0 \leq^x_{\lambda,\chi} M_1$ and $M_0 \leq^x_{\lambda,\chi} M_2$ then there are $N_\ell(\ell < 4)$ and $h_\ell(\ell < 3)$ such that:*

(a) $\text{NF}^x_{\lambda,\chi}(N_0, N_1, N_2, N_3)$

(b) *for $\ell = 0, 1, 2, h_\ell$ is an isomorphism from M_ℓ onto N_ℓ and $h_0 \subseteq h_1, h_0 \subseteq h_2$.*

This follows by:

3.21 Claim. *Assume $\lambda = \chi \geq \chi_{\mathfrak{s}}$.*

Suppose $x = i$ and $\{M_t : t \in I\}$ is independent over M, for the framework $\mathfrak{s}$, of course, $t \in I \Rightarrow M \leq^x_{\lambda,\chi} M_t$, and for $J \subseteq I$ let $M^*_J = \langle \bigcup_{t \in J} M_t \rangle^{\text{gn}}_{\mathfrak{C}}$.

Then:

(a) $M^*_{J_1} \leq^x_{\lambda,\chi} M^*_{J_2}$ *if $J_1 \subseteq J_2 \subseteq I$*

(b) $\text{NF}^x_{\lambda,\chi}(M^*_{J_1 \cap J_2}, M^*_{J_1}, M^*_{J_2}, M^*_I)$ *if $J_1, J_2 \subseteq I$.*

Remark. This will be called $\text{Ax}(\text{C2})^+$, formally:

3.22 Definition. Assume the framework $\mathfrak{t}$ satisfies Axioms (A0)-(A3),(B0)-(B3) and (C0),(C1),(C3).

1) We say $\{M_t : t \in I\}$ is locally independent[3] over M inside N when $\{M_t : t \in J\}$ is independent inside N over M for every finite $J \subseteq I$.

1A) For finite J, we say $\{M_t : t \in J\}$ is independent over M inside N when:

(a) $M \leq_{\mathfrak{t}} M_t \leq_{\mathfrak{t}} N$ for $t \in J$

[3]Of course, if $\mathfrak{t}$ satisfies (AxFr_1) this is equivalent to being independent

(*b*) there is a list $\langle t_\alpha : \alpha < \alpha(*)\rangle$ of the members of J and $\leq_{\mathfrak{t}}$-increasing continuous sequences, $\bar{N} = \langle N_\alpha : \alpha \leq \alpha(*)\rangle$ such that $M_0 = M, N_{\alpha(*)} \leq_{\mathfrak{t}} N$ and $\mathrm{NF}_{\mathfrak{t}}(M, N_\alpha, M_{t_\alpha}, N_{\alpha+1})$ and $N_{\alpha+1} = \langle N_\alpha \cup M_{t_\alpha}\rangle^{\mathrm{gn}}_{N_{\alpha+1}}$.

1B) We say $\langle M_{t_\alpha} : \alpha < \alpha_*\rangle$ is independent over M inside N if (a) + (a) of part (1A) holds for $J = \{t_\alpha : \alpha < \alpha_*\}$ which is with no repetitions (and the enumeration $\langle t_\alpha : \alpha < \alpha_*\rangle$).
2) We say $\mathfrak{t}$ satisfies the axiom $(\mathrm{C2})^+$ <u>when</u> given $M \in K_{\mathfrak{s}}$, an index set I and $M_t \in K_{\mathfrak{s}}$ which $\leq_{\mathfrak{s}}$-extends M for $t \in I$, we can find $\bar{f}, N$ such that

(*a*) $M \leq_{\mathfrak{s}} N$
(*b*) $\bar{f} = \langle f_t : t \in I\rangle$
(*c*) f_t is a $\leq_{\mathfrak{s}}$-embedding of M_t into M over M
(*d*) $\langle f_t(M_t) : t \in I\rangle$ is independent over M inside N
(*e*) $N = \langle \cup\{f_t(M_t) : t \in I\} \cup M\rangle^{\mathrm{gn}}_N$
(*f*) letting $M^*_J = \langle \cup\{f_t(M_t) : t \in J\} \cup M\rangle^{\mathrm{gn}}_N$ for $J \subseteq I$, we have
   (α) $M^*_{J_1} \leq_{\mathfrak{t}} M^*_{J_2}$ for any $J_1 \subseteq J_2 \subseteq I$
   (β) $\mathrm{NF}_{\mathfrak{t}}(M^*_{J_1\cap J_2}, M^*_{J_1}, M^*_{J_2}, M^*_{J_1\cup J_2})$ for $J_1, J_2 \subseteq I$.

3) We say $\mathfrak{t}$ satisfies $\mathrm{Ax}(\mathrm{C2})^{\oplus}$ <u>when</u>: if $\langle M_t : t \in I\rangle$ is locally independent over M inside N <u>then</u>

($*$) $M^*_I \leq_{\mathfrak{s}} N$.

4) Let $\mathrm{Ax}(\mathrm{C2})^*$ be defined like $(\mathrm{C2})^+$ adding to clause (f) also

(γ) if $M \leq_{\mathfrak{s}} M'_t \leq_{\mathfrak{s}} M_t$ for $t \in I$ then $M' \leq_{\mathfrak{s}} M^*_I$ where $M' = \langle \cup\{M'_t : t \in I\} \cup M\}\rangle^{\mathrm{gn}}_N$.

5) Let $\mathrm{Ax}(\mathrm{C2})^{\otimes}$ be $\mathrm{Ax}(\mathrm{C2})^* + \mathrm{Ax}(\mathrm{C2})^{\oplus}$.

Remark. In 3.22(3) the demand $M^*_J \leq_{\mathfrak{s}} N$ is a weak form of smoothness.

Proof of 3.21. We use V.C§4, V.D§3.

Clause (a) follows by clause (b). For clause (b) let $J_3 = J_1 \cup J_2, J_0 = J_1 \cap J_2, \bar{M}^* = \langle M^*_{J_\ell} : \ell \le 3\rangle$.

By 3.11(1)+(3) it suffices to assume $\circledast_{\bar{N},\bar{M}^*}$ and prove the existence of an embedding as there.
By Definition V.C.4.8, Claim V.C.4.10, there are J'_ℓ ($\ell \le 3$), M', N', N'_t ($t \in J'_3$) such that

$(*)_1$ (a) $J'_\ell \subseteq J_\ell$ has cardinality $\le \chi$
(b) $J'_3 = J'_1 \cup J'_2$
(c) $J'_0 = J'_1 \cap J'_2$
(d) $M', N', N'_t(t \in J'_3)$ are from $K^{\mathfrak{s}}_{\le\chi}$
(e) $\mathrm{NF}_{\mathfrak{s}}(M', M, N'_t, M_t)$ for $t \in J'_3$
(f) $\mathrm{NF}_{\mathfrak{s}}(M', M, N'_\ell, M^*_{J_\ell})$
(g) $N_0 \le_{\mathfrak{s}} M'$ and $N' \le_{\mathfrak{s}} M^*_{J_3}$
(h) $N_\ell \subseteq \langle \cup\{N'_t : t \in J'_\ell\}\rangle^{\mathrm{gn}}_{M^*_{J_3}}$ for $\ell = 1, 2, 3$
(i) $\{N'_t : t \in J'_\ell\}$ is independent over M' inside N'_ℓ
(j) $N'_\ell = \langle \cup\{N'_t : t \in J_\ell\rangle^{\mathrm{gn}}_{M^*_{J_3}}$.

Next let $\langle t_\alpha : \alpha < \alpha_* \le \chi\rangle$ list J'_3 and choose f_{t_α} by induction on α such that

$(*)_2$ (a) f_{t_α} is a $\le_{\mathfrak{s}}$-embedding of N'_t into M over N'
(b) $\{f_{t_\beta}(N'_{t_\beta}) : \beta \le \alpha\}$ is independent over N'.

There is no problem to carry the induction as $\{f_t(N'_t) : t \in J'_3\}$ is independent over M' inside M_0 and clearly $\{N_t : t \in J_\ell\} \cup \{M\}$ is independent over N', we can extend $f'_\ell = \cup\{f_t : t \in J_\ell\}$ to a $\le_{\mathfrak{s}}$-embedding of f_ℓ of $N'_{J'_\ell}$ into M. Now we can extend $f'_1 \cup \mathrm{id}_{N'_{J'_2}}$ to an $\le_{\mathfrak{s}}$-embedding f of $N'_{J'_3}$ into $M^*_{J_2}$ hence clearly $f \restriction N_3$ is as required. $\square_{3.21}$

3.23 Observation. Let $\mathfrak{t}$ be as in Definition 3.22.
1) Assume $\mathfrak{t}$ satisfies $\mathrm{Ax(C2)}^+$, see Definition 3.28(5) below. If $\langle M_t : t \in I\rangle$ is locally independent over M inside N and letting $M^*_J =$

$\langle \cup\{M_t : t \in J\} \cup M\rangle^{\rm gn}_N$ for $J \subseteq I$, <u>then</u> clauses $(\alpha) + (\beta)$ of (f) of 3.22(2) holds.
2) Assume $\mathfrak{t}$ satisfies Ax(C4). If $\{M_t : t \in I\}$ is locally independent over M inside N and $M \leq_{\mathfrak{t}} M'_t \leq_{\mathfrak{t}} M_t$ for $t \in I$ <u>then</u> $\{M'_t : t \in I\}$ is locally independent over M inside N.
3) Assume $\mathfrak{t}$ satisfies Ax(C4). If $\mathfrak{t}$ satisfies Ax(C2)$^\oplus$ <u>then</u> $\mathfrak{t}$ satisfies Ax(C2)* hence Ax(C2)$^\oplus$.
4) In part (1), $\langle M_t : t \in I\rangle$ is independent over M inside M^*_I, as witnessed by an enumeration $\langle t_i : i < \alpha\rangle$ of I.

Proof. 1) By Ax(C2)$^+$ we can find $N', \bar f$ such that the demands (a)-(e) of 3.22(2) hold with N there replaced by N' here.

For $J \subseteq I$ let $\mathscr{G}_j = \{g : g$ is an isomorphism from $\langle \cup\{M_t : t \in J\} \cup M\rangle^{\rm gn}_N$ onto $\langle \cup\{f_t(M^-_t) : t \in J\} \cup M\rangle^{\rm gn}_{N'}$ and ${\rm id}_M \subseteq g$ and $f_t \subseteq g$ for $t \in J\}\rangle^{\rm gn}_N$.

Now we prove by induction on $\mu \leq |I|$ that

$\circledast_\mu$ (*a*) if $J \subseteq I$ and $|J| \leq \mu$ then $\mathscr{G}_J \neq \emptyset$

(*b*) if $J_1 \subseteq J_2 \subseteq I$ and $|J_2| \leq \mu$ then every $g_1 \in \mathscr{G}_{J_1}$ can be extended to some $g_2 \in \mathscr{G}_{J_2}$

(*c*) if $\langle J_\alpha : \alpha \leq \delta\rangle$ is $\subseteq$-increasing continuous, $J_\delta \subseteq I, |J_\delta| \leq \mu$ and $g_\alpha \in \mathscr{G}_{J_\alpha}$ for $\alpha < \delta$ is $\subseteq$-increasing with α <u>then</u> $\cup\{g_\alpha : \alpha < \delta\} \in \mathscr{G}_{J_\delta}$.

There is no problem to carry the induction and for $\mu = |I|$ we are done.
2) Without loss of generality I is finite; as $\{M_t : t \in I\}$ is independent over M inside N, clearly for some enumeration $\langle t_\ell : \ell < n\rangle$ of I there are $\langle N_\ell : \ell \leq n\rangle$ such that ${\rm NF}_{\mathfrak{t}}(M, N_\ell, M_{t_\ell}, N_{\ell+1})$ for $\ell < n$ and $N_n \leq_{\mathfrak{s}} N$. So without loss of generality $N_{\ell+1} = \langle N_\ell \cup M_{t_\ell}\rangle^{\rm gn}_N$ for $\ell < n$ and $N_\ell = N_n$. Now we can find $N'_\ell \leq_{\mathfrak{t}} N'_\ell$ such that for $\ell \leq n$ such that $N'_\ell = N_0 = M$ and ${\rm NF}_{\mathfrak{t}}(M, N'_\ell, M'_{t_\ell}, N_{\ell+1})$ and $N'_{\ell+1} = \langle N'_\ell \cup M'_{t_\ell}\rangle$. This is done by induction on ℓ.
3),4) Left to the reader. $\square_{3.23}$

3.24 Claim. *Let $x = i, j, \mathrm{nc}$ and assume $\lambda \geq \chi \geq \chi_{\mathfrak{s}}$.*
1) If $\langle M_i : i < \delta \rangle$ is $\leq^x_{\lambda,\chi}$-increasing and $\mathrm{cf}(\delta) > \lambda + \chi$, then $M_i <^x_{\lambda,\chi} \bigcup\limits_{j<\delta} M_j$ for every $i < \alpha$, i.e. union existence.
2) If $\langle M_i : i \leq \delta \rangle$ is $\leq^x_{\lambda,\chi}$-increasing and $\mathrm{cf}(\delta) > \lambda + \chi$, then $\bigcup\limits_{i<\delta} M_\delta \leq^x_{\lambda,\chi} M_\delta$ (i.e. smoothness).

Proof. Easy. $\square_{3.24}$

3.25 Exercise: Assume $x = i, j, \mathrm{nc}$ and $\lambda \geq \chi \geq \chi_{\mathfrak{s}}$. If

- (a) I is a λ^+-directed partial order,
- (b) $M_s \leq^x_{\lambda,\chi} M_t$ for $s \leq_I t$
- (c) $M = \cup\{M_s : s \in I\}$.

Then

- (α) $s \in I \Rightarrow M_s \leq^x_{\mathfrak{s}} M$
- (β) $M \leq^x_{\lambda,\chi} N$ when $(\forall s \in I)(M_0 \leq^x_{\lambda,\chi} N)$.

3.26 Main Conclusion. Suppose $\theta^* = \beth_\omega(\chi_{\mathfrak{s}})$ hence restating Hypothesis 3.1, for every $\chi < \theta^*$ but $\geq \chi_{\mathfrak{s}}$, $\mathfrak{s}$ does not have the $(\Phi_{\mathfrak{s}}, (2^{2^\chi})^+)$-order property. Let K^+ be $K_{\mathfrak{s}}$ (or $K_{\geq \beth_\omega(\chi_{\mathfrak{s}})}$, little difference) and $\leq^+ = <^i_{<\beth_\omega(\chi_{\mathfrak{s}}), <\beth_\omega(\chi_{\mathfrak{s}})}$ and $\mathfrak{K}^+ = (K^+, \leq^+)$, $(\mathrm{A4})_{>\theta^*}$ and $\mathfrak{s}^+ = \mathfrak{s}(+) = (\mathfrak{K}^+, \mathrm{NF}_{\mathfrak{s}(+)}, <>^{\mathrm{gn}})$ where $\mathrm{NF}_{\mathfrak{s}(+)} = \mathrm{NF}^i_{<\theta^*,<\theta^*}$.
1) $\mathfrak{s}^+ = \mathfrak{s}(+)$ satisfies AxFr_1^- which means it satisfies the following axioms (see Ch.II,§1):$M \in K^+, M \leq^+ N$ are preserved by isomorphisms, (A0),(A1),(A2),(A3) and (B0) - (B3), see below and (C1),(C2),(C2)$^+$,(C3),(C4),(C5),(C6),(C7),(C9) and (C10), see 3.14 above, (C11), see 3.28(5) below and the λ-LSP for every λ satisfying $\lambda = \lambda^{\beth_\omega(\chi_{\mathfrak{s}})}$.
2) $\mathfrak{s}^+$ satisfies also $(\mathrm{A4})_{>\beth_\omega(\chi_{\mathfrak{s}})}$ and (A6), and even (A6)$^+$, see below in 3.28(2)(2B), and smoothness for cofinality $> \beth_\omega(\chi_{\mathfrak{s}})$.
3) The pair $(\mathfrak{s}^+, \mathfrak{s})$ has the λ-LSP for $\not\leq_{\mathfrak{s}(+)}$ whenever $\lambda = \lambda^{\beth_\omega(\chi_{\mathfrak{s}})}$,

(see below Definition 3.31(1) below).
4) $\chi^*_{\mathfrak{s}(+)}$ is well defined and $\leq \beth_\omega(\chi_{\mathfrak{s}})$ in fact $\chi = \beth_\omega(\chi_{\mathfrak{s}})$ satisfies the demand on χ in Definition 3.28(3) and even in 3.28(4), (and has rigidity (Ax(C10)) as $\mathfrak{s}$ has it).

Before proving 3.26:

3.27 Claim. *Assume $M_0 \leq_{\mathfrak{s}(+)} M_\ell \leq_{\mathfrak{s}(+)} M_{\mathfrak{s}}$ for $\ell = 1, 2$.*
Then $\mathrm{NF}_{\mathfrak{s}(+)}(M_0, M_1, M_2, M_3)$ *<u>iff</u>* $\mathrm{NF}_{\mathfrak{s}}(M_0, M_1, M_2, M_3)$ *and* $\langle M_1 \cup M_2\rangle^{\mathrm{gn}}_{M_{\mathfrak{s}}} \leq_{\mathfrak{s}(+)} M_{\mathfrak{s}}$.

Remark. Can phrase it for $<^*_{\lambda,\chi}$'s.

Proof. Straight (recalling clause (β) of 3.21(2)(f)). $\square_{3.27}$

3.28 Definition. 1) AxFr^-_1 is what $\mathfrak{s}^+ = \mathfrak{s}(+) := (\mathfrak{K}^+, \mathrm{NF} \restriction \mathfrak{K}^+, <>^{\mathrm{gn}})$ satisfies from the axioms from V.B§1 as listed in 3.26(1) except possibly Ax(C10),(C11), i.e. (A0)-(A3),(B0)-(B3),C(1)-(C7) and $(\mathrm{C2})^+$,(C9) (so not the cases of $(\mathrm{A4})_\theta$, $(\mathrm{A6}))_\theta)$.
2) We say $\mathfrak{s}$ satisfies Ax(A6) when: if $\langle M_\alpha : \alpha \leq \delta\rangle$ is $\leq_{\mathfrak{s}}$-increasing and $M'_\delta = \cup\{M_\alpha : \alpha < \delta\}$ <u>then</u> $M'_\delta \in K_{\mathfrak{s}}$ and $\alpha < \delta \Rightarrow M_\alpha \leq_{\mathfrak{s}} M'_\delta$ (but $M'_\delta \leq_{\mathfrak{s}} M_\delta$ is not required).
2A) We define $\mathrm{Ax(A6)}_\theta$, $\mathrm{Ax(A6)}_{>\theta}$, etc., similarly.
2B) We say $\mathfrak{s}$ satisfies $\mathrm{Ax(A6)}^+$ when: if I is a directed partial order, $\langle M_t : t \in I\rangle$ is $\leq_{\mathfrak{s}}$-increasing, and $t \in I \Rightarrow M_t \leq_{\mathfrak{s}} N$ and $M = \cup\{M_t : t \in I\}$ <u>then</u> $M \in K_{\mathfrak{s}}$ and $t \in I \Rightarrow M_t \leq_{\mathfrak{s}} M$. Let $\mathrm{Ax(A6)}^+_{\geq\theta}$ be defined similarly restricting ourselves to θ-directed partial orders I.
3) Let $\chi^*_{\mathfrak{s}}$ be the minimal χ satisfying the following (and ∞ if there is no one):

- (a) $\mathfrak{s}$ has the LSP_θ for every $\theta = \theta^\chi$
- (b) $\mathfrak{s}$ is θ-based for every $\theta = \theta^\chi$ (see Ex V.B.1.19), also if $\langle M_t : t \in I\rangle$ is locally independent over M inside N, see Definition 3.22(1) and $A \subseteq N$ and $\theta^\chi = \theta \geq \|M\| + |A| + \sup\{\|M_t\| : t \in I\}$ then we can find $N_* \leq_{\mathfrak{s}} N$ of cardinality $\leq \theta$ and

$J \subseteq I$ of cardinality $\le \theta$ such that $\cup\{M_t : t \in J\} \subseteq N_*$ and $\{M_t : t \in I \backslash J\} \cup \{N_*\}$ is independent over M inside N

(c) $\text{Ax(A4)}_{\ge\chi}$ and $(\ge \chi)$-smoothness and $\text{Ax(A6)}_{\ge\chi}$ holds

(d) if $\text{NF}_{\mathfrak{s}}(M_0, M_1, M_2, M_3)$ then the cardinality of $\langle M_1 \cup M_2\rangle^{\text{gn}}_{M_3}$ is at most $\|M_1\| + \|M_2\| + \chi$

(e) if Ax(A4)^*_θ fails, then this is exmplified by a sequence $\langle M_i : i < \theta\rangle$ of models of cardinality $\le 2^\chi$.

4) We write $\chi^{**}_{\mathfrak{s}}$ when we add:

(f) if $M \subseteq N$ are both from $K_{\mathfrak{s}}$, but $M \not\le_{\mathfrak{s}} N$ then there is a $\mathscr{P}$ such that

  (α) $\mathscr{P} \subseteq \mathscr{P}(N)$

  (β) if $A \in [N]^{\le\chi}$ then for some $B \in [N]^{\le\chi}$ we have $A \subseteq B \in \mathscr{P}$

  (γ) $\mathscr{P}$ is closed under increasing unions

  (δ) if $A \in \mathscr{P}$ then $\neg(M \restriction A \le_{\mathfrak{s}} N \restriction A)$.

5) Ax(C11) says[4] that: if $\mathscr{T}$ is a subtree of some ${}^{\alpha>}\lambda$ (so closed under initial segments) and for $\ell = 1, 2$ we have $N^\ell_t \le_{\mathfrak{s}} N_\ell$ for $t \in \mathscr{T}$ increasing continuously with t and $\langle N^\ell_\eta : t \in \mathscr{T}'\rangle$ is independent in N_ℓ for every finite $\mathscr{T}' \subseteq \mathscr{T}$ closed under intersections and f_t is an isomorphism from N^1_t onto N^2_t increasing with t and N'_ℓ is the submodel of N_t with universe $\cup\{\langle\cup\{N^\ell_t : t \in \mathscr{T}'\}\rangle^{\text{gn}}_{\mathfrak{C}} : \mathscr{T}' \subseteq \mathscr{T}$ is finite closed under intersections$\}$ then there is an isomorphism f from N'_ℓ onto N'_2 extending every $f_t\,(t \in \mathscr{T}')$; note that $N_\ell \in K_{\mathfrak{s}}$ is not required.
6) Ax(C11)^+ is defined similarly but we require that $\langle N_s : s \le_I t\rangle$ is $\le_{\mathfrak{s}}$-increasing but we do not require that it is continuous.
7) Let Ax(C8)^- be (if $\mathfrak{s}$ has Ax(A4) + smoothness then is equivalent to Ax(C8) see V.C.1.9): if $\langle M_{1,i} : i \le \delta\rangle$ is $\le_{\mathfrak{s}}$-increasing, continuous in δ and $M_\delta \le_{\mathfrak{s}} N$ and $\text{NF}_{\mathfrak{s}}(M_0, M_{1,i}, M_{2,i}, M)$ for $i < \delta$ then $\text{NF}_{\mathfrak{s}}(M_0, M_{1,\delta}, M_{2,\delta}, M)$

[4]it is really closely related to Ax(C10)

3.29 Remark. 1) Compared to AxFr_1, in AxFr_1^- we lose (A4) (existence of union) and get only 3.21's conclusion (called $\mathrm{Ax(C2)}^+$) and the cases of Ax(A4) and smoothness for large cofinality as a weak compensation. Reasonable additions are (A6),$(\mathrm{A6})^+$ which also compensate.
2) Question: Does (A6) help us to get $(\mathrm{A4})_{\aleph_0}$? Still §4 uses less.
3) Note that $\mathrm{NF}_{\mathfrak{s}(+)}$ is very close to $\mathrm{NF}_{\mathfrak{s}}$, in fact it is the maximal set of quadruples from $\mathrm{NF}_{\mathfrak{s}}$ satisfying Ax(C1) for $\leq_{\mathfrak{s}(+)}$.

3.30 Observation. Assume $\mathfrak{t}$ satisfies AxFr_1^- (or just as in 3.23 and $\mathrm{Ax(C4)}^+$).

Then Ax(C10) implies $\mathrm{Ax(C11)}^+$ which implies Ax(C11).

Proof. Easy, or see 5.15(1). $\square_{3.30}$

Proof of 3.26. 1) Membership in $K^+ = K_{\mathfrak{s}(+)}$ is preserved by isomorphisms.

Trivial as this holds for $\mathfrak{s}$.

The order $\leq_{\mathfrak{s}(+)}$ is preserved under isomorphisms:

Trivial by the definition of the order.

Axioms (A0),(A1),(A2),(A3)

First, Ax(A0), $M \leq_{\mathfrak{s}(+)} M$, is trivial by the definition. Second, Ax(A1), $M \leq_{\mathfrak{s}(+)} N \Rightarrow M \subseteq N$ holds as $M \leq_{\mathfrak{s}(+)} N \Rightarrow M \leq_{\mathfrak{s}} N$ while $\mathfrak{s}$ satisfies Ax(A1). Third, Ax(A2), $\leq_{\mathfrak{s}(+)}$ is transitive holds by Observations 3.7(2). Lastly, Ax(A3), if $M_\ell \leq_{\mathfrak{s}(+)} N$ for $\ell = 1, 2$ and $M_1 \subseteq M_2$ then $M_1 \leq_{\mathfrak{s}(+)} M_2$, holds by part (4) of Observation 3.7.

Axiom $(\mathrm{A4})_{\geq\theta}$ for $\theta > \beth_\omega(\chi_{\mathfrak{s}})$ which is regular, of course

By 3.24(1).

Axiom (B0)-(B3)

Follows from this holding for $\mathfrak{s}$.

Axiom (C1)

By clause $(d)_1(\beta)$ of Definition 3.9(1) of $\mathrm{NF}_{\mathfrak{s}(+)} = \mathrm{NF}^i_{<\theta^*,<\theta^*}$.

Axiom (C2), existence

Holds by Claim 3.20.

<u>Axiom (C2)$^+$, see Definition 3.22</u>

Holds by 3.21 and properties of $\mathfrak{s}$

<u>Axiom (C3), monotonicity</u>

By Observation 3.10(2),(3)

<u>Axiom (C4), base enlargement</u>

Assume $\mathrm{NF}_{\mathfrak{s}(+)}(M_0, M_1, M_2, M_4)$ and $M_0 \le_{\mathfrak{s}(+)} M_0^* \le_{\mathfrak{s}(+)} M_1$. The result follows by 3.19.

<u>Axiom (C5), uniqueness</u>

Should be clear as by claim 3.15 we get the stronger version V.C.1.2 and uniqueness for $\mathfrak{s}$.

<u>Axiom (C6), symmetry</u>

The definition is symmetric; alternatively if $\mathrm{NF}_{\mathfrak{s}(+)}(M_0, M_1, M_2, M_3)$ let $M_3^* = \langle M_1 \cup M_2\rangle^{\mathrm{gn}}_{M_3}$ so by Claim 3.19 we have $M_3^* \le_{\mathfrak{s}(+)} M_3$ and by monotonicity obviously $\mathrm{NF}_{\mathfrak{s}}(M_0, M_1, M_2, M_3)$ hence $\mathrm{NF}_{\mathfrak{s}}(M_0, M_1, M_2, M_3)$. By claim 3.21 we have $\mathrm{NF}_{\mathfrak{s}(+)}(M_0, M_2, M_1, M_3^*)$ and by Ax(C3) we get $\mathrm{NF}_{\mathfrak{s}(+)}(M_0, M_1, M_2, M_3)$.

<u>Axiom(C7), continuity</u>

Should be clear (as nothing new for $<>^{\mathrm{gn}}$).

<u>Axiom(C9), transitivity</u>

By Claim V.C.1.3 and Definition V.C.1.4 it says: if $\mathrm{NF}_{\mathfrak{s}(+)}(M_0, M_1, M_2, M_3)$ and $\mathrm{NF}_{\mathfrak{s}(+)}(M_2, M_3, M_4, M_5)$ then $\mathrm{NF}_{\mathfrak{s}(+)}(M_0, M_1, M_2, M_3)$. We finish by noting that the proof of V.C.1.3 use only axioms which have been proved above.

<u>Axiom(C10)</u>

Obvious by Ax(C10) holding for $\mathfrak{s}$.

<u>Axiom(C11)</u>

See §5.

<u>$\mathfrak{s}(+)$ has the LSP(λ) if $\lambda = \lambda^{\beth_\omega(\chi_\mathfrak{s})}$</u>

By Observation 3.7(3).

<u>Axiom (A4)$_{>\beth_\omega(\chi_\mathfrak{s})}$+ smoothness for cofinality $> \beth_\omega(\chi_\mathfrak{s})$</u>

By claim 3.24(2)

<u>Axiom (A6),(A6)$^+$</u>

Obvious by the definition of $\le_{\mathfrak{s}(+)}$ as we use $x = i$.

2), 3), 4) Should be clear. $\square_{3.26}$

Paying a debt from 3.26(3) let:

3.31 Definition. 1) For frameworks $\mathfrak{t}_1, \mathfrak{t}_2$ satisfying $K_{\mathfrak{t}_1} = K_{\mathfrak{t}_2}$ and $\le_{\mathfrak{t}_2} \subseteq \le_{\mathfrak{t}_1}$ to say that the pair $(\mathfrak{t}_2, \mathfrak{t}_1)$ has the $(< \theta)$-LSP for $M_1 \nleq_{\mathfrak{t}_2} M_2$ means: if $M_1 \le_{\mathfrak{t}_1} M_2$ but $\neg[M_1 \le_{\mathfrak{t}_2} M_2]$ <u>then</u> there are $\chi < \theta$ and $A_\ell \subseteq M_\ell, |A_\ell| \le \chi$ such that for no $N_1 \le_{\mathfrak{t}_1} N_2$ satisfying $A_\ell \subseteq N_\ell \le_{\mathfrak{t}_1} M_\ell$ for $\ell = 1,2$ do we have $N_1 \le_{\mathfrak{t}_2} N_2$; instead $(< \theta^+)$ we may write θ.
2) We say that the framework $\mathfrak{t}$ satisfies the χ-LSP for $\nleq_{\mathfrak{t}}$ when: if $M_1 \subseteq M_2$ but $M_1 \nleq_{\mathfrak{t}} M_2$ then for the χ-majority of $X \subseteq M_2$ we have $M_1 \restriction X \nleq_{\mathfrak{t}} M_2$.

<u>3.32 Exercise</u>: Prove the obvious implication and what holds in 3.26, 3.31(2),(3).

§4 Regaining existence of ω-unions

We assume

4.1 Hypothesis. $\mathfrak{s}$ satisfies AxFr_1^- (see Definition 3.28(1)).

4.2 Remark. 1) Note that for $\mathfrak{s}$, "[locally] independence" has some properties proved as in Chapter V.B. Alternatively, we can assume that for some framework $\mathfrak{s}_*$ satisfying AxFr_1 is as in §3 and $\mathfrak{s}$ derived from it; then those properties are immediate and our loss is not serious (this applies in particular to Exercise 4.5).
2) No harm in assuming that $\mathfrak{s}$ satisfies whatever is proved in 3.26.

4.3 Definition. For a sequence $\bar{M} = \langle M_n : n < \omega \rangle$ satisfying $\bigwedge_n M_n \le_{\mathfrak{s}} M_{n+1}$ and $\le_{\mathfrak{s}}$-embedding $f : M_n \to N$ (we let $n_{\bar{M}}(f) = n(f, \bar{M}) := n$) and cardinal λ, where if $\lambda = 1$ we may omit it, we define $\mathrm{rk}^{\mathrm{emb},\lambda}_{\bar{M}}(f, N)$, an ordinal (or infinity) as follows:

$\mathrm{rk}^{\mathrm{emb},\lambda}_{\bar{M}}(f, N) \geq \alpha$ iff ($\bar{M}, f, N$ are as above and) for every $\beta < \alpha$ there are f_i(for $i < \lambda$) such that:

(i) $\mathrm{rk}^{\mathrm{emb},\lambda}_{\bar{M}}(f_i, N) \geq \beta$

(ii) $n_{\bar{M}}(f_i) = n_{\bar{M}}(f) + 1$

(iii) $\{f_i(M_{n_{\bar{M}}(f_i)}) : i < \lambda\}$ is locally independent over $f(M_{n_{\bar{M}}(f)})$ inside N

(iv) $f \subseteq f_i$ for $i < \lambda$.

So we let $\mathrm{rk}^{\mathrm{emb},\lambda}_{\bar{M}}(f, N) = \alpha$ if $\mathrm{rk}^{\mathrm{emb},\lambda}_{\bar{M}}(f, N) \geq \alpha$ but $\not\geq \alpha + 1$, and $\mathrm{rk}^{\mathrm{emb},\lambda}_{\bar{M}}(f, N) = \infty$ if for every $\mathrm{rk}^{\mathrm{emb},\lambda}_{\bar{M}}(f, N) \geq \alpha$. If λ is clear from context, we may omit it.

4.4 Definition. 1) Let $\mathbf{T}_{\aleph_0}$ be the class of trees I with $\leq \omega$ levels, a root $\mathrm{rt}(I)$, i.e. minimal element and no ω-branch. For $\eta \in I$ let $\mathrm{lev}_I(\eta)$ be the level of η in I. Let $I_1 \leq_{\mathbf{T}_{\aleph_0}} I_2$ means that $I_1 \subseteq I_2$ (as partial orders) and $s \leq_{I_2} t \in I_1 \Rightarrow s \in I_1$.
1A) Let $\mathbf{T}^{\mathrm{st}}_{\aleph_0} = \{I : I$ is a non-empty set of decreasing sequences of ordinals closed under initial segments$\}$, where st stands for standard, so $\mathrm{lev}_I(\eta) = \ell g(\eta)$ for $\eta \in I \in \mathbf{T}^{\mathrm{st}}_{\aleph_0}$.
2) For a sequence $\bar{M} = \langle M_n : n < \omega \rangle$ such that for $n < \omega$ $M_n \leq_{\mathfrak{s}} M_{n+1}$ and ordinal α and tree $I \in \mathbf{T}_{\aleph_0}$, we define when is $\mathbf{n} = \langle N, N_\eta, f_\eta : \eta \in I \rangle = \langle N_{\mathbf{n}}, N^{\mathbf{n}}_\eta, f^{\mathbf{n}}_\eta : \eta \in I \rangle$ an $(\mathfrak{s}, I)$-tree of models for $\bar{M}$, as follows:

(a) f_η is an isomorphism from $M_{\ell g(\eta)}$ onto N_η

(b) $\nu \triangleleft \eta \Rightarrow f_\nu \subseteq f_\eta$ (hence $\nu \triangleleft \eta \Rightarrow N_\nu \leq_{\mathfrak{s}} f_\nu$)

(c) $N_\eta \leq_{\mathfrak{s}} N$ for $\eta \in I$.

3) We say $\mathbf{n}$ is a weakly independent $(\mathfrak{s}, I)$-tree (of models) for $\bar{M}$ when clauses (a),(b),(c) of part (2) and

(d) if $\eta \in I$ then $\langle N_\nu : \nu \in \mathrm{suc}_I(\eta) \rangle$ is locally independent over N_η inside N, recall 3.22(1).

4) We say $\mathbf{n}$ is a canonical $(\mathfrak{s}, I)$-tree of models for M when clauses (a),(b),(c) of part (2) and

- (d) $\langle N_\eta : \eta \in I \rangle$ freely generates N which means: there are M_η (for $\eta \in I$) such that:
  - (α) if η is $<_I$-maximal then $M_\eta = N^{\mathbf{n}}_\eta$
  - (β) if η is not $<_I$-maximal <u>then</u> $\{M_\nu : \nu \in \text{suc}_I(\eta)\}$ is independent[5] over M_η inside $N^{\mathbf{n}}_\eta$
  - (γ) $M_\eta = \langle \cup \{M_\nu : \nu \in \text{suc}_I(\eta)\} \rangle^{\text{gn}}_{M_\eta}$ for any non-$<_I$-maximal $\eta \in I$
  - (δ) $N^{\mathbf{n}}_\eta \leq_{\mathfrak{s}} M_\eta$
  - (ε) $M_\eta \leq_{\mathfrak{s}} N_{\mathbf{n}}$ for $\eta \in I$
  - (ζ) $M_{\text{rt}(I)} = N$.

5) We say $\mathbf{n}$ an locally independent $(\mathfrak{s}, I)$-tree of models for $\bar{M}$ when $\mathbf{n} = \langle N_{\mathbf{n}}, N^{\mathbf{n}}_\eta, f^{\mathbf{n}}_\eta : \eta \in I \rangle$ and for every finite $J \leq_{\mathbf{T}_{\aleph_0}} I$, for some $(\mathfrak{s}, J)$-canonical tree $\mathbf{m}$ of models we have $N_{\mathbf{m}} \leq_{\mathfrak{s}} N$ and $(N^{\mathbf{n}}_\eta, f^{\mathbf{n}}_\eta) = (N^{\mathbf{m}}_\eta, f^{\mathbf{n}}_\eta)$ for every $\eta \in J$.
6) For $\bar{M}$ as in Definition 4.3, let $\mathbf{N}_{\aleph_0}$ [let $\mathbf{N}^{\text{st}}_{\aleph_0}$] be the class set of canonical $(\mathfrak{s}, I)$-trees of models for $\bar{M}$ with $I \in \mathbf{T}_{\aleph_0}$ (with $I \in \mathbf{T}^{\text{st}}_{\aleph_0}$). If $\bar{M}$ is not clear from the context we may write $\mathbf{N}_{\aleph_0}[\bar{M}], \mathbf{N}^{\text{st}}_{\aleph_0}[\bar{M}]$.
7) Let $\mathbf{n}_1 \leq_{\mathbf{N}_{\aleph_0}} \mathbf{n}_2$ mean that $I_{\mathbf{n}_1} \leq_{\mathbf{T}_{\aleph_0}} I_{\mathbf{n}_2}$ and $(N^{\mathbf{n}_1}_\eta, f^{\mathbf{n}_1}_\eta) = (N^{\mathbf{n}_2}_\eta, f^{\mathbf{n}_2}_\eta)$ for $\eta \in I_{\mathbf{n}_1}$ and $N_{\mathbf{n}_1} \leq_{\mathfrak{s}} N_{\mathbf{n}_2}$.
8) Let $\text{des}_\lambda(\alpha)$ be the tree I such that:

- (a) $\eta \in I$ <u>iff</u> for some ℓ we have
  - (α) $\eta = \langle (\alpha_\ell, \varepsilon_\ell) : \ell < n)$
  - (β) $\alpha > \alpha_0 > \alpha_1 > \alpha > \ldots > \alpha_{n-1}$
  - (γ) $\varepsilon_\ell < \lambda$
- (b) I is ordered by $\triangleleft$.

[5]assuming clause (γ), this is equivalent to locally independent

9) Let des(α) be defined similarly omitting the ε's.

4.5 Exercise: 1) Assume Ax(A6) and (C8) or Ax(C2)$^\oplus$. If $M \le_{\mathfrak{s}} M_\alpha \le_{\mathfrak{s}} N$ for $\alpha < \alpha_*$ and $\langle M_\alpha : \alpha \in u\rangle$ is independent over M inside N for every finite $u \subseteq \alpha_*$, see Definition 3.22(1) then $\langle M_\alpha : \alpha < \alpha_*\rangle$ is independent over M inside N.
2) Assume $\mathfrak{s}$ satisfies Ax(C8) or Ax(C2)$^\oplus$. If $I \in \mathbf{T}_{\aleph_0}$ and $\mathbf{n}$ is a canonical $(\mathfrak{s}, I)$-tree of models, $N_{\mathbf{n}} \subseteq N$ and $t \in I_{\mathbf{n}} \Rightarrow N^{\mathbf{n}}_t \le_{\mathfrak{s}} N$ then $N_{\mathbf{n}} \le_{\mathfrak{s}} N$. If $\mathbf{n}$ is a locally independent $(\mathfrak{s}, I)$-tree of models, then $\langle \cup\{N^{\mathbf{n}}_t : t \in I\rangle^{\rm gn}_{N_{\mathbf{n}}} \le_{\mathfrak{s}} N_{\mathbf{n}}$.
3) The results of V.C§4 for $(< \aleph_0)$-stable constructions (where $(< \mu)$-stable construction are defined as in Definition V.C.4.2(3)) except replacing "$|W_i| + |B_i| < \mu$" by $|W_i| < \mu$.
4) If $\mathbf{n}$ is a canonical $(\mathfrak{s}, I)$-tree of models then $t \in I \Rightarrow N^{\mathbf{n}}_t \le_{\mathfrak{s}} N_{\mathbf{n}}$.

4.6 Exercise: 1) If $\mathbf{n}_\ell$ is a canonical $(\mathfrak{s}, I_\ell)$-tree of models for $\bar M$ for $\ell = 1, 2$ and $I_{\mathbf{n}_1} \le_{\mathbf{T}_{\aleph_0}} I_{\mathbf{n}_2}$ then there is a $\le_{\mathfrak{s}}$-embedding g of $N_{\mathbf{n}_1}$ into $N_{\mathbf{n}_2}$ such that $\eta \in I_{\mathbf{n}_1} \Rightarrow f^{\mathbf{n}_2}_\eta = g \circ f^{\mathbf{n}_1}_\eta$.
2) Moreover in (1) the range of the embedding is unique.
3) If $\mathbf{n}$ is a canonical $(\mathfrak{s}, I)$-tree of models, then $N_{\mathbf{n}} = \cup\{\langle \cup\{N^{\mathbf{n}}_t : n \le J\}\rangle^{\rm gn} : J \in I \text{ is finite}\}$.
4) If $\mathfrak{s}$ satisfies Ax(C10), rigidity, then in part (1) the embedding is unique.

4.7 Claim. *Assume $\bar M = \langle M_n : n < \omega\rangle$ is $\le_{\mathfrak{s}}$-increasing (as in Definition 4.3, 4.4(2).*
1) If $\mathrm{Rang}(f) \subseteq N^1 \le_{\mathfrak{s}} N^2$ *and $n < \omega, \lambda \ge 1$ and f is a $\le_{\mathfrak{s}}$-embedding of M_n into N_1 then* $\mathrm{rk}^{\mathrm{emb},\lambda}_{\bar M}(f, N^1) \le \mathrm{rk}^{\mathrm{emb},\lambda}_{\bar M}(f, N^2)$.
2)

(a) *If $I \in \mathbf{T}_{\aleph_0}$ (e.g. $I =$ des(α)) then there is a canonical $(\mathfrak{s}, I)$-tree $\mathbf{n} = \langle N, N_\eta, f_\eta : \eta \in I\rangle$ for $\bar M$ (unique up to isomorphism, really).*
(b) *If $\mathbf{n}$ is a locally independent $(\mathfrak{s}, I)$-tree then for some $N \subseteq N_{\mathbf{n}}$ we have $\langle N, N^{\mathbf{n}}_\eta, f^{\mathbf{n}}_\eta : \eta \in I\rangle$ is a canonical $(\mathfrak{s}, I)$-tree. Actually $N = \langle \cup\{N^{\mathbf{n}}_\eta : \eta \in I\}\rangle^{\rm gn}_{N_{\mathbf{n}}}$. If Ax(C2)$^\oplus$ then $N \le_{\mathfrak{s}} N_{\mathbf{n}}$*

(c) *Assume $\chi = \chi^*_{\mathfrak{s}}$ is well defined (see Definition 3.28(3)) and $\chi \geq \chi^*_{\mathfrak{s}}$ or just LSP_χ holds. If $\chi \geq |I| + \chi^*_{\mathfrak{s}} + \Sigma\{\|M_n\|\} : n < \omega\}$ <u>then</u> in part (a), $\|N\| \leq \chi$.*

3) Assume $\lambda_* = \chi^*_{\mathfrak{s}} + \Sigma\{\|M_n\| : n < \omega\}$ and $\mathrm{rk}^{\mathrm{emb},\lambda}_{\bar{M}}(f, N^*) \geq (\|N^*\|^\lambda)^+$

(a) *it is ∞ if $\lambda \geq \lambda_*$*

(b) *if $(\forall \alpha < \lambda)(|\alpha|^{\chi^*_{\mathfrak{s}}} + \lambda_* < \lambda)$ and $n(f, \bar{M}) = 0$ for simplicity, <u>then</u> we can find (N_η, f_η) for $\eta \in {}^{\omega>}\lambda$ such that $\langle N^*, N_\eta, f_\eta : \eta \in {}^{\omega>}\lambda\rangle$ is a locally independent $(\mathfrak{s}, I)$-tree of models (for $\bar{M}$).*

4) Assume $n_{\bar{M}}(f) = 0$ and λ_*, λ are as in (3)(b), <u>then</u>:
$\mathrm{rk}^{\mathrm{emb},\lambda}_{\bar{M}}(f, N^*) \geq \alpha$ <u>iff</u> we can find a tree $\mathbf{n} = \langle N^*, N_\eta, f_\eta : \eta \in \mathrm{des}_\lambda(\alpha)\rangle$ for $\bar{M} \restriction [n, \omega)$ such that $f = f_{<>}$ hence $N_{<>} = f(M_n)$ and such that

⊙ *for every $\eta = \langle(\alpha_\ell, \varepsilon_\ell) : \ell < n\rangle \in \mathrm{des}_\lambda(\alpha)$ and $\alpha_\ell < \alpha_{\ell-1}$ (stipulating $\alpha_{-1} = \alpha$) the sequence $\langle N_{\eta\hat{}<(\alpha_n,\varepsilon)>} : \varepsilon < \lambda\rangle$ is locally independent over N_η inside N^*.*

5) Suppose $\lambda \geq |\bigcup_{n<\omega} M_n| + \chi^*_{\mathfrak{s}}$, so in particular, $\chi^*_{\mathfrak{s}}$ is well defined. If $(*)^\alpha_{\bar{M}}$ holds for every $\alpha < \lambda^+$, <u>then</u> for every ordinal α we have $(*)^\alpha_{\bar{M}}$ where:

$(*)^\alpha_{\bar{M}}$ *for some canonical $(\mathfrak{s}, \mathrm{des}(\alpha)))$-tree $\langle N, N_\eta, f_\eta : \eta \in \mathrm{des}(\alpha)\rangle$ of models for $\bar{M}$ we have $\mathrm{rk}^{\mathrm{emb}}_{\bar{M}}(f_{<>}, N) < \infty$.*

6) If $\bar{M}, \lambda$ are as in (5)'s assumption <u>then</u> $\alpha < \beta$ & $(*)^\beta_{\bar{M}} \Rightarrow (*)^\alpha_{\bar{M}}$ hence $(\forall \alpha < \lambda^+)[(*)^\alpha_{\bar{M}}] \Leftrightarrow (*)^{\lambda^+}_{\bar{M}} \Leftrightarrow \forall \alpha[(*)^\alpha_{\bar{M}}]$.
7) If $\bar{M}, \lambda$ are as in (5) and $(*)^{\lambda^+}_{\bar{M}}$ and $\mathfrak{s}$ satisfies the LSP_χ <u>then</u>

$$\chi = \chi^\lambda \Rightarrow \dot{I}(\chi, \mathfrak{K}_{\mathfrak{s}}) > \chi.$$

8) If $\lambda = 1$ <u>then</u> $\mathrm{rk}^{\mathrm{emb},\lambda}_{\bar{M}}(f, N^*) = \infty$ iff we can find $\leq_{\mathfrak{s}}$-embedding $f_n : M_n \to N^*$ for $n \in [n_{\bar{M}}(f), \omega)$ such that $f_n \subseteq f_{n+1}$ and

$f_{n(f,\bar{M})} = f$.
9) If $\mathbf{n}$ *is an independent* $(\mathfrak{s}, \mathrm{des}_\lambda(\alpha))$*-tree for* $\bar{M}$ *<u>then</u>* $\mathrm{rk}^{\mathrm{emb},\lambda}_{\bar{M}}(f_{<\lambda}, N_{\mathbf{n}})$ $\geq \alpha$.
10) If $\mathbf{n}$ *is a canonical* $(\mathfrak{s}, I)$*-tree of models for* $\bar{M}$ *and* $(\forall\alpha)[(*)^\alpha_{\bar{M}}]$ *<u>then</u> for some infinite* u *we have* $\|N_{\mathbf{n}}\| \geq \Sigma\{\eta \in I : \ell g(\eta) \in u\}$*, in fact* u *does not depend on* $\mathbf{n}$*.*

Remark. 1) On 4.7(7), see more later.
2) We can continue 4.7(9), e.g. $\mathrm{des}_\lambda(\alpha)$ can be embedded into $\mathrm{des}(\lambda\alpha)$ which can be embeded into $\mathrm{des}_\lambda(\lambda\alpha)$.

Proof. 1) By induction on the ordinal γ we prove that $\mathrm{rk}^{\mathrm{emb},\lambda}_{\bar{M}}(g, N^1) \geq \gamma \Rightarrow \mathrm{rk}^{\mathrm{emb},\lambda}_{\bar{M}}(g, N_2)$ for every $n < \omega$ and $\leq_{\mathfrak{s}}$-embedding g of M_n into N_1.
2) By induction on the ordinal γ we prove that: if $\eta \in I$, $\mathrm{Dp}(\eta) = \cup\{\mathrm{Dp}(\nu)+1 : \eta \triangleleft \nu \in I\} \leq \gamma$ then we can find a canonical $(\mathfrak{s}, I^{[\eta]})$-tree for $\bar{M}$ where $I^{[\eta]} = \{\nu \in I : \nu \trianglelefteq \eta \text{ or } \eta \trianglelefteq \nu\}$. The induction step is by $\mathrm{Ax(C2)}^+$. This suffices for the first clause, (a). The second clause (b) is by 4.5(2) using $\mathrm{Ax(C2)}^+$ and we can prove it by induction on $\mathrm{rk}_I(\nu)$ where ν is the $\trianglelefteq$-maximal member of I such that $I^{[\nu]} = I$. Also the last sentence of clause (b) and clause (c) are obvious.
3) The first clause is obvious. For the second, let $\langle \eta_\alpha : \alpha < \lambda \rangle$ list ${}^{\omega>}\lambda$ such that $\eta_\alpha \triangleleft \eta_\beta \Rightarrow \alpha < \beta$.

Now we choose $\mathbf{n}_\alpha$ by induction on $\alpha \leq \lambda$ such that

⊛ (a) $\mathbf{n}_\alpha, M_\alpha$ is a locally independent $(\mathfrak{s}, \{\eta_\beta : \eta < \alpha\})$-tree of models for $\bar{M}$ with $\beta < \alpha \Rightarrow N^{\mathbf{n}_\alpha}_{\eta_\beta} \leq_{\mathfrak{s}} N^*$
(b) $\beta < \alpha \Rightarrow \mathbf{n}_\alpha \leq_{\mathbf{N}_{\aleph_0}} \mathbf{n}_\beta$
(c) $\mathrm{rk}^{\mathrm{emb},\lambda}_{\bar{M}}(f^{\mathbf{n}_\alpha}_\eta, N^*) \geq (\|N^*\|^{\lambda_*})^+$ if $\eta \in \{\eta_\beta : \beta < \alpha\} = I_{\mathbf{n}_\alpha}$
(d) $f^{\bar{M}}_{<>} = f$
(e) $M_\alpha \leq_{\mathfrak{s}} N^*$ and $\|M_\alpha\| \leq (|\alpha| + \lambda_*)^{\chi^*_{\mathfrak{s}}} < \lambda$
(f) if $\beta < \alpha$ then $M_\beta \leq_{\mathfrak{s}} M_\alpha$
(g) if $\alpha = \beta+1$ and $\eta_\beta \in \mathrm{suc}(\eta_\gamma)$ then $\mathrm{NF}_{\mathfrak{s}}(N^{\mathbf{n}_\alpha}_\gamma, N^{\mathbf{n}_\alpha}_\beta, M_\beta, M_\alpha)$.

For the induction step assume $\alpha = \beta + 1$ and $\eta_\beta \in \operatorname{suc}_{\mathscr{T}_\alpha}(\eta_\gamma)$. By the definition of $\mathrm{rk}^{\mathrm{emb},\lambda}_{\bar{M}}$ we can find $\langle f_{\eta_\beta,\varepsilon} : \varepsilon < \lambda\rangle$ such that:

$\oplus$ (a) $f_{\eta_\beta,i}$ is an $\le_{\mathfrak{s}}$-embedding of $M_{\ell g(\eta_\beta)}$ into N^* extending $f^{\mathbf{n}_\beta}_{\eta_\gamma}$ such that $\langle f_{\eta_\beta,\varepsilon}(M_{\ell g(\eta_\beta)} : \varepsilon < \lambda\rangle$ is independent in N^*.

Now first choose M_α (possible as $\mathfrak{s}$ is $(|\alpha|+\lambda_*)^{\chi^*_{\mathfrak{s}}}$-based, see Definition 3.28(3)(a), as required such that by Definition 3.28(3)(b), for at least one $\varepsilon < \lambda$ the choice $f^{\mathbf{n}_\alpha}_{\eta_\beta} = f_{\eta_\beta,\varepsilon}$ will be O.K. (note if $\mathfrak{s} = \mathfrak{s}'(+)$ we can work for $\mathfrak{s}'$).

4) Let $\langle \eta_\alpha : \alpha < \alpha_*\rangle$ list $\mathrm{des}_\lambda(\alpha)$ such that $\eta_\alpha \triangleleft \eta_\beta \Rightarrow \alpha < \beta$.
Let

$$\begin{aligned} I_\alpha = \{<>\} \cup \{\eta : & \eta \text{ has the form } \langle(\alpha_\ell,\varepsilon_\ell) : \ell \le n\rangle, \text{ and for some} \\ & \beta < \alpha, \eta_\beta \text{ has the form} \\ & \langle(\alpha_\ell,\varepsilon_\ell) : \ell < n\rangle{}^\frown\langle(\alpha_n,\zeta)\rangle; \\ & \text{for some } \zeta < \lambda\}. \end{aligned}$$

Now we proceed as in the proof of part (3) only replacing (c) by

$(c)'$ if $\eta = \langle(\alpha_\ell,\varepsilon_\ell) : \ell < n\rangle \in I_\alpha$ and $n > 0$ then $\mathrm{rk}^{\mathrm{emb},\lambda}_{\bar{M}}(f_\eta, N^*) \ge \alpha_{n-1}$.

5) By 4.6(1),(2),(3).
6) For the first implication it is easy and the second by part (5).
7) For every $N \in K^{\mathfrak{s}}_\lambda$ let

$\circledast_1$ $\alpha^\lambda_{\bar{M}}(N) = \sup\{\mathrm{rk}^{\mathrm{emb},\lambda}_{\bar{M}}(f,N) : f$ is a $\le_{\mathfrak{s}}$-embedding of M_0 into N satisfying $\infty > \mathrm{rk}^{\mathrm{emb},\lambda}_{\bar{M}}(f,N)\}$.

Now

$\circledast_2$ $\alpha^\lambda_{\bar{M}}(N) < \chi^+$.

[Why? It is the supremum on a set of $\le \|N\|^{\|M_0\|} = \chi^\lambda = \chi < \chi^+$ ordinals $< \chi^+$.]

By the assumption and part (5) we have $(*)^{\chi^+}_{\bar{M}}$, hence for every $\alpha \in [\chi, \chi^+)$ there is a canonical $(\mathfrak{s}, \text{des}_\lambda(\alpha))$-tree $\mathbf{n}_\alpha$ of models for $\bar{M}$. Clearly

$(*)_1$ $N_{\mathbf{n}_\alpha} \in K^{\mathfrak{s}}_\chi$ if $\alpha \geq \chi \in [\chi, \chi^+)$.
[Why? Read the Definition 4.4. By clause (c) of part (2) we have $\|N_{\mathbf{n}_\alpha}\| \leq \chi$. By part (10) we get that $\|N_{\mathbf{n}_\alpha}\| \geq \chi$ when $\alpha \geq \chi$.]

$(*)_2$ $\text{rk}^{\text{emb},\lambda}_{\bar{M}}(f^{\mathbf{n}_\alpha}_{<>}, N_{\mathbf{n}_\alpha}) \geq \alpha$
[Why? By Definition 4.4, i.e. by part (9).]

$(*)_3$ $\infty >\ \text{rk}^{\text{emb},\lambda}_{\bar{M}}(f^{\mathbf{n}_\alpha}_{<>}, N_{\mathbf{n}_\alpha})$
[Why? Because $(*)^{\chi^+}_{\bar{M}}$ holds.]

$(*)_4$ $\alpha^\lambda_{\bar{M}}(N_{\mathbf{n}_\alpha}) \in [\alpha, \chi^+)$
[Why? By $(*)_2 + (*)_3 +$ the Definition of $\alpha^\lambda_{\bar{M}}(N_{\mathbf{n}_\alpha})$.]

Hence

$(*)_5$ $\{\alpha^\lambda_{\bar{M}}(N) : N \in K^{\mathfrak{s}}_\chi\}$ is an unbounded subset of χ^+.

Now $\dot{I}(\chi, K^s_{\mathfrak{s}}) \geq \chi^+$ follows.
8),9), 10) Left to the reader. $\square_{4.7}$

4.8 Claim. *Assume $Ax(A6)_{\aleph_0}$. If $\mathfrak{s}$ fails $Ax(A4)_{\aleph_0}$ as exemplified by $\bar{M} = \langle M_n : n < \omega \rangle$ <u>then</u> for every ordinal α the statement $(*)^\alpha_{\bar{M}}$ from 4.7 holds.*

Proof. By 4.7(2) there is a canonical $(\mathfrak{s}, \text{des}(\alpha))$-tree $\mathbf{n}$ of models for $\bar{M}$. If $\text{rk}^{\text{emb}}_{\bar{M}}(f^{\mathbf{n}}_{<>}) < \infty$ we are done, so assume that $\text{rk}^{\text{emb}}_{\bar{M}}(f^{\mathfrak{s}}_{<>}) = \infty$ hence by 4.7(3) we can find $\langle f_n : n < \omega \rangle$ such that f_n is a $\leq_{\mathfrak{s}}$-embedding of M_n into $N_{\mathbf{n}}$ satisfying $f_n \subseteq f_{n+1}$ for $n < \omega$. By $\text{Ax(A6)}_{\aleph_0}$ applied to $\langle f_n(M_n) : n < \omega \rangle\hat{\ }\langle N_{\mathbf{n}} \rangle$ we know that $n < \omega \Rightarrow f_n(M_n) \leq_{\mathfrak{s}} \cup\{f_k(M_k) : n < \omega\}$.

By preservation under isomorphisms we get $n < \omega \Rightarrow M_n \leq_{\mathfrak{s}} \cup\{M_k : k < \omega\}$, contradicting the choice of $\bar{M}$. So we are done. $\square_{4.8}$

4.9 Conclusion. 1) Assume $\mathfrak{s}$ satisfies $\text{Ax(A6)}_{\aleph_0}$ and $\chi^*_{\mathfrak{s}}$ is well defined.

If $\mathfrak{s}$ fails $\text{Ax(A4)}_{\aleph_0}$ then there is a counterexample $\bar{M}$ with $\lambda := 2^{\chi^*_{\mathfrak{s}}} \geq \Sigma\{\|M_n\| : n < \omega\}$ then $\chi = \chi^\lambda \Rightarrow \dot{I}(\chi, K_{\mathfrak{s}}) > \chi$.
2) Assume $(\mathfrak{t}, \mathfrak{t}^+, \chi^*_{\mathfrak{t}+})$ are like $(\mathfrak{s}, \mathfrak{s}^+, \chi_{\mathfrak{s}}, \chi^*_{\mathfrak{s}(+)})$ in 3.26 and $\mathfrak{t}^+$ fail $\text{Ax(A4)}_{\aleph_0}$ so $\chi^*_{\mathfrak{s}(+)} \leq \beth_\omega(\chi_{\mathfrak{s}})$. Then for some $n < \omega$ we have $\chi = \chi^{\beth_n(\chi_{\mathfrak{t}})} \Rightarrow \dot{I}(\chi, K_{\mathfrak{t}}) = \dot{I}(\chi, K_{\mathfrak{t}^+}) > \chi$.

Proof. 1) By clause (e) of Definition 3.28 there is a counterexample $\bar{M} = \langle M_n : n < \omega \rangle$ to Ax(A4) such that $\lambda := 2^{\chi^*_{\mathfrak{s}}} \geq \Sigma\{\|M_n\| : n < \omega\}$. By 4.8 we know that $(\forall \alpha)[(*)^\alpha_{\bar{M}}]$ from 4.7(5) holds. Hence by 4.7(2)(b) for every $\alpha < \chi^+$ there is a $(\mathfrak{s}, \text{des}(\alpha))$-tree $\mathbf{n}_\alpha$ of models for $\bar{M}$. By 4.7(7)(b) we know that $\chi = \chi^\lambda \Rightarrow \dot{I}(\chi, K_{\mathfrak{s}}) > \chi$ as required.
2) Similarly. $\square_{4.9}$

§5 Non-existence of unions Implies Non-structure

Our aim is to prove a non-structure theorem for $\mathfrak{s}$ satisfying $\text{AxFr}_1^- + \chi^*_{\mathfrak{s}}$ well defined (i.e. as gotten in §3) but $\mathfrak{s}$ not satisfying AxFr_1. The first step was done in §4, so we can assume $\text{Ax(A4)}_{\aleph_0} = \text{Ax(A4)}^*_{\aleph_0}$. We shall deal with the two variants of Ax(A4) in this section. Both deal with the union being a $\leq_{\mathfrak{s}}$-increasing chain, continuous for $k = 1$, not necessarily for $k = 2$.

For $k = 1$ we get non-structure in ZFC, and for $k = 2$ under some extra set theoretic assumption by Chapter V.C.

5.1 Definition. 1) Let Ax(A4)^k_θ be Ax(A4)_θ if $k = 2$ and Ax(A4)^*_θ if $k = 1$.
1A) Let Ax(A4)_k be $(\forall \theta)[\text{Ax(A4)}^k_\theta]$. So $(\text{A4})_2$ is (A4),$(\text{A4})_1$ is $(\text{A4})^*$.
2) Let Ax(C11)_k be Ax(11) if $k = 1$, $(\text{C11})^+$ if $k = 2$.

So assume $\theta = \theta_k$ is minimal such that Ax(A4)^k_θ fails. As $\chi^*_{\mathfrak{s}}$ is well defined clearly $\theta = \text{cf}(\theta) \leq \chi^*_{\mathfrak{s}}$ and by clause (e) of Definition 3.28(3) there is a counterexample $\bar{M}^* = \langle M^*_i : i < \theta \rangle$ such that $\chi^* = i < \theta \Rightarrow \|M^*_i\| \leq 2^{\chi^*_{\mathfrak{s}}}$; so the sequence is $\leq_{\mathfrak{s}}$-increasing and continuous if $k = 1$.

We like to construct models in $K^{\mathfrak{s}}_\mu$ from say trees of copies of M^*_i using as index sets subtrees of ${}^{\theta>}\mu$ assuming $\text{Ax(A4)}^k_{<\theta}$ and Ax(C2)^+

and sometimes $\mathrm{Ax(A6)}^+$, (C10) or $(\mathrm{C11})_k$. So for which trees $\mathscr{T}$ can we find N "freely" generated by $\{N_\eta = f_\eta(M_{\ell g(\eta)}) : \eta \in \mathscr{T}\}$ in particular when f_η is a $\leq_{\mathfrak{s}}$-embedding of $M_{\ell g(\eta)}$ into N; of course N_η is increasing with η, continuous when $k = 1$. Well, as usually $\theta > \aleph_0$, of course "no $\eta \in {}^\theta\mu$ is a θ-branch of $\mathscr{T}$" is necessary, but it is not clear whether it is sufficient (i.e. to find such $N \in K_{\mathfrak{s}}$, etc.).

Now $\mathbf{T}^{\mathrm{nc}}_\theta$, defined below is the class of trees for which we clearly can construct, allowing increasing continuous unions of length $< \theta$. It would be nice, i.e. helpful for non-structure to be able to show that for some limit $\delta < \theta$ for any $\mathscr{T} \subseteq {}^{\delta+1\geq}\mu$ which includes ${}^{\delta\geq}\mu$ (so $\mathscr{T} \in \mathbf{T}^{\mathrm{nc}}_\theta$), we can in $N = N_{\mathbf{T}}$ distinguish among the $\eta \in {}^\delta\mu$ between those for which $(\exists\alpha)(\eta\hat{}\langle\alpha\rangle \in \mathscr{T})$ and those for which $(\forall\alpha)(\eta\hat{}\langle\alpha\rangle \notin \mathscr{T})$, e.g. no automorphism g of $N_{\mathscr{T}}, g \circ f_{\eta_1} = f_{\eta_2}$ for some $\eta_1, \eta_2 \in {}^\delta\mu$ from different sets.

To prove this we tried to show that failure gives an increasing sequence $\langle f_i : i < \theta\rangle, f_i$ a $\leq_{\mathfrak{s}}$-embedding of M_i into one $N \in K_{\mathfrak{s}}$. But this has not converged to a proof.

For $k = 1$ we use another possible avenue: to imitate §4. For any $M \in K_{\mathfrak{s}}$ let $\mathscr{T}_M = \{\bar f : \bar f = \langle f_j : i < 1 + i\rangle, f_j$ is a $\leq_{\mathfrak{s}}$-embedding of M_j into M increasing with j for some $i < \theta\}$ is naturally a tree with θ levels.

Now $\mathscr{T}_M$ has no θ-branch and if we can build for it a model $N_{\mathscr{T}}$ as above, it cannot be $\leq_{\mathfrak{s}}$-embedded into M. Such argument is enough in order to prove $\dot I(\mu, K_{\mathfrak{s}}) > \mu$ for many μ's but can we find such $N_{\mathscr{T}}$? In general it is not clear, so we may try to restrict M to models of the form $M_{\mathscr{T}}$. Analyzing lead to defining a class $\mathbf{N}_\theta$ of models constructed from suitable $\mathscr{T}$'s such that the construction above leads to models which again are of this form.

How much do we care about using failure of $\mathrm{Ax(A4)}^1_\theta$ or $\mathrm{Ax(A4)}^2_\theta$? In the first case, the sequence $\langle M_i : i < \theta\rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous, so it is easier to build. But there is a price: we like to prove that failure of smoothness implies non-structure. But, see Chapter V.C we need some form of Ax(A4). Specifically, if $\mathrm{Ax(A4)} = \mathrm{Ax(A4)}_2$ holds, the non-structure from failure of smoothness is proved in ZFC, whereas if we assume only $\mathrm{Ax(A4)}_* = \mathrm{Ax(A4)}_1$, we need, e.g. the existence of a quite non-reflecting stationary set. See more in V.G§1.

Naturally if we are dealing with $\mathrm{Ax(A4)}^1_\theta$, it is reasonable to assume $\mathrm{Ax(A4)}^1_{<\theta}$. Also note that some of the claims are proved for any tree of model indexed by $\mathscr{T} \in \mathbf{T}_\theta$ not connected to $\bar{M}^*$.

5.2 Hypothesis. 1) We assume $k \in \{1,2\}, \mathfrak{s}$ satisfies AxFr_1^- and $\chi^*_{\mathfrak{s}}$ is well defined and $\mathrm{Ax(A4)}^k_{<\theta}$ where θ is regular.
2) It is natural to use $\mathrm{Ax(A6)}^+$ and Ax(C10) or just $\mathrm{Ax(C11)}_k$ but we try to mention it.

5.3 Remark. 1) We may assume $\mathfrak{s} = \mathfrak{s}_1^+, \mathfrak{s}_1$ as in §3, this saves in some arguments and clarify.
2) We have two versions.
3) We see no harm in assuming Ax(A6),$\mathrm{(C11)}^+$ all the time.
4) No harm in adding $\theta > \aleph_0$ by §4, but for $\theta = \aleph_0$ we look again at $\mathbf{N}_\theta, \mathscr{T}_\theta$.

5.4 Definition. 1) Let $\mu^k_\theta(\mathfrak{s})$ be $\min\{\Sigma\{\|M_\varepsilon\| : \varepsilon < \theta\} : \langle M_\varepsilon : \varepsilon < \theta\rangle$ a counterexample to $\mathrm{Ax(A4)}^k_\theta\}$ when $\mathrm{Ax(A4)}^k_\theta$ fail.
2) Let $\theta_k(\mathfrak{s})$ be the first cardinal θ such that $\mathrm{Ax(A4)}^k_\theta$ fails (hence $\mathrm{Ax(A4)}^k_{<\theta}$ holds and so $\theta = \mathrm{cf}(\theta), \aleph_0 \le \theta < \chi^*_{\mathfrak{s}}$).
3) let $\bar{M} = \langle M_i : i < \alpha\rangle$ be $\le_{\mathfrak{s}}$-increasing; we say that it is k-continuous <u>if</u> $k = 2$ or it is continuous and $k = 1$.

Remark. If $\theta = \theta_k(\mathfrak{s})$ then θ is also minimal such that there is a $\le_{\mathfrak{s}}$-increasing k-continuous sequence $\langle M_\varepsilon : \varepsilon < \theta\rangle$ such that $M_0 \not\le_{\mathfrak{s}} \cup\{M_\varepsilon : \varepsilon < \theta\}$.

5.5 Claim. *1) If $\theta = \theta_k(\mathfrak{s})$ <u>then</u> $\mu^k_\theta(\mathfrak{s}) \le 2^{\chi^*_{\mathfrak{s}}}$; i.e. there is a $\le_{\mathfrak{s}}$-increasing k-continuous sequence $\langle M^*_\varepsilon : \varepsilon < \theta\rangle$ such that $\|M_\varepsilon\| \le 2^{\chi^*_{\mathfrak{s}}}$ and $M_0 \not\le_{\mathfrak{s}} M^*_\theta := \cup\{M_i : i < \theta\}$.*
2) For any θ, we have $\mu^k_\theta(\mathfrak{s}) \le 2^{\chi^*_{\mathfrak{s}}+\theta}$. $\square_{5.5}$

Proof. By the properties of $\mathfrak{s}$ related to $\chi^*_{\mathfrak{s}}$. $\square_{5.5}$

5.6 Convention. Let $\bar{M}^* = \langle M^*_i : i \le \theta\rangle$ be such that

(a) if $i \leq j < \theta$ <u>then</u> $M_i \leq_{\mathfrak{s}} M_j \in K_{\mathfrak{s}}$ and moreover $M_i <_{\mathfrak{s}} M_j$

(b) if $k = 1$ then $\bar{M}^*$ is $\subseteq$-increasing continuous if $k = 2$ then just $M^*_\theta = \cup\{M^*_i : i < \theta\}$

(c) if $\neg$ AX$(A4)^k_\theta$ then $M^*_0 \nleq_{\mathfrak{s}} \cup\{M^*_i : i < \theta\}$ so is an example for the failure of Ax(A4)$^k_\theta$, fixed for this section and $\|M^*_i\| \leq \mu^k_\theta(\mathfrak{s})$.

Remark. 1) We do not use $M^*_0 \nleq_{\mathfrak{s}} M^*_\theta$ till 5.18.
2) Note that possibly $M^*_\theta = \cup\{M^*_i : i < \theta\} \notin K_{\mathfrak{s}}$.

5.7 Definition. 1) $\mathbf{T}_\theta$ is the class $\mathscr{T} = (\mathscr{T}, <)$ which satisfies:

(a) $(\mathscr{T}, <)$ is a partial order with a minimal element, the root

(b) $(\mathscr{T}, <)$ is a normal well founded tree, that is: for every $t \in \mathscr{T}$, $\mathscr{T}_{<t} = \{s : s <_I t\}$ is well ordered (so in particular linearly ordered) and if it has no last element then x is its unique least upper bound in $\mathscr{T}$.

(c) For $t \in \mathscr{T}$, otp$\{s : s <_{\mathscr{T}} t\}$ is $< \theta$ and we call it lev$_{\mathscr{T}}(t)$ moreover

(d) there is no $<_{\mathscr{T}}$-increasing sequence of length θ of members of $\mathscr{T}$.

2) $\mathscr{T}_1 \leq_{\mathbf{T}_\theta} \mathscr{T}_2$ (or $\mathscr{T}_2$ extends $\mathscr{T}_1$) <u>when</u> $\mathscr{T}_1 \subseteq \mathscr{T}_2$ are from $\mathbf{T}_\theta$ and $s <_{\mathscr{T}_2} t \in \mathscr{T}_1 \Rightarrow s \in \mathscr{T}_1$.
3) $\mathscr{T}_1 \leq^{c\ell}_{\mathbf{T}_\theta} \mathscr{T}_2$ <u>when</u> $\mathscr{T}_1 \leq_{\mathbf{T}_\theta} \mathscr{T}_2$ and if $t \in \mathscr{T}_2$ and lev$_{\mathscr{T}_2}(t)$ is a limit ordinal then $(\forall s)(s <_{I_2} t \rightarrow s \in \mathscr{T}_1) \Rightarrow t \in \mathscr{T}_1$.
4) For $\mathscr{T}_* \in \mathbf{T}_\theta$ let $\mathbf{T}_\theta[\mathscr{T}_*]$ be the class $\{\mathscr{T} \in \mathbf{T}_\theta : \mathscr{T}_* \leq_{\mathbf{T}_\theta} \mathscr{T}\}$.
5) For $\mathscr{T}_* \in \mathbf{T}_\theta$ let sp$(\mathscr{T}_1) := \{t \in \mathscr{T}$: suc$_{\mathscr{T}_1}(t)$ has at least two members$\}$.
6) For $\mathscr{T}_1 \in \mathbf{T}_\theta$ let fin$(\mathscr{T}_1) = \{\mathscr{T} : \mathscr{T} <_{\mathbf{T}_\theta} \mathscr{T}_1$ and $t \in$ sp$(\mathscr{T}_1) \Rightarrow$ suc$_{\mathscr{T}}(t)$ is finite and sp$(\mathscr{T}_1)$ is finite$\}$.
7) For $\mathscr{T}_1 \leq_{\mathbf{T}_\theta} \mathscr{T}_2$ let fin$(\mathscr{T}_2, \mathscr{T}_1) = \{\mathscr{T}_2 \restriction (\mathscr{T}_1 \cup \mathscr{T}) : \mathscr{T} \in$ fin$(\mathscr{T}_2)\}$.

Remark. Presently 5.8, 5.9, 5.10 are not used.

5.8 Definition. 1) $\mathbf{T}_\theta^{\rm nc}$ is the minimal subclass of $\mathbf{T}_\theta$, satisfying the following closure properties (a)-(e):

- (a) $(\alpha, <) \in \mathbf{T}_\theta^{\rm nc}$ for any ordinal $\alpha < \theta$
- (b) $\mathscr{T}_\theta^{\rm nc}$ is closed under isomorphism
- (c) $\mathscr{T} \in \mathbf{T}_\theta^{\rm nc}$ <u>when</u>: $\mathscr{T} \in \mathbf{T}_\theta$, and there is $A \subseteq \mathscr{T}$, a maximal set of pairwise $<_{\mathscr{T}}$-incomparable elements of $\mathscr{T}$ such that:
  - (α) $\mathscr{T}_{\leq A} := \mathscr{T} \restriction \{t \in \mathscr{T} : \neg(\exists s \in A)(s < t)\}$ is in $\mathbf{T}_\theta^{\rm nc}$ and
  - (β) for each $t \in A$, $\mathscr{T}^{[t]} = \mathscr{T} \restriction \{s \in \mathscr{T} : t \leq_{\mathscr{T}} s \text{ or } s \leq_{\mathscr{T}} t\}$ is in $\mathbf{T}_\theta^{\rm nc}$
- (d) If $\mathscr{T} \in \mathbf{T}_\theta, \mathscr{T} = \bigcup\limits_{i<\delta} A_i, \delta < \theta, [i < j < \delta \Rightarrow A_i \subseteq A_j]$, each A_i is (non-empty and) $<_{\mathscr{T}}$-downward closed and $i < \delta \Rightarrow \mathscr{T} \restriction A_i \in \mathbf{T}_\theta^{\rm nc}$ <u>then</u> $\mathscr{T} \in \mathbf{T}_\theta^{\rm nc}$
- (e) if $\mathscr{T}_1 \leq_{\mathbf{T}_\theta} \mathscr{T}_2$, every $t \in \mathscr{T}_2 \backslash \mathscr{T}_1$ is the $\leq_{\mathscr{T}_2}$-lub in $\mathscr{T}_2$ of $\{s \in \mathscr{T}_1 : s \leq_{\mathscr{T}_2} t\}$ and $\mathscr{T}_1 \in \mathbf{T}_\theta^{\rm nc}$ then $\mathscr{T}_2 \in \mathbf{T}_\theta^{\rm nc}$.

2) For $\mathscr{T}_* \in \mathbf{T}_\theta$ let $\mathbf{T}_\theta^{\rm nc}[\mathscr{T}_*]$ be the minimal subclass of $\mathbf{T}_\theta$ satisfing the following closure conditions:

- (a) $\mathscr{T}_* \in \mathbf{T}_\theta^{\rm nc}[\mathscr{T}_*]$ and if $\mathscr{T}_* \leq_{\mathbf{T}_\theta} \mathscr{T}$ and $\mathscr{T} \backslash \mathscr{T}_*$ is linearly ordered by $\leq_{\mathscr{T}}$ then $\mathscr{T} \in \mathscr{T}_*$
- $(b) - (e)$ as above.

3) For $\mathscr{T}_* \in \mathbf{T}_\theta$ we define $\mathbf{T}_\theta[\mathscr{T}_*] := \{\mathscr{T} \subseteq \mathbf{T}_\theta : \mathscr{T}_* \leq_{\mathbf{T}_\theta} \mathscr{T}\}$.

Remark. The superscript nc stands for “nice”.

5.9 Definition. 1) We define $\mathbf{T}_\theta^\gamma$ by induction on $\gamma \leq \infty$, increasing with γ:

- (a) $\mathbf{T}_\theta^0$ is the class of $\mathscr{T} \in \mathbf{T}_\theta$ isomorphic to $(\alpha, <)$ for some $\alpha < \theta$ (so satisfies clause (a) + (b) of Definition 5.8
- (b) for limit $\gamma, \mathbf{T}_\theta^\gamma = \bigcup\limits_{\beta<\theta} \mathbf{T}_\theta^\beta$, and

(*c*) for $\gamma = \beta + 1, \beta$ a non-limit even ordinal, $\mathbf{T}^\gamma_\theta$ is class of $\mathscr{T} \in \mathbf{T}_\theta$ (which are from $\mathbf{T}^\beta_\theta$ or) such that:

$(*)_1$ there is a set $A \subseteq T$ of pairwise incomparable elements of T such that:

(α) $\mathscr{T}_{\leq A} := \mathscr{T} \restriction \{t \in \mathscr{T} : \neg(\exists s \in A)[s < t]\}$ is in $\mathbf{T}^\beta_\theta$

(β) $\mathscr{T}^{[t]} = \mathscr{T} \restriction \{s \in \mathscr{T} : t \leq_\mathscr{T} s \text{ or } s \leq_\mathscr{T} t\}$ is in $\mathbf{T}^\beta_\theta$ for each $t \in A$

(*d*) for $\gamma = \beta + 1, \beta$ a limit ordinal, $\mathbf{T}^\gamma_\theta$ is the class of $\mathscr{T} \in \mathbf{T}_\theta$ (which are from $\mathbf{T}^\beta_\theta$ or) such that

$(*)_2$ $\mathscr{T} = \bigcup\limits_{i<\delta} A_i, \delta < \theta, [i < j \Rightarrow A_i \subseteq A_j], A_i$ downward closed non-empty and $i < \delta \Rightarrow \mathscr{T} \restriction A_i \in \mathbf{T}^\beta_\theta$

(*e*) if $\gamma \in \beta+1, \beta$ an odd ordinal is the class of $\mathscr{T}_2 \in \mathbf{T}_\theta$ such that for some $\mathscr{T}_1 \in \mathbf{T}^\beta_\theta, (\mathscr{T}_1, \mathscr{T}_2)$ are as in clause (e) of Definition 5.8(1) above.

Clearly

5.10 Observation. 1) $\mathbf{T}^\gamma_\theta$ increase with γ and $\mathbf{T}^{\rm nc}_\theta = \cup\{\mathbf{T}^\gamma_\theta : \gamma \text{ an ordinal}\}$.
2) $\mathbf{T}_\theta = \mathbf{T}^{\rm nc}_\theta$ if $\theta = \aleph_0$.
3) Similarly for $\mathbf{T}^\gamma_\theta[\mathscr{T}_*]$ for any $\mathscr{T}_* \in \mathbf{T}_\theta$.
4) $\mathbf{T}^{\rm nc}_\theta[\mathscr{T}_*] \subseteq \{\mathscr{T} \in \mathbf{T}_\theta : \mathscr{T}_* \leq_{\mathbf{T}_\theta} \mathscr{T}\}$.
5) If $\mathscr{T} \in \mathbf{T}_\theta$ and $\mathscr{T}_0 = \mathscr{T} \restriction \{\text{rt}(\mathscr{T})\}$ then $\text{fin}(\mathscr{T}) = \text{fin}(\mathscr{T}, \mathscr{T}_0)$.
6) If $\mathscr{T}_1 \leq_{\mathbf{T}_\theta} \mathscr{T}_2$ then $\text{fin}(\mathscr{T}_2, \mathscr{T}_1)$ is a directed (under $\leq_{\mathbf{T}_\theta}$) family of members of $\mathbf{T}_\theta$ with union (also direct limit) $\mathscr{T}_2$.

5.11 Observation. 1) $\leq_{\mathbf{T}_\theta}$ partial orders $\mathbf{T}_\theta$.
2) If $\langle \mathscr{T}_\alpha : \alpha < \delta \rangle$ is $\leq_{\mathbf{T}_\theta}$-increasing continuous sequence of members of $\mathbf{T}_\theta$ [of $\mathbf{T}^{\rm nc}_\theta$] and $\delta < \theta$ <u>then</u> $\mathscr{T}_\delta := \cup\{\mathscr{T}_\alpha : \alpha < \delta\}$ belongs to $\mathbf{T}_\theta$ [to $\mathbf{T}^{\rm nc}_\theta$] and $\alpha < \delta \Rightarrow \mathscr{T}_\alpha \leq_{\mathbf{T}_\theta} \mathscr{T}_\delta$. Similarly for $\mathbf{T}^{\rm nc}_\theta[\mathscr{T}^*]$.
3) [amalgamation] If $\mathscr{T}_0 \leq^{c\ell}_{\mathbf{T}_\theta} \mathscr{T}_\ell$ for $\ell = 1, 2$ and $\mathscr{T}_1 \cap \mathscr{T}_2 = \mathscr{T}_0$ <u>then</u> we can find $\mathscr{T}_3 \in \mathbf{T}_\theta$ such that $\mathscr{T}_\ell \leq_{\mathbf{T}_\theta} \mathscr{T}_3$ for $\ell = 1, 2$ and

$\mathscr{T}_3 = \mathscr{T}_1 \cup \mathscr{T}_2$ (equivalently $t \in \mathscr{T}_3 \Rightarrow t \in \mathscr{T}_1 \vee t \in \mathscr{T}_2$).
4) If $\mathscr{T}_1 \leq_{\mathbf{T}_\theta} \mathscr{T}_2$ and $\mathscr{T}_2 \in \mathbf{T}^{\rm nc}_\theta$ then $\mathscr{T}_1 \in \mathbf{T}^{\rm nc}_\theta$, moreover $\gamma < \infty \wedge \mathscr{T}_2 \in \mathbf{T}^\gamma_\theta \Rightarrow I_1 \in \mathbf{T}^\gamma_\theta$. Similarly for $\mathbf{T}^{\rm nc}_\theta[\mathscr{T}^*]$.
5) If $\mathscr{T} \in \mathbf{T}_\theta$ and $\gamma < \theta$, then $\mathscr{T} \in \mathbf{T}^{\rm nc}_\theta$ iff $\eta \in \mathscr{T} \cap {}^\gamma\mathrm{Ord} \Rightarrow \{\nu : \eta\hat{}\nu \in \mathscr{T}\} \in \mathbf{T}^{\rm nc}_\theta$.
6) In Definition 5.9, clause (d) without loss of generality $\langle A_i : i < \delta\rangle$ is $\subseteq$-increasing continuous.

Proof. Easy. $\square_{5.11}$

5.12 Convention. Obviously omitting k means k is as in 5.2. Several of the definitions and claims below work for both versions (from (0) and from (1)) so then we do not mention the version by writing $\mathscr{T}$ instead of $(\mathscr{T}, *)$ or $(\mathscr{T}, \bar{M}^*)$.

5.13 Definition. 0) We say that $\mathbf{n}$ is a $(\mathscr{T}, \bar{M}^*) - k$-tree of models (for $\bar{M}^*$, we may omit k as it is constant) when

- (a) $\mathbf{n} = \langle N_{\mathbf{n}}, N^{\mathbf{n}}_t, f^{\mathbf{n}}_t : t \in \mathscr{T}\rangle$ and let $\mathscr{T} = \mathscr{T}_{\mathbf{n}}$
- (b) $N^{\mathbf{n}}_t \leq_{\mathfrak{s}} N_{\mathbf{n}}$
- (c) if $s \leq_{\mathscr{T}} t$ then $N^{\mathbf{n}}_s \leq_{\mathfrak{s}} N^{\mathbf{n}}_t$ and $f^{\mathbf{n}}_s \subseteq f^{\mathbf{n}}_t$
- (d) $f^{\mathbf{n}}_t$ is an isomorphism from $M^*_{\mathrm{lev}_{\mathscr{T}}(t)}$ onto $N^{\mathbf{n}}_t$.

1) We say $\mathbf{n}$ is a $(\mathscr{T}, *) - k$-tree of models or $(\mathscr{T}, *) - k$-tree of models when:

- (a) $\mathscr{T} \in \mathbf{T}_\theta$
- (b) $\mathbf{n} = \langle N_{\mathbf{n}}, N^{\mathbf{n}}_\eta : \eta \in \mathscr{T}\rangle$
- (c) $N^{\mathbf{n}}_\eta \leq_{\mathfrak{s}} N_{\mathbf{n}}$ if $\eta \in \mathscr{T}$
- (d) $N^{\mathbf{n}}_\eta \leq_{\mathfrak{s}} N^{\mathbf{n}}_\nu$ if $\eta \leq_{\mathscr{T}} \nu$
- (e) if $k = 1$ then $N^{\mathbf{n}}_\eta = \cup\{N^{\mathbf{n}}_\nu : \nu <_{\mathscr{T}} \eta\}$ when $\eta \in \mathscr{T}$ is of limit level.

1A) We say $\mathbf{n}$ is a $\mathbf{T} - k$-tree of models when this holds for some $\mathscr{T} = \mathscr{T}_{\mathbf{n}} \in \mathbf{T}$. Omitting $\mathbf{T}$ we mean $\mathbf{T}_\theta$; so we have $(\mathbf{T}, *) - k$-trees and $(\mathscr{T}, \bar{M}^*) - k$-trees.

1B) We say $\mathbf{n}$ is locally independent <u>when</u>: if $\mathscr{T}_1 \in \text{fin}(\mathscr{T})$ <u>then</u> $\langle N^{\mathbf{n}}_\eta : \eta \in \mathscr{T}_1\rangle$ is independent inside $N_{\mathbf{n}}$, which means, (similarly to 3.22(1),(1A)): we can find $\langle M_\eta : \eta \in \mathscr{T}_1\rangle$ such that:

- (α) $N^{\mathbf{n}}_\eta \leq_{\mathfrak{s}} M_\eta \leq_{\mathfrak{s}} N_{\mathbf{n}}$
- (β) if $\neg(\exists \nu)(\eta \leq_{\mathscr{T}_1} \nu \in \text{sp}(\mathscr{T}_1))$ then $M_\eta = \cup\{N^{\mathbf{n}}_\nu : \eta \leq_{\mathscr{T}_1} \nu\}$
- (γ) if $\eta \in \text{sp}(\mathscr{T}_1)$ then $\{M_\nu : \nu \in \text{suc}_{\mathscr{T}_1}(\eta)\}$ is independent over $N^{\mathbf{n}}_\eta$ inside $N_{\mathbf{n}}$ and $M_\eta = \langle \cup\{M_\nu : \nu \in \text{suc}_{I_1}(\eta)\rangle^{\text{gn}}_{M_\eta}$
- (δ) in the remaining case $M_\eta = M_\nu$ when ν is $<_{\mathscr{T}}$-minimal such that $\eta <_{\mathscr{T}_1} \nu \in \text{sp}(\mathscr{T}_1)$.

1C) Let $\mathbf{N}^{\text{gn}}_\theta[*]$ be the class of $(\mathbf{T}_\theta, *)$-trees and $\mathbf{N}^{\text{gn}}_\theta[\bar{M}^*]$ be the class of $(\mathbf{T}_\theta, \bar{M}^*)$-trees. Writing N^{gn}_θ means it works for both cases of models. We define the two-place relation (actually partial order) $\leq = \leq_{\mathbf{N}^{\text{gn}}_\theta}$ on $\mathbf{N}^{\text{gn}}_\theta$ as follows $\mathbf{n}_1 \leq_{\mathbf{N}^{\text{gn}}_\theta} \mathbf{n}_2$ when $\mathscr{T}_{\mathbf{n}_1} \leq_{\mathbf{T}_\theta} \mathscr{T}_{\mathbf{n}_2}$ and $N_{\mathbf{n}_1} \leq_{\mathfrak{s}} N_{\mathbf{n}_2}$ and $(N^{\mathbf{n}_1}_\eta, f^{\mathbf{n}_1}_\eta) = (N^{\mathbf{n}_2}_\eta, f^{\mathbf{n}_2}_\eta)$ for every $\eta \in \mathscr{T}_{\mathbf{n}_1}$.
1D) If $\langle \mathbf{n}_\varepsilon : \varepsilon < \delta\rangle$ is $\leq_{\mathbf{N}^{\text{gn}}_\theta}$-increasing we define $\mathbf{n}_\delta := \cup\{\mathbf{n}_\varepsilon : \varepsilon < \delta\}$ by $\mathscr{T}_{\mathbf{n}_\delta} = \cup\{\mathscr{T}_{\mathbf{n}_\varepsilon} : \varepsilon < \delta\}$ and $(N^{\mathbf{n}}_{\delta_\eta}, f^{\mathbf{n}_\delta}_\eta)$ is $(N^{\mathbf{n}_\varepsilon}_\eta, f^{\mathbf{n}_\varepsilon}_\eta)$ whenever $\varepsilon < \delta$ is such that $\eta \in \mathscr{T}_{\mathbf{n}_\varepsilon}$ and $N_{\mathbf{n}_\varepsilon} = \cup\{N_{\mathbf{n}_\varepsilon} : \varepsilon < \delta\}$; note that in general maybe $\mathscr{T}_{\mathbf{n}_\delta} \notin \mathbf{T}_\theta, N_{\mathbf{n}_\delta} \notin K_{\mathfrak{s}}$.
1E) We define when $\mathbf{n} = \langle N, N_\eta, f_\eta : \eta \in \mathscr{T}\rangle = \langle N_{\mathbf{n}}, N^{\mathbf{n}}_\eta, f^{\mathbf{n}}_\eta : \eta \in \mathscr{T}_{\mathbf{n}}\rangle$ is a canonical $(\mathscr{T}, \bar{M}^*) - k$-tree of models for $\mathscr{T} \in \mathbf{T}_\theta$, i.e. for $\mathbf{n}$ such that $\mathscr{T}_{\mathbf{n}} \in \mathbf{T}_\theta$.
Now $\mathbf{n}$ is a canonical $\mathscr{T}$-tree of models if:[6]

- (i) $\mathscr{T} \in \mathbf{T}^{\text{nc}}_\theta$
- (ii) $\mathbf{n}$ is locally independent $(\mathscr{T}, \bar{M}^*) - k$-tree of models hence if $\mathscr{T}' \in \text{fin}(T)$ then $N^{\mathbf{n}}_{\mathscr{T}'} := N_{\mathbf{n}} \restriction \langle \cup\{N^{\mathbf{n}}_t : t \in \mathscr{T}'\}\rangle^{\text{gn}}_{N_{\mathbf{n}}}$ is well defined and $\leq_{\mathfrak{s}} N_{\mathbf{n}}$, really is as in part (1B)
- (iii) $N_{\mathbf{n}}$ is equal to $\cup\{N^{\mathbf{n}}_{\mathscr{T}'} : \mathscr{T}' \in \text{fin}(\mathscr{T})\}$.

1F) Let $\mathbf{T}^{\text{cn}}_\theta = \mathbf{T}^{\text{cn}}_\theta[\bar{M}^*]$ is the set of $\mathscr{T} \in \mathbf{T}$ such that there is a canonical $\mathbf{T}$-free of models for $\bar{M}^*$.
1G) We define when $\mathbf{n} = \langle N, N_\eta : \eta \in \mathscr{T}\rangle = \langle N_{\mathbf{n}}, N^{\mathbf{n}}_\eta; \eta \in \mathscr{T}_{\mathbf{n}}\rangle$ is a canonical $(\mathscr{T}, *) - k$-tree of models as in part (1E).

[6] remember $\bar{M}^*$ is from 5.6

2) Let $\mathbf{N}_\theta = \mathbf{N}_\theta^{\rm cn}$ be the class of canonical $\mathbf{T}_\theta$-trees of models so actually we define $\mathbf{N}_\theta[*] = \mathbf{N}_\theta^{\rm cn}[*]$ and $\mathbf{N}_\theta[\bar{M}^*] = \mathbf{N}_\theta^{\rm cn}[\bar{M}^*]$.
3) We define $\leq_{\mathbf{N}_\theta} = \leq_{\mathbf{N}_\theta^{\rm gn}} \restriction \mathbf{N}_\theta$, i.e. the two-place relation on $\mathbf{N}_\theta$ as follows: $\mathbf{m} \leq_{\mathbf{N}_\theta} \mathbf{n}$ means ($\mathbf{m}, \mathbf{n}$ are $\mathbf{T}_\theta$-trees of models and):

- (a) $\mathscr{T}_{\mathbf{m}} \leq_{\mathbf{T}_\theta} \mathscr{T}_{\mathbf{n}}$
- (b) $N_t^{\mathbf{m}} = N_t^{\mathbf{n}}$ or $(N_t^{\mathbf{m}}, f_t^{\mathbf{m}}) = (N_t^{\mathbf{n}}, f_t^{\mathbf{n}})$ for $t \in \mathscr{T}_{\mathbf{m}}$ according to the case
- (c) $N_{\mathbf{m}} \leq_{\mathfrak{s}} N_{\mathbf{m}}$ hence
- (d) if $\mathscr{T} \in \text{fin}(\mathscr{T}_{\mathbf{n}}, \mathscr{T}_m)$, then $N_{\mathscr{T}}^{\mathbf{m}} = \langle \cup\{N_t^{\mathbf{n}} : t \in \mathscr{T} \backslash \mathscr{T}_m\} \cup N_{\mathbf{m}} \rangle_{N_{\mathbf{n}}}^{\rm gn}$ is well defined and $\leq_{\mathfrak{s}} N_{\mathbf{m}}$.

3A) We define $\leq_{\mathbf{N}_\theta}^{c\ell}$ similarly.
4) For $\leq_{\mathbf{N}_\theta}$-increasing sequence $\langle \mathbf{n}_\varepsilon : \varepsilon < \delta \rangle$ such that $\mathscr{T} := \cup\{\mathscr{T}_{\mathbf{n}_\varepsilon} : \varepsilon < \delta\} \in \mathbf{T}_\theta$ we define $\mathbf{n} = \cup\{\mathbf{n}_\varepsilon : \varepsilon < \delta\}$ as in part (1D).
5) We write $\mathbf{m} = \mathbf{n} \restriction \mathscr{T}$ <u>when</u>: $\mathbf{n} \in \mathbf{N}_\theta, \mathscr{T} \leq_{\mathbf{T}_\theta} \mathscr{T}_{\mathbf{n}}$ and $\mathbf{m}$ is the unique canonical $\mathscr{T}$-tree of models such that $\mathbf{m} \leq_{\mathbf{N}_\theta} \mathbf{n}$; see 5.15(3) below.

5.14 Observation. 1) $\leq_{\mathbf{N}_\theta^{\rm gn}}$ partially ordered $\mathbf{N}_\theta^{\rm gn}$.
2) The "hence" in Definition 5.13(1E)(ii), 5.13(3)(d) holds.

Proof. Straight.

5.15 Claim.
1) [Ax(C10) or just Ax(C11)$_k$] If $\mathscr{T} \in \mathbf{T}_\theta^{\rm cn}$ and $\mathbf{n}, \mathbf{m}$ are canonical $(\mathscr{T}, \bar{M}^*)$-trees of models <u>then</u> $N_{\mathbf{n}}, N_{\mathbf{m}}$ are isomorphic, moreover there is an isomorphism g from $N_{\mathbf{m}}$ onto $N_{\mathbf{n}}$ such that $\eta \in \mathscr{T} \Rightarrow f_\eta^{\mathbf{n}} = g \circ f_\eta^{\mathbf{m}}$.
1A) Similarly for canonical $(\mathscr{T}, *)$-free of models so assuming g_η is an isomorphism from $N_\eta^{\mathbf{n}}$ onto $N_\eta^{\mathbf{m}}$ increasing with η.
2) [Ax(A6)$^+$] If $\mathscr{T}_1 \leq_{\mathbf{T}_\theta} \mathscr{T}_2$ and $\mathscr{T}_2 \in \mathbf{T}_\theta^{\rm cn}$ <u>then</u> $\mathscr{T}_1 \in \mathbf{T}_\theta^{\rm cn}$.
3) [Ax(A6)$^+$] Moreover in (2), if $\mathbf{n}_2$ is a canonical $\mathscr{T}_2$-tree of models <u>then</u> there is a unique canonical $\mathscr{T}_1$-tree of models $\mathbf{n}_1$ denoted by $\mathbf{n}_2 \restriction \mathscr{T}_1$ such that:

$\mathscr{T}_{\mathbf{n}_1} = \mathscr{T}_1$ *and* $N^{\mathbf{n}_2}_\eta = N^{\mathbf{n}_1}_\eta$ *or* $(N^{\mathbf{n}_2}_\eta, f^{\mathbf{n}_2}_\eta) = (N^{\mathbf{n}_1}_\eta, f^{\mathbf{n}_1}_\eta)$ *for* $\eta \in \mathscr{T}_1$

and $N_{\mathbf{n}_1}$ *is the submodel of* $N_{\mathbf{n}_2}$ *with universe* $\langle \cup\{N^{\mathbf{n}_2}_{\mathscr{T}} : \mathscr{T} \in \mathrm{fin}(\mathscr{T}_1)\}\rangle$.

4) If $\mathscr{T}_1 \le_{\mathbf{T}_\theta} \mathscr{T}_2 \in \mathbf{T}_\theta$ *and* $\mathscr{T}_1 \supseteq \mathscr{T}_2 \restriction \{\eta \in \mathscr{T}_2$*: if* $\mathrm{lev}_{\mathscr{T}_2}(\eta)$ *is a limit ordinal then* η *is not* $<_{\mathscr{T}_2}$*-maximal*$\}$ *then*

(a) $\mathscr{T}_1 \le_{\mathbf{T}_\theta} \mathscr{T}_2$
(b) $\mathscr{T}_1 \in \mathbf{T}^{\mathrm{cn}}_\theta \Leftrightarrow \mathscr{T}_2 \in \mathbf{T}^{\mathrm{cn}}_\theta$ *when* $k = 1$.

5) If $\mathscr{T}_1 \le_{\mathbf{T}_\theta} \mathscr{T}_2 \in \mathbf{T}_\theta$ *and* $\mathbf{n}_2$ *is a* $\mathscr{T}_2$*-tree of models and* $\mathscr{T}_1 \supseteq \{\eta \in \mathscr{T}_2$*: if* $\mathrm{lev}_{\mathscr{T}_2}(\eta)$ *is a limit ordinal and* $M_\eta = \cup\{M_\nu : \nu <_{\mathscr{T}_2} \eta\}$ *then* η *is not* $<_{\mathscr{T}_2}$*-maximal*$\}$ *and*

(c) *if* $\mathscr{T}_1 \in \mathbf{T}^{\mathrm{cn}}_\theta$ *and* $\mathbf{n}_2$ *is a canonical* $\mathscr{T}_2$*-tree of models and* $\mathbf{n}_1$ *defined from it then* $\mathbf{n}_1$ *is a* $\mathscr{T}_1$*-canonical tree of models and* $N_{\mathbf{n}_1} = N_{\mathbf{n}_2}$
(d) *if* $\mathbf{n}_1$ *is a canonical* $\mathscr{T}_1$*-model then we can define* $\mathbf{n}_2$ *from it.*

Proof. 1) First we prove it for finite $\mathscr{T}$ by induction on $|\mathscr{T}|$ by the uniqueness of $\mathrm{NF}_{\mathfrak{s}}$-amalgamation.

In general, by induction on m, we choose $\langle g_{\mathscr{T}'} : \mathscr{T}' \in \mathrm{fin}(\mathscr{T}), |\mathrm{sp}(\mathscr{T}'| \le m\rangle$ such that

(a) $g_{\mathscr{T}'}$ is an isomorphism from $N^{\mathbf{m}}_{\mathscr{T}'}$ onto $N^{\mathbf{m}}_{\mathscr{T}'}$
(b) if $\eta \in \mathscr{T}$ and $\mathscr{T}' = \{\nu \restriction \ell : \ell \le \ell g(\eta)\}$ then $(N^{\mathbf{m}}_{\mathscr{T}'} = N^{\mathbf{m}}_{\mathscr{T}'} = N^{\mathbf{m}}_\eta, N^{\mathbf{n}}_{\mathscr{T}'} = N^{\mathbf{n}}_\eta$ and) $g_{\mathscr{T}} = f^{\mathbf{n}}_\eta \circ (f^{\mathbf{m}}_\eta)^{-1}$
(c) if $\mathscr{T}' \subseteq \mathscr{T}''$ are both from $\mathrm{fin}(\mathscr{T})$ and $|\mathscr{T}''| \le m$ then $g_{\mathscr{T}'} \subseteq g_{\mathscr{T}''}$.

There are no problems if Ax(C10) holds. If not use $\mathrm{Ax(C11)}_k$ but it just say that this holds (so we can prove (C10) $\Rightarrow$ $(\mathrm{C11})_k$). Note alternatively if $\mathfrak{s}$ is above $\mathfrak{t}, \mathfrak{t}$ an AxFr_1 this is also clear.
2),3) Let $\mathbf{n}_2$ be a canonical $\mathscr{T}_2$-tree of models. We define $\mathbf{n}_1$ as in part (3). Now $N_{\mathbf{n}_1} \in K_{\mathfrak{s}}$ by $\mathrm{Ax(A6)}^+$.
4),5) Left to the reader. $\square_{5.15}$

5.16 Claim. $\mathbf{T}_\theta^{\rm cn}$ *has the closure properties of* $\mathbf{T}_\theta^{\rm nc}$.

Proof. Straight. $\square_{5.16}$

5.17 Claim. *[*Ax*(C10) or at least*[7] Ax*(C11)$_k$] Assume that* $\mathbf{n}_2$ *is a canonical* $\mathscr{T}_2$*-tree of models.*
1) $\mathscr{T}_1 \le^{c\ell}_{\mathbf{T}_\theta} \mathscr{T}_2$ *or* $k=1$ *and* $\mathscr{T}_1 \le_{\mathbf{T}_\theta} \mathscr{T}_2$ *and* $\mathbf{n}_1 = \mathbf{n}_2 \restriction \mathscr{T}_1$ *is defined as in 5.15(3)* <u>*then*</u> $N_{\mathbf{n}_1} \le_{\mathfrak{s}} N_{\mathbf{n}_2}$.
2) If $\mathscr{T}_{1,\ell} \le_{\mathbf{T}_\theta} \mathscr{T}_2$ *for* $\ell = 0,1,2$ *and* $\mathscr{T}_{1,1} \cap \mathscr{T}_{1,2} = \mathscr{T}_{1,0}$ *and* $\mathbf{n}_{1,\ell} = \mathbf{n}_2 \restriction \mathscr{T}_{1,\ell}$ *is defined as in 5.15(3)* <u>*then*</u> $\mathrm{NF}_{\mathfrak{s}}(N_{\mathbf{n}_{1,0}}, N_{\mathbf{n}_{1,1}}, N_{\mathbf{n}_{1,2}}, N_{\mathbf{n}_2})$.
3) If $\mathscr{T}_{1,\ell} \le_{\mathbf{T}_\theta} \mathscr{T}_2$ *for* $\ell = 1,2$ *and* $\mathscr{T}_{1,0} := \mathscr{T}_{1,1} \cap \mathscr{T}_{1,2}$, $\mathscr{T}_{1,3} = \mathscr{T}_{1,1} \cup \mathscr{T}_{1,2}$ <u>*then*</u> $\mathscr{T}_{1,\ell} \le_{\mathbf{T}_\theta} \mathscr{T}_2$ *for* $\ell = 0,3$ *and* $NF_{\mathfrak{s}}(N_{\mathbf{n}_{1,0}}, N_{\mathbf{n}_{1,1}}, N_{\mathbf{n}_{1,2}}, N_{\mathbf{n}_{1,3}})$ *and* $N_{\mathbf{n}_{1,3}} = \langle N_{\mathbf{n}_{1,1}} \cup N_{\mathbf{n}_{1,2}} \rangle^{\rm gn}_{N_{\mathbf{n}_2}}$.

Proof. 1) By 5.15(4) without loss of generality $\mathscr{T}_1$ is closed in $\mathscr{T}_2$.

Let $A = \{\eta \in \mathscr{T}_2 : \mathrm{suc}_{\mathscr{T}_2}(\eta) \nsubseteq \mathscr{T}_1\}$ and for each $\eta \in \mathscr{T}_1$ let $\mathscr{T}_{1,\eta} = \{\nu : \nu \trianglelefteq \eta$ or $\eta \triangleleft \nu \in \mathscr{T}_2 \wedge \nu \restriction (\ell g(\eta)+1) \notin \mathscr{T}_1\}$ and $\mathscr{T}_{2,\eta} = \mathscr{T}_{1,\eta} \cup \mathscr{T}_1$ and $\mathscr{T}_{0,\eta} = \{\nu : \nu \trianglelefteq \eta\}$ and $\mathbf{n}_{\ell,\eta} = \mathbf{n}_2 \restriction \mathscr{T}_{\ell,\eta}$.

For $\mathscr{T} \le_{\mathbf{T}_\theta} \mathscr{T}_2$ let $N_{\mathscr{T}} = N[\mathscr{T}] = \langle \cup\{N_{\mathscr{T}_1} : \mathscr{T}' \in \mathrm{fin}(\mathscr{T})\rangle$. So $N_{\mathscr{T}} \subseteq N_{\mathbf{n}_2}, \mathbf{n}_{\mathscr{T}} = \mathbf{n}_2 \restriction \mathscr{T}$ is well defined as in 5.15 and $\mathscr{T}' \le_{\mathbf{T}_\theta} \mathscr{T}'' \le_{\mathbf{T}_\theta} \mathscr{T}_2 \Rightarrow N_{\mathscr{T}'} \subseteq N_{\mathscr{T}''}$.

We prove in two stages: the first from a special case and then will use it in the second stage which deal with the general case.

<u>Stage 1</u>: The case $\mathscr{T}_1$ is linearly ordered, closed inside $\mathscr{T}_2$.

Obviously, with η varying on $\mathscr{T}_1$

(a) $\mathbf{n}_{\mathscr{T}_{0,\eta}} \le_{\mathbf{N}^{\rm cn}_\theta} \mathbf{n}_{\mathscr{T}_{1,\eta}}$ and $N_{\mathscr{T}_{0,\eta}} \le_{\mathfrak{s}} N_{\mathscr{T}_{1,\eta}}$.

[Why? As $N^{\mathbf{m}}_\eta \le_{\mathfrak{s}} N_{\mathbf{m}}$ for every tree of models $\mathbf{m}$ and $\eta \in \mathscr{T}_{\mathbf{m}}$.]

(b) $\mathbf{n}_{\mathscr{T}_{0,\eta}} \le_{\mathbf{N}^{\rm cn}_\theta} \mathbf{n}_{\mathscr{T}_1}$.

[7] Alternatively we require suitable continuity for $\mathrm{NF}_{\mathfrak{s}}$. Note that if $\mathfrak{s} = \mathfrak{t}^+, \mathfrak{t}$ as in, this thing are more transparent

[Why? As $\mathrm{otp}(\mathscr{T}_1)$ is $< \theta$ this means $N_\eta^{\mathbf{n}_1} \leq_{\mathfrak{s}} N_\nu^{\mathbf{n}_1}$ for appropriate $\eta \trianglelefteq \nu < \theta$.]

(*c*) There is a model N'_η such that $\mathrm{NF}_{\mathfrak{s}}(N_{\mathscr{T}_{0,\eta}}, N_{\mathscr{T}_1}, N_{\mathscr{T}_{1,\eta}}, N'_\eta)$.

[Why? As $\mathfrak{s}$ satisfies existence for $\mathrm{NF}_{\mathfrak{s}}$, pedantically we use $\mathrm{NF}_{\mathfrak{s}}$-amalgamation is disjoint.]

(*d*) There is an isomorphism f_η from N'_η onto $N_{\mathscr{T}_{2,\eta}}$ over $N_{\mathscr{T}_1} \cup N_{\mathscr{T}_{1,\eta}}$.

[Why? Both are the direct limit of essentially the same directed system.]

(*e*) $N_{\mathscr{T}_{0,\eta}} \leq_{\mathfrak{s}} N_{\mathscr{T}_1} \leq_{\mathfrak{s}} N_{\mathscr{T}_{2,\eta}}$ and $N_{\mathscr{T}_{0,\eta}} \leq_{\mathfrak{s}} N_{\mathscr{T}_{1,\eta}} \leq_{\mathfrak{s}} N_{\mathscr{T}_{2,\eta}}$.

[Why? By (c) + (d).]

(*f*) $\mathscr{T}_2 = \cup\{\mathscr{T}_{2,\eta} : \eta \in \mathscr{T}_2\}$.

[Why? As $\mathscr{T}_1$ is closed.]

(*g*) There is a model $N'_{\mathscr{T}_2} \in K_{\mathfrak{s}}$ which is the $\mathrm{NF}_{\mathfrak{s}}$-amalgamation of $\langle N_{\mathscr{T}_{2,\eta}} : \eta \in \mathscr{T}_1\rangle$ over $N_{\mathscr{T}_1}$.

[Why? By $\mathrm{Ax(C2)}^+$.]

(*h*) Without loss of generality $N'_{\mathscr{T}_2} = N_{\mathscr{T}_2} = N_{\mathbf{n}_2}$.

[Why? The first equality by (f) as in proving claim (d).]

(*i*) $N_{\mathscr{T}_1} \leq_{\mathfrak{s}} N_{\mathscr{T}_2} = N_{\mathbf{n}_2}$.

[Why? Follows.]

<u>Stage 2</u>: The general case.

Similar: for $\eta \in A$ prove $N_{\mathscr{T}_{0,\eta}} \leq_{\mathfrak{s}} N_{\mathscr{T}_2}$ as $\mathscr{T}_{0,\eta}$ has the form $\{\nu : \nu \leq_{\mathscr{T}_1} \eta\}$, so $N_{\mathscr{T}_{0,\eta}} \leq_{\mathfrak{s}} N_{\mathscr{T}_1}, N_{\mathscr{T}_{0,\eta}} \leq_{\mathfrak{s}} N_{\mathscr{T}_{1,\eta}}$, and as in Stage 1, $\mathrm{NF}_{\mathfrak{s}}(N_{\mathscr{T}_{0,\eta}}, N_{\mathscr{T}_{1,\eta}}, N_{\mathscr{T}_1}, N_{\mathscr{T}_{2,\eta}})$.

Then $N_{\mathscr{T}_2}$ is the $\mathrm{NF}_{\mathfrak{s}}$-amalgamation of $\langle N_{\mathscr{T}_{2,\eta}} : \eta \in A\rangle$ over $N_{\mathscr{T}_2}$, so by $\mathrm{Ax(C2)}^+$ we are done.

2),3) Included in the proof of part (1). $\square_{5.17}$

5.18 Theorem. *Assume Ax(A6)$^+$ and $k = 1$. We have $\dot{I}(\lambda, K_{\mathfrak{s}}) \geq \lambda^+$ when:*

(a) *Ax(A4)$^*_\theta$ fail*

(b) $\lambda = \lambda^{\chi^*_{\mathfrak{s}}}$ *and* $\alpha < \lambda \Rightarrow |\alpha|^{\mu_\theta(\mathfrak{s})^{\chi^*_s}} < \lambda$.

Proof.

$\circledast_1$ For $\mathscr{T} \in \mathbf{T}_\theta$ let $\mathscr{T}' := \mathscr{T} *^{\theta>} \lambda$ be the following partial order

(a) set of elements is $\{(t, \eta) : t \in \mathscr{T}$ and $\eta \in {}^{\mathrm{lev}_{\mathscr{T}}(t)}\lambda\}$

(b) $(t_1, \eta_1) <_{\mathscr{T}'} (t_2, \eta_2)$ iff $t_2 \leq_{\mathscr{T}} t_2 \wedge \eta_1 \triangleleft \eta_2$

$\circledast_2$ let $\mathbf{T}'_{\theta,\lambda} = \{\mathscr{T} *^{\theta>} \lambda : \mathscr{T} \in \mathbf{T}_\theta$ has cadinality $\lambda\}$

$\circledast_3$ if $\mathscr{T}' \in \mathbf{T}'_{\theta,\lambda}$ then $\mathscr{T}' \in \mathbf{T}_\theta$.

Let $K'_\lambda = \{N_{\mathbf{n}} : \mathbf{n}$ is a canonical $(\mathscr{T}, \bar{M}^*)$-tree of models for some $\mathscr{T} \in \mathbf{T}'_{\theta,\lambda}$ hence $N_{\mathbf{n}}$ has cardinality $\lambda\}$, it suffices to find $\geq \lambda^+$ pairwise non-isomorphic models among the members of K'_λ. So let $\mathbf{n}_\zeta$ be a canonical $\mathscr{T}^*_\zeta$-tree of models and $\mathscr{T}^*_\zeta \in \mathbf{T}'_{\theta,\lambda}$ hence $\|N_{\mathbf{n}_\zeta}\| = \lambda$ for $\zeta < \lambda$ and we shall find a member of K'_λ not isomorphic to any of them.

We shall prove more:

$\oplus$ there is $N \in K'_\lambda$ not $\leq_{\mathfrak{s}}$-embeddable into $N_{\mathbf{n}_\zeta}$ for $\zeta < \lambda$.

For $\zeta < \lambda$ and $i < \theta$ let

$\circledast_4$ $\mathscr{W}_{\zeta,i} = \{(f, \mathscr{T}) : f$ is a $\leq_{\mathfrak{s}}$ -embedding of M^*_i into $N_{\mathbf{n}_\zeta}$ and $\mathscr{T} \leq_{\mathbf{T}_\theta} \mathscr{T}^*_\zeta$ has cardinality $\leq \mu_\theta(\mathfrak{s})$ satisfies $\mathrm{Rang}(f) \subseteq N_{\mathbf{n}_\zeta} \restriction \mathscr{T}\}$, recalling Definition 5.13(3)(d),(5).

Let for $\zeta < \lambda, i < \theta$

$\circledast_5$ $\mathscr{T}'_{\zeta,i}$ is the set of η such that

(a) η is a sequence of length i

(b) if $j < i$ then $\eta(j)$ has the form $(\zeta_{\eta,j}, f_{\eta,j}, \mathscr{T}_{\eta,j}, \gamma_{\eta,j})$

(c) $\zeta_{\eta,j} = \zeta$

(d) $(f_{\eta,j}, \mathscr{T}_{\eta,j}) \in \mathscr{W}_{\zeta,j}$
(e) $\gamma_{\eta,j} < \lambda$
(f) $\langle f_{\eta,j} : j < i\rangle$ is $\subseteq$-increasing continuous
(g) $\langle \mathscr{T}_{\eta,j} : j < i\rangle$ is $\leq_{\mathscr{T}_\theta}$-increasing continuous
(h) if $j_1 < j_2$ then $\mathrm{NF}_{\mathfrak{s}}(f_{\eta,j_1}(M^*_{j_1})), N_{\mathbf{n}_\zeta[\mathscr{T}_{\eta,j_1}]}, f_{\eta,j_2}(M^*_{j_2}), N_{\mathbf{n}_\zeta[\mathscr{T}_{\eta,j_1}]})$.

Then we let (all ordered by $\triangleleft$)

$(*)_1$ $\mathscr{T}^+_\zeta := \cup\{\mathscr{T}'_{\zeta,i} : i < \theta\}$
$(*)_2$ $\mathscr{T}' = \cup\{\mathscr{T}'_\zeta : \zeta < \lambda\}$
$(*)_3$ $\mathscr{T}'_\zeta = \{\eta : \eta$ is not maximal in $\mathscr{T}^+_\zeta\}$
$(*)_4$ $\mathscr{T}^+$ is $\cup\{\mathscr{T}^+_\zeta : \zeta < \lambda\}$
$(*)_5$ $\mathscr{T}' = \cup\{\mathscr{T}'_\zeta : \zeta < \lambda\}$.

Obviously

$(*)_6$ $\mathscr{T}^+$ is closed under increasing unions of length $< \theta$.

The crux of the matter is

$\otimes$ $\mathscr{T}^+ \in \mathbf{T}^{\mathrm{cn}}_\theta$.

Why $\otimes$ is sufficient?

Let $\mathbf{n}^+$ be a canonical $\mathscr{T}^+$-tree of models, exists because $\mathscr{T}^+ \in \mathbf{T}^{\mathrm{cn}}_\theta$. Easily $|\mathscr{T}^+| = \|N_{\mathbf{n}^+}\|$, so $N_{\mathbf{n}^+} \in K'_\lambda$ and we should prove that $N_{\mathbf{n}^+}$ is not $\leq_{\mathfrak{s}}$-embeddable in N_ζ for $\zeta < \lambda$. So assume $\zeta < \lambda$ and let g be a $\leq_{\mathfrak{s}}$-embedding of $N_{\mathbf{n}^+}$ into $N_{\mathbf{n}_\zeta}$. We now by induction on $i < \theta$ choose $\eta_i \in \mathscr{T}'_\zeta$ of length i increasing with i such that $j < i \Rightarrow f_{\eta,j} = g \circ f^{\mathbf{n}^+}_{\eta\restriction j}$. For $i = 0$ and i limit we have no problem (and no real choice). For $i = j + 1$, we let $f_j = g \circ f^{\mathbf{n}^+}_{\eta\restriction j}$ and then by the definition of χ^*_s as $\mu_\theta(\mathfrak{s}) \geq \|M^*_j\|$ there is $\mathscr{T}'_j$ as required, i.e. $\mathscr{T}_j \leq_{\mathbf{T}_\theta} \mathscr{T}^*_\zeta, |\mathscr{T}_j| \leq \mu_\theta(\mathfrak{s}), j_i < j \Rightarrow \mathscr{T}_{\eta,j_1} \subseteq \mathscr{T}_j$. We shall choose η_i as $\eta_j\hat{\ }\langle(\zeta, f_j, \mathscr{T}_j, \gamma)\rangle$ for some $\gamma < \lambda$.

The only problematic point is the demand on $\mathrm{NF}_{\mathfrak{s}}(f^{\mathbf{n}^+}_{\eta_i}(M^*_j), N_{\mathbf{n}_\zeta[\mathscr{T}_j]}, f^{\mathbf{n}^+}_{\eta_i}(M^*_i), N_{\mathbf{n}_\zeta[\mathscr{T}_i]})$. This does not necessarily hold but it holds for all but $\le \mu_\theta(\mathfrak{s})^{\chi^*_{\mathfrak{s}}}$ of the ordinal γ because $\chi^*_{\mathfrak{s}}$ is well defined, see Definition 3.28(3), the "also ..." in clause (b).

Having carried the induction on i, we get that $\mathscr{T}'_\zeta$ has a θ-branch, contradiction (as if $\langle \eta_i : i < \theta \rangle$) is such a branch, then $g := \cup\{f_{\eta_{i+1},i} : i < \theta\}$ is an embedding of M^*_θ into $N_{\mathbf{n}_\zeta}$ such that $i < \theta \Rightarrow g(M^*_i) \le_{\mathfrak{s}} N_{\mathbf{n}_\zeta}$. By Ax(A6) this gives $i < \theta \Rightarrow g(M^*_*) \le_{\mathfrak{s}} g(M^*_\theta)$, contradiction to the choice of $\bar{M}^*$).

Why is ⊗ true?

By 5.16 it suffices to prove that $\mathscr{T}'_\zeta \in \mathbf{T}^{\rm cn}_\theta$ for a fixed $\zeta < \lambda$.

At first glance, there is a natural $\le_{\mathfrak{s}}$-embedding of $N_{\mathbf{n}^+}$ into $N_{\mathbf{n}_\zeta}$, using the $f_{\eta,j}$'s and then we can use Ax(A6)$^+$; however, their images are not locally independent. So we shall thin the tree $\mathscr{T}'_\zeta$ to $\mathscr{T}''_\zeta$ such that this will hold, i.e. we get an isomorphic subtree, for which we have an embedding. Because $\mathscr{T}'_\zeta \in \mathbf{T}'_{\theta,\lambda}, |\mathscr{T}'_\zeta| = \lambda$ hence it suffices to show

- $\odot_1$ there is $\mathscr{T}''_\zeta \le_{\mathbf{T}_\theta} \mathscr{T}'_\zeta$ isomorphic to $\mathscr{T}'_\zeta$ such that: if $\eta_1, \eta_2 \in \mathscr{T}''_\zeta$ then $(\cup\{\mathscr{T}_{\eta_1,j} : j < \ell g(\eta_1)\}) \cap (\cup\{\mathscr{T}_{\eta_2,j} : j < \ell g(\eta_2)\}) = \cup\{\mathscr{T}_{\eta_1,j} : j < \ell g(\eta_1) \text{ and } \eta_1 \restriction (j+1) = \eta_2 \restriction (j+1)\})$.

Let $\langle S_\eta : \eta \in \mathscr{T}'_\zeta \rangle$ be a sequence of pairwise disjoint subsets of λ each of cardinality λ.

As $\mathscr{T}^*_\zeta \in \mathbf{T}'_{\theta,\lambda}$ there is $\mathscr{T}^{**}_\zeta \in \mathscr{T}_\theta$ of cardinality λ such that $\mathscr{T}^*_\zeta = \mathscr{T}^{**}_\zeta * {}^{\theta>}\lambda$ so for $t \in \mathscr{T}^*_\zeta$ let $t = (s_t, \varrho_t)$ where $s_t \in \mathscr{T}^{**}_\zeta$ of level $\mathrm{lev}_{\mathscr{T}^*_\zeta}(t)$ and $\varrho_t \in {}^{\mathrm{lev}_{\mathscr{T}^*_\zeta}(t)}\lambda$.

Lastly, let

$$\mathscr{T}''_\zeta = \{\eta \in \mathscr{T}'_\zeta : \text{if } j < \ell g(\eta) \text{ and } t \in \mathscr{T}_{\eta,j} \text{ and } s <_{\mathscr{T}^*_\zeta} t,\ \mathrm{lev}_{T^*_\zeta}(s) = i+1 \le \ell g(\rho) \text{ and } s \notin t \cup \{\mathscr{T}_{\eta,j_1} : j_1 < j\} \text{ then } \varrho_s(i) \in S_{\eta \restriction j}\}.$$

$\square_{5.18}$

UNIVERSAL CLASSES: CHANGING THE FRAMEWORK – SH300G

§0 Introduction

First, we phrase what we have accomplished, when we vary $\mathfrak{s}$ on $\mathfrak{S}_1$, i.e., on frameworks satisfying enough axioms. Discarding appropriate non-structure cases we can assume $\mathfrak{s}$ is smooth and LS($\mathfrak{s}$)-based (similarly to Chapter V.C, Chapter V.D) and $\mathfrak{s}^+$ is well defined and also belongs to $\mathfrak{S}_1$, though LS($\mathfrak{s}^+$) is somewhat larger. We can continue to define $\mathfrak{s}_\alpha$ by induction on the ordinal α and there are no problems in the limit. The hope is that for suitable limit $\delta, \mathfrak{s}_\delta$ is similar enough to elementary class, because passing from $\mathfrak{s}$ to $\mathfrak{s}^+$ is like allowing one existential quantification (or we can derive $\mathfrak{s}'$ which is an "interpolation" between $\mathfrak{s}$ and $\mathfrak{s}_\delta$).

Unfortunately, discarding the non-smooth cases is not done in ZFC, we need the existence of somewhat non-reflecting sets, which is easily forced by $(< \lambda)$-complete forcing for any λ. Still as argued in Chapter N this suffices to show that in ZFC.

Note that having produced $\langle \mathfrak{s}_\alpha : \alpha < \delta \rangle$ we can try to prove that for $K_\mathfrak{s}$, the quantifier depth cannot be too large (in suitable infinitary logics). A step toward this is done in §2, we show that if we consider existential quantifiers only (no negations) then the quantifier depth is quite small, this supposedly helps in the dream above.

§1 On the family of $\mathfrak{s}$'s

1.1 Definition. 1) Let $\mathfrak{S}_1$ be the class of frameworks satisfying (AxFr$_1$), i.e. quadruple as in V.B.1.6, with $\mathfrak{s} = (K_\mathfrak{s}, \leq_{\mathfrak{K}_\mathfrak{s}}, \langle\rangle^{\mathfrak{s},\text{gn}}, \text{NF}_\mathfrak{s})$ and for simplicity $\mathfrak{s}$, LS($\mathfrak{s}$) is well defined so has the LSP(LS($\mathfrak{s}$)) and if $\lambda = \lambda^{\text{LS}(\mathfrak{s})}$ then λ^+ is $\leq_\mathfrak{s}$-inaccessible.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

1A) Let $\mathfrak{S}$ be the class of quadruples as in V.B.1.6 satisfying Ax(A0)-(A3),(B0)-(B3) and (C1).
2) Let $\mathfrak{S}^1_{\rm sm}$ be the class of $\mathfrak{s} \in \mathfrak{S}_1$ satisfying smoothness. Let $\mathfrak{S}^1_{{\rm sb},\chi}$ be the class of $\mathfrak{s} \in \mathfrak{S}^1_{\rm sm}$ which are χ-based (and $\mathrm{LS}(K_\mathfrak{s}) \le \chi$). We omit χ when this holds for some χ. So $\chi_\mathfrak{s}$ is well defined when $\mathfrak{s} \in \mathfrak{S}^1_{\rm sb}$.
3) Let $\mathfrak{S}_0$ be the class of quadruples $\mathfrak{s}$ satisfying AxFr^-_1. Let $\mathfrak{s} \in \mathfrak{S}^0_{{\rm bs},\chi}$ when $\mathfrak{s} \in \mathfrak{S}_0$ and $\chi^*_\mathfrak{s}$ is well defined $\le \chi$, see V.F.3.28(3). Let $\mathfrak{S}^0_{{\rm bs},\chi,\theta}$ be the class of $\mathfrak{s} \in \mathfrak{S}^0_{{\rm bs},\chi}$ which satisfies $\mathrm{Ax(A4)}^*_{<\theta}$ and $(< \theta)$-smoothness.
4) We define a two place relation $\le$ on $\mathfrak{S} : \mathfrak{s}_1 \le \mathfrak{s}_2$ when:

- (a) $K_{\mathfrak{s}_1} = K_{\mathfrak{s}_2}$
- (b) $M \le_{\mathfrak{s}_2} N \Rightarrow M \le_{\mathfrak{s}_1} N$
- (c) $\mathrm{NF}_{\mathfrak{s}_2} \subseteq \mathrm{NF}_{\mathfrak{s}_1}$
- (d) if $\mathrm{NF}_{\mathfrak{s}_1}(M_0, M_1, M_2, M_3)$ and $M_0 \le_{\mathfrak{s}_2} M_\ell \le_{\mathfrak{s}_2} M_3$ for $\ell = 1, 2$,
  then $\mathrm{NF}_{\mathfrak{s}_2}(M_0, M_1, M_2, M_3)$ and $\langle M_1 \cup M_2\rangle^{\rm gn}_{\mathfrak{s}_2} = \langle M_1 \cup M_2\rangle^{\rm gn}_{\mathfrak{s}_1}$
- (e) in $\mathfrak{s}_1$ we have $\langle A\rangle^{\rm gn}_M = B$ iff this holds in $\mathfrak{s}_2$, too.

1.2 Claim. *1)* $\mathfrak{S} \supseteq \mathfrak{S}_0 \supseteq \mathfrak{S}_1 \supseteq \mathfrak{S}^1_m \supseteq \mathfrak{S}^1_{\rm sb}$ *and* $\mathfrak{S}^0 \supseteq \mathfrak{S}^0_{{\rm bs},\chi}$ *(and* $\mathfrak{S}^1_{{\rm sb},\chi}$ *increase with* χ*, etc.).*
2) On the family $\mathfrak{S}, \le$ *is a partial order.*

Proof. Obvious. $\square_{1.2}$

1.3 Definition. If $\mathfrak{s}_\alpha \in \mathfrak{S}$, for $\alpha < \delta$, and $\alpha < \beta < \delta \Rightarrow \mathfrak{s}_\alpha \le \mathfrak{s}_\beta$, then the limit of $\mathfrak{s}_\delta$ of $\langle \mathfrak{s}_\alpha : \alpha < \delta\rangle$ is defined as the following quadruple:

- (a) $K_{\mathfrak{s}_\delta} = K_{\mathfrak{s}_\alpha}$ for every $\alpha < \delta$ (equivalently, some $\alpha < \delta$)
- (b) $M \le_{\mathfrak{s}_\delta} N$ iff $M \le_{\mathfrak{s}_\alpha} N$ for every $\alpha < \delta$
- (c) $\mathrm{NF}_{\mathfrak{s}_\delta}(M_0, M_1, M_2, M_3)$ iff $\bigwedge\limits_{\alpha<\delta} \mathrm{NF}_{\mathfrak{s}_\alpha}(M_0, M_1, M_2, M_3)$
- (d) $\langle A\rangle^{\rm gn}_M = B$ in $\mathfrak{s}_\delta$ iff this holds in every $\mathfrak{s}_\alpha$ $(\alpha < \delta)$, equivalently some $\alpha < \delta$.

1.4 Claim. *1) If $\mathfrak{s}_\alpha \in \mathfrak{S}_1$ for $\alpha < \delta$ and $\langle \mathfrak{s}_\alpha : \alpha \le \delta \rangle$ is increasing, $\mathfrak{s}_\delta$ as in Definition 1.3, then $\mathfrak{s} \in \mathfrak{S}_1$.*
2) If for $\alpha < \delta, \mathfrak{s}_\alpha \in \mathfrak{S}^1_{\rm sb}$, then $\mathfrak{s}_\delta$ satisfies it too. Similarly for $\mathfrak{S}^1_{\rm sm}$.
3) Similarly for $\mathfrak{S}, \mathfrak{S}_0, \mathfrak{S}^0_{{\rm bs},\chi}$ and $\mathfrak{S}^0_{{\rm bs},\chi,\theta}$ and $\mathfrak{S}_0+$ Ax(C11) + (A6)$^+$.

Proof. Check.

1.5 Claim. *1) If $\mathfrak{s} \in \mathfrak{S}^1_{\rm sb}$, i.e. $\mathfrak{s}$ satisfies* AxFr_1 *+ smoothness +* $\text{NF}_\mathfrak{s}$ *is $\chi_\mathfrak{s}$-based and $\lambda = 2^{\chi(\mathfrak{s})}$ then the following classes (A)-(E) below are* $\text{PC}_{\lambda,\chi(\mathfrak{s})}$ *(hence the class of reducts of some $\mathbb{L}_{\lambda^+,\omega}$-sentence).*
2) If in addition $\mathfrak{s} = \mathfrak{s}_1^+$ where $\mathfrak{s}_1 \in (\text{AxFr})_1$ is as in V.F§3 and $\mu = \beth_\omega(\chi_\mathfrak{s}) \ge \chi^*_\mathfrak{s}$ then the classes (A)-(E) defined below are PC_λ *(hence the class of reducts of some $\mathbb{L}_{\lambda^+,\omega}$-sentence where:*

(A) $K_\mathfrak{s}$
(B) $\{(M,N) : M \le_\mathfrak{s} N\}$
(C) $\{(M_0,M_1,M_2,M_3)$*:* $\text{NF}_\mathfrak{s}(M_0,M_1,M_2,M_3)\}$
(D) $\{(M_0,M_1,M_2,M_3)$*:* $\text{NF}_\mathfrak{s}(M_0,M_1,M_2,M_3)$ *and* $M_3 = \langle M_1 \cup M_2, M_3\rangle^{\rm gn}\}$
(E) $\{(M_0,M_1,a,M_3) : M_0 \le_\mathfrak{s} M_1 \le_\mathfrak{s} M_3$ *and* $a \in M_3$*, and* $\mathbf{tp}_\mathfrak{s}(a,M_1,M_3)$
in $\mathfrak{s}$'s sense does not fork over $M_0\}$.

Proof. Obvious.
1) Recall the "for the λ-majority $A \subseteq B$, see Definition V.C.4.8. For Clauses (A),(B) use V.C.4.9(3) or I§1; for clause (C) use V.D.1.18(1); for clause (D) use V.D.1.18(2) and for clause (E) use V.D.1.18(3).
2) Use $\mathfrak{s}^i_{\mu,\theta}$ for $\mu \ge \theta$ from $[\chi,\lambda]$. $\square_{1.5}$

1.6 Claim. *Assume $\mathfrak{s}_1 \le \mathfrak{s}_2$ are from $\mathfrak{S}$.*
1) If $M_0 \le_{\mathfrak{s}_2} M_1, A \subseteq M_1$ and[1] $\mathbf{tp}_{\mathfrak{s}_2}(M_1, M_0 \cup A) \pm_{\rm wk} M_0$ *in $\mathfrak{s}_2$ then this holds for $\mathfrak{s}_1$, too.*

[1]still well defined

2) If $M_0 \le_{\mathfrak{s}_2} M_\ell, \bar a_\ell \subseteq M_\ell$ for $\ell = 1,2$ and $\mathbf{tp}_{\mathfrak{s}_2}(\bar a_1, M_0, M_1) = \mathbf{tp}_{\mathfrak{s}_2}(\bar a_2, M_0, M_2)$ then $\mathbf{tp}_{\mathfrak{s}_1}(\bar a_1, M_0, M_1) = \mathbf{tp}_{\mathfrak{s}_1}(\bar a_2, M_0, M_2)$.

Proof. Easy.

From the previous chapters

1.7 The Successor Framework Theorem. *Assume $\mathfrak{s} \in \mathfrak{S}_1$ satisfies Ax(C11) + (A6)$^+$ and $\chi = \mathrm{LS}(\mathfrak{K}_{\mathfrak{s}})$. If for some regular $\mu = \mu^{\beth_{\omega+1}(\chi_{\mathfrak{s}})}$ we have $\dot I(\mu^+, K_{\mathfrak{s}}) < \mu^+$ then*

- (a) *$\mathfrak{s}$ is smooth and χ-based so $\chi_{\mathfrak{s}} = \mathrm{LS}(\mathfrak{s})$ is well defined*
- (b) *$\mathfrak{s}^+$ is well defined and $\chi^*_{\mathfrak{s}(+)}$ is well defined $\le \beth_\omega(\chi_{\mathfrak{s}})$*
- (c) *assume $(*)_\mu$ below then $\mathfrak{s}^+ \in \mathfrak{S}_1$ satisfies Ax(A6) + (C11)$^+$ and even moreover is χ_1-based where $\chi_1 = \beth_\omega(\chi)$*
  - $(*)_\mu$ *for any regular $\theta \le \beth_\omega(\chi_{\mathfrak{s}})$ there is a square on $S^{\mu^+}_{\le \beth_\omega(\chi)}$ or just a stationary $S \subseteq S^{\mu^+}_\theta$ not reflecting in any $\delta \in S^{\mu^+}_{\le \beth_\omega(\chi_{\mathfrak{s}})}$).*

Remark. 1) We can weaken the assumption to: for every $n < \omega$ for some regular $\mu = \mu^{\beth_n(\chi_{\mathfrak{s}})}$ we have $\dot I(\mu^+, K_{\mathfrak{s}}) < \mu^+$.

Proof. First, recall $\chi = \mathrm{LS}(\mathfrak{K}_{\mathfrak{s}})$, hence by Definition 1.1(1), we know that $\mu^\chi = \mu$ implies μ^+ is $\le_{\mathfrak{s}}$-inaccessible.

Hence by V.D.1.15, if $\mathfrak{s}$ is not $(\le \chi^+, \le \chi^+)$-smoooth or $\mathfrak{s}$ is not (χ^+, χ)-based and $\mu = \mu^\chi \ge \chi^{++}$ then (μ^+ is $\le_{\mathfrak{s}}$-inaccessible and) $\dot I(\mu^+, K_{\mathfrak{s}}) = 2^{\mu^+}$. This contradicts an assumption of 1.7 hence $\mathfrak{s}$ is $(\le \chi^+, \le \chi^+)$-smooth and is (χ^+, χ)-based. Hence by lemma V.D.1.12 we deduce: $\mathfrak{s}$ is smooth and μ-based for every $\mu \ge \chi$. So $\chi_{\mathfrak{s}}$ is well defined and $\le \chi = \mathrm{LS}(\mathfrak{K}_{\mathfrak{s}})$, so is $= \chi = \mathrm{LS}(\mathfrak{K}_{\mathfrak{s}})$ in other words $\mathfrak{s} \in \mathfrak{S}^1_{\mathrm{sb},\chi}$.

Second, for every $\theta < \beth_\omega(\chi)$, if $\mathfrak{s}$ has the $(\Lambda^{\mathfrak{s}}_\theta, (2^\theta)^+)$-order property (see V.F.2.3) and $\lambda = \lambda^\chi + \theta^{++} + \chi_{\mathfrak{s}}$ then by V.F.3.2 we have

$\dot{I}(\lambda, K_{\mathfrak{s}}) = 2^\lambda$, contradicting an assumption of 1.7. Hence the Hypothesis V.F.3.1 holds.

Third, by V.F.3.26, the framework $\mathfrak{s}^+ = \mathfrak{s}(+)$ is well defined and satisfies AxFr_1^-, so $\mathfrak{s}^+ \in \mathfrak{S}_0$ and easily $\mathfrak{s} \le \mathfrak{s}^+$ moreover $\mathfrak{s}^+ \in \mathfrak{S}^0_{\mathrm{bs},\chi_*}$ where $\chi_* = \beth_\omega(\chi_{\mathfrak{s}})$.

Fourth, toward contradiction assume $\mathfrak{s}^+$ fails $\mathrm{Ax(A4)}_*$ or smoothness then by recalling Definition V.F.3.28(2) for some regular $\theta \le \chi^*_{\mathfrak{s}(+)}$ it belongs to $\mathfrak{S}^0_{\mathrm{bs},\chi_*,\theta}$ but not to $\mathfrak{S}^0_{\mathrm{bs},\chi_*,\theta^+}$, i.e. it satisfies $\mathrm{Ax(A4)}^*_{<\theta}$ and $(<\theta)$-smoothness but fail $\mathrm{Ax(A4)}^*_\theta$ or fail θ-smoothness.

Now, assume it fails $\mathrm{Ax(A4)}^*_\theta$. If $\theta = \aleph_0$ by Theorem V.F.4.9 we get contradiction to an assumption of 1.7. If $\theta > \aleph_0$ we get a contradiction to the same assumption by V.F§5. So $\mathrm{Ax(A4)}^*_\theta$ holds hence $\mathfrak{s}$ fails θ-smoothness, so essentially we get a contradiction by Theorem V.C.2.6; well there are some cheating.

First, a minor point: there in V.C§1,§2 in the main presentation we assume Ax(A4) whereas here we have gotten only $\mathrm{Ax(A4)}_*$, but as remarked in V.C.2.7(3) this is O.K.; see more in [Sh:E54].

Second, we need there a square on $S^{\mu^+}_{<\mathrm{cf}(\chi^*_{\mathfrak{s}})}$ from $\check{I}[\mu^+]$ but we assume this.

Third, most serious, we just know that $\mathfrak{s} \in \mathfrak{S}_{\mathrm{bs},\chi_*,\theta}$ and $\theta = \mathrm{cf}(\theta) \in [\chi, \chi_*)$ but we can replace the use of stable constructions by the ones in V.F§5, see more in [Sh:E54]. So $\mathfrak{s}^+ \in \mathfrak{S}^1_{\mathrm{bs},\chi^*_{\mathfrak{s}}}$ and we are done.

In fact, careful checking shows that: there is a good stationary $S \subseteq S^{\mu^+}_\theta$ which reflects in no $\delta \in S^{\mu^+}_{<\mathrm{cf}(\chi^*_{\mathfrak{s}})}$ suffice.

§2 From large enough $\mathrm{rk}^{\mathrm{emb},2}_{\bar{M}}(f, N)$ to every ordinal

This is a continuation of V.F§4. Here we are quite close to the combinatorial theorems of Komjath Shelah [KoSh 796], in particular 2.6 is a case of it.

2.1 Hypothesis. $\mathfrak{s}$ as in $\in \mathfrak{S}^1_{\mathrm{sb}}$.

2.2 Claim. *Suppose* $\bar M^\ell = \langle M^\ell_n : n < \omega\rangle$ *for* $\ell = 0,1, M^\ell_n \le_{\mathfrak{s}} M^\ell_{n+1}, M^0_n \le_{\mathfrak{s}} M^1_n$ *and* $f : M^1_n \to N$ *is a* $\le_{\mathfrak{s}}$*-embedding.* *Then* $\mathrm{rk}^{\mathrm{emb},2}_{\bar M^1}(f,N) \le \mathrm{rk}^{\mathrm{emb},2}_{\bar M^0}(f \restriction M^0_n, N)$.

Proof. Straightforward.

2.3 Lemma. *Suppose* $\lambda \ge \chi_{\mathfrak{s}} + |\bigcup\limits_{n<\omega} M_n|$, $\mu^\lambda = \mu$, $M_n \le_{\mathfrak{s}} M_{n+1}$, $\bar M = \langle M_n : n < \omega\rangle$. *Suppose for some* $N^* \in K$ *and* f *we have* $\mathrm{rk}^{\mathrm{emb},2}_{\bar M}(f,N^*)$ *is* $< \infty$ *but* $\ge \beth_2(\mu)^+$. *Then we can find* $\bar M' = \langle M'_n : n < \omega\rangle$, $\|M'_n\| \le \lambda$, $M'_n \le M'_{n+1}$, $M_n \le_{\mathfrak{s}} M'_n$ *such that:*

$(*)^\alpha_{\bar M}$ *holds for every* α *(see V.F.4.7(5)).*

Proof. Without loss of generality $\mathrm{Dom}(f) = M_0$ and assume first $\mu > (2^\lambda)^+$. By the definition we can find $\bar f = \langle f_\eta : \eta \in \mathrm{des}(\beth_2(\mu)^+)\rangle$ such that:

(i) f_η is a $\le_{\mathfrak{s}}$-embedding of $M_{\ell g(\eta)}$ into N
(ii) $\nu \triangleleft \eta \Rightarrow f_\nu \subseteq f_\eta$.

Let χ be regular large enough such that $\beth_2(\lambda)^+$, N, $\bar f$, $\bar M$ belongs to $\mathscr{H}(\chi)$. For $\eta \in \mathrm{des}(\beth_2(M)^+)$ let the model $\mathfrak{B}_\eta$ be the minimal elementary submodel of $(\mathscr{H}(\chi), \in, <^*_\chi)$ such that

(a) $\mathfrak{B}_\eta$ includes

$$\{\mathfrak{x}_\eta\} \cup \{i : i \le \mu\} \cup \cup\{\mathfrak{B}_{\eta\restriction\ell} : \ell < \ell g(\eta)\}$$

(b) $Y \subseteq \mathfrak{B}_\eta$ & $|Y| \le \lambda \Rightarrow Y \in \mathfrak{B}_\eta$
where $\mathfrak{x}^*_\eta = \langle \eta, N, \bar f, \bar M, \lambda, M\rangle$.

Let us define a function c with domain $(\mathrm{des}(\beth_2(M)^+)$ as follows: $c(\eta)$ is the isomorphism type $\big(\mathfrak{B}_\eta, \mathfrak{B}_{\eta\restriction(\ell g\,\eta - 1)}, \ldots, \mathfrak{B}_{\eta\restriction 0}, \mathfrak{x}^*_\eta, i\big)$. So note that $|\mathrm{Rang}(c)| = \beth_1(\mu)$, $\beth_2(\mu)^{\beth_1^{(\mu)}} = \beth_2(\mu)$; hence by 2.6(2) below

there is $\mathscr{T} \subseteq \operatorname{des}(\beth_2(\mu)^+)$ closed under initial segments such that $\operatorname{rk}_{\mathscr{T}}(<>) \geq \beth_2(\mu)^+$. Now we can find $\mathfrak{B}'_0 \prec \mathfrak{B}'_1 \prec \cdots \prec \mathfrak{B}'_n \prec \cdots$ and $\mathfrak{x}'_n \in \mathfrak{B}'_n$, and for $\eta \in \mathscr{T}$ an isomorphism g_η from $\mathfrak{B}'_{\ell g(\eta)}$ onto $\mathfrak{B}_\eta$ satisfying $g^*_\eta(\mathfrak{x}'_{\ell g(\eta)}) = \mathfrak{x}_\eta, g_\eta(\mathfrak{B}'_\ell) = \mathfrak{B}_{\eta \restriction \ell}$.
Let $\mathfrak{x}'_n = (\eta'_n, N', \bar{f}', \bar{M}', \lambda', \mu')$; so without loss of generality $\bar{M}' = \bar{M}$ and let N'_n be the interpretation of N' in $\mathfrak{B}'_n$. Now $M'_n \in K$ as K is a $\mathrm{PC}_{(2^{\chi(\mathfrak{s})})^+,\omega}$-class (see (A) of 1.5). Similarly $N'_n \leq_{\mathfrak{s}} N'_{n+1}, N'_{n+1} \cap \mathfrak{B}'_n = N'_n$. Now using 1.5 we can find $M'_n (n < \omega)$ such that:

$\circledast$ (a) $M'_n \in K^{\mathfrak{s}}_\mu$

(b) $M'_n \leq_{\mathfrak{s}} N'_n$

(c) $M'_n \leq_{\mathfrak{s}} M'_{n+1}$

(d) $M'_{n+1} \bigcup\limits_{M'_n} N'_n$

(e) $\operatorname{Rang}(f_{\eta'_n}) \subseteq M'_n$.

Let f'_η be $g_\eta \restriction M'_{\ell g(\eta)}$ for $\eta \in \mathscr{T}$. We shall now prove by induction on $\alpha < \mu$ that

$$\bigotimes_\alpha \text{ if } \eta \in \mathscr{T}, \operatorname{rk}_{\mathscr{T}}(\eta) \geq \alpha \text{ then } \operatorname{rk}^{\mathrm{emb},\mu^+}_{\langle M'_n : n < \omega\rangle}(f'_\eta, N) \geq \alpha.$$

This suffices by V.F.4.7(5) as $\mu > 2^\lambda$ (except that μ's? do not extend M_n only $f^{\mathfrak{B}_n}_{\eta'_n}$ is $\leq_{\mathfrak{s}}$-embedding M_n into M'_n; by chasing arrows we can finish).

Proof of $\otimes_\alpha$. For $\alpha = 0$ or α limit this should be clear. For $\alpha = \beta + 1, \eta \in \mathscr{T}$ of length n there is $\nu \in \mathscr{T}$ such that $\eta \triangleleft \nu$, $\ell g(\nu) = n+1$ and $\operatorname{rk}_{\mathscr{T}}(\nu) \geq \beta$. Let $\varphi^\alpha_\eta\left(\bar{x}_{M'_{n+1} \setminus M'_n}, \bar{x}_{M_n}\right)$ be the formula such that $\models \varphi^\alpha_\eta\left(\bar{a}_{M'_{n+1} \setminus M'_n}, \bar{a}_{M'_n}\right)$ iff the mapping f, $f(c) = a_c$ for $c \in M'_{n+1}$ satisfies $\operatorname{rk}^{\mathrm{emb},\mu^+}_{\langle M'_k : k < \omega\rangle}(f, N) \geq \beta$. Now $\varphi^\alpha_\eta \in \mathfrak{B}'_n$ and is satisfied by $\langle g_\nu(c) : c \in M'_{n+1}\rangle$ (in the model N). The rest is as in V.F.4.8.

Now what about the case $\mu = 2^\lambda$? For each $\zeta < (2^\lambda)^+$ we can repeat the above proof with $\zeta + 1 \subseteq \mathfrak{B}_0$, and then prove $\otimes_\alpha$ for $\alpha \leq \zeta$. For $\langle M'_{\zeta,n} : n < \omega\rangle$; now one isomorphism type of $\langle M'_{\zeta,n} : n < \omega\rangle$ over

$\langle M_n, f^{\mathfrak{B}_n}_{\eta'_n} : n < \omega\rangle$ appears for $(2^\lambda)^+$ ordinals.
This is enough. $\square_{2.3}$

2.4 Definition. For a tree $\mathscr{T}$ with ω levels and $\eta \in \mathscr{T}$, we define $\mathrm{rk}_{\mathscr{T}}(\eta) = \mathrm{rk}(\eta, \mathscr{T})$ as $\cup\{\mathrm{rk}_{\mathscr{T}}(\nu) + 1 : \nu \in \mathrm{Suc}_{\mathscr{T}}(\eta)\}$ and $\mathrm{rk}_{\mathscr{T}}(\eta) = \infty$ iff there is a ω-branch of $\mathscr{T}$ to which η belongs.

For $\eta \in \mathscr{T}$ we let $\mathrm{lev}_{\mathscr{T}}(\eta) = \mathrm{otp}\{\nu : \nu < \eta\}$, $\eta \restriction \ell$ the unique $\nu <_{\mathscr{T}} \eta$ such that $\mathrm{lev}_{\mathscr{T}}(\eta) = \ell$ and $\mathrm{suc}_{\mathscr{T}}(\eta) = \{\nu : \nu <_{\mathscr{T}} \eta$ and $\rho <_{\mathscr{T}} \eta \Rightarrow \rho \le_{\mathscr{T}} \eta\}$ and let $\mathrm{rk}_{\mathscr{T}}$ be the root of T (if $\mathscr{T}$ is standard then ?).

2.5 Observation. 1) If $\mathscr{T}_1 \subseteq \mathscr{T}_2, \eta \in \mathscr{T}_1$ then $\mathrm{rk}_{T_1}(\eta) \le \mathrm{rk}_{\mathscr{T}_2}(\eta)$.
2) If $f : M_0 \to N$ is a $\le_{\mathfrak{s}}$-embedding and $M_n \le_{\mathfrak{s}} M_{n+1}$, <u>then</u> $\mathrm{rk}^{\mathrm{emb},2}_{\bar M}(f, N)$ is $\mathrm{rk}(f, \mathscr{T})$ where

$$\mathscr{T} = \{g : \text{for some } n < \omega, g \text{ is a } \le_{\mathfrak{s}} \text{-embedding of } M_n \text{ into } N\}$$

ordered by inclusion.

2.6 Claim. *Assume* $\mathrm{rk}(\eta, \mathscr{T}) \ge \lambda^+$ *(and* $\mathrm{lev}_{\mathscr{T}}(\eta) = 0$ *for simplicity) and* $\mathbf{c}$ *is a function from* $\mathscr{T}$ *to* κ.
1) If $\kappa \le \lambda$ <u>*then*</u> *for some sequence* $\bar\alpha = \langle \alpha_n : n < \omega\rangle$, *for every* n *there is* $\nu \in \mathscr{T}$, $\mathrm{lev}_{\mathscr{T}}(\nu) = n$ *and* $\bigwedge_{\ell \le n} \mathbf{c}(\nu \restriction \ell) = \alpha_\ell$.
2) If $\lambda^{(\kappa^{\aleph_0})} = \lambda$ <u>*then*</u> *for some* $\bar\alpha = \langle \alpha_n : n < \omega\rangle$ *we have* $\mathrm{rk}(\mathrm{rt}_{\mathscr{T}}, \mathscr{T}^{c,\bar\alpha}) \ge \lambda^+$ *where* $\mathscr{T}^{\mathbf{c},\bar\alpha} = \{\nu \in \mathscr{T} : c(\nu \restriction \ell) = \alpha_\ell$ *for every* $\ell \le lev_{\mathscr{T}}(\nu)\}$.

Proof. Without loss of generality $(\forall \nu \in \mathscr{T})\eta \trianglelefteq \nu$.

Part (1) is old: choose by induction on n, $\bar\alpha \restriction n$ such that for every $\gamma < \lambda^+$ there is $\nu^n_\gamma \in \mathscr{T}$ of length n such that: $\mathrm{rk}(\nu_\gamma, \mathscr{T}) \ge \gamma$ and $\bigwedge_{\ell \le n} \mathbf{c}(\nu^n_\gamma \restriction \ell) = \alpha_\ell$. for $n+1$ for every $\gamma < \lambda$ there is $\rho_\gamma \in Suc_{\mathscr{T}}(\nu^n_\gamma)$ and $\mathrm{rk}(\rho_\gamma, T) \ge \gamma$, so for some α_{n+1} the set $\{\gamma < \lambda^+ : \mathbf{c}(\rho_\gamma, \mathscr{T}) = \alpha_{n+1}\}$ is unbounded in λ^+, now define the ν^{n+1}_γ's. We now prove part (2): apply part (1) to $\mathbf{c}'$ where for $\nu \in \mathscr{T}$ we let $\mathbf{c}'(\nu) :=$

$\langle \text{rk}(\nu, \mathscr{T}^{\mathbf{c},\bar\beta}) : \bar\beta \in {}^\omega\kappa\rangle$. If $\gamma_\beta = \text{rk}(\text{rt}_{\mathscr{T}}, \mathscr{T}^{c,\bar\beta}) < \lambda^+$ for each $\nu \in T$ and $\bar\beta \in {}^\omega\kappa$, then $|\text{Rang}(\mathbf{c}')| \le \Pi\{|\gamma_\beta + 1| : \bar\beta \in {}^\omega\kappa\} \le \lambda^{\kappa^{\aleph_0}} = \lambda$.

Also $\mathbf{c}'', \mathbf{c}''(\nu) := \langle \mathbf{c}(\eta), \mathbf{c}'(\eta)\rangle$ is a function from $\mathscr{T}$ with range of cardinality $\le \lambda$.

So by part (1) we can find an ω-sequence $\langle e_n : n < \omega\rangle$, $(e_n = \langle < \gamma^n_{\bar\beta} : \bar\beta \in {}^\omega\kappa >, \alpha_n\rangle)$ such that for every n for some $\nu = \nu_n \in \mathscr{T}$, $\text{lev}_{\mathscr{T}}(\nu_n) = n$ and $\bigwedge\limits_{\ell \le n} \mathbf{c}(\nu_n \restriction \ell) = e_\ell$. So $\bigwedge\limits_n \bigwedge\limits_{\ell\le n} \mathbf{c}(\nu_n \restriction \ell) = \alpha_\ell$ and for every $\bar\beta \in {}^\omega\kappa$ we have $\bigwedge\limits_n \bigwedge\limits_{\ell \le n} \text{rk}_{\mathscr{T}}(\nu_n \restriction \ell, \mathscr{T}^{c,\bar\beta}) = \gamma^\ell_{\bar\beta}$, in particular for $\bar\alpha = \langle \alpha_n : n < \omega\rangle$, so $\text{rk}(\nu_n \restriction \ell, \mathscr{T}^{\mathbf{c},\bar\alpha}) = \gamma^\ell_{\bar\alpha}$ hence, by definition of rk, $\gamma^\ell_{\bar\alpha} > \gamma^{\ell+1}_{\bar\alpha}$, for each ℓ, contradiction to "the ordinals are well ordered"). $\square_{2.6}$

In fact in 2.3 we prove, in V.F§1's style, (we can use $\mu = 2^\lambda$ by the μ^+ in the definition of rk).

2.7 Claim. *If* $N_\eta <_{\mathfrak{s}} N$ *fo* $\eta \in \text{des}(\beth_3(\lambda)^+)$, $\lambda \ge \chi(\kappa) + \sup\limits_\eta \|N_\eta\|$, $[\nu \triangleleft \eta \Rightarrow N_\nu \le_{\mathfrak{s}} N_\eta]$ *<u>then</u> we can find* $\mathscr{T} \subseteq \text{des}(\beth_2(\lambda)^+)$ *closed under initial segments and* $M_\eta <_{\mathfrak{s}} N$ *for* $\eta \in \mathscr{T}$ *such that:*

- $\circledast$ (*a*) $\text{rk}_{\mathscr{T}}(<>) = \beth_1(\lambda)$
  - (*b*) $N_\eta \le_{\mathfrak{s}} N_\eta$
  - (*c*) $\lambda \ge \|M_\eta\|$
  - (*d*) $\langle M_\eta : \eta \in \mathscr{T}\rangle$ *is independent inside* N.

Proof. Should be clear. $\square_{2.7}$

CATEGORICITY OF AN ABSTRACT ELEMENTARY CLASS IN TWO SUCCESSIVE CARDINALS REVISITED E46

§0 Introduction

Our primary concern is:

Problem 0.1. Can we have some (not necessarily much) classification theory for an a.e.c., with no uses of even traces of compactness and only mild set theoretic assumptions?

Let me try to clarify the meaning of Problem 0.1.
What is the meaning of "mild set theoretic assumptions?" We are allowing requirements on cardinal arithmetic like GCH and weaker relatives. Preferably, assumptions like diamonds and squares and even mild large cardinals will not be used (so all is provable in ZFC, or in ZFC plus allowable assumptions).

In fact we try to continue Chapter I, where results about the number of non-isomorphic models in $\aleph_1$ and $\aleph_2$ of a sentence $\psi \in \mathbb{L}_{\omega_1,\omega}$ are obtained, replacing $\aleph_0$ by some $\lambda \geq \mathrm{LS}(\mathfrak{K})$. Now in Chapter I the theorem parallel to the present one is proved assuming $2^{\aleph_0} < 2^{\aleph_1}$ and to a large extent is provably necessary, so it is quite natural to use such assumptions here.

What is the meaning of "some classification theory?" While the dream is to have a classification theory as "full" as the one obtained in [Sh:c], we will be glad to have theorems speaking just on having few models in some cardinals or even categoricity and at least one model in others. E.g. by Chapter I if $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ satisfies $1 \leq$

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

$\dot{I}(\aleph_1, \psi) < 2^{\aleph_1}$ (and $2^{\aleph_0} < 2^{\aleph_1}$) then $\dot{I}(\aleph_2, \psi) > 0$, confirming the Baldwin conjecture.

What is the meaning of "uses traces of compactness?" For non-first order classes we cannot use the powerful compactness theorem, but there are many ways to get weak forms of it: one way is using large cardinals (compact cardinals in Makkai Shelah [MaSh 285], or just measurable cardinals as in Kolman Shelah [KlSh 362], and in [Sh 472]). Another way is to use "non-definability of well ordering" which follows from the existence of Ehrenfeucht-Mostowski models, and also from $\psi \in \mathbb{L}_{\omega_1,\omega}$ having uncountable models (used extensively in Chapter I). Our aim is to use none of those and we would like to see if any theory is left.

Above all, we hope the proofs will initiate classification theory in this case, so we hope the flavour will be one of introducing and investigating notions of a model theoretic character. Proofs of, say, a descriptive set theory character, will not satisfy this hope.

It seems to us that this goal is met.

In [Sh 576] it was proved that if $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ and an a.e.c. K with $\mathrm{LS}(\mathfrak{K}) \le \lambda$ is categorical in λ, λ^+ and neither zero nor too large number of models in λ^{++} then it has a model in λ^{+3}. This was used in II§3 to get good λ^+-frames $\mathfrak{s}$ with $\mathfrak{K}_{\mathfrak{s}} = \mathfrak{K}_{\lambda^+}$ and investigate the models of cardinality $> \lambda^{+3}$; subsequently [Sh 603] improve the results on density of minimal pairs. The non-structure side is all redone in Chapter VII and II§6 is related to the proof in [Sh 576, §8], but one was not a special case of the other. Now here + Chapter VII we do better than in [Sh 576] + [Sh 603]:

(a) this is essentially an improved version of [Sh 576], getting enough structure results so justify the existence of good λ^+-frames and here the existence of almost good λ-frames used in Chapter VII

(b) (i) we omit here the pure non-structure part [Sh 576, §1,§3]

(ii) we do it "better" in §3, §4, §8 [meaning that the set theoretic assumption is $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ only, without the additional "$\mathrm{WDmId}_{\lambda^+}$ is λ^{++}-saturated"] (*iii*) still the work in §3,§4 (a little in §2 (2.18) and in §6) rely on, i.e.

quote Chapter VII

(*c*) we would like to use: $(\mathrm{slm})_{\lambda^+}$, which means "there is a superlimit model in λ^+" instead of using categoricity in λ^+; we try to do this, that is we delay as much as possible using the assumption "$\mathfrak{K}$ is categorical in λ^+"

(*d*) in [Sh 576] at one point $\lambda > \aleph_0$ was used, this was eliminated in [Sh 603] and incorporated here

(*e*) in [Sh 576], having density of uniqueness triples (and earlier results) we succeed to define non-forking amalgamation of models; this is now done in II§6 (or see VII.8.12(4)).

In particular we shall prove (part (1) in 6.13, and part (2) in 8.4).

Theorem 0.2. $(2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}})$. *Let $\mathfrak{K}$ be an abstract elementary class with* $\mathrm{LS}(\mathfrak{K}) \leq \lambda$.
1) If $\mathfrak{K}$ is categorical in λ, λ^+ and λ^{++} then $I(\lambda^{+3}, \mathfrak{K}) > 0$.
2) Moreover, if $\mathfrak{K}$ is categorical in λ, λ^+ and
$1 \leq \dot{I}(\lambda^{+2}, \mathfrak{K}) < \mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$, *see below then:*

(*a*) $K_{\lambda^{+3}} \neq \emptyset$

(*b*) *there is an almost good λ-frame $\mathfrak{s}$ such that $\mathfrak{K}_{\mathfrak{s}} = \mathfrak{K}_\lambda$ hence $\mathfrak{K}^{\mathfrak{s}} = \mathfrak{K}_{\geq\lambda}$".*

On $\mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$, see Definition VII.0.4(6),(7), Claim VII.9.4 it is essentially equal to $2^{\lambda^{++}}$. So during our investigation we consider weaker versions of the assumptions (introduced mainly in §1, listed in 1.1). To let the reader see how we treat them we classify them by:

Group A: cheap ones: $(\mathrm{cat})_\lambda, (\mathrm{nmx})_{\lambda^+}, (\mathrm{ext})_{\lambda^{+2}}$ where cat, nmx, ext stand for categorical, has no $\leq_{\mathfrak{K}}$-maximal member, existence of member of K in this cardinal.

Group B: reasonable ones: $(\mathrm{slm})_{\lambda^+}$, $(\mathrm{amg})_\lambda$, $(\mathrm{mdn})^1_{\lambda^+}$, i.e. $1 \leq \dot{I}(\lambda^+, K) < 2^{\lambda^+}$, $(\mathrm{mdn})^2_{\lambda^{++}}$, i.e. $1 \leq \dot{I}(\lambda^{++}, K) < \mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$

where slm stands for superlimit model (exist), amg stands for amalgamation and mdn stands for medium number (of models up to isomorphism).

Group C: expensive: $(\mathrm{cat})_{\lambda^+}$.

Group D: very expensive: $K_{\lambda^{+3}} = \emptyset$.

Why do we consider $(\mathrm{cat})_\lambda$ cheap and $(\mathrm{cat})_{\lambda^+}$ expensive? Because if $M_* \in K_\lambda$ is superlimit, then $\mathfrak{K}^{[M_*]}$ is categorical in λ; for our present purposes the restriction of the class is reasonable, but we do not know to do that for λ and λ^+ simultaneously and still have an a.e.c. Also without $(\mathrm{cat})_\lambda$ we cannot start our investigation, whereas $(\mathrm{cat})_{\lambda^+}$ is needed later. Also in Chapter I the $(\mathrm{cat})_\lambda$ was "cheap".

0.3 Remark. Let us stress again
1) Our main case is the case "$2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$, $\mathrm{LS}(\mathfrak{K}) \le \lambda, \mathfrak{K}$ categorical in λ, λ^+ and $1 \le \dot{I}(\lambda^{++}, \mathfrak{K}) < 2^{\lambda^{++}}$" or just "$< \mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$ but ≥ 1", recalling $\mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$ is essentially equal to 2^{λ^+}, see VII.9.4; we conclude amalgamation in λ and λ^+.
2) We hope eventually to weaken those assumptions (in particular, $(\mathrm{cat})_{\lambda^+}$ to $(\mathrm{slm})_{\lambda^+}$), but for now we try at least to say what is needed in each proof when the proof is not too long.
3) Note: we cannot simply hope to just replace $(\mathrm{cat})_{\lambda^+}$ by $(\mathrm{slm})_{\lambda^+}$ as in the latter case the minimal types are not necessarily inevitable. They exist but there are others, hopefully analyzable by induction; maybe see in [Sh:F888].

0.4 Conjecture. 1) $K_{\lambda^{+3}} \ne \emptyset$ if: when ($2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ and)

- (a) $\mathfrak{K}$ is an a.e.c. with $\mathrm{LS}(\mathfrak{K}) \le \lambda$
- (b) $\mathfrak{K}$ has a superlimit model in λ
- (c) $\mathfrak{K}$ has a superlimit model in λ^+
- (d) $1 \le \dot{I}(\lambda^{++}, K) < \mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$.

2) Similarly at least when in addition we assume

- (e) $\dot{I}(\lambda^+, K) < 2^{\lambda^+}$.

<u>Notation</u>:

$\mathscr{D}_\lambda$ club filter

$\mathbf{tp}(a, M, N)$ means $\mathbf{tp}_{\mathfrak{K}}(a, M, N)$ when $\mathfrak{K}$ is clear from the context.

$\mathscr{S}(M)$ means $\mathscr{S}_{\mathfrak{K}}(M)$ if $\mathfrak{K}$ is clear from the context.

0.5 Remark. 1) Note that $\dot{I}(\mu, K) \le 2^{\mu+|\tau_K|}$ hence if $|\tau_{\mathfrak{K}}| \le \mu$ then $\dot{I}(\mu, \mathfrak{K}) \le 2^\mu$ so our problem is to prove the lower bound but we may "forget" this point and write $\dot{I}(\mu, \mathfrak{K}) = 2^\mu$ instead $\dot{I}(\mu, \mathfrak{K}) \ge 2^\mu$.
2) Alternatively we can use a weaker context: assume $|\tau_{\mathfrak{K}}| \le \lambda$, a very reasonable assumption.

§1 Basic properties

We use freely basic facts on an a.e.c., see the definitions and easy claims of II§1.

The aim of this section (and §2) is to show that we can start to analyze such classes and introduce some basic notions: the class $K^{3,\mathrm{na}}_\lambda$ of triples (M, N, a), minimal triples, reduced triples, and the (weak) extension property.

Given amalgamation in $\mathfrak{K}_\lambda$ (cf. I.2.7(2) or II.1.9(4A), justified in 1.4(1)) we try to define and analyze types $p \in \mathscr{S}(M) = \mathscr{S}_{\mathfrak{K}_\lambda}(M)$ for $M \in K_\lambda$. But types here (as in [Sh 300]), i.e. Chapter V.B or II§1 are not sets of formulas, they are orbital. Types may be represented by triples (M, N, a) with $M \le_{\mathfrak{K}} N$ and $a \in N \backslash M$ ordered naturally by $\le_{K^{3,\mathrm{na}}_\lambda}$; the advantage of representing types in this way is the existence of upper bounds even union of increasing chains. We look for "nice" types (i.e. triples) and try to prove mainly the weak extension property. The major claim (for this section) says that if $\mathscr{S} \subseteq \mathscr{S}_{\mathfrak{K}_\lambda}(N)$ has cardinality $> \lambda^+$ then we can find $N^* <_{\mathfrak{K}_*} N_i$ for $i < \lambda^{++}$ such that the set $\{\mathbf{tp}_{\mathfrak{K}}(a, N^*, N_i) : a \in N_i \backslash N^*\}$ are non-empty and pairwise disjoint and more.

1.1 Hypothesis. 1) $\mathfrak{K}$ is an abstract elementary class with $\mathrm{LS}(\mathfrak{K}) \le \lambda$.
2) The following may be assumed (but mentioned explicitly, always $\mu \ge \lambda$, mostly $\mu = \lambda, \lambda^+, \lambda^{++}$)

(cat)$_\mu$ means: $\mathfrak{K}$ is categorical in μ, i.e. $\dot{I}(\mu, \mathfrak{K}) = 1$

(ext)$_\mu$ means: there is $M \in K_\mu$, i.e. $K_\mu \neq \emptyset$

(nmx)$_\mu$ means: $(K_\mu, \leq_{\mathfrak{K}})$ has no maximal member and is non-empty

(slm)$_\mu$ means: $\mathfrak{K}_\mu$ has a superlimit model (so of cardinality μ, see Definition II.1.13(1))

(lsl)$_\mu$ means: $\mathfrak{K}_\mu$ has a locally superlimit model

(mdn)$^1_\mu$ means: $1 \leq \dot{I}(\mu, K) < 2^\mu$

(mdn)$^2_\mu$ means: $1 \leq \dot{I}(\mu, K) < \mu_{\rm unif}(\mu, 2^{<\mu})$

(amg)$_\mu$ means: $\mathfrak{K}_\mu$ has the amalgamation property (so in μ)

(unv)$_\mu$ means: there is a $\leq_{\mathfrak{K}}$-universal $M \in K_\mu$

(stb)$_\mu$ means: stability in μ of $\mathfrak{K}$, i.e. $M \in \mathfrak{K}_\mu \Rightarrow |\mathscr{S}_{\mathfrak{K}}(M)| \leq \mu$

(mst)$_\mu$ means: stability for minimal types

(jep)$_\mu$ means: $\mathfrak{K}_\mu$ has the joint embedding property

(iev)$_\lambda$ means: some triple in $K^{3,\rm na}_\lambda$ is minimal and inevitable (will be defined in Definition 5.2)

(iev)$^-_\lambda$ means: for every $M \in K_\lambda$ the set $\mathscr{S}^{\min}_{\mathfrak{K}_\lambda}(M) := \{p \in \mathscr{S}_{\mathfrak{K}_\lambda}(M) : p$ is minimal$\}$ is an inevitable set (of types)

(dmn)$_\lambda$ means: density of minimal types, i.e. the minimal triples $(M, N, a) \in K^{3,\rm na}_\lambda$ are dense in $(K^{3,\rm na}_\lambda, \leq_{\rm na})$

1.2 Definition. 1) $K^{\rm slm} = \{M \in K : M$ is superlimit$\}, K^{\rm lsl} = \{M : M$ is locally superlimit$\}$, so $K^{\rm slm}_\mu = K^{\rm slm} \cap K_\mu$ and, of course, $\leq_{\mathfrak{K}^{\rm slm}_\mu} = \leq_{\mathfrak{K}} \restriction K^{\rm slm}_\mu$.
2) For locally (or just pseudo) superlimit $M \in \mathfrak{K}_\mu$ let $K_{[M]} = \{N \in K_\mu : N \cong M\}$ and let $\mathfrak{K}_{[M]} = (K_{[M]}, \leq_{\mathfrak{K}} \restriction K_{[M]})$ and $\mathfrak{K}^{[M]} = \mathfrak{K}^{\rm up}_{[M]}$; see Definition II.1.25, Claim II.1.26; or IV.0.5, IV.0.6.

Easy properties are

1.3 Claim. *Let* $\mu \geq \lambda$.
1) (slm)$_\mu$ $\Rightarrow$ (lsl)$_\mu$ $\Rightarrow$ (ext)$_\mu$ *and* (slm)$_\mu$ $\Rightarrow$ (unv)$_\lambda$ $\wedge$ (nmx)$_\mu$ *and* $\mathfrak{K}^{\rm slm}_\mu \subseteq \mathfrak{K}^{\rm lsl}_\mu$ *and* $\mathfrak{K}^{\rm slm}_\mu$ *is categorical.*
2) If (jep)$_\mu$ <u>*then*</u>*:* $K^{\rm lsl}_\mu = K^{\rm slm}_\mu$ *hence has at most one member (up to isomorphism) so* $M \in K_\mu$ *is locally superlimit iff it is (globally)*

superlimit, i.e. $(\mathrm{slm})_\mu \Leftrightarrow (\mathrm{lsl})_\mu$.
3) $(\mathrm{cat})_\mu \Rightarrow (\mathrm{mdn})^2_\mu \Rightarrow (\mathrm{mdn})^1_\mu \Rightarrow (\mathrm{ext})_\mu$ *and* $(\mathrm{nmx})_\mu \Rightarrow (\mathrm{ext})_\mu$.
4) $(\mathrm{nmx})_\mu \Rightarrow (\mathrm{ext})_{\mu^+} \Rightarrow <_{\mathfrak{K}_\lambda} \neq \emptyset$ *(the last one is equivalent to* $K^{3,\mathrm{na}}_\lambda \neq \emptyset$*, see 1.14(0)(b) below).*
5) If $(\mathrm{cat})_\mu$ *<u>then</u>* $(\mathrm{nmx})_\mu \Leftrightarrow (\mathrm{ext})_{\mu^+} \Leftrightarrow <_{\mathfrak{K}_\lambda} \neq \emptyset \Rightarrow (\mathrm{slm})_\mu \Leftrightarrow (\mathrm{lsl})_\mu$ *also* $(\mathrm{jep})_\lambda$.
6) If $M \in K^{\mathrm{slm}}_\mu$ *or even just* $M \in K^{\mathrm{lsl}}_\mu$ *<u>then</u>* $\mathfrak{K}^{[M]}_\mu = \mathfrak{K}_{[M]}$ *and* $\mathfrak{K}^{[M]}$ *is an a.e.c. with* $\mathrm{LS}(\mathfrak{K}^{[M]}) = \mu$ *and* $\mathfrak{K}^{[M]}$ *is categorical in* μ *and* $\mathfrak{K}^{[M]}_{\mu^+} \neq \emptyset$*. Also* $\mathfrak{K}^{[M]}$ *has amalgamation in* μ *<u>iff</u>* M *is an amalgamation base in* $\mathfrak{K}$ *(i.e.,* $\mathfrak{K}_\mu$*) recalling that this means that: if* $M \leq_{\mathfrak{K}_\mu} M_\ell$ *for* $\ell = 1, 2$ *then for some* $N \in K_\mu$ *there are* $\leq_{\mathfrak{K}}$*-embedding* f_ℓ *of* M_ℓ *into* N *for* $\ell = 1, 2$ *such that* $f_1 \restriction M = f_2 \restriction M$.

Proof. Easy, but
4) Assume $(\mathrm{nmx})_\mu$ and we shall prove $(\mathrm{ext})_{\mu^+}$. By the definition of $(\mathrm{nmx})_\mu$ there is $M_0 \in K_\mu$, we try to choose $M_\alpha \in K_\lambda$ by induction on $\alpha < \mu^+$, which is $<_{\mathfrak{K}}$-increasing continuous. For $\alpha = 0$ the model M_α has already been chosen. For α limit take union and for $\alpha = \beta + 1$ use $(\mathrm{nmx})_\mu$. So $M = \cup\{M_\alpha : \alpha < \mu^+\}$ is as required. For the other implication read the definitions in 1.1(2).

So $(\mathrm{nmx})_\mu \Rightarrow (\mathrm{ext})_{\mu^+}$ and $(\mathrm{ext})_{\mu^+} \Rightarrow <_{\mathfrak{K}_\lambda} \neq \emptyset$ because $\mathrm{LS}(\mathfrak{K}) \leq \lambda$ by 1.1(1) recalling $\lambda \leq \mu$ by the claim's assumption.
5) Note $(\mathrm{cat})_\lambda$ does imply that $(\mathrm{jep})_\mu$ and $<_{\mathfrak{K}_\mu} \neq \emptyset \Rightarrow (\mathrm{nmx})_\mu \Rightarrow (\mathrm{slm})_\mu$ and use earlier parts.
6) See Definition II.1.25 and Claim II.1.26 except amalgamation which is straight. $\square_{1.3}$

1.4 Claim. *1) Assume* $(\mathrm{cat})_\lambda$ *and* $(\mathrm{mdn})^1_{\lambda^+} \vee (\mathrm{unv})_{\lambda^+}$ *and* $2^\lambda < 2^{\lambda^+}$.
<u>Then</u> the λ*-a.e.c.* $\mathfrak{K}_\lambda$ *has amalgamation (i.e.* $(\mathrm{amg})_\lambda$*).*
2) In part (1) we can replace the assumption $(\mathrm{mdn})^1_{\lambda^+} \vee (\mathrm{unv})_{\lambda^+}$ *with there a set of* $< \mu_{\mathrm{wd}}(\lambda^+)$ *members of* K_{λ^+} *which is* $\leq_{\mathfrak{K}}$*-universal, i.e. any member of* K_{λ^+} *is* $\leq_{\mathfrak{K}}$*-embeddable into at least one of them.*
3) Assume $(\mathrm{ext})_\lambda$*. Then* $(\mathrm{nmx})_\lambda$ *<u>iff</u> every* $M_0 \in K_{\lambda^+}$ *has a* $\leq_{\mathfrak{K}}$*-extension* M *of cardinality* λ^+*. Also for every* $M \in K_\lambda$ *has a* $\leq_{\mathfrak{K}}$*-extension in* K_{λ^+} *or has a* $\leq_{\mathfrak{K}}$*-maximal extension in* K_λ*.*

4) If $\dot{I}(\lambda^{++}, K) = 0$, *i.e.* $\neg(\mathrm{ext})_{\lambda^{++}}$ *then any* $M_0 \in K_{\lambda^+}$ *has a* $\leq_{\mathfrak{K}}$*-extension* $M \in K_{\lambda^+}$ *which is* $\leq_{\mathfrak{K}}$*-maximal.*
4A) If $\neg(\mathrm{ext})_\mu$ *then any* $M_0 \in K_{<\mu}$ *has a* $\leq_{\mathfrak{K}}$*-extension which is* $<_{\mathfrak{K}}$*-maximal.*
5) Assume $(\mathrm{amg})_\lambda$*; if* $M \in K_{\lambda^+}$ *is* $<_{\mathfrak{K}}$*-maximal then*

(*i*) *M is* $(\mathbb{D}_{\mathfrak{K}_\lambda}, \lambda^+)$*-homogeneous for* $\mathfrak{K}_{\geq\lambda}$*, i.e. above* λ*, (I.2.3(1B))*
(*ii*) *M is saturated above* λ *(see Definition II.1.13(2)).*

6) Assume $\neg(\mathrm{ext})_{\lambda^{++}}$ *and* $(\mathrm{amg})_\lambda$*. If* $N \in K_\lambda$ *then* $|\mathscr{S}(N)| \leq \lambda^+$*.*
7) If $M \in K_{\lambda^+}$ *is locally superlimit and* $N \in K_{\lambda^+}$ *is saturated above* λ *then* $M \cong N$*.*

Proof. 1) If amalgamation fails in K_λ, then say as exemplified by $M_0, M_1, M_2 \in K_\lambda$ by the definition $M_0 <_{\mathfrak{K}_\lambda} M_\ell$ for $\ell = 1, 2$, so $<_{\mathfrak{K}_\lambda} \neq \emptyset$ hence the assumption $(\mathrm{cat})_\lambda$ by 1.3(5) we have $(\mathrm{nmx})_\lambda$ so by 1.3(5) any $M \in K_\lambda$ is a superlimit for $\mathfrak{K}_\lambda$ and by the definitions, is not an amalgamation base hence by I.3.8 recalling that we assume $2^\lambda < 2^{\lambda^+}$, we get $\dot{I}(\lambda^+, K) = 2^{\lambda^+}$, contradiction to the assumption $(\mathrm{mdn})^1_{\lambda^+}$. Hence $(\mathrm{amg})_\lambda$ holds.
2) Similar proof.
3) By the proof of 1.3(4).
4) We try to repeat the proof of 1.3(4), with μ there standing for λ^+ here, i.e. to define M_α for $\alpha < \lambda^{++}$, starting with the given M_0. As we are assuming $K_{\lambda^{++}} \neq \emptyset$ we should fail; but neither for $\alpha = 0$ nor for limit α, so for some $\alpha = \beta + 1, M_\beta$ is defined but we cannot choose M_α. So M_β is as required.
4A) Similarly.
5) If M is not $(\mathbb{D}_{\mathfrak{K}_\lambda}, \lambda^+)$-homogeneous above λ then there are $N_0, N_1 \in K_\lambda$ with $N_0 \leq_{\mathfrak{K}} M$ and $N_0 \leq_{\mathfrak{K}} N_1$ such that N_1 cannot be $\leq_{\mathfrak{K}}$-embedded into M over N_0. Use I.2.11(1) to get a contradiction, so clause (i) of $(*)$ holds. Now clause (ii), saturation follows by II.1.14(1).
6) By the second phrase of part (3), one of the following cases occurs. In the first, M has a $\leq_{\mathfrak{K}}$-extension M' in K_{λ^+}, so by part (3) the model M' has a $\leq_{\mathfrak{K}}$-maximal extension M^+ in K_{λ^+}, and by clause (ii) of part (5) every $p \in \mathscr{S}(M)$ is realized in M^+, so

$|\mathscr{S}(M)| \leq \|M^+\| = \lambda^+$. In the second M has a $\leq_{\mathfrak{K}}$-maximal extension $M^+ \in K_\lambda$ and the proof is even easier (by $(\text{amg})_\lambda$).
7) Easy (or see the proof of 2.8(4)). $\square_{1.4}$

1.5 Exercise: 1) Assume $(\text{slm})_\lambda$, $(\text{mdn})^1_{\lambda^+}$ and $2^\lambda < 2^{\lambda^+}$.

Then any $M \in K^{\text{slm}}_\lambda$ is an amalgamation base in $\mathfrak{K}_\lambda$.
2) If $M \in K^{\text{slm}}_\lambda$ then M is an amalgamation base in $\mathfrak{K}_\lambda$ iff M is an amalgamation base in $\mathfrak{K}^{\text{slm}}_\lambda$.

1.6 Definition. 1) $K^{3,\text{na}}_\lambda = \{(M_0, M_1, a) : M_0 \leq_{\mathfrak{K}} M_1$ are both in K_λ and $a \in M_1 \backslash M_0\}$ and actually we should write $K^{3,\text{na}}_{\mathfrak{K}_\lambda}$, but as $\mathfrak{K}$ is constant we usually ignore this.
2) We define a two-place relation $\leq_{K^{3,\text{na}}_\lambda} = \leq_{\text{na}}$ on $K^{3,\text{na}}_\lambda$ by:

$$(M_0, M_1, a) \leq_{\text{na}} (M'_0, M'_1, a') \text{ if } a = a', M_0 \leq_{\mathfrak{K}} M'_0, M_1 \leq_{\mathfrak{K}} M'_1.$$

3) $(M_0, M_1, a) \leq^{\text{na}}_h (M'_0, M'_1, a')$ when $h(a) = a'$ and for $\ell = 0, 1$ we have:

$$h \restriction M_\ell \text{ is a } \leq_{\mathfrak{K}} \text{-embedding of } M_\ell \text{ into } M'_\ell.$$

4) $(M_0, M_1, a) <_{\text{na}} (M'_0, M'_1, a')$ if $(M_0, M_1, a) \leq_{\text{na}} (M'_0, M'_1, a)$ and $M_0 \neq M'_0$.
4A) (M_0, M_1, a) is maximal in $K^{3,\text{na}}_\lambda$ if (it belongs to $K^{3,\text{na}}_\lambda$ and) for no (M'_0, M'_1, a') do we have $(M_0, M_1, a) <_{\text{na}} (M'_0, M'_1, a')$.
5) We define similarly $<^{\text{na}}_h$.

1.7 Observation. 1) $(K^{3,\text{na}}_\lambda, \leq_{\text{na}})$ is a λ-a.e.c., in particular $\leq_{\text{na}}$ is a partial order on $K^{3,\text{na}}_\lambda$, preserved under isomorphism (see more in 1.14(2)(a)).
2) Similarly $<_{\text{na}}$ except smoothness (note that $<_{\text{na}}$ is not the partial order derived from $\leq_{K^{3,\text{na}}_\lambda}$).
3) The parallel statements hold for $\leq_h, <_h$ using compositions.

Proof. 1) Check the axioms of a.e.c., see II.1.4, the main point, smoothness, is repeated in 1.14(2)(a) below.
2),3) Similarly. $\square_{1.7}$

Central notions here for some time are:

1.8 Definition. 1) $(M_0, M_1, a) \in K^{3,\mathrm{na}}_\lambda$ has the weak extension property <u>when</u> there is $(M'_0, M'_1, a) \in K^{3,\mathrm{na}}_\lambda$ such that $(M_0, M_1, a) <_{\mathrm{na}} (M'_0, M'_1, a)$, that is such that $(M_0, M_1, a) \le_{\mathrm{na}} (M'_0, M'_1, a)$ and $M_0 \neq M'_0$.
2) $(M_0, M_1, a) \in K^{3,\mathrm{na}}_\lambda$ has the extension property <u>when</u>: for every $N_0 \in \mathfrak{K}_\lambda$ and
$\le_{\mathfrak{K}}$-embedding f of M_0 into N_0 there are N_1, b and g such that: $(M_0, M_1, a) \le^{\mathrm{na}}_g (N_0, N_1, b) \in K^3_\lambda$ and $g \supseteq f$ (so $g(a) = b$ and g is a $\le_{\mathfrak{K}}$-embedding of M_1 into N_1).
3) We say $K^{3,\mathrm{na}}_\lambda$ has the extension property/weak extension property <u>when</u> every $(M, N, a) \in K^{3,\mathrm{na}}_\lambda$ has it.

Clearly the extension/weak extension property will help to build models of cardinality λ^+, and even λ^{++}.
Under a strong hypothesis the weak version holds by:

1.9 Claim. *Assume* $(\mathrm{cat})_\lambda$ *and* $(\mathrm{nmx})_{\lambda^+}$
Every $(M_0, M_1, a) \in K^{3,\mathrm{na}}_\lambda$ *has the weak extension property.*

Proof. By $(\mathrm{cat})_\lambda$ we can choose (N_i, a_i, h_i) by induction on $i < \lambda^+$ such that:

- (a) $N_i \in K_\lambda$ is $\le_{\mathfrak{K}}$-increasing continuous in i;
- (b) h_i is an isomorphism from M_1 onto N_{i+1} such that $h_i(M_0) = N_i$ and $h_i(a) = a_i$.

Now as $a \in M_1 \backslash M_0$ clearly $i < j < \lambda^+ \Rightarrow a_i \in N_{i+1} \le_{\mathfrak{K}} N_j$ & $a_j \notin N_i$ hence $\bigcup\{N_i : i < \lambda^+\} \in K_{\lambda^+}$. Let $M'_0 = \cup\{N_i : i < \lambda^+\}$, by the hypothesis $(\mathrm{nmx})_{\lambda^+}$ there is $M'_1 \in K_{\lambda^+}$ satisfying $M'_0 <_{\mathfrak{K}} M'_1$; choose $b \in M'_1 \backslash M'_0$. Let χ be large enough and $\mathfrak{B} \prec (\mathscr{H}(\chi) \in, <^*_\chi)$ be such that $\lambda + 1 \subseteq \mathfrak{B}$, $\|\mathfrak{B}\| = \lambda$ and $\langle N_i, a_i, h_i : i < \lambda^+\rangle, M'_0, M'_1, b$ and the definition of $\mathfrak{K}$ belong to $\mathfrak{B}$, e.g. $\le_{\mathfrak{K}}\restriction \{M \in K_\lambda : M$ has universe $\subseteq \lambda\} \in \mathfrak{B}$.
Let $\delta = \mathfrak{B} \cap \lambda^+$, so $\delta \in (\lambda, \lambda^+)$ is a limit ordinal and clearly (using e.g. I.1.12)

- (c) $N_\delta \le_{\mathfrak{K}} N_{\delta+1} \le_{\mathfrak{K}} M'_1$

(d) $N_\delta \leq_{\mathfrak{K}} (M'_1 \restriction \mathfrak{B}) \leq_{\mathfrak{K}} M'_1$
(e) $M'_0 \restriction \mathfrak{B} = N_\delta$
(f) $N_{\delta+1} \cap (M'_1 \restriction \mathfrak{B}) = N_\delta$.

so, recalling $\mathrm{LS}(\mathfrak{K}) \leq \lambda$, see II§1 or I.1.7, I.1.12, for some N we have:

(g) $N \in \mathfrak{K}_\lambda, N \leq_{\mathfrak{K}} M'_1$, and $(N_{\delta+1} \cup (M'_1 \restriction \mathfrak{B})) \subseteq N$

so (see Definition 1.6(1) above)

(h) $(N_\delta, N_{\delta+1}, a_\delta) \leq_{\mathrm{na}} (M'_1 \cap \mathfrak{B}, N, a_\delta)$,

and b witnesses that $N_\delta \neq M'_1 \cap \mathfrak{B}$.
As h_δ exemplifies that $(M_0, M_1, a) \cong (N_\delta, N_{\delta+1}, a_\delta)$, the result follows. $\square_{1.9}$

1.10 Claim. *1) Assume* $(\mathrm{amg})_\lambda$.

If $(M_0, M_1, a) \leq_{\mathrm{na}} (M'_0, M'_1, a)$ *are from* $K^{3,\mathrm{na}}_\lambda$, *and the second has the extension property, then so does the first.*
2) If $(M_0, M_1, a) \leq_{\mathrm{na}} (M'_0, M'_1, a)$ *are from* $K^{3,\mathrm{na}}_\lambda$ *and the second has the weak extension property then so does the first.*
3) If $(M, N, a) \in K^{3,\mathrm{na}}_\lambda$ *has the extension property then it has the weak extension property.*
4) If $\mathfrak{K}_\lambda$ *has disjoint amalgamation then* $K^{3,\mathrm{na}}_\lambda$ *has the extension property.*

Proof. 1) Use amalgamation over M_0: if $M_0 \leq_{\mathfrak{K}} N_0 \in K_\lambda$ then we can find N'_0 such that $M'_0 \leq_{\mathfrak{K}} N'_0 \in K_\lambda$ and there is a $\leq_{\mathfrak{K}}$-embedding of N_0 into N'_0 over M_0. Now use "(M'_0, M'_1, a) has the extension property" for N'_0 and $\leq_{K^{3,\mathrm{na}}_\lambda}$ being a partial order.
2) By the definitions (and, of course, transitivity of $\leq_{\mathrm{na}}$).
3) As (M, N, a) has the extension property and $M <_{\mathfrak{K}} N$ because $(M, N, a) \in K^{3,\mathrm{na}}_\lambda$, we can apply it to the choice $M' := N$, so we can find (M_1, N_1, f) such that $(M, N, a) \leq_{\mathrm{na}} (M_1, N_1, a)$ and f is a $\leq_{\mathfrak{K}}$-embedding of N into M_1 over M.

So (M_1, N_1, a) exemplifies that the triple (M, N, a) has the weak extension property.
4) Easy. $\square_{1.10}$

Additional central notions are defined below. In §1-§4 our long term aim will be to prove that the minimal triples are dense and the extension property (later we prove reduced implies minimal).

The following definition will be used almost always only when $(\mathrm{amg})_\lambda$ holds .

1.11 Definition. 1) $(M_0, M_1, a) \in K^{3,\mathrm{na}}_\lambda$ is <u>minimal</u> when:

$$\begin{aligned}&\text{if } (M_0, M_1, a) \le_{\mathrm{na}} (M'_0, M^\ell_1, a) \in K^{3,\mathrm{na}}_\lambda \text{ for } \ell = 1, 2,\\ &\quad \text{then we can find } N \in K_\lambda \text{ and } \le_{\mathfrak{K}}\text{-embeddings}\\ &\quad h_\ell \text{ of } M^\ell_1 \text{ into } N \text{ for } \ell = 1, 2 \text{ such that}\\ &\quad h_1 \restriction M'_0 = h_2 \restriction M'_0 \text{ and } h_1(a) = h_2(a)\end{aligned}$$

(but maybe for every such triple (M'_0, M'_1, a) we have $M'_0 = M_0$).
1A) We say that the minimal members of $K^{3,\mathrm{na}}_\lambda$ are dense <u>when</u>: for every $(M_1, N_1, a) \in K^{3,\mathrm{na}}_\lambda$ there is a minimal $(M_2, N_2, a) \in K^{3,\mathrm{na}}_\lambda$ such that $(M_1, N_1, a) \le_{\mathrm{na}} (M_2, N_2, a)$.
2) $(M_0, M_1, a) \in K^{3,\mathrm{na}}_\lambda$ is called <u>reduced</u> when:

$$\text{if } (M_0, M_1, a) \le_{\mathrm{na}} (M'_0, M'_1, a) \in K^{3,\mathrm{na}}_\lambda \text{ then } M'_0 \cap M_1 = M_0.$$

3) We say $p \in \mathscr{S}(M_0)$ is <u>minimal</u>, where $M_0 \in K_\lambda$, if for some a, M_1 we have: $p = \mathbf{tp}(a, M_0, M_1)$ and $(M_0, M_1, a) \in K^{3,\mathrm{na}}_\lambda$ is minimal.
4) We say $p \in \mathscr{S}(M_0)$ is <u>reduced</u> where $M_0 \in K_\lambda$, if for some a, M_1 we have
$p = \mathbf{tp}(a, M_0, M_1)$ and $(M_0, M_1, a) \in K^{3,\mathrm{na}}_\lambda$ is reduced.
5) We say $p \in \mathscr{S}(M)$ is <u>algebraic</u> where $M \in K_\lambda$ if for some $c \in M$ we have $p = \mathbf{tp}(c, M, M)$.
6) For $M \in K_\lambda$ let $\mathscr{S}^{\min}(M) = \mathscr{S}^{\min}_{\mathfrak{K}}(M) = \mathscr{S}^{\min}_{\mathfrak{K}_\lambda}(M) := \{p \in \mathscr{S}_{\mathfrak{K}_\lambda}(M) : p \text{ is minimal}\}$.
7) For $M \in K_\lambda$ let $\mathscr{S}^{\mathrm{red}}(M) = \mathscr{S}^{\mathrm{red}}_{\mathfrak{K}}(M) = \mathscr{S}^{\mathrm{red}}_{\mathfrak{K}_\lambda}(M) := \{p \in \mathscr{S}_{\mathfrak{K}_\lambda}(M) : p \text{ is reduced}\}$.
8) For $M \in K_\lambda$ let $\mathscr{S}^{\mathrm{na}}(M) = \mathscr{S}^{\mathrm{na}}_{\mathfrak{K}}(M) = \mathscr{S}^{\mathrm{na}}_{\mathfrak{K}_\lambda}(M) := \{\mathbf{tp}(a, M, N) : M \le_{\mathfrak{K}} N \text{ and } a \in N \backslash M\}$.
9) Let $p \le q$, in words q extends p, mean that for some $M \le_{\mathfrak{K}} N$ we

have $p \in \mathscr{S}_{\mathfrak{K}}(M), q \in \mathscr{S}_{\mathfrak{K}}(N)$ and $p = q \restriction M$, i.e. for some N', a we have $q = \mathbf{tp}(a, N, N'), p = \mathbf{tp}(a, M, N')$ so $N \leq_{\mathfrak{K}} N'$; see Definition II.1.9(6) (usually we deal with $M, N \in K_\lambda$).

Note that in Definition 1.12(1) below the main case is "$\mathfrak{K}_\lambda$ has amalgamation and we are dealing with $\mathfrak{K}_\lambda$".

1.12 Definition. 1) We say $\mathscr{S}_*$ is a $\leq_{\mathfrak{K}_\lambda}$-type-kind (and similarly for $\leq_{\mathfrak{K}}$) <u>when</u>:

- (a) $\mathscr{S}_*$ is a (class) function with domain $\mathfrak{K}_\lambda$
- (b) $\mathscr{S}_*(M) \subseteq \mathscr{S}(M)$ for $M \in \mathfrak{K}_\lambda$
- (c) $\mathscr{S}_*$ commutes with isomorphisms.

2) We say $\mathscr{S}_1$ is hereditarily in $\mathscr{S}_2$ <u>when</u>: for $M \leq_{\mathfrak{K}} N$ from $\mathrm{Dom}(\mathscr{S}_1)$, normally K_λ, and $p \in \mathscr{S}_2(N)$ we have $p \restriction M \in \mathscr{S}_1(M) \Rightarrow p \in \mathscr{S}_1(N)$.
3) $M \in K_{\geq\lambda}$ is $(\mathscr{S}_*, \lambda^+)$-saturated above λ <u>when</u>: if $M_0 \leq_{\mathfrak{K}} M$ and $M_0 \in K_\lambda$ and $p \in \mathscr{S}_*(M_0)$ then p is realized in M; we may say M is λ^+-saturated above λ for $\mathscr{S}_*$-types.
4) $\mathscr{S}_*$ is λ-stable <u>when</u>: if $M \in K_\lambda$ then $\mathscr{S}_*(M)$ has cardinality $\leq \lambda$.
5) In part (2) if $\mathscr{S}_2 = \mathscr{S}^{\mathrm{na}}$ we may omit it and if $\mathscr{S}_* = \mathscr{S}^{\mathrm{na}}$ in part (4) we may say $\mathfrak{K}$ or $\mathfrak{K}_\lambda$ is λ-stable.

1.13 Claim. *1) $\mathscr{S}_{\mathfrak{K}_\lambda}, \mathscr{S}^{\min}_{\mathfrak{K}_\lambda}, \mathscr{S}^{\mathrm{na}}_{\mathfrak{K}_\lambda}, \mathscr{S}^{\mathrm{red}}_{\mathfrak{K}_\lambda}$ are $\leq_{\mathfrak{K}_\lambda}$-type-kinds.*
2) $\mathscr{S}^{\min}_{\mathfrak{K}_\lambda}$ is hereditary (i.e. in $\mathscr{S}^{\mathrm{na}}$).
3) $p \in \mathscr{S}_{\mathfrak{K}_\lambda}(M)$ is algebraic <u>iff</u> $p \notin \mathscr{S}^{\mathrm{na}}_{\mathfrak{K}_\lambda}(M)$.
4) If $M \leq_{\mathfrak{K}_\lambda} N$ and $p \in \mathscr{S}^{\mathrm{na}}(N)$ then $p \restriction M \in \mathscr{S}^{\mathrm{na}}(M)$.

Proof. Obvious. $\square_{1.13}$

Basic facts are

1.14 Fact. Assume $(\mathrm{amg})_\lambda$; actually need only in parts (5),(10),(13) and the assumption of part (8) implies $(\mathrm{amg})_\lambda$.
0)

(a) If $(\mathrm{nmx})_\lambda$ <u>then</u> $K^{3,\mathrm{na}}_\lambda \neq \emptyset$

(b) $K^{3,\mathrm{na}}_\lambda \neq \emptyset$ <u>iff</u> $<_{\mathfrak{K}_\lambda} \neq \emptyset$ <u>iff</u> there are $N_0 <_{\mathfrak{K}_\lambda} N_1$ <u>if</u> $(\mathrm{ext})_{\lambda^+}$

(c) if $M \leq_{\mathfrak{K}} N$ and $a \in N$ <u>then</u> $\mathbf{tp}(a, M, N) \in \mathscr{S}^{\mathrm{na}}(M)$ iff $(M, N, a) \in K^{3,\mathrm{na}}_\lambda$ iff $a \notin M$

(d) for $M \in K_\lambda$ we have $\mathscr{S}^{\mathrm{na}}(M) = \{\mathbf{tp}(a, M, N) : (M, N, a) \in K^{3,\mathrm{na}}_\lambda\} = \{p \in \mathscr{S}(M) : p \text{ not algebraic}\}$.

1) For every $(M_0, M_1, a) \in K^{3,\mathrm{na}}_\lambda$ there[1] is a reduced (M'_0, M'_1, a) such that: $(M_0, M_1, a) \leq_{\mathrm{na}} (M'_0, M'_1, a) \in K^{3,\mathrm{na}}_\lambda$.

2) Assume $\langle (M_{0,\alpha}, M_{1,\alpha}, a) : \alpha < \delta \rangle$ is an $\leq_{\mathrm{na}}$-increasing sequence of members of $K^{3,\mathrm{na}}_\lambda$

(a) if $\delta < \lambda^+$ <u>then</u>[2] $(M_{0,\alpha}, M_{1,\alpha}, a) \leq_{\mathrm{na}} (\bigcup_{\beta<\delta} M_{0,\beta}, \bigcup_{\beta<\delta} M_{1,\beta}, a) \in K^{3,\mathrm{na}}_\lambda$ for $\alpha < \delta$.

(b) If $\delta = \lambda^+$ the result (i.e. union) may be in $K^{3,\mathrm{na}}_{\lambda^+}$ but may $\notin K^{3,\mathrm{na}}_\lambda \cup K^{3,\mathrm{na}}_{\lambda^+}$ however if $\{\alpha < \delta : M_{0,\alpha} \neq M_{0,\alpha+1}\}$ is cofinal in δ then the union $\in K^{3,\mathrm{na}}_{\lambda^+}$

(c) If $\delta < \lambda^+$ and each $(M_{0,\alpha}, M_{1,\alpha}, a)$ is reduced <u>then</u> so is $(\bigcup_{\beta<\delta} M_{0,\beta}, \bigcup_{\beta<\delta} M_{1,\beta}, a)$.

3) If $(M_0, M_1, a) \leq_{\mathrm{na}} (M'_0, M'_1, a)$ are in $K^{3,\mathrm{na}}_\lambda$ and the first triple is minimal <u>then</u> so is the second.

4) If $(M_0, M_1, a) \leq_{\mathrm{na}} (M'_0, M'_1, a)$ are in $K^{3,\mathrm{na}}_\lambda$ <u>then</u> $\mathbf{tp}(a, M_0, M_1) \leq \mathbf{tp}(a, M'_0, M'_1)$ and $\mathbf{tp}(a, M_0, M_1) = \mathbf{tp}(a, M'_0, M'_1) \restriction M_0$; (see Definition 1.11(9)).

5) For $(M_0, M_1, a) \in K^{3,\mathrm{na}}_\lambda$ the following are equivalent:

(a) (M_0, M_1, a) is a minimal triple

(b) if $(M_0, M_1, a) \leq^{\mathrm{na}}_{h_\ell} (M'_0, M'_1, a_\ell) \in K^{3,\mathrm{na}}_\lambda$ for $\ell = 1, 2$ and $h_1 \restriction M_0 = h_2 \restriction M_0$ <u>then</u> $\mathbf{tp}(a_1, M'_0, M'_1) = \mathbf{tp}(a_2, M'_0, M'_1)$

---

[1]see more in Exercise 2.4

[2]if we deal with an increasing sequence of types, the existence of union is not clear, for cofinality $\aleph_0$ there is one but not necesarily unique. See on this subject Baldwin-Shelah [BlSh 862].

(c) the type $p = \mathbf{tp}(a, M_0, M_1)$ satisfies: if $M_0 \leq_{\mathfrak{K}_\lambda} N$ <u>then</u> p has at most one extension in $\mathscr{S}^{\rm na}(N)$

(d) the type $\mathbf{tp}(a, M_0, M_1)$ is minimal (so in Definition 1.11, we can replace "for some" by "for every").

6) If there is no $<_{\rm na}$-maximal member[3] of $K^{3,\rm na}_\lambda$ and there are $N_0 <_{\mathfrak{K}} N_1$ in K_λ (i.e. $K^{3,\rm na}_\lambda \neq 0$), <u>then</u> there are $N^0 <_{\mathfrak{K}} N^1$ in $\mathfrak{K}_{\lambda^+}$. If in addition $(\rm cat)_{\lambda^+}$ then $(\rm ext)_{\lambda^{++}}$, i.e. $K_{\lambda^{++}} \neq \emptyset$.
7) If every triple in $K^{3,\rm na}_\lambda$ has the weak extension property, and there are $N_0 <_{\mathfrak{K}} N_1$ in $\mathfrak{K}_\lambda$ <u>then</u> there are $N^0 <_{\mathfrak{K}} N^1$ in $\mathfrak{K}_{\lambda^+}$.
8) If every triple in $K^{3,\rm na}_\lambda$ has the extension property and $K^{3,\rm na}_\lambda \neq \emptyset$ <u>then</u>

(a) $K_{\lambda^+} \neq \emptyset$ and no $M \in K_{\lambda^+}$ is $<_{\mathfrak{K}}$-maximal hence $K_{\lambda^{++}} \neq \emptyset$

(b) if $M \leq_{\mathfrak{K}_\lambda} N$ and $p \in \mathscr{S}^{\rm na}(M)$ <u>then</u> p has an extension in $\mathscr{S}^{\rm na}(N)$.

9) If $K_{\lambda^+} \neq \emptyset$, <u>then</u> $K^{3,\rm na}_\lambda \neq \emptyset$. If $(\rm cat)_\lambda$ then: $K_{\lambda^+} \neq \emptyset$ (i.e. $(\rm ext)_{\lambda^+}$) iff $K^{3,\rm na}_\lambda \neq \emptyset$ iff $\leq_{\mathfrak{K}}$ is not the equality.
10) The following are equivalent:

(a) all triples $(M, N, a) \in K^{3,\rm na}_\lambda$ have the extension property

(b) if $M \leq_{\mathfrak{K}_\lambda} N$ and $p \in \mathscr{S}^{\rm na}(M)$ then p has at least one extension in $\mathscr{S}^{\rm na}(N)$.

11) If $K^{3,\rm na}_\lambda \neq \emptyset$ and $(\rm jep)_\lambda$ and every triple in $K^{3,\rm na}_\lambda$ has the weak extension property, <u>then</u> there is no maximal model in $\mathfrak{K}_\lambda$.
12) If every triple from $K^{3,\rm na}_\lambda$ has the weak extension property and $(M_0, M, a) \in K^{3,\rm na}_\lambda$ then M is not $\leq_{\mathfrak{K}}$-maximal.
13) If $(M, M_\ell, a_\ell) \in K^{3,\rm na}_\lambda$ is reduced for $\ell = 1, 2$ and $\mathbf{tp}(a_2, M, M_\ell)$ is not realized in M_1 then the sets $\Gamma_\ell = \{\mathbf{tp}(b, M, M_\ell) : b \in M_\ell \backslash M\}$ for $\ell = 1, 2$ are disjoint.

Proof. Easy. Note that the assumption of parts (6) and (7) are equivalent, part (8) is dual to 1.9. Parts (6) and (7) are essentially

[3] will be applied for λ^+

proved in the proofs of I.3.11, I.3.13 stages (a),(b),(c). Part (8)(a) is proved similarly; part (8)(b) is proved by the definitions; in part (9) the first sentence holds as $\mathrm{LS}(\mathfrak{K}) \leq \lambda$ by Hypothesis 1.1(1) and for the second sentence use part (0)(b) and 1.3(5), and we elaborate the proofs of parts (11),(12).

11) Assume toward contradiction that $M' \in K_\lambda$ is $\leq_{\mathfrak{K}}$-maximal; by an assumption there is $(M_0, M, a) \in K^{3,\mathrm{na}}_\lambda$.

As $(\mathrm{jep})_\lambda$ without loss of generality for some M'' we have $M' \leq_{\mathfrak{K}_\lambda} M'' \wedge M \leq_{\mathfrak{K}} M''$ hence $M' = M''$ by M' being $\leq_{\mathfrak{K}_\lambda}$-maximal so we can replace M by M'. So $(M_0, M, a) \in K^{3,\mathrm{na}}_\lambda$ and M is a maximal model in $\mathfrak{K}_\lambda$. Also without loss of generality there is no M'_0 such that $M_0 <_{\mathfrak{K}} M'_0 \leq_{\mathfrak{K}} M \wedge a \notin M'_0$. Now as $K^{3,\mathrm{na}}_\lambda$ has no maximal member there are N_0, N such that $(M_0, M, a) <_{\mathrm{na}} (N_0, N, a)$, but by the last sentence $N_0 \nsubseteq M$ hence necessarily $N \nsubseteq M$ so $M <_{\mathfrak{K}} N$. Recalling $M' = M$ clearly M' is not maximal.

12) Included in the proof of part (11). $\square_{1.14}$

1.15 Definition. 1) If $p \in \mathscr{S}^{\mathrm{na}}(N), N \in K_\lambda$ and $N' \in K_\lambda$ <u>then</u> we let

(a) $\mathscr{S}_p(N') = \mathscr{S}_{\mathfrak{K},p}(N') = \mathscr{S}_{\mathfrak{K}_\lambda,p}(N') := \{f(p) : f$ is an isomorphism from N onto $N'\}$
(so if $N' \ncong N$ this is empty, but if $(\mathrm{cat})_\lambda$ then $N' \cong N$)

(b) $\mathscr{S}_{\geq p}(N') = \mathscr{S}_{\mathfrak{K},\geq p}(N') = \{q \in \mathscr{S}(N') : q$ not algebraic (i.e. not realized by any $c \in N'$) and, for some $N'' \in K_\lambda$ satisfying $N'' \leq_{\mathfrak{K}} N'$, we have $q \restriction N'' \in \mathscr{S}_p(N'')\}$

(c) we may write (M, N, a) instead of p when $p = \mathbf{tp}(a, M, N)$ so $(M, N, a) \in K^{3,\mathrm{na}}_\lambda$.

2) We say the type $p \in \mathscr{S}(N)$ is μ-algebraic <u>when</u> for every M such that $N \leq_{\mathfrak{K}} M$ we have: $\mu \geq |\{c \in M : \mathbf{tp}(c, N, M) = p\}|$.

3) For $M \in K_\lambda$ let $\mathscr{S}^{\mathrm{nb}}(M) = \mathscr{S}^{\mathrm{nb}}_{\mathfrak{K}}(M)$ be $\{p \in \mathscr{S}^{\mathrm{na}}(M) : p$ is not 1-algebraic$\}$.

4) For $M \in K_\lambda$ let $\mathscr{S}^{\mathrm{nc}}(M) = \mathscr{S}^{\mathrm{nc}}_{\mathfrak{K}}(M)$ be the set of $p \in \mathscr{S}^{\mathrm{na}}(M)$ such that for no pair (N, q) do we have $M \leq_{\mathfrak{K}} N \in K_\lambda$ and $q \in \mathscr{S}^{\mathrm{na}}(N)$ is 1-algebraic and extends p.

5) For $M \in K_\lambda$ let $\mathscr{S}^{\mathrm{nm}}(M) = \{\mathscr{S}^{\mathrm{nm}}_{\mathfrak{K}}(M) = \mathscr{S}^{\mathrm{nm}}_{\mathfrak{K}_\lambda}(M)$ be the

set of $p \in \mathscr{S}^{\rm na}(M)$ which has no minimal extension (i.e. in any $N \in \mathfrak{K}_\lambda$ which $\le_{\mathfrak{K}}$-extends M) and $K^{3,\rm nm}_\lambda = \{(M,N,a) \in K^{3,\rm na}_\lambda : \mathbf{tp}(a,M,N) \in \mathscr{S}^{\rm nm}(M)\}$.

1.16 Definition. 1) For $M \in K_\lambda$ let $\mathscr{S}^{\rm sn}(M) = \mathscr{S}^{\rm sn}_{\mathfrak{K}}(M)$ be the set of $p \in \mathscr{S}(M)$ which are not λ-algebraic; see Definition 1.15(2).
2) Let $K^{3,\rm sn}_\lambda$ be the family of triples $(M,N,a) \in K^{3,\rm na}_\lambda$ such that $\mathbf{tp}(a,M,N) \in \mathscr{S}^{\rm sn}(M)$.
3) Let $\mathscr{S}^{\rm al}(M) = \mathscr{S}(M) \backslash \mathscr{S}^{\rm sn}(M)$.

Remark. Note that λ-algebraic types are the obstacle to having the extension property (which we desire; assuming $(\rm amg)_\lambda$), see Claim 2.7(3).

1.17 Claim. *Assume* $(\rm amg)_\lambda$.
0) Assume $M_0 \le_{\mathfrak{K}_\lambda} M_1 \le_{\mathfrak{K}_\lambda} M_2$ and $a \in M_2, p_\ell = \mathbf{tp}(a,M_\ell,M_2)$ for $\ell = 0,1$ and $N_1 \le_{\mathfrak{K}_\lambda} N_2$. Then $p_1 \in \mathscr{S}_{p_1}(M_1)$ and $p_1 \in \mathscr{S}_{\ge p_0}(M_1)$ and $\mathscr{S}_{p_1}(N_1) \subseteq \mathscr{S}_{\ge p_1}(N_1)$ and $\mathscr{S}_{\ge p_1}(N_1) \subseteq \mathscr{S}_{\ge p_0}(N_1)$. If $q \in \mathscr{S}_{\ge p_1}(N_1)$ then $(\exists r \in \mathscr{S}_{\ge p_1}(N_2))[(q \le r)$ or q is realized in $N_2]$. If $q_1 \in \mathscr{S}_{p_1}(N_1)$ then $N \in K_\lambda \Rightarrow \mathscr{S}_{\ge q_1}(N) \subseteq \mathscr{S}_{\ge p_1}(N)$.
1) If $N_1 \le_{\mathfrak{K}} N_2$ are from K_λ and $p_1 \in \mathscr{S}^{\rm na}(N_1)$ is minimal and is omitted by N_2 then p_1 has a one and only one extension in $\mathscr{S}(N_2)$, call it p_2, and p_2 is minimal and $p_1 \in \mathscr{S}_{\ge p^*}(N_1) \Rightarrow p_2 \in \mathscr{S}_{\ge p^*}(N_2)$ for any relevant p^*.
2) If $N_1 \le_{\mathfrak{K}} N_2$ are in K_λ and $p_1 \in \mathscr{S}(N_1)$ is minimal, then p_1 has at most one non-algebraic extension in $\mathscr{S}(N_2)$ called p_2; if it exists it is minimal and $p_1 \in \mathscr{S}_{\ge p^*}(N_1) \Rightarrow p_2 \in \mathscr{S}_{\ge p^*}(N_2)$.
3) (Continuity) if $\langle N_i : i \le \alpha \rangle$ is a $\le_{\mathfrak{K}}$-increasing continuous sequence of members of $K_\lambda, p_0 \in \mathscr{S}(N_0)$ is minimal, $p_i \in \mathscr{S}(N_i)$ extends p_0 and is non-algebraic for $i \le \alpha$ then $\langle p_i : i \le \alpha \rangle$ is increasing continuous, the continuity means that for any limit ordinal $\delta \le \alpha, p_\delta$ is the unique $p \in \mathscr{S}(M_\delta)$ such that $(\forall i < \delta)(p \restriction M_i = p_i)$.
3A) Assume $\langle N_i : i \le \delta \rangle$ is $\le_{\mathfrak{K}_\lambda}$-increasing continuous, $\delta < \lambda^+$ a limit ordinal. If $\bar p = \langle p_i : i < \delta \rangle, p_i \in \mathscr{S}^{\rm na}(N_i), p_i \restriction N_0 = p_0$ for $i < \delta$ and p_0 is minimal, then:

(a) *there is $p_\delta \in \mathscr{S}(N_\delta)$ such that $i < \delta \Rightarrow p_i \le p_\delta$*

(b) p_δ *is minimal so* $\in \mathscr{S}^{\rm na}(N_\delta)$

(c) p_δ *is unique in clause (a).*

4) If $p \in \mathscr{S}(N_1)$ *is minimal not* λ*-algebraic and* $N_1 \le_{\mathfrak{K}_\lambda} N_2$ *<u>then</u>* p_1 *has one and only one extension* $p_2 \in \mathscr{S}^{\rm na}(N_2)$ *and* p_2 *is minimal not* λ*-algebraic.*
5) $\mathscr{S}_p$ *is a* $\le_{\mathfrak{K}}$*-type-kind,* $\mathscr{S}_{\ge p}$ *is a hereditarily* $\le_{\mathfrak{K}}$*-type-kind (in* $\mathscr{S}^{\rm na}$*).*
6) If $M \in K_\lambda$ *and* $p \in \mathscr{S}^{\rm na}(M)$ *is 1-algebraic <u>then</u>* p *is minimal.*
7) For $M \in K_\lambda$ *we have* $\mathscr{S}^{\rm nm}(M) \subseteq \mathscr{S}^{\rm nc}(M) \subseteq \mathscr{S}^{\rm nb}(M) \subseteq \mathscr{S}^{\rm na}(M)$ *and* $\mathscr{S}^{\rm nm}(M) \subseteq \mathscr{S}(M)\backslash\mathscr{S}^{\rm nb}(M)$.

Remark. 1) Interesting mainly if $\mathfrak{K}$ is categorical in λ.
2) On a non-λ-algebraic type and having the extension property see later.
3) Concerning 1.17 note that $\mathscr{S}^{\rm nm}(M) \subseteq \mathscr{S}^{\rm sn}(M)$ by 2.3(4).

Proof of 1.17. Easy. E.g.,
3) If $i < j \le \alpha$ then $p_j \restriction N_i$ is well defined, it belongs to $\mathscr{S}(N_i)$, also it is non-algebraic (as p_j is) and extends p_0 hence by the uniqueness (=1.17(2)) we have $p_i = p_j \restriction N_i$, so indeed $\langle p_i : i \le \alpha\rangle$ is increasing. If $\delta \le \alpha$ is a limit ordinal, then: $p_\delta \in \mathscr{S}(N_\delta)$ extends p_i for $i < \delta$; if $p'_\delta \in \mathscr{S}(N_\delta)$ extends each $p_i (i < \delta)$ then it extends p_0 and is non-algebraic (as each p_i is) hence by uniqueness $p'_\delta = p_\delta$.
3A) Let (N'_0, a) be such that $(N_0, N'_0, a) \in K^{3,\rm na}_\lambda$ and $p_0 := \mathbf{tp}(a, N_0, N'_0)$. We choose (N'_i, f_i) by induction on $i \le \delta$ such that $\langle N'_j : j \le i\rangle$ is $\le_{\mathfrak{K}}$-increasing continuous, f_i is a $\le_{\mathfrak{K}}$-embedding of N_i into N'_i such that $a \notin \mathrm{Rang}(f_i)$ and $j < i \Rightarrow f_j \subseteq f_i, N'_0$ as above and $f_0 = \mathrm{id}_{N_0}$. For $i = 0$ the pair (N'_0, f_0) is already determined and for limit take unions recalling 1.14(2)(a). For $i = j + 1$, by $(\mathrm{amg})_\lambda$+ the definition of type, let N^+_i, a_i be such that $(N_i, N^+_i, a_i) \in K^{3,\rm na}_\lambda$ and $p_i = \mathbf{tp}(a_i, N_i, N^+_i)$, so clearly $f_j(p_i \restriction N_j) = f_j(p_j) = \mathbf{tp}(a, f_j(N_j), N'_j)$. So by $(\mathrm{amg})_\lambda$ and the definition of types there is a pair (N'_i, f'_i) such that $N'_j \le_{\mathfrak{K}_\lambda} N'_i, f'_i$ is a $\le_{\mathfrak{K}}$-embedding of N^+_i into N'_i extending f_j and satisfying $f'_i(a_i) = a$. So we have chosen N'_i and let $f_i = f'_i \restriction N_i$. So we have carried the induction and for

$i = \delta$ by renaming $f_\delta = \mathrm{id}_{N_\delta}$ so $p_\delta = \mathbf{tp}(a, N_\delta, N'_\delta)$ is as required in clause (a). Clauses (b),(c) are easy, too, e.g. by part (3). $\square_{1.17}$

The following will help to prove that we have a non-structure result or density of minimal types.

1.18 Claim. *Assume* $(\mathrm{amg})_\lambda + \lambda > \aleph_0$.

If $N \in K_\lambda, \mathscr{S} \subseteq \mathscr{S}(N)$ *and* $|\mathscr{S}| > \lambda^+$, *then we can find* N^* *and* N_i *for* $i < \lambda^{++}$ *such that:*

- (α) $N \le_{\mathfrak{K}} N^* <_{\mathfrak{K}} N_i \in K_\lambda$
- (β) *for no* $i_0 < i_1 < \lambda^{++}$ *and* $c_\ell \in N_{i_\ell} \backslash N^*$ *(for* $\ell = 0, 1$*) do we have* $\mathbf{tp}(c_0, N^*, N_{i_0}) = \mathbf{tp}(c_1, N^*, N_{i_1})$
- (γ) *there are* $a_i \in N_i$ *(for* $i < \lambda^{++}$*) such that* $\mathbf{tp}(a_i, N, N_i) \in \mathscr{S}$ *is not realized in* N^* *(hence* $a_i \notin N^*$*) and those types are pairwise distinct; moreover* $\mathbf{tp}(a_i, N, N_i)$ *is not realized in* N_j *for* $j < i$.

1.19 Remark. In clause (γ) above we can add moreover for $j \neq i, j < \lambda^{++}$, the type $\mathbf{tp}(a_i, N, N_i)$ is not realized (by thinning $\langle N_i : i < \lambda^{++} \rangle$); of course this applies to 1.20, too.

Proof.

Stage A:

Without loss of generality $|N| = \lambda$; now by induction on $\alpha < \lambda^{++}$ choose $\bar{N}^\alpha, N_\alpha, a_\alpha, E_\alpha$ such that:

- (A) $N_\alpha \in K_{\lambda^+}$ as a set of elements is an ordinal $\gamma_\alpha \le \lambda^+ \times (1+\alpha)$ and N_α is $\le_{\mathfrak{K}}$-increasing continuous with α
- (B) $\bar{N}^\alpha = \langle N^\alpha_i : i < \lambda^+ \rangle$ is $\le_{\mathfrak{K}_\lambda}$-increasing sequence with union N_α
- (C) for $\alpha < \lambda^{++}$ successor, if $i < j < \lambda^+$ and $p \in \mathscr{S}(N^\alpha_i)$ is realized in N^α_j and is λ-algebraic (see Definition 1.15(2)) then for no pair (N', b) do we have $N^\alpha_j \le_{\mathfrak{K}} N' \in K_\lambda$ and $b \in N' \backslash N^\alpha_j$ realizes p (actually not used)

(D) $N \leq_{\mathfrak{K}} N_0^\alpha \leq_{\mathfrak{K}} N_\alpha$ and $a_\alpha \in N_{\alpha+1} \backslash N_\alpha$ realizes in $N_{\alpha+1}$ some $p_\alpha \in \mathscr{S}$ not realized in N_α.

(E) for each $\alpha < \lambda^{++}$,

- (α) E_α is a club of λ^+
- (β) $\langle N_i^\alpha : i \in E_\alpha \rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous
- (γ) for each $i \in E_\alpha$ the triple $(N_i^\alpha, N_i^{\alpha+1}, a_\alpha)$ belongs to $K_\lambda^{3,\mathrm{na}}$ and is "relatively reduced", that is:

  $(*)_\alpha$ if $i \in E_\alpha$ and $b \in N_i^{\alpha+1} \backslash N_i^\alpha$ <u>then</u> the type $\mathbf{tp}(b, N_i^\alpha, N_i^{\alpha+1})$ is
  not realized in N_α (a key point).

(F) If $\aleph_0 < \mathrm{cf}(\alpha) \leq \lambda$ <u>then</u> for $j < \lambda^+$ let:
$N_j^\alpha := \bigcup_{\beta \in e} N_j^\beta$ for any club e of α such that for any $\beta_1 < \beta_2$ from e, $N_j^{\beta_1} = N_j^{\beta_2} \cap N_{\beta_1}$ (any two such e's gives the same result) <u>provided that</u> there is such e

Let us carry the construction.

<u>Case 1</u>: $\alpha = 0$
Trivial.

<u>Case 2</u>: α limit, $\mathrm{cf}(\alpha) = \aleph_0 \vee \mathrm{cf}(\alpha) = \lambda^+$.
Easy. We let $N_\alpha = \cup\{N_\beta : \beta < \alpha\}$ so the universe of N_α is $\gamma_\alpha := \cup\{\gamma_\beta : \beta < \alpha\} < \lambda^{++}$ and let $\langle N_j^\alpha : j < \lambda^+ \rangle$ be any $\leq_{\mathfrak{K}}$-representation of N_α such that $N_0^\alpha = N$.

<u>Case 3</u>: α limit, $\lambda \geq \mathrm{cf}(\alpha) > \aleph_0$.
Let $N_\alpha := \cup\{N_\beta : \beta < \alpha\}$. Let $S_\alpha = \{j < \lambda^+$: there is a club e of α such that for any $\beta_1 < \beta_2$ from e we have $N_j^{\beta_1} = N_j^{\beta_2} \cap N_{\beta_1}\}$.
If $S_\alpha = \emptyset$ proceed as in Case 2, so assume $S_\alpha \neq \emptyset$.
For $j \in S_\alpha$ choose a club e_j^α of α witnessing it and let $N_j^\alpha = \cup\{N_j^\beta : \beta \in e_j^\alpha\}$.
If $j \leq \sup(S_\alpha)$, let $N_j^\alpha = \cup\{N_i^\alpha : i \in S_\alpha \wedge i \leq j$ <u>or</u> $i = \min(S_\alpha)\}$.

If $\sup(S_\alpha) < \lambda^+$ we choose $\langle N^\alpha_j : j > \sup(S_\alpha), j < \lambda^+\rangle$ as in Case 2 only demanding that $N^\alpha_{\sup(S_\alpha)+1}$ include $N^\alpha_{\sup(S_\alpha)}$; actually does not occur.

Note that

(a) if $j_1 < j_2$ are from S_α then $N^\alpha_{j_1} \le_{\mathfrak{K}} N^\alpha_{j_2}$.

[Why? If $j_1 < j_2 < \lambda^+$ are from S_α then $e^\alpha_{j_1} \cap e^\alpha_{j_2}$ is unbounded in α and if $\beta \in e^\alpha_{j_1}$ then there is $\gamma \in e^\alpha_{j_1} \cap e^\alpha_{j_2} \backslash \beta$ hence $N^\beta_{j_1} \le_{\mathfrak{K}} N^\gamma_{j_1} \le_{\mathfrak{K}} N^\gamma_{j_2} \le_{\mathfrak{K}} N^\alpha_{j_2}$, so as $M^\alpha_{j_\ell} = \cup\{N^\beta_{j_\ell} : \beta \in e^\alpha_{j_\ell}\}$ for $\ell = 1, 2$ we get $M^\alpha_{j_1} \le_{\mathfrak{K}} M^\alpha_{j_2}$.]

(b) if $j_1 < j_2 < \lambda^+$ then $N^\alpha_{j_1} \le_{\mathfrak{K}} N^\alpha_{j_2}$.

[Why? Check our choices.]

Let e_α be a club of α of order type $\mathrm{cf}(\alpha)$, and let $E_\alpha := \{i < \lambda^+$: for every $\beta < \gamma$ from e_α we have $i \in E_\beta \cap E_\gamma$ and $N^\gamma_i \cap N_\beta = N^\beta_i\}$, easily

(c) (α) E_α is a club of λ^+
  (β) $S_\alpha \supseteq E_\alpha$ and
  (γ) $\langle N^\alpha_j : j \in E_\alpha\rangle$ is $\le_{\mathfrak{K}_\lambda}$-increasing continuous.

[Why? By the definitions.]

(d) $\cup\{N^\alpha_j : j < \lambda^+\} = N_\alpha$.

[Why? As E_α is a club of λ^+, including $\bigcap\limits_{\beta \in e_\alpha} E_\beta$, the choice of $N^\alpha_j, j \in S_\alpha$ and the induction hypothesis.]

(e) clause (F) holds.

[Why? By our choice of S_α and N^α_i for $i \in S_\alpha$.]

(f) $N \le_{\mathfrak{K}} N^\alpha_0$.

[Why? Use the induction hypothesis and our choices.]

<u>Case 4</u>: $\alpha = \beta + 1$.

Let $p_\beta \in \mathscr{S} \subseteq \mathscr{S}(N)$ be a type not realized in N_β, and let a_β, N^*_α be such that $N^\beta_0 \le_{\mathfrak{K}} N^*_\alpha$ hence $N \le_{\mathfrak{K}} N^*_\alpha$ and $p_\beta = \mathbf{tp}_{\mathfrak{K}_\lambda}(a_\beta, N, N^*_\alpha)$ and by induction on $i < \lambda^+$ we choose a triple $(N_{\alpha,i}, j_i, f_i)$ such that

$\circledast$ (a) $N_{\alpha,i} \in K_\lambda$ is $\le_{\mathfrak{K}}$-increasing continuous with i

(b) $j_i < \lambda^+$ is increasing continuous (so $j_i < j_{i+1}$)

(c) f_i is a $\le_{\mathfrak{K}}$-embedding of $N^\beta_{j_i}$ into $N_{\alpha,i}$

(d) $N_{\alpha,0} = N^*_\alpha$ and $j_0 = 0, f_0 = \mathrm{id}_{N^\beta_0}$ so $N \le_{\mathfrak{K}} N_{\alpha,0}$

(e) for limit i, if there is $c \in N_\beta \backslash N^\beta_{j_i}$ such that $f_i(\mathbf{tp}(c, N^\beta_{j_i}, N_\beta)) \in \mathscr{S}(f_i(N^\beta_{j_i}))$ is realized in $N_{\alpha,i}$ <u>then</u> there is $c \in N^\beta_{j_{i+1}} \backslash N^\beta_{j_i}$ such that $f_{i+1}(c) \in N_{\alpha,i} \backslash f_i(N^\beta_{j_i})$; this is helpful for clause (E)(γ)

(f) if $j \le i$ and $p \in \mathscr{S}(N_{\alpha,j})$ is λ-algebraic and is realized in $N_{\alpha,i+1}$ <u>then</u> for no M, c do we have $N_{\alpha,i+1} \le_{\mathfrak{K}} M, c \in M \backslash N_{\alpha,i+1}$ and $\mathbf{tp}(c, N_{\alpha,j}, M) = p$.

Using amalgamation and the definition of types there is no problem to carry the induction on i. By renaming without loss of generality $i < \lambda^+ \Rightarrow f_i = \mathrm{id}_{N^\beta_{j_i}}$. Let $N^\alpha_i = N_{\alpha,i}, N_\alpha = \cup\{N_{\alpha,i} : i < \lambda^+\}$ and by renaming without loss of generality the universe of N_α is an ordinal $\gamma_\alpha \le \lambda^+ \times (1+\alpha)$ and let E_α be any club of λ^+, we shall need that it is disjoint to the non-stationary set S dealt with below and we should check clauses (A)-(E).

Now clauses (A),(B) are trivial and clause (C) holds by (f) above and clause (D) holds by the choice of a_β.

Toward proving clause (E), subclause (α) is trivial and subclause (β) holds by clause (a) above. For subclause (γ) let $S = \{i < \lambda^+ : i$ is a limit ordinal and $i = j_i$ and for some $c \in N_\beta \backslash N^\beta_{j_i}$ and $b \in N^\alpha_i \backslash N^\beta_i$ we have $\mathbf{tp}(c, N^\beta_i, N_\beta) = \mathbf{tp}(b, N^\beta_i, N^\alpha_i)\}$. As $\{i : j_i = i\}$ is a club of λ^+ it is enough (for proving subclause (γ) of clause (E)) to show that S is not stationary; toward contradiction assume that S is stationary. Clearly for each $i \in S$ by clause (e) there is c as required there hence we can choose (c_i, b_i) which exemplifies this, i.e. such that $f_{i+1}(c_i) = b_i$ actually $c_i = b_i$ as $f_i = \mathrm{id}_{N^p_{j_i}}$.

By Fodor's lemma for some $i_0 < i_1$ in S we have $b_{i_0} = b_{i_1}$, but this gives contradiction so we have proved clause (E). Now clause (F) by its formulation is trivial (for α successor) so the construction is done.

Stage B:

For $i < \lambda^+, \alpha < \lambda^{++}$ let $w_i^\alpha := \{\beta < \alpha : N_i^{\alpha+1} \cap N_{\beta+1} \nsubseteq N_\beta\}$, so necessarily $|w_i^\alpha| \le \|N_i^{\alpha+1}\| = \lambda, w_i^\alpha$ is increasing continuous with $i < \lambda^+$ and $\alpha = \bigcup_{i<\lambda^+} w_i^\alpha$; lastly, for $\beta < \alpha$ let $\mathbf{i}(\beta,\alpha) = \text{Min}\{i : \beta \in w_i^\alpha\}$.
Now for every $\alpha \in S^* := \{\delta < \lambda^{++} : \text{cf}(\delta) = \lambda^+\}$, the set E^*_α is a club of λ^+ where (some clauses are redundant):

$$E^*_\alpha := \Big\{ i<\lambda^+ : i \text{ limit and belongs to } E_\alpha \text{ (from clause (E))}, N \le_{\mathfrak{K}} N_i^{\alpha+1},$$
$$a_\alpha \in N_i^{\alpha+1}, \text{ and for every } \beta < \alpha \text{ if}$$
$$\beta \in w_i^\alpha \text{ then } i \in E_\beta \cap E_{\beta+1} \text{ and}$$
$$N_i^\beta = N_i^\alpha \cap N_\beta \text{ and for } j < i$$
$$\text{the closure of } w_j^\alpha \text{ (in } \alpha) \text{ is included in } w_i^\alpha$$
$$\text{and } \sup(w_j^\alpha) < \sup(w_i^\alpha) \text{ and}$$
$$\beta_1 < \beta_2 \ \& \ \beta_1 \in w_j^\alpha \ \& \ \beta_2 \in w_j^\alpha \Rightarrow \mathbf{i}(\beta_1,\beta_2) < i \Big\}.$$

Recall that we are assuming $\lambda > \aleph_0$, so we can choose $j_\alpha \in \text{acc}(E^*_\alpha)$ such that $\text{cf}(j_\alpha) = \aleph_1$ and let $\delta_\alpha = \sup(w^\alpha_{j_\alpha})$, now $w^\alpha_{j_\alpha}$ is closed under ω-limits (as $\langle w_j^\alpha : j < \lambda^+\rangle$ is increasing continuous and $j < j_\alpha \Rightarrow \text{closure}(w_j^\alpha) \subseteq w^\alpha_{j_\alpha}$ and $\aleph_1 = \text{cf}(j_\alpha)$ as $j < j_\alpha \Rightarrow \sup(w_j^\alpha) < \sup(w^\alpha_{j_\alpha})$ and obviously $\delta_\alpha < \alpha$ because $\text{cf}(\alpha) = \lambda^+$). So there is an increasing continuous sequence $\langle \beta_\varepsilon : \varepsilon < \omega_1\rangle$ with limit δ_α satisfying $\beta_\varepsilon \in w^\alpha_{j_\alpha}$ so $\varepsilon < \omega_1 \Rightarrow N^\alpha_{j_\alpha} \cap N_{\beta_\varepsilon} = N^{\beta_\varepsilon}_{j_\alpha}$ and hence $\varepsilon < \zeta < \omega_1 \Rightarrow N^{\beta_\varepsilon}_{j_\alpha} = N^{\beta_\zeta}_{j_\alpha} \cap N_{\beta_\varepsilon}$ and so recalling clause (F) from Stage A

$(*)_1$ $N^{\delta_\alpha}_{j_\alpha} = \cap\{\bigcup_{j\in C} N_j^\beta : C \text{ a club of } \delta_\alpha\}$

$(*)_2$ $N^\alpha_{j_\alpha} = N^{\delta_\alpha}_{j_\alpha}$ [4]

By Fodor's lemma for some j^*, δ^* and stationary $S \subseteq S^*$ we have $\alpha \in S \Rightarrow j_\alpha = j^* \ \& \ \delta_\alpha = \delta^*$. So for all $\alpha \in S, N^\alpha_{j_\alpha}$ is the same, say $N^* = N^{\delta^*}_{j^*}$; we shall show that N^* and $\langle N^{\alpha+1}_{j_*} : \alpha \in S\rangle$ are as promised, thus finishing. Clause (α) is obvious. So $N^* \in K_\lambda$ and for $\alpha \in S$ the type $q_\alpha = \mathbf{tp}(a_\alpha, N^*, N^{\alpha+1}_{j_\alpha})$ extend $p_\alpha(\in \mathscr{S})$. Also if $r \in \mathscr{S}(N^*)$ is realized in $N^{\alpha+1}_{j_\alpha}$ say by b (for some $\alpha \in S$) then no member of $\bigcup\{N^{\beta+1}_{j^*} \backslash N^\beta_{j^*} : \beta \in S \cap \alpha\}$ realizes it (holds by clause (D)).

Let $\Gamma_\alpha = \{\mathbf{tp}(b, N^{\delta^*}_{j^*}, N^{\alpha+1}_{j^*}) : b \in N^{\alpha+1}_{j^*} \backslash N^*\}$ for $\alpha \in S$ so Γ_α has a member extending $p_\alpha \in \mathscr{S}$ (as exemplified by a_α), also p_α is not extended by any $p \in \bigcup\limits_{\beta<\alpha} \Gamma_\beta$ (as p_α is not realized in N_α)). Also each triple $(N^{\delta^*}_{j^*}, N^{\alpha+1}_{j^*}, a_\alpha)$ is "relatively reduced", see $(E)(\gamma)$, so by a variant of 1.14(13) the sequence $\bar\Gamma = \langle \Gamma_\alpha : \alpha \in S\rangle$ is a sequence of pairwise disjoint sets (each of cardinality $\leq \|N^{\alpha+1}_{j^*}\| = \lambda$, of course). More fully if $\alpha_1 < \alpha_2$ are from S and $p \in \Gamma_{\alpha_2}$ then p is not realized in N_{α_2} by clause $(E)(\gamma)$ but $N^{\alpha_1+1}_{j^*} \leq_{\mathfrak{K}} N_{\alpha_2}$ so necessarily $p \notin \Gamma_{\alpha_1}$, as required. So clause (β) of the desired conclusion holds and clause (γ) was proved above. So we are done.

$\square_{1.18}$

Now 1.18 is superceded by 1.20, still (the older) 1.18's proof may be useful elsewhere.

1.20 Claim. *We can allow $\lambda = \aleph_0$ in Claim 1.18.*

Proof. Assume toward contradiction that the conclusion fails. First

$\circledast_1$ if $N \leq_{\mathfrak{K}_\lambda} N_1 \leq_{\mathfrak{K}} M_1 \in K_{\leq\lambda^+}$, <u>then</u> we can find a pair $(M_2, \mathscr{S}_1)$ such that:

(a) $M_1 \leq_{\mathfrak{K}} M_2 \in K_{\leq\lambda^+}$

(b) $\mathscr{S}_1 \subseteq \mathscr{S}$ has cardinality $\leq \lambda^+$

[4] see [Sh 351, §4] on similar proof.

(c) if the pair (N_2, b) satisfies $N_1 \leq_{\mathfrak{K}_\lambda} N_2, b \in N_2 \backslash N_1$ and $\mathbf{tp}(b, N, N_2) \in \mathscr{S} \backslash \mathscr{S}_1$ then for some $c \in N_2 \backslash N_1$ and $d \in M_2 \backslash N_1$ we have $\mathbf{tp}(c, N_1, N_2) = \mathbf{tp}(d, N_1, M_2)$.

[Why $\circledast_1$ holds? We try to choose a triple $(p_\varepsilon, N_{1,\varepsilon}, a_\varepsilon)$ by induction on $\varepsilon < \lambda^{++}$ such that:

$\odot_1(\alpha)$ $p_\varepsilon \in \mathscr{S}$

(β) $N_1 \leq_{\mathfrak{K}_\lambda} N_{1,\varepsilon}$ and $a_\varepsilon \in N_{1,\varepsilon} \backslash N_1$

(γ) $p_\varepsilon = \mathbf{tp}(a_\varepsilon, N, N_{1,\varepsilon})$

(δ) if $c \in N_{1,\varepsilon} \backslash N_1, \zeta < \varepsilon$ and $d \in N_{1,\zeta} \backslash N_1$ then $\mathbf{tp}(c, N_1, N_{1,\varepsilon}) \neq \mathbf{tp}(d, N_1, N_{1,\zeta})$

(ε) p_ε is not realized in N_1 and in $N_{1,\zeta}$ when $\zeta < \varepsilon$.

If we succeed then $N_1, \langle N_{1,i} : i < \lambda^{++} \rangle$ are as required on $N^*, \langle N_i : i < \lambda^{++} \rangle$ in the claim. So by our assumption toward contradiction for some $\varepsilon(*) < \lambda^{++}$ we are stuck. As $\mathfrak{K}_\lambda$ has amalgamation, recalling I.2.11(1) we can find $(M_2, \bar{f})$ such that

$\odot_2$ (a) $M_1 \leq_{\mathfrak{K}} M_2 \in K_{\leq \lambda^+}$

(b) $\bar{f} = \langle f_\varepsilon : \varepsilon < \varepsilon(*) \rangle$

(c) f_ε is a $\leq_{\mathfrak{K}}$-embedding of $N_{1,\varepsilon}$ into M_2 over N_1.

Now M_2 and $\mathscr{S}_1 := \{p \in \mathscr{S}$: for some $b \in M_2$ we have $p = \mathbf{tp}(b, N, M_2)\}$ are as required in $\circledast_1$.]

Now we choose $M^\alpha, \bar{M}^\alpha, \mathscr{S}_\alpha$ by induction on $\alpha < \lambda^+$ such that:

$\circledast_2$ (a) $\bar{M}^\alpha = \langle M_i^\alpha : i < \lambda^+ \rangle$ is a $\leq_{\mathfrak{K}_\lambda}$-increasing continuous sequence and $M_0^0 = N$

(b) $M^\alpha = \cup\{M_i^\alpha : i < \lambda^+\} \in \mathfrak{K}_{\leq \lambda^+}$

(c) $\langle M^\beta : \beta < \lambda^+ \rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous

(d) if $\beta < \alpha$ then $i < \lambda^+ \Rightarrow M_i^\beta \leq_{\mathfrak{K}} M_i^\alpha$

(e) if $i < \lambda^+$ and $\alpha < \lambda^+$ is a limit ordinal then $M_i^\alpha = \cup\{M_i^\beta : \beta < \alpha\}$

(f) $\mathscr{S}_\alpha \subseteq \mathscr{S}$ has cardinality $\leq \lambda^+$

(g) $\mathscr{S}_\alpha$ is $\subseteq$-increasing continuous with α

(h) if $\alpha < \lambda^+$ and $M_{\alpha,\alpha} \le_{\mathfrak{K}_\lambda} N_2$ and $b \in N_2, \mathbf{tp}(b,N,N_2) \in \mathscr{S} \backslash \mathscr{S}_{\alpha+1}$ then for some $c \in N_2 \backslash M_{\alpha,\alpha}$ and $d \in M_{\alpha+1} \backslash M_{\alpha,\alpha}$ we have $\mathbf{tp}(c, M_{\alpha,\alpha}, N_2) = \mathbf{tp}(c, M_{\alpha,\alpha}, M_{\alpha+1})$.

[Why can we carry out the induction? For $\alpha = 0$ trivial (e.g. $M_i^\alpha = N$ for $i < \lambda^+$). For α successor say $\alpha = \beta + 1$ use $\circledast_1$ where $N_1, M_1, M_2, \mathscr{S}_1$ there standing for $M_{\alpha,\alpha}, M_\alpha, M_{\alpha+1}, \mathscr{S}_{\alpha+1}$ here. Lastly, for α limit $< \lambda^+$ take unions.]

Now let $M := \cup\{M_\alpha : \alpha < \lambda^+\}$, so clearly $\langle M_{\alpha,\alpha} : \alpha < \lambda^+\rangle$ is a $\le_{\mathfrak{K}}$-representation of M and let $p_* \in \mathscr{S} \backslash \cup \{\mathscr{S}_\alpha : \alpha < \lambda^+\}$, possible by cardinality consideration. Now let the pair (N^+, b) be such that $N \le_{\mathfrak{K}_\lambda} N^+, b \in N^+ \backslash N$ and $p_* = \mathbf{tp}(b, N, N^+)$, clearly exists.

We choose $(j_\alpha, f_\alpha, N_\alpha^+)$ by induction on $\alpha \le \lambda^+$ such that

$\circledast_3$ (*a*) $N_\alpha^+ \in \mathfrak{K}_\lambda$ is $\le_{\mathfrak{K}}$-increasing continuous, $N_0^+ = N^+$

(*b*) $j_\alpha < \lambda^+$ is increasing continuous and $j_0 = 0$

(*c*) f_α is a $\le_{\mathfrak{K}}$-embedding of M_{j_α,j_α} into N_α^+ and $f_0 = \mathrm{id}_N$

(*d*) f_α is $\subseteq$-increasing continuous

(*e*) if there are $c \in N_\alpha^+ \backslash f_\alpha(M_{j_\alpha,j_\alpha})$ and $d_\alpha \in M$ such that $f_\alpha(\mathbf{tp}(d_\alpha, M_{j_\alpha,j_\alpha}, M)) = \mathbf{tp}(c, f_\alpha(M_{j_\alpha,j_\alpha}), N_\alpha^+)$ then $f_{\alpha+1}(M_{j_{\alpha+1},j_{\alpha+1}}) \cap N_\alpha^+ \ne f_\alpha(M_{j_\alpha,j_\alpha})$.

There is no problem to carry out the induction. Having carried out the induction let $N_{\lambda^+}^+ := \cup\{N_\alpha^+ : \alpha < \lambda^+\}$, so easily $f = \cup\{f_\alpha : \alpha < \lambda^+\}$ is a $\le_{\mathfrak{K}}$-embedding of M into $N_{\lambda^+}^+$. By renaming without loss of generality $f = \mathrm{id}_M$ and $b \notin M$ by the choice of p_*.

Let $E = \{\alpha < \lambda^+ : \alpha$ is a limit ordinal and $M \cap N_\alpha^+ = M_{\alpha,\alpha}\}$. Clearly E is a club of λ^+. By clause (e) of $\circledast_3$ it follows that for every $\alpha \in E$ we have $c \in N_\alpha^+ \backslash M_{\alpha,\alpha} \wedge d \in M \backslash M_{\alpha,\alpha} \Rightarrow \mathbf{tp}(c, M_{\alpha,\alpha}, N_{\lambda^+}^+) \ne \mathbf{tp}(d, M_{\alpha,\alpha}, N_{\lambda^+}^+)$.

But recalling $b \in N^+ \backslash M$ realizes in N^+ the type $p_* \in \mathscr{S} \subseteq \mathscr{S}(N)$ and $b \in N_\alpha^+ \backslash M_{\alpha,\alpha}$, by clause (h) of $\circledast_2$ the previous sentence implies that $\mathbf{tp}(b, N, N_{\lambda^+}^+) \in \mathscr{S}_{\alpha+1}$. But $\mathbf{tp}(b, N, N_{\lambda^+}^+) = p_* \notin \cup\{\mathscr{S}_\beta : \beta < \lambda^+\}$, hence in particular $p_* \notin \mathscr{S}_{\alpha+1}$, contradiction. $\square_{1.20}$

1.21 Conclusion. Assume $(\mathrm{amg})_\lambda + (\mathrm{cat})_\lambda$.

If $\mathscr{S}(M)$ has cardinality $> \lambda^+$ where $M \in K_\lambda$, and $(M, N, a) \in K^{3,\mathrm{na}}_\lambda$ <u>then</u> this triple has the weak extension property.

Remark. Compared with 1.9: there we assume something on $\mathfrak{K}_{\lambda^+}$, i.e. $(\mathrm{nmx})_{\lambda^+}$ here on $\mathfrak{K}_\lambda$, i.e. $(\mathrm{amg})_\lambda$.

Proof. Straight, but we elaborate.

The assumption of 1.18 or 1.20 with $M, \mathscr{S}(M)$ here standing for $N, \mathscr{S}$ there holds hence its conclusion. So by $(\mathrm{cat})_\lambda$ we can find $\bar{M} = \langle M_i : i < \lambda^{++} \rangle$ such that $i < \lambda^+ = M <_{\mathfrak{K}_\lambda} M_i$ and the sets $\Gamma_i := \{\mathbf{tp}(b, M, M_i) : b \in M_i \backslash M\}$ for $i < \lambda^{++}$ are pairwise disjoint but we use them only for $i < \lambda^+$. By $(\mathrm{amg})_\lambda$ there are $M_* \in K_{\lambda^+}$ which $\le_{\mathfrak{K}}$-extends M and $\le_{\mathfrak{K}}$-embedding g of N into M_* over M and $\le_{\mathfrak{K}}$-embedding f_i of M_i into M over M for $i < \lambda^+$. Clearly $\langle f_i(M_i) \backslash M : i < \lambda^+ \rangle$ are pairwise disjoint sets. So for some $i < \lambda^+$ we have $f_i(M_i) \cap g(N) = M$.

Now as $\mathrm{LS}(\mathfrak{K}) \le \lambda$ there is $N' \le_{\mathfrak{K}} M$ of cardinality λ such that $f_i(M_i) \cup g(N) \subseteq N'$, clearly $(M, g(N), g(a)) \le_{\mathrm{na}} (f_i(M_i), N', a)$, so the triple $(M, g(N), g(a))$ has the weak extension property. But g is an isomorphism from (M, N, a) onto $(M, g(N), g(a))$ and the weak extension property is preserved by isomorphisms so the triple (M, N, a) has the weak extension property as required. $\square_{1.21}$

<u>1.22 Exercise</u>: Phrase and prove a generalization 1.20 reasonably replacing λ^+ by μ with cofinality $> \lambda$.

[Hint: We assume

$\boxtimes$ (*a*) $\mu > \lambda$, $\mathrm{cf}(\mu) > \lambda$ and $\alpha < \mu \Rightarrow \mathrm{cov}(|\alpha|, \lambda^+, \lambda^+, 2) < \mu$

(*b*) $\mathfrak{K}$ has $(\lambda, \lambda, < \mu)$-amalgamation

(*c*) $N \in K_\lambda$

(*d*) $\mathscr{S} \subseteq \mathscr{S}(N)$ has cardinality $\ge \mu$

(*e*) $\mathfrak{K}$ has $(\mathrm{amg})_\theta$ if $\theta = \lambda$ if $\lambda \le \theta$ & $\theta^+ < \lambda$.

We deduce

- $\circledast$ for some $N^*, \langle N_i : i < \mu \rangle$ we have $N \leq_{\mathfrak{K}} N^* \leq_{\mathfrak{K}_\lambda} N_i$ for $i < \mu$
  - (α) for no $i_0 < i_1 < \mu$ and $c_\ell \in N_{i_\ell} \backslash N^*$ (for $\ell = 0, 1$) do we have $\mathbf{tp}(c_0, N^*, N_{i_0}) = \mathbf{tp}(c_1, N^*, N_{i_1})$
  - (β) there are $a_i \in N_i$ (for $i < \mu$) such that $\mathbf{tp}(a_i, N, N_i) \in \mathscr{S}$ is not realized in N^* (hence $a_i \notin N^*$) and those types are pairwise distinct; moreover $\mathbf{tp}(a_i, N, N_i)$ is not realized in N_j for $j < i$.

Imitating the proof of 1.20:

- $\circledast_1$ if $N \leq_{\mathfrak{K}} N_1 \leq_{\mathfrak{K}} M, N_1 \in K_\lambda, M_1 \in K_{<\mu}$ <u>then</u> we cn find a pair $(M_2, \mathscr{S}_1)$ as there (so $M_2 \in K_{<\mu}$).]

<u>1.23 Exercise</u>: Assume $\mathfrak{K}_\lambda$ is a λ-a.e.c., $K' \subseteq K_\lambda$ is dense (i.e. $(\forall M \in K)(\exists N \in K')(M \leq_{\mathfrak{K}} N)$ and $\mathfrak{K}' = (K', \leq_{\mathfrak{K}_\lambda} \restriction K')$ too is a λ-a.e.c. Further $M \in K'_\lambda$ is an amalgamation base in $\mathfrak{K}_\lambda$. <u>Then</u> $\mathscr{S}_{\mathfrak{K}_\lambda}(M), \mathscr{S}_{\mathfrak{K}'_\lambda}(M)$ are essentially equal.

§2 Toward the extension property and toward density of minimal types

2.1 Hypothesis. $\mathfrak{K}$ is an a.e.c. and $\mathrm{LS}(\mathfrak{K}) \leq \lambda, K_\lambda \neq \emptyset$.

We are interested in proving the extension property (for triples from $K^{3,\mathrm{na}}_\lambda$) and the density of minimal types.

The first is proved in 2.23(1), requiring an expensive assumption, but reasonable for our aim: categoricity in λ^+, i.e. the assumption of Theorem 0.2 are enough.

Concerning the density of minimal types, to simplify matters we (in 2.25) allow uses of stronger assumptions than are desired and ultimately used (i.e. $2^{\lambda^+} > \lambda^{++}$ and $K_{\lambda^{+3}} = \emptyset$), we use them for a "shortcut", but the extra assumptions will be eliminated later. However the first extra assumption is still a "mild set theoretic assumption", and the second is harmless if we think only of proving our main theorem 0.2(1) and 0.2(2)(a) and not on subsequent continuations for which 0.2(2)(b) is helpful.

We shall construct models $M \in K_{\lambda^+}$ which are saturated at least in some restricted form (see 2.2), deduce the extension property for triples $<_{\rm na}$-above which there is no minimal ones (in 2.3, 2.5) which gives more, give sufficient conditions for disjoint amalgmation and for the extension property (via λ-algebraic types, see 2.7). Using a weak form of stability or the existence of an inevitable $\Gamma \subseteq \mathscr{S}^{\rm na}(M)$ of cardinality $\le \lambda^+$, we give sufficient conditions for the existence of $M \in K_{\lambda^+}$ saturated above λ, (see 2.8, also 2.11). We define μ-minimal types, observe their properties (in 2.12, 2.13) and prove for λ^+-minimal p that $|\mathscr{S}_{\ge p^*}(M)| \le \lambda^+$ when $M \in K_\lambda$ in 2.14, 2.15, 2.16 and give sufficient conditions for non-structure in λ^+, see 2.18.

We then deduce version of saturation for every $M \in K^{\rm slm}_{\lambda^+}$ (in 2.19), get the extension property and disjoint amalgamation (2.21, 2.23) and deduce density of minimal types (in 2.24 - 2.25) and the existence of a saturated $M \in K_{\lambda^+}$ (in 2.26).

2.2 Claim. *1) Assume* $(\mathrm{amg})_\lambda$.

If $M_0 \le_{\mathfrak{K}} N_0 \in K_\lambda$ and $(M_0, M_1, a) \in K^{3,\rm na}_\lambda$ <u>then</u> there is $N \in K_{\le\lambda^+}$ such that: $N_0 \le_{\mathfrak{K}} N$ and for every $c \in N$ satisfying $\mathbf{tp}(c, M_0, N) = \mathbf{tp}(a, M_0, M_1)$, there is a $\le_{\mathfrak{K}}$-embedding h of M_1 into N extending id_{M_0} such that $h(a) = c$ and $N \notin K_{\lambda^+} \Rightarrow N$ is a $<_{\mathfrak{K}}$-maximal member of K_λ.
2) Assume $M_0 \le_{\mathfrak{K}} N_0 \in K_\lambda$ and $(M_0, M_1, a) \in K^{3,\rm na}_\lambda$. <u>Then</u> there is $N \in K_{\le\lambda^+}$ such that: $N_0 \le_{\mathfrak{K}} N$ and for every $c \in N$ either for some $N' \in K_\lambda$ we have $N_0 \cup \{c\} \subseteq N' \le_{\mathfrak{K}} N$ and c does not strongly realize $\mathbf{tp}(a, M_0, M_1)$ in N' <u>or</u> there is an $\le_{\mathfrak{K}}$-embedding h of M_1 into N extending id_{M_0} such that $h(a) = c$.
3) If $(\mathrm{cat})_\lambda$ or $(M_0, M_1, a) \in K^{3,\rm na}_\lambda$ has the extension property or $(\mathrm{nmx})_\lambda$, <u>then</u> in parts (1),(2) we can add $N \in K_{\lambda^+}$.
4) Part (1) holds also when $N_0 \in K_{\le\lambda^+}$.

Proof. 1),2) We choose by induction on $\alpha < \lambda^+$, a model $N_\alpha \in K_\lambda$

increasing (by $\leq_{\mathfrak{K}}$) continuous such that: N_0 is given, for α even $N_\alpha \neq N_{\alpha+1}$ if N_α is not $\leq_{\mathfrak{K}}$-maximal, and for α odd let $\beta_\alpha =$ Min$\{\beta : \beta = \alpha+1$ or $\beta \leq \alpha$ and there is $c \in N_\beta$ such that there is no $\leq_{\mathfrak{K}}$-embedding h of M_1 into N_α extending id_{M_0} satisfying $h(a) = c$ but for some $N \in K_\lambda, N_\alpha \leq_{\mathfrak{K}} N$ and there is a $\leq_{\mathfrak{K}}$-embedding h of M_1 into N extending id_{M_0} satisfying $h(a) = c\}$. Note that for part (1) the "but for some $N \in K_\lambda$" is equivalent to c realizes in N_α the type $\mathbf{tp}(a, M_0, M_1)$. Now if $\beta_\alpha = \alpha + 1$ then let $N_{\alpha+1} = N_\alpha$ and if $\beta_\alpha \leq \alpha$ then choose N exemplifying this and let $N_{\alpha+1} = N$. By the definition of type (and "strongly realizes" when $\neg(\mathrm{amg})_\lambda$) we are done.

3) Same proof, note that the non-$\leq_{\mathfrak{K}}$-maximality of N_α (and hence $N \in K_{\lambda^+}$) follows from $(\mathrm{nmx})_\lambda$ and also by "(M_0, M_1, a) has the extension property" applied to $M_0 \leq_{\mathfrak{K}} N_\alpha$. This proves the assertion, for the second part assume $(\mathrm{cat})_\lambda$ so as $(M_0, M_1, a) \in K^{3,\mathrm{na}}_\lambda$ exemplified that $M_0 \in K_\lambda$ is not $\leq_{\mathfrak{K}_\lambda}$-maximal, also N_α, which is isomorphic to M_0, is not $\leq_{\mathfrak{K}_\lambda}$-maximal. We also can use 1.14(10).

4) Easy because $\mathfrak{K}$ has $(\lambda, \lambda, \lambda^+)$-amalgamation, see Definition I.2.7(2) and Claim I.2.11(1). $\square_{2.2}$

2.3 Claim. *Assume* $(\mathrm{amg})_\lambda$ *and* $(M, N, a) \in K^{3,\mathrm{nm}}_\lambda$, *i.e. above* $(M, N, a) \in K^{3,\mathrm{na}}_\lambda$ *there is no minimal triple.*

1) We can find $\langle (M^0_\eta, M^1_\eta, a) : \eta \in {}^{\lambda^+ >}2 \rangle$ *such that*

- *(i)* $(M^0_\eta, M^1_\eta, a) \in K^{3,\mathrm{na}}_\lambda$
- *(ii)* $\nu \triangleleft \eta \Rightarrow (M^0_\nu, M^1_\nu, a) <_{\mathrm{na}} (M^0_\eta, M^1_\eta, a)$
- *(iii)* $M^0_{\eta\hat{}\langle \ell \rangle}$ *for* $\ell = 0, 1$ *are equal*
- *(iv)* $\mathbf{tp}(a, M^0_{\eta\hat{}\langle 0 \rangle}, M^1_{\eta\hat{}\langle 0 \rangle}) \neq \mathbf{tp}(a, M^0_{\eta\hat{}\langle 1 \rangle}, M^1_{\eta\hat{}\langle 1 \rangle})$*; this makes sense as* $M^0_{\eta\hat{}\langle 0 \rangle} = M^0_{\eta\hat{}\langle 1 \rangle}$
- *(v) if* $\eta \in {}^\delta 2$ *and* $\delta < \lambda^+$ *is a limit ordinal,* <u>*then*</u> $M^\ell_\eta = \{M^\ell_{\eta \restriction \alpha} : \alpha < \delta\}$ *for* $\ell = 0, 1$
- *(vi)* $(M^0_{<>}, M^1_{<>}, a_{<>}) = (M, N, a)$.

2) Assume that $\mathscr{T}$ *is a tree with* $\delta < \lambda^+$ *levels and* $\leq \lambda$ *nodes and* $|\lim_\delta(\mathscr{T})| > \lambda$ *for simplicity and let* $p = \mathbf{tp}(a, M, N)$.

Then we can find $\mathscr{Y} \subseteq \lim_\delta(\mathscr{T})$ *of cardinality* $\leq \lambda$, *and* $M^* \in K_\lambda$ *and* $(M^*, N_\eta, a) \in K^{3,\mathrm{na}}_\lambda$ *above* (M, N, a) *for* $\eta \in \lim_\delta(\mathscr{T})\backslash\mathscr{Y}$ *such that* $\mathbf{tp}(a, M^*, N_\eta)$ *for* $\eta \in \lim_\delta(\mathscr{T})\backslash\mathscr{Y}$ *are pairwise distinct and* $|\mathscr{S}_{\geq p}(M^*)| \geq |\mathscr{S}^{\mathrm{nm}}(M)| \geq |\lim_\delta(\mathscr{T})|$, *see 1.15(4), 1.16(1).*
3) If $M \in K_\lambda$ *is universal (in* $\mathfrak{K}_\lambda$*) then* $\mathscr{S}_{\geq p}(M) \geq \sup\{\lim_\delta(\mathscr{T}) : \mathscr{T}$ *a tree with* $\leq \lambda$ *nodes and* $\delta < \lambda^+$ *levels*$\}$.
4) Stability in λ, $(\mathrm{stb})_\lambda$, *fails. Also if* $M \in K_\lambda$ *the* $\mathscr{S}^{\mathrm{nm}}(M) \subseteq \mathscr{S}^{\mathrm{sn}}(M)$.
5) If $2^\lambda > \lambda^+$ *then for some* $M \in K_\lambda$ *we have* $|\mathscr{S}^{\mathrm{nc}}(M)| > \lambda^+$, *in fact* $|\mathscr{S}^{\mathrm{nc}}(M)| \geq |\mathscr{S}^{\mathrm{sn}}(M)| \geq |\mathscr{S}^{\mathrm{nm}}(M)| > \lambda$.

Remark. Used in 3.13 and in the proof of 2.5, 2.25, 4.5, 4.13.

Proof. 1) We choose $\langle (M^0_\eta, M^1_\eta, a) : \eta \in {}^\alpha 2\rangle$ by induction on $\alpha < \lambda^+$.

This is straightforward: for $\alpha = 0$ choose $(M^0_{<>}, M^1_{<>}, a)$ as the given triple $(M, N, a) \in K^{3,\mathrm{na}}_\lambda$ above which there is no minimal triple; in limit α take limits, i.e. unions; in successor α, use non-minimality and its definition (see Definition 1.11(1) or noting that 1.14(5) apply)).

2) First without loss of generality $\mathscr{T}$ is a subtree of ${}^{\lambda^+ >}2$ and even, after changing $\mathscr{T}$ a little, of ${}^{\lambda \times \mathrm{cf}(\delta) >}2$ so $\delta = \lambda \times \mathrm{cf}(\delta)$ and as usual identify $\lim_\delta(\mathscr{T})$ with $\{\eta \in {}^\delta 2 : (\forall i < \delta)\eta \restriction i \in \mathscr{T}\}$. Second, by 1.11(4), 1.14(1), without loss of generality (M, N, a) is reduced.

Now let $\langle \eta_\alpha : \alpha < \alpha_* \rangle$ list $\mathscr{T}$ without repetitions such that $\eta_\alpha \triangleleft \eta_\beta \Rightarrow \alpha < \beta$ and $\eta_\alpha = \nu\hat{}\langle 0\rangle \Rightarrow \eta_{\alpha+1} = \nu\hat{}\langle 1\rangle$. Now we choose $N_\alpha, f_{\eta_\alpha}$ by induction on $\alpha \leq \alpha_*$ such that (where the M^0_η are from part (1)):

$\circledast$ (*a*) $N_\alpha \in K_\lambda$ is $\leq_{\mathfrak{K}}$-increasing continuous for $\alpha \leq \alpha_*$

(*b*) $N_0 = M^0_{<>}$

(*c*) f_{η_α} is a $\leq_{\mathfrak{K}}$-embedding of $M^0_{\eta_\alpha}$ into $N_{\alpha+1}$ for $\alpha < \alpha_*$

(*d*) if $\eta_\beta \triangleleft \eta_\alpha$ then $f_{\eta_\beta} \subseteq f_{\eta_\alpha}$

(*e*) if $\eta_\alpha = \nu\hat{}\langle 0\rangle$ then $f_{\eta_{\alpha+1}} = f_{\nu\hat{}<1>}$ is equal to f_{η_α}.

By $(\mathrm{amg})_\lambda$ we can carry the definition. Now for every $\eta \in \lim_\delta(\mathscr{T})$ let $f_\eta := \cup\{f_{\eta\restriction i} : i < \delta\}$ and let $N^0_\eta = f_\eta(M^0_\eta)$ so $N^0_\eta \leq_{\mathfrak{K}} N_{\alpha_*}$ and by $(\mathrm{amg})_\lambda$ there are N^1_η, g_η such that: $N_{\alpha_*} \le_{\mathfrak{K}} N^1_\eta \in K_\lambda$ and g_η is a $\le_{\mathfrak{K}}$-embedding of M^1_η into N^1_η extending f_η. Let $a_\eta = g_\eta(a)$ for $\eta \in \lim_\delta(\mathscr{T})$, so we do not know to exclude the possibility that $a_\eta \in N_{\alpha_*}$.

Now obviously

- $(*)$ if $\eta_0, \eta_1 \in \lim_\delta(\mathscr{T}), i < \delta, \eta_0 \restriction i = \eta_1 \restriction i, \eta_0(i) = 0, \eta_1(i) = 1$ <u>then</u>
  - (a) $\mathbf{tp}(a_\eta, f_{\eta_\ell \restriction (i+1)}(M^0_{\eta_\ell \restriction (i+1)}), g_{\eta_\ell \restriction (i+1)}(M^1_{\eta \restriction (i+1)}))$ for $\ell = 0, 1$ are distinct
  - (b) $f_{\eta_0 \restriction (i+1)}(M^0_{\eta_0 \restriction (i+1)}) = f_{\eta_1 \restriction (i+1)}(M^0_{\eta_1 \restriction (i+1)}) \le_{\mathfrak{K}} N_{\alpha_*}$
  - (c) $\mathbf{tp}(a_{\eta_0}, N_{\alpha_*}, N^1_{\eta_0}) \neq \mathbf{tp}(a_{\eta_1}, N_{\alpha_*}, N^1_{\eta_1})$.

Let $\mathscr{Y} = \{\eta \in \lim_\delta(\mathscr{T}) : a_\eta \in N_{\alpha_*}\}$, so by clause (c) of $(*)$ we have $\eta_0 \neq \eta_1 \in \mathscr{Y} \Rightarrow a_{\eta_0} \neq a_{\eta_1}$ hence $|\mathscr{Y}| \le \lambda$. Clearly $\eta \in \lim_\delta(\mathscr{T}) \backslash \mathscr{Y} \Rightarrow (N_{\alpha_*}, N^1_\eta, a_\eta) \in K^{3,\mathrm{na}}_\lambda$ also $g_\eta(N) \cap N_{\alpha_*} = M$, recalling that without loss of generality (M, N, a) is reduced, (not really necessary) hence by renaming $\eta \in \lim_\delta(\mathscr{T}) \backslash \mathscr{Y} \Rightarrow (M, N, a) \le_{\mathrm{na}} (N_{\alpha_*}, N^1_\eta, a_\eta)$. Also clearly $|\mathscr{S}_{\ge p}(N_{\alpha_*})| \ge |\{\mathbf{tp}(a_\eta, N_{\alpha_*}, N'_u) : \eta \in \lim_\delta(\mathscr{T}) \backslash \mathscr{Y}\}| = |\lim_\delta(\mathscr{T}) \backslash \mathscr{Y}| = |\lim_\delta(\mathscr{T})|$ as $|\mathscr{Y}| \le \lambda < |\lim_\delta(\mathscr{T})|$ and $N_{\alpha_*} \in K_\lambda$, so we are done.
3) As for any $N \in K_\lambda$ there is a model $N' \le_{\mathfrak{K}} M$ isomorphic to N, now $p \mapsto p \restriction N'$ is a function from $\mathscr{S}(M)$ onto $\mathscr{S}(N')$ by II.1.11(5) hence $|\mathscr{S}_{\ge p}(M)| \ge |\mathscr{S}_{\ge p}(N')| = |\mathscr{S}_{\ge p}(N)|$. Now we can use part (2).
4) Should be clear as there is a tree $\mathscr{T}$ as in part (2), e.g. ${}^{\theta>}2$ if $\theta = \min\{\theta : 2^\theta > \lambda\}$.
5) If $2^\lambda > \lambda^+$ then for some such $\mathscr{T}$ and $\delta, \lambda^+ < |\lim_\delta(\mathscr{T})|$; see $(*)$ inside the proof of 4.13. Now apply part (2), recalling 1.17(7). $\square_{2.3}$

<u>2.4 Exercise</u>: If $\circledast$ then $\boxtimes$ where:

- $\circledast$ (a) $\mathscr{T}$ a subtree of ${}^{\lambda^+>}(\lambda^+)$ of cardinality $\le \lambda$
  - (b) $\bar{M} = \langle M_\eta : \eta \in \mathscr{T} \rangle$
  - (c) $\langle M_{\eta \restriction i} : i \le \ell g(\eta) \rangle$ is $\le_{\mathfrak{K}}$-increasing[5] for each $\eta \in \mathscr{T}$

[5]yes! we do not require continuity

(d) $(M_\eta, M_{\eta\hat{}<\alpha>}, a_{\eta\hat{}<\alpha>}) \in K^{3,\mathrm{na}}_\lambda$ for every $\eta\hat{}\langle\alpha\rangle \in \mathscr{T}$

$\boxtimes$ (a) $\bar{M}^* = \langle M^*_\eta : \eta \in \mathscr{T}\rangle$

(b) $\langle M^*_{\eta\restriction i} : i < \ell g(\eta)\rangle$ is $\leq_{\mathfrak{K}}$-increasing for each $\eta \in \mathscr{T}^*$

(c) $(M^*_\eta, M^*_{\eta\hat{}<\alpha>}, a_{\eta\hat{}<\alpha>}) \in K^{3,\mathrm{na}}_\lambda$ is reduced for every $\eta\hat{}\langle\alpha\rangle \in \mathscr{T}$

(d) $\bar{f} = \langle f_\eta : \eta \in \mathscr{T}\rangle$

(e) f_η is a $\leq_{\mathfrak{K}}$-embedding of M_η into M^*_η

(f) $f_\nu \subseteq f_\eta$ if $\nu \triangleleft \eta \in \mathscr{T}$

(g) $(M_\eta, M_{\eta\hat{}<\alpha>}, a_{\eta\hat{}<\alpha>}) \leq^{\mathrm{na}}_{f_{\eta\hat{}<\alpha>}} (a_{\eta\hat{}<\alpha>}))$.

[Hint: Straight; you may use lots of mapping or imitate Definition 3.7(3).]

2.5 Claim. *Assume* (amg)$_\lambda$.
1) If above $(M_0, M_1, a) \in K^{3,\mathrm{na}}_\lambda$ there is no minimal member of $K^{3,\mathrm{na}}_\lambda$ <u>then</u> (M_0, M_1, a) itself has the extension property.
2) If $(M_0, M_1, a) \in K^{3,\mathrm{na}}_\lambda, M_0 \leq_{\mathfrak{K}} N \in K$ and the number of $c \in N$ such that $\mathbf{tp}(c, M_0, N) = \mathbf{tp}(a, M_0, M_1)$ is $> \lambda$ <u>then</u> (M_0, M_1, a) has the extension property.
2A) Given $(M_0, M_1, a) \in K^{3,\mathrm{na}}_\lambda$ there is N as required in part (2) <u>iff</u> $\mathbf{tp}(a, M_0, M_1)$ is not λ-algebraic, i.e. $\in \mathscr{S}^{\mathrm{sn}}(M_0)$ iff for every $M'_0 \in K_\lambda$ which $\leq_{\mathfrak{K}}$-extends M_0 then are M''_0, b such that $M'_0 <_{\mathfrak{K}} M''_0$ and $b \in M''_0 \backslash M'_0$ realizes $\mathbf{tp}(a, M'_0, M''_0)$.
2B) A triple $(M, N, a) \in K^{3,\mathrm{na}}_\lambda$ has the extension property <u>iff</u> $\mathbf{tp}(a, M, N)$ is not λ-algebraic.
3) Assume that above $(M_0, M_1, a) \in K^{3,\mathrm{na}}_\lambda$ there is no minimal member of $K^{3,\mathrm{na}}_\lambda$ <u>then</u>

$(*)_1$ *For some $M^+_0, M_0 \leq_{\mathfrak{K}} M^+_0 \in K_\lambda$ and $\mathbf{tp}(a, M_0, M_1)$ has $> \lambda$ extensions in $\mathscr{S}(M^+_0)$ (in fact $\geq \min\{2^\mu : 2^\mu > \lambda\}$ and even $|\lim_\delta(\mathscr{T})|$ for any tree $\mathscr{T}$ with $\delta < \lambda^+$ levels and $\leq \lambda$ nodes).*

3A) If $(M_0, M_1, a) \in K^{3,\mathrm{na}}$ satisfies $(*)_1$ above <u>then</u>

$(*)_2$ *for some N we have: $M_0 \leq_{\mathfrak{K}} N$ and N is as required in part (2).*

4) In 2.3(2) we can add "and each (M^*, N_η, a_η) *is reduced. That is, we can find* M_*, $\langle((N_\eta, a_\eta, f_\eta) : \eta \in \lim_\delta(\mathscr{T})\rangle$ *such that* $(M_*, N_\eta,\ f_\eta(a)) \in K^{3,\mathrm{na}}_\lambda$ *is reduced,* $(M, N, a) \le^{\mathrm{na}}_{f_\eta} (M_*, N_\eta, f_\eta(a))$, *and* $\langle \mathbf{tp}(f_\eta(a), M_*, N_\eta) : \eta \in \lim_\delta(\mathscr{T})\rangle$ *are pairwise distinct when:*

- (a) $(\mathrm{amg})_\lambda$
- (b) *above* $(M, N, a) \in K^{3,\mathrm{na}}_\lambda$ *there is no minimal triple*
- (c) $\delta < \lambda^+$ *is a limit ordinal*
- (d) $\mathscr{T}$ *is a tree with* δ *levels and* $\le \lambda$ *nodes.*

Proof. 1) Follows by parts (2),(3) and (3A).
2) Trivially N has cardinality $\ge \lambda^+$. By II.1.8(3) and II.1.11(1), i.e. by the LS property without loss of generality N has cardinality λ^+ and also is as in 2.2(1).
By I.2.11(1) for any M'_0 such that $M_0 \le_{\mathfrak{K}} M'_0 \in K_\lambda$ there are $N_1, N \le_{\mathfrak{K}} N_1 \in K_{\lambda^+}$ and a $\le_{\mathfrak{K}}$-embedding h of M'_0 into N_1 extending id_{M_0}. Now some $c \in N \backslash h(M'_0)$ realizes $\mathbf{tp}(a, M_0, M_1)$ and by the assumption on "N is as in 2.2(1)" there is g, a $\le_{\mathfrak{K}}$-embedding of M_1 into N over M_0 such that $g(a) = c$. Let $N^-_1 \le_{\mathfrak{K}} N_1$ be such that $N_1 \in K_\lambda$ include $g(M_1) \cup h(M'_0) \cup \{c\}$. So modulo chasing arrows we have proved that (M_0, M_1, a) has the extension property for the case $M'_0 \in K_\lambda, M_0 \le_{\mathfrak{K}} M'_0$, which was arbitrary so we are done.
2A) Obviously, the first iff holds by the definitions, the first phrase implies the third by the proof of part (2) which implies the second by (2B).
2B) The "if" direction holds by part (2A) and the definition. For the other direction recall $(M, N, a) \in K^{3,\mathrm{na}}_\lambda$ and let $p = \mathbf{tp}(a, M, N)$. We try to choose $N_i \in K_\lambda$ which is $\le_{\mathfrak{K}}$-increasing with i such that: there is $b_i \in N_{i+1} \backslash N_i$ realizing p. We can continue by the third phrase and $\cup\{N_i : i < \lambda\}$ is as required.
3) Note that there are limit $\delta < \lambda^+$ and a tree $\mathscr{T}$ with δ levels, $\le \lambda$ nodes such that the set of δ-branches, $\lim_\delta(\mathscr{T})$, has cardinality $> \lambda$ (e.g. $({}^{\theta>}2, \triangleleft)$ for first θ such that $2^\theta > \lambda$). Clearly $(*)_1$ holds by 2.3(2).
3A) We choose by induction on $i < \lambda^+, N_i \in K_\lambda$ which is $\le_{\mathfrak{K}}$-increasing continuous, $N_0 = M^+_0$ (M^+_0 is from $(*)_1$ above) and for

each i some $c_i \in N_{i+1}$ realizes over $N_0 = M_0^+$ an extension of $p = \mathbf{tp}(a, M_0, M_1)$ not realized in N_i. There is such type by clause $(*)_1$ above and there is such an N_{i+1} as $\mathfrak{K}$ has amalgamation in λ. Clearly $c_i \notin N_i$ and so $\cup\{N_i : i < \lambda^+\}$ is as required.

4) So as in the proof of 2.3(2) let $\mathscr{T}$ be a subtree of ${}^{\delta>}2$ with $\le \lambda$ nodes let $\mathscr{T}_\alpha$ be $\mathscr{T} \cap {}^\alpha 2$ if $\alpha < \delta$ and $\lim_\delta(\mathscr{T}) = \{\eta \in {}^\delta 2 : (\forall \alpha < \delta)(\eta \restriction \alpha \in \mathscr{T})\}$. We choose $N_\alpha, \langle N_{\eta,\alpha}, a_\eta : \eta \in \mathscr{T}_\delta\rangle, \langle g_{\eta,\nu,\alpha} : \eta, \nu \in \mathscr{T}_\delta, \alpha \le \delta, \eta \restriction \alpha = \nu \restriction \alpha\rangle$ by induction on $\alpha \le \delta$ such that:

⊛ (α) $N_\alpha \in K_\lambda$ is $\le_{\mathfrak{K}}$-increasing continuous

(β) $(N_\alpha, N_{\eta,\alpha}, a_\eta) \in K_\lambda^{3,\mathrm{na}}$ is reduced for $\eta \in \mathscr{T}_\delta$

(γ) $\langle N_{\eta,\beta} : \beta \le \alpha\rangle$ is $\le_{\mathfrak{K}_\lambda}$-increasing continuous

(δ) if $\eta, \nu \in \mathscr{T}_\delta$ and $\eta \restriction \alpha = \nu \restriction \alpha$ then $g_{\eta,\nu,\alpha}$ is an isomorphism from $N_{\nu,\alpha}$ onto $N_{\eta,\alpha}$ over N_α maping a_ν to a_η and being the identity on N_α

(ε) if $\beta < \alpha$ and $\eta, \nu \in {}^\delta 2, \eta \restriction \alpha = \nu \restriction \alpha$ then $g_{\eta,\nu,\beta} \subseteq g_{\eta,\nu,\alpha}$

(ζ) for every (equivalently some) $\eta \in \mathscr{T}_\delta$ the triple $(N_0, N_{\eta,0}, a_\eta)$ is isomorphic to the given triple (M, N, a).

There is no problem to carry the induction: for $\alpha = 0$ just note clause (ζ), e.g. let $N_0 = M$ and $\langle (N_{\eta,0}) : \eta \in \mathscr{T}_\delta\rangle$ pairwise disjoint copies of N over M.

For $\alpha = \beta + 1$ let $\mathscr{U} \subseteq \mathscr{T}_\beta$ be maximal such that $\eta \ne \nu \in \mathscr{U} \Rightarrow \eta \restriction \alpha \ne \nu \restriction \alpha$ and we use the assumption on $(M, N, a) \in K_\lambda^{3,\mathrm{nm}}$ to get a "first version" of $\langle N_\alpha, N_{\eta\hat{}\langle \ell\rangle,\alpha} : \eta \in \mathscr{U},$ and $\ell < 2\rangle$, <u>and</u> then use Exercise 2.6 to get "reduced".

$\square_{2.5}$

2.6 Exercise: Assume $(\mathrm{amg})_\lambda$.
1) We can find $M_*, \langle N_t : t \in I\rangle$ such that for each $t \in I$ we have $(M_t, N_t, a_t) \le_{\mathrm{na}} (M_*, N_t^+, a_t)$ and (M_*, N_t^+, a_t) is reduced [and if (e) also $\langle N_t \backslash M_* : t \in I\rangle$ are pairwise disjoint] when:

$\circledast$ (a) I is a set of cardinality $\le \lambda$
(b) $(M_t, N_t, a_t) \in K^{3,\mathrm{na}}_\lambda$ is reduced
(c) above (M_t, N_t, a_t) there is no minimal triple (or just every $\le_{\mathrm{na}}$-extension of it has the extension property)
(d) for every $s, t \in I$ we have $M_t \le_{\mathfrak{K}} M_s \vee M_s \le_{\mathfrak{K}} M_t$
(e) (optional) $N_t \backslash M_t$ is disjoint to $\cup\{N_s : s \in I\}$.

2) If we omit the "is reduced" in clause (b) then we get only $(M_t, N_t, a_t) \le^{\mathrm{na}}_{f_t} (M_*, N_2^+, a_t^+)$ for some f_t extending id_{M_t}.

* * *

Concerning disjoint amalgamation

2.7 Claim. *Assume* $(\mathrm{amg})_\lambda + (\mathrm{cat})_\lambda + (\mathrm{cat})_{\lambda^+}$, *i.e.* $\mathfrak{K}$ *has amalgamation in* λ *and is categorical in* λ, λ^+.
1) If $M \le_{\mathfrak{K}_\lambda} N$ *then we can find* $\bar M$ *such that*

(a) $\bar M = \langle M_\alpha : \alpha \le \alpha^*\rangle$ *is* $\le_{\mathfrak{K}_\lambda}$*-increasing continuous*
(b) $\alpha^* < \lambda^+$
(c) $M_0 = M$ *and* $N \le_{\mathfrak{K}_\lambda} M_{\alpha^*}$
(d) $(M_\alpha, M_{\alpha+1}, a_\alpha) \in K^{3,\mathrm{na}}_\lambda$ *is reduced.*

2) Assume that: if $p \in \mathscr{S}^{\mathrm{na}}_{\mathfrak{K}_\lambda}(M)$ *then* p *is non-*λ*-algebraic. Then* $\mathfrak{K}_\lambda$ *has disjoint amalgamation.*
3) Every $(M, N, a) \in K^{3,\mathrm{na}}_\lambda$ *has the extension property iff for no* $M \in K_\lambda$ *and* $p \in \mathscr{S}^{\mathrm{na}}(M)$ *is* p *a* λ*-algebraic type.*

Proof. 1) If not, we shall contradict categoricity in K_{λ^+}.

Clearly without loss of generality $M \ne N$. By induction on $i < \lambda^+$ we choose $N^0_i \in K_\lambda$, $\le_{\mathfrak{K}}$-increasing continuous such that $(N^0_i, N^0_{i+1}) \cong (M, N)$ (possible by the categoricity of K in λ). Let $N^0 = \bigcup\limits_{i<\lambda^+} N^0_i$.

By induction on $i < \lambda^+$ we choose $N^1_i \in K_\lambda, \le_{\mathfrak{K}}$-increasing continuous and a_i such that $(N^1_i, N^1_{i+1}, a_i) \in K^{3,\mathrm{na}}_\lambda$ is reduced (possible by 1.14(1) and the categoricity of K in λ) and let $N^1 = \bigcup\limits_{i<\lambda^+} N^1_i$. So by the categoricity in λ^+ without loss of generality $N^1 = N^0$, hence for some $\delta_1 < \delta_2 < \lambda^+$ we have

$$N^0_{\delta_1} = N^1_{\delta_1}, N^0_{\delta_2} = N^1_{\delta_2}.$$

By changing names $(N^0_{\delta_1}, N^0_{\delta_1+1}) = (M, N)$ and so $\langle N_{\delta_1+i} : i \le \delta_2 - \delta_1\rangle, \langle a_{\delta_1+i} : i < \delta_2 - \delta_1\rangle$ are as required.
2) So assume $M_0 \le_{\mathfrak{K}_\lambda} M_\ell$ for $\ell = 1, 2$. By part (1) we can find $\alpha < \lambda^+$, a $\le_{\mathfrak{K}}$-increasing continuous $\bar{M}_1 = \langle M_{1,i} : i \le \alpha\rangle$ such that $M_{1,0} = M_0, M_1 \le_{\mathfrak{K}_\lambda} M_{1,\alpha}$ and $(M_{1,i}, M_{1,i+1}, a_i) \in K^{3,\mathrm{na}}_\lambda$ is reduced for $i < \alpha$. Now we choose $(M_{2,i}, h_i)$ by induction on $i \le \alpha$ such that:

⊛ (a) $M_{2,i}$ is $\le_{\mathfrak{K}_\lambda}$-increasing continuous
  (b) $M_{2,0} = M_2$
  (c) h_i embeds $M_{1,i}$ into $M_{2,i}$
  (d) $h_0 = \mathrm{id}_{M_{1,0}} = \mathrm{id}_{M_0}$
  (e) if $i = j + 1$ then $h_i(a_j) \notin M_{2,i}$.

There is no problem to carry the definition as $\mathbf{tp}(a_i, M_{1,i}, M_{1,i+1})$ is not λ-algebraic and 2.5(2A). Now we can prove $h_i(M_{1,i}) \cap M_2 = M_0$ by induction on i.

For $i = 0$ this is trivial, for i limit obvious and for i successor say $i = j + 1$ use $(M_{1,j}, M_{1,j+1}, a)$ is reduced to show $h_i(M_{1,i}) \cap M_{2,j} = h_j(M_{1,j})$ so together with the induction hypothesis we are done.

Now $(h_\alpha \restriction M_1, M_{2,\alpha})$ exemplifies the existence of the disjoint amalgamation of M_1, M_2 over M_0.
3) By 2.5(2B). $\square_{2.7}$

2.8 Claim. *Assume* $(\mathrm{amg})_\lambda$ *and* $(\mathrm{nmx})_\lambda$*, i.e.,* $\mathfrak{K}$ *has no* $\le_{\mathfrak{K}_\lambda}$*-maximal member.*
1) There is $M \in K_{\lambda^+}$ *saturated above* λ *(above any* $N_0 \in K_\lambda$*; so* $N \in K_\lambda \Rightarrow |\mathscr{S}(N)| \le \lambda^+$*)* _when_

⊛ *small dense set of types: for* $M \in K_\lambda$ *there is* $\Gamma_M \subseteq \mathscr{S}^{\mathrm{na}}(M)$ *of cardinality* $\le \lambda^+$ *which is inevitable, see below.*

2) [Local version] Assume

$\circledast$ (a) $\mathscr{S}_* \subseteq \mathscr{S}^{\rm na}$ *is a* $\leq_{\mathfrak{K}_\lambda}$*-type-kind, see Definition 1.12*

(b) $|\mathscr{S}_*(M)| \leq \lambda^+$ *for every* $M \in K_\lambda$.

Then there is M such that:

$\odot_{M,\mathscr{S}_*}$ $M \in K_{\lambda^+}$ *and if* $\langle M_\alpha : \alpha < \lambda^+\rangle$ *is a* $\leq_{\mathfrak{K}}$*-representation of* M *then for a club of* $\delta < \lambda^+$, *every* $p \in \mathscr{S}_*(M_\delta)$ *is realized in* M.

3) If $\odot_{M,\mathscr{S}_*}$ *above holds and* $\mathscr{S}_* = \mathscr{S}_{\geq q}$, *or just* $\mathscr{S}_*$ *is hereditary then:* M *is* $\mathscr{S}_*$*-saturated, i.e. if* $M' <_{\mathfrak{K}} M, M' \in K_\lambda$ *and* $p \in \mathscr{S}_*(M')$, *then* p *is realized in* M.
4) Every $M \in K^{\rm lsl}_{\lambda^+}$ *satisfies the conclusion of parts (1),(2),(3) if the respective assumptions holds. So if* $\mathfrak{K}_{\lambda^+}$ *has a superlimit model and a model saturated above* λ *then the superlimit* $M \in \mathfrak{K}_{\lambda^+}$ *is saturated above* λ.
5) In part (2) we can replace clause (b) by

$(b)'$ (α) $\mathscr{S}_*$ *is hereditary*

(β) *for every* $M \in K_\lambda$ *there is an* $\mathscr{S}_*$*-inevitable set* $\Gamma \subseteq \mathscr{S}(M)$ *of cardinality* $\leq \lambda^+$.

2.9 Definition. 1) For $M \in K$ and $\Gamma \subseteq \mathscr{S}_{\mathfrak{K}_*}(M)$ we say that Γ is inevitable (for $\mathfrak{K}$) when: if $M <_{\mathfrak{K}} N$ then for some $a \in N \backslash M$ we have $\mathbf{tp}(a, M, N) \in \Gamma$.
2) We say $\Gamma \subseteq \mathscr{S}(M)$ is $\mathscr{S}_*$-inevitable when: if $M <_{\mathfrak{K}} N$ and some $p \in \mathscr{S}_*(M)$ is realized in N then some $q \in \Gamma$ is realized in N, so if $\mathscr{S}_* = \mathscr{S}^{\rm na}$ we get back "Γ is inevitable".

Proof. 1) Let $\langle S_i : i < \lambda^+\rangle$ be a partition of λ^+ such that $S_i \cap i = \emptyset$ and $|S_i| = \lambda$. Now we choose M_i and $\langle p^i_\alpha : \alpha \in S_i\rangle$ by induction on $i < \lambda^+$ such that:

$\circledast$ (a) $M_i \in \mathfrak{K}_\lambda$ is $\leq_{\mathfrak{K}}$-increasing continuous

(b) $\Gamma_i \subseteq \mathscr{S}^{\rm na}_{\mathfrak{K}}(M_i)$ is as guaranteed by clause $\circledast$ of the assumption; necessarily $\Gamma_i \neq \emptyset$ because $({\rm nmx})_\lambda$ holds

(c) $\langle p^i_\alpha : \alpha \in S_i \rangle$ list Γ_i, possibly with repetitions

(d) if $i \in S_j$ hence $j \leq i$ then the type p^i_j is realized in M_{i+1}.

As $\mathfrak{K}$ has amalgamation in λ there is no problem to carry the induction noting that Γ_i is necessarily non-empty. Let $M := \cup\{M_i : i < \lambda^+\}$ and we shall prove that M is saturated above λ, but we assume $(\mathrm{nmx})_\lambda$ hence M has cardinality λ^+.

So assume $M' \leq_{\mathfrak{K}} M, M' \in K_\lambda$ and $q \in \mathscr{S}_{\mathfrak{K}}(M')$ and we shall prove that q is realized in M. We can find $\alpha_0 < \lambda^+$ such that $M' \leq_{\mathfrak{K}} M_{\alpha_0}$ and $q_0 \in \mathscr{S}(M_{\alpha_0})$ extending q; hence we can find a pair (N_0, a) satisfying $(M_{\alpha_0}, N_0, a) \in K^{3,\mathrm{na}}_\lambda$, such that $q_0 = \mathbf{tp}(a, M_{\alpha_0}, N_0)$. We now try to choose $(N_\varepsilon, f_\varepsilon, \alpha_\varepsilon)$ by induction on $\varepsilon < \lambda^+$ such that

$\boxtimes$ (a) N_0, α_0 are chosen above, $f_0 = \mathrm{id}_{M_{\alpha_0}}$

(b) $N_\varepsilon \in K_\lambda$ is $\leq_{\mathfrak{K}}$-increasing continuous

(c) $\alpha_\varepsilon < \lambda^+$ is increasing continuous

(d) f_ε is a $\leq_{\mathfrak{K}}$-embedding of M_{α_ε} into N_ε

(e) f_ε is increasing continuous with ε

(f) for each ε, for some $c \in M_{\alpha_{\varepsilon+1}}$ realizing some $p \in \Gamma_{\alpha_\varepsilon}$ we have $f_{\varepsilon+1}(c) \in N_{\alpha_\varepsilon} \backslash f_\varepsilon(M_{\alpha_\varepsilon})$.

If we succeed to carry the induction, by clause (f) and Fodor lemma we get a contradiction. So we are stuck in some ε. For $\varepsilon = 0$ we apply clause (a) and all the demands hold. For limit ε we define by continuity. So necessarily for some $\varepsilon = \zeta + 1$ we have defined $(N_\zeta, f_\zeta, \alpha_\zeta)$ but cannot choose for ε.

If $f_\zeta(M_{\alpha_\zeta}) = N_\varepsilon$ then $f_\zeta^{-1}(a)$ realizes $\mathbf{tp}(a_0, M_{\alpha_0}, N_0)$ which is q_0 hence it realizes q in M so we are done. Otherwise as the property of $(M_{\alpha_\zeta}, \Gamma_{\alpha_\zeta})$ is preserved by isomorphisms, for some $p \in \Gamma_{\alpha_\zeta}$, the type $f_\zeta(p)$ is realized in N_ε say by $c' \in N_{\alpha_\zeta}$ and $c' \notin f_\zeta(M_{\alpha_\zeta})$ as $\Gamma_{\alpha_\zeta} \subseteq \mathscr{S}^{\mathrm{na}}_{\mathfrak{K}}(M_{\alpha_\zeta})$. By the construction of $\langle M_\alpha : \alpha < \lambda^+ \rangle$ some $c \in M$ realizes p. Let $\alpha_\varepsilon \in (\alpha_\zeta, \lambda^+)$ be such that $c \in M_{\alpha_\varepsilon}$. Now by basic properties of types (see II.1.11) there are $(N_\varepsilon, f_\varepsilon)$ such that $N_\zeta \leq_{\mathfrak{K}} N_\varepsilon \in K_\lambda$ and f_ε is a $\leq_{\mathfrak{K}}$-embedding of N_{α_ε} into N_ε extending f_ζ and $f_\varepsilon(c) = c'$. So we have not been stuck in ζ, contradiction.

2) Similarly to the proof of (1) noting that the desired conclusion is equivalent to "for some $\leq_{\mathfrak{K}}$-representation $\langle M_\alpha : \alpha < \lambda^+ \rangle$ of M, for

a club of $\alpha < \lambda^+$, every $p \in \mathscr{S}_*(M_\alpha)$ is realized in M".
3) So $\langle M_\alpha : \alpha < \lambda^+ \rangle$ is a $\leq_{\mathfrak{K}_\lambda}$-increasing sequence with union $M \in K_{\lambda^+}$ and for some club E of λ^+ (or just $E \subseteq \lambda^+ = \sup(E)$) we have $[\alpha \in E \ \& \ p \in \mathscr{S}_*(M_\alpha) \Rightarrow p$ is realized in $M]$.

Let $N^* \leq_{\mathfrak{K}} M, N^* \in K_\lambda$ and $p \in \mathscr{S}_*(N^*)$ and we should prove that p is realized in M. We can find $\alpha \in E$ such that $N^* \leq_{\mathfrak{K}} M_\alpha$. As $(\text{amg})_\lambda$ clearly there is $q \in \mathscr{S}(M_\alpha)$ extending p. Now first if $q \in \mathscr{S}^{\text{na}}(M_\alpha)$ as $\mathscr{S}_*$ is hereditary, necessarily $q \in \mathscr{S}_*(M_\alpha)$ hence q is realized in M by some $c \in M$, but then c also realizes p in M so we are done. Second, if $q \notin \mathscr{S}^{\text{na}}(M_\alpha)$ then q is algebraic hence realized by some $c \in M_\alpha$ and we are done.
4) It suffices to prove the conclusion of part (2) as the conclusion of part (1) holds by 1.3(6) and the conclusion of part (3) follows from the conclusion of part (2). Let $\langle S_i : i < \lambda^+ \rangle$ be as in part (1). We imitate the proof of part (1) but now we choose $M_i, M_i^*, \langle p'_\alpha : \alpha \in S_i \rangle$ by induction on $i < \lambda^*$ such that:

$\circledast$ (a) $M_i \in K_\lambda$ is $\leq_{\mathfrak{K}}$-increasing continuous
(b) $M_i^+ \in K_{\lambda^+}$ is $<_{\mathfrak{K}}$-increasing continuous
(c) $M_i \leq_{\mathfrak{K}} M_i^+$
(d) $M_i^+ \cong M$ has universe $\cup \{S_j : j < 1 + i\}$
(e) $\langle p^i_\alpha : \alpha \in S_i \rangle$ list $\mathscr{S}_*(M_i)$, possibly with repetitions
(f) if $i \in S_j$ hence $j \leq i$ then the type p^i_j is realized in M_{i+1}
(g) $M_j^+ \cap i \subseteq M_i$ for $j < i$.

This is easy, noting that we can preserve clause (d) as M is locally superlimit. Now $\cup\{M_{1,i} : i < \lambda\}$ is as required in part (2) and $\cup\{M_i^+ : i < \lambda\}$ is equal to it and is isomorphic to M so we are done.
5) Similarly. $\square_{2.8}$

Remark. More on the proof of 2.8, see hopefully in [Sh:F888].

Let us rephrase 2.8(4).

2.10 Conclusion. Assume $(\text{amg})_\lambda + (\text{nmx})_\lambda$.

If $\mathscr{S}_* \subseteq \mathscr{S}^{\rm na}$ is a hereditary $\le_{\mathfrak{K}_\lambda}$-type-kind (e.g. $\mathscr{S}^{\rm na}, \mathscr{S}_{\ge p}$) and $M \in K_\lambda \Rightarrow |\mathscr{S}_*(M)| \le \lambda^+$ <u>then</u> any $M \in K^{\rm lsl}_{\lambda^+}$ is $\mathscr{S}_*$-saturated, (so if $\mathscr{S} = \mathscr{S}^{\rm na}$, M is saturated above λ).

A sufficient condition for the demand in 2.10 is

2.11 Claim. *1) For every $M \in K_\lambda$ we have $|\mathscr{S}_*(M)| \le \lambda^+$ <u>when</u>:*

⊛ (a) $(\mathrm{amg})_\lambda$, *i.e., $\mathfrak{K}$ has amalgamation in λ*

(b) *$\mathscr{S}_* = \mathscr{S}_{\ge q}$ for some $q \in \mathscr{S}_{\mathfrak{K}_\lambda}(M^*)$ (or just $\mathscr{S}_*$ is a hereditary $\le_{\mathfrak{K}_\lambda}$-type-kind, see Definition 1.12)*

(c) *for every $M_0 \in K_\lambda$ there is an $\mathscr{S}_*$-inevitable $\Gamma_M \subseteq \mathscr{S}(M_0)$ of cardinality $\le \lambda^+$ (i.e. see Definition 2.9) or equivalently*

(c)' *for every $M_0 \in K_\lambda$ for some M_1 we have $M_0 <_{\mathfrak{K}} M_1 \in K_{\le \lambda^+}$ and if $M_0 <_{\mathfrak{K}} M_2$ and some $p \in \mathscr{S}_*(M_0)$ is realized in M_2 then we can find $b_1 \in M_1 \backslash M_0$ and $b_2 \in M_2 \backslash M_0$ such that $\mathbf{tp}(b_1, M_0, M_2) = \mathbf{tp}(b_2, M_0, M_2)$.*

2) If we strengthen clause (c) such that $|\Gamma_M| \le \lambda$, or in clause (c)′ demand $M_1 \in K_\lambda$ <u>then</u> $|\mathscr{S}_*(M)| \le \lambda$.

Proof. The same proof as 2.8. $\square_{2.11}$

2.12 Definition. 1) For $M \in K_\lambda$ and $1 \le \mu \le 2^\lambda$ we say that $p \in \mathscr{S}^{\rm na}(M)$ is μ-minimal <u>when</u>: for every N satisfying $M \le_{\mathfrak{K}} N \in K_\lambda$ the set $\{q \in \mathscr{S}^{\rm na}(N) : q \restriction M = p\}$ has $\le \mu$ members. Similarly $(< \mu)$-minimal.
2) For $M \in K_\lambda$, the type $p \in \mathscr{S}(M)$ is pseudo-minimal <u>if</u> it is λ^+-minimal.
3) $\mathscr{S}^{\mu\text{-minimal}}(M) = \{p \in \mathscr{S}(M) : p \text{ is } \mu\text{-minimal}\}$ for $M \in K_\lambda$.

2.13 Observation. 1) If $M \le_{\mathfrak{K}} N \in K_\lambda$ and $q \in \mathscr{S}^{\rm na}(N)$ and $q \restriction M$ is μ-minimal <u>then</u> q is μ-minimal.
2) The function $\mathscr{S}^{\mu\text{-minimal}}$ is a hereditary $\le_{\mathfrak{K}_\lambda}$-type-kind.
3) $p \in \mathscr{S}^{\rm na}_{\mathfrak{K}_\lambda}(M)$ is minimal iff it is 1-minimal.

4) If $p \in \mathscr{S}_{\mathfrak{K}_\lambda}(M)$ is μ_1-minimal and $1 \le \mu_1 \le \mu_2$ <u>then</u> p is μ_2-minimal.

Remark. So in particular minimal are pseudo minimal as $1 \le \lambda^+$.

Proof. Obvious. $\square_{2.13}$

2.14 Claim. *Assume* $(\text{amg})_{\lambda^+}$ $(\text{cat})_\lambda + (\text{mdn})^1_{\lambda^+}$ *and* $2^\lambda < 2^{\lambda^+}$.
1) If $p \in \mathscr{S}(M_0)$ is pseudo minimal, $M_0 \in K_\lambda$, <u>then</u> $N \in K_\lambda \Rightarrow |\mathscr{S}_{\ge p}(N)| \le \lambda^+$.
2) If in addition $N \in K^{\text{slm}}_{\lambda^+}$ <u>then</u> N is $\mathscr{S}_{\ge p}$-saturated.

Proof. 1) We shall use 2.16 below (so indirectly use 1.18 + 1.20 and 2.15). Choose $M_* \in K_\lambda$. Note that $\mathscr{S}_{\ge p^*}(N)$ has the same cardinality for every $N \in K_\lambda$ because of $(\text{cat})_\lambda$. So if $\mathscr{S}_{\ge p^*}(M_*)$ has cardinality $\le \lambda^+$ we are done. Otherwise, the assumptions of 2.16 below holds: first, $(\text{amg})_{\lambda^+}$ $(\text{cat})_\lambda$ and clause (a) there, are assumptions of 2.14, second, clause $(b)^-$ is an assumption of 2.14(1) and, third, clause (c) holds by the "otherwise" above. But the conclusion of 2.16 is $\dot{I}(\lambda^+, K) = 2^{\lambda^+}$ contradicting the assumptions $(\text{mdn})^1_{\lambda^+}$ of 2.14 so we are done.
2) Apply the conclusion of part (1) in 2.10, in equivalently in 2.8(2), (3),(4). $\square_{2.14}$

2.15 Claim. *Assume* $(\text{amg})_{\lambda^+}$ $(\text{cat})_\lambda, M^* \in K_\lambda$ *and* $p^* \in \mathscr{S}^{\text{na}}(M^*)$, *$\mathscr{S}_*$ is $\mathscr{S}_{\ge p^*}$ or just is a hereditary $\le_{\mathfrak{K}_\lambda}$-type-kind and lastly $|\mathscr{S}_*(M^*)| > \lambda^+$.*

If $N \in K_\lambda, \Gamma \subseteq \mathscr{S}_*(N), |\Gamma| \le \lambda^+$ <u>then</u>

$$\{p \in \mathscr{S}_*(N) : \textit{for some } N', N \le_{\mathfrak{K}} N' \in K_\lambda \textit{ and some } b \in N' \textit{ realizes } p \textit{ but no } b \in N' \textit{ realizes any } q \in \Gamma\}$$

has cardinality $\ge \lambda^{++}$.

Proof. Apply 1.18 + 1.20 with N and $\mathscr{S}_*(N)$ here standing for N and $\mathscr{S}$ there, so we get $N^*, N_i(i < \lambda^{++})$ such that $N \le_{\mathfrak{K}} N^* \le_{\mathfrak{K}} N_i \in \mathfrak{K}_\lambda$, and $\Gamma_i = \{\mathbf{tp}(a, N^*, N_i) : a \in N_i \backslash N^*\}$ are pairwise disjoint and there are $p_i \in \Gamma_i$ such that $q_i := p_i \restriction N \in \mathscr{S}_*(N)$ are pairwise distinct. Now p_i is from Γ_i is not algebraic and $\mathscr{S}_*$ is hereditary hence $p_i \in \mathscr{S}_*(N^*)$. As K is categorical in λ, there is an isomorphism f from N^* onto N, and we can find (N'_i, f'_i) such that $N \le_{\mathfrak{K}} N'_i$ and $f_i \supseteq f$ is an isomorphism from N_i onto N'_i, so all but $\le \lambda^+$ of the models N_i can serve as the required N'.

$\square_{2.15}$

2.16 Claim. *Assume* $(\mathrm{amg})_\lambda + (\mathrm{cat})_\lambda$.

We have $\dot{I}(\lambda^+, \mathfrak{K}) \ge \mu_{\mathrm{wd}}(\lambda^+)$ *and moreover* $= 2^{\lambda^+}$ *<u>when</u> (for some or any $M_* \in K_\lambda$):*

- $\circledast$ (a) $2^\lambda < 2^{\lambda^+}$
  - (b) $\mathscr{S}_* = \mathscr{S}_{\ge p^*}, p^* \in \mathscr{S}_{\mathfrak{K}_\lambda}(M_*)$ *is pseudo minimal or just*
  - $(b)^-$ $\mathscr{S}_*$ *is a hereditary $\le_{\mathfrak{K}_\lambda}$-type-kind and* $\mathscr{S}_* \subseteq \mathscr{S}^{\lambda^+\text{-minimal}}$
  - (c) $\mathscr{S}_*(M_*)$ *has cardinality* $> \lambda^+$ *so* $M_* \in K_\lambda$.

2.17 Remark. 0) On $\mu_{\mathrm{wd}}(\lambda^+)$, see VII.0.3(6), if $2^\lambda < 2^{\lambda^+}$ it is close to 2^{λ^+}.
1) We may consider replacing the assumption (c) by $\boxtimes$ from the proof.
2) We can replace "$\le \lambda^+$" by "$< \mu$" if $|\mathscr{S}_{\ge p^*}(M^*)| \ge \mu$ and $\mathrm{cf}(\mu) \ge \lambda^+$.
3) In 2.16, can we get $\dot{I}\dot{E}(\lambda^+, \mathfrak{K}) = 2^{\lambda^+}$, see probably [Sh:E45].

Proof. Now by 2.15 we have

- $\boxtimes$ if $M \in K_\lambda$ and $\Gamma \subseteq \mathscr{S}_*(M)$ is of cardinality λ^+ <u>then</u> for some N, b we have $M <_{\mathfrak{K}_\lambda} N, b \in N \backslash M$ realizes some type from $\mathscr{S}_*(M)$ but no $q \in \Gamma$ is realized in N.

Now we shall apply 2.18 for $\mu = \lambda^{++}$. Clearly the conclusion of 2.18 is the desired conclusion of 2.16. Now $(\mathrm{amg})_\lambda$ is assumed in 2.16, clause (a),(c),(d) of 2.18 holds by clauses (a),(b) $\vee$ (b)$^-$,(c) of 2.16 respectively, clause (b) of 2.18 holds as $\mathrm{cf}(\mu) = \mathrm{cf}(\lambda^{++}) \geq \lambda^+$, and clause (e) of 2.18 holds by $\boxtimes$ above. $\square_{2.16}$

2.18 Claim. *Assume* $(\mathrm{amg})_\lambda$. *We have* $\dot{I}(\lambda^+, \mathfrak{K}) = 2^{\lambda^+}$ *when*

⊛ (a) $2^\lambda < 2^{\lambda^+}$

(b) $\mathrm{cf}(\mu) \geq \lambda^+$

(c) $\mathscr{S}_* \subseteq \mathscr{S}^{(<\mu)\text{-minimal}}$

(d) $\mathscr{S}_*(M_*)$ *has cardinality* $\geq \mu$ *for some* $M_* \in K_\lambda$

(e) *if* $M_* \leq_{\mathfrak{K}} M \in K_\lambda, \Gamma \subseteq \mathscr{S}_*(M)$ *has cardinality* $< \mu$ *then for some* N, b *we have* $M <_{\mathfrak{K}_\lambda} N, b \in N \backslash M$ *realizes a type from* $\mathscr{S}_*(M)$ *but no* $q \in \Gamma$ *is realized in* N.

Proof. Let $\mathbf{M} = \{(M, \Gamma) : M \in K_\lambda$ and $\Gamma \subseteq \cup\{\mathscr{S}_*(M') : M' \leq_{\mathfrak{K}_\lambda} M\}$ has cardinality $\leq \lambda$ and no $p \in \Gamma$ is realized in $M\}$ and let $\leq_{\mathbf{M}}$ be the following two-place relation on $\mathbf{M}$: $(M_1, \Gamma_1) \leq (M_2, \Gamma_2)$ iff $M_1 \leq_{\mathfrak{K}_\lambda} M_2$ and $\Gamma_1 \subseteq \Gamma_2$.

Easily

$(*)_1$ $\leq_{\mathbf{M}}$ partially order $\mathbf{M}$

$(*)_2$ if $\langle (M_\alpha, \Gamma_\alpha) : \alpha < \delta \rangle$ is $\leq_{\mathbf{M}}$-increasing and δ is a limit ordinal $< \lambda^+$ and $M := \cup\{M_\alpha : \alpha < \delta\}$ and $\Gamma_\delta = \cup\{\Gamma_\alpha : \alpha < \delta\}$ then $(M_\delta, \Gamma_\delta) \in \mathbf{M}$ is the $\leq_{\mathbf{M}}$-lub of the sequence.

[Why? E.g. does M_δ omit every $p \in \Gamma_\delta$? Clearly for some $\alpha < \delta$ we have $p \in \Gamma_\alpha$ hence for some $M' \leq_{\mathfrak{K}} M_\alpha$ we have $p \in \mathscr{S}_*(M')$. Also if $c \in M_\delta$ then for some $\beta < \delta$ we have $c \in M_\beta$ and without loss of generality $\alpha < \beta < \delta$, but $p \in \Gamma_\alpha \subseteq \Gamma_\beta$ and $(M_\beta, \Gamma_\beta) \in \mathbf{M}$ hence M_β omits p hence c does not realize p in M_β but $M_\beta \leq_{\mathfrak{K}} M_\delta$ hence c does not realize p in M_δ, as required. Also $|\Gamma_\delta| \leq \lambda$ as $\mathrm{cf}(\mu) \geq \lambda^+$.]

Now by induction on $i < \lambda^+$, choose $\langle (N_\eta, \Gamma_\eta) : \eta \in {}^i 2 \rangle$ such that:

⊛ (a) $(N_\eta, \Gamma_\eta) \in \mathbf{M}$

(b) $(N_\nu, \Gamma_\nu) \leq_{\mathbf{M}} (N_\eta, \Gamma_\eta)$ for ever $\nu \triangleleft \eta$

(c) if i is a limit ordinal, then $N_\eta = \bigcup_{j<i} N_{\eta \restriction j}$ and $\Gamma_\eta = \bigcup_{j<i} \Gamma_{\eta \restriction j}$

(d) (α) some $p \in \Gamma_{\eta\char`^\langle 0 \rangle}$ is from $\mathscr{S}_*(N_\eta)$ and is realized in $N_{\eta\char`^\langle 1 \rangle}$

(β) similarly for $\Gamma_{\eta\char`^\langle 1 \rangle}, N_{\eta\char`^\langle 0 \rangle}$

For $i = 0$ let $M_{<>} = M_*$ and $\Gamma_{<>} = \emptyset$.

For i limit, for $\eta \in {}^i 2$ we choose (N_η, Γ_η) by clause (c) of $\circledast$. Now the demands in $\circledast$ holds: for clauses (e) as if $p \in \Gamma_\eta$ recall $(*)_2$. The other clauses are even easier.

Lastly, assume $i < \lambda^+$ is a successor ordinal say $i = j + 1$ and let $\eta \in {}^j 2$. We define $\Gamma'_\eta = \{q \in \mathscr{S}(N_\eta) : q$ extends some $p \in \Gamma_\eta\}$.
So clearly $\Gamma'_\eta \subseteq \mathscr{S}^{\mathrm{na}}(N_\eta)$ because no $p \in \Gamma_\eta$ is realized in N_η by clause (d) of the induction hypothesis but $\mathscr{S}_*$ is hereditary so

$(*)_3$ $\Gamma'_\eta \subseteq \mathscr{S}_*(N_\eta)$.

Also

$(*)_4$ Γ'_η has cardinality $< \mu$.

[Why? Because

$$|\Gamma'_\eta| \leq \Sigma\{|\{q \in \mathscr{S}(N_\eta) : q \restriction N_{\eta \restriction j_1} = p\} : p \in \Gamma_\eta \cap \mathscr{S}(N_{\eta \restriction j_1}) \text{ and } j_1 \leq j\}| < \mu$$

The inequalities are justified as follows: the first inequality by the definition of Γ'_η, the second inequality as each $p \in \Gamma_\eta$ is $(< \mu)$-minimal, so the middle term is the sum of $\leq |\Gamma_\eta| \leq \lambda$ sets each of cardinality $< \mu$, but $\mathrm{cf}(\mu) > \lambda$ so the sum is $< \mu$.]

By clause (e) of the assumption applied to (N_η, Γ'_η) we can find $(N_{\eta,0}, b_{\eta,0})$ such that $N_\eta \leq_{\mathfrak{K}_\lambda} N_{\eta,0}$, $b_{\eta,0} \in N_{\eta,0}$ and $p_{\eta,0} := \mathbf{tp}(b_{\eta,0}, N_\eta, N_{\eta,0})$ belongs to $\mathscr{S}_*(N_\eta) \backslash \Gamma'_\eta$ and no $c \in N_{\eta,0}$ realizes any $q \in \Gamma'_\eta$.

Let $\Gamma'_{\eta,1} = \Gamma'_\eta \cup \{\mathbf{tp}(b, N_\eta, N_{\eta,0}) : b \in N_{\eta,0}$ and $\mathbf{tp}(b, N_\eta, N_{\eta,0}) \in \mathscr{S}_*(N_\eta)\}$, so $p_{\eta,0} \in \Gamma'_{\eta,1}$.

By clause (e) of the assumption applied to the pair $(N_\eta, \Gamma'_{\eta,1})$ we can find $(N_{\eta,1}, b_{\eta,1})$ such that $N_\eta \leq_{\mathfrak{K}_\lambda} N_{\eta,1}$, no member of $N_{\eta,1}$ realizes any type from $\Gamma'_{\eta,1}$ and $b_{\eta,1} \in N_{\eta,1}$ and $p_{\eta,1} := \mathbf{tp}(b_{\eta,1}, N_\eta, N_{\eta,1})$ belongs to $\mathscr{S}_*(N_\eta) \backslash \Gamma'_{\eta,1}$.

Next, let $\Gamma_{\eta,0} = \Gamma_\eta \cup \{\mathbf{tp}(b, N_\eta, N_{\eta,1}) : b \in N_{\eta,0}$ and $\mathbf{tp}(b, N_\eta, N_{\eta,1}) \in \mathscr{S}_*(N_\eta)\}$ and $\Gamma_{\eta,1} = \Gamma_\eta \cup \Gamma'_{\eta,1}$.

Lastly, let $N_{\eta\hat{}<\ell>} = N_{\eta,\ell}, \Gamma_{\eta\hat{}<\ell>} = \Gamma_{\eta,\ell}$ for $\ell = 0, 1$. Now it is easy to check that they are as required, e.g. $N_{\eta\hat{}<\ell>}$ omit every $p \in \Gamma_\eta$ as if c realizes p in $N_{\eta\hat{}<\ell>}$ then $p' := \mathbf{tp}(c, N_\eta, N_{\eta\hat{}<\ell>}) \in \mathscr{S}_*(N_{\eta\restriction\varepsilon})$ hence $p' \in \Gamma'_\eta \subseteq \Gamma'_{\eta,1}$, contradiction.

For $\eta \in {}^{\lambda^+}2$ let $N_\eta = \bigcup\limits_{i<\lambda^+} N_{\eta\restriction i}$.

Clearly

(∗) if $\eta\hat{}\langle\ell\rangle \lhd \nu_\ell \in {}^{\lambda^+}2$ for $\ell = 1, 2$ <u>then</u> N_{ν_1}, N_{ν_2} are not isomorphic over N_η.

This is enough to get $\dot{I}(\lambda^+, K)$ large, in fact $\geq |\{N_\eta/\cong : \eta \in {}^{\lambda^+}2\}| \geq \mu_{\rm wd}(\lambda)$ by VII.9.6; this is enough for 2.16. But the situation is similar to the one in I.3.8 hence the set $\{N_\eta/\cong : \eta \in {}^{\lambda^+}2\}$ has cardinality 2^{λ^+} by VII.9.7, but we give details. We consider the demand (not necesary when it suffices for us to prove $\dot{I}(\lambda^+, K) \geq \mu_{\rm wd}(\lambda^+)$)

$\odot_{(M_1,\Gamma_1),(M_2,\Gamma_2)}$ (a) $(M_1, \Gamma_1) \leq_{\mathbf{M}} (M_2, \Gamma_2)$

(b) if $(M_2, \Gamma_2) \leq_{\mathbf{M}} (M^\ell, \Gamma^\ell)$ for $\ell = 1, 2$ then we can find $(M^{\ell+2}, \Gamma^{\ell+2})$ for $\ell = 1, 2$ such that

(α) $(M^\ell, \Gamma^\ell) \leq (M^{\ell+2}, \Gamma^{\ell+2})$ for $\ell = 1, 2$

(β) $\{p \in \mathscr{S}_*(M_1) : p$ realized in $M^3\} = \{p \in \mathscr{S}_*(M_1) : p$ realized in $M^4\}$.

The proof below will make a distinction to two cases.

<u>Case 1</u>: For some $(M_1, \Gamma_1) \leq_{\mathbf{M}} (M_2, \Gamma_2)$, for every (M_3, Γ_3) which is $\leq_{\mathbf{M}}$-above (M_2, Γ_2) we have $\neg\odot_{(M_1,\Gamma_1),(M_3,\Gamma_3)}$.

In this case we can in ⊗ demand $(M_{<>}, \Gamma_{<>}) = (M_1, \Gamma_1)$ and $(M_{<k>}, \Gamma_{<k>}) = (M_2, \Gamma_2)$ for $k = 0, 1$ and replace clause (d) of ⊛ above by

$(d)'$ if $i = j + 1 \geq 2$ and $\eta \in {}^j2$ and $(M_{\eta\hat{}<k>}, \Gamma_{\eta\hat{}<k>}) \leq_{\mathbf{M}} (M^k, \Gamma^k)$ for $k = 0, 1$ then $\{p \in \mathscr{S}_*(M_1) : p$ realizes in $M^1\} \neq \{p \in \mathscr{S}_*(M_1) : p$ realizes in $M^2\}$.

Easily done and sufficient by VII.9.5.

Case 2: Note Case 1.

So for every $(M_1, \Gamma_2) \leq_{\mathbf{M}} (M_2, \Gamma_3)$ there is no (M_3, Γ_3) as above. Now in $\circledast$ we add

(e) if $i = j+1, \nu \in {}^i 2, \eta = \nu \restriction j$ then $\odot_{(M_\nu, \Gamma_\nu),(M_\eta, \Gamma_\eta)}$.

Now we can apply VII.9.7, as in the proof of 3.11 below. $\square_{2.18}$

2.19 Claim. *Assume* $(\mathrm{amg})_\lambda + (\mathrm{cat})_\lambda + (\mathrm{slm})_{\lambda^+}$ *and* $2^\lambda + \dot{I}(\lambda^+, K) < 2^{\lambda^+}$.

1) If $M_0 \in K_\lambda, M_1 \in K^{\mathrm{slm}}_{\lambda^+}$, *and* $M_0 \leq_{\mathfrak{K}} M_1$ *then every* λ^+*-minimal type* $p \in \mathscr{S}(M_0)$ *is realized in* M.

2) Every $M_1 \in K^{\mathrm{slm}}_{\lambda^+}$ *is saturated at least for* λ^+*-minimal types (i.e. if* $M_0 \leq_{\mathfrak{K}} M_1$,
$M_0 \in K_\lambda$ *and* $M_1 \in K^{\mathrm{slm}}_{\lambda^+}$ *then every* λ^+*-minimal* $p \in \mathscr{S}(M_0)$ *is realized in* M_1*).*

3) If $M \in K_\lambda$ *then* $\{p \in \mathscr{S}(M) : p$ *is* λ^+*-minimal*$\}$ *has cardinality* $\leq \lambda^+$.

4) The above holds for minimal types, too.

2.20 Remark. 1) Compare with 5.3.

2) Instead $\dot{I}(\lambda^+, \mathfrak{K}) < 2^{\lambda^+}$ we can assume $|\mathscr{S}_{\geq p}(M)| \leq \lambda^+$ when $M \in \mathfrak{K}_\lambda, p \in \mathscr{S}(M)$ is λ^+-minimal.

Proof. 1) Let $N \in K_\lambda, p \in \mathscr{S}(N)$ be λ^+-minimal (if there is no such N then the desired conclusion holds vaccuously).

We have $(\mathrm{mdn})^1_{\lambda^+}$ because $\dot{I}(\lambda^+, \mathfrak{K}) \geq 1$ follows from $(\mathrm{slm})_{\lambda^+}$ and $\dot{I}(\lambda^+, \mathfrak{K}) < 2^{\lambda^+}$ which are assumptions; so the assumptions of 2.14 holds, hence its conclusions.

In particular by 2.14(2) any $N'_{\lambda^+} \in K^{\mathrm{slm}}_{\lambda^+}$ is $\mathscr{S}_{\geq p}$-saturated hence this holds for N_{λ^+}.

So this holds for every λ^+-minimal p as required.

2) By part (1).

3) Follows by part (1).

4) Trivially. $\square_{2.19}$

Now in 2.21 - 2.24 we shall use stronger hypothesis, $(\text{cat})_{\lambda^+}$ which is enough for our main result, still "expensive", in particular not enough for Conjecture 0.4, see §0.

Later in 2.25 we use $K_{\lambda^{+3}} = \emptyset$, but also a cardinal arithmetic assumption which is stronger than WGCH is assumed.

We use categoricity in λ^+ to deduce that every triple from $K^{3,\text{na}}_\lambda$ has the extension property. By earlier results we can concentrate on minimal types which we investigate. We prefer not to use $(\text{cat})_{\lambda^+}$, categoricity in λ^+ but just the existence superlimit but we do not succeed to avoid it.

Also we show how $K_{\lambda^{+3}} = \emptyset +$ cardinal arithmetic helps to prove "minimal types are dense". This we do not consider acceptable assumption and eventually we shall avoid those assumptions (by the harder proofs in §3,§4).

2.21 Claim. *Assume* $(\text{amg})_\lambda + (\text{cat})_\lambda + (\text{cat})_{\lambda^+} + (\text{nmx})_{\lambda^+} + 2^\lambda < 2^{\lambda^+}$.

If $(M_0, M_1, a) \in K^{3,\text{na}}_\lambda$ *is minimal, then it has the extension property.*

Remark. 1) We have gotten the same conclusion from different assumptions in 2.7.
2) The proof uses categoricity in λ^+ not just "intermediate number of models". Recall that $(\text{mdn})^1_{\lambda^+}$ follows from $(\text{cat})_{\lambda^+}$.

Proof. Let $p^* = \mathbf{tp}(a, M_0, M_1)$, and assume $(M_0, M_1, a) \in K^{3,\text{na}}_\lambda$ is a counter-example. We note:

- $\otimes_1$ if $p \in \mathscr{S}_{\geq p^*}(N)$ and $N \in K_\lambda$ and $N \leq_{\mathfrak{K}} N^* \in K$, then the set of elements of $b \in N^*$ realizing p has cardinality $\leq \lambda$. [Why? By 2.5(2)]
- $\otimes_2$ if $N \in K_\lambda$, then $|\mathscr{S}_{\geq p^*}(N)| > \lambda^+$.

Proof of $\otimes_2$. If N forms a counterexample, as K is categorical in λ we have $M \in K_\lambda \Rightarrow |\mathscr{S}_{\geq p^*}(M)| \leq \lambda^+$. On the one hand by $(\text{cat})_{\lambda^+}$, any $N \in K_{\lambda^+}$ belongs to $K^{\text{lsl}}_{\lambda^+}$ hence by 2.10 is $\mathscr{S}_{\geq p^*}$-saturated.

On the other hand, we can choose by induction on $\alpha < \lambda^+$ a triple $(N_{0,\alpha}, N_{1,\alpha}, a) \in K^{3,\mathrm{na}}_\lambda$ which is $\le_{\mathrm{na}}$-increasing continuous in α such that $(N_{0,0}, N_{1,0}, a) = (M_0, M_1, a)$ and $N_{0,\alpha} \ne N_{0,\alpha+1}$ (we can carry the induction for $\alpha = 0$ trivially, for α, limit by 1.14(2)(a) and for $\alpha = \beta + 1$ by the weak extension property which holds by 1.9 because its assumptions $(\mathrm{cat})_\lambda+$ $(\mathrm{nmx})_{\lambda^+}$ are assumed here). Now $N_{0,\lambda^+} = \bigcup\limits_{\alpha<\lambda^+} N_{0,\alpha} \in K_{\lambda^+}$ hence by the previous paragraph it is $\mathscr{S}_{\ge p^*}$-saturated, but $\langle N_{0,\alpha} : \alpha < \lambda^+\rangle$ is a $\le_{\mathfrak{K}}$-representation of N_{0,λ^+}, and for every $\alpha, \mathbf{tp}(a, N_{0,\alpha}, N_{1,\alpha})$ extend $\mathbf{tp}(a, N_{0,0}, N_{1,0}) = \mathbf{tp}(a, M_0, M_1) = p^*$ hence $p_\alpha = \mathbf{tp}(a, N_{0,\alpha}, N_{1,\alpha}) \in \mathscr{S}_{\ge p^*}(N_{0,\alpha}) \subseteq \mathscr{S}^{\mathrm{na}}(N_{0,\alpha})$ is realized in N_{0,λ^+} say by $b_\alpha \in N_{0,\lambda^+}$ but $p_\alpha \in \mathscr{S}^{\mathrm{na}}(N_{0,\alpha})$ so $b_\alpha \notin N_\alpha$ and it realizes p^* so $\lambda^+ = |\{b_\alpha : \alpha < \lambda^+\}|$. But this contradicts $\otimes_1$. So $\otimes_2$ holds. $\square_{\otimes_2}$

To finish the proof of 2.21 we shall use 2.14(1) for $p = p^*$.

Its conclusion "$N \in K_\lambda \Rightarrow |\mathscr{S}_{\ge p^*}(N)| \le \lambda^+$", contradiction to $\otimes_2$. Among its assumptions, "$(\mathrm{amg})_\lambda+$ $(\mathrm{cat})_\lambda$ $+ 2^\lambda < 2^{\lambda^+}$" are also assumptions of the present claim and "$(\mathrm{mdn})^1_{\lambda^+}$" follows from $(\mathrm{cat})_{\lambda^+}$.

Lastly, "p is pseudo minimal" holds for p^* as p^* is minimal by 2.13(4) recalling Definition 2.12; contradiction, so (M, N, a) cannot be a counterexample. So we are done. $\square_{2.21}$

2.22 Claim. *Assume* $(\mathrm{amg})_\lambda$, $(\mathrm{cat})_\lambda$, $(\mathrm{cat})_{\lambda^+}$ *and* $(\mathrm{nmx})_{\lambda^+}$. *If* $(M, N, a) \in K^{3,\mathrm{na}}_\lambda$ *fails the extension property and* $p = \mathbf{tp}(a, M, N)$ *is minimal, then* $|\mathscr{S}_{\ge p}(M)| > \lambda^+$.

Proof. Similar to an initial segment of the proof of 2.21, i.e. $\otimes_1 + \otimes_2$ there. $\square_{2.22}$

2.23 Conclusion. Assume as in 2.21 or for parts (1),(2) just the conclusion of 2.21 and $(\mathrm{amg})_\lambda$ and for part (3) the conclusion of 2.21 + $(\mathrm{amg})_\lambda+$ $(\mathrm{nmx})_\lambda$.

1) Every triple $(M, N, a) \in K^{3,\mathrm{na}}_\lambda$ has the extension property.

2) If $M \le_{\mathfrak{K}_\lambda} N$ and $p \in \mathscr{S}^{\min}(M)$ then p has one and only one

extension in $\mathscr{S}^{\rm na}(M)$ and it is minimal.
3) $\mathfrak{K}_\lambda$ has disjoint amalgamation.

Proof. 1) Let $(M,N,a) \in K^{3,\rm na}_\lambda$ be given.

Case 1: There is a minimal $(M',N',a) \in K^{3,\rm na}_\lambda$ which is $\le_{\rm na}$-above (M,N,a).

By 2.21 the triple (M',N',a) has the extension property, and by 1.10(1) this implies that also the triple (M,N,a) has the extension property.

Case 2: Not Case 1.

By 2.5(1) the result follows.
2) We know that there is at most one extension in $\mathscr{S}^{\rm na}(N)$ and by the extension property there is at least one, by 1.14(8)(b)).
3) Follows from part (1) by 2.7(2),(3) or recalling 2.5(2B). $\square_{2.23}$

2.24 Claim. *[*$({\rm amg})_\lambda + ({\rm cat})_\lambda + ({\rm cat})_{\lambda^+}$.*]*

If $p \in \mathscr{S}_{\mathfrak{K}_\lambda}(M^*)$ *is minimal*[6] *then the set* $\mathscr{S}_{\ge p}(M^*)$ *is inevitable, i.e. if* $M <_{\mathfrak{K}} N$ *in* K_λ *then for some* $a \in N \backslash M$ *is* $\mathbf{tp}(a,M,N) \in \mathscr{S}_{\ge p}(M)$.

Proof. Otherwise, we can find $\langle M^1_i : i < \lambda^+\rangle$, a $\le_{\mathfrak{K}}$-representation of a model $M^1 \in K_{\lambda^+}$ such that: $a \in M^1_{i+1}\backslash M^1_i \Rightarrow \mathbf{tp}(a,M^1_i,M^1_{i+1}) \notin \mathscr{S}_{\ge p}(M^1_i)$. This implies $i < \lambda^+ \wedge a \in M^1 \backslash M^1_i \Rightarrow \mathbf{tp}(a,M^1_i,M^1) \notin \mathscr{S}_{\ge p}(M^1_i)$ (as for some $j \in [i,\lambda^+)$ we have $a \in M^1_{j+1}\backslash M^1_j$, so $(M^1_i, M^1_{j+1},a) \le_{\rm na} (M^1_j, M^1_{j+1},a)$ and the latter is not in $\mathscr{S}_{\ge p}(M^1_j)$). But we can build another $\le_{\mathfrak{K}}$-representation $\langle M^2_i : i < \lambda^+\rangle$ of a model $M^2 \in K_{\lambda^+}$ such that for each $i < \lambda^+$ for some $a \in M^1_{i+1}\backslash M^1_i$ we have $\mathbf{tp}(a,M^1_i,M^1_{i+1}) \in \mathscr{S}_p(M^1_i) \subseteq \mathscr{S}_{\ge p}(M^1_i)$; (why? as K is categorical in λ hence M^*, M^1_i are isomorphic hence there is $q \in \mathscr{S}_p(M^1_i)$ so by the definition of type there are M^1_i, a_i such that $M^1_i \le_{\mathfrak{K}_\lambda} M^1_{i+1}$ and $q = \mathbf{tp}_{\mathfrak{K}_\lambda}(a_i,M^1_i,M^1_{i+1})$ so M^1_{i+1} is as required). So $M^1 \not\cong M^2$ contradicting the assumption $({\rm cat})_{\lambda^+}$. $\square_{2.24}$

[6]Note that every $q \in \mathscr{S}_{\ge p}(M^*)$ is minimal but still it is not clear if $\mathscr{S}_{\ge p}(M^*) = \mathscr{S}_p(M^*)$

2.25 Claim. *Assume*

(*a*) $(\text{amg})_{\lambda^+}$ $(\text{amg})_{\lambda^{++}}$ $(\text{jep})_\lambda$ *hence* $(\text{jep})_{\lambda^+}$
(*b*) $K_{\lambda^{++}} \neq \emptyset$
(*c*) $2^{\lambda^+} > \lambda^{++}$
(*d*) $K_{\lambda^{+3}} = \emptyset$.

Then in $K^{3,\text{na}}_\lambda$ the minimal triples are dense (i.e., $\leq_{\text{na}}$-above every triple in $K^{3,\text{na}}_\lambda$ there is a minimal one).

Remark. We do not intend to adopt the hypotheses "$2^{\lambda^+} > \lambda^{++}$", "$K_{\lambda^{+3}} = \emptyset$". They will be eliminated in §3,§4.

Proof. As $K_{\lambda^{++}} \neq \emptyset$ and $\text{LS}(\mathfrak{K}) \leq \lambda$ it follows that $K_\lambda \neq \emptyset \neq K_{\lambda^+}$. Toward contradiction assume that the desired conclusion fails. By 2.3(1) we can choose by induction on $\alpha < \lambda^+$, for $\eta \in {}^\alpha 2$ a triple $(M^0_\eta, M^1_\eta, a_\eta)$ such that:

⊛ (i) $(M^0_\eta, M^1_\eta, a_\eta) \in K^{3,\text{na}}_\lambda$
(ii) $\nu \triangleleft \eta \Rightarrow (M^0_\nu, M^1_\nu, a) \leq_{\text{na}} (M^0_\eta, M^1_\eta, a_\eta)$
(iii) $M^0_{\eta\hat{}<0>} = M^0_{\eta\hat{}<1>}$
(iv) $\mathbf{tp}(a, M^0_{\eta\hat{}\langle 0\rangle}, M^1_{\eta\hat{}\langle 0\rangle})) \neq \mathbf{tp}(a, M^0_{\eta\hat{}\langle 1\rangle}, M^1_{\eta\hat{}\langle 1\rangle})$
(v) if $\eta \in {}^\delta 2$ and $\delta < \lambda^+$ is a limit ordinal, then $M^\ell_\eta = \bigcup_{\alpha<\delta} M^\ell_{\eta\restriction\alpha}$ for $\ell = 0, 1$,
(vi) $(M^0_{<>}, M^1_{<>}, a) \in K^{3,\text{mn}}_\lambda$, i.e. is a triple $\in K^{3,\text{na}}_\lambda$ above which there is no minimal one.

Let $M^* \in \mathfrak{K}_{\lambda^{++}}$ be $\leq_{\mathfrak{K}}$-maximal (exists by 1.4(3) noting $K_{\lambda^{++}} \neq \emptyset$). So by 1.4(4) the model M^* is necessarily homogeneous universal above λ^+; (note: λ there stands for λ^+ here). As $\text{LS}(\mathfrak{K}) \leq \lambda$ and $\mathfrak{K}_\lambda$ has amalgamation, clearly M^* is saturated also above λ.

We choose by induction on $\alpha < \lambda^+$ for every $\eta \in {}^\alpha 2$, a $\leq_{\mathfrak{K}}$-embedding g_η of M^0_η into M^* such that:

$(*)$ (a) $\nu \triangleleft \eta \Rightarrow g_\nu \subseteq g_\eta$

(b) $g_{\eta\char`\^\langle 0\rangle} = g_{\eta\char`\^\langle 1\rangle}$.

This is clearly possible. For $\eta \in {}^{\lambda^+>}2$ let $N^0_\eta = M^* \restriction \mathrm{Rang}(g_\eta) = g_\eta(M^0_\eta)$. For $\eta \in {}^{\lambda^+}2$ let $N^0_\eta = M^* \restriction \cup\{\mathrm{Rang}(g_{\eta\restriction\alpha}) : \alpha < \lambda^+\}$ and let $g_\eta = \cup\{g_{\eta\restriction\alpha} : \alpha < \lambda^+\}$ and let $M^\ell_\eta = \cup\{M^0_{\eta\restriction\alpha} : \alpha < \lambda^+\}$ for $\ell = 0,1$ hence $M^0_\eta \leq_{\mathfrak{K}_{\lambda^+}} M^1_\eta$ and g_η is a $\leq_{\mathfrak{K}}$-embedding of M^0_η into M^*. But $\mathfrak{K}_{\lambda^+}$ has amalgamation and M^* is homogeneous univeral above λ^+ so we can extend f_η to f^+_η, a $\leq_{\mathfrak{K}}$-embedding of M^1_η into M^*.

Let $a^*_\eta = f^+_\eta(a)$ for $\eta \in {}^{\lambda^+}2$.

As $2^{\lambda^+} > \lambda^{++}$, necessarily for some $\eta_0 \neq \eta_1$ from λ^+ we have $a^*_{\eta_0} = a^*_{\eta_1}$. So for some $\alpha < \lambda^+, \eta_0 \restriction \alpha = \eta_1 \restriction \alpha$ but $\eta_0(\alpha) \neq \eta_1(\alpha)$, and without loss of generality $\eta_\ell(\alpha) = \ell$ for $\ell = 0,1$ and by clause (iv) of $\circledast$ above (recalling clause (iii) of $\circledast$ above) we get a contradiction. $\square_{2.25}$

2.26 Conclusion. Assume $(\mathrm{amg})_\lambda$, $(\mathrm{nmx})_\lambda$ and $K^{3,\mathrm{min}}_\lambda$ is dense in $K^{3,\mathrm{na}}_\lambda$ or just $K^{3,\mathrm{min}}_\lambda \neq \emptyset$.

1) Assume further $(\mathrm{cat})_\lambda + (\mathrm{cat})_{\lambda^+}, 2^\lambda < 2^{\lambda^+}$. Then there is $M \in K_{\lambda^+}$ saturated above λ.

2) If the conclusions of 2.19(3), 2.24 then the conclusion of part (1) holds.

Proof. 1) Clearly the assumptions of 2.24 are assumed and $M \in K_\lambda \Rightarrow \mathscr{S}^{\mathrm{min}}(M) \neq \emptyset$ by an assumption of 2.26, so by 2.24 for every $M \in K_\lambda$ the set $\mathscr{S}^{\mathrm{min}}(M)$ is inevitable and by 2.19(3) it has cardinality $\leq \lambda^+$.

[Why the assumptions of 2.19 holds? $(\mathrm{cat})_\lambda + (\mathrm{amg})_\lambda$ are assumed here and $(\mathrm{slm})_{\lambda^+} + (\mathrm{mdn})^1_{\lambda^+}$ follows from $(\mathrm{cat})_{\lambda^+}$ and $2^\lambda < 2^{\lambda^+}$.]

Now we shall apply 2.8(1) so we have to check its assumptions, i.e. $(\mathrm{amg})_\lambda + (\mathrm{nmx})_\lambda$ and $\circledast$ there. Now $(\mathrm{amg})_\lambda$ holds by the assumptions of 2.26 and $(\mathrm{nmx})_\lambda$ again is an assumption of 2.26. Lastly, $\circledast$ there says that for any $M \in K_\lambda$ there is an inevitable $\Gamma \subseteq \mathscr{S}^{\mathrm{na}}(M)$ of cardinality $\le \lambda^+$; we choose $\Gamma = \mathscr{S}^{\min}(M)$ and it is as required as deduced above.

Hence the conclusion of 2.8(1) holds, which says that there is $M \in K_{\lambda^+}$ which is saturated above λ, as desired.
2) By the same proof. $\square_{2.26}$

§3 ON UQ TRIPLES OF MODELS WITH UNIQUE AMALGAMATION

We shall quote here Chapter VII but in a black box way; from §1 + §2 we use (the basic statements on a.e.c. and their types, superlimit models and) $K^{3,\mathrm{na}}_\lambda$, minimal types and minimal triples, $\mathscr{S}^{\mathrm{nc}}(M)$.

3.1 Hypothesis.

(*a*) $\mathfrak{K}$ abstract elementary class with $\mathrm{LS}(\mathfrak{K}) \le \lambda$ and $K_\lambda \ne \emptyset$

(*b*) $\mathfrak{K}$ has amalgamation in λ, i.e. $(\mathrm{amg})_\lambda$.

3.2 Remark. We shall use mostly $x = a$ from the definition below.

The following definition is very natural but reflect an extreme situation, and will be central in §3 + §4. Still it gives some "positive theory".

3.3 Definition. 1) For $x \in \{a, d\}$ we say $\mathrm{UQ}^x_\lambda(M_0, M_1, M_2, M_3)$ <u>when</u>:

(*a*) $M_\ell \in K_\lambda$ for $\ell \le 3$

(*b*) $M_0 \le_{\mathfrak{K}} M_\ell \le_{\mathfrak{K}} M_3$ for $\ell = 1, 2$

(*c*) if for $i \in \{1, 2\}$ we have $M^i_\ell \in K_\lambda$ for $\ell < 4$ and $M^i_0 \le_{\mathfrak{K}} M^i_\ell \le_{\mathfrak{K}} M^i_3 \in K_\lambda$ for $\ell = 1, 2$ and $[x = d \Rightarrow M^i_1 \cap M^i_2 = M^i_0]$ and f^i_ℓ is an isomorphism from M_ℓ onto M^i_ℓ for $\ell < 3$ and $f^i_0 \subseteq f^i_1, f^i_0 \subseteq f^i_2$ <u>then</u> there are M'_3, f_3 such that $M^2_3 \le_{\mathfrak{K}} M'_3$ and f_3 is a $\le_{\mathfrak{K}}$-embedding of M^1_3 into M'_3 extending $(f^2_1 \circ (f^1_1)^{-1}) \cup (f^2_2 \circ (f^1_2)^{-1})$ i.e. $f_3 \circ f^1_1 = f^2_1$ & $f_3 \circ f^1_2 = f^2_2$

(*d*) $x = d \Rightarrow M_1 \cap M_2 = M_0$.

2) We say $\mathrm{UQ}^x_\lambda(M_0, M_1, M_2)$ if $\mathrm{UQ}^x_\lambda(M_0, M'_1, M'_2, M_3)$ for some M_3 and M'_1, M'_2 isomorphic to M_1, M_2 over M_0 respectively.
Let $\mathrm{NUQ}^x_\lambda(M_0, M_1, M_2)$ be the negation of $\mathrm{UQ}^x_\lambda(M_0, M_1, M_2)$. If the identity of $\mathfrak{K}$ is not clear we may write $\mathrm{UQ}^x_{\mathfrak{K}_\lambda}$, $\mathrm{NUQ}^x_{\mathfrak{K}_\lambda}$.
3) If we omit x, we mean $x = a$.
4) $K^{3,\mathrm{nmr}}_\lambda$ is the family of reduced members of $K^{3,\mathrm{nm}}_\lambda$; where nm stands for "no minimal"; recall that $K^{3,\mathrm{nm}}_\lambda$ is the family of triples $(M, N, a) \in K^{3,\mathrm{na}}_\lambda$ such that there is no minimal triple above it by $\leq_{\mathrm{na}}$ in 1.15(4).
5) $K^{2,\mathrm{nm}}_\lambda$ is the family $\{(M, N) :$ for some $a, (M, N, a) \in K^{3,\mathrm{nm}}_\lambda\}$.
6) For $M \in K_{\lambda^+}$ let[7] $\mathscr{S}^{\mathrm{nm}}_*(M) = \{\mathbf{tp}(a, M, N) : (M, N, a) \in K^{3,\mathrm{nm}}_{\lambda^+,*}\}$ where $K^{3,\mathrm{nm}}_{\lambda^+,*} = \{(M, N, a) \in K^{3,\mathrm{na}}_{\lambda^+}$: for some $M_0 \leq_{\mathfrak{K}} N_0 \leq_{\mathfrak{K}} N$ we have $M_0, N_0 \in K_\lambda, M_0 \leq_{\mathfrak{K}} M$ and $(M_0, N_0, a) \in K^{3,\mathrm{nm}}_\lambda\}$.
7) For $M \in K_{\lambda^+}$ let $\mathscr{S}^{\mathrm{nmr}}_*(M) = \{\mathbf{tp}(a, M, N) : (M, N, a) \in K^{3,\mathrm{nmr}}_{\lambda^+,*}\}$ where $K^{3,\mathrm{nmr}}_{\lambda^+,*} = \{(M, N, a) \in K^{3,\mathrm{na}}_{\lambda^+}$: there is a $\leq_{\mathrm{na}}$-increasing continuous sequence $\langle (M_\alpha, N_\alpha, a) : \alpha < \lambda^+\rangle$ of members of $K^{3,\mathrm{nmr}}_\lambda$ with union $(M, N, a)\}$.

The reader may find it helpful to look at the following example.

3.4 Exercise. Let K be the class of models $M = (|M|, E^M)$, $\|M\| \geq \lambda$ such that E^M is an equivalence relation on $|M|$ and let $\leq_{\mathfrak{K}}$ be being a submodel.
1) We have $\mathrm{UQ}_\lambda(M_0, M_1, M_2)$ <u>iff</u>

- (a) $M_\ell \in K_\lambda$ for $\ell = 0, 1, 2$
- (b) $M_0 \subseteq M_\ell$ for $\ell = 1, 2$
- (c) if $aE^{M_1}c, bE^{M_2}d$ and $a, b \in M_0$ and $c \in M_1 \backslash M_0$ and $d \in M_2 \backslash M_0$ then $\neg(aE^{M_0}b)$
- (d) for at least one $\ell \in \{1, 2\}$ for every $c \in M_\ell$ we have $(c/E^{M_\ell}) \cap M_0 \neq \emptyset$.

2) We have $\mathrm{UQ}^d_\lambda(M_0, M_1, M_2)$ <u>iff</u> clauses (a),(b),(d) above holds.

[7]recall that $\mathscr{S}^{\mathrm{nm}}(M)$ for $M \in K_\lambda$ is defined in Definition 1.15(4).

3.5 Claim. *0)* UQ^x_λ *is preserved by isomorphisms.*
1) Symmetry: assuming $x \in \{a, d\}$ *we have* $\mathrm{UQ}^x_\lambda(M_0, M_1, M_2, M_3) \Rightarrow \mathrm{UQ}^x_\lambda(M_0, M_2, M_1, M_3)$*; we also can omit* M_3*. Moreover in Definition 3.3(1), clause (c) itself is symmetric.*
2) If M_1, M_2 *has a disjoint* $\le_{\mathfrak{K}_\lambda}$*-amalgamation over* M_0 *and* $\mathrm{UQ}^a_\lambda(M_0, M_1, M_2)$ *then* $\mathrm{UQ}^d_\lambda(M_0, M_1, M_2)$*; moreover for any* M_3*,*
$\mathrm{UQ}^a_\lambda(M_0, M_1, M_2, M_3) \Rightarrow \mathrm{UQ}^d_\lambda(M_0, M_1, M_2, M_3)$*.*
2A) Assume $M_0 \le_{\mathfrak{K}} M_\ell$ *for* $\ell = 1, 2$ *and* $a_1 \in M_1 \backslash M_0$ $\&$ $a_2 \in M_2 \backslash M_0 \Rightarrow \mathbf{tp}(a_1, M_0, M_1) \ne \mathbf{tp}(a_2, M_0, M_1)$*. Then* $\mathrm{UQ}^a_\lambda(M_0, M_1, M_2)$ *iff* $\mathrm{UQ}^d_\lambda(M_0, M_1, M_2, M_3)$*.*
3) $\mathrm{UQ}^x_\lambda(M_0, M_1, M_2, M_3)$ *iff clauses (a),(b),(d) of Definition 3.3(1), (2) holds and also* $(c)^-$*, i.e., clause (c) restricted to the case* $M^1_\ell = M_\ell$ *for* $\ell \le 3$*.*
4) Monotonicity in M_3*: if* $\mathrm{UQ}^x_\lambda(M_0, M_1, M_2, M_3)$ *and* $M_3 \le_{\mathfrak{K}} M'_3 \in K_\lambda$ *then* $\mathrm{UQ}^x_\lambda(M_0, M_1, M_2, M'_3)$*; and also the inverse: if* $\mathrm{UQ}^x_\lambda(M_0, M_1, M_2, M'_3)$ *and* $M_1 \cup M_2 \subseteq M_3 \le_{\mathfrak{K}} M'_3$ *then* $\mathrm{UQ}^x_\lambda(M_0, M_1, M_2, M_3)$*.*
5) Assume $(M, N, a) \in K^{3,\mathrm{na}}_\lambda$ *and it is not minimal (even less) then* $\mathrm{NUQ}_\lambda(M, N, N)$*.*
6) Assume $M_0 \le_{\mathfrak{K}_\lambda} M_1 \le_{\mathfrak{K}_\lambda} M_3$*. Then* $\mathrm{UQ}_\lambda(M_0, M_1, M_1, M_3)$ *iff for every* $a \in M_1 \backslash M_0$ *the type* $\mathbf{tp}(a, M_0, M_1)$ *is 1-algebraic, i.e. is realized at most by one element in any* $\le_{\mathfrak{K}}$*-extension of* M_0*.*
7) If $M_1 \le_{\mathfrak{K}_\lambda} N_1 \le_{\mathfrak{K}_\lambda} N_2 \le_{\mathfrak{K}} M_2$ *and* $\mathrm{NUQ}(N_1, N_2, N_2)$ *then* $\mathrm{NUQ}(M_1, M_2, M_2)$*.*

Proof. 0),1),2),2A) Trivial (for the second sentence in part (1) use amalgamation in $\mathfrak{K}_\lambda$).
3) Done by chasing arrows. As trivially $(c) \Rightarrow (c)^-$ it is enough to assume clause $(c)^-$ and prove clause (c) of Definition 3.3(1). Assume we are given sequences $\langle M^1_\ell : \ell < 4\rangle, \langle M^2_\ell : \ell < 4\rangle, \langle f^i_\ell : \ell < 3\rangle$ as there for $i = 1, 2$. First for $i = 1, 2$ apply clause $(c)^-$ to $\langle M^i_\ell : \ell < 4\rangle, \langle f^i_\ell : \ell < 3\rangle$. So there are N^i_3, f^i_3 such that: $M^i_3 \le_{\mathfrak{K}} N^i_3 \in K_\lambda$, and f^i_3 a $\le_{\mathfrak{K}}$-embedding of M_3 into N^i_3 extending $f^i_1 \cup f^i_2$. As $\mathfrak{K}$ has amalgamation in λ (by 3.1(b)) there are $N \in K_\lambda$ and $\le_{\mathfrak{K}}$-embeddings $g^i : N^i \to N$ such that $g^1 \circ f^1_3 = g^2 \circ f^2_3$, so we are done.
4) Again by the amalgamation i.e., 3.1(b).
5) So we can find (M', N'_ℓ, a) such that $(M, N, a) \le_{\mathrm{na}} (M', M'_\ell, a)$ for $\ell = 1, 2$ and $\mathbf{tp}(a, M', M'_1) \ne \mathbf{tp}(a, M', M'_2)$. By $(\mathrm{amg})_\lambda$ we can find

(N', f_1, f_2) such that $M' \leq_{\mathfrak{K}} N' \in K_\lambda$ and f_ℓ is a $\leq_{\mathfrak{K}}$-embedding of M'_ℓ into N' over M' for $\ell = 1, 2$. So necessarily $a_1 \neq a_2$. Both realizes $\mathbf{tp}(a, M, N)$ in N' hence this type is not 1-algebraic. Now we can apply part (6).

6) First, assume that $a \in M_1 \backslash M_0$ exemplify the failure of the second phrase. So by $(\mathrm{amg})_\lambda$ we can find (N_0, a_1, a_2) such that $M_0 \leq_{\mathfrak{K}} N_0 \in K_\lambda$ and $a_1 \neq a_2$ both realizes $\mathbf{tp}(a, M_0, M_1)$. By $(\mathrm{amg})_\lambda$ and basic properties of types we can find (N_1, f_1, f_2) such that $N_0 \leq_{\mathfrak{K}} N_1 \in K_\lambda$ and f_ℓ is a $\leq_{\mathfrak{K}}$-embedding of M_3 into N_1 over M_0 mapping a_ℓ to a.

Now for $i = 1, 2$ we choose $\langle M^i_\ell : \ell \leq 3\rangle, \langle f^i_\ell : \ell \leq 3\rangle$ as follows: for $i = 1, 2$ let $M^i_0 = M_0, f^i_0 = \mathrm{id}_M, M^i_3 = N_1, M^i_1 = f_1(M_1), f^i_1 = f_i$ and (M^i_2, f^i_2) is $(f_i(M_1), f_i)$ for $i = 1, 2$.

Clearly we have gotten two contradictory amalgamations, so we are done.

7) Easy. $\square_{3.5}$

3.6 Claim. *1) transitivity: If* $\mathrm{UQ}_\lambda(M_\ell, N_\ell, M_{\ell+1}, N_{\ell+1})$ *for* $\ell = 0, 1$ *then*
$\mathrm{UQ}_\lambda(M_0, N_0, M_2, N_2)$.
2) Long transitivity: if $\theta = \mathrm{cf}(\theta) < \lambda^+$, *and* $\langle M_i : i \leq \theta\rangle$ *is* $\leq_{\mathfrak{K}}$*-increasing continuous and* $\langle N_i : i \leq \theta\rangle$ *is* $\leq_{\mathfrak{K}}$*-increasing and* $\mathrm{UQ}_\lambda(M_i, N_i, M_{i+1}, N_{i+1})$ *for each* $i < \theta$ *then*
$\mathrm{UQ}_\lambda(M_0, N_0, M_\theta, N_\theta)$.
3) Assume:

- (a) $\alpha, \beta < \lambda^+$
- (b) $M_{i,j} \in K_\lambda$ *for* $i \leq \alpha, j \leq \beta$
- (c) $i_1 \leq i_2 \leq \alpha$ & $j_1 \leq j_2 \leq \beta \Rightarrow M_{i_1,j_1} \leq_{\mathfrak{K}} M_{i_2,j_2}$
- (d) $\langle M_{i,j} : i \leq \alpha\rangle$ *is* $\leq_{\mathfrak{K}}$*- increasing continuous for each* $j \leq \beta$
- (e) $\langle M_{i,j} : j \leq \beta\rangle$ *is* $\leq_{\mathfrak{K}}$*-increasing continuous for each* $i \leq \alpha$
- (f) $\mathrm{UQ}_\lambda(M_{i,j}, M_{i+1,j}, M_{i,j+1}, M_{i+1,j+1})$ *for every* $i < \alpha, j < \beta$.

Then $\mathrm{UQ}_\lambda(M_{0,0}, M_{\alpha,0}, M_{0,\beta}, M_{\alpha,\beta})$.
4) Monotonicity in M_1, M_2*: if* $\mathrm{UQ}_\lambda(M_0, M_1, M_2)$ *and* $M_0 \leq_{\mathfrak{K}} M'_1 \leq_{\mathfrak{K}} M_1$ *and* $M_0 \leq_{\mathfrak{K}} M'_2 \leq_{\mathfrak{K}} M_2$ *then* $\mathrm{UQ}_\lambda(M_0, M'_1, M'_2)$.
5) If $M \leq_{\mathfrak{K}} N_\ell$ *for* $\ell = 1, 2$ *and* N_1 *can be* $\leq_{\mathfrak{K}}$*-embedded into* N_2 *over* M*, then* $\mathrm{UQ}_\lambda(M, N_2, M')$ *implies* $\mathrm{UQ}_\lambda(M, N_1, M')$.

Proof. Chasing arrows, and for part (3) use also symmetry, and long transitivity, i.e. 3.5(2) (note: $\mathrm{UQ} = \mathrm{UQ}^a$ is easier than UQ^d, for UQ^d the parallel claim is not clear at this point, i.e. the straightforward proof of transitivity fails). $\square_{3.6}$

The following definition will be widely used in our non-structure proofs here in §3 + §4.

3.7 Definition. 1) $\mathbf{M}^{\mathrm{dis}}_\lambda[\mathfrak{K}] = \{(M,\Gamma) : M \in K_\lambda, \Gamma \subseteq K_\lambda, |\Gamma| \leq \lambda$ satisfies $N \in \Gamma \Rightarrow M <_{\mathfrak{K}} N$ and $N_1 \neq N_2 \in \Gamma \Rightarrow N_1 \cap N_2 = M\}$.
The last demand $N_1 \cap N_2 = M$ is for technical reasons.
2) Let $(M_1,\Gamma_1) \leq^{\mathrm{or}}_f (M_2,\Gamma_2)$ mean that

- (*a*) $(M_\ell,\Gamma_\ell) \in \mathbf{M}^{\mathrm{dis}}_\lambda[\mathfrak{K}]$ for $\ell = 1,2$
- (*b*) f is a function with domain $\cup\{N : N \in \Gamma_1\} \cup M_1$
- (*c*) $f \restriction M_1$ is a $\leq_{\mathfrak{K}}$-embedding of M_1 into M_2
- (*d*) for every $N_1 \in \Gamma_1$ we have
  - (α) $\mathrm{UQ}(f(M_1), f(N_1), M_2)$, this is the main point
  - (β) if $f(N_1) \nsubseteq M_2$ <u>then</u> for some (unique) $N_2 \in \Gamma_2$ the function $f \restriction N_1$ is a $\leq_{\mathfrak{K}}$-embedding of N_1 into N_2
  - (γ) if $f(N_1) \subseteq M_2$ then $f \restriction N_1$ is a $\leq_{\mathfrak{K}}$-embedding of N_1 into M_2.

2A) Above we say f is simple in $(M_1,\Gamma_2) \leq^{\mathrm{or}}_f (M_2,\Gamma_2)$ <u>when</u>, in addition $N \in \Gamma_1 \Rightarrow f(N) \cap M_2 = M_1$.
3) $(M_1,\Gamma_1) \leq_{\mathrm{or}} (M_2,\Gamma_2)$ means that $(M_1,\Gamma_1) \leq^{\mathrm{or}}_f (M_2,\Gamma_2)$ for some f such that:

- (*a*) f is simple and $f = \cup\{\mathrm{id}_N : N \in \Gamma_1\} \cup \{\mathrm{id}_M\}$
  <u>or</u> generally
- (*b*) $f \restriction M = \mathrm{id}_M$ and $a \in N \in \Gamma_1 \wedge a \notin M_1 \wedge f(a) \notin M_2 \Rightarrow f(a) = a$.

3A) We say $(M_1,\Gamma_1) \leq_{\mathrm{or}} (M_2,\Gamma_2)$ simply <u>when</u> there is f witnessing it which is simple, in fact, f is unique.
4) $(M_1,\Gamma_1) <^*_{\mathrm{or}} (M_2,\Gamma_2)$ means that $(M_1,\Gamma_1) \leq_{\mathrm{or}} (M_2,\Gamma_2)$ and

$\mathrm{NUQ}_\lambda(M_1, M_2, M_2)$ hence $M_1 <_{\mathfrak{K}} M_2$ (so it is not the order derived from $\le_{\mathrm{or}}$) and $\le^*_{\mathrm{or}}$ means equality or $<^*_{\mathrm{or}}$.
5) We say that $\langle (M_i, \Gamma_i) : i < i_* \rangle$ is $\le_{\mathrm{or}}$-increasing continuous <u>when</u>:

- (a) if $i < j < i_*$ then $(M_i, \Gamma_i) \le_{\mathrm{or}} (M_j, \Gamma_j)$ so both from $\mathbf{M}^{\mathrm{dis}}_\lambda[\mathfrak{K}]$
- (b) if $\delta < i_*$ is a limit ordinal then $(M_\delta, \Gamma_\delta) := \cup\{(M_i, \Gamma_i) : i < \delta\}$ which means: $M_\delta = \cup\{M_i : i < \delta\}$ and

<u>Case 1</u>: $(M_i, \Gamma_i) \le_{\mathrm{or}} (M_j, \Gamma_j)$ simply for $i < j < \delta$.

We let $\Gamma_\delta = \{N$: for some $i < \delta$ and $N_i \in \Gamma_i$ we have $N_i \nsubseteq M_\delta$ and letting $N_j \in \Gamma_j$ be the unique $N' \in \Gamma_j$ such that $N_i \le_{\mathfrak{K}} N'$ for $j \in [i, \delta)$, (necessarily well defined) we have $N = \cup\{N_j : j \in [i, \delta)\}\}$.

<u>Case b</u>: In general:

For $i < j < \delta$ let $f_{j,i}$ be the unique function witnessing $(M_i, \Gamma_i) \le_{\mathrm{or}} (M_j, \Gamma_j)$, see 3.8(0) below, we let $\mathbf{X} = \{(i, N, c) : i < \delta, N \in \Gamma_i, c \in N \backslash M_i$ and $j \in (i, \delta) \Rightarrow f_{j,i}(c) \notin M_j\}$, for $\mathbf{x} \in \mathbf{X}$ let $\mathbf{x} = (i_{\mathbf{x}}, N_{\mathbf{x}}, c_{\mathbf{x}})$ and let $\mathbf{Y} = \{\mathbf{x} \in \mathbf{X}$: for no $\mathbf{y} \in \mathbf{X}$ do we have $i_{\mathbf{y}} < i_{\mathbf{x}}$ and $f_{i_{\mathbf{x}}, i_{\mathbf{y}}}(c_{\mathbf{y}}) = c_{\mathbf{x}}$ so necessarily $c_{\mathbf{y}} = c_{\mathbf{x}}\}$.

For $\mathbf{y} \in \mathbf{X}$ and $j \in [i_{\mathbf{y}}, \delta)$ let $N_{\mathbf{y},j}$ be the unique $N \in \Gamma_j$ such that $f_{j,i_{\mathbf{y}}}(c_{\mathbf{y}}) \in N \backslash M_j$ and let $f_{\mathbf{y},\delta}$ be the function with domain $N_{\mathbf{y}}$ such that: $f_{\mathbf{y},\delta}(a)$ is $f_{j,i_{\mathbf{y}}}(a)$ <u>if</u> j is the minimal $j \in [i_{\mathbf{y}}, \delta)$ such that $f_{j,i}(a) \in M_j$ and is a <u>if</u> there is no such j.

For $\mathbf{y} \in \mathbf{X}$ and $j \in [i_{\mathbf{y}}, \delta)$ let $\mathbf{x}_{\mathbf{y},j}$ be the unique $\mathbf{x} \in \mathbf{X}$ such that $c_{\mathbf{x}} = c_{\mathbf{y}} \wedge i_{\mathbf{x}} = j$. Now for $\mathbf{y} \in \mathbf{X}$ let $g_{\mathbf{y}} = \cup\{f_{\mathbf{x}_{\mathbf{y},j},\delta} : j \in [i_{\mathbf{y}}, \delta)\}$. Now let $N^-_{\mathbf{y},\delta}$ be the unique $\tau_{\mathfrak{K}}$-model such that $g_{\mathbf{y},\delta}$ is an isomorphism from $N_{\mathbf{y}}$ onto $N_{\mathbf{y},\delta}$.

Clearly for any $\mathbf{y} \in \mathbf{X}$ the sequence $\langle N^-_{\mathbf{x}_{\mathbf{y},j},\delta}$: for $j \in [i_{\mathbf{y}}, \delta)\rangle$ is a $\le_{\mathfrak{K}}$-increasing continuous and let $N_{\mathbf{y},\delta} = \cup\{N^-_{\mathbf{x}_{\mathbf{y},j}} : j \in \pi_{\mathbf{y}}, \delta)\}$.

Lastly, let $\Gamma_\delta := \{N_{\mathbf{y},\delta} : \mathbf{y} \in \mathbf{X}$ and $N_{\mathbf{y}} \nsubseteq M_\delta\}$. Of course we may well have $\mathbf{y}_1 \ne \mathbf{y}_2, N_{\mathbf{y}_1} = N_{\mathbf{y}_2}$ but $N_{\mathbf{y}_1} \ne N_{\mathbf{y}_1} \Rightarrow N_{\mathbf{y}_1,\delta} \cap N_{\mathbf{y}_2,\delta} = M_\delta$.

3.8 Observation. 0) In Definition 3.7(3), the function f is unique, i.e. determined by $(M_1, \Gamma_1, M_2, \Gamma_2)$.
1) $\le_{\mathrm{or}}$ is a partial order of $\mathbf{M}^{\mathrm{dis}}_\lambda[\mathfrak{K}]$.
2) $\mathbf{M}^{\mathrm{dis}}_\lambda[\mathfrak{K}]$ is non-empty, e.g., $(M, \emptyset) \in \mathbf{M}^{\mathrm{dis}}_\lambda[\mathfrak{K}]$ when $M \in K_\lambda$.

3) If $\delta < \lambda^+$ is a limit ordinal and $\langle (M_i, \Gamma_i) : i < \delta \rangle$ is $\leq_{\rm or}$-increasing continuous <u>then</u> this sequence has an upper bound $(M_\delta, \Gamma_\delta) := \cup\{(M_i, \Gamma_i) : i < \delta\}$, see clause (b) of Definition 3.7(5).
4) If $(M_0, \Gamma_0) \leq_{\rm or} (M_1, \Gamma_1) <^*_{\rm or} (M_2, \Gamma_2) \leq_{\rm or} (M_3, \Gamma_3)$
<u>then</u> $(M_0, \Gamma_0) <^*_{\rm or} (M_3, \Gamma_3)$.

Proof. Easy, e.g.
3) The main point is that we use the long transitivity, i.e. 3.6(2).
4) By part (1) and the definition noting that $\mathrm{NUQ}(M_0, M_3, M_3)$ by 3.5(7). $\square_{3.8}$

Below in 3.9 - 3.12 we investigate the non-structure consequences of "there is no maximal member of $(\mathbf{M}^{\rm dis}_\lambda[\mathfrak{K}], <^*_{\rm or})$".
First we get $\mu_{\rm wd}(\lambda^+)$ pairwise non-isomorphic models in K_{λ^+}, see Definition VII.0.3(6), later we work more to get more.

3.9 Claim. *Assume* $2^\lambda < 2^{\lambda^+}$.

If $(*)_\lambda$ *or at least* $(*)'_\lambda$ *below holds,* <u>*then*</u> $\dot{I}(\lambda^+, K) \geq \mu_{\rm wd}(\lambda^+)$, *where*

- $(*)_\lambda$ *for every* $(M, \Gamma) \in \mathbf{M}^{\rm dis}_\lambda[\mathfrak{K}]$ *for some* M', Γ' *we have* $(M, \Gamma) <^*_{\rm or} (M', \Gamma')$
  or just
- $(*)'_\lambda$ *for some* $(M_*, \Gamma_*) \in \mathbf{M}^{\rm dis}_\lambda[\mathfrak{K}]$, *if* $(M_*, \Gamma_*) \leq^*_{\rm or} (M, \Gamma)$ *then for some* M', Γ' *we have* $(M, \Gamma) <^*_{\rm or} (M', \Gamma')$.

Remark. From where some freedom, i.e. failure of amalgamation, comes? E.g. if $M_0 <_{\mathfrak{K}_\lambda} M$ <u>then</u> $(M, \emptyset), (M_0, \{M\})$ are contradictory $\leq_{\rm or}$-above $(M_0, \emptyset)$, in $\mathbf{M}$.

Proof. Note that as $\mathbf{M}^{\rm dis}_\lambda[\mathfrak{K}] \neq \emptyset$, see 3.8(2), clearly $(*)_\lambda \Rightarrow (*)'_\lambda$ hence we can assume $(*)'_\lambda$.
We choose $\langle (M_\eta, \Gamma_\eta, \Gamma^+_\eta) : \eta \in {}^\alpha 2 \rangle$ by induction on $\alpha < \lambda^+$ such that:

$\circledast$ (a) $M_\eta \in K_\lambda$ has universe $\gamma_\eta < \lambda^+$

(b) $(M_\eta, \Gamma_\eta) \in \mathbf{M}^{\rm dis}_\lambda[\mathfrak{K}]$

(c) $N \in \Gamma_\eta \Rightarrow (N \backslash M_\eta) \cap \lambda^+ = \emptyset$

(d) $\nu \triangleleft \eta \Rightarrow (M_\nu, \Gamma_\nu) <^*_{\rm or} (M_\eta, \Gamma_\eta)$;
moreover $\langle (M_{\eta \restriction \beta}, \Gamma_{\eta \restriction \beta}) : \beta \leq \alpha \rangle$ is $<^*_{\rm or}$-increasing continuous

(e) $(M_\eta, \Gamma^+_\eta) \in \mathbf{M}^{\rm dis}_\lambda[\mathfrak{K}]$ and $N \in \Gamma^+_\eta \Rightarrow (N \backslash M_\eta) \cap \lambda^+ = \emptyset$

(f) $\Gamma_\eta \subseteq \Gamma^+_\eta$

(g) $(M_\eta, \Gamma^+_\eta) \leq_{\rm or} (M_{\eta\hat{}\langle 0 \rangle}, \Gamma_{\eta\hat{}\langle 0 \rangle})$

(h) for some $N \in \Gamma^+_\eta$ we have $N \cong_{M_\eta} M_{\eta\hat{}<1>}$.

There is no serious problem to carry the induction with Γ^+_η (for $\eta \in {}^\alpha 2$) chosen in the $(\alpha+1)$-th step but we elaborate. For $\alpha = 0$ let $(M_{<>}, \Gamma_{<>})$ be the (M_*, Γ_*) from $(*)'_\lambda$ except that we rename the elements to make the relevant parts of clauses (a), (c) true. For α limit use 3.8(3). For $\alpha = \beta + 1, \eta \in {}^\beta 2$, by $(*)'_\lambda$ we can find $(M_{\eta\hat{}\langle 1 \rangle}, \Gamma_{\eta\hat{}\langle 1 \rangle})$ such that $(M_{\eta\hat{}\langle 1 \rangle}, \Gamma_{\eta\hat{}\langle 1 \rangle}) \in \mathbf{M}^{\rm dis}_\lambda[\mathfrak{K}]$ and $(M_\eta, \Gamma_\eta) <^*_{\rm or} (M_{\eta\hat{}\langle 1 \rangle}, \Gamma_{\eta\hat{}\langle 1 \rangle})$.

By renaming without loss of generality the universe of $M_{\eta\hat{}\langle 1 \rangle}$ is some $\gamma_{\eta\hat{}\langle 1 \rangle} \in (\gamma_\eta, \lambda^+)$ and clause (c) holds. Let N_η be isomorphic to $M_{\eta\hat{}\langle 1 \rangle}$ over M_η with $N_\eta \backslash M_\eta$ disjoint to $\lambda^+ \cup \bigcup \{|N| : N \in \Gamma_\eta\}$ and let $\Gamma^+_\eta = \Gamma_\eta \cup \{N_\eta\}$, so $(M_\eta, \Gamma^+_\eta) \in \mathbf{M}^{\rm dis}_\lambda[\mathfrak{K}]$ and $N \in \Gamma^+_\eta \Rightarrow (N \backslash M_\eta) \cap \lambda^+ = \emptyset$ as required in clause (d), now apply $(*)'_\lambda$ to this pair and get some $(M_{\eta\hat{}\langle 0 \rangle}, \Gamma_{\eta\hat{}\langle 0 \rangle}) \in \mathbf{M}^{\rm dis}_\lambda[\mathfrak{K}]$ such that $(M_\eta, \Gamma^+_\eta) <^*_{\rm or} (M_{\eta\hat{}\langle 0 \rangle}, \Gamma_{\eta\hat{}\langle 0 \rangle})$, and by renaming without loss of generality the universe of $M_{\eta\hat{}\langle 0 \rangle}$ is some $\gamma_{\eta\hat{}\langle 0 \rangle} \in (\gamma_\eta, \lambda^+)$ and clause (c) holds. Note that by 3.8(4) we have $(M_\eta, \Gamma_\eta) <^*_{\rm or} (M_{\eta\hat{}\langle 1 \rangle}, \Gamma_{\eta\hat{}\langle 1 \rangle})$. Why does clause (h) hold? By the choice of N_η and Γ^+_η. So $M_\eta, \Gamma_\eta, \Gamma^+_\eta$ $(\eta \in {}^{\lambda^+ >}2)$ are defined.
Note:

$\oplus$ if $\eta\hat{}\langle 0 \rangle \triangleleft \nu \in {}^{\lambda^+ >}2$, <u>then</u> $M_{\eta\hat{}<1>}$ is not $\leq_{\mathfrak{K}}$-embeddable into M_ν over M_η.

[Why? Toward contradiction assume that there is such an embedding h. By clause (d) and Definition 3.7(4) we have $\mathrm{NUQ}_\lambda(M_\eta, M_{\eta\hat{}<1>}, M_{\eta\hat{}<1>})$ hence by Claim 3.6(5) applied to h we get $\mathrm{NUQ}_\lambda(M_\eta, M_\nu, M_{\eta\hat{}<1>})$. Now by clause (g) we have $(M_\eta, \Gamma^+_\eta) \leq_{\rm or}$

(M_ν, Γ_ν) so by Definition 3.7(4) we have $N \in \Gamma^+_\eta \Rightarrow \mathrm{UQ}_\lambda(M_\eta, M_\nu, N)$, so by preservation under isomorphisms and clause (h) we have $\mathrm{UQ}_\lambda(M_\eta, M_\nu, M_{\eta\hat{}<1>})$, contradicting the previous sentence.]

By VII.9.6 we get the desired conclusion (really, usually also on $\dot{I}\dot{E}$). $\square_{3.9}$

3.10 Claim. *1) An equivalent condition to $(*)_\lambda$ from 3.9 is:*

$(**)_\lambda$ *for every $M \le_{\mathfrak{K}} N$ from K_λ for some M' we have $M <_{\mathfrak{K}} M' \in K_\lambda$ and $\mathrm{UQ}_\lambda(M, M', N)$ but $\mathrm{NUQ}_\lambda(M, M', M')$.*

2) Also, the condition $(*)'_\lambda$ from 3.9 is equivalent to $(**)'_\lambda$ and follows from $(**)''$ where:

$(**)''_\lambda$ *for some $M_* \in K_\lambda$ if $M_* \le_{\mathfrak{K}} M \le_{\mathfrak{K}} N \in K_\lambda$ <u>then</u> for some M', $M <_{\mathfrak{K}} M' \in K_\lambda$ and $\mathrm{UQ}_\lambda(M, M', N)$ but $\mathrm{NUQ}_\lambda(M, M', M')$*

$(**)'_\lambda$ *for some $M_* <_{\mathfrak{K}_\lambda} N_*$, if $M_* \le_{\mathfrak{K}} M \in K_\lambda, N_* \le_{\mathfrak{K}} N$ and $\mathrm{UQ}_\lambda(M_*, N_*, M, N)$ <u>then</u> for some M' we have $M <_{\mathfrak{K}} M' \in K_\lambda$ and $\mathrm{UQ}_\lambda(M, M', N)$ but $\mathrm{NUQ}_\lambda(M, M', M')$.*

3) We have $(*)_\lambda \Rightarrow (*)'_\lambda$. If $\mathfrak{K}$ is categorical in λ <u>then</u> $(*)_\lambda \Leftrightarrow (*)'_\lambda \Leftrightarrow (**)'_\lambda \Leftrightarrow (**)''_\lambda$.

Proof. 1),2) For any $(M, \Gamma) \in \mathbf{M}^{\mathrm{dis}}_\lambda[\mathfrak{K}]$, by "$\mathfrak{K}$ has amalgamation in λ and $\mathrm{LS}(\mathfrak{K}) \le \lambda$" (and properties of abstract elementary classes), there are $N^*, \langle f_N : N \in \Gamma\rangle$ such that:

(*a*) $M \le_{\mathfrak{K}} N^* \in K_\lambda$

(*b*) for $N \in \Gamma, f_N$ is a $\le_{\mathfrak{K}}$-embedding of N into N^* over M.

Now if $M' \in K_\lambda$ satisfies $\mathrm{UQ}_\lambda(M, N^*, M')$ then $N \in \Gamma \Rightarrow \mathrm{UQ}_\lambda(M, N, M')$ by 3.6(5). This shows that $(**)_\lambda \Rightarrow (*)_\lambda$ and also $(**)'_\lambda \Rightarrow (*)'_\lambda$.

The other direction is deduced by applying $(*)_\lambda$ (or $(*)'_\lambda$) to the pair $(M, \{N\})$.

3) Should be clear. $\square_{3.10}$

We continue 3.9 getting a sharper result, (note that for proving just Theorem 0.2, this improvement is not necessary).

3.11 Claim. $\dot{I}(\lambda^+, \mathfrak{K}) = 2^{\lambda^+}$ *<u>when</u>:*

$\boxtimes_0$ $2^\lambda < 2^{\lambda^+}$

$\boxtimes_1$ $(M_*, \Gamma_*) \in \mathbf{M}^{\rm dis}_\lambda[\mathfrak{K}]$

$\boxtimes_2$ *if* $(M_*, \Gamma_*) \le^*_{\rm or} (M, \Gamma)$ *<u>then</u> for some* M', Γ' *we have* $(M, \Gamma) <^*_{\rm or} (M', \Gamma')$

$\boxtimes_3$ *if* $(M_*, \Gamma_*) \le_{\rm or} (M_1, \Gamma_1) \in \mathbf{M}^{\rm dis}_\lambda[\mathfrak{K}]$ *<u>then</u> we can find* (M_2, Γ_2) *such that*

- (a) $(M_1, \Gamma_1) \le_{\rm or} (M_2, \Gamma_2) \in \mathbf{M}^{\rm dis}_\lambda[\mathfrak{K}]$
- (b) *if* $(M_2, \Gamma_2) \le_{\rm or} (M^\ell, \Gamma^\ell)$ *for* $\ell = 3, 4$ *then we can find* (M^ℓ, Γ^ℓ) *for* $\ell = 5, 6$ *such that* $(M^\ell, \Gamma^\ell) \le_{\rm or} (M^{\ell+2}, \Gamma^{\ell+2})$ *for* $\ell = 3, 4$ *and* M^5, M^6 *are isomorphic over* M_1.

Remark. 1) This corresponds to case B in the proofs of I.3.8, VII.9.7.
2) The gain over 3.9 is in some sense not large but here we get the "right" number.

Proof. Without loss of generality the universe of M_* is $\gamma_{<>} < \lambda^+$ and $N \in \Gamma_* \Rightarrow N_* \cap \lambda^+ = M_*$.

By induction on $\alpha < \lambda$, we choose $\langle (M_\eta, \Gamma_\eta, \Gamma^+_\eta) : \eta \in {}^\alpha 2\rangle$ such that:

$\circledast_1$(a) $M_\eta \in K_\lambda$ has universe $\gamma_\eta < \lambda^+$

(b) $(M_\eta, \Gamma_\eta) \in \mathbf{M}^{\rm dis}_\lambda[\mathfrak{K}]$

(c) $N \in \Gamma_\eta \Rightarrow (N \backslash M_\eta) \cap \lambda^+ = \emptyset$

(d) $\nu \triangleleft \eta \Rightarrow (M_\nu, \Gamma_\nu) <^*_{\rm or} (M_\eta, \Gamma_\eta)$

(e) $(M_{<>}, \Gamma_{<>}) = (M_*, \Gamma_*)$ and $(M_\eta, \Gamma^+_\eta) \in \mathbf{M}^{\rm dis}_\lambda[\mathfrak{K}]$ and $N \in \Gamma^+_\eta \Rightarrow (N \backslash M_\eta) \cap \lambda^+ = \emptyset$

(f) $\Gamma_\eta \subseteq \Gamma^+_\eta$

(g) $(M_\eta, \Gamma^+_\eta) \le_{\rm or} (M_{\eta\hat{}\langle 0\rangle}, \Gamma_{\eta\hat{}\langle 0\rangle})$

(h) for some $N \in \Gamma^+_\eta$ we have $N \cong_{M_\eta} M_{\eta\hat{}<1>}$

(i) if $\eta \in {}^{\lambda^+>}2$, $\nu \in \{\eta\hat{}\langle 0\rangle, \eta\hat{}\langle 1\rangle\}$ and $(M_\nu, \Gamma_\nu) \le_{\rm or} (M^\ell, \Gamma^\ell) \in \mathbf{M}^{\rm dis}_\lambda[\mathfrak{K}]$ for $\ell = 1,2$ then we can find $(M^{\ell+2}, \Gamma^{\ell+2})$ for $\ell = 1,2$ such that $(M^\ell, \Gamma^\ell) \le_{\rm or} (M^{\ell+2}, \Gamma^{\ell+2})$ and M^3, M^4 are isomorphic over M_η.

There is no problem to carry the induction as in the proof of 3.9, to guarantee clause (i) we use assumption $\boxtimes_3$.
Clearly, by clauses (g) + (h), as in the proof of 3.9

$\circledast_2$ if $\eta \in {}^{\lambda^+>}2$ and $(M_{\eta\hat{}<\ell>}, \Gamma_{\eta\hat{}<\ell>}) \le_{\rm or} (M_\ell, \Gamma_\ell) \in \mathbf{M}^{\rm dis}_\lambda[\mathfrak{K}]$ for $\ell = 0,1$ <u>then</u> we cannot find $(M'_\ell, \Gamma'_\ell) \in \mathbf{M}$ for $\ell = 0,1$ such that $(M_\ell, \Gamma_\ell) \le_{\rm or} (M'_\ell, \Gamma'_\ell)$ for $\ell = 0,1$ and M'_0, M'_1 are isomorphic over M_η, equivalently: we cannot find $(M, \Gamma) \in \mathbf{M}^{\rm dis}_\lambda[\mathfrak{K}]$ and functions f_0, f_1 such that $(M_\ell, \Gamma_\ell) \le^{\rm or}_{f_\ell} (M, \Gamma)$ for $\ell = 0,1$ and $f_0 \restriction M_\eta = f_1 \restriction M_\eta$.

Now we would like to apply VII.9.7, so we have to check it's assumption.

The main point is proving $(*)_1$ there, a stronger version of $\circledast_2$ above which says

$\circledast_3$ the following is impossible

- (α) $\alpha < \beta < \lambda^+$
- (β) $\eta_1, \eta_2 \in {}^\alpha 2$
- (γ) $\eta_1\hat{}\langle 0\rangle \triangleleft \nu_1 \in {}^\beta 2$ and $\eta_1\hat{}\langle 1\rangle \triangleleft \rho_1 \in {}^\beta 2$
- (δ) $\eta_2\hat{}\langle 0\rangle \triangleleft \nu_2 \in {}^\beta 2$ and[8] $\eta_2\hat{}\langle 0\rangle \triangleleft \rho_2 \in {}^\beta 2$
- (ε) f is an isomorphism from M_{ν_1} onto M_{ν_2} mapping M_{η_1} onto M_{η_2}
- (ζ) g is an isomorphism from M_{ρ_1} onto M_{ρ_2} mapping M_{η_1} onto M_{η_1}
- (η) $g \restriction M_{\eta_1} = f \restriction M_{\eta_1}$.

[8] yes! again $\eta_2\hat{}\langle 0\rangle$

Note: $\nu_1, \rho_1, \nu_2, \rho_2$ here stands for $\nu_0, \nu_1, \nu_0', \nu_1'$ in $\boxtimes_1$ in VII.9.7.

Proof of $\circledast_3$. Toward contradiction assume that α, β, η_1, η_2, ν_1, ν_2, ρ_1, ρ_2 form a counterexample.

We can find $\Gamma^*_{\nu_2}$, $\Gamma^*_{\rho_2}$ such that

(*i*) $(M_{\nu_2}, \Gamma_{\nu_2}) \leq_{\rm or} (M_{\nu_2}, \Gamma^*_{\nu_2}) \in \mathbf{M}^{\rm dis}_\lambda[\mathfrak{K}]$

(*ii*) $(M_{\rho_2}, \Gamma_{\rho_2}) \leq_{\rm or} (M_{\rho_2}, \Gamma^*_{\rho_2}) \in \mathbf{M}^{\rm dis}_\lambda[\mathfrak{K}]$

(*iii*) if $N \in \Gamma_{\nu_1}$ then f can be extended to an isomorphism from N onto some $N' \in \Gamma^*_{\nu_2}$

(*iv*) if $N \in \Gamma_{\rho_1}$ then g can be extended to an isomorphism from N onto some $N' \in \Gamma^*_{\rho_2}$.

So $(M_{\eta_2\hat{}<0>}, \Gamma_{\eta_2\hat{}<0>}) \leq_{\rm or} (M_{\nu_2}, \Gamma_{\nu_2}) \leq_{\rm or} (M_{\nu_2}, \Gamma^*_{\nu_2})$ and $(M_{\eta_2\hat{}<0>}, \Gamma_{\eta_2\hat{}<0>}) \leq_{\rm or} (M_{\rho_2}, \Gamma_{\rho_2}) \leq_{\rm or} (M_{\rho_2}, \Gamma^*_{\rho_2})$.
Now by applying clause (i) of $\circledast_1$ with η_2, $\eta_2\hat{}<0>$, $(M_{\nu_2}, \Gamma^*_{\nu_2})$, $(M_{\rho_2}, \Gamma^*_{\rho_2})$ here standing for $\eta, \nu, (M^1, \Gamma^1), (M^2, \Gamma^2)$ there, we can find (N^ℓ, Γ^ℓ_*) for $\ell = 1, 2$ and h such that

(*v*) $(M_{\nu_2}, \Gamma^*_{\nu_2}) \leq_{\rm or} (N^1, \Gamma^1_*)$

(*vi*) $(M_{\rho_2}, \Gamma^*_{\rho_2}) \leq_{\rm or} (N^2, \Gamma^2_*)$

(*vii*) h is an isomorphism from N^1 onto N^2 over M_{η_2}.

But now the mapping $h \circ f$ and g contradict the choice of $(M_{\eta_1\hat{}<0>}, \Gamma_{\eta_1\hat{}<0>})$ and $(M_{\eta_1\hat{}<1>}, \Gamma_{\eta_1\hat{}<1>})$, that is $\circledast_2$ above, in particular clauses (g),(h) there.

Having proved $\circledast_3$ we have justified applying VII.9.7 which gives the desired conclusion. $\square_{3.11}$

3.12 Conclusion. If $2^\lambda < 2^{\lambda^+}$ and $(**)''_\lambda$ or just $(**)'_\lambda$ from 3.10(2) then $\dot{I}(\lambda^+, K) = 2^{\lambda^+}$.

Proof. First, clearly $\boxtimes_0$ from 3.11 which says $2^\lambda < 2^{\lambda^+}$ holds by our assumption. Second, by our assumption $(**)'_\lambda$ and Claim 3.10(2) clearly the statement $(*)'_\lambda$ from 3.9 holds which says that we can choose (M_*, Γ_*) from $\mathbf{M}^{3,\mathrm{dis}}_\lambda[\mathfrak{K}]$ which satisfy $\boxtimes_1 + \boxtimes_2$ from 3.11. If $\boxtimes_3$ there holds too, then by 3.11 we are done so assume that it fails for $(M_1, \Gamma_1) \in \mathbf{M}^{\mathrm{dis}}_\lambda[\mathfrak{K}]$. Now we choose (M_η, Γ_η) as the proof of 3.9 except that:

$(a)-(d)$ as in $\circledast$ there

$(e)'$ we choose $(M_{<>}, \Gamma_{<>}) = (M_1, \Gamma_1)$

$(f)'$ for each η, for no $(M^1, \Gamma^1), (M^2, \Gamma^2)$ do we have:

(i) $(M_{\eta\hat{}<0>}, \Gamma_{\eta\hat{}<0>}) \le_{\mathrm{or}} (M^1, \Gamma^1)$

(ii) $(M_{\eta\hat{}<1>}, \Gamma_{\eta\hat{}<1>}) \le_{\mathrm{or}} (M^2, \Gamma^2)$

(iii) M^1, M^2 are isomorphic over $M_{<>}$.

So we can carry the definition as there because we assume that $\boxtimes_3$ of 3.9 fail. Now $\dot{I}(\lambda^+, K) \ge 2^{\lambda^+}$ by VII.9.5, (or elaborating, note that $\langle M_\eta : \eta \in {}^{\lambda^+}2\rangle$ are pairwise non-isomorphic over $M_{<>}$ and this suffices, see I.0.3). $\square_{3.12}$

Complimentary to 3.12 is

3.13 Claim. *$\dot{I}(\lambda^{++}, K) \ge \mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$, see below, when $\boxtimes$ below holds except possibly when $\circledast_\lambda$ holds <u>where</u>:*

$\boxtimes$ (a) *$(**)'_\lambda$ of 3.10 fails so in particular (equivalent by (c) below recalling 3.10(3)): for some $M^* \le_{\mathfrak{K}_\lambda} N^*$ we have $M^* <_{\mathfrak{K}_\lambda} M' \in K_\lambda$ & $\mathrm{NUQ}_\lambda(M^*, M', M') \Rightarrow \mathrm{NUQ}(M^*, N^*, M')$*

(b) *$M \in K_\lambda \Rightarrow |\mathscr{S}^{\mathrm{nc}}(M)| > \lambda^+$ (or $\mathscr{S}^{\mathrm{nc}}$, see Definition 1.15(4); follows from "$(\mathrm{cat})_{\lambda^+}$ above some $(M, N, a) \in K^{3,\mathrm{na}}_\lambda$ there is no minimal triple" $+ 2^\lambda > \lambda^+$ see 2.3(5))*

(c) *K is categorical in λ, hence $(\mathrm{jep})_\lambda$ (or strengthen (a) to: for any $M^* \in K_\lambda$ there is N^* as there and retain $(\mathrm{jep})_\lambda$)*

(d) *there is a superlimit $M \in K_{\lambda^+}$*

(e) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$.

$\circledast_\lambda$ *$\mathrm{WDmId}(\lambda^+)$ is λ^{++}-saturated, (it is a normal ideal on λ^+).*

3.14 Remark. 1) On $\mu_{\text{unif}}(\lambda^{++}, 2^{\lambda^+})$, see VII.0.4(6),(7) and VII.9.4 it is usually $2^{\lambda^{++}}$ and on "WDmId(λ^+) is λ^{++}-saturated", see Definition VII.0.7.
2) This claim and VII§4(A) and more specifically VII.4.3(1) gives the two halfs of the proof of one statement.
3) We may weaken the model theoretic assumption (d) in 3.13 to "there is a (λ^+, λ^+)-superlimit $M \in K_{\lambda^+}$" if we strengthen the set theoretic assumptions, e.g.

$(*)_1$ for some stationary $S \subseteq S^{\lambda^{++}}_{\lambda^+}$ we have $S \in \check{I}[\lambda]$ but $S \notin \text{WDmId}(\lambda^{++})$.

In this case we can use VII.4.3(2) instead VII.4.3(1).
4) Note that: if $\lambda = \lambda^{<\lambda}$ and $\mathbf{V} = \mathbf{V}^{\mathbb{Q}}$ where $\mathbb{Q}$ is adding λ^+-Cohen set (3.1 and) the minimal types are not dense then $\dot{I}(\lambda^+, K) = 2^{\lambda^+}$. More generally, this is connected to replacing WDmId(μ^+) by the "definable weak diamond ideal on μ^+" for $\mu = \lambda, \lambda^+$, but we do not pursue this here.

Proof. This holds by VII.4.3(1). So we have to check the demands $\odot(a), (b), (c)$ from VII.4.1 and $(d)'$ from VII.4.3(1).

<u>Clause (a)</u>: $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$.

Obvious by clause (e) of $\boxtimes$ which we are assuming.

<u>Clause (b)</u>: WdmId(λ^+) is not λ^{++}-saturated.

Holds by $\neg\circledast_\lambda$ from our claim.

<u>Clause (c)</u>: $\mathfrak{K}$ is an a.e.c. with LS($\mathfrak{K}$) $\le \lambda$ and has $(\text{amg})_\lambda$+ $(\text{jep})_\lambda$ and $K_{\lambda^+} \ne \emptyset$.

Clearly: "$\mathfrak{K}$ is an a.e.c. with LS($\mathfrak{K}$) $\le \lambda$" as well as $(\text{amg})_\lambda$ holds by the hypothesis of the section, $(\text{jep})_\lambda$ by $\boxtimes(c)$ here and lastly $K_{\lambda^+} \ne \emptyset$ holds by $\boxtimes(d)$.

<u>Clause (d)′</u>:

So let $M \in K_{\lambda^+}$ be superlimit and $\langle M_\alpha : \alpha < \lambda^+\rangle$ be a $<_{\mathfrak{K}}$-representation of M and we should find (α_0, N_0, a) as in clause (d) of VII.4.3, i.e. satisfying clauses $(\alpha), (\beta), (\gamma)$ from clause (d) of VII.4.1. As $\mathfrak{K}$ has amalgamation in λ and M is superlimit, clearly without loss of generality for each even $\alpha < \lambda^+$, the pair $(M_\alpha, M_{\alpha+1})$ is isomorphic to the pair (M^*, N^*) from clause $\boxtimes(a)$ here, so

(∗) if α is even and $M_\alpha <_{\mathfrak{K}_\lambda} M'$ & $\mathrm{NUQ}_\lambda(M_\alpha, M', M')$ then $\mathrm{NUQ}_\lambda(M_\alpha, M_{\alpha+1}, M')$.

Now by $\boxtimes(b)$ here there is $N \in K_\lambda$ satisfying $\mathscr{S}^{\mathrm{nc}}_{\mathfrak{K}}(N)$ has cardinality $> \lambda^+$. By $(\mathrm{jep})_\lambda+$ $(\mathrm{amg})_\lambda+$ $(\mathrm{slm})_{\lambda^+}$ there is $M' \in K^{\mathrm{slm}}_{\lambda^+}$ such that $N \le_{\mathfrak{K}} M'$, but $M' \cong M$ so without loss of generality $M' = M$ so $N \le_{\mathfrak{K}} M$ but $\|M\| \le \lambda^+ < |\mathscr{S}^{\mathrm{nc}}_{\mathfrak{K}}(N)|$ hence some $p \in \mathscr{S}^{\mathrm{nc}}_{\mathfrak{K}}(N)$ is not realized in M and choose $\alpha_0, (N_0, a)$ such that $\alpha_0 < \lambda^+, N \le_{\mathfrak{K}} M_{\alpha_0}, M_{\alpha_0} \le_{\mathfrak{K}_\lambda} N_0$ and $a \in N_0 \backslash M_0$ and $p = \mathbf{tp}_{\mathfrak{K}_\lambda}(a, N, N_0)$. It is enough to show that the triple (M_0, N_0, a) is as required in subclauses $(\alpha), (\beta), (\gamma)$ of clause (d) of VII.4.1.

But clause (α) says $M_{\alpha_0} \le_{\mathfrak{K}} N_0$ which holds and clause (β) says that $p' = \mathbf{tp}(a, M_{\alpha_0}, N_0)$ is not realized in M but even $p' \restriction N$ is not realized in M. We are left with clause (γ).

So assume $\alpha_1 < \lambda^+$ is a limit ordinal $> \alpha_0, M_{\alpha_1} \le_{\mathfrak{K}_\lambda} N_1$ and f is a $\le_{\mathfrak{K}_\lambda}$-embedding of N_0 into N_1 over M_{α_0}. Let $\alpha_2 = \alpha_1 + 1$ and now $M_{\alpha_1} <_{\mathfrak{K}} N_1$ by the choice of p, i.e. by (β) there which we have proved, i.e. $\mathbf{tp}(f(a), M_{\alpha_1}, N_1)$ extends p hence is not algebraic (reclling M omit p) and is not 1-algebraic as $p \in \mathscr{S}^{\mathrm{nc}}_{\mathfrak{K}_\lambda}(N)$, see Definition 1.15(4), hence $\mathrm{NUQ}(M_{\alpha_1}, N_1, N_1)$ holds by Claim 3.5(6). So as $(M_{\alpha_1}, M_{\alpha_2}) = (M_{\alpha_1}, M_{\alpha_1+1}) \cong (M^*, N^*)$ it follows that $\mathrm{NUQ}(M_{\alpha_1}, M_{\alpha_2}, N_1)$, i.e. is as required. $\square_{3.13}$

3.15 Remark. 1) We can get more abstract results.
2) Note that $\neg\circledast_\lambda$ of 3.13 is a "cheap", "light" assumption, in fact, e.g. its negation has reasonably high consistency strength and we have "to work" to get it to hold in forcing extensions.

§4 Density of minimal types

We deduce the density of minimal types from reasonable assumption (so weaker than in 2.25). We rely on results from Chapter VII, but not on the understanding of Chapter VII.

From §1,§2 we use as in §3 (in particular $K^{3,\mathrm{na}}_\lambda$) and quote 1.9, 2.3 in proof of 4.5(1); in 4.2 we quote VII.11.1, which we extensively use below (and also VII.11.4). We shall use freely 2.5(1), the extension property for $K^{3,\mathrm{nm}}_\lambda$.

4.1 Remark. Recall that

- $(*)_1$ WDmId(λ^+), the weak diamond is an ideal on λ^+, which is normal $\neq \mathscr{P}(\lambda^+)$ when $2^\lambda < 2^{\lambda^+}$, see Definition VII.0.3(4)(a)
- $(*)_2$ $\mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$, see Definition VII.0.4(6),(7)
- $(*)_3$ remember that

$$\mathrm{cov}(\chi,\mu,\theta,\sigma) = \chi + \ \mathrm{Min}\big\{|\mathscr{P}| : \mathscr{P} \subseteq [\chi]^{<\mu}, \text{ and every member of } [\chi]^{<\theta} \text{ is included in the union of } <\sigma \text{ members of } \mathscr{P}\big\}.$$

4.2 Claim. *Assume $2^\lambda < 2^{\lambda^+}$.*

<u>Then</u> one of the following cases occurs: (clauses $(\alpha) - (\lambda)$ appear later)

- $(A)_\lambda$ $\chi^* = 2^{\lambda^+}$ *and for some μ clauses $(\alpha) - (\varepsilon)$ hold*
- $(B)_\lambda$ *for some $\chi^* > 2^\lambda$ and μ clauses $(\alpha) - (\kappa)$ hold (note: μ appear only in $(\alpha) - (\varepsilon)$)*
- $(C)_\lambda$ $\chi^* = 2^\lambda$ *and clauses $(\eta) - (\mu)$ hold*

  <u>where</u>

  - (α) $\lambda^+ < \mu \le 2^\lambda$ *and* $\mathrm{cf}(\mu) = \lambda^+$
  - (β) $\mathrm{pp}(\mu) = \chi^*$, *moreover* $\mathrm{pp}(\mu) =^+ \chi^*$
  - (γ) $(\forall \mu')(\mathrm{cf}(\mu') \le \lambda^+ < \mu' < \mu \Rightarrow \mathrm{pp}(\mu') < \mu)$ *hence* $\mathrm{cf}(\mu') \le \lambda^+ < \mu' < \mu \Rightarrow \mathrm{pp}_{\lambda^+}(\mu') < \mu$
  - (δ) *for every regular cardinal χ in the interval $(\mu, \chi^*]$ there is an increasing sequence $\langle \lambda_i : i < \lambda^+ \rangle$ of regular cardinals $> \lambda^+$ with limit μ such that* $\chi = \mathrm{tcf}\left(\prod\limits_{i<\lambda^+} \lambda_i / J^{\mathrm{bd}}_{\lambda^+}\right)$, *and* $i < \lambda^+ \Rightarrow \max \mathrm{pcf}\{\lambda_j : j < i\} < \lambda_i < \mu$

(ε) *for some regular* $\kappa \le \lambda$, *for any* $\mu' < \mu$ *there is a tree* $\mathscr{T}$ *with* $\le \lambda$ *nodes,* κ *levels and* $|\lim_\kappa(\mathscr{T})| \ge \mu'$ *(in fact e.g.* $\kappa = \text{Min}\{\theta : 2^\theta \ge \mu\}$ *is appropriate; without loss of generality* $\mathscr{T} \subseteq {}^{\kappa>}\lambda$*)*

(ζ) *there is no normal* λ^{++}*-saturated ideal on* λ^+

(η) *there is* $\langle \mathscr{T}_\zeta : \zeta < \chi^* \rangle$ *such that:* $\mathscr{T}_\zeta \subseteq {}^{\lambda^+>}2$, *a subtree of cardinality* λ^+ *and* ${}^{\lambda^+}2 = \{\lim_{\lambda^+}(\mathscr{T}_\zeta) : \zeta < \chi^*\}$

(θ) $\chi^* < 2^{\lambda^+}$ *moreover* $\chi^* < \mu_{\text{unif}}(\lambda^+, 2^\lambda)$, *but* $< \mu_{\text{unif}}(\lambda^+, 2^\lambda)$ *is not used here,*

(ι) *for some* $\zeta < \chi^*$ *we have* $\lim_{\lambda^+}(\mathscr{T}_\zeta) \notin \text{UnfmTId}_{(\chi^*)^+}(\lambda^+)$, *not used here*

(κ) $\text{cov}(\chi^*, \lambda^{++}, \lambda^{++}, \aleph_1) = \chi^*$ *or* $\chi^* = \lambda^+$, *equivalently* $\chi^* = \sup[\{\text{pp}(\chi) : \chi \le 2^\lambda, \aleph_1 \le \text{cf}(\chi) \le \lambda^+ < \chi\} \cup \{\lambda^+\}]$ *by [Sh:g, Ch.II,5.4]; <u>note</u> that clause* (κ) *trivially follows from* $\chi^* = 2^{\lambda^+}$

(λ) *for no* $\mu \in (\lambda^+, 2^\lambda]$ *do we have* $\text{cf}(\mu) \le \lambda^+$, $\text{pp}(\mu) > 2^\lambda$*; equivalently* $2^\lambda > \lambda^+ \Rightarrow \text{cf}([2^\lambda]^{\lambda^+}, \subseteq) = 2^\lambda$

(μ) *if there is a normal* λ^{++}*-saturated ideal on* λ^+*, moreover the ideal* $\text{WDmId}(\lambda^+)$ *is, <u>then</u>* $2^{\lambda^+} = \lambda^{++}$ *(so as* $2^\lambda < 2^{\lambda^+}$ *clearly* $2^\lambda = \lambda^+$*).*

Proof. By VII.11.1. $\square_{4.2}$

4.3 Definition. 1) We say K' is (λ, λ^+)-dense in $\mathfrak{K}$ when $K' \subseteq K_{\lambda^+}$ and in the following game the odd player has a winning strategy. A play last λ^+ moves, in the α-th move a model $M_\alpha \in K_\lambda$ is chosen, $<_{\mathfrak{K}}$-increasing continuous with α and we can add with universe $\subseteq \lambda^+$.

Naturally, M_α is chosen by the even/odd player when α is even/odd. Lastly, the odd player wins a play when $\cup\{M_\alpha : \alpha < \lambda^+\} \in K'$. To avoid the use of global choice we may require the universe of M_α (or just $M_\alpha \backslash M_0$) is $\subseteq \lambda^+$.

2) Writing above "$\mathfrak{K}'$ is" we mean $\mathfrak{K}' = (K', \le_{\mathfrak{K}} \restriction K')$ and K' is as

above. We say "...dense over S..." when $S \subseteq \lambda^+$ and for some club E of λ^+, for $\delta \in S \cap E$ we let the even player choose $M_{\delta+1}$.

4.4 Claim. *Let $\mathfrak{K}$ be an a.e.c., $\mathrm{LS}(\mathfrak{K}) \leq \lambda$ and $(\mathrm{nmx})_\lambda$.*
1) $\mathfrak{K}' = \mathfrak{K}_{\lambda^+}$ is (λ, λ^+) dense in $\mathfrak{K}_{\lambda^+}$.
2) If $\mathfrak{K}$ has $(\mathrm{amg})_\lambda$ and $K' = K^{\mathrm{slm}}_{\lambda^+} \neq \emptyset$ <u>then</u> K' is (λ, λ^+)-dense in $\mathfrak{K}$.
3) Assume $(\mathrm{amg})_\lambda + (\mathrm{jep})_\lambda$. If $S \subseteq \lambda^+$ is satisfied $|S| = \lambda^+ = |\lambda^+ \backslash S|$ and the game ∂_S is defined as in 4.3 but we let the odd player choose $M_{\alpha+1}$ iff $\alpha \in S$, <u>then</u> "odd win in ∂_S" does not depend on S (and $\partial_{\{2\alpha:\alpha<\lambda^+\}}$ is the original game).
4) If there is a (λ^+, λ^+)-superlimit model in $\mathfrak{K}_{\lambda^+}$ <u>then</u> $\{N : N \cong M\}$ is (λ, λ^+)-dense in $\mathfrak{K}$.
5) K' is (λ, λ^+)-dense in $\mathfrak{K}$ <u>when</u> $\mathfrak{K}$ has $(\mathrm{amg})_\lambda$ and $K' \subseteq K_{\lambda^+}$ is closed under isomorphisms, is dense in $\mathfrak{K}_{\lambda^+}$ (i.e. $(\forall M \in K_{\lambda^+})(\exists N \in K')[M \leq_{\mathfrak{K}} N]$) and K' is closed under unions $\leq_{\mathfrak{K}}$-increasing continuous chains of length $< \lambda^{++}$.

Proof. 1) Trivial.
2),3) Straight.
4),5) Obvious. $\square_{4.4}$

4.5 Claim. *1) Assume*

- (a) *$2^\lambda < 2^{\lambda^+}$ and case $(A)_\lambda$ or $(B)_\lambda$ of Claim 4.2 holds for μ, χ^* (or just the conclusions there)*
- (b) *$\mathfrak{K}$ is an abstract elementary class with $\mathrm{LS}(\mathfrak{K}) \leq \lambda$*
- (c) *$K_{\lambda^+} \neq 0$*
- (d) *$\mathfrak{K}$ has amalgamation in λ*
- (e) *in $K^{3,\mathrm{na}}_\lambda$, the minimal triples are not dense.*

<u>Then</u>

- $(*)_1$ *for any regular $\chi < \mu$ we have:*
- $(*)^1_\chi$ *there is $M \in K_\lambda$ such that $|\mathscr{S}_{\mathfrak{K}_\lambda}(M)| > \chi$.*

2) If in part (1) we strengthen clause (*d*) *to* (*d*)$^+$, *then we get* $(*)^+_1$ *where:*

- (*d*)$^+$ $(\mathrm{amg})_{\lambda^+}$ $(\mathrm{unv})_\lambda$, *i.e.* $\mathfrak{K}$ *has amalgamation in* λ *and has a universal member in* λ *(the second follows from* $(\mathrm{slm})_\lambda$*)*
- $(*)^+_1$ *for some* $M \in K_\lambda$ *we have* $|\mathscr{S}_{\mathfrak{K}_\lambda}(M)| \geq \mu$, *in fact every* $\leq_{\mathfrak{K}_\lambda}$*-universal* M *is O.K.*

3) Assume (a), (b), (c), (e) of part (1) and (*d*)$^+$ *of part (2) then:*

- $(*)_2$ $\dot{I}(\lambda^+, K) \geq \chi^*$ *and if* $(2^\lambda)^+ < \chi^*$ *then* $\dot{I}\dot{E}(\lambda^+, \mathfrak{K}) \geq \chi^*$
- $(*)_3$ *there is no universal model in* $\mathfrak{K}_{\lambda^+}$
- $(*)_4$ *if* $\mathfrak{K}'$ *is* (λ, λ^+)*-dense in* $\mathfrak{K}$ *then* $\dot{I}(\lambda^+, K') \geq \chi^*$ *and if in addition* $(2^\lambda)^+ < \chi^*$ *then* $\dot{I}\dot{E}(\lambda^+, \mathfrak{K}') \geq \chi^*$.

4) If in clause (*a*) *of part (1) we restrict ourselves to Case* $(A)_\lambda$ *of Fact 4.2, then* $\chi^* = 2^{\lambda^+}$ *and in part (3) we get*

- $(*)^+_2$ $\dot{I}(\lambda^+, K) = 2^{\lambda^+}$ *and* $(2^\lambda)^+ < 2^{\lambda^+} \Rightarrow \dot{I}\dot{E}(\lambda^+, K) \geq 2^{\lambda^+}$.

4.6 Remark. 1) We can restrict clause (b) of 4.5(1) to $\mathfrak{K}_\lambda$, interpreting in (c) + (e) the class K_{λ^+} as $\{\bigcup_{i<\lambda^+} M_i : M_i \in K_\lambda$ is $<_{\mathfrak{K}}$-increasing (strictly and) continuous$\}$, but see II.1.23.
2) Part (3) of 4.5 and 4.7 below generalize [Sh:g, Ch.II,4.10E] and Kojman-Shelah [KjSh 409, §2].
3) We can apply this with λ^+ standing for λ here.
4) We can state the part of (A) from 4.2 which we actually use (and can replace 2^{λ^+} by smaller cardinals).
5) We can replace λ^+ by a weakly inaccessible cardinal with suitable changes.

Proof. 1) Note that μ is singular (as by clause (α) of 4.2 (which holds if $(A)_\lambda$ or $(B)_\lambda$) we have $\mathrm{cf}(\mu) = \lambda^+ < \mu$).

We can apply 2.3, its assumption, because Hypothesis 2.1 holds by clause (b) here, and $(\mathrm{amg})_\lambda$ and non-density of minimal triples

in $K^{3,\mathrm{na}}_\lambda$ hold by clauses (d),(e) here. Now by 2.3(2) to prove $(*)_1$ it suffices for each $\mu' < \mu$ to show that there are $\delta < \lambda^+$ and a tree with $\le \lambda$ nodes and δ levels and $\ge \mu'$ δ-branches. They exist by clause (ε) of 4.2 (which holds as $(A)_\lambda$ or $(B)_\lambda$ hold).
2) Let $M^0_* \in K_\lambda$ be $\le_{\mathfrak{K}}$-universal. So for every $\chi < \mu$ by part (1) applied to χ^+ which is regular $< \mu$, there is $M_\chi \in K_\lambda$ such that $\mathscr{S}(M_\chi)$ has cardinality $\ge \chi$. As M_* is $\le_{\mathfrak{K}}$-universal, without loss of generality $M_\chi \le_{\mathfrak{K}} M_*$. But by II.1.11(5) we know that $p \mapsto p \restriction M_\chi$ is a mapping from $\mathscr{S}(M_*)$ onto $\mathscr{S}(M_\chi)$ hence $|\mathscr{S}(M_*)| \ge |\mathscr{S}(M_\chi)| \ge \chi$. As μ is a limit ordinal we are done.
3) By part (2) we can find $M^* \in K_\lambda$ such that $\mathscr{S}_{\mathfrak{K}_\lambda}(M^*)$ has cardinality $\ge \mu$, and apply 4.7 below. Let us elaborate, all the assumptions of 4.7 hold because:

Clause (a): holds by (β) of 4.2; recalling that $\mathrm{cf}(\mu) = \theta < \mu \Rightarrow \mathrm{cf}([\mu]^\theta, \subset) \ge \mathrm{pp}(\mu)$, apply it with μ, λ^+ standing for μ, θ and increasing χ^* is O.K.

Clause (b): holds by clause (b) of 4.5.

Clause (c): holds by the choice of M^* above as $\in K_\lambda$ (recalling $\mathfrak{K}_\lambda$ has amalgamation by clause (d) of 4.5).

Clause (d): holds by the choice of M^*.

To prove $(*)_2 + (*)_3$ of 4.5(3) we let $\mathfrak{K}' = \mathfrak{K}_{\lambda^+}$ so all the assumptions of 4.7 holds, i.e. also clause (e) by 4.4(1) hence also the conclusion of 4.7 holds. Now $(*)_2$ holds by clause $(\alpha) + (\beta)$ of 4.7 because χ^* here is $\le \mathrm{cf}([\mu]^{\lambda^+}, \subseteq)$ which is χ^* in 4.7. Next $(*)_3$ holds by clause (γ) of 4.7. To prove $(*)_4$ of 4.5(3), we have $\mathfrak{K}'$ in it, then assumption clause (e) of 4.7 holds by the assumption of $(*)_4$ and we can apply 4.7.
4) Should be clear from the proof of part (3). $\square_{4.5}$

4.7 Claim. *Assume*

- (a) $\mu > \lambda^+$ *and* $\chi^* := \mathrm{cf}([\mu]^{\lambda^+}, \subseteq) > 2^\lambda$
- (b) $\mathfrak{K}$ *is an a.e.c. with* $\mathrm{LS}(\mathfrak{K}) \le \lambda$
- (c) $M^* \in K_\lambda$ *is an amalgamation base in* $\mathfrak{K}_\lambda$
- (d) $M^* \in K_\lambda$ *satisfies* $|\mathscr{S}_{\mathfrak{K}_\lambda}(M^*)| \ge \mu$ *and for simplicity* $(\mathrm{nmx})_\lambda$ *is satisfied by* $\mathfrak{K}$

(e) *$\mathfrak{K}' \subseteq \mathfrak{K}_{\lambda^+}$ is (λ, λ^+)-dense in $\mathfrak{K}$ (if $K' = K_{\lambda^+}$ we ignore it).*

Then

(α) *$\dot{I}(\lambda^+, K) \geq \chi^*$ moreover $\dot{I}(\lambda^+, \mathfrak{K}') \geq \chi^*$*

(β) *if $(2^\lambda)^+ < \chi^*$ then $\dot{I}\dot{E}(\lambda^+, \mathfrak{K}') \geq \chi^*$*

(γ) *there is no $\leq_{\mathfrak{K}}$-universal $N^* \in K_{\lambda^+}$ (in fact $\mathrm{cov}(\mathfrak{K}_{\lambda^+}, \leq_{\mathfrak{K}})$ is $\geq \chi^*$ when defined reasonably; also there is no $\mathfrak{K}'_{\lambda^+}$ universal model even in $\mathfrak{K}_{\lambda^+}$).*

4.8 Remark. We add $(\mathrm{nmx})_\lambda$ just to justify using K' in (e), see 4.4(1). But this is not a real assumption as even not assuming it, $\mathfrak{K} \restriction \{M : M^* \text{ is } \leq_{\mathfrak{K}}\text{-embeddable into } M\}$ satisfies all the assumptions and $(\mathrm{nmx})_\lambda$ and each of its conclusions implies the parallel for the original $\mathfrak{K}$. This applies also to 4.9.

Proof. Let $p_\eta \in \mathscr{S}(M^*)$ for $\eta \in Z$ be pairwise distinct, $|Z| \geq \mu$ and let (N_η, a_η) for $\eta \in Z$ be such that $M^* \leq_{\mathfrak{K}} N_\eta \in K_\lambda$ and $p_\eta = \mathbf{tp}(a_\eta, M^*, N_\eta)$.
Now for every $X \in [Z]^{\lambda^+}$, as M^* is an amalgamation base in $\mathfrak{K}_\lambda$ there is $M_X \in K_{\leq \lambda^+}$ such that $M^* \leq_{\mathfrak{K}} M_X$ and $\eta \in X \Rightarrow N^*_\eta$ is embeddable into M_X over M^* (hence p_η is realized in M_X). Moreover as "K' is (λ, λ^+)-dense in $\mathfrak{K}$", it is easy to guarantee $M_X \in K'$, [ignoring K', $(\mathrm{nmx})_\lambda$ we can argue that as $\langle p_\eta : \eta \in Z \rangle$ is without repetitions it follows that $\|M_X\| \geq \lambda^+$ so $M_X \in K_{\lambda^+}$]. Let $Y[X] = \{\eta \in Z : p_\eta \text{ is realized in } M_X\}$. So $X \subseteq Y[X] \in [Z]^{\lambda^+}$, hence $\{Y[X] : X \in [Z]^{\lambda^+}\}$ is a cofinal subset of $[Z]^{\lambda^+}$, hence (see clause (a) of the assumption)

$$\begin{aligned} |\{(M_X, c)_{c \in M^*}/\cong : X \in [Z]^{\lambda^+}\}| \geq \\ &|\{Y[X] : X \in [Z]^{\lambda^+}\}| \geq \mathrm{cf}([Z]^{\lambda^+}, \subseteq) \geq \\ &\mathrm{cf}([\mu]^{\lambda^+}, \subseteq) = \chi^*. \end{aligned}$$

As $2^\lambda < \chi^*$ also $|\{M_X/\cong : X \in [Z]^{\lambda^+}\}| \geq \chi^*$, see I.0.3 because $\|M_X\| = \lambda^+, \|M^*\| = \lambda$ and $(\lambda^+)^\lambda = 2^\lambda < \chi^*$ by clause (a) in 4.5. But $\dot{I}(\lambda^+, K)$ is $\geq$ than the former so we have proved clause (α).

Second, we shall prove $(2^\lambda)^+ < 2^{\lambda^+} \Rightarrow \dot{I}\dot{E}(\lambda^+, K) \geq \chi^*$, i.e. claues (β).
For each $X \in [\mu]^{\lambda^+}$, let $\mathscr{F}_X = \{f : f \text{ is a } \leq_{\mathfrak{K}}\text{-embedding of } M^* \text{ into } M_X\}$. We would like to apply claim VII.11.4; so for simplicity without loss of generality $Z = \mu$. We define the function $\mathbf{F}$ with domain $[\mu]^{\lambda^+}$ by $\mathbf{F}(X) = \{\{\eta \in Z : f(p_\eta) \text{ is realized in } M_X\} : f$ belongs to $\mathscr{F}_X\}$, so $\mathbf{F}(X)$ is a subset of $[\mu]^{\lambda^+}$ of cardinality $\leq |\mathscr{F}_X| \leq 2^\lambda$.

Now we apply VII.11.4 with $\kappa, \mu, \theta, \chi^*, \mathbf{F}, F$ there standing for $\lambda^+, \mu, 2^\lambda, \chi^*, \mathbf{F}, X \mapsto Y[X]$ here, the assumptions clearly holds hence we can find $X_i \in [Z]^{\lambda^+}$ for $i < \chi^*$ such that $i \neq j < \chi^*$ & $X \in \mathbf{F}(X_j) \Rightarrow Y[X_i] \nsubseteq X$. Clearly $\langle M_{X_i} : i < \chi^* \rangle$ is as required in clause (β) of the conclusion and it holds.
Lastly, toward proving clause (γ) assume (toward contradiction) that $N_* \in K_{\lambda^+}$ is $\leq_{\mathfrak{K}}$-universal, then for every $X \in [Z]^{\lambda^+}$ there is a $\leq_{\mathfrak{K}}$-embedding f_X of M_X into M^*. Let $\mathscr{F} = \{f_X \restriction M : X \in [Z]^{\lambda^+}\}$ and for $f \in \mathscr{F}$ let $\mathscr{X}_f = \{X \in [Z]^{\lambda^+} : f_X \restriction M = f\}$, so $[Z]^{\lambda^+} = \cup\{\mathscr{X}_f : f \in \mathscr{F}\}$. Note that $|\mathscr{F}| \leq (\lambda^+)^\lambda = 2^\lambda < \chi^*$.

Let $Y^+[X] = \{\eta \in Z : f_X(p_\eta) \text{ is realized in } N_*\}$, so $Y[X] \subseteq Y^+[X] \subseteq [Z]^{\lambda^+}$ and $f_{X_1} = f_{X_2} \Rightarrow Y^+[X_1] = Y^+[X_2]$, hence $\{Y^+[X] : X \in [Z]^{\lambda^+}\}$ is cofinal in $[Z]^{\lambda^+}$ and has cardinality $\leq |\mathscr{F}| \leq 2^\lambda$, contradiction. So also clause (γ) holds and we are done. $\square_{4.7}$

4.9 Claim. *1) Assume*

- (a) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{+2}}$ *and case* $(B)_\lambda$ *or* $(C)_\lambda$ *of Claim 4.2 for* λ *occurs*
  (so $\chi^* < 2^{\lambda^+}$ *and* $\mathscr{T}_\zeta$ *for* $\zeta < \chi^*$ *are determined)*
- (b) $\mathfrak{K}$ *is an abstract elementary class,* $\mathrm{LS}(\mathfrak{K}) \leq \lambda$
- (c) $K_{\lambda^{++}} \neq \emptyset$,
- (d) $\mathfrak{K}$ *has amalgamation in* λ
- (e) *in* $K^{3,\mathrm{na}}_\lambda$, *the minimal triples are not dense.*

<u>*Then*</u>

- $(*)$ *for each* $\zeta < \chi^*$ *for some* $M \in K_{\lambda^+}$ *we have* $|\mathscr{S}^{\mathrm{nm}}_*(M)| \geq |\lim_{\lambda^+}(\mathscr{T}_\zeta)|$

(the tree is from clause (η) of 4.2; on $\mathscr{S}_*^{\rm nm}$ see Definition 3.3(6), make sense even without amalgamation in λ^+) moreover

$(**)$ *for each $\zeta < \chi^*$ there are $M \in K_{\lambda^+}$ and $(M, M_\varepsilon, a_\varepsilon) \in K^{3,\rm nm}_{\lambda^+,*}$ for $\varepsilon < |\lim_{\lambda^+}(\mathscr{T}_\zeta)|$ such that $\langle \mathbf{tp}(a_\varepsilon, M, M_\varepsilon)/\mathbb{E}_\lambda : \varepsilon < |\lim_{\lambda^+}(\mathscr{T}_\zeta)|\rangle$ is without repetitions, i.e. if $\varepsilon_1 < \varepsilon_2 < |\lim_{\lambda^+}(\mathscr{T}_\zeta)|$ then for some $M' \leq_{\mathfrak{K}} M$ satisfying $M' \in K_\lambda$ we have $\mathbf{tp}_{\mathfrak{K}}(a_{\varepsilon_1}, M', M_{\varepsilon_1}) \neq \mathbf{tp}_{\mathfrak{K}}(a_{\varepsilon_2}, M', M_{\varepsilon_2})$*

$(***)$ *if K' is (λ, λ^+)-dense in K then we can add in $(**)$ that $M, M_\varepsilon \in K'$.*

2) Assume K satisfies clauses (a)-(e) and

$\boxtimes$ *at least one of the following occurs:*

- (α) *K is categorical in λ^+*
- (β) *$\mathfrak{K}$ has a universal member in λ^+ and amalgamation in λ^+*
- (γ) *there is a (λ, λ^+)-dense $K' \subseteq \mathfrak{K}_{\lambda^+}$ which is categorical*
- (δ) *there is a (λ, λ^+)-dense $K' \subseteq \mathfrak{K}_{\lambda^+}$ such that $(\mathfrak{K}', \leq_{\mathfrak{K}} \restriction K')$ has amalgamation and has a universal member.*

<u>Then</u> for some $M \in K_{\lambda^+}$, and for clauses (γ) of (δ) moreover $M \in K'_{\lambda^+}$, i.e. M is as in $\boxtimes$ we have $|\mathscr{S}_*^{\rm rmr}(M)| = 2^{\lambda^+}$ and if (α) or (γ) of $\boxtimes$ holds, moreover $|\mathscr{S}_*^{\rm nmr}(M)/\mathbb{E}_\lambda| = 2^{\lambda^+}$.

Remark. Recall that $p\mathbb{E}_\lambda q$ iff $(\forall M' \leq_{\mathfrak{K}} M)(M' \in K_\lambda \Rightarrow p \restriction M' = q \restriction M')$.

Before we prove 4.5, we state a conclusion to be proved later.

4.10 Conclusion. If clauses $(a) - (g)$ below holds, <u>then</u> $\dot{I}(\lambda^{++}, K) \geq \mu_{\rm unif}(\lambda^{++}, 2^{\lambda^+})$, i.e. we assume:

- (a) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{+2}}$
- (b) (α) $\mathfrak{K}$ is an abstract elementary class $\mathrm{LS}(\mathfrak{K}) \leq \lambda$

(β) $|\tau_K| \le \lambda$

(c) $K_{\lambda^{++}} \neq \emptyset$,

(d) $\mathfrak{K}$ has amalgamation in λ

(e) in $K^{3,\mathrm{na}}_\lambda$, the minimal triples are not dense

(f) $\mathfrak{K}$ is categorical in λ

(g) (α) $\mathfrak{K}$ is categorical in λ^+ <u>or</u> at least $\mathfrak{K}$ has a superlimit member in λ^+

(β) $\dot{I}(\lambda^+, K) < 2^{\lambda^+}$

(γ) $\mathfrak{K}$ has $(\mathrm{amg})_{\lambda^+}$ <u>or</u> at least the $M \in K^{\mathrm{slm}}_{\lambda^+}$ is an amalgamation basis in $\mathfrak{K}_{\lambda^+}$.

4.11 Remark. 1) Note that 4.10 put 4.5 + 4.9 together. Clause (b)(β) is used in the end of case (2), clause (g)(β) is used in case (1B).
1A) Can we eliminate (g)(β), i.e. $(\mathrm{mdn})^1_{\lambda^+}$ or replace it by $(\mathrm{unv})_{\lambda^+}$?
2) Note that 4.13 below is essentially another presentation, in it we do not use 4.9.
3) We would like, compared to [Sh 603], to weaken in the assumption "$\mathfrak{K}$ categorical in λ^+" to "no maximal model in $\mathfrak{K}_\lambda$". So by 4.7 for amalgamation bases $M^* \in K_{\lambda^+}, \mathscr{S}(M^*)$ cannot be too large (used in the proof of 4.13) and as $\dot{I}(\lambda^{++}, K) < \mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$, there are many amalgamation basis and by 4.9(1) there are many $M \in K_{\lambda^+}$ with $\mathscr{S}(M)$ large. But we have to put them together.

Proof of 4.9. 1) Let $\zeta < \chi^*$. Recall that $\mathscr{T}_\zeta$ is a subtree of ${}^{\lambda^+ >}2$ of cardinality $\le \lambda^+$, hence let $\mathscr{T}_\zeta = \cup\{\mathscr{T}^\zeta_\alpha : \alpha < \lambda^+\}$ where the $\mathscr{T}^\zeta_\alpha$ are pairwise disjoint for $\alpha < \lambda^+$, each $\mathscr{T}^\zeta_\alpha$ has cardinality $\le \lambda, \mathscr{T}^\zeta_0 = \{<>\}$ and $\eta \in \mathscr{T}^\zeta_\alpha \ \ \& \ \ \beta < \ell g(\eta) \Rightarrow \eta \restriction \beta \in \bigcup\limits_{\gamma<\alpha} \mathscr{T}^\zeta_\gamma$, and $\eta \in \mathscr{T}^\zeta_\alpha \Rightarrow \bigwedge\limits_{\ell<2} \eta\hat{\ }\langle \ell \rangle \in \mathscr{T}^\zeta_{\zeta,\alpha+1}$ and $\mathscr{T}_{\zeta,\alpha+1} = \{\eta\hat{\ }\langle \ell \rangle : \eta \in \mathscr{T}^\zeta_\alpha$ and $\ell < 2\}$. For $\eta \in \mathscr{T}^\zeta_\delta, \delta$ a limit ordinal, necessarily both $\ell g(\eta)$ and $\alpha(\eta) = \sup\{\gamma : \text{for some } \varepsilon < \ell g(\eta), \eta \restriction \varepsilon \in \mathscr{T}^\zeta_\gamma\}$ are limit ordinals $\le \delta$, clearly $\eta \in \mathscr{T}_\alpha \Rightarrow \ell g(\eta) \le \alpha$.

Let $(M, N, a) \in K_\lambda^{3,\mathrm{na}}$ be such that there is no minimal triple above it, i.e. $\in K_\lambda^{3,\mathrm{nm}}$; without loss of generality it is reduced. If K' is not well defined, let $K' = K_{\lambda^+}$.

If we have K', i.e. for $(***)$, then let **st** be a winning strategy for the odd player recalling Definition 4.3.
We now by induction on $\alpha < \lambda^+$ choose $\langle M_\alpha^\zeta, N_\eta^\zeta : \eta \in \mathscr{T}_\alpha^\zeta \rangle$ such that:

(a) $(M_\alpha^\zeta, N_\eta^\zeta, a) \in K_\lambda^{3,\mathrm{na}}$

(b) $(M_\alpha^\zeta, N_\eta^\zeta, a)$ is reduced (see Definition 1.11(4)) <u>if</u> $\eta \in \mathscr{T}_\alpha^\zeta, \alpha$ non-limit

(c) $(M_0^\zeta, N_{<>}^\zeta, a) = (M, N, a)$

(d) if $\nu \in \mathscr{T}_\beta^\zeta, \eta \in \mathscr{T}_\alpha^\zeta, \nu \triangleleft \eta, \beta < \alpha$ and α, β are non-limit ordinals then $(M_\beta^\zeta, N_\nu^\zeta, a) \leq_{\mathrm{na}} (M_\alpha^\zeta, N_\eta^\zeta, a)$

(e) if δ is a limit ordinal then:

(α) $M_\delta^\zeta = \bigcup_{\beta<\delta} M_\beta^\zeta$

(β) if $\eta \in \mathscr{T}_\delta$ and $\delta = \sup\{\beta < \delta : \eta \restriction \gamma \in \mathscr{T}_\beta^\zeta$ for some $\gamma < \ell g(\eta)\}$ then $N_\eta^\zeta = \cup\{N_{\eta\restriction\gamma}^\zeta : \gamma < \ell g(\eta)\}$

(f) if $\eta \in \mathscr{T}_\alpha^\zeta$ then $\mathbf{tp}_{\mathfrak{K}_\lambda}(a, M_{\alpha+1}, N_{\eta\hat{}<0>}) \neq \mathbf{tp}_{\mathfrak{K}_\lambda}(a, M_{\alpha+1}, N_{\eta\hat{}<1>})$

(g) $M_\alpha^\zeta \neq M_{\alpha+1}^\zeta$

(h) if we are proving $(***)$, i.e. have K', **st** <u>then</u> in stage α a successor we also choose $\langle M_\beta^{\zeta,*} : \beta \leq 2\alpha \rangle, f_\alpha, \langle N_{\eta,\beta}^{\zeta,*} : \beta \leq 2\ell g(\eta)\rangle, f_\eta^\zeta$ for $\eta \in \mathscr{T}_\alpha^\zeta$ such that:

(α) $\langle M_\beta^{\zeta,*} : \beta \leq 2\alpha \rangle$ is an initial segment of a play of the game from 4.3 in which the odd player uses its winning strategy **st**

(β) f_α is an isomorphism from M_α onto $M_{2\alpha}^{\zeta,*}$, increasing with α

(γ) $\langle N_{\eta,\beta}^{\zeta,*} : \beta \leq 2\ell g(\eta)\rangle$ is an initial segment of a play of the game from 4.3 in which the odd player uses its winning strategy **st**

(δ) f^ζ_α is an isomorphism from N^ζ_η onto $N^{\zeta,*}_{\eta,2\ell g(\eta)}$

(ε) if $\nu \triangleleft \eta, \nu \in \mathscr{T}_\beta, \eta \in \mathscr{T}_\alpha$, and β (as well as α) are successor ordinals and $\gamma \leq 2\ell g(\nu)$ then $N^{\zeta,*}_{\nu,\gamma} = N^{\zeta,*}_{\eta,\gamma}$ and $f^\zeta_\nu \subseteq f^\zeta_\eta$

There is no problem to carry the definition, recalling that for α successor we can use Exercise 2.6 and for α limit, for any $\eta \in \mathscr{T}^\zeta_\alpha$ the triple $(\cup\{M^\zeta_{\eta\restriction\varepsilon} : \eta \restriction \varepsilon \in \mathscr{T}^\zeta_\beta$ for some $\beta < \alpha\}, \cup\{N^\zeta_{\eta\restriction\varepsilon} : \eta \restriction \varepsilon \in \mathscr{T}^\zeta_\beta$ for some $\beta \in \alpha\}, a)$ is a reduced member of $K^{3,\mathrm{nm}}_\lambda$. Let $M_\zeta = \bigcup\limits_{\alpha<\lambda^+} M^\zeta_\alpha \in K_{\lambda^+}$, and for each $\nu \in \lim_{\lambda^+}(\mathscr{T}_\zeta)$ let $N^\zeta_\nu = \bigcup\limits_{\alpha<\lambda^+} N^\zeta_{\nu\restriction\alpha}$, clearly $M_\zeta \leq_{\mathfrak{K}} N^\zeta_\nu$ are both from K_{λ^+} and even from K' in the relevant case. Also $a \in N^\zeta_\nu$ and $\langle \mathbf{tp}(a, M_\zeta, N^\zeta_\nu) : \nu \in \lim_{\lambda^+}(\mathscr{T}_\zeta)\rangle$ are pairwise distinct members of $\mathscr{S}(M_\zeta)$ and even of $\mathscr{S}^{\mathrm{nm}}_*(M_\zeta)$ moreover (particularly if $\mathfrak{K}_{\lambda^+}$ fails the amalgamation property, it is better to add) pairwise non-$\mathbb{E}_\lambda$-equivalent. Now $M_\zeta, \langle (M_\zeta, N^\zeta_\nu, a) : \nu \in \lim(\mathscr{T}_\zeta)\rangle$ exemplifies clauses $(*)$ and $(**)$ and $(***)$ of part (1).
2) Note that ${}^{\lambda^+}2 = \cup\{\lim_{\lambda^+}(\mathscr{T}_\zeta) : \zeta < \chi_*\}$, by clause ($\eta$) of 4.2 and $\chi^* < 2^{\lambda^+}$ by clause (θ) of 4.2. So for every regular $\chi \leq 2^{\lambda^+}$ for some $\zeta < \chi_*$ we have $|\lim_{\lambda^+}(\mathscr{T}_\zeta)| \geq \chi$ hence by part (1) for some $M_* \in K_{\lambda^+}$ we have $|\mathscr{S}(M_\chi)| \geq |\mathscr{S}(M_\chi)/\mathbb{E}_\lambda| \geq |\lim_{\lambda^+}(\mathscr{T}_\zeta)|$ and $M' \in K'$ when clause (γ) holds. If 2^{λ^+} is regular we are done, if it is singular the proof splits according to the case, i.e. which of $(\alpha), (\beta), (\gamma), (\delta)$ from $\boxtimes$ we assume. If $\mathfrak{K}$ is categorical in λ^+, i.e. clause (α) holds, this is obvious. If $\mathfrak{K}_{\lambda^+}$ has a universal member and has amalgamation, i.e. clause (β) holds the result follows as in the proof of 4.5(2). The proof when clause (γ) or clause (δ) holds is similar using $(**)$ of part(1). $\square_{4.9}$

Proof of 4.10. Assume toward contradiction that the conclusion fails. Let $\mathfrak{K}' = \mathfrak{K}^{\mathrm{slm}}_{\lambda^+}$ so $\dot{I}(K') = \dot{I}(\lambda^+, K') = 1$ and K' is (λ, λ^+)-dense in $\mathfrak{K}$, see 4.4(2) justified by clause (g)(α) of the assumption. Now if $(A)_\lambda \vee (B)_\lambda$ of 4.2 of course, let χ^* be as there, then the assumptions of 4.5(3) holds. [Why? Clause (a) there by clause (a) of 4.10 and our present assumption $(A)_\lambda \vee (B)_\lambda$; clauses (b),(c) there by $(b)(\alpha)$ and

(c) here; clause (e) there by clause (e) here, and clause $(d)^+$ there by clauses (d),(g)(α) here). So the conclusions of 4.5(3) holds and so by $(*)_3$ from 4.5(3) we get that there is no $\leq_{\mathfrak{K}}$-universal $M \in K_{\lambda^+}$ contradicting clause (g)(α) of the assumption; we can use K' instead of no universal. Hence we cannot assume $(A)_\lambda \vee (B)_\lambda$, so in 4.2,

$(*)_1$ case $(C)_\lambda$ holds, so let $\langle \mathscr{T}_\zeta : \zeta < \chi^* \rangle$ be as there.

Now we can apply 4.9(2) because:

$\odot_1$ clauses (b),(c),(d),(e) of 4.9 holds by clauses (b)(α),(c),(d),(e) of 4.10 respectively

$\odot_2$ clause (a) of 4.9 holds by combining clause (a) of 4.10 and $(*)_1$

$\odot_3$ among the four possibilities in 4.9(2), clause (γ) holds by 4.4(2) recalling (g)(α) of 4.10.

So its conclusion holds, i.e. for some $M \in K' \subseteq K_{\lambda^+}$ we have $|\mathscr{S}_*^{\rm nmr}(M)/\mathbb{E}_\lambda| = 2^{\lambda^+}$. We try to apply 4.7 with $\lambda^+, 2^{\lambda^+}, 2^{\lambda^{++}}, \mathfrak{K}' = \mathfrak{K}^{\rm slm}_{\lambda^+}, \mathfrak{K}_{\lambda^{++}}, M$ here standing for $\lambda, \mu, \chi^*, \mathfrak{K}, \mathfrak{K}', M^*$ pedantically use $(\mathfrak{K}^{\rm slm}_{\lambda^+})^{\rm up}$ there.

Now clause (b) there means "$\mathfrak{K}$ is an a.e.c. with $\mathrm{LS}(\mathfrak{K}) \leq \lambda^+$", obviously holds. Clause (c) there means "$M \in K_{\lambda^+}$ is an amalgamation base" holds by clause (g)(γ) of the assumption here as $M \in K' = K^{\rm slm}_{\lambda^+}$. Clause (d) of the assumption of 4.7 means here: "$|\mathscr{S}_{\mathfrak{K}'_{\lambda^+}}(M)| \geq 2^{\lambda^+}$ which holds by the way M was chosen because $\mathscr{S}_{\mathfrak{K}_{\lambda^+}}(M), \mathscr{S}_{\mathfrak{K}'_{\lambda^+}}(M)$ are essentially the same as M is an amalgamation base in $\mathfrak{K}_{\lambda^+}$, see 1.23.

Lastly, clause (e) there is trivial as K' there stands for $\mathfrak{K}_{\lambda^{++}}$ here. If the conclusions of this instance of 4.7 hold, in particular by clause (α) there, we have $\dot{I}(\lambda^{++}, K) \geq 2^{\lambda^{++}}$ which is more than promised in 4.10. But we are assuming toward contradiction that it fails. So we deduce that the remaining assumption of 4.7, i.e. clause (a) fails, i.e. $\neg(2^{\lambda^{++}} = \mathrm{cf}([2^{\lambda^+}]^{\lambda^{++}}, \subseteq) > 2^{\lambda^+}\ \ \&\ \ 2^{\lambda^+} > \lambda^{++})$. It follows that

$(*)_2$ $2^{\lambda^+} > \lambda^{++} \Rightarrow \mathrm{cf}([2^{\lambda^+}]^{\lambda^{++}}, \subseteq) < 2^{\lambda^{++}}$.

Also note that

$(*)_3$ context 3.1 holds.

Why? By clauses $(b)(\beta), (d)$ of 4.10. We shall use $(*)_3$ freely below.

Case 1: $(2^\lambda > \lambda^+)+$ (WDmId(λ^+) is not λ^{++}-saturated).

Subcase 1A: $\neg(**)'_\lambda$ of 3.10 holds.

We shall try to apply Claim 3.13, so let us check its assumptions. Clause (a) of 3.13 says that $(**)'_\lambda$ of 3.13 fail, but this is the assumption of the present subcase.

Clause (b) says that "$|\mathscr{S}^{\rm nc}(M)| > \lambda^+$ for $M \in K_\lambda$" which as said there follows from "$(\mathrm{cat})_\lambda+$ the minimal triple in $K^{3,\rm na}_\lambda$ are not dense $+2^\lambda > \lambda^+$" and: $(\mathrm{cat})_\lambda$ holds of clause (f) by 4.10, and the non-density of the minimal triple holds by clause (e) of the present claim 4.10 and $2^\lambda > \lambda^+$ holds as we are in Case 1 of the proof. Next, clause (c) of 3.13 says $(\mathrm{cat})_\lambda$ which holds by clause (f) of 4.10. Now clause (d) of 3.13 says "there is a superlimit $M \in K_{\lambda^+}$" and it holds by clause (g)(α) of 4.10.

Next, clause (e) of 3.13 says "$2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$" which hold by clause (a) of 4.10. Lastly, $\circledast_\lambda$ of 3.13 fails as we are in Case 1 of the proof.

Together by 3.13 we get $\dot{I}(\lambda^{++}, K) \geq \mu_{\rm unif}(\lambda^{++}, 2^{\lambda^+})$ as required.

Subcase 1B: $(**)'_\lambda$ of 3.12 holds.

By 3.12 we get a contradiction to $\dot{I}(\lambda^+, K) < 2^{\lambda^+}$, i.e. clause $(g)(\beta)$.

So we can assume that

Case 2: "$(2^\lambda > \lambda^+) +$ (WDmId(λ^+) is not λ^{++}-saturated" fails.

If the second clause fails (i.e. WdmId(λ^+) is λ^{++}-saturated), as we are in case $(C)_\lambda$ (see $(*)_1$ in the beginning of the proof), so by clause (μ) of 4.2 we have $2^{\lambda^+} = \lambda^{++}$ hence by (a) of 4.10 we have $2^\lambda = \lambda^+$.

If the first clause fails, i.e. $\neg(2^\lambda > \lambda^+)$ then $2^\lambda = \lambda^+$. So in both cases $2^\lambda = \lambda^+$. However, once we know $2^\lambda = \lambda^+$, from $(\mathrm{amg})_\lambda+$ $\mathrm{LS}(\mathfrak{K}) \leq \lambda$, as in clause $(b)(\beta)$ we are assuming $|\tau_{\mathfrak{K}}| \leq \mathrm{LS}(\mathfrak{K}) \leq \lambda$, clearly $M \in K_\lambda \Rightarrow |\mathscr{S}(M)| \le \lambda^+$. Now by Theorem I.2.8 with λ^+, λ^+ here standing for κ, λ there, there is $M \in K_{\lambda^+}$ which is model-homogeneous for $\mathfrak{K}_{\ge\lambda}$, so by II.1.14 this model is λ^+-saturated above λ and we apply the claim 4.12 below. We are allowed to do it as the assumption (h) of 4.12 was just proved above. $\square_{4.10}$

Remark. 1) Before Case 1, if it suffices for us to conclude ($\dot I(\lambda^{++}, K) > 2^{\lambda^+}$ so the case left is $\mathrm{cf}((2^{\lambda^+})^{\lambda^{++}}, \subseteq) = 2^{\lambda^+}$, does this help?
2) In Subcase (1B), cannot we use "weak coding"? This means re-thinking §3. We can try to apply it to $(K^{\mathrm{slm}}_{\lambda^+})^{\mathrm{up}}$.

4.12 Claim. *If the assumptions (a)-(e) + (g)(α) of 4.10 and clause (h) below* __then__ $\neg(\mathrm{mdn})^2_{\lambda^{++}}$, *i.e.* $\dot I(\mathfrak{K}, \lambda^{++}) \ge \mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$ *where*

- (*h*) *there is* $M \in K_{\lambda^+}$ *saturated above* λ *(so it is universal).*

Proof. We apply Theorem VII.4.10, why is this justified? The assumptions (a)-(e) (i.e. there and here of 4.10) not only are the same but the clauses names are equal except that (b)(β) there is not mentioned here; the assumption (f) there holds by (g)(α) of 4.10, and the assumption (g) there is the assumption (h) here. But the conclusion of VII.4.10 is the desired conclusion here so we are done. $\square_{4.12}$

Essentially another presentation of 4.10 is

4.13 Claim. *Above* $(M^*, N^*, a) \in K^{3,\mathrm{na}}_\lambda$ *the minimal triples are dense* __*when*__*:*

- (a) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$
- (b) (α) $\mathfrak{K}$ *is an a.e.c.,* $\mathrm{LS}(\mathfrak{K}) \le \lambda$,
  - (β) $(\mathrm{cat})_\lambda$,
  - (γ) $(\mathrm{amg})_\lambda$,
  - (δ) $(\mathrm{slm})_{\lambda^+}$, *(hence* $(\mathrm{unv})_{\lambda^+}$*)*
  - (ε) $|\tau_K| \le \lambda$ __*or just*__ *if* $2^\lambda = \lambda^+$ *then* $\mathfrak{K}$ *has a saturated model in* λ^+
- (c) $1 \le \dot I(\lambda^{++}, K) < \mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$
- (d) $(\mathrm{mdn})^1_{\lambda^+}$,

Remark. Weak extensions for $K_\lambda^{3,\mathrm{na}}$ holds by 1.9 because we assume $(\mathrm{cat})_\lambda$ and have $(\mathrm{nmx})_\lambda$ which follows from the assumptions $(\mathrm{slm})_{\lambda^+}$, see 1.3(1).

Proof. Assume toward contradiction that above $(M^*, N^*, a) \in K_\lambda^{3,\mathrm{na}}$ there is no minimal type. If $2^\lambda = \lambda^+$, then as we have $\mathrm{LS}(\mathfrak{K}) \leq \lambda, |\tau_K| \leq \lambda$ there is a $M \in K_{\lambda^+}$ saturated above λ, in fact, the $M \in K_{\lambda^+}^{\mathrm{slm}}$ is by 1.4(6), hence we finish by 4.12 above.

So we can assume $2^\lambda > \lambda^+$, hence

(∗) there are $\kappa \leq \lambda$ and tree $\mathscr{T}$ with $\leq \lambda$ nodes and κ levels with $|\lim_\kappa(\mathscr{T})| > \lambda^+$.

[Why? Considerably more than this holds by 4.2, just check each of the cases. Alternatively, use [Sh 430, 6.3] or directly $\kappa = \mathrm{Min}\{\theta : 2^\theta > \lambda^+\}$, so if $2^{<\kappa} \leq \lambda$ then $({}^{\kappa>}2, \triangleleft)$ is okay, otherwise by our present assumption $2^\lambda > \lambda^+$ hence $\kappa \leq \lambda$ so ${}^{\kappa>}2 = \bigcup\limits_{i<\lambda^+} \mathscr{T}_i, |\mathscr{T}_i| \leq \lambda, \mathscr{T}_i$ increasing with i so for some $i, |\{\eta \in {}^\kappa 2 : \bigwedge\limits_{\alpha<\kappa} \eta \restriction \alpha \in \mathscr{T}_i\}| > \lambda^+$].

Hence by 2.3(2) + 2.5(4) for some $M \in K_\lambda, |\mathscr{S}^{\mathrm{nmr}}(M)| > \lambda^+$. If $\mathrm{WDmId}(\lambda^+)$ is not λ^{++}-saturated, we try to apply Claim 3.13; in it assumption (b) holds by the previous sentence by 1.17(7), and assumptions (c) + (d) + (e) holds by $(b)(\beta), (b)(\delta), (a)$ from the assumptions of the present claim but the conclusion of 3.13 fails by the assumption (c) of the present Claim 4.13, so one of the following occurs.

<u>Case 1</u>: Clause (a) from 3.13 fails,

That is $(**)'_\lambda$ of 3.10 holds hence by 3.12 we have $\dot{I}(\lambda^+, K) = 2^{\lambda^+}$ which contradicts clause (d) of the assumption of the present claim.

<u>Case 2</u>: $\mathrm{WDmId}(\lambda^+)$ is λ^{++}-saturated.

Hence the assumption of clause (μ) of 4.2 occurs, but not its conclusions as we are assuming $2^\lambda > \lambda^+$ so clause (μ) there fail, hence Cases $(B)_\lambda, (C)_\lambda$ of 4.2 do not occur and hence Case $(\mathrm{A})_\lambda$ for some μ and $\chi^* = 2^{\lambda^+}$ occurs. Now all the assumptions of 4.5 holds. So by $(*)_3$ of 4.5(3) we get a contradiction to $(\text{unv})_{\lambda^+}$ in λ^+, see clause $(b)(\delta)$ of our assumption. $\square_{4.13}$

We have used $|\tau(\mathfrak{K})| \leq$ LS$(\mathfrak{K})$ in the end of the proof of 4.10 and in the beginning of 4.13; actually we use just $2^\lambda = \lambda^+ +$ $(\text{amg})_\lambda \Rightarrow$ there is a saturated $M \in K_{\lambda^+}$; this demand is usually irrelevant and can be eliminated there.

4.14 Exercise: Assume

- (a) $\mathfrak{K}$ is an a.e.c., LS$(\mathfrak{K}) \leq \lambda$ but $|\tau_K| > \lambda$ and $\mathfrak{K} = \mathfrak{K}_{\geq\lambda}$ for notational simplicity
- (b) $2^\lambda = \lambda^+$ but $\mathfrak{K}_\lambda$ has no saturated model in λ^+
- (c) $\mathfrak{K}$ satisfies $(\text{amg})_\lambda +$ $(\text{jep})_\lambda$.

1) For some $M \in K_\lambda$ we have $\mu_M > \lambda^+$ where $\mu_M = |\{(N,c)_{c\in M}/\cong : N$ satisfies $M \leq_{\mathfrak{K}} N \in K_\lambda\}|$.
2) If $M_1 \leq_{\mathfrak{K}_\lambda} M_2$ and $N \in K_\lambda$ then there is N^+ such that: $N \leq_{\mathfrak{K}_\lambda} N^+ \in K_{\lambda^+}$ and N^+ is specifically (M_0, M_1)-homogeneous which means that every $\leq_{\mathfrak{K}}$-embedding of M_0 into N^+ can be extended to a $\leq_{\mathfrak{K}}$-embedding of M_1 into N^+.
3) If $N \in K_{\lambda^+}$ then N has a $\leq_{\mathfrak{K}}$-extension $N^+ \in K_{\lambda^+}$ such that: for some $<_{\mathfrak{K}_\lambda}$-representation $\langle M_\alpha : \alpha < \lambda^+\rangle$ of N^+, the model N^+ is specifically (M_α, M_β)-homogeneous for every $\alpha < \beta < \lambda$.
3A) If $N \in K_{\lambda^+}$ and $M_1 \in K_\lambda$ then the set $\{M_2 : M_1 \leq_{\mathfrak{K}} M_2 \in K_\lambda$ and the model N is specifically (M_1, M_2)-homogeneous$\}$ modulo isomorphism over M_1 has cardinality $\leq \lambda^+$.
4) For $M \in K_\lambda$ such that $\mu_M > \lambda^+$ we have $\dot{I}(\lambda^+, K) \geq \mu_M$ and moreover $\dot{I}(\lambda^+, K) \geq$ cf$([\mu]^{\lambda^{++}}, \subseteq)$, in fact, this holds for any (λ, λ^+)-dense in $K' \subseteq \mathfrak{K}_{\lambda^+}$.
5) $\mathfrak{K}$ has no superlimit model in λ^+ (and even weaker relatives).

§5 Inevitable types and stability in λ

5.1 Hypothesis. 1) $\mathfrak{K}$ is an a.e.c., $\lambda \geq$ LS$(\mathfrak{K})$ and $K_\lambda \neq \emptyset$.
2) $(\text{cat})_\lambda +$ $(\text{amg})_\lambda + K_{\lambda^{+2}} \neq \emptyset$.

Similarly to Definition 2.9:

5.2 Definition. 1) We call $p \in \mathscr{S}(N)$ inevitable <u>when</u>:
$N \leq_{\mathfrak{K}} M \ \& \ N \neq M \Rightarrow$ some $c \in M$ realizes p.
2) We call $(M, N, a) \in K^{3,\mathrm{na}}_\lambda$ inevitable <u>when</u> the type $\mathbf{tp}(a, M, N)$ is inevitable.
3) Let $(\mathrm{iev})_\lambda = (\mathrm{iev})^4_\lambda$ means there is an inevitable minimal member of $K^{3,\mathrm{na}}_\lambda$.
4) Let $(\mathrm{iev})^3_\lambda$ means that there is an inevitable member of $K^{3,\mathrm{na}}_\lambda$.
5) Let $(\mathrm{iev})^1_\lambda$ means that for every $M \in K_\lambda$ the set $\Gamma^{\min}_M = \{p \in S(M) : p \text{ minimal}\}$ is inevitable (see Definition 2.9).
6) Let $(\mathrm{iev})^2_\lambda$ means that for every $M \in K_\lambda$ some $\Gamma \subseteq \mathscr{S}^{\min}(M)$ is inevitable and is of cardinality $\leq \lambda$.

5.3 Claim. *Assume* $(\mathrm{cat})_{\lambda^+}$ *and* $2^\lambda < 2^{\lambda^+}$.
1) If there is a minimal triple in $K^{3,\mathrm{na}}_\lambda$, <u>*then*</u> *there is an inevitable minimal* $p \in \mathscr{S}_{\mathfrak{K}_\lambda}(M)$.
2) Moreover, if $p_0 \in \mathscr{S}(N_0)$ *is minimal,* $N_0 \in K_\lambda$ <u>*then*</u> *we can find* $N_1, N_0 \leq_{\mathfrak{K}} N_1 \in K_\lambda$ *such that there is an extension* p_1 *of* p_0 *in* $\mathscr{S}^{\mathrm{na}}(N_1)$, *of course it is unique, and it is inevitable and, of course, minimal.*
3) $(\mathrm{iev})^4_\lambda$ *implies* $(\mathrm{iev})^3_\lambda$ *which implies* $(\mathrm{iev})^2_\lambda$ *which implies* $(\mathrm{iev})^1_\lambda$.

Proof of 5.3. 1) Follows by part (2) and the categoricity of K in λ.
2) First we verify that all the assumptions of 2.19 holds (and Hypothesis 2.1 holds by Hypothesis 5.1(1)); consider the assumptions of 2.19, now $(\mathrm{amg})_\lambda + (\mathrm{cat})_\lambda$ hold by 5.1(2), $(\mathrm{slm})_{\lambda^+}$ holds by $(\mathrm{cat})_{\lambda^+}$ assumed here (i.e. in 5.3) and $\dot{I}(\lambda^+, K) < 2^{\lambda^+}$ holds as $(\mathrm{cat})_{\lambda^+}$ is assumed and $2^\lambda < 2^{\lambda^+}$ holds as it is assumed here. So the conclusion of 2.19(2) holds; i.e. every $M \in K^{\mathrm{slm}}_{\lambda^+} = K_{\lambda^+}$ is saturated for minimal types, (and even λ^+-minimal types, see Definition 2.12(1), recalling Definition 2.12 and Observation 2.13(3),(4).

Let $(M_0, M_1, a) \in K^{3,\mathrm{na}}_\lambda$ be minimal and $p_0 = \mathbf{tp}(a, M_0, M_1)$. We try to choose by induction on i a model N_i such that: $N_0 = M_0, N_i \in K_\lambda$ is $\leq_{\mathfrak{K}}$-increasing continuous and N_i omits $p_0, N_i \neq N_{i+1}$. If we succeed, $\cup\{N_i : i < \lambda^+\}$ is a member of K_{λ^+} which is non-saturated for minimal types, contradicting 2.19(2). As for $i = 0$, i limit we can define, necessarily for some i we have N_i but not N_{i+1}. Now p_0 has at least one extension in $\mathscr{S}(N_i)$ by $(\text{amg})_\lambda$, has at most one non-algebraic extension in $\mathscr{S}(N_i)$ which we call p_i and p_0 has no algebraic extension in $\mathscr{S}(N_i)$. [Why? As N_i omits p_0 and by amalgamation it has an extension in $\mathscr{S}(N_i)$]. So p_i exists (i.e. is well defined and) is the unique extension of p_0 in $\mathscr{S}(N_i)$ [by 1.17(1)], and so

(∗) if $N_i \leq_{\mathfrak{K}} N' \in K_\lambda$ and $N' \neq N$, <u>then</u> p_i is realized in N'.

By L.S. we can omit "$N' \in K_\lambda$", so (N_i, p_i) are as required.
3) Read the definition. $\square_{5.3}$

5.4 Fact. 1) Being inevitable is preserved by isomorphisms.
2) Inevitable types have few ($\leq \lambda$) conjugates (i.e. for inevitable $p \in \mathscr{S}(M_0), M_0 \in K_\lambda, M_1 \in K_\lambda$ we have $|\mathscr{S}_p(M_1)| \leq \lambda$), moreover $|\{p \in \mathscr{S}(N) : p \text{ inevitable}\}| \leq \lambda$ for $N \in K_\lambda$.

Proof. 1) Being inevitable is preserved by isomorphisms is proved by chasing arrows.
2) Trivial as if $N_0 <_{\mathfrak{K}_\lambda} N_1$ then every inevitable $p \in \mathscr{S}(N_0)$ is realized in N_1 hence their number is $\leq \|N_1\| = \lambda$. $\square_{5.4}$

The following construction shall play a central role here, it is assumed there is a minimal inevitable p, so essentially it relies on 5.3. Note in particular that we shall use the construction to show stability in λ, in 5.8.

5.5 Claim. *Assume $M \in K_\lambda$ and $p \in \mathscr{S}_{\mathfrak{K}_\lambda}(M)$ is minimal and inevitable.*
1) For any limit $\alpha < \lambda^+$, we can find $\bar{N} = \langle N_i : i \leq \alpha \rangle$ and $\bar{p} = \langle p_i : i \leq \alpha \rangle$ satisfying $\boxtimes^\alpha_{\bar{N},\bar{p}}$ below.
2) Moreover, if $\bar{N} = \langle N^0_\alpha : \alpha < \lambda^+ \rangle$ is a $<_{\mathfrak{K}_\lambda}$-increasing continuous sequence and $N^0 = \cup\{N^0_\alpha : \alpha < \lambda^+\}$ is not $<_{\mathfrak{K}_{\lambda^+}}$-maximal, <u>then</u> for some club E of λ, for any $\alpha \in \text{acc}(E)$, letting $\bar{N}' = \bar{N} \restriction ((\alpha+1) \cap E)$ for some sequence $\bar{p}$ we have $\boxtimes_{\bar{N}',\bar{p}}$ and $p_0 \in \mathscr{S}_p(N_{\min(E)})$
<u>*where:*</u>

$\boxtimes_{\bar{N},\bar{p}}$ *means that* $\boxtimes^{\alpha}_{\bar{N},\bar{p}}$ *for some limit* $\alpha < \lambda^+$ *where*

$\boxtimes^{\alpha}_{\bar{N},\bar{p}}$ $\bar{N} = \langle N_i : i \leq \alpha \rangle, \bar{p} = \langle p_i : i \leq \alpha \rangle$ *satisfy*

(i) $N_i \in K_\lambda$,

(ii) N_i *is* $\leq_{\mathfrak{K}}$*-increasing continuous*

(iii) $p_i \in \mathscr{S}(N_i)$ *is minimal*

(iv) p_i *is increasing continuous with* i *(see 1.17(3))*

(v) $p_0 \in \mathscr{S}_p(M_0)$ *so is inevitable*

(vi) $p_\alpha \in \mathscr{S}_p(M_\alpha)$ *so is inevitable*

(vii) $N_i \neq N_{i+1}$

(viii) *if* $i < \alpha$ *then some* $c \in N_{i+1} \backslash N_i$ *realizes* p_0 *(hence* p_i*).*

5.6 Remark. 1) Why not try to build a non-saturated model in λ^+ in order to prove 5.5? Works, too. Note that $M \in K_\lambda \Rightarrow 1 \leq |\mathscr{S}_p(M)| \leq \lambda$, the first inequality by $(\text{cat})_\lambda$, the second inequality by 5.4(2).
2) For 5.5(1) we can imitate the proof of II.4.2 so get any $\alpha < \lambda^+$ divisible by λ hence any α.
3) If $p^* \in \mathscr{S}(N^*)$ is minimal and $N^* \in K_\lambda$ then in 5.1, forgetting p, we can demand $p_i \in \mathscr{S}_{\geq p^*}(N_i)$; also and if $\mathfrak{K}$ is stable in λ then in 5.5(1) in addition we can demand that N_α is $(\lambda, \text{cf}(\alpha))$-brimmed over N_0, but see below.

Proof. For part (1) choose $N^0 <_{\mathfrak{K}} N^1$ in K_{λ^+} (so $N^0 \neq N^1$), such a pair exists as $K_{\lambda^{+2}} \neq \emptyset$ (and $\text{LS}(\mathfrak{K}) \leq \lambda$), see 5.1(2). Let $N^\ell = \bigcup\limits_{i<\lambda^+} N^\ell_i$ with $N^\ell_i \in K_\lambda$ being $\leq_{\mathfrak{K}}$-increasing continuous in i for $\ell = 0,1$. For part (2), $N^0, \langle N^0_i : i < \lambda^+ \rangle$ are given and we can choose $N^1, \langle N^1_i : i < \lambda^+ \rangle$ because it is assumed that N^0 is not $<_{\mathfrak{K}_{\lambda^+}}$-maximal.

Now the set

$$E_0 = \Big\{\delta < \lambda^+ : \begin{array}{ll} (a) & N^1_\delta \neq N^0_\delta \text{ and} \\ (b) & N^1_\delta \cap N^0 = N^0_\delta \text{ and} \\ (c) & N^0_\delta \neq N^0_i \text{ for every } i < \delta \text{ and} \\ (d) & \text{if } i < \delta \text{ and } p \in \mathscr{S}(N^0_i) \text{ is minimal} \\ & \text{realized in } N^1_i \text{ and in } N^0 \\ & \text{and } i < \delta \text{ then } p \text{ is realized in } N^0_\delta \Big\} \end{array}$$

is a club of λ^+.

For each $c \in N^1 \backslash N^0$, obviously the set

$$X_c := \{i < \lambda^+ : c \in N^1_i \text{ and } (N^0_i, N^1_i, c) \text{ is minimal}\}$$

is empty or is an end segment of λ^+ hence

$$E_1 = \Big\{\delta \in E_0 : \begin{array}{l} \text{(i) } \delta \text{ is a limit ordinal and } N^1_\delta \not\subseteq N^0_\delta \\ \text{(ii) if } i < \delta \text{ and } q \in \mathscr{S}^{\text{na}}(N^0_\delta), \\ \quad q \restriction N^0_i \text{ is minimal inevitable} \\ \quad \text{and realized in } N^0 \backslash N^0_\delta \text{ then for any} \\ \quad j \in (i, \delta), q \restriction N^0_j \text{ is realized in } N^0_\delta \backslash N^0_i \\ \text{(iii) if } c \in N^1_\delta \backslash N^0 \text{ (hence } \exists i < \delta, c \in N^1_i) \text{ and } X_c \\ \quad \text{is non-empty } \underline{\text{then}}\ \delta \in X_c \text{ and } \min(X_c) < \delta \Big\} \end{array}$$

is a club of λ^+ (see 5.4; concerning (ii) note that if $q' \in \mathscr{S}^{\text{na}}(N^0_\delta)$ is an extension of q then q' is minimal but we do not know that it is inevitable; however recalling 5.4(2), the number of such q for a given i is $\leq \lambda$).

Now for $\delta \in E_1$, we have $N^0_\delta <_{\mathfrak{K}} N^1_\delta$, so as $\mathfrak{K}$ is categorical in λ, by the assumption on p there is $c_\delta \in N^1_\delta \backslash N^0_\delta$ such that:

$(*)$ (a) $(N^0_\delta, N^1_\delta, c_\delta)$ is minimal

(b) $\mathbf{tp}(c_\delta, N^0_\delta, N^1_\delta)$ is inevitable, in fact $\in \mathscr{S}_p(N^0_\delta)$.

As δ is a limit ordinal, for some $i < \delta, c_\delta \in N^1_i$, also $\delta \in X_{c_\delta}$ hence by clause (iii) in the definition of E_1, there is j such that: $i < j < \delta \;\;\&\;\; j \in X_{c_\delta}$ hence $(N^0_j, N^1_j, c_\delta) \in K^{3,\mathrm{na}}_\lambda$ is minimal; choose such j_δ, c_δ. Let us concentrate on part (1), part (2) is similar, let $\kappa = \mathrm{cf}(\kappa) := \mathrm{cf}(\alpha) \leq \lambda$ where α is from the claim, so for some j^*, c^* the set

$$S = \{\delta \in E_1 : \mathrm{cf}(\delta) = \kappa, j_\delta = j^*, c_\delta = c^*\}$$

is stationary in λ^+. Let $E_2 = \{\delta \in E_1 : \delta = \sup(S \cap E_1)\}$, clearly a club of λ^+.

Choose a closed set $e \subseteq E_2$ of order type $\alpha + 1$ with first element and last element in S; for $\zeta \in [j^*, \lambda^+)$ let $p_\zeta = \mathbf{tp}(c^*, N^0_\zeta, N^1_\zeta)$. (In fact, we could have: all non-accumulation member of e are in S; no real help.)

Now $\langle N^0_\zeta, p_\zeta : \zeta \in e\rangle$ is as required (up to re-indexing), in particular clause (viii) of $\boxtimes^\alpha_{\bar N, \bar f}$ holds as there is a member of S in the open interval determined by any successive members of e recalling $e \subseteq E_2$, and also by clause (ii) in the definition of E_1. $\square_{5.5}$

5.7 Claim. *Assume the pair $(\bar N, \bar p)$, i.e. $\langle N_i, p_i : i \leq \alpha\rangle$ is as in 5.5(1), i.e. $\boxtimes^\alpha_{\bar N, \bar p}$ holds, so in particular p_0, p_α are inevitable and in addition the ordinal $\alpha < \lambda^+$ is divisible by λ. Then any $p \in \mathscr{S}(N_0)$ is realized in N_α, moreover N_α is universal in K_λ over N_0.*

Proof. (Similar to the proofs of II.1.14, II.4.2, II.1.28 II.1.16.)

Let $N_0 <_{\mathfrak{K}} M_0$ be from K_λ and we shall show that M_0 is $\leq_{\mathfrak{K}}$-embeddable into N_α over N_0.

Let $\alpha = \cup\{S_i : i < \alpha\}, \langle S_i : i < \alpha\rangle$ pairwise disjoint, each S_i unbounded in α, λ divides $\mathrm{otp}(S_i)$ and $\mathrm{Min}(S_i) \geq i$. We choose by induction on $i \leq \alpha$ the following:

⊛ $N^1_i, M^1_i, h_i, \langle a_\zeta : \zeta \in S_i\rangle$ (the last one only if $i < \alpha$) such that:
- (a) $N^1_i \leq_{\mathfrak{K}} M^1_i$ are in K_λ
- (b) N^1_i is $\leq_{\mathfrak{K}}$-increasing continuous in i

(c) M^1_i is $\leq_{\mathfrak{K}}$-increasing continuous in i

(d) $(N^1_0, M^1_0) = (N_0, M_0)$

(e) $\langle a_\zeta : \zeta \in S_i \rangle$ is a list of $A_i := \{c \in M^1_i : c \text{ realizes } p_0\}$, possibly with repetitions, note that A_i is non-empty by $\boxtimes_{\bar{N},\bar{p}}(v)$ as $N_0 \neq M_0$

(f) h_i is an isomorphism from N_i onto N^1_i

(g) $j < i \Rightarrow h_j \subseteq h_i$ and $h_0 = \text{id}_{N_0}$

(h) $a_i \in N^1_{i+1}$ (note: $M^1_i \cap N^1_{i+1} \neq N^1_i$ at least when $a_i \notin N^1_i$).

<u>For $i = 0$</u>: See clauses $(d), (g)$ so we choose

$$N^1_0 = N_0,\ M^1_0 = M_0,\ h_0 = \text{id}_{N_0}.$$

<u>For i limit</u>: Let $N^1_i = \bigcup_{j<i} N^1_j$ and $M^1_i = \bigcup_{j<i} M^1_j$ and $h_i = \bigcup_{j<i} h_j$ and lastly choose $\langle a_\zeta : \zeta \in S_i \rangle$ by clause (e).

<u>For $i = j + 1$</u>: Note that a_j is already well defined because if $j \in S_\varepsilon$ then $j \geq \text{Min}(S_\varepsilon) \geq \varepsilon$ and by clause (e) clearly a_j belongs to M^1_j and it realizes p_0.

<u>Case 1</u>: $a_j \in N^1_j$ (so clause (h) is no problem).
Use amalgamation on N_j, N_i, M^1_j and the mappings id_{N_j}, h_i, i.e.

$$\begin{array}{ccc} N_i & \longrightarrow & M^1_i \\ {\scriptstyle \text{id}_{N_j}}\uparrow & & \uparrow{\scriptstyle \text{id}_{N^1_0}} \\ N_j & \xrightarrow{h_i} & N^1_j \end{array}$$

<u>Case 2</u>: $a_j \notin N^1_j$.

Then $\mathbf{tp}(a_j, N^1_j, M^1_j)$ is not algebraic and (see clause (e) of $\circledast$) it extends the minimal type $p_0 \in \mathscr{S}(N_0)$.

Also by clause (*viii*) of 5.5 there is $c \in N_i \backslash N_j$ which realizes p_0. As $p_0 \in \mathscr{S}(N)$ is minimal it follows that

$$h_j\left(\mathbf{tp}(c, N_j, N_i)\right) = \mathbf{tp}(a_j, N_j^1, M_j^1)$$

so (by (amg)$_\lambda$ and definition of types) we can also guarantee $h_i(c) = a_j$, so $a_j \in \text{Rang}(h_i) = N_i^1$ as required. So we have carried the induction.

In the end we have $N_\alpha^1 \leq_{\mathfrak{K}} M_\alpha^1$. If $N_\alpha^1 = M_\alpha^1$, then $h_\alpha^{-1} \restriction M_0 = h_\alpha^{-1} \restriction M_0^1 \subseteq h_\alpha^{-1} \restriction N_\alpha^1$ show that M_0 can be $\leq_{\mathfrak{K}}$-embedded into N_α over N_0 as required. So assume $N_\alpha^1 <_{\mathfrak{K}} M_\alpha^1$. Now $p_\alpha \in \mathscr{S}(N_\alpha)$ is inevitable hence $h_\alpha(p_\alpha) \in \mathscr{S}(N_\alpha^1)$ is inevitable. Hence some $d \in M_\alpha^1 \backslash N_\alpha^1$ realizes $h_\alpha(p_\alpha)$ hence d realizes $h_\alpha(p_\alpha) \restriction N_0^1 = p_0$; also α is a limit ordinal so for some $i < \alpha, d \in M_i^1$ and so $d \in M_i^1 \backslash N_\alpha^1 \subseteq M_i^1 \backslash N_i^1$ realizes p_0 hence for some $\zeta \in S_i$ we have $a_\zeta = d$, hence by clause (h) of ⊛

$$d = a_\zeta \in N_{\zeta+1}^1 \subseteq N_\alpha^1,$$

contradicting the choice of d.
So we are done. $\square_{5.7}$

5.8 Conclusion. [Assume (iev)$_\lambda$ = (iev)$_\lambda^4$ or just the conclusion of 5.5(1), i.e. for some $\bar{N}, p$ we have $\boxtimes_{\bar{N},\bar{p}}$ of 5.5.]

If $N \in K_\lambda$ then:

- (*a*) $|\mathscr{S}(N)| = \lambda$, i.e. (stb)$_\lambda$
- (*b*) there is $N_1, N <_{\mathfrak{K}} N_1 \in K_\lambda$ such that N_1 is universal over N in K_λ
- (*c*) for any regular $\kappa \leq \lambda$ we can demand that N_1 is (λ, κ)-brimmed over N (see Definition II.1.15(1)

Proof. Recalling Definition 5.2(3), if there are $M \in K_\lambda$ and inevitable minimal $p \in \mathscr{S}(M)$, then the assumption of 5.5(1) holds, hence its conclusion so it holds in any case.

Let $(\bar{N}, \bar{p})$ be as in $\boxtimes$ of 5.5(1), for α such that λ divides α where $\alpha + 1 = \ell g(\bar{p})$, but $N \cong N_0$ by 5.1(2) so by renaming without loss of generality $N = N_0$. Now by 5.7, clearly $|\mathscr{S}(N)| = |\mathscr{S}(N_0)| \leq \|N_\alpha\| \leq \lambda$.

So clause (a) holds. But, in fact, amalgamation in λ (which we are assuming in 5.1) and stability in λ (i.e., clause (a) of 5.8) implies clauses (b) and (c) of 5.8 by Claim II.1.16(1)(a) and Claim II.1.16(1)(b), respectively; (for clause (b) we can use again 5.7). So we are done. $\square_{5.8}$

5.9 Conclusion. Assume $(\mathrm{iev})_\lambda$.

Every $N \in K_{\lambda^+}$ is saturated above λ (i.e. over models in K_λ!) hence $(\mathrm{cat})_{\lambda^+}$ holds.

Proof. Let $N_* \in K_\lambda$ and let $q \in \mathscr{S}(N_*)$ be minimal inevitable.

First assume $N^0 \in K_{\lambda^+}$ is not $<_{\mathfrak{K}_{\lambda^+}}$-maximal, let $\langle N^0_i : i < \lambda^+ \rangle$ be $<_{\mathfrak{K}_\lambda}$-increasing with union N^0. So for every $\alpha < \lambda^+$ by 5.5(2) applied to $\langle N^0_{\alpha+i} : i < \lambda^+ \rangle, N^0$ we know that for some $\beta \in (\alpha, \lambda^+)$ for some $\bar{N}', \bar{p}', \delta$ and $\alpha' \in [\alpha, \beta)$ we have $\boxtimes^\delta_{\bar{N}',\bar{p}'}$ where $N'_0 = N_{\alpha'}$ and $N'_\delta = N^0_\beta$ hence by 5.7 we know that the model N^0_β is $\leq_{\mathfrak{K}_\lambda}$-universal over $N^0_{\alpha'}$ hence over N^0_α. As this holds for every $\alpha < \lambda^+$ clearly N^0 is saturated above λ as required. Now for any $N \in K_{\lambda^+}$ we can find $N^0 <_{\mathfrak{K}_{\lambda^+}} N^1$ because $K_{\lambda^{++}} \neq \emptyset$ by 5.1(2) hence by the above N^0 is saturated above λ hence (recalling $(\mathrm{amg})_\lambda+$ $(\mathrm{jep})_\lambda$ by 5.1(2)), there is a $\leq_{\mathfrak{K}}$-embedding h of N into N^0; without loss of generality $N \leq_{\mathfrak{K}} N^0$ hence N^1 witness that N is not $\leq_{\mathfrak{K}_{\lambda^+}}$-maximal hence is saturated above λ as required. Hence $(\mathrm{cat})_{\lambda^+}$ follows.

$\square_{5.9}$

5.10 Claim. *[$(\mathrm{iev})_\lambda$]*

Assume $\kappa = \mathrm{cf}(\kappa) \leq \lambda$.

There are $N_0, N_1, a, N^+_0, N^+_1$ such that:

(*i*) $(N_0, N_1, a) \in K^{3,\mathrm{na}}_\lambda$

(*ii*) $(N_0, N_1, a) \leq_{\mathrm{na}} (N^+_0, N^+_1, a) \in K^{3,\mathrm{na}}_\lambda$

(*iii*) N^+_0 *is (λ, κ)-brimmed over N_0, see Definition II.1.15(1)*

(*iv*) $\mathbf{tp}(a, N_0, N_1)$ *is minimal inevitable and*

(*v*) $\mathbf{tp}(a, N^+_0, N^+_1)$ *is minimal inevitable.*

Proof. As in the proof of 5.5 but we start with $N^0 <_{\mathfrak{K}} N^1$ from K_{λ^+} which are saturated, justified by 5.9 so clearly recalling 5.8

$$E_* = \{\delta < \lambda^+ : \delta \text{ is a limit ordinal and for every } i < \delta, N^\ell_\delta \text{ is brimmed over } N^\ell_i \text{ for } \ell = 0,1\}$$

is a club of λ^+. $\square_{5.10}$

5.11 Claim. *[*$(\text{iev})_\lambda$*]*
1) In $\mathfrak{K}_\lambda$ we have disjoint amalgamation.
2) If $M \leq_{\mathfrak{K}} N$ are from K_λ and $p \in \mathscr{S}(M)$ is non-algebraic then *for some N', c we have: $N \leq_{\mathfrak{K}} N' \in K_\lambda$ and $c \in N' \backslash N$ realizes p.*
3) For every $M \leq_{\mathfrak{K}} N \in K_\lambda$ we can find a sequence $\langle M_i : i \leq \alpha\rangle$ which is $\leq_{\mathfrak{K}}$-increasing continuous in $K_\lambda, M_0 = M, N \leq_{\mathfrak{K}} M_*$ and the triple (M_i, M_{i+1}, a_i) is minimal and reduced for some a_i, for every $i < \alpha$.

Remark. 1) Note that Part (1) was proved in 2.7(2) under an additional assumption on p and in 2.23(3) under other assumptions on K.
2) Note that part (2) follows from part (1) but seemingly not part (3).
3) Note that part (2) says that p is not λ-algebraic.

Proof. 1) We first prove a consequence of the disjoint amalgamation, i.e. the extension property (see Definition 1.8(2),(3), it is a consequence of disjoint amalgamation by 1.10(4)). Given $(M, M, a) \in K^{3,\text{na}}_\lambda$, first if there is no minimal triple above it we apply 2.5(1). There, the assumptions are $(\text{amg})_\lambda +$ 2.1 and they hold by 5.1.

Second, if not, then by 1.10(1) without loss of generality (M, N, a) is minimal hence we can apply 2.22, i.e. we repeat the proof in 2.21. Now its assumptions are $(\text{cat})_\lambda$, $(\text{amg})_\lambda$ which holds by 5.1(2) (and 2.1 which holds by 5.1(1)), and $(\text{cat})_{\lambda^+}$ which holds by 5.9 which can be applied as its assumption $(\text{iev})_\lambda$ is assumed here; and $(\text{nmx})_{\lambda^+}$ which 1.3(5) says follows from $(\text{cat})_{\lambda^+}$ & $K_{\lambda^{++}} \neq \emptyset$, the first holds by 5.9, the second by 5.1(2). So the conclusion of 2.22 holds, i.e. letting $p^* = \mathbf{tp}(a, M, N)$ we have $|\mathscr{S}_{\geq p^*}(M)| > \lambda^+$ for $M \in K_\lambda$, but this contradicts 5.8.

Having proved the extension property by 2.7(3) + 2.7(2) we are done.

2) Follows from part (1).

3) As we are assuming $(\text{iev})_\lambda$, there is a minimal $(M, N, a) \in K^{3,\text{na}}_\lambda$, hence by 1.14(1) there is a reduced (M', N', a) which is $\leq_{\text{na}}$-above (M, N, a), and necessarily is minimal, hence we can find $\langle M_i : i < \lambda^+\rangle$ which is $<_{\mathfrak{K}}$-increasing continuous, (M_i, M_{i+1}, a_i) minimal reduced and $M = M_0$. So by 5.9 we know $\bigcup\limits_{i<\lambda^+} M_i \in K_{\lambda^+}$ is saturated, hence we can embed N into $\bigcup\limits_{i<\lambda^+} M_i$ over M so this is a $\leq_{\mathfrak{K}}$- embedding is into some $M_\alpha, \alpha < \lambda^+$. $\square_{5.11}$

<u>5.12 Question</u>: If $M \in K_\lambda, p \in \mathscr{S}(M)$ is minimal, is it reduced? Or at least, if $M_0 \leq_{\mathfrak{K}} M_1$ are in $K_\lambda, p_1 \in \mathscr{S}(M_\ell)$ is non-algebraic, $p_0 = p_1 \restriction M_0, p_0$ is minimal and reduced is also p_1 reduced?
(Probably true and would somewhat simplify our work, but we shall go around it fulfulling our aims (here and in VII§4, II§3). Now 5.5 is an approximation.)

3) Can we prove it if $\lambda < \lambda^{\aleph_0}$ or there are E.M. models in K? We hope to return to this in [Sh:F888].

5.13 Claim. *[$(\text{iev})_\lambda$]*
If $\boxtimes^\delta_{\bar{N},\bar{p}}$ from 5.5 and δ is divisible by λ, <u>then</u> N_δ is $(\lambda, \text{cf}(\delta))$-brimmed over N_0.

Proof. As in the proof of 5.7 above by exhausting.

But we elaborate, letting $\alpha = \delta$ we repeat the proof of 5.7 but in $\circledast$ we add:

(*i*) M^1_{i+1} is $\leq_{\mathfrak{K}_\lambda}$-universal over M^1_i.

This causes no problem by clause (b) of 5.8. In the end we prove there that $M^1_\alpha = N^1_\alpha$ but obviously M^1_α is $(\lambda, \text{cf}(\alpha))$-brimmed over N^0_0

by clause (i) and the definition, hence $N_\alpha = h_\alpha^{-1}(M^1_\alpha)$ is $(\lambda, \mathrm{cf}(\alpha))$-brimmed over $h^{-1}(N^0_0) = N_0$ as required. $\square_{5.13}$

§6 Density of uniqueness and Proving for $\mathfrak{K}$ categorical in λ^{+2}

6.1 Hypothesis. 1) $\mathfrak{K}$ is an a.e.c., $\lambda \geq$ LS($\mathfrak{K}$) and $K_\lambda \neq \emptyset$.
2) $(\mathrm{cat})_\lambda + (\mathrm{amg})_\lambda + (\mathrm{nmx})_\lambda$.
3) $(\mathrm{stb})_\lambda +$ disjoint amalgamation in λ.
4) $(\mathrm{slm})_{\lambda^+}$.
5) For every $M \in K_\lambda$ the set $\mathscr{S}^{\min}_{\mathfrak{K}}(M)$ is inevitable.

6.2 Remark. 1) We can justify the hypothesis above, i.e. show that it follows from the assumptions of 0.2(1) and even of 0.2(2) which has weaker assumptions.

Now 6.1(1) is obvious by the first phrase in 0.2, recalling $(\mathrm{cat})_\lambda \Rightarrow K_\lambda \neq \emptyset$; in 6.1(2), $(\mathrm{cat})_\lambda$ is an assumption of 0.2(2), $(\mathrm{amg})_\lambda$ holds by 1.4(1) and $(\mathrm{nmx})_\lambda$ holds by 1.3(5) because we have $(\mathrm{cat})_\lambda + (\mathrm{ext})_{\lambda^+}$ where $(\mathrm{ext})_{\lambda^+}$ holds by $(\mathrm{cat})_{\lambda^+}$. Concerning 6.1(4), clearly $(\mathrm{slm})_{\lambda^+}$ follows by $(\mathrm{cat})_{\lambda^+}$, so we are left with 6.1(3),(5). They are proved in §5, $(\mathrm{stb})_\lambda$ in 5.8(a) and disjoint amalgamation in 5.11(1), both assuming $(\mathrm{iev})_\lambda$ which implies $\mathscr{S}^{\min}_{\mathfrak{K}_\lambda}(M)$ is inevitable, but $(\mathrm{iev})_\lambda$ is proved in 5.3 assuming $(\mathrm{cat})_{\lambda^+}$ and $2^\lambda < 2^{\lambda^+}$ both of which are assumed in 0.2(1). Still we have to justify quoting §5 by proving Hypothesis 5.1, but $(\mathrm{ext})_{\lambda^{++}}$ is assumed in 0.2(2) and all its other parts were proved above.
2) We use 2.23, 2.3, 4.13.

6.3 Definition. Let $K^{3,\mathrm{uq}}_\lambda$ be the class of triples $(M, N, a) \in K^{3,\mathrm{na}}_\lambda$ such that:

- $(*)$ if $(M, N, a) \leq_{\mathrm{na}} (M', N_\ell, a)$ for $\ell = 1, 2$ <u>then</u> we can find (N', f_1, f_2) such that:
  - (a) $N' \in K_\lambda$
  - (b) f_ℓ is a $\leq_{\mathfrak{K}}$-embedding of N_ℓ into N' for $\ell = 1, 2$
  - (c) $f_1 \restriction (M' \cup N) = f_2 \restriction (M' \cup N)$.

6.4 Claim. *1) In 6.3 we can add $f_1 = \mathrm{id}_{N_1}$.*
2) For every $(M,N,a) \in K^{3,\mathrm{na}}_\lambda$ there is a minimal reduced $(M',N',a) \in K^{3,\mathrm{na}}_\lambda$ which is $\le_{\mathrm{na}}$-above it.
3) If $(M,N,a) \in K^{3,\mathrm{uq}}_\lambda$ then (M,N,a) is minimal and reduced.
4) $(M,N,a) \in K^{3,\mathrm{uq}}_\lambda$ is equiavlent to: $(M,N,a) \in K^{3,\mathrm{na}}$ is reduced and for some M' which is $\le_{\mathfrak{K}_\lambda}$-universal over M and $M' \cap N = M$, the statement $(*)$ of 6.3 holds (i.e. for this M' any N_1,N_2).
4A) In part (4), if M' is brimmed over M then in $(*)$ of 6.3 we can add $(f_\ell(M'),N',f_\ell(a))$ is reduced for $\ell = 1,2$.
5) If $(M_0,M_2,a) \in K^{3,\mathrm{uq}}_\lambda$ hence is minimal and $M_0 \le_{\mathfrak{K}_\lambda} M_1$ then $\mathrm{UQ}^d_\lambda(M_0,M_1,M_2)$.

Proof. Easy, for part (2) to get minimality use $(\mathrm{stb})_\lambda$, i.e. 2.3(4), justified by 6.1(3); for parts (3) and (5) recall $\mathfrak{K}$ has disjoint amalgamation in λ, justified by 6.1(3).

As for aprt (4A), first find (M',N_ℓ,a) such that $(M,N,a) \le_{\mathrm{na}} (M',N'_\ell,a)$ for $\ell = 0,1$ as in $(*)$ of 6.3. Now we can choose $(M_\alpha,N^0_\alpha,N^1_\alpha)$ by induction on $\alpha < \lambda^+$ such that $(M_0,N^\ell_0,a) = (M',N_\ell)$ and $\langle (M_\beta,N^\ell_\beta,a) : \beta \le \alpha\rangle$ is $\le_{\mathrm{na}}$-increasing continuous and $M_{\beta+1}$ is $\le_{\mathfrak{K}_\lambda}$-universal over N_β if $\beta = 3\gamma, (M_{\beta+1},N^\ell_{\beta+1},a)$ is reduced if $\beta = 3\gamma+1+\ell$ for $\ell = 0,1$. If $\delta < \lambda^+$ is limit then M_δ is as required in (4A) and is $(\lambda,\mathrm{cf}(\delta))$-brimmed over M (so M' is isomorphic over M to some such M_δ). $\square_{6.4}$

6.5 Claim. *1) Assume $(M,N,a) \in K^{3,\mathrm{na}}_\lambda$ is such that $\le_{\mathrm{na}}$-above it there is no member of $K^{3,\mathrm{uq}}_\lambda$ and $M_* \in K_{\lambda^+}$ is saturated above λ; and let $\langle M^*_\alpha : \alpha < \lambda^+\rangle$ be a $<_{\mathfrak{K}_\lambda}$-representation of M_*, so without loss of generality each M^*_α is brimmed and $M^*_{\alpha+1}$ is brimmed over M^*_α. Then we can find $\langle N_\eta : \eta \in {}^{\lambda^+>}2\rangle$ such that (for $\eta \in {}^{\lambda^+>}2$):*

- *(a) $N_\eta \in K_\lambda$*
- *(b) $\langle N_{\eta\restriction\alpha} : \alpha \le \ell g(\eta)\rangle$ is $\le_{\mathfrak{K}}$-increasing continuous*

(c) $M^*_{\ell g(\eta)} \le_{\mathfrak{K}_\lambda} N_\eta$ *and* $M_* \cap N_\eta = M_{\ell g(\eta)}$ *for* $\eta \in {}^{\lambda^+ >}2$

(d) $N_{\eta\hat{}<0>}, N_{\eta\hat{}<1>}$ *are* $\le_{\mathfrak{K}}$*-incompatible,* $\le_{\mathfrak{K}}$*-amalgamations of* $M^*_{\ell g(\eta)+1}, N_\eta$ *over* $M^*_{\ell g(\eta)}$

(e) *for some* $\le_{\mathfrak{K}}$*-embedding* f *of* N *into* $N_{<>}$ *we have* $(f(M), f(N), f(a)) \le_{\rm na} (M^*_0, N_{<>}, f(a))$

(f) $(M^*_{\ell g(\eta)}, N_\eta, f(a))$ *is reduced.*

2) In part (1), if $M_* \le_{\mathfrak{K}} M \in K_\mu$ *and* $\mathrm{cf}([\mu]^{\lambda^+}, \subseteq) < \mu_{\rm wd}(\lambda^+)$ *__then__ for some* $\eta \in {}^{\lambda^+}2$ *the model* $N_\eta := \cup\{N_{\eta \restriction \alpha} : \alpha < \lambda^+\rangle$ *cannot be* $\le_{\mathfrak{K}}$*-embedded into* M *over* M_* *(though, of course,* $M_* \le_{\mathfrak{K}_{\lambda^+}} N_\eta$*).*

Proof. 1) Easy using 6.4(4),(4A).
2) By part (1) and the definition of $\mu_{\rm wd}(\lambda^+)$, see VII.0.3(6) exactly as in the proof of 1.4(1A). $\square_{6.5}$

6.6 Lemma. *If* $\otimes$ *then* $\boxplus$ *where:*

$\otimes$ $K^{3,\rm uq}_\lambda \ne \emptyset$

$\boxplus$ *there are* $N^0 <_{\mathfrak{K}} N^1$ *in* K_{λ^+} *such that:*

- (a) $N^0 \ne N^1$
- (b) *for every* $c \in N^1 \backslash N^0$ *there is* $M = M_c$ *satisfying* $N^0 \le_{\mathfrak{K}} M \le_{\mathfrak{K}} N^1$ *and* $N^0 \ne M$ *and* $c \in N^1 \backslash M$ *so* $M \ne N^1$.

Proof. Let $(M_0, M_2, a) \in K^{3,\rm uq}_\lambda$.

Choose $\langle N^0_i : i < \lambda^+\rangle$, a sequence of members of K_λ which is $\le_{\mathfrak{K}}$-increasing continuous, such that:

$$(N^0_i, N^0_{i+1}) \cong (M_0, M_2).$$

So $N^0_i \ne N^0_{i+1}$ hence $N_0 = \bigcup_{i<\lambda^+} N^0_i \in K_{\lambda^+}$ and without loss of generality $|N_0| = \lambda^+$.

We now choose by induction on $i < \lambda^+$, N^1_i and $M_{i,c}$ for $c \in N^1_i \backslash N^0_i$ such that:

- (a) $N^0_i \leq_{\mathfrak{K}} N^1_i \in K_\lambda$ and $N^0_i \neq N^1_i$
- (b) N^1_i is $\leq_{\mathfrak{K}}$-increasing continuous in i
- (c) $j < i \Rightarrow N^1_j \cap N^0_i = N^0_j$; moreover $N^1_i \cap N_0 = N^0_i$
- (d) $N^0_i \leq_{\mathfrak{K}} M_{i,c} \leq_{\mathfrak{K}} N^1_i$
- (e) $c \notin M_{i,c}$
- (f) $N^0_i \neq M_{i,c}$
- (g) if $j < i$ and $c \in N^1_j \backslash N^0_j$ then $M_{i,c} \cap N^1_j = M_{j,c}$ hence $M_{j,c} \leq_{\mathfrak{K}} M_{i,c}$.

For $i = 0$: Choose N^1_i such that $N^0_i \leq_{\mathfrak{K}} N^1_i, N^1_i \cap N_0 = N^0_i, N^1_i$ brimmed over N^0_i (any cofinality will do). Then by disjoint amalgamation, recalling 6.1(3), it is easy to define the $M_{0,c}$ for every $c \in N^0_i$ (remembering clause (c) and our knowledge on "N^1_i brimmed over N^0_i"), see Claim II.1.16(6).

For i limit: Straightforward, take unions.

For $i = j + 1$: First we disjointly amalgamate N^1_j, N^0_i over N^0_j getting $N'_i \in K_\lambda$ such that, so $N^0_i \leq_{\mathfrak{K}} N'_i, N^1_j \leq_{\mathfrak{K}} N'_i$ and we can demand $|N'_i| \cap |N_0| = |N^0_i|$ (as set of elements).

Let N^1_i be such that:

$(*)$ (a) $N'_i \leq_{\mathfrak{K}} N^1_i \in K_\lambda$

(b) N^1_i is brimmed over N'_i, (any cofinality will do)

(c) $|N^1_i| \cap |N_0| = |N^0_i|$.

Lastly, we shall find the $M_{i,c}$'s, the point is that $(N^0_j, N^0_i, N^1_j) \in \mathrm{UQ}^d_\lambda$ (by 6.4(5)).
First, let $c \in N^1_j \backslash N^0_j$. By the disjoint amalgamation we can find $M'_{i,c} \in K$ such that $N^0_i \leq_{\mathfrak{K}} M'_{i,c}$ and $M'_{j,c} \leq_{\mathfrak{K}} M'_{i,c}$ and without loss of generality $|M'_{i,c}| \cap |N^1_j| = M_{j,c}$. Again by disjoint amalgamation there is $M''_{i,c} \in K_\lambda$ such that $M'_{i,c} \leq_{\mathfrak{K}} M''_{i,c}, N^1_j \leq_{\mathfrak{K}} M''_{i,c}$. But $\mathrm{UQ}^d_\lambda(N^0_j, N^0_i, N^1_j)$ here there is a pair $(N'_{i,c}, f_{i,c})$ such that $N'_i \leq_{\mathfrak{K}}$

$N'_{i,c}$ and $f_{i,c}$ is a $\le_{\mathfrak{K}}$-embedding of $M''_{i,c}$ into $N'_{i,c}$ over $|N^0_i| \cup |N^1_j|$. Also there is a $\le_{\mathfrak{K}}$-embedding of $N'_{i,c}$ into N^1_i over N'_i. Lastly, let $M_{i,c} = f'_{i,c}(f_{i,c}(M'_{i,c}))$, easy to check.

Second, if $c \in (N^1_i \backslash N^0_i)$ but $c \notin N^1_j \backslash N^0_j$, then there is $M^0_{i,c}$ such that $N^0_i \cup \{c\} \subseteq M^0_{i,c} <_{\mathfrak{K}} N'_i$ and N^1_i is brimmed over $M^0_{i,c}$ by $(*)(b)$ and the definition of brimmed over. By disjoint amalgamation there is $M'_{i,c} \le_{\mathfrak{K}_\lambda} M^2_{i,c}$ such that $M^0_{i,c} \le_{\mathfrak{K}} M^2_{i,c}, N^0_i <_{\mathfrak{K}} M^1_{i,c}$ and $c \notin M^1_{i,c}$.

Let f^0_i be a $\le_{\mathfrak{K}}$-embedding of $M^2_{i,c}$ into N^1_i over $M^0_{i,c}$ and let $M_{i,c} = f^0_i(M^1_{i,c})$.

Now let $N_1 := \bigcup\limits_{1<\lambda^+} N^1_i$ and for $c \in N_1 \backslash N_0$ let $M_c = \bigcup\{M_{i,c} : c \in N'_i\}$ they are as required. $\square_{6.6}$

6.7 Remark. 1) The proof of 6.8 below is similar to parts of Stages (c),(d) of the proof of I.3.13. The aim is to contradict under $K^{3,\mathrm{uq}}_\lambda \neq \emptyset$, the existence of maximal triples in $K^{3,\mathrm{na}}_{\lambda^+}$.
2) The assumption "$\mathfrak{K}$ has amalgamation in λ^+" is O.K. in our circumstances, i.e. for proving Theorem 0.2, see 1.4(1).

6.8 Conclusion. Assume $(\mathrm{cat})_{\lambda^+} + (\mathrm{amg})_{\lambda^+}$.
If $K^{3,\mathrm{uq}}_\lambda \neq \emptyset$, i.e. $\otimes$ of 6.6 or just $\boxplus$ of 6.6, <u>then</u> there is no maximal triple in $K^{3,\mathrm{na}}_{\lambda^+}$.

Proof. We can assume $\boxplus$ of 6.6.
Toward contradiction, assume $(N_0, N_2, a) \in K^{3,\mathrm{na}}_{\lambda^+}$ is $<_{\mathrm{na}}$-maximal, and $(N^0, N^1), \langle M_c : c \in N' \backslash N^0 \rangle$ are as in $\boxplus$ of 6.6, i.e. they satisfy clauses (a) + (b) of 6.6. By categoricity in λ^+ without loss of generality $N_0 = N^0$ and let $N_1 = N^1$. Now $\mathfrak{K}$ has amalgamation for λ^+ so there are $N \in K_{\lambda^+}$ and f such that $f : N_2 \to N$ is a $\le_{\mathfrak{K}}$-embedding of N_2 into N over N_0 and $N_1 \le_{\mathfrak{K}} N$. If $f(a) \notin N_1$, <u>then</u> $(N_0, N_2, a) <^{\mathrm{na}}_f (N_1, N, f(a))$ contradicting the maximality of (N_0, N_2, a) under $K^{3,\mathrm{na}}_{\lambda^+}$.

If $f(a) \in N_1$, then $M_{f(a)}$ is well defined (see clause (b) of $\boxplus$ from 6.6) and $(N_0, N_2, a) <^{\mathrm{na}}_f (M_{f(a)}, N, f(a))$ contradicts the choice of (N_0, N_2, a). $\square_{6.8}$

A similar claim is

6.9 Claim. *Assume* $(\text{cat})_{\lambda^+}$ *and* "$\dot I(\lambda^{+2},K) < 2^{\lambda^{+2}}$".

If $\otimes$ *or just* $\boxplus$ *of 6.6 holds, then there is no maximal triple in* $K^{3,\text{na}}_{\lambda^+}$.

Proof. Assume toward contradiction that $(N_0,N_2,a) \in K^{3,\text{na}}_{\lambda^+}$ is $<_{\text{na}}$-maximal, and as K is categorical in λ^+ and $\otimes$ or just $\boxplus$ of 6.6 holds, there are $N_1, \langle M_c : c \in N_0 \backslash N_1\rangle$ such that $N_0 <_{\mathfrak{K}_\lambda} N_1, N_0 <_{\mathfrak{K}_\lambda} M_{1,c} <_{\mathfrak{K}} N_1, c \notin M_c$. We build, for every $S \subseteq \lambda^{+2}$, a sequence $\langle M^S_\alpha : \alpha < \lambda^{+2}\rangle$ of members of K_{λ^+}, which is $\le_{\mathfrak{K}}$-increasing continuous, and $\alpha \in S \Rightarrow (M^S_\alpha, M^S_{\alpha+1}, a^S_\alpha) \cong (N_0,N_2,a)$, and $\alpha \in \lambda^{+2}\backslash S \Rightarrow (M^S_\alpha, M^S_{\alpha+1}) \cong (N_0,N_1)$. Let $M^S := \cup\{M^S_\alpha : \alpha < \lambda^{+2}\} \in K_{\lambda^{+2}}$ and from $M^S/\cong$ we can reconstruct $S/\mathscr{D}_{\lambda^{+2}}$. $\square_{6.9}$

6.10 Remark. So here we use $\dot I(\lambda^{+2},K) < 2^{\lambda^{+2}}$ but no need for (any version of) the weak diamond for λ^{++}.
2) Note that if $2^{\lambda^+} < 2^{\lambda^{++}}$ then the assumption of 6.9 implies the assumption of 6.8 by 1.4(1).

6.11 Claim. $\dot I(\lambda^{+2},K) \ge \mu_{\text{unif}}(\lambda^{++},2^{\lambda^+})$ *when:*

- $(*)_1$ $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$
- $(*)_2$ (α) $\text{WDmId}(\lambda^+)$ *is not a* λ^{++}*-saturated ideal, or*
  - (β) $K_{\lambda^{+3}} = \emptyset$
- $(*)_3$ $K^{3,\text{uq}}_\lambda = \emptyset$, *i.e.* $\otimes$ *of 6.6 fails*
- $(*)_4$ $(\text{amg})_{\lambda^+}$
- $(*)_5$ $\lambda = \aleph_0 \Rightarrow 2^\lambda > \lambda^+$.

Proof. We shall mention in each case if $(*)_2$ or a part of it is assumed. Clearly cases 1,2,3,4 below cover all possibilities.

Case 1: $\text{WDmId}(\lambda^+)$ is not λ^{++}-saturated, i.e., $(*)_2(\alpha)$ holds.

Then by VII.4.20 we get the conclusion but we have to check the assumptions.
For the set theoretic part of the assumption, (A)(a) says $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ holds by $(*)_1$ and (A)(b) which says "WDmId(λ^+) is not λ^{++} saturated" is the present case assumption. As for the model theoretic part, (B)(a) holds by 6.1(1), (B)(b)(α) which says $(\mathrm{cat})_\lambda$, holds by 6.1(2). Concerning (B)(b)(β), it holds as by 6.1(4) we have $(\mathrm{slm})_{\lambda^+}$ and (B)(c) which says $(\mathrm{amg})_\lambda+$ $(\mathrm{stb})_\lambda$, it holds by 6.1(2),(3). Then (B)(d)(α), "the minimal types are dense" holds by 2.3(4) which apply because its assumptions $(\mathrm{amg})_\lambda+$ $(\mathrm{stb})_\lambda$ holds by 6.1(2),(3).

Clause (B)(d)(β), "for $M \in K_\lambda$ the set $\mathscr{S}^{\min}_{\mathfrak{K}_\lambda}(M)$ is inevitable" holds by 6.1(5).

Clause (B)(d)(γ), "the $M \in K^{\mathrm{slm}}_{\lambda^+}$ is saturated above λ" holds by $(\mathrm{amg})_\lambda+$ $(\mathrm{stb})_\lambda$ and 2.8(4).

Lastly, Clause (B)(e) holds by $(*)_3$.

Case 2: $K_{\lambda^{+3}} = \emptyset$ and $2^{\lambda^+} > \lambda^{++}$.

As $K_{\lambda^{+3}} = \emptyset$, by 1.4(4) there is $M_2 \in K_{\lambda^{++}}$ which is $\leq_{\mathfrak{K}}$-maximal hence by 1.4(5) is $(\mathbb{D}_{\mathfrak{K}_\lambda}, \lambda^{++})$-homogeneous above λ hence is saturated (above λ^+ and above λ as $\mathfrak{K}_{\lambda^+}$ and $\mathfrak{K}_\lambda$ has amalgamation), and let $M_1 \leq_{\mathfrak{K}} M_2, M_1 \in K_{\lambda^+}$. There is a saturated $M_1' \in K_\lambda$ by $(\mathrm{amg})_\lambda+$ $(\mathrm{stb})_\lambda$ so M_1 is $\leq_{\mathfrak{K}}$-embeddable into M_1' so without loss of generality $M_1 \leq_{\mathfrak{K}} M_1'$. By the choice of M_2 we can embed M_1' into M_2 so without loss of generality $M_1 = M_1'$. Now 6.5(2) is applicable by $(*)_3$. So as easily $\mathrm{cf}([\lambda^{++}]^{\lambda^+}, \subseteq) = \lambda^{++}$ and (as $2^\lambda < 2^{\lambda^+}$, by $(*)_1$ and $\lambda^{++} < 2^{\lambda^+}$ by the case assumption), by VII.0.5(2) we have $\lambda^{++} < \mu_{\mathrm{wd}}(\lambda^+)$ and $\lambda^+ \notin$ WDmId$_\mu(\lambda^+)$ so there is a $\leq_{\mathfrak{K}}$-extension of M_1 in K_{λ^+} not $\leq_{\mathfrak{K}}$-embeddable into M_2, contradiction.

Case 3: $2^{\lambda^+} = \lambda^{++}$ and $\lambda \geq \aleph_1$.

Then as $\lambda < 2^\lambda < 2^{\lambda^+} = \lambda^{++}$, by Cantor Theorem, by assumption $(*)_1$ and by the case assumption respectively necessarily $2^\lambda = \lambda^+$ so by [Sh:922], because in the present case $\lambda \geq \aleph_1$, it follows that $\diamondsuit_{\lambda^+}$ holds hence easily WDmId(λ^+) is not λ^{++}-saturated, a possibility we have dealt with in Case 1.

Case 4: $2^{\lambda^+} = \lambda^{++}$ and $\lambda = \aleph_0$.

As in Case 3 this implies $2^\lambda = \lambda^+$ (and $2^{\lambda^+} = \lambda^{++}$), but this contradicts the assumption $(*)_5$. $\square_{6.11}$

6.12 Discussion. 1) We can get $\dot{I}\dot{E}(\lambda^{+2},\mathfrak{K}) = 2^{\lambda^{+2}}$ when $(2^{\lambda^+})^+ < 2^{\lambda^{+2}}$.
2) The assumption $(*)_5$ of 6.11 looks unreasonable but anyhow this is just a shortcut and it is treated separately when proving 6.13.

We now prove 0.2(1)

6.13 Theorem. *Assume* $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$. *If* $(\text{cat})_{\lambda^+}$ *and* $(\text{cat})_{\lambda^{++}}$, *then* $(\text{ext})_{\lambda^{+3}}$, *i.e.* $\dot{I}(\lambda^{+3},K) > 0$.

6.14 Remark. 1) This has a parallel in I.3.11, I.3.13.
2) To show that this proves Theorem 0.2(1) we have to show that both the assumptions of 6.13 and Hypothesis 6.1 follows from the assumptions of 0.2(1). About 6.1, see Remark 6.2(1), and about the assumptions of 6.13, clearly $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ appears in 0.2(1) and also $(\text{cat})_{\lambda^+}$ and $(\text{cat})_{\lambda^{++}}$.

Proof. By 1.14(9) it is enough to show that for some $M \in K_{\lambda^{++}}$ there is M' satisfying $M <_{\mathfrak{K}} M' \in K_{\lambda^{++}}$.
[Why? As then we can choose by induction on $i < \lambda^{+3}$ models $M_i \in K_{\lambda^{+2}}$,
$\leq_{\mathfrak{K}}$-increasing continuous, $M_i \neq M_{i+1}$; for $i = 0$ use $K_{\lambda^{+2}} \neq \emptyset$, for i limit take union, for $i = j + 1$ use the present assumption and $(\text{cat})_{\lambda^{+2}}$; so $M_{\lambda^{+3}} = \cup\{M_i : i < \lambda^{+3}\} \in K_{\lambda^{+3}}$ as required.]

We try to apply 6.11.
Now from the assumption of 6.11, the first, $(*)_1$ there holds by the assumptions of 6.13. The second, $(*)_2$, without loss of generality the possibility (β) holds, otherwise we already get the desired conclusion. Also $(*)_4$ holds as $(\text{amg})_{\lambda^+}$ follows by 1.4(1).

Now the final conclusion of 6.11, that is $\dot{I}(\lambda^{+2},K) \geq \mu_{\text{unif}}(\lambda^{++},2^{\lambda^+})$ fail as we are assuming $(\text{cat})_{\lambda^{+2}}$, i.e. $\dot{I}(\lambda^{+2},K) = 1$. So necessarily $(*)_3$ or $(*)_5$ of 6.11 fails.

Case A: $\lambda > \aleph_0$.

So also $(*)_5$ of 6.11 holds hence necessarily the assumption $(*)_3$ of 6.11 fails, that is, the statement $K^{3,\text{uq}}_\lambda = \emptyset$ fail, so $\otimes$ of 6.6 holds hence by Claim 6.6 the statement $\boxplus$ there holds so we can find (N^0, N^1) as there so by 6.8 there is in $K^{3,\mathrm{na}}_{\lambda^+}$ no maximal member. This implies (easy, see 1.14(6)) that there are $M^* \leq_{\mathfrak{K}} N^*$ from $K_{\lambda^{+2}}$ such that $M^* \neq N^*$ which as said above (by categoricity in λ^{++}), suffices.

Case B: $\lambda = \aleph_0$.

As in stage (d) of the proof of I.3.13 because of $(\mathrm{ext})_{\lambda^{+2}}$. That is, first let $M_* \in K$ have universe λ^{++} and let E be a club of λ^{++} such that $\min(E) \geq \lambda^+$ and $\delta \in E \Rightarrow M_* \restriction \delta \leq_{\mathfrak{K}} M_*$, let χ be large enough and $\mathfrak{A}_3 = (\mathscr{H}(\chi), \in)$.
Second, choose $\mathfrak{A}_0 \prec \mathfrak{A}_3$ countable such that M_*, E and the definition of $\mathfrak{K}$ belongs to $\mathfrak{A}_0$, e.g. $\leq_{\mathfrak{K}}\restriction \{M \in K : |M| \subseteq \lambda\} \in \mathfrak{A}_0$, see Conclusion I.1.12.
Third, let $\mathfrak{A}_1$ be such that: $\mathfrak{A}_0 \prec \mathfrak{A}_1, \mathfrak{A}_1$ is countable, $\omega^{\mathfrak{A}_1} = \omega^{\mathfrak{A}_0}$ and in $\omega_2^{\mathfrak{A}_1} \backslash \omega_2^{\mathfrak{A}_0}$ under $<^{\mathfrak{A}_1}$ there is no first element, see I.0.4(3).

Hence there are $\langle a_n : n < \omega \rangle$ such that $\mathfrak{A}_1 \models$ "a_n is an ordinal from E and $a_{n+1} < a_n$".

Lastly, let $\mathfrak{A}_2$ be such that $\mathfrak{A}_1 \prec \mathfrak{A}_2$ and $\omega^{\mathfrak{A}_1} = \omega^{\mathfrak{A}_2}$ and $\omega_1^{\mathfrak{A}_2}$ has cardinality $\aleph_1$, i.e. $\{b \in \mathfrak{A}_2 : \mathfrak{A}_2 \models$ "b is a countable ordinal"$\}$ has cardinality $\aleph_1$. If $\mathfrak{A}_2 \models$ "$a \in E$" let $M_a = M_*^{\mathfrak{A}_2} \restriction \{b : b <^{\mathfrak{A}_2} a\}$, so $M_a \in K_{\lambda^+}$ and $a <^{\mathfrak{A}_2} b \Rightarrow M_a <_{\mathfrak{K}} M_b$ by I.1.12. Let $N^1 = \cup\{M_a : a \in E^{\mathfrak{A}_2}\}, N^0 = \cup\{M_a : a \in E^{\mathfrak{A}_2}$ and $n < \omega \Rightarrow \mathfrak{A}_2 \models$ "$a < a_n$"$\}$, this union is on a non-empty set, linearly ordred by $<^{\mathfrak{A}_2}$. So clearly $N^0 \leq_{\mathfrak{K}} N^1$ are from $K_{\lambda^+}, N^0 \leq_{\mathfrak{K}} M_{a_{n+1}} \leq_{\mathfrak{K}} M_{a_n} \leq_{\mathfrak{K}} N^1$ for $n < \omega$ and $\cap\{M_{a_n} : n < \omega\} = N^0$ hence the conclusion $\boxplus$ of 6.6 holds as exemplified by (N^0, N^1), (though we are assuming that its assumption fail, see $(*)_3$).

Hence so we can finish as in as Case A. $\square_{6.13}$

§7 Extensions and conjugacy

7.1 Hypothesis.

(a) $\mathfrak{K}$ is an abstract elementary class with $\mathrm{LS}(\mathfrak{K}) \leq \lambda$

(b) $(\mathrm{amg})_\lambda, \mathfrak{K}$ has amalgamation in λ (see 1.4(1))

(c) $(\mathrm{cat})_\lambda, \mathfrak{K}$ is categorical in λ (can be weakened)

(d) $(\mathrm{stb})_\lambda, \mathfrak{K}$ is stable in λ (see 5.8, clause (a))

(e) $(\mathrm{iev})^3_\lambda$, there is an inevitable $p \in \mathscr{S}(N)$ for some, equivalent every $N \in K_\lambda$

(f) $K^{3,\mathrm{na}}_\lambda$ has the extension property.

7.2 Remark. Those assumptions are justified by the earlier sections, (see 6.2 which does more, recalling $(\mathrm{iev})_\lambda \Rightarrow (\mathrm{iev})^3_\lambda$, see Definition 5.2, 5.3(3) and that the disjoint amalgamation (see 6.1(3)) implies that $K^{3,\mathrm{na}}_\lambda$ has the extension property; <u>or</u> see the proof of 8.1(1)). In §8 we shall use 7.3, 7.5.

We now continue toward better understanding of $\mathfrak{K}$. We first deal with the nice types in $\mathscr{S}(N), N \in K_\lambda$ in particular the parallel to the realize/materialize problem from I§5; see I.4.3(5), the discussion after I.5.12 and Claim I.5.23; this problem here means: if $N_1 \le_{\mathfrak{K}} N_2$ are in $K_\lambda, p_\ell \in \mathscr{S}(N_\ell)$ is minimal, $p_1 \le p_2$, are they conjugate? (i.e., does $p_2 \in \mathscr{S}_{p_1}(N_2)$?), see $(*)$ of 7.5.

7.3 Claim. *If $N \in K_\lambda$ and $p \in \mathscr{S}(N)$ is minimal and reduced or just p is reduced (see Definition 1.11(2)(4)), <u>then</u> p is inevitable.*

Remark. Compare with 5.3 and 2.24.

Proof. Suppose N, p form a counterexample. As p is reduced, see Definition 1.11(2),(4) we can then find N_1 and a such that $N \le_{\mathfrak{K}} N_1 \in K_\lambda, a \in N_1 \backslash N$ and $p = \mathbf{tp}(a, N, N_1)$ and the triple (N, N_1, a) is reduced. As p is not inevitable, there is N_2 such that: $N \le_{\mathfrak{K}} N_2 \in K_\lambda, N \ne N_2$ but no element of N_2 realizes p. By amalgamation in K_λ, without loss of generality there is $N_3 \in K_\lambda$ such that $\ell \in \{1,2\} \Rightarrow N_\ell \le_{\mathfrak{K}} N_3$. By Hypothesis 7.1(e) there is $q \in \mathscr{S}(N)$ which is inevitable so there are $c_\ell \in N_\ell$ with $q = \mathbf{tp}(c_\ell, N, N_\ell)$ for $\ell \in \{1,2\}$. By the equality of types (and amalgamation in $\mathfrak{K}_\lambda$) there is $N^+ \in K$, a $\le_{\mathfrak{K}}$-extension of N_1 and a $\le_{\mathfrak{K}}$-embedding f of N_2

into N^+ over N such that $f(c_2) = c_1$; so without loss of generality $N^+ = N_3$ and f is the identity, hence $c_1 = c_2$. Now $a \notin N_2$ as $p = \mathbf{tp}(a, N, N_1)$ is not realized in N_2. So $(N, N_1, a) \leq_{\rm na} (N_2, N_3, a)$ and $c_2 = c_1 \in N_2 \cap N_1 \backslash N \neq \emptyset$ contradicting "(N, N_1, a) is reduced". $\square_{7.3}$

7.4 Conclusion. For every $M \in K_\lambda$ there is a minimal reduced inevitable $p \in \mathscr{S}^{\rm na}_{\mathfrak{K}_\lambda}(M)$.

Proof. Let $M \in K_\lambda$ and $p \in \mathscr{S}^{\rm na}(M)$ be inevitable, exists by clause (e) of 7.1, or use any $p \in \mathscr{S}^{\rm na}(M)$. So we can find (N, a) such that $(M, N, a) \in K^{3,{\rm na}}_\lambda$ and $p = \mathbf{tp}(a, M, N)$. As $\mathfrak{K}$ is stable in λ (and $K^{3,{\rm na}}_\lambda$ has the extension property by 7.1(d), 7.1(f), respectively) by Claim 2.3(4) there is $(M', N', a) \in K^{3,{\rm na}}_\lambda$ which is $\leq_{\rm na}$-above (M, N, a) and is minimal. By 1.14(1) there is $(M'', N'', a) \in K^{3,{\rm na}}_\lambda$ which is $\leq_{\rm na}$-above (M', N', a) and is reduced, clearly it is still minimal and by 7.3 it is inevitable. As $\mathfrak{K}$ is categorical in λ by 7.1(c), for every $N \in K_\lambda$ there is such $p \in \mathscr{S}^{\rm na}(N)$. $\square_{7.4}$

7.5 Claim. *1) If $\kappa = {\rm cf}(\kappa) \leq \lambda$ and $\bar{N} = \langle N_i : i \leq \omega\kappa \rangle$ is an $\leq_{\mathfrak{K}}$-increasingly continuous sequence, $N_i \in K_\lambda, N_{i+1}$ universal over N_i, and $p \in \mathscr{S}(N_{\omega\kappa})$ is minimal reduced (or just minimal inevitable) <u>then</u> for some $i < \omega\kappa$ we have $p \restriction N_i \in \mathscr{S}(N_i)$ is minimal (so p is the unique a non-algebraic extension of $p \restriction N_i$ in $\mathscr{S}(N_{\omega\kappa})$ (and of course, there is one)).*
2) If $\lambda \geq \kappa = {\rm cf}(\kappa), \bar{N} = \langle N_i : i \leq \kappa \rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous in K_λ and
$p \in \mathscr{S}(N_\kappa)$ is minimal and reduced and the set $Y := \{i < \kappa : N_{i+1}$ is (λ, κ)-brimmed over $N_i\}$ is unbounded in κ <u>then</u> for every large enough $i \in Y$ there is an isomorphism f from N_{i+1} onto N_κ which is the identity on N_i and

(∗) *f maps $p \restriction N_{i+1} \in \mathscr{S}(N_{i+1})$ to $p \in \mathscr{S}(N_\kappa)$.*

Hence as p is minimal reduced, so is $p \restriction N_{i+1}$.
3) Assume $\delta < \lambda^+$ is a limit ordinal, $\langle M_i : i \leq \delta \rangle$ is $\leq_{\mathfrak{K}_\lambda}$-increasing

continuous and $\delta = \sup\{i : M_{i+1}$ *is* $\leq_{\mathfrak{K}_\lambda}$*-universal over* $M_i\}$*. If* $p \in \mathscr{S}(M_\delta)$ *is minimal and inevitable* *then* *for some* $i < \delta$ *the type* $p \restriction M_i$ *is minimal.*

Proof. 1) We can choose for $i < \lambda^+$ a reduced $(N^0_i, N^1_i, a) \in K^{3,\mathrm{na}}_\lambda$ which is $\leq_{\mathrm{na}}$-increasing continuous such that N^ℓ_{i+1} is $\leq_{\mathfrak{K}_\lambda}$-universal over N^ℓ_i for $i < \lambda^+$ and $\ell = 0, 1$; exists by 7.1(f) and 7.1(b),(d), of course. Let $N_\ell = \bigcup\limits_{i<\lambda^+} N^\ell_i$. As in the proof of 5.5 for $c \in N_1 \backslash N_0$ the set I^*_c is empty or is an end segment of λ^+ where

$$I^*_c = \{j < \lambda^+ : c \in N^1_j \text{ and } \mathbf{tp}(c, N^0_j, N^1_j) \text{ is minimal}\}$$

by a basic property of minimal type, i.e. if $M \leq_{\mathfrak{K}_\lambda} N$ and $p \in \mathscr{S}_{\mathfrak{K}}(M)$ is minimal then any extension of p in $\mathscr{S}^{\mathrm{na}}_{\mathfrak{K}}(N)$ is minimal, see 1.13(2).

Also the set

$$\begin{aligned} E = \{\delta < \lambda^+ : \ & \delta \text{ is a limit ordinal; } N^1_\delta \cap N_0 = N^0_\delta; \\ & \underline{\text{if}}\ c \in N^1_\delta \text{ and } I^*_c \neq \emptyset\ \underline{\text{then}}\ I^*_c \cap \delta \\ & \text{is an unbounded subset of } \delta; \\ & \text{and if } Pr \text{ is one of the properties reduced and/or} \\ & \text{inevitable and/or minimal and there is } i \geq \delta \\ & \text{such that } (N^0_i, N^1_i, c) \text{ has } Pr, \\ & \text{then there are arbitrarily large such } i < \delta\} \end{aligned}$$

is a club of λ^+.

Let $\delta \in \operatorname{acc}(\operatorname{acc}(E))$ be such that $\operatorname{cf}(\delta) = \kappa$, let $\langle \alpha_\zeta : \zeta < \omega\kappa \rangle$ be an increasing continuous sequence of ordinals from E with limit δ, now set $\alpha_{\omega\kappa} = \delta$.

So by the uniqueness for "N is (λ, θ)-brimmed over M" there is an isomorphism f from $N_{\omega\kappa}$ onto $N^0_{\alpha_{\omega\kappa}}$ such that for every $\zeta < \omega\kappa$ we have $N^0_{\alpha_{2\zeta}} \leq_{\mathfrak{K}} f(N_{\alpha_{2\zeta}}) \leq_{\mathfrak{K}} N^0_{\alpha_{2\zeta+1}}$ (so if ζ is a limit ordinal, then $N^0_{\alpha_\zeta} = N^0_{\alpha_{2\zeta}} = f(N_\zeta)$), so without loss of generality f is the identity. As $p \in \mathscr{S}(N_{\omega\kappa})$ is inevitable (by assumption or by 7.3) and $N_{\omega\kappa} = N^0_{\alpha_{\omega\kappa}} <_{\mathfrak{K}} N^1_{\alpha_{\omega\kappa}}$, for some $c \in N^1_{\alpha_{\omega\kappa}} \backslash N^0_{\alpha_{\omega\kappa}}$ we have $p = \mathbf{tp}(c, N^0_{\alpha_{\omega\kappa}}, N^1_{\alpha_{\omega\kappa}})$, so for some $\beta < \alpha_{\omega\kappa}$ we have $c \in N^1_\beta$. As p is minimal (by assumption) clearly $\delta \in I^*_c$, but $\delta \in E$ so $\mathrm{Min}(I_c) < \delta$, but I_c is an end segment of λ^+ hence without loss of generality for some $\zeta < \omega\kappa$ we have $\beta = \alpha_\zeta \in I_c$. So for $\xi \in (\zeta, \omega\kappa)$, both $p \in \mathscr{S}(N_{\omega\kappa})$ and $p \restriction N_\xi \in \mathscr{S}(N_\xi)$ are non-algebraic extensions of the minimal $p \restriction N^0_{\alpha_\zeta} \in \mathscr{S}(N^0_{\alpha_\zeta})$ and $N^0_{\alpha_\zeta} \leq_{\mathfrak{K}} N_\xi \leq_{\mathfrak{K}} N_{\omega\kappa}$, all in K_λ, so we have proved part (1). In fact by this argument we get $p \restriction N_i$ is minimal and inevitable for arbitrarily large $i < \delta$.

2) By renaming without loss of generality every $\zeta < \kappa$ is in Y. So again (as in the proof of part (1)) choose N^ℓ_i for $i < \lambda^+, \ell = 1, 2$ as there and choose E, δ and $\langle \alpha_\zeta : \zeta \leq \omega\kappa \rangle$ as there and without loss of generality $i \leq \kappa \Rightarrow N_i = N^0_{\alpha_{\omega i}}$. Choose also c, β as there, as we can increase β without loss of generality for some $\zeta < \kappa$ we have $\beta = \alpha_{\omega\zeta} \in I_c$. Clearly by the uniqueness of (λ, κ)-brimmed models there is an isomorphism f from $N^0_{\omega(\zeta+1)}$ onto $N_\kappa = N^0_{\omega\kappa}$ over $N_\zeta = N^0_{\omega\zeta}$, and $f(p \restriction N_{\zeta+1}) = p$ is proved by the uniqueness of the non-algebraic extension.

3) Letting $\kappa = \mathrm{cf}(\delta)$ by the definition of "$\leq_{\mathfrak{K}}$-universal over" we can find a $\leq_{\mathfrak{K}}$-increasing continuous sequence $\langle N_i : i \leq \omega\kappa \rangle$ as in part (1) such that $N_{\omega\kappa} = M_\delta$ and $(\forall i < \omega\kappa)(\exists j < \delta)(N_i \leq_{\mathfrak{K}} M_j)$ and $(\forall j < \delta)(\exists i < \omega\kappa)[M_j \leq_{\mathfrak{K}} N_i]$. By part (1) for some $j < \omega\kappa$ the type $p \restriction N_j$ is minimal so for some $i < \delta$ we have $N_j \leq_{\mathfrak{K}} M_i \leq_{\mathfrak{K}} M_\delta$ so by 1.17(2) also $p \restriction M_i$ is minimal as required. $\square_{7.5}$

7.6 Claim. *1) If $M_0 \leq_{\mathfrak{K}} M_1$ are in K_λ and the types $p^*_\ell \in \mathscr{S}(M_\ell)$ are minimal and reduced, for $\ell = 0, 1$ and $p^*_0 = p^*_1 \restriction M_0$ then p^*_0, p^*_1 are conjugate; (i.e., there is an isomorphism f from M_0 onto M_1 such that $f(p^*_0) = p^*_1$).*
2) If in addition $M \leq_{\mathfrak{K}} M_0$ and M_0, M_1 are (λ, κ)-brimmed over M, then p_0, p_1 are conjugate over M.

Remark. Recall that p minimal (or reduced) implies that p is not algebraic.

Proof. 1) Let $\langle (N^0_i, N^1_i, a) : i < \lambda^+ \rangle$ and E be as in the proof of 7.5(1) and $\kappa = \mathrm{cf}(\kappa) \le \lambda$. For each $\delta \in S_\kappa := \{\alpha < \lambda^+ : \alpha \in \mathrm{acc}(\mathrm{acc}(E))$ and $\mathrm{cf}(\alpha) = \kappa\}$, and minimal and reduced $p \in \mathscr{S}(N^0_\delta)$, we know that for some $i_p < \delta$ the type $p \restriction N^0_{i_p}$ is minimal and reduced [why? by 7.5(1),(2)] and some $q_p \in \mathscr{S}(N^0_{i_p})$ is conjugate to p say by an isomorphism g_p from N^0_δ onto $N^0_{i_p}$. For $\kappa = \mathrm{cf}(\kappa) \le \lambda$ minimal $q \in \mathscr{S}(N^0_i), i < \lambda^+$ and minimal $r \in \mathscr{S}(N^0_i)$ let

$A^{\kappa,i}_{q,r} = \{\delta < \lambda^+ :$ there is a type p such that $r \le p \in \mathscr{S}(N^0_\delta)$, p non-algebraic (this determines p), p minimal and reduced, $i_p = i, q_p = q$ and clearly $p \restriction N^0_i = r)$ and $\mathrm{cf}(\delta) = \kappa\}$.

Next let

$$\begin{aligned} E_1 = \{\delta < \lambda^+ : \delta = \sup(S \cap \delta) &\text{ and for every } \kappa = \mathrm{cf}(\kappa) \le \lambda, \\ &\text{and minimal } r, q \in \mathscr{S}(N^0_i) \text{ and } i < \delta, \\ &\underline{\text{if}}\ A^{\kappa,i}_{q,r} \text{ is well defined and unbounded in } \lambda^+ \\ &\underline{\text{then}} \text{ it is unbounded in } \delta\}. \end{aligned}$$

So if $\delta_1 \in E_1, \kappa = \mathrm{cf}(\delta_1)$ and $p_1 \in \mathscr{S}(N^0_{\delta_1})$ is minimal reduced, <u>then</u> we can find
$\delta_0 < \delta_1$ satisfying $\delta_0 \in S$ so $\mathrm{cf}(\delta_0) = \kappa$, and $p_0 \in \mathscr{S}(N^0_{\delta_0})$ minimal reduced with
$q_{p_1} = q_{p_0}, i_{p_1} = i_{p_0}, p_0 \restriction N^0_{i_{p_0}} = p_1 \restriction N^0_{i_{p_1}}$ call it r, it is necessarily minimal.

As p_1, p_0 extend $r, N^0_{i_{p_0}} = N^0_{i_{p_1}} \le_{\mathfrak{K}} N^0_{\delta_0} \le_{\mathfrak{K}} N^0_{\delta_1}$, necessarily $p_1 = p_0 \restriction N^0_{\delta_0}$, and also they are both conjugate to $q_{p_0} = q_{p_1}$ hence they are conjugate.
Next we prove

(∗) if $M_0 <_{\mathfrak{K}} M_1$ are in K_λ, M_1 is (λ,κ)-brimmed over M_0, $p'_0 \in \mathscr{S}(M_0)$ is minimal and reduced and $p'_0 \le p'_1 \in \mathscr{S}(M_1), p'_1$ non-algebraic, <u>then</u> p'_0, p'_1 are conjugate.

Above we have a good amount of free choice in choosing $p_1 \in \mathscr{S}(N^0_{\delta_1})$ (it should be minimal and reduced) so we could have chosen p_1 to be conjugate to p'_0, i.e., is in $\mathscr{S}_{p'_0}(N^0_{\delta_1})$; now also the corresponding p_0

is conjugate to p_1 hence p_0 is conjugate to p'_0, hence we can find an isomorphism f_0 from M_0 onto $N^0_{\delta_0}$ satisfying $f_0(p'_0) = p_0$, and extend it to an isomorphism f_1 from M_1 onto $N^0_{\delta_1}$, exists as $M_1, N^0_{\delta_1}$ is (λ,κ)-brimmed over $M_0, N^0_{\delta_0}$ respectively; so necessarily $f_1(p'_1) = p_1$ (as p_1 is the unique non-algebraic extension of p_0 in $\mathscr{S}(M_{\delta_1})$). As p_0, p_1 are conjugate (see a paragraph above), through $(g_{p_1})^{-1} \circ g_{p_0}$, also p'_0, p'_1 are conjugate. So $(*)$ holds.

Now assume just

> $\odot$ $M_0 \le_{\mathfrak{K}} M_1$ are in $K_\lambda, p_0 \in \mathscr{S}(M_0)$ minimal and reduced, $p_1 \in \mathscr{S}(M_1)$ the unique non-algebraic extension of p_0 and it is reduced (and necessarily minimal).

There is M_2 such that $M_1 \le_{\mathfrak{K}} M_2 \in K_\lambda$ and M_2 is (λ,κ)-brimmed over M_1 hence also over M_0 and let p_2 be the unique non-algebraic extension of p_1 in $\mathscr{S}(M_2)$ hence p_2 is also the unique non-algebraic extension of p_0 in $\mathscr{S}(M_2)$ (we could choose M_2 such that p_2 is reduced (see 5.6(2) but not needed).
Using $(*)$ on (M_0, M_2, p_0, p_2) and on (M_1, M_2, p_1, p_2) and get that p_0, p_2 are conjugate and that p_1, p_2 are conjugate respectively, hence p_1, p_2 are conjugate, the required result.
2) Similar proof. $\square_{7.6}$

7.7 Claim. *1) Assume $M_1 \le_{\mathfrak{K}} M_2$ are in K_λ and M_2 is (λ,κ)-brimmed over M_1. If $p_1 \in \mathscr{S}(M_1)$ is minimal and reduced, then p_2, the unique non-algebraic extension of p_1 in $\mathscr{S}(M_2)$, is reduced (and, of course, minimal).*
2) There is no need to assume "p_1 reduced".

Proof. 1) We choose (N^1_i, N^2_i) by induction on $i \le \kappa$ such that:

- (a) $(N^1_i, N^2_i, a) \in K^{3,\mathrm{na}}_\lambda$
- (b) (N^1_i, N^2_i, a) is $\le_{\mathrm{na}}$-increasing continuous
- (c) $N^1_0 = M_1$ and $\mathbf{tp}(a, N^1_i, N^2_i) = p_1$
- (d) if $i = 2j+1$ then N^1_i is $\le_{\mathfrak{K}_\lambda}$-universal over N^1_j
- (e) if $i = 2j+2$ then (N^1_i, N^2_i, a) is reduced.

There is no problem to do this. Now N^2_κ is (λ,κ)-brimmed over $N^1_0 = M_1$ so by the uniqueness of (λ,κ)-brimmed extensions of M_1 without loss of generality $N^2_\kappa = M_2$. Also $\mathbf{tp}(a, N^1_\kappa, N^2_\kappa) \in \mathscr{S}^{\rm na}(N^1_\kappa) = \mathscr{S}^{\rm na}(M_2)$, extends the minimal p_1 and is not algebraic hence is equal to p_2.

Lastly, as $\langle (N^1_{2i+2}, N^2_{2i+2}, a) : i < \kappa\rangle$ is an $\le_{\rm na}$-increasing sequence of reduced members of $K^{3,{\rm na}}_\lambda$ also their union is reduced by 1.14(2)(c), so $(N^1_\kappa, N^2_\kappa, a) \in K^{3,{\rm na}}_\lambda$ is reduced but this means that $\mathbf{tp}(a, N^1_\kappa, N^2_\kappa) = p_2$ is a reduced type so we are done.
2) Easy as we can find $N, M_1 \le_{\mathfrak{K}} N, q \in \mathscr{S}^{\rm na}(N)$ extend p_1 and is reduced; necessarily q is minimal; without loss of generality $N \le_{\mathfrak{K}} M_2$ and M_2 is (λ,κ)-brimmed over N, and apply part (1). $\square_{7.7}$

§8 To almost-good λ-frame

Here we sum up our results, for this we define "$\mathfrak{s}$ is a pre-λ-frame", "$\mathfrak{s}$ is an almost good λ-frame" (see Definition 8.2) and $\mathfrak{s}[\mathfrak{K}_\lambda]$ (in Definition 8.3).

In Theorem 8.1 we prove that we get such frames using the earlier results and from it Theorem 0.2 can be deduced via Chapter VII + Chapter II, of course an overkill (which we could have avoided). Of course, 8.1 is close to II.3.7.

8.1 Theorem. *1) If $\mathfrak{K}_\lambda$ is a λ-a.e.c. <u>then</u> $\mathfrak{s}_{\mathfrak{K}_\lambda}$ is a pre-λ-frame, see Definitions 8.2, 8.3 below.*
2) The pre-λ-frame $\mathfrak{s}_{\mathfrak{K}_\lambda}$ is an almost good λ-frame <u>when</u> $\mathfrak{K}$ and λ satisfies:

$\circledast$ (*a*) *$\mathfrak{K}$ is an a.e.c. with* $\mathrm{LS}(\mathfrak{K}) \le \lambda$
(*b*) $(\mathrm{cat})_\lambda$
(*c*) $(\mathrm{amg})_\lambda$
(*d*) $(\mathrm{ext})_{\lambda^+}$ *hence* $(\mathrm{ext})_\lambda$
(*e*) $(\mathrm{iev})^3_\lambda$, *i.e. there is inevitable $p \in \mathscr{S}_{\mathfrak{K}_\lambda}(M)$.*
(*f*) $(\mathrm{stb})_\lambda$
(*g*) *$K^{3,{\rm na}}_\lambda$ has the extension property.*

3) The assumptions hence the conclusion of (2) hold <u>*when*</u> $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{+2}}$ *and*

- (α) $\mathfrak{K}$ *is an abstract elementary class with* $\mathrm{LS}(\mathfrak{K}) \le \lambda$
- (β) $\mathfrak{K}$ *is categorical in* λ *and in* λ^+
- (γ) $\mathfrak{K}$ *has a model in* λ^{++}
- (δ) $\dot{I}(\lambda^{+2}, K) < \mu_{\mathrm{unif}}(\lambda^{+2}, 2^{\lambda^+})$.

Recall

8.2 Definition. 1) We say $\mathfrak{s}$ is a pre-λ-frame when $\mathfrak{s} = (K_{\mathfrak{s}}, \mathscr{S}^{\mathrm{bs}}, \bigcup)$ satisfies axioms (A),(D)(a),(b),(E)(a),(b), see II.2.1.
2) $\mathfrak{s}$ is an almost good λ-frame is defined as in II.2.1 except that we weaken Ax(E)(c) and strengthen Ax(D)(d) as follows:

<u>Ax(E)(c)$^-$</u>: the weak local character: <u>if</u> $\delta < \lambda^+$ is a limit ordinal $\langle M_i : i \le \delta + 1\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous and $\{i < \delta : N_{i+1}$ is universal of $N_i\}$ is unbounded in δ <u>then</u> for some $a \in M_{\delta+1}$, the type $\mathbf{tp}(a, M_\delta, M_{\delta+1})$ does not fork over M_i for some $i < \delta$ hence belongs to $\mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_\delta)$

$(D)(d)^+$ if $M \in K_{\mathfrak{s}}$ then $\mathscr{S}_{\mathfrak{K}_{\mathfrak{s}}}(M)$ has cardinality $\le \lambda$ (for a good λ-frame this holds by II.4.2).

8.3 Definition. For $\mathfrak{K}_\lambda$ a λ-a.e.c. we define a pre-λ-frame $\mathfrak{s} = \mathfrak{s}_{\mathfrak{K}_\lambda} = \mathfrak{s}[\mathfrak{K}_\lambda]$ by:

- (a) $\mathfrak{K}_{\mathfrak{s}} = \mathfrak{K}_\lambda$
- (b) $\mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M) = \{\mathbf{tp}(a, M, N)$: for some M, N, a we have $(M, N, a) \in K^{3,\mathrm{na}}_\lambda$ and $\mathbf{tp}(a, M, N) \in \mathscr{S}(M')$ is minimal$\}$ (see Definitions 1.6, 1.11) and
- (c) $\bigcup = \bigcup_{\mathfrak{s}}$ be defined by: $\bigcup(M_0, M_1, a, M_3)$ <u>iff</u> $M_0 \le_{\mathfrak{K}} M_1 \le_{\mathfrak{K}} M_3$ are from $K_\lambda, a \in M_3 \backslash M_1$ and the type $\mathbf{tp}(a, M_0, M_3) \in \mathscr{S}(N)$ is minimal.

In other words

$(c)'$ $p \in \mathscr{S}^{\rm na}(M_1)$ does not fork over M_0 iff $M_0 \le_{\mathfrak{K}_\lambda} M_1$ and $p \restriction M_0$ is minimal.

Remark. But first we show that Theorem 8.1 suffice to prove the theorem promised in the introduction.

8.4 Proof of 0.2. 1) Now part (1) follows from part (2) and we have already proved it in 6.13.
2) Now part (3) of 8.1 is proved below. Its assumptions hold by the assumptions of 0.2(2), in fact, they are the same. Hence by 8.1(3) it follows that $\mathfrak{s} = \mathfrak{s}[\mathfrak{K}_\lambda]$, defined in 8.3, is an almost good λ-frame (and $K^{\mathfrak{s}} = \mathfrak{K}$). By VII.4.32(2) recalling we are assuming $\dot I(\lambda^{++}, \mathfrak{K}) < \mu_{\rm nif}(\lambda^{++}, 2^{\lambda^+})$ it follows that $\mathfrak{s}$ is a good λ–frame as its conclusion fails and its assumptions hold except possibly (B)(c) (in more detail, by VII§6, mainly VII.6.17, the almost good λ-frame $\mathfrak{s}$ has the so called almost existence for $K^{3,\rm up}_{\mathfrak{s}}$ and even existence, hence the Hypothesis VII.7.1 holds and by VII§7, mainly VII.7.19, we get the $\mathfrak{s}$ is a good λ-frame; still this is better exaplained in VII.4.32(2)). Now we can apply the results of Chapter II, e.g. the Main Lemma II.9.1, so we are done (of course, this is an overkill, we could as in [Sh 576] work less, but it is easier to quote). $\square_{0.2}$

Proof of 8.1. Part (1) is obvious. Part (3) follows from part (2) and the previous sections and part (2) holds by the previous sections. Let us elaborate.

<u>Proof of part (3) from part (2)</u>: We have to show that $\mathfrak{K}$ satisfies clauses (a)-(g) of $\circledast$ from 8.1.

First, "$\mathfrak{K}$ is an a.e.c. with ${\rm LS}(\mathfrak{K}) \le \lambda$" holds by assumption (α) of 8.1(3), so <u>clause (a)</u> of $\circledast$ holds.

Second, $({\rm ext})_\lambda$, $({\rm ext})_{\lambda^+}$, $({\rm ext})_{\lambda^{++}}$ holds by assumption (γ) and using ${\rm LS}(\mathfrak{K}) \le \lambda$, so <u>clause (d)</u> of $\circledast$ holds.

Third, $(\mathrm{cat})_\lambda$ holds by assumption (β) of 8.1(3) so clause (b) of ⊛ holds and by 1.3(5) and $(\mathrm{ext})_{\lambda^+} + (\mathrm{ext})_{\lambda^{++}}$, see "second" above, it follows that we also have $(\mathrm{slm})_\lambda$.

Fourth, $(\mathrm{mdn})^2_{\lambda^+}$, $(\mathrm{mdn})^2_{\lambda^{++}}$ holds by the assumptions $(\beta), (\gamma) + (\delta)$ respectively.

Fifth, $(\mathrm{amg})_\lambda + (\mathrm{amg})_{\lambda^+}$ holds by applying 1.4(1) to λ and to λ^+ respectively, this is justified as its assumptions are assumed or have already been proved, i.e., $(\mathrm{cat})_\lambda + (\mathrm{cat})_{\lambda^+}$ are assumed in 8.1(3)(β) so O.K., $(\mathrm{ext})_{\lambda^+} + (\mathrm{ext})_{\lambda^{++}}$ hold by "the second" above; $(\mathrm{mdn})^1_{\lambda^+} + (\mathrm{mdn})^1_{\lambda^{++}}$ holds by $(\mathrm{cat})_{\lambda^+}$, i.e. 8.1(3)$(\beta)$ and 8.1(3)(δ) respectively and $2^\lambda < 2^{\lambda^+}, 2^{\lambda^+} < 2^{\lambda^{++}}$ are assumed in 8.1(3). In particular we get clause (c) of ⊛ of 8.1(2).

Sixth, $K^{3,\mathrm{na}}_\lambda$ has the weak extension property by 1.9; which we can apply as its assumptions hold: on $(\mathrm{cat})_\lambda$ see 8.1(3)(β) above, $(\mathrm{nmx})_{\lambda^+}$ follows by "$(\mathrm{cat})_{\lambda^+} + (\mathrm{ext})_{\lambda^{++}}$" assumed in (β) of 8.1(3) and proved in "second" above respectively and similarly $(\mathrm{nmx})_\lambda$.

Seventh, all the assumptions of 2.21 holds (they are $(\mathrm{amg})_\lambda$, $(\mathrm{cat})_\lambda$, $(\mathrm{cat})_{\lambda^+}$, $(\mathrm{nmx})_{\lambda^+}$ they appear in "fifth", 8.1(3)(β), 8.1(3)(β), and inside "six" respectively; also $2^\lambda < 2^{\lambda^+}$; and of course also Hypothesis 2.1), so this applies to 2.23(1) too, so its conclusion "every triple from $K^{3,\mathrm{na}}_\lambda$ has the extension property" holds so clause (g) of ⊛ holds.

Eighth, all the assumptions of claim 4.13 holds. [Why? Clause (a) there saying $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ is assumed in 8.1(3); clause (b)(α) there saying "$\mathfrak{K}$ is an a.e.c. and $\mathrm{LS}(\mathfrak{K}) \le \lambda$" by 8.1(3)$(\alpha)$; clause (b)$(\beta)$ there saying "$(\mathrm{cat})_\lambda$", holds by 8.1(3)(β); clause (b)(γ) there, saying "$(\mathrm{amg})_\lambda$", holds by "fifth" above; clause (b)(δ) there saying "$(\mathrm{slm})_{\lambda^+}$" holds by 8.1(3)$(\beta)$, i.e., $(\mathrm{cat})_{\lambda^+}$ recalling $(\mathrm{nmx})_{\lambda^+}$, see "sixth" above; clause (c) there saying $(\mathrm{mdn})^2_{\lambda^{++}}$, holds by 8.1(3)$(\gamma)$, (δ); and clause (d) there saying $(\mathrm{mdn})^1_{\lambda^+}$ holds by 8.1(3)(β), i.e., $(\mathrm{cat})_{\lambda^+}$. But what about clause (b)(ε)? We may reasonably restrict ourselves to the case $|\tau_K| \le \lambda$, otherwise, if this clause fails then $2^\lambda = \lambda^+$ but there is no $M \in K_{\lambda^+}$ saturated (above λ), but then by Exercise 4.14(5) we get $\neg(\mathrm{slm})_{\lambda^+}$; contradiction.]

Hence conclusion 4.13 which says: the minimal triples are dense in $K^{3,\mathrm{na}}_\lambda$, and in particular there are minimal triples in $K^{3,\mathrm{na}}_\lambda$; alternatively use 4.10. As $K^{3,\mathrm{na}}_\lambda \neq \emptyset$ because $(\mathrm{nmx})_\lambda$ holds, it follows that there is a minimal triple in $K^{3,\mathrm{na}}_\lambda$ (see proof of $\oplus_3$ below).

Now all the requirements in 5.1 holds: 5.1(1) saying "$\mathfrak{K}$ is an a.e.c., $\lambda \geq$ LS($\mathfrak{K}$) and $K_\lambda \neq \emptyset$" by 8.1(3)(α) and "second" above; and 5.1(2) saying "(cat)$_\lambda$+ (amg)$_\lambda$ + $K_{\lambda^{++}}$" by 8.1(3)(β), "fifth" above and "second" above respectively. Also the assumptions in Claim 5.3 holds, i.e. (cat)$_{\lambda^+}$ and $2^\lambda < 2^{\lambda^+}$: by 8.1(3)($\beta$) and by 8.1(3)'s assumption respectively. Also the additional assumption of 5.3(1) holds as said in "eighth" above, i.e. there is a minimal triple in $K^{3,\mathrm{na}}_\lambda$, hence its conclusion, i.e. there is minimal and inevitable member of $K^{3,\mathrm{na}}_\lambda$ so (iev)$_\lambda$, see Definition 5.2(3), holds, which means that clause (e) of $\circledast$ holds.

Now the assumption of 5.8 holds (and as said above 5.1, too) hence the conclusions of 5.8, in particular clause (a) there, which says $\mathfrak{K}_\lambda$ is stable, i.e. (stb)$_\lambda$, which means clause (f) of $\circledast$ holds.

So we have checked all clauses of $\circledast$ hence has finished deducing the second phrase of part (3) of Theorem 8.1 from part (2).

The rest will be a

Proof of part (2) of Theorem 8.1: Also note that , as we are assuming $\circledast$ of 8.1(2) obviously

- $\oplus_0$ the hypothesis 1.1(1), 2.1 of §1,§2 hold (so we can apply the results of §1,§2).

[Why? By Clause (a) of $\circledast$ except $K_\lambda \neq \emptyset$ which holds by clause (d) of $\circledast$.]

- $\oplus_1$ the hypothesis of 7.1 holds hence we can use claims of §7.

[Why? Compare the demands in 7.1 with $\circledast$ of 8.1, i.e. clauses (a),(b),(c),(d),(e),(f) of 7.1 holds by clauses (a),(b),(c),(f),(e),(g) of $\circledast$ of 8.1(2) respectively.]

- $\oplus_2$ there is a minimal $p \in \mathscr{S}_{\mathfrak{K}_\lambda}(M)$.

[Why? By clause (d) of our assumption $\circledast$ we have (ext)$_{\lambda^+}$ hence by 1.3(4), 1.14(0)(b) we get $K^{3,\mathrm{na}}_\lambda \neq \emptyset$. Now the conclusion of 2.3(4) contradict (stb)$_\lambda$ which is clause (f) of $\circledast$ which we are assuming. So at least one of the assumptions of 2.3 fail, but the first, (amg)$_\lambda$ holds by clause (c) of $\circledast$, hence the second fails. This means that above every triple from $K^{3,\mathrm{na}}_\lambda$ there is a minimal one, but as we have noted above that $K^{3,\mathrm{na}}_\lambda \neq \emptyset$ we can find a minimal $(M, N, a) \in K^{3,\mathrm{na}}_\lambda$. So $\mathbf{tp}(a, M, N) \in \mathscr{S}(M)$ is as required.]

$\oplus_3$ we have $(\mathrm{iev})_\lambda = (\mathrm{iev})^4_\lambda$, i.e. there is $p \in \mathscr{S}^{\mathrm{na}}_{\mathfrak{K}_\lambda}(M)$ which is minimal and inevitable.

[Why? By $\oplus_2$ there is a minimal type, so a minimal triple $(M, N, a) \in K^{3,\mathrm{na}}_\lambda$, so by Fact 1.14(2) there is a reduced triple (M', N', a) above it and, of course, it too is minimal (1.14(3)). So the assumptions of 7.3 holds, recalling that by $\oplus_1$ the hypothesis of 7.1 of §7 holds, hence the conclusion of 7.3 holds, i.e. the triple (M', N', a) is inevitable, so $\mathbf{tp}(a, M', N')$ is as required in $\oplus_3$.]

$\oplus_4$ $(\mathrm{nmx})_{\lambda^+}$ and $K_{\lambda^{++}} \neq \emptyset$.

[Why? By $\oplus_0$ hypothesis 1.1(1) holds, hence we can apply 1.14(8)(a). Its assumptions hold: the first say "$K^{3,\mathrm{na}}_\lambda$ has the extension property" which holds by clause (g) of $\circledast$ of 8.1, and the second says "$K^{3,\mathrm{na}}_\lambda \neq \emptyset$" which holds as $\mathrm{LS}(\mathfrak{K}) \leq \lambda$ and $(\mathrm{ext})_{\lambda^+}$ by clause (d) of $\circledast$ of 8.1 by 1.14(0)(b). Hence the conclusions of 1.14(8) holds and in particular clause (a) there says $K_{\lambda^+} \neq \emptyset$, $(\mathrm{nmx})_{\lambda^+}$ and $K_{\lambda^{++}} \neq \emptyset$, so we are done.]

$\oplus_5$ the hypothesis of 5.1 holds hence we may use the claims of §5.

[Why? Part (1) of 5.1 holds by $\oplus_0$, i.e. as we proved Hypothesis 2.1. In part (2) of 5.1, the demands $(\mathrm{cat})_\lambda +$ $(\mathrm{amg})_\lambda$ holds by clauses (b),(c) of $\circledast$ respectively and the demand $K_{\lambda^{+2}} \neq \emptyset$ holds by $\oplus_4$.] Let us check each axiom of Definition II.2.1 as revised in Definition 8.2(2) above.

Now

Clause (A):

This is by clause (a) of 8.1(2).

Clause (B):

As K is categorical in λ by clause (b) of the assumption of 8.1(2), the existence of superlimit $M \in K_\lambda$ follows by 1.3(5) with μ there standing for λ here, recalling that $\mathrm{LS}(\mathfrak{K}) \leq \lambda$ & $K_{\lambda^+} \neq \emptyset$ by clause (d) of 8.1(2) and $(\mathrm{ext})_{\lambda^+}$ means $K_{\lambda^+} \neq \emptyset$.

Clause (C):

$\mathfrak{K}_\lambda$ has the amalgamation property by assumption (c) of 8.1(2) and has JEP in λ by categoricity in λ, i.e. by clause (b) of the assumption.

Clause (D):

Subclause $(D)(a),(b)$:

By the definition of $\mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ and of minimal types (in $\mathscr{S}(N), N \in K_\lambda$), this should be clear.

Subclause $(D)(c)$: [density of basic types]

Suppose $M \leq_{\mathfrak{K}} N$ are from K_λ and $M \neq N$; there is a minimal inevitable $p \in \mathscr{S}_{\mathfrak{s}}(M)$ by clause $\oplus_3$ proved above and categoricity of K in λ; so for some $a \in N \backslash M$ we have $p = \mathbf{tp}(a, M, N)$. So $\mathbf{tp}(a, M, N) \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ as required.

Subclause $(D)(d)^+$:

Holds by stability of $\mathfrak{K}_\lambda$, see clause (f) of 8.1(2).

Clause (E):

Subclause $(E)(a)$:

Follows by the definition.

Subclause $(E)(b)$: (Monotonicity)

Obvious properties of minimal types in $\mathscr{S}(M), M \in K_\lambda$. I.e. if $\bigcup(M_0, M_1, a, M_3)$ then changing M_3 does not change the type of a over M, decreasing M_1 preserve $a \notin M_1$, i.e. the type being not algebraic and increasing M_0 (inside M_1) preserve the type is minimal, see 1.17(2).

Subclause $(E)(c)^-$: (Weak local character)

Let $\delta < \lambda^+$ be a limit ordinal and $M_i \in K_\lambda$ be $\leq_{\mathfrak{K}}$-increasing continuous for $i \leq \delta + 1, M_{i+1}$ is $\leq_{\mathfrak{K}_\lambda}$-universal over M_i for unbounded many $i < \delta$. By clause (e) of 8.1(2) or more exactly by $\oplus_3$ some $p \in \mathscr{S}(M_\delta)$ is minimal and inevitable hence by 7.5(3), legitimatized by $\oplus_1$, for some $i < \delta$ the type $p \restriction M_i$ is minimal and as p is inevitable, clearly it is realized by some $a \in M_{\delta+1}$, so we are done.

Subclause $\underline{(E)(d)}$: (Transitivity)

Easy by the definition of "a type does not fork".

Subclause $\underline{(E)(e)}$: (Uniqueness)

By the definitions of minimal and of $\mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$.

Subclause $\underline{(E)(f)}$: (Symmetry)

Assume $M_0 \le_{\mathfrak{s}} M_3$ and $a_\ell \in M_3, p_\ell = \mathbf{tp}(a_\ell, M_0, M_3) \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_0)$. By the symmetry it is enough to assume $(*)_1$ and prove $(*)_2$ where:

- $(*)_\ell$ there are M_ℓ, M'_3 in $K_{\mathfrak{s}}$ satisfying that $M_0 \le_{\mathfrak{s}} M_\ell \le_{\mathfrak{s}} M'_3$, $M_3 \le_{\mathfrak{s}} M'_3$ such that $a_\ell \in M_\ell$ and a $\mathbf{tp}(a_{3-\ell}, M_\ell, M'_3)$ does not fork over M_0 which here just means $a_{3-\ell} \notin M_\ell$.

So let M_1, M'_3 be as in $(*)_1$. We can apply 5.11(1) by $\oplus_5$ as its assumptions, $(\mathrm{iev})_\lambda$ holds by clause (e) of $\circledast$ of 8.1(2), more exactly by $\oplus_3$. So the conclusion 5.11(1) holds, i.e. $K_{\mathfrak{s}}$ has the disjoint amalgamation property.

As we are assuming $(*)_1$ we can find (f, M_4) such that $M'_3 \le_{\mathfrak{K}} M_4$ and f is a $\le_{\mathfrak{K}}$-embedding of M'_3 into M_4 over M_0 such that $f(M'_3) \cap M_3 = M_0$. As $p_2 \in \mathscr{S}(M_0)$ is minimal and $a_2 \in M_3 \backslash M_1 \subseteq M_4 \backslash M_1$ and $f(a_2) \in f(M'_3) \backslash f(M_0) = f(M'_3) \backslash M_0 \subseteq M_4 \backslash M'_3 \subseteq M_4 \backslash M_1$ necessarily $\mathbf{tp}(a_2, M_1, M_4) = \mathbf{tp}(f(a_2), M_1, M_4)$. Hence there is a pair (M_5, g) such that $M_4 \le_{\mathfrak{K}_\lambda} M_5$ and g is a $\le_{\mathfrak{K}}$-embedding of M_4 into M_5 over M_1 mapping $f(a_2)$ to a_2. Together

- (a) $M_0 \le_{\mathfrak{K}} M_3 \le_{\mathfrak{K}_\lambda} M'_3 \le_{\mathfrak{K}_\lambda} M_4 \le_{\mathfrak{K}_\lambda} M_5$
- (b) g is a $\le_{\mathfrak{K}}$-embedding of M_4 into M_5
- (c) $gf(a_2) = g(f(a_2)) = a_2$
- (d) $f(M'_3) \le_{\mathfrak{K}} M_4$
- (e) $M_0 \le_{\mathfrak{K}} g(f(M'_3)) \le_{\mathfrak{K}_\lambda} M_5$
- (f) $a_2 = g(f(a_2)) \in g(f(M'_3))$
- (g) $a_1 \in M_1 \le_{\mathfrak{K}_\lambda} M'_3 \le_{\mathfrak{K}_\lambda} M_5$.

As $M_3' \cap f(M_3') = M_0$ by their choice and $a_1 \in M_1 \leq_{\mathfrak{K}} M_3' =$ Dom(f) but $a_1 \notin M_0$ clearly

(*h*) $a_1 \notin f(M_3')$.

Lastly, $g \supseteq \text{id}_{M_1}$ and so $g^{-1}(a_1) = a_1 \notin f(M_3')$ hence

(*i*) $a_1 \notin g(f(M_3'))$.

Obviously

(*j*) $M_0 \leq_{\mathfrak{K}_\lambda} gf(M_3') \leq_{\mathfrak{K}_\lambda} M_5$.

So together $(M_5, gf(M_3'))$ exemplify $(*)_2$ standing for M_3', M_2 there.

Subclause $(E)(g)$: (Extension existence)

By the extension property of $K_\lambda^{3,\text{na}}$ which holds by clause (g) of $\circledast$ of 8.1 (and basic properties of types).

Subclause $(E)(h)$: (Continuity)

By existence and uniqueness for minimal types, i.e. by 1.17(3A).

Subclause $(E)(i)$: (Non-forking amalgamation)

Like $(E)(f)$ or use II.2.16 which does not depend on Ax(E)(c) as said there explicitly. $\square_{8.1}$

8.5 Question: If $\mathfrak{K}$ is categorical in λ and in μ and $\mu > \lambda \geq \text{LS}(\mathfrak{K})$, can we conclude categoricity in $\chi \in (\mu, \lambda)$?

NON-STRUCTURE IN λ^{++} USING INSTANCES OF WGCH SH838

§0 Introduction

Our aim is to prove the results of the form "build complicated/many models of cardinality λ^{++} by approximation of cardinality λ" assuming only $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$, which are needed in developing classification of a.e.c., i.e. in this book and the related works (this covers [Sh 87b], [Sh 88] redone in Chapter I and [Sh 576], [Sh 603] which are redone in Chapter VI + Chapter and [Sh 87b] which is redone by Chapter II, Chapter III, [Sh 842], so we ignore, e.g. [Sh 576] now), fulfilling promises, uniformizing and correcting inaccuracies there and doing more. But en-route we spend time on the structure side.

As in [Sh 576, §3] we consider a version of construction framework, trying to give sufficient conditions for constructing many models of cardinality ∂^+ by approximations of cardinality $< \partial$ so λ^+ above correspond to ∂. Compared to [Sh 576, §3], the present version is hopefully more transparent.
We start in §1,§2 (and also §3) by giving several sufficient conditions for non-structure, in a framework closer to the applications we have in mind than [Sh 576, §3]. The price is delaying the actual proofs and losing some generality. Later (mainly in §4, but also in §6 and §8) we do the applications, usually each is quoted (in some way) elsewhere. Of course, it is a delicate question how much should we repeat the background which exists when the quote was made.

The "many" is interpreted as $\geq \mu_{\text{unif}}(\lambda^{++}, 2^{\lambda^+})$, see 9.4 why this is almost equal to $2^{\lambda^{++}}$. Unfortunately, there is here no one theorem covering all cases. But if a "lean" version suffice for us, which means that we assume the very weak set theoretic assumption "$\mathrm{WDmId}_{\lambda^+}$ (a normal ideal on λ^+) is not λ^{++}-saturated", (and, of course, we are content with getting $\geq \mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$) then all the results can be deduced from weak coding, i.e. Theorem 2.3. In this case, some parts are redundant and the paper is neatly divided to two: structure part and non-structure part and we do now describe this.

First, in §1 we define a (so called) nice framework to deal with such theorems, and in the beginning of §2 state the theorem but we replace λ^+ by a regular uncountable cardinal ∂. This is done in a way closed to the applications we have in mind as deduced in §4. In Theorem 2.3 the model in ∂^+ is approximated triples by $(\bar{M}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta)$ for $\eta \in {}^{\partial^+>}2$ increasing with η where $\bar{M}^\eta$ is an increasing chain of length ∂ of models of cardinality $< \partial$ and for each $\eta \in {}^{\partial^+}2$ the sequence $\langle \cup\{M^{\eta\restriction\varepsilon}_\alpha : \alpha < \partial\} : \varepsilon < \partial^+\rangle$ is increasing; similarly in the other such theorems.

Theorem 2.3 is not proved in §2. It is proved in §9, §10, specifically in 10.10. Why? In the proof we apply relevant set theoretic results (see in the end of §0 and more in §9 on weak diamond and failure of strong uniformization), for this it is helpful to decide that the universe of each model (approximating the desired one) is $\subseteq \partial^+$ and to add commitments $\bar{\mathbb{F}}$ on the amalgamations used in the construction called amalgamation choice function.

So model theoretically they look artificial though the theorems are stronger.

Second, we deal with the applications in §4, actually in §4(A),(C), (D),(E), so we have in each case to choose $\mathfrak{u}$, the construction framework and prove the required properties. By our choice this goes naturally.

But we would like to eliminate the extra assumption "$\mathrm{WDmId}_{\lambda^+}$ is λ^{++}-saturated". So in §2 and §3 there are additional "coding" theorems. Some still need the "amalgamation choice function", others, as we have a stronger model theoretic assumption do not need such function so their proof is not delayed to §10.

Probably the most interesting case is proving the density of $K^{3,\mathrm{uq}}_{\mathfrak{s}}$, i.e. of uniqueness triples $(M, N, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ for $\mathfrak{s}$ an (almost) good λ-frame, a somewhat weaker version of the (central) notion of Chapter II. Ignoring for a minute the "almost" this is an important step in (and is promised in) II§5. The proof is done in two stages. In the first stage we consider, in §6, a wider class $K^{3,\mathrm{up}}_{\mathfrak{s}} \subseteq K^{3,\mathrm{bs}}_{\mathfrak{s}}$ than $K^{3,\mathrm{uq}}_{\mathfrak{s}}$ and prove that its failure to be dense implies non-structure. This is done in §6, the proof is easier when $\lambda > \aleph_0$ or at least "$\mathscr{D}_{\lambda^+}$ is not λ^{++}-saturated"; (but this is unfortunate for an application to Chapter I). But for the proofs in §6 we need before this in §5 to prove some "structure positive theory" claims even if $\mathfrak{s}$ is a good λ-frame; we need more in the almost good case.

So naturally we assume (categoricity in λ and) density of $K^{3,\mathrm{up}}_{\mathfrak{s}}$ and prove (in §7) that $\mathrm{WNF}_{\mathfrak{s}}$, a weaker relative of $\mathrm{NF}_{\mathfrak{s}}$, is a weak $\mathfrak{s}$-non-forking relation on $\mathfrak{K}_{\mathfrak{s}}$ respecting $\mathfrak{s}$ and that $\mathfrak{s}$ is actually a good λ-frame (see 7.19(1)); both results are helpful.

The second stage (in §8) is done in two substages. In the first substage we deal with a delayed version of uniqueness, proving that its failure implies non-structure. In the second substage we assume delayed uniqueness but $K^{3,\mathrm{uq}}_{\mathfrak{s}}$ is not dense and we get another non-structure <u>but</u> relying on a positive consequence of density of $K^{3,\mathrm{up}}_{\mathfrak{s}}$ (that is, on a weak form of NF, see §7).

Why do we deal with almost good λ-frames? By II§3(E) from an a.e.c. $\mathfrak{K}$ categorical in λ, λ^+ which have $\in [1, \mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+}))$ non-isomorphic models in λ^{++} we construct a good λ^+-frame $\mathfrak{s}$ with $\mathfrak{K}^{\mathfrak{s}} \subseteq \mathfrak{K}$. The non-structure theorem stated (and used) there is fully proven in §4. However, not only do we use the λ^{++}-saturation of the weak diamond ideal on λ^+, but it is a good λ^+-frame rather than a good λ-frame.

This does not hamper us in Chapter III but still is regretable. Now we "correct" this but the price is getting an <u>almost</u> good λ-frame, noting that such $\mathfrak{s}$ is proved to exist in VI§8, the revised version of [Sh 576]. However, to arrive to those points in Chapter VI, [Sh 576] we have to prove the density of minimal types under the weaker assumptions, i.e. without the saturation of the ieal $\mathrm{WDmId}_{\lambda^+}$ together Chapter + VI§3,§4 gives a full proof. This requires again on developing some positive theory, so in §5 we do here some positive theory. Recall that [Sh 603], [Sh 576] are subsumed by them.

We can note that in building models $M \in K^{\mathfrak{s}}_{\lambda^{++}}$ for $\mathfrak{s}$ an almost good λ-frame, for convenience we use disjoint amalgamation. This may seem harmless but proving the density of the minimal triples this is not obvious; without assuming this we have to use $\langle M_\alpha, h_{\alpha,\beta} : \alpha < \beta < \partial\rangle$ instead of increasing $\langle M_\alpha : \alpha < \partial\rangle$; a notationally cumbersome choice. So we use a congruence relation $=_\tau$ but the models we construct are not what we need. We have to take their quotient by $=_\tau$, which has to, e.g. have the right cardinality. But we can take care that $|M|/\equiv_\tau$ has cardinality λ^{++} and $a \in M \Rightarrow |a/\equiv_\tau| = \lambda^{++}$. For the almost good λ-frame case this follows if we use not just models M which are λ^+-saturated above λ but if $M_0 \leq_{\mathfrak{K}[\mathfrak{s}]} M_1 \leq_{\mathfrak{K}[\mathfrak{s}]} M, M_0 \in K_\mathfrak{s}, M_1 \in K^\mathfrak{s}_{\leq\lambda}$ and $p \in \mathscr{S}^{\rm bs}_\mathfrak{s}(M_0)$ then for some $a \in M$, for every $M'_1 \leq_{\mathfrak{K}[\mathfrak{s}]} M_1$ of cardinality λ including M_0, the type $\mathbf{tp}_\mathfrak{s}(a, M'_1, M)$ is the non-forking extension of p, so not a real problem.

Reading Plans: The miser model - theorist Plan A:

If you like to see only the results quoted elsewhere (in this book), willing to assume an extra weak set theoretic assumption, this is the plan for you.

The results are all in §4, more exactly §4(A),(C),(D),(E). They all need only 2.3 relying on 2.2, but the rest of §2 and §3 are irrelevant as well as §5 - §8.

To understand what 2.3 say you have to read §1 (what is $\mathfrak{u}$; what are $\mathfrak{u}$-free rectangles; assuming $\mathfrak{K}$ is categorical in λ^+ you can ignore the "almost"). You may take 2.3 on belief, so you are done; otherwise you have to see §9 and 10.1 - 10.10.

The pure model - theorist Plan B:

Suitable if you like to know about the relatives of "good λ-frames". Generally see §5 - §8.

In particular on "almost good λ-frames" see §5; but better first read 1.1 - 1.14, which deal with a related framework called "nice construction framework" and in §6 learn of the class $K^{3,\rm up}_\mathfrak{s} \subseteq K^{3,\rm bs}_\mathfrak{s}$ with a weak version of uniqueness. By quoting we get non-structure if they fail density. Then in §7 learn on weak non-forking relations WNF on $\mathfrak{K}_\lambda$ which respects $\mathfrak{s}$, it is interesting when we assume $K^{3,\rm up}_\mathfrak{s}$ has density or reasonably weak existence assumption, because then we can prove that the definition given such existence, and this implies that $\mathfrak{s}$ is a good λ-frame (not just almost). In §8 we prove density of uinqueness triples ($K^{3,\rm uq}_\mathfrak{s}$) in $K^{3,\rm bs}_\mathfrak{s}$, so quote non-structure theorems.

The set theorist Plan C:

Read §1,§2,§3,§9,§10,§11 this presents construction in ∂^+ by approximation of cardinality $\leq \lambda$.

0.1 Notation:

1) $\mathfrak{u}$, a construction framework, see §1, in particular Definition 1.2

2) Triples $(M, N, \mathbf{J}) \in \text{FR}^\ell_{\mathfrak{u}}$, see Definition 1.2

2A) $\mathbf{J}$ and $\mathbf{I}$ are $\subseteq M \in K_{\mathfrak{u}}$

3) $\mathbf{d}$, and also $\mathbf{e}$, a $\mathfrak{u}$-free rectangle (or triangle), see Definitions 1.4, 1.6

4) $K^{\text{qt}}_{\mathfrak{u}}$, the set of $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f})$, see Definition 1.15, where, in particular:
 4A) $\mathbf{f}$ a function from ∂ to ∂
 4B) $\bar{\mathbf{J}} = \langle \mathbf{J}_\alpha : \alpha < \partial \rangle, \mathbf{J}_\alpha \subseteq M_{\alpha+1} \backslash M_\alpha$
 4C) $\bar{M} = \langle M_\alpha : \alpha < \partial \rangle$ where $M_\alpha \in \mathfrak{K}_{<\partial}$ is $\leq_{\mathfrak{K}}$-increasing continuous

5) Orders (or relations) on $K^{\text{qt}}_{\mathfrak{u}} : \leq^{\text{at}}_{\mathfrak{u}}, \leq^{\text{qt}}_{\mathfrak{u}}, \leq^{\text{qs}}_{\mathfrak{u}}$

6) $\mathbf{c}$, a colouring (for use in weak diamond)

7) $\mathbb{F}$ (usually $\bar{\mathbb{F}}$), for amalgamation choice functions, see Definition 10.3

8) $\mathfrak{g}$, a function from $K^{\text{qt}}_{\mathfrak{u}}$ to itself, etc., see Definition 1.22
 (for defining "almost every ...")

9) Cardinals $\lambda, \mu, \chi, \kappa, \theta, \partial$, but here $\partial = \text{cf}(\partial) > \aleph_0$, see 1.8(2) in 1.8(1B), $\mathscr{D}_\partial$ is the club filter on ∂

10) $\mathbf{F}$ in the definition of limit model, see Definition Chapter I, marginal.

0.2 Definition. 1) For K a set or a class of models let $\dot{I}(K) = \{M/\cong : M \in K\}$, so it is a cardinality or ∞.
2) For a class K of models let $\dot{I}(\lambda, K) = \dot{I}(K_\lambda)$ where $K_\lambda = \{M \in K : \|M\| = \lambda\}$.
3) For $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ let $\dot{I}(\mathfrak{K}) = I(K)$ and $\dot{I}(\lambda, \mathfrak{K}) = \dot{I}(\lambda, K)$.

Remark. We shall use in particular $\dot{I}(\mathfrak{K}^{\mathfrak{u},\mathfrak{h}}_{\partial^+})$, see Definition 1.23.

We now define some set theoretic notions (we use mainly the ideal WDmTId_∂ and the cardinals $\mu_{\mathrm{wd}}(\partial), \mu_{\mathrm{unif}}(\partial^+, 2^\partial)$).

0.3 Definition. Fix ∂ regular and uncountable.
1) For ∂ regular uncountable, $S \subseteq \partial$ and $\bar\chi = \langle \chi_\alpha : \alpha < \partial\rangle$ but only $\bar\chi \restriction S$ matters so we can use any $\bar\chi = \langle \chi_\alpha : \alpha \in S'\rangle$ where $S \subseteq S'$ let

$$\begin{aligned}\mathrm{WDmTId}(\partial, S, \bar\chi) = \Big\{ A : A \subseteq \prod_{\alpha\in S} \chi_\alpha,\ &\text{and for some function}\\ &(= \text{colouring})\ \mathbf{c} \text{ with domain } \bigcup_{\alpha<\partial} {}^\alpha(2^{<\partial})\\ &\text{mapping } {}^\alpha(2^{<\partial}) \text{ into } \chi_\alpha,\\ &\text{for every } \eta \in A,\ \text{for some } f \in {}^\partial(2^{<\partial})\\ &\text{the set} \{\delta \in S : \eta(\delta) = \mathbf{c}(f \restriction \delta)\}\\ &\text{is not stationary (in } \partial)\Big\}.\end{aligned}$$

(Note: WDmTId stands for weak diamond target ideal; of course, if we increase the χ_α we get a bigger ideal); the main case is when $\alpha \in S \Rightarrow \chi_\alpha = 2$ this is the weak diamond, see below.
1A) Here we can replace $2^{<\partial}$ by any set of this cardinality, and so we can replace $f \in {}^\partial(2^{<\partial})$ by $f_1, \ldots, f_n \in {}^\partial(2^{<\partial})$ and $f \restriction \delta$ by $\langle f_1 \restriction \delta, \ldots, f_n \restriction \delta\rangle$ and $\mathbf{c}(f \restriction \delta)$ by $\mathbf{c}'(f_1 \restriction \delta, \ldots, f_n \restriction \delta)$ so with $\mathbf{c}'$ being an n-place function; justified in [Sh:f, AP,§1].
2)[1]

$$\mathrm{cov}_{\mathrm{wdmt}}(\partial, S, \bar\chi) = \mathrm{Min}\Big\{ |\mathscr{P}| : \mathscr{P} \subseteq \mathrm{WDmTId}(\partial, S, \bar\chi) \text{ and } \prod_{\alpha\in S} \chi_\alpha \subseteq \bigcup_{A\in\mathscr{P}} A \Big\}$$

[1]in [Sh:b, AP,§1], [Sh:f, AP,§1] we express $\mathrm{cov}_{\mathrm{wdmt}}(\partial, S) > \mu^*$ by allowing $f(0) \in \mu^* < \mu$

3)

$$\mathrm{WDmTId}_{<\mu}(\partial, S, \bar{\chi}) = \Big\{A \subseteq \prod_{\alpha \in S} \chi_\alpha : \text{for some } i^* < \mu \text{ and } A_i \in \mathrm{WDmTId}(\partial, S, \bar{\chi}) \text{ for } i < i^* \text{ we have } A \subseteq \bigcup_{i<i^*} A_i\Big\}$$

4) (a) $$\mathrm{WDmTId}(\partial) = \mathrm{WDmTId}(\partial, \partial, 2)$$

4) (b) $$\mathrm{WDmId}_{<\mu}(\partial, \bar{\chi}) = \{S \subseteq \partial : \mathrm{cov}_{\mathrm{wdmt}}(\partial, S, \bar{\chi}) < \mu\}.$$

5) Instead of "$< \mu^+$" we may write $\leq \mu$ or just μ; if we omit μ we mean $(2^{<\partial})$. If $\bar{\chi}$ is constantly 2 we may omit it, see below, if $\chi_\alpha = 2^{|\alpha|}$ we may write pow instead of $\bar{\chi}$; all this in the parts above and below.

6) Let $\mu_{\mathrm{wd}}(\partial, \bar{\chi}) = \mathrm{cov}_{\mathrm{wdmt}}(\partial, \partial, \bar{\chi})$.

7) We say that the weak diamond holds on λ if $\partial \notin \mathrm{WDmId}(\partial)$.

Remark. This is used in 3.9 VI.3.9, VI.6.11. Note that by 0.5(1A) that $\mu_{\mathrm{wd}}(\lambda^+)$ is large (but $\leq 2^{\lambda^+}$, of course).

A relative is

0.4 Definition. Fix ∂ regular and uncountable.
1) For ∂ regular uncountable, $S \subseteq \partial$ and $\bar{\chi} = \langle \chi_\alpha : \alpha < \partial \rangle$ let

$$\mathrm{UnfTId}(\partial, S, \bar{\chi}) = \Big\{A : A \subseteq \prod_{\alpha \in S} \chi_\alpha, \text{ and for some function } (= \text{colouring})\ \mathbf{c} \text{ with domain } \bigcup_{\alpha<\partial} {}^{\alpha}(2^{<\partial}) \text{ mapping } {}^{\alpha}(2^{<\partial}) \text{ into } \chi_\alpha, \text{ for every } \eta \in A, \text{ for some } f \in {}^{\partial}(2^{<\partial}) \text{ the set } \{\delta \in S : \eta(\delta) \neq \mathbf{c}(f \restriction \delta)\} \text{ is not stationary (in } \partial)\Big\}.$$

(Note: UnfTId stands for uniformization target ideal; of course, if we increase the χ_α we get a smaller ideal); when $\alpha \in S \Rightarrow \chi_\alpha = 2$ this is the weak diamond, i.e. as in 0.3(1), similarly below.
1A) Also here we can replace $2^{<\partial}$ by any set of this cardinality, and so we can replace $f \in {}^{\partial}(2^{<\partial})$ by $f_1, \ldots, f_n \in {}^{\partial}(2^{<\partial})$ and $f \restriction \delta$ by $\langle f_1 \restriction \delta, \ldots, f_n \restriction \delta \rangle$ and $\mathbf{c}(f \restriction \delta)$ by $\mathbf{c}'(f_1 \restriction \delta, \ldots, f_n \restriction \delta)$ so with $\mathbf{c}'$ being an n-place function, justified in [Sh:f, AP,§1].

2)
$$\mathrm{cov}_{\mathrm{unf}}(\partial, S, \bar{\chi}) = \mathrm{Min}\Big\{ |\mathscr{P}| : \mathscr{P} \subseteq \mathrm{UnfTId}(\partial, S, \bar{\chi}) \text{ and } \prod_{\alpha \in S} \chi_\alpha \subseteq \bigcup_{A \in \mathscr{P}} A \Big\}$$

3)
$$\mathrm{UnfTId}_{<\mu}(\partial, S, \bar{\chi}) = \Big\{ A \subseteq \prod_{\alpha \in S} \chi_\alpha : \text{for some } i^* < \mu \text{ and } A_i \in \mathrm{UnfTId}(\partial, S, \bar{\chi}) \text{ for } i < i^* \text{ we have } A \subseteq \bigcup_{i < i^*} A_i \Big\}$$

4)
$$\mathrm{UnfId}_{<\mu}(\partial, \bar{\chi}) = \Big\{ S \subseteq \partial : \mathrm{cov}_{\mathrm{unf}}(\partial, S, \bar{\chi}) < \mu \Big\}.$$

5) Instead of "$< \mu^+$" we may write $\leq \mu$ or just μ, if we omit μ we mean $(2^{<\partial})$. If $\bar{\chi}$ is constantly 2 we may omit it, if $\chi_\alpha = 2^{|\alpha|}$ we may write pow instead of $\bar{\chi}$; all this in the parts above and below.
6) $\mu_{\mathrm{unif}}(\partial, \bar{\chi})$ where $\bar{\chi} = \langle \chi_\alpha : \alpha < \partial \rangle$ is $\mathrm{Min}\{|\mathscr{P}| : \mathscr{P}$ is a family of subsets of $\prod_{\alpha<\partial} \chi_\alpha$ with union $\prod_{\alpha<\partial} \chi_\alpha$ and for each $A \in \mathscr{P}$ there is a function $\mathbf{c}$ with domain $\bigcup_{\alpha<\partial} \prod_{\beta<\alpha} \chi_\beta$ such that $f \in A \Rightarrow \{\delta \in S : \mathbf{c}(f \restriction \delta) = f(\delta)\}$ is not stationary$\}$.
7) $\mu_{\mathrm{unif}}(\partial, \chi) = \mu_{\mathrm{unif}}(\partial, \bar{\chi})$ where $\bar{\chi} = \langle \chi : \alpha < \partial \rangle$ and $\mu_{\mathrm{unif}}(\partial, < \chi)$

means $\sup\{\mu_{\rm unif}(\partial,\chi_1) : \chi_1 < \chi\}$; similarly in the other definitions above. If $\chi = 2$ we may omit it.

By Devlin Shelah [DvSh 65], [Sh:b, XIV,1.5,1.10](2);1.18(2),1.9(2) (presented better in [Sh:f, AP,§1] we have:

0.5 Theorem.
1) If $\partial = \aleph_1, 2^{\aleph_0} < 2^{\aleph_1}, \mu \le (2^{\aleph_0})^+$ *then* $\partial \notin {\rm WDmId}_{<\mu}(\partial)$.
2) If $2^\theta = 2^{<\partial} < 2^\partial, \mu = (2^\theta)^+$, *or just: for some* $\theta, 2^\theta = 2^{<\partial} < 2^\partial, \mu \le 2^\partial$, *and* $\chi^\theta < \mu$ *for* $\chi < \mu$, *then* $\partial \notin {\rm WDmId}_\mu(\partial)$ *equivalently* ${}^\partial 2 \notin {\rm WDmTId}_\mu(\partial)$. *So* $(\mu_{\rm wd}(\partial))^\theta = 2^\partial$.
3) Assume $2^\theta = 2^{<\partial} < 2^\partial, \mu \le 2^\partial$ *and*

- (a) $\mu \le \partial^+$ *or* ${\rm cf}([\mu_1]^{\le\partial}, \subseteq) < \mu$ *for* $\mu_1 < \mu$ *and*
- (b) ${\rm cf}(\mu) > \partial \,{\rm or}\, \mu \le (2^{<\partial})^+$.

Then ${\rm WDmId}_{<\mu}(\partial,\bar\chi)$ *is a normal ideal on* ∂ *and* ${\rm WDmTId}_{<\mu}(\partial,\chi)$ *is a* ${\rm cf}(\mu)$*-complete ideal on* ${}^\partial 2$. *[If this ideal is not trivial, then* $\partial = {\rm cf}(\partial) > \aleph_0, 2^{<\partial} < 2^\partial$*.]*
4) ${\rm WDmTId}_{<\mu}(\partial,S,\bar\chi)$ *is* ${\rm cf}(\mu)$*-complete ideal on* $\prod\limits_{\alpha\in S}\chi_\alpha$.

0.6 Remark. 0) Compare to §9,§10, mainly 9.6.
1) So if ${\rm cf}(2^\partial) < \mu$ (which holds if 2^∂ is singular and $\mu = 2^\partial$) then 0.5(3) implies that there is $A \subseteq {}^\partial 2, |A| < 2^\partial, A \notin {\rm WDmTId}(\partial)$.
2) Some related definitions appear in [Sh:E45, §1], mainly ${\rm DfWD}_{<\mu}(\partial)$, but presently we ignore them.
3) We did not look again at the case $(\forall\sigma < \lambda)(2^\sigma < 2^{<\partial} < 2^\partial)$.
4) Recall that for an a.e.c. $\mathfrak{K}$:

- (a) if $K_\lambda \ne \emptyset$ but $\mathfrak{K}$ has no $\le_{\mathfrak{K}}$-maximal model in K_λ then $K_{\lambda^+} \ne \emptyset$
- (b) if $\mathfrak{K}$ is categorical in λ and ${\rm LS}(\mathfrak{K}) \le \lambda$ then $K_{\lambda^+} \ne \emptyset$ iff K has no $\le_{\mathfrak{K}}$-maximal model in K_λ.

5) About $\mu_{\rm wd}(\partial)$ see VI.1.4, VI.2.8, VI.6.5.

0.7 Definition. 1) We say that a normal ideal $\mathbb{I}$ on a regular uncountable cardinal λ is μ-saturated <u>when</u> we cannot find a sequence $\bar{A} = \langle A_i : i < \mu \rangle$ such that $A_i \subseteq \lambda, A_i \notin \mathbb{I}$ for $i < \mu$ and $A_i \cap A_j \in \mathbb{I}$ for $i \neq j < \mu$; if $\mu \leq \lambda^+$ without loss of generality $A_i \cap A_j \in [\lambda]^{<\lambda}$.
2) Similarly for a normal filter on a regular uncountable cardinal λ.

§1 Nice construction framework

We define here when $\mathfrak{u}$ is a nice construction framework. Now $\mathfrak{u}$ consists of an a.e.c. $\mathfrak{K}$ with $\mathrm{LS}(\mathfrak{K}) < \partial_{\mathfrak{u}} = \mathrm{cf}(\partial_{\mathfrak{u}})$, and enables us to build a model in K_{∂^+} by approximations of cardinality $< \partial := \partial_{\mathfrak{u}}$.

Now for notational reasons we prefer to use increasing sequences of models rather than directed systems, i.e., sequences like $\langle M_\alpha, f_{\beta,\alpha} : \alpha \leq \beta < \alpha^* \rangle$ with $f_{\beta,\alpha} : M_\alpha \to M_\beta$ satisfying $f_{\gamma,\beta} \circ f_{\beta,\alpha} = f_{\alpha,\gamma}$ for $\alpha \leq \beta \leq \gamma < \alpha^*$. For this it is very desirable to have disjoint amalgamation; however, in one of the major applications (the density of minimal types, see here in §4(A),(B) or in [Sh 576, §3] used in VI§3,§4) we do not have this. In [Sh 576, §3] the solution was to allow non-standard interpretation of the equality (see Definition 1.10 here). Here we choose another formulation: we have $\tau \subseteq \tau(\mathfrak{u})$ such that we are interested in the non-isomorphism of the τ-reducts $M^{[\tau]}$ of the M's constructed, see Definition 1.8. Of course, this is only a notational problem.

The main results on such $\mathfrak{u}$ appear later; a major theorem is 2.3, deducing non-structure results assuming the weak coding property. This and similar theorems, assuming other variant of the coding property, are dealt with in §2,§3. They all have (actually lead to) the form "if most triples $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ has, in some sense 2 (or many, say $2^{<\partial}$) extensions which are (pairwise) incompatible in suitable sense, <u>then</u> we build a suitable tree $\langle (\bar{M}_\eta, \bar{\mathbf{J}}_\eta, f_\eta) : \eta \in {}^{\partial^+>}2 \rangle$ and letting $M_\eta = \cup\{M^\eta_\alpha : \alpha < \lambda\}$ for $\eta \in {}^{\partial^+>}2$ and $M_\nu := \cup\{M_{\nu \restriction \alpha} : \alpha < \partial^+\}$ for $\nu \in {}^{\partial^+}2$ we have: among $\langle M_\nu : \nu \in {}^{\partial^+}2 \rangle$ many are non-isomorphic (and in K_{∂^+}). Really, usually the indexes are $\eta \in {}^{\partial^+>}(2^\partial)$ and the conditions speak on amalgamation in $\mathfrak{K}_{\mathfrak{u}}$, i.e. on models of cardinality $< \partial$ but using FR_1, FR_2, see below.

As said earlier, in the framework defined below we (relatively) prefer transparency and simplicity on generality, e.g. we can weaken "$\mathfrak{K}_{\mathfrak{u}}$ is an a.e.c." and/or make FR^+_ℓ is axiomatic and/or use more than atomic successors (see 10.14 + 10.16).

In 1.1 - 1.6 we introduce our frameworks $\mathfrak{u}$ and $\mathfrak{u}$-free rectangles/triangles; in 1.7 - the dual of $\mathfrak{u}$, and in 1.8 - 1.12 we justify the disjoint amalgamation through "τ is a $\mathfrak{u}$-sub-vocabulary", so a reader not bothered by this point can ignore it, then in 1.13 we consider another property of $\mathfrak{u}$, monotonicity and in 1.14 deal with variants of $\mathfrak{u}$.

In 1.15 - 1.26 we introduce a class $K^{\mathrm{qt}}_{\mathfrak{u}}$ of triples $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f})$ serving as approximations of size ∂, some relations and orders on it and variants, and define what it means "for almost every such triple" (if $K_{\mathfrak{u}}$ is categorical in ∂ this is usually easy and in many of our applications for most $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f})$ the model $\cup\{M_\alpha : \alpha < \partial\}$ is saturated (of cardinality ∂).

1.1 Convention: If not said otherwise, $\mathfrak{u}$ is as in Definition 1.2.

1.2 Definition. We say that $\mathfrak{u}$ is a nice construction framework when (the demands are for $\ell = 1, 2$ and later (D) means $(\mathrm{D})_1$ and $(\mathrm{D})_2$ and (E) means $(\mathrm{E})_1$ and $(\mathrm{E})_2$):

- (A) $\mathfrak{u}$ consists of $\partial, \mathfrak{K} = (K, \leq_{\mathfrak{K}}), \mathrm{FR}_1, \mathrm{FR}_2, \leq_1, \leq_2$ (also denoted by $\partial_{\mathfrak{u}}, \mathfrak{K}^{\mathfrak{u}} = \mathfrak{K}^{\mathrm{up}}_{\mathfrak{u}} = (K^{\mathrm{up}}_{\mathfrak{u}}, \leq_{\mathfrak{u}}), \mathrm{FR}^{\mathfrak{u}}_1, \mathrm{FR}^{\mathfrak{u}}_2, \leq^1_{\mathfrak{u}}, \leq^2_{\mathfrak{u}}$) and let $\tau_{\mathfrak{u}} = \tau_{\mathfrak{K}}$. The indexes 1 and 2 can be replaced by ver (vertical[2], direction of ∂) and hor (horizontal, direction of ∂^+) respectively
- (B) ∂ is regular uncountable
- (C) $\mathfrak{K} = \mathfrak{K}^{\mathrm{up}}_{\mathfrak{u}} = (K, \leq_{\mathfrak{K}})$ is an a.e.c., $K \neq \emptyset$ of course with $\mathrm{LS}(\mathfrak{K}) < \partial$ (or just $(\forall M \in K)(\forall A \in [M]^{<\partial})(\exists N)(A \subseteq N \leq_{\mathfrak{K}}$

[2]Hard but immaterial choice. We construct a model of cardinality ∂^+ by a sequence of length ∂^+ approximations, each of the form $\langle M_\alpha, \mathbf{J}_\alpha : \alpha < \partial\rangle, M_\alpha \in K_{<\partial}$ is $\leq_{\mathfrak{K}_{<\partial}}$-increasing and $(M_\alpha, M_{\alpha+1}, \mathbf{J}_\alpha) \in \mathrm{FR}_2$. If $\langle M'_\alpha, \mathbf{J}'_\alpha : \alpha < \partial\rangle$ is an immediate successor in the ∂^+-direction of $\langle M_\alpha, \mathbf{J}_\alpha : \alpha < \partial\rangle$ then for most $\alpha, M_\alpha \leq_{\mathfrak{u}} M'_\alpha$ and $(M_\alpha, M'_\alpha, \mathbf{I}_\alpha) \in \mathrm{FR}_1$ for suitable $\mathbf{I}_\alpha$, increasing with α and $(M'_\alpha, M'_{\alpha+1}, \mathbf{J}'_\alpha) \in \mathrm{FR}_2$ is $\leq^2_{\mathfrak{u}}$-above $(M_\alpha, M_{\alpha+1}, \mathbf{J}_\alpha)$. Now the natural order on FR_2 leads in the horizontal direction.

$M \wedge \|N\| < \partial)$). Let $\mathfrak{K}_{\mathfrak{u}} = \mathfrak{K}_{<\partial} = (\mathfrak{K}_{<\partial}, \leq_{\mathfrak{K}}\restriction K_{<\partial})$ where $K_{<\partial} = K_{\mathfrak{u}} \restriction \{M : M \in K^{\rm up}_{\mathfrak{u}}$ has cardinality $< \partial\}$ and $\mathfrak{K}[\mathfrak{u}] = K^{\rm up}_{\mathfrak{u}}$.

(To prepare for weaker versions we can start with $\mathfrak{K}_{\mathfrak{u}}$, a $(< \partial)$-a.e.c.[3]; this means K is a class of models of cardinality $< \partial$, and in AxIII, the existence of union we add the assumption that the length of the union is $< \partial$ (here equivalently the union has cardinality $< \partial$) and we replace "LS($\mathfrak{K}_{<\partial}$) exists" by $K_{<\partial} \neq \emptyset$ and let $\mathfrak{K}^{\rm up}_{\mathfrak{u}}$ be its lifting up, as in II.1.23 and we assume $K \neq \emptyset$ so $\mathfrak{K} = \mathfrak{K}^{\rm up}_{\mathfrak{u}}$ and we write $\tau_{\mathfrak{u}} = \tau_{\mathfrak{K}}$ and $\leq_{\mathfrak{K}}$ for $\leq_{\mathfrak{K}^{\rm up}}$ and $\leq_{\mathfrak{u}}$ for $\leq_{\mathfrak{K}_{<\partial}}$)

$(D)_\ell$ (a) FR_ℓ is a class of triples of the form $(M, N, \mathbf{J})$, closed under isomorphisms, let $\mathrm{FR}^+_\ell = \mathrm{FR}^{\mathfrak{u},+}_\ell$ be the family of $(M, N, \mathbf{J}) \in \mathrm{FR}_\ell$ such that $\mathbf{J} \neq \emptyset$;

(b) if $(M, N, \mathbf{J}) \in \mathrm{FR}_\ell$ <u>then</u> $M \leq_{\mathfrak{u}} N$ hence both are from $K_{\mathfrak{u}}$ so of cardinality $< \partial$

(c) if $(M, N, \mathbf{J}) \in \mathrm{FR}_\ell$ <u>then</u>[4] $\mathbf{J}$ is a set of elements of $N \backslash M$

(d) if $M \in K_{\mathfrak{u}}$ <u>then</u>[5] for some N, $\mathbf{J}$ we have $(M, N, \mathbf{J}) \in \mathrm{FR}^+_\ell$

(e) if $M \leq_{\mathfrak{u}} N \in K_{\mathfrak{u}}$ <u>then</u>[6] $(M, N, \emptyset) \in \mathrm{FR}^{\mathfrak{u}}_\ell$

$(E)_\ell$ (a) $\leq_\ell = \leq^\ell_{\mathfrak{u}}$ is a partial order on FR_ℓ, closed under isomorphisms

(b)(α) if $(M_1, N_1, \mathbf{J}_1) \leq_\ell (M_2, N_2, \mathbf{J}_2)$ then $M_1 \leq_{\mathfrak{u}} M_2, N_1 \leq_{\mathfrak{u}} N_2$ and $\mathbf{J}_1 \subseteq \mathbf{J}_2$

(β) moreover $N_1 \cap M_2 = M_1$ (disjointness)

(c) if $\langle (M_i, N_i, \mathbf{J}_i) : i < \delta \rangle$ is $\leq_\ell$-increasing continuous (i.e. in limit we take unions) and $\delta < \partial$ <u>then</u>

[3] less is used, but natural for our applications, see §9

[4] If we use the a.e.c. $\mathfrak{K}'$ defined in 1.10 we, in fact, weaken this demand to "$\mathbf{J} \subseteq N$". This is done, e.g. in the proof of 4.1 that is in Definition 4.5.

[5] We can weaken this and in some natural example we have less, but we circumvent this, via 1.10, see 4.1(c); this applies to (E)(b)(β), too; see Example 2.8

[6] not a great loss if we demand $M = N$; but then we have to strengthen the amalgamation demand (clause (F)); this is really needed only for $\ell = 2$

the union $\bigcup_{i<\delta}(M_i, N_i, \mathbf{J}_i) := (\bigcup_{j<\delta} M_i, \bigcup_{i<\delta} N_i, \bigcup_{i<\delta} \mathbf{J}_i)$
belongs to FR_ℓ and
$j < \delta \Rightarrow (M_j, N_j, \mathbf{J}_j) \le_\ell (\bigcup_{i<\delta} M_i, \bigcup_{i<\delta} N_i, \bigcup_{i<\delta} \mathbf{J}_i)$

(d) if $M_1 \le_{\mathfrak{u}} M_2 \le_{\mathfrak{u}} N_2$ and $M_1 \le_{\mathfrak{u}} N_1 \le_{\mathfrak{u}} N_2$ then $(M_1, N_1, \emptyset) \le_\ell (M_2, N_2, \emptyset)$

(F) (amalgamation) if $(M_0, M_1, \mathbf{I}_1) \in \mathrm{FR}_1, (M_0, M_2, \mathbf{J}_1) \in \mathrm{FR}_2$ and $M_1 \cap M_2 = M_0$ then we can find[7] $M_3, \mathbf{I}_2, \mathbf{J}_2$ such that $(M_0, M_1, \mathbf{I}_1) \le_1 (M_2, M_3, \mathbf{I}_2)$ and $(M_0, M_2, \mathbf{J}_1) \le_2 (M_1, M_3, \mathbf{J}_2)$ hence $M_\ell \le_{\mathfrak{u}} M_3$ for $\ell = 0, 1, 2$.

1.3 Claim. *1) $\mathfrak{K}_{\mathfrak{u}}$ has disjoint amalgamation.*
2) If $\ell = 1, 2$ and $(M_0, M_1, \mathbf{I}_1) \in \mathrm{FR}_\ell$ and $M_0 \le_{\mathfrak{u}} M_2$ and $M_1 \cap M_2 = M_0$ then we can find a pair $(M_3, \mathbf{I}_2^*)$ such that: $(M_0, M_1, \mathbf{I}) \le_{\mathfrak{u}}^\ell (M_2, M_3, \mathbf{I}_2^*) \in \mathrm{FR}_\ell^{\mathfrak{u}}$.

Proof. 1) Let $M_0 \le_{\mathfrak{K}_{\mathfrak{u}}} M_\ell$ for $\ell = 1, 2$ and for simplicity $M_1 \cap M_2 = M_0$. Let $\mathbf{I}_1 = \emptyset$, so by condition $(\mathrm{D})_1$(e) of Definition 1.2 we have $(M_0, M_1, \mathbf{I}_1) \in \mathrm{FR}_1$. Now apply part (2), (for $\ell = 1$).
2) By symmetry without loss of generality $\ell = 1$. Let $\mathbf{J}_1 := \emptyset$, so by Condition $(\mathrm{D})_2$(e) of Definition 1.2 we have $(M_0, M_2, \mathbf{J}_1) \in \mathrm{FR}_2^{\mathfrak{u}}$. So $M_0, M_1, \mathbf{I}_1, M_2, \mathbf{J}_1$ satisfies the assumptions of condition (F) of Definition 1.2 hence there are $M_3, \mathbf{I}_2, \mathbf{J}_2$ as guaranteed there so in particular $(M_0, M_1, \mathbf{I}_1) \le_{\mathfrak{u}}^2 (M_2, M_3, \mathbf{I}_2)$ so the pair $(M_3, \mathbf{I}_2)$ is as required. $\square_{1.3}$

1.4 Definition. 1) We say that $\mathbf{d}$ is a $\mathfrak{u}$-free (α, β)-rectangle or is $\mathfrak{u}$-non-forking (α, β)-rectangle (we may omit $\mathfrak{u}$ when clear from the context) when:

(a) $\mathbf{d} = (\langle M_{i,j} : i \le \alpha, j \le \beta\rangle, \langle \mathbf{J}_{i,j} : i < \alpha, j \le \beta\rangle, \langle \mathbf{I}_{i,j} : i \le \alpha, j < \beta\rangle)$,

[7] we can ask for $M_3' \le_{\mathfrak{K}} M_3''$ and demand $(M_0, M_1, \mathbf{J}_1) \le_2 (M_2, M_3', \mathbf{J}_1')$, $(M_0, M_2, \mathbf{I}_1) \le (M_1, M_3'', \mathbf{I}_2')$, no real harm here but also no clear gain

(we may add superscript $\mathbf{d}$, the "$i < \alpha$", "$j < \beta$" are not misprints)

(*b*) $\langle (M_{i,j}, M_{i,j+1}, \mathbf{I}_{i,j}) : i \leq \alpha \rangle$ is $\leq_1$-increasing continuous for each $j < \beta$

(*c*) $\langle (M_{i,j}, M_{i+1,j}, \mathbf{J}_{i,j}) : j \leq \beta \rangle$ is $\leq_2$-increasing continuous for each $i < \alpha$

(*d*) $M_{\alpha,\beta} = \cup \{M_{i,\beta} : i < \alpha\}$ if α, β are limit ordinals.

2) For $\mathbf{d}^1$ a $\mathfrak{u}$-free (α, β)-rectangle and $\alpha_1 \leq \alpha, \beta_1 \leq \beta$ let $\mathbf{d}^2 = \mathbf{d}^1 \restriction (\alpha_1, \beta_1)$ means:

(*a*) $\mathbf{d}^2$ is a $\mathfrak{u}$-free (α_1, β_1)-rectangle, see 1.5 below

(*b*) the natural equalities: $M^{\mathbf{d}^1}_{i,j} = M^{\mathbf{d}^2}_{i,j}, \mathbf{J}^{\mathbf{d}^1}_{i,j} = \mathbf{J}^{\mathbf{d}^2}_{i,j}, \mathbf{I}^{\mathbf{d}_1}_{i,j} = \mathbf{I}^{\mathbf{d}_2}_{i,j}$ when both sides are well defined.

3) $\mathbf{d}^2 = \mathbf{d}^1 \restriction ([\alpha_1, \alpha_2], [\beta_1, \beta_2])$ when $\alpha_1 \leq \alpha_2 \leq \alpha, \beta_1 \leq \beta_2 \leq \beta$ is defined similarly.
4) For $\mathbf{d}$ as above we may also write $\alpha_{\mathbf{d}}, \alpha(\mathbf{d})$ for α and $\beta_{\mathbf{d}}, \beta(\mathbf{d})$ for β and if $\mathbf{I}^{\mathbf{d}}_{i,j}$ is a singleton we may write $\mathbf{I}^{\mathbf{d}}_{i,j} = \{a^{\mathbf{d}}_{i,j}\}$ and may just write $(M^{\mathbf{d}}_{i,j}, M^{\mathbf{d}}_{i,j+1}, a^{\mathbf{d}}_{i,j})$ and if $\mathbf{J}^{\mathbf{d}}_{i,j}$ is a singleton we may write $\mathbf{J}^{\mathbf{d}}_{i,j} = \{b^{\mathbf{d}}_{i,j}\}$ and may write $(M^{\mathbf{d}}_{i,j}, M^{\mathbf{d}}_{i+1,j}, b^{\mathbf{d}}_{i,j})$. Similarly in Definition 1.6 below.
5) We may allow $\alpha = \partial$ and or $\beta \leq \partial$, but we shall say this.

1.5 Observation. 1) The restriction in Definition 1.4(2) always gives a $\mathfrak{u}$-free (α_1, β_1)-rectangle.
2) The restriction in Definition 1.4(3) always gives a $\mathfrak{u}$-free $(\alpha_2 - \alpha_1, \beta_2 - \beta_1)$-rectangle.
3) If $\mathbf{d}$ is a $\mathfrak{u}$-free (α, β)-rectangle, <u>then</u>

(*e*) $\langle M^{\mathbf{d}}_{i,j} : i \leq \alpha \rangle$ is $\leq_{\mathfrak{u}}$-increasing continuous for $j \leq \beta$

(*f*) $\langle M^{\mathbf{d}}_{i,j} : j \leq \beta \rangle$ is $\leq_{\mathfrak{u}}$-increasing continuous for each $i \leq \alpha$.

4) In Definition 1.6 below, clause (d), when $j < \beta$ or $j = \beta \wedge (\beta$ successor) follows from (b). Similarly for the pair of clauses (e),(c).
5) Assume that $\alpha_1 \leq \alpha_2, \beta_1 \geq \beta_2$ and $\mathbf{d}_\ell$ is $\mathfrak{u}$-free $(\alpha_\ell, \beta_\ell)$-rectangle for $\ell = 1, 2$ and $\mathbf{d}_1 \restriction (\alpha_1, \beta_2) = \mathbf{d}_2 \restriction (\alpha_1, \beta_2)$ and $M^{\mathbf{d}_1}_{\alpha_1,\beta_1} \cap M^{\mathbf{d}_2}_{\alpha_2,\beta_2} =$

$M^{\mathbf{d}_\ell}_{\alpha_1,\beta_2}$. Then we can find a $\mathfrak{u}$-free (α_2,β_1)-rectangle $\mathbf{d}$ such that $\mathbf{d} \restriction (\alpha_\ell,\beta_\ell) = \mathbf{d}_\ell$.

Proof. Immediate, e.g. in (5) we use clause (F) of Definition 1.2 for each $\alpha \in [\alpha_1,\alpha_2), \beta \in [\beta_2,\beta_1)$ in a suitable induciton. $\square_{1.5}$

1.6 Definition. We say that $\mathbf{d}$ is a $\mathfrak{u}$-free $(\bar\alpha,\beta)$-triangle or $\mathfrak{u}$-non-forking $(\bar\alpha,\beta)$-triangle when $\bar\alpha = \langle \alpha_i : i \le \beta\rangle$ is a non-decreasing[8] sequence of ordinals and (letting $\alpha := \alpha_\beta$):

- (a) $\mathbf{d} = (\langle M^{\mathbf{d}}_{i,j} : i \le \alpha_j, j \le \beta\rangle, \langle \mathbf{J}_{i,j} : i < \alpha_j, j \le \beta\rangle, \langle \mathbf{I}_{i,j} : i \le \alpha_j, j < \beta\rangle)$
- (b) $\langle (M^{\mathbf{d}}_{i,j}, M^{\mathbf{d}}_{i,j+1}, \mathbf{I}_{i,j}) : i \le \alpha_j\rangle$ is $\le_1$-increasing continuous for each $j < \beta$
- (c) $\langle (M^{\mathbf{d}}_{i,j}, M^{\mathbf{d}}_{i+1,j}, \mathbf{J}_{i,j}) : j \le \beta$ and j is such that $i+1 \le \alpha_j\rangle$ is $\le_2$-increasing continuous for each $i < \alpha$
- (d) for each $j \le \beta$ the sequence $\langle M_{i,j} : i \le \alpha_j\rangle$ is $\le_{\mathfrak{u}}$-increasing continuous
- (e) for each $i_* \in [\alpha_{j_*},\alpha), j_* \le \beta$ the sequence $\langle M_{i_*,j} : j \in [j_*,\beta]\rangle$ is $\le_{\mathfrak{K}}$-increasing continuous.

1.7 Definition/Claim. 1) For nice construction framework $\mathfrak{u}_1$ let $\mathfrak{u}_2 = \text{dual}(\mathfrak{u}_1)$ be the unique nice construction framework $\mathfrak{u}_2$ such that: $\partial_{\mathfrak{u}_2} = \partial_{\mathfrak{u}_1}, \mathfrak{K}_{\mathfrak{u}_2} = \mathfrak{K}_{\mathfrak{u}_1}$ (hence $\mathfrak{K}^{\text{up}}_{\mathfrak{u}_2} = K^{\text{up}}_{\mathfrak{u}_1}$, etc) and $(\text{FR}^{\mathfrak{u}_2}_\ell, \le^\ell_{\mathfrak{u}_2}) = (\text{FR}^{\mathfrak{u}_1}_{3-\ell}, \le^{3-\ell}_{\mathfrak{u}_1})$ for $\ell = 1,2$.
2) We call $\mathfrak{u}_1$ self-dual when $\text{dual}(\mathfrak{u}_1) = \mathfrak{u}_1$.
3) In part (1), if addition if $\mathbf{d}_1$ is $\mathfrak{u}_1$-free rectangle then there is a unique $\mathbf{d}_2 = \text{dual}(\mathbf{d}_1)$ which is a $\mathfrak{u}_2$-free rectangle such that:

$$\alpha_{\mathbf{d}_2} = \beta_{\mathbf{d}_1}, \beta_{\mathbf{d}_2} = \alpha_{\mathbf{d}_1}, M^{\mathbf{d}_2}_{i,j} = M^{\mathbf{d}_1}_{j,i}$$

$$\mathbf{I}^{\mathbf{d}_2}_{i,j} = \mathbf{J}^{\mathbf{d}_1}_{j,i} \text{ and } \mathbf{J}^{\mathbf{d}_2}_{i,j} = \mathbf{I}^{\mathbf{d}_1}_{i,j}.$$

[8]not unreasonable to demand $\bar\alpha$ to be increasing continuous

1.8 Definition. 1) We say τ is a weak $\mathfrak{u}$-sub-vocabulary <u>when</u>:

(a) $\tau \subseteq \tau_{\mathfrak{u}} = \tau_{\mathfrak{K}_{\mathfrak{u}}}$ except that $=_\tau$ is, in $\tau_{\mathfrak{u}}$, a two-place predicate such that for every $M \in \mathfrak{K}_{\mathfrak{u}}$ hence even $M \in \mathfrak{K}^{\rm up}_{\mathfrak{u}}$, the relation $=^M_\tau$ is an equivalence relation on $\mathrm{Dom}(=^M_\tau) = \{a : a =_\tau b \vee b =_\tau a$ for some $b \in M\}$ and is a congruence relation for all $R^M \restriction \mathrm{Dom}(=^M_\tau), F^M \restriction \mathrm{Dom}(=^M_\tau)$ for $R, F \in \tau$ and F^M maps[9] the set $\mathrm{Dom}(=^M_\tau)$ into itself for any function symbol $F \in \tau$.

So
1A) For $M \in K^{\rm up}_{\mathfrak{u}}$, the model $M^{[\tau]}$ is defined naturally, e.g. with universe
$\mathrm{Dom}(=^M_\tau)/=^M_\tau$ and K^τ, K^τ_μ are defined accordingly. Let $M_1 \cong_\tau M_2$ means $M^{[\tau]}_1 \cong M^{[\tau]}_2$.
1B) Let $\dot{I}_\tau(\lambda, K^{\rm up}_{\mathfrak{u}}) = \{M^{[\tau]}/\cong : M \in K^{\mathfrak{u}}_\lambda$ and $M^{[\tau]}$ has cardinality $\lambda\}$.
1C) We say that τ is a strong $\mathfrak{u}$-sub-vocabulary <u>when</u> we have clause (a) from above and[10]

(b) if $(M, N, \mathbf{I}) \in \mathrm{FR}^+_1$ <u>then</u> for some $c \in \mathbf{I} \cap \mathrm{Dom}(=^N_\tau)$ we have $N \models$ "$\neg(c =_\tau d)$" for every $d \in M$

(c) if $(M_1, N_1, \mathbf{I}_1) \le_1 (M_2, N_2, \mathbf{I}_2)$ and $c \in \mathbf{I}_1$ is as in clause (b) for $(M_1, N_1, \mathbf{I}_1)$ <u>then</u> $c \in \mathbf{I}_2$ is as in (b) for $(M_2, N_2, \mathbf{I}_2)$.

2) We say that N_1, N_2 are τ-isomorphic over $\langle M_i : i < \alpha\rangle$ <u>when</u>: $\ell \in \{1, 2\} \wedge i < \alpha \Rightarrow M_i \le_{\mathfrak{u}} N_\ell$ and there is a τ-isomorphism f

---

[9] we may better ask less: for $F \in \tau$ a function symbol letting $n = \mathrm{arity}_\tau(F)$, so $F^{M^{[\tau]}}$ is a function with n-place from $\mathrm{Dom}(=^M_\tau)/=^M_\tau$ to itself and F^M is $\{(a_0, a_1, \ldots, a_n) : (a_0/=^M = F(a_1/=^M), \ldots, a_n/=^M)\}$, i.e. the graph of $M^{[\tau]}$, so we treat F as an $(\mathrm{arity}_\tau(F)+1)$-place predicate; neither real change nor a real gain

[10] note that it is important for us that the model we shall construct will be of cardinality ∂^+; this clause will ensure that the approximations will be of cardinality ∂ for $\alpha < \partial^+$ large enough and the final model (i.e. for $\alpha = \partial^+$) will be of cardinality ∂^+. This is the reason for a preference to $\le_1$, however there is no real harm in demanding clauses (b) + (c) for $\ell = 2$, too. But see 1.9, i.e. if $|\tau| \le \partial$ and we get $\mu > 2^\partial$ pairwise non-isomorphic models of cardinality $\le \partial^+$, clearly only few (i.e. $\le 2^\partial$) of them have cardinal $< \partial^+$; so this problem is not serious to begin with.

of N_1 onto N_2 over $\bigcup_{i<\alpha} M_i$ which means: f is an isomorphism from $N_1^{[\tau]}$ onto $N_2^{[\tau]}$ which is the identity on the universe of $M_i^{[\tau]}$ for each $i < \alpha$.
2A) In part (2), if $\alpha = 1$ we may write M_0 instead $\langle M_i : i < 1\rangle$ and we can replace M_0 by a set $\subseteq N_1 \cap N_2$. If $\alpha = 0$ we may omit "over $\bar{M}$".
3) We say that N_1, N_2 are τ-incompatible extensions of $\langle M_i : i < \alpha\rangle$ when:

- (a) $M_i \leq_{\mathfrak{u}} N_\ell$ for $i < \alpha, \ell = 1, 2$
- (b) if $N_\ell \leq_{\mathfrak{u}} N'_\ell$ for $\ell = 1, 2$ then N'_1, N'_2 are not τ-isomorphic over $\langle M_i : i < \alpha\rangle$.

4) We say that N_2^1, N_2^2 are τ-incompatible (disjoint) amalgamations of N_1, M_2 over M_1 when ($N_1 \cap M_2 = M_1$ and):

- (a) $M_1 \leq_{\mathfrak{u}} N_1 \leq_{\mathfrak{u}} N_2^\ell$ and $M_1 \leq_{\mathfrak{u}} M_2 \leq_{\mathfrak{u}} N_2^\ell$ for $\ell = 1, 2$ (equivalently $M_1 \leq_{\mathfrak{u}} N_1 \leq_{\mathfrak{u}} N_2^\ell, M_1 \leq_{\mathfrak{u}} M_2 \leq_{\mathfrak{u}} N_2^\ell$)
- (b) if $N_2^\ell \leq_{\mathfrak{u}} N_2^{\ell,*}$ for $\ell = 1, 2$ then $(N_2^{1,*})^{[\tau]}, (N_2^{2,*})^{[\tau]}$ are not τ-isomorphic over $M_2 \cup N_1$, i.e. over $M_2^{[\tau]} \cup N_1^{[\tau]}$.

5) We say τ is a $\mathfrak{K}$-sub-vocabulary or K-sub-vocabulary when clause (a) of part (1) holds replacing $K_{\mathfrak{u}}$ by K; similarly in parts (1A),(1B).

1.9 Observation. Concerning 1.8(1B) we may be careless in checking the last condition, $= \lambda$, i.e. $\leq \lambda$ usually suffice, because if $|\{M^{[\tau]}/\cong: M \in K_\lambda, \|M^{[\tau]}\| < \lambda\}| < \mu$ then in proving $\dot{I}_\tau(\lambda, K_{\mathfrak{u}}^{\rm up}) \geq \mu$ we may omit it.

Remark. 1) But we give also remedies by FR_ℓ^+, i.e., clause (c) of 1.8(1).
2) We also give reminders in the phrasing of the coding properties.
3) If $|\tau| < \lambda$ and $2^{<\lambda} < \lambda$ the demand in 1.9 holds.

Proof. Should be clear. $\square_{1.9}$

1.10 Definition. 1) For any $(< \partial)$-a.e.c. $\mathfrak{K}$ let $\mathfrak{K}'$ be the $(< \partial)$-a.e.c. defined like $\mathfrak{K}$ only adding the two-place predicate $=_\tau$, demanding it to be a congruence relation, i.e.

- (a) $\tau' = \tau(\mathfrak{K}') = \tau \cup \{=_\tau\}$ where $\tau = \tau(\mathfrak{K})$
- (b) $K' = \{M : M$ is a τ'-model, $=^M_\tau$ is a congruence relation and $M/=^M_\tau$ belongs to κ and $\|M\| < \partial\}$
- (c) $M \le_{\mathfrak{K}'} N$ <u>iff</u> $M \subseteq N$ and the following function is a $\le_{\mathfrak{K}}$-embedding of $M/=^M_\tau$ into $N/=^N_\tau$: $f(a/=^M_\tau) = a/=^N_\tau$

(see Definition 1.8(1)).
1A) Similarly for $\mathfrak{K}$ an a.e.c. or a λ-a.e.c.
2) This is a special case of Definition 1.8.
3) We can interpret $M \in K$ as $M' \in K'$ just letting $M' \restriction \tau = M, =^{M'}_\tau$ is equality on $|M|$.
4) A model $M' \in \mathfrak{K}$ is called $=_\tau$-full when $a \in M' \Rightarrow \|M'\| = |\{b \in M' : M' \models a =_\tau b\}|$.
5) A model $M' \in \mathfrak{K}$ is called $(\lambda, =_\tau)$-full <u>when</u> $a \in M' \Rightarrow \lambda \le |\{b \in M' : M' \models \text{“}a =_\tau b\text{”}\}|$.
6) A model M' is called $=_\tau$-fuller <u>when</u> it is $=_\tau$-full and $\|M'\|$ is the cardinality of $M'/=^{M'}_\tau$.

1.11 Claim. *Assume $\mathfrak{K}$ is*[11] *a $(< \partial)$-a.e.c. and*[12] *$\mathfrak{K}'$ is from 1.10 and $\lambda < \partial$.*
0) $\mathfrak{K}'_\lambda$ is a λ-a.e.c.
1) If $M', N' \in K'_\lambda$ <u>then</u> $(M'/=^{M'}_\tau) \in K_{\le\lambda}$ and $(N'/=^{N'}_\tau) \in K_{\le\lambda}$ and if in addition $M' \le_{\mathfrak{K}'_\lambda} N'$ <u>then</u> (up to identifying $a/=^{M'}_\tau$ with $q/=^{N'}_\tau$) we have $(M'/=^{M'}_\tau) \le_{\mathfrak{K}_{\le\lambda}} (N'/=^{N'}_\tau)$, i.e. $(M')^{[\tau]} \le_{\mathfrak{K}} (N')^{[\tau]}$.
2) If $M' \subseteq N'$ are $\tau'_{\mathfrak{K}}$-models of cardinality λ and $=^{M'}_\tau, =^{N'}_\tau$ are congruence relation on $M' \restriction \tau_{\mathfrak{K}}, N' \restriction \tau_{\mathfrak{K}}$ respectively, <u>then</u>

- (a) $M' \in K'_{<\partial}$ <u>*iff*</u> $(M'/=^{M'}_\tau) \in K_{<\partial}$

[11] We can use $\mathfrak{K}$ is an a.e.c. and have similar results.
[12] now pedantically $\mathfrak{K}$ may be both a $(< \partial_1)$-a.e.c. and a $(< \partial_2)$-a.e.c., e.g. if $\partial_2 = \partial_1^+, K_{\partial_1} = \emptyset$, so really ∂ should be given

(b) *$M' \le_{\mathfrak{K}'_{<\partial}} N'$ iff $(M'/=^{M'}_\tau) \le_{\mathfrak{K}_{<\partial}} (N'/=^{N'}_\tau)$; pedantically*[13] *$(M'/=^{N'}_\tau) \le_{\mathfrak{K}_\lambda} (N'/=^{N'}_\tau)$ or make the natural identification*

(c) *if M', N' are $=_\tau$-fuller then in clauses (a),(b) we can replace "$\le \lambda$" by "$= \lambda$"*

(d) *if $K_{<\lambda} = \emptyset$ then in clauses (a),(b) we can replace "$\le \lambda$" by "$= \lambda$".*

3) $\mathfrak{K}'$ has disjoint amalgamation if $\mathfrak{K}$ has amalgamation.
4) $K_\lambda \subseteq K'_\lambda$ and $\le_{\mathfrak{K}_\lambda} = \le_{\mathfrak{K}'_\lambda} \restriction K_\lambda$.
5) $(\mathfrak{K}^{\rm up}_{<\mu})' = (\mathfrak{K}'_{<\mu})^{\rm up}$ for any μ so we call it $K'_{<\mu}$ and $\mathfrak{K}'_\mu = (\mathfrak{K}'_{<\mu^+})_\mu$.
6) For every μ,

(a) *$\dot{I}(\mu, K) = |\{M'/\cong : M' \in K'_\mu$ is $=_\tau$-fuller$\}|$*

(b) *$K_\mu = \{M'/=^{M'}_\tau : M' \in K'_\mu$ is μ-fuller$\}$ under the natural identification*

(c) *$K_{\le\mu} = \{M/=^M_\tau : M \in K'_\mu\}$ under the natural identification*

(d) *if $M', N^1 \in K'_\mu$ are μ-full then $M' \cong N' \Leftrightarrow (M'/=^{M'}_\tau) \cong (N'/=^{N'}_\tau)$.*

Proof. Straight. $\square_{1.11}$

1.12 Exercise: Assume $\mathfrak{K}, \mathfrak{K}'$ are as in 1.10.
1) If $\lambda \ge |\tau_{\mathfrak{K}}|$ and $2^\lambda < 2^{\lambda^+}$ then $\dot{I}(\lambda^+, \mathfrak{K}) + 2^\lambda = \dot{I}(\lambda^+, \mathfrak{K}') + 2^\lambda$, so if $\dot{I}(\lambda^+, \mathfrak{K}) > 2^\lambda$ or $\dot{I}(\lambda^+, \mathfrak{K}') > 2^\lambda$ then they are equal.
2) If $\lambda > |\tau_{\mathfrak{K}}|$ and $2^{<\lambda} < 2^\lambda$ then $\dot{I}(\lambda, \mathfrak{K}) + 2^{<\lambda} = \dot{I}(\lambda^+, \mathfrak{K}') + 2^{<\lambda}$ (and as above).

Remark. Most of our examples satisfies montonicity, see below. But not so $\mathrm{FR}_1, \le_1$ in §4(C).

1.13 Exercise: Let $\mathfrak{u}$ be a nice construction framework, as usual.
1) [Definition] We say $\mathfrak{u}$ satisfies $(\mathrm{E})_\ell(\mathrm{e})$, monotonicity, when:

[13] or define when f is a $\le_{\mathfrak{K}'_{<\partial}}$-embedding of M^1 into N'

$(E)_\ell(e)$ if $(M, N, \mathbf{J}) \in \mathrm{FR}^\ell_{\mathfrak{u}}$ and $N \le_{\mathfrak{u}} N'$
then $(M, N, \mathbf{J}) \le^\ell_{\mathfrak{u}} (M, N, \mathbf{J}') \in \mathrm{FR}^\ell_{\mathfrak{u}}$.

1A) Let (E)(e) mean $(\mathrm{E})_1(\mathrm{e})$ + $(\mathrm{E})_2(\mathrm{e})$.
2) [Claim] Assume $\mathfrak{u}$ has monotonicity.

Assume $\mathbf{d}$ is a $\mathfrak{u}$-free (α_2, β_2)-rectangle, h_1, h_2 is an increasing continuous function from α_1+1, β_1+1 into α_2+1, β_2+1 respectively. Then $\mathbf{d}'$ is a $\mathfrak{u}$-free rectangle where we define $\mathbf{d}'$ by:

- (a) $\alpha(\mathbf{d}') = \alpha_1, \beta(\mathbf{d}') = \beta_1$
- (b) $M^{\mathbf{d}'}_{i,j} = M^{\mathbf{d}}_{h_1(i),h_2(1)}$ if $i \le \alpha_1, j \le \beta_1$
- (c) $\mathbf{J}^{\mathbf{d}'}_{i,j} = \mathbf{J}^{\mathbf{d}}_{h_1(i),h_2(j)}$ for $i \le \alpha_1, j < \beta_1$
- (d) $\mathbf{J}^{\mathbf{d}'}_{i,j} = \mathbf{J}^{\mathbf{d}}_{h_1(i),h_2(j)}$ for $i < \alpha_1, j \le \beta_1$.

3) [Claim] Phrase and prove the parallel of part (2) for $\mathfrak{u}$-free triangles.

1.14 Observation. Assume $\mathfrak{u}$ is a nice construction framework except that we omit clauses $(\mathrm{D})_\ell(\mathrm{e})$ + $(\mathrm{E})_\ell(\mathrm{d})$ for $\ell = 1, 2$ but satisfying Claim 1.3. We can show that $\mathfrak{u}'$ is a nice construction framework where we define $\mathfrak{u}'$ like $\mathfrak{u}$ but, for $\ell = 1, 2$, we replace $\mathrm{FR}_\ell, \le_\ell$ by $\mathrm{FR}'_\ell, \le'_\ell$ defined as follows:

- (a) $\mathrm{FR}'_\ell = \{(M_1, M_2, \mathbf{J}) : (M_1, M_2, \mathbf{J}) \in \mathrm{FR}_\ell$ or $M_1 \le_{\mathfrak{u}} M_2 \in K_{<\partial}$ and $\mathbf{J} = \emptyset\}$
- (b) $\le'_\ell = \{((M_1, N_1, \mathbf{J}'), (M_2, N_2, \mathbf{J}'')) : (M_1, N_1, \mathbf{J}') \le^\ell_{\mathfrak{u}} (M_2, N_2, \mathbf{J}'')$ <u>or</u> $M_1 \le_{\mathfrak{u}} N_1$, $\mathbf{J}' = \emptyset$, $M_1 \le_{\mathfrak{u}} M_2$, $N_1 \le_{\mathfrak{u}} N_2$, $N_1 \cap M_2 = M_1$ and $(M_2, N_2, \mathbf{J}'') \in \mathrm{FR}_\ell$ <u>or</u> $M_1 \le_{\mathfrak{u}} N_1 \le_{\mathfrak{u}} N_2$, $M_1 \le_{\mathfrak{u}} M_2 \le_{\mathfrak{u}} N_2$, $N_1 \cap M_2 = M_1$ and $\mathbf{J}' = \emptyset = \mathbf{J}''\}$.

Proof. Clauses (A),(B),(C) does not change, most subclauses of $(\mathrm{D})_\ell$(a),(b),(d),$(\mathrm{E})_\ell$(a),(b) hold by the parallel for $\mathfrak{u}$ and the choice of $\mathrm{FR}'_\ell, \le'_\ell$; clauses $(\mathrm{D})_\ell(\mathrm{e})$ and $(\mathrm{E})_\ell(\mathrm{d})$ holds by the choice of $(\mathrm{FR}'_\ell, \le'_\ell)$; and clause (F) holds by clause (F) for $\mathfrak{u}$ and Claim 1.3. Lastly

<u>Condition $(\mathrm{E})_\ell(\mathrm{c})$</u>:

So assume $\langle (M_i, N_i, \mathbf{J}_i) : i < \delta \rangle$ be increasing continuous, where δ is a limit ordinal; and let $(M_\delta, N_\delta, \mathbf{J}_\delta) = (\cup\{M_i : i < \delta\}, \cup\{N_i : i < \delta\}, \cup\{\mathbf{J}_i : i < \delta\})$.

First, assume $i < \delta \Rightarrow \mathbf{J}_i = \emptyset$ hence $\mathbf{J}_\delta = \emptyset$ and the desired conclusion holds trivially (by the properties of a.e.c. and our definition of $\mathfrak{u}'$).

Second, assume $i < \delta \not\Rightarrow \mathbf{J}_i \neq \emptyset$ hence $j :=\ \min\{i : \mathbf{J}_i \neq \emptyset\}$ is well defined and let $\delta' = \delta - j$, it is a limit ordinal. Now use the "$\mathfrak{u}$ satisfies the Condition $(\mathrm{E})_\ell(\mathrm{c})$" for the sequence $\langle (M_{j+i}, N_{j+i}, \mathbf{J}_{j+i} : i < \delta' \rangle$ and $\leq^\ell_{\mathfrak{u}}$ being transitive.

$\square_{1.14}$

*   *   *

Now we define the approximations of size ∂; note that the notation $\leq_{\mathrm{qt}}$ and the others below hint that they are quasi orders, this will be justified later in 1.19(2), <u>but</u> not concerning $\leq^{\mathrm{at}}_{\mathfrak{u}}$. On the existence of canonical limits see 1.19(4).

1.15 Definition. 1) We let $K^{\mathrm{qt}}_\partial = K^{\mathrm{qt}}_{\mathfrak{u}}$ be the class of triples $(\bar M, \bar{\mathbf{J}}, \mathbf{f})$ such that

- (a) $\bar M = \langle M_\alpha : \alpha < \partial \rangle$ is $\leq_{\mathfrak{u}}$-increasing continuous, so $M_\alpha \in K_{\mathfrak{u}}\,(= K^{\mathfrak{u}}_{<\partial})$
- (b) $\bar{\mathbf{J}} = \langle \mathbf{J}_\alpha : \alpha < \partial \rangle$
- (c) $\mathbf{f} \in {}^\partial\partial$
- (d) $(M_\alpha, M_{\alpha+1}, \mathbf{J}_\alpha) \in\ \mathrm{FR}_2$ for $\alpha < \partial$.

1A) We call $(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ non-trivial <u>if</u> for stationarily many $\delta < \partial$ for some $i < \mathbf{f}(\delta)$ we have $\mathbf{J}_{\delta+i} \neq \emptyset$ that is $(M_{\delta+i}, M_{\delta+i+1}, \mathbf{J}_{\delta+i}) \in \mathrm{FR}^+_2$.

1B) If $\mathscr{D}$ is a normal filter on ∂ let $K^{\mathrm{qt}}_{\mathscr{D}} = K^{\mathrm{qt}}_{\mathfrak{u},\mathscr{D}}$ be the class of triples $(\bar M, \bar J, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ such that

- (e) $\{\delta < \partial : \mathbf{f}(\delta) = 0\} \in \mathscr{D}$.

1C) When we have $(\bar M^x, \bar{\mathbf{J}}^x, \mathbf{f}^x)$ then $M^x_\alpha, \mathbf{J}^x_\alpha$ for $\alpha < \partial$ has the obvious meaning and M^x_∂ or just M^x is $\cup\{M^x_\alpha : \alpha < \partial\}$

2) We define the two-place relation $\leq_{\rm qt} = \leq^{\rm qt}_{\mathfrak u}$ on $K^{\rm qt}_{\mathfrak u}$ as follows: $(\bar M^1, \bar{\mathbf J}^1, \mathbf f^1) \leq^{\rm qt}_{\mathfrak u} (\bar M^2, \bar{\mathbf J}^2, \mathbf f^2)$ <u>if</u> they are equal (and $\in K^{\rm qt}_{\mathfrak u}$) or for some club E of ∂ (a witness) we have:

- (a) $(\bar M^k, \bar{\mathbf J}^k, \mathbf f^k) \in K^{\rm qt}_{\mathfrak u}$ for $k = 1, 2$
- (b) $\delta \in E \Rightarrow \mathbf f^1(\delta) \leq \mathbf f^2(\delta)$
- (c) $\delta \in E \ \& \ i \leq \mathbf f^1(\delta) \Rightarrow M^1_{\delta+i} \leq_{\mathfrak u} M^2_{\delta+i}$
- (d) $\delta \in E \ \& \ i < \mathbf f^1(\delta) \Rightarrow$
  $\Rightarrow (M^1_{\delta+i}, M^1_{\delta+i+1}, \mathbf J^1_{\delta+i}) \leq_2 (M^2_{\delta+i}, M^2_{\delta+i+1}, \mathbf J^2_{\delta+i})$
- (e) $\delta \in E \ \& \ i \leq \mathbf f^1(\delta) \Rightarrow M^2_{\delta+i} \cap \bigcup\limits_{\alpha<\partial} M^1_\alpha = M^1_{\delta+i}$, disjointness.

3) We define the two place relation $\leq_{\rm at} = \leq^{\rm at}_{\mathfrak u}$ on $K^{\rm qt}_{\mathfrak u}$: $(\bar M^1, \bar{\mathbf J}^1, \mathbf f^1) \leq^{\rm at}_{\mathfrak u} (\bar M^2, \bar{\mathbf J}^2, \mathbf f^2)$ <u>if</u> for some club E of ∂ and $\bar{\mathbf I}$ (the witnesses) we have (a)-(e) as in part (2) and

- (f) $\bar{\mathbf I} = \langle \mathbf I_\alpha : \alpha < \partial\rangle$ and $\langle (M^1_\alpha, M^2_\alpha, \mathbf I_\alpha) : \alpha \in \cup\{[\delta, \delta + \mathbf f^1(\delta)] : \delta \in E\}\rangle$ is $\leq_1$-increasing continuous, so we may use $\langle \mathbf I_\alpha : \alpha \in \cup\{[\delta, \delta + \mathbf f^1(\delta)] : \delta \in E\rangle$ only.

3A) We say $(\bar M^1, \bar{\mathbf J}^1, \mathbf f^1), (\bar M^2, \bar{\mathbf J}^2, \mathbf f^2)$ are equivalent when for a club of $\delta < \partial$ we have $\mathbf f^1(\delta) = \mathbf f^2(\delta)$ and $i \leq \mathbf f^1(\delta) \Rightarrow M^1_{\delta+i} = M^2_{\delta+i}$ and $i < \mathbf f^1(\delta) \Rightarrow \mathbf J^1_{\delta+i} = \mathbf J^2_{\delta+i}$.
3B) Let $(\bar M^1, \bar{\mathbf J}^1, \mathbf f^1) <^{\rm at}_{\mathfrak u} (\bar M^2, \bar{\mathbf J}^2, \mathbf f^2)$ mean that in part (3) in addition

- (g) for some $\alpha \in \cup\{[\delta, \delta + \mathbf f(\delta)] : \delta \in E\}$ the triple $(M^1_\alpha, M^2_\alpha, \mathbf I_\alpha)$ belongs to ${\rm FR}^+_1$.

4) We say $(\bar M^\delta, \bar{\mathbf J}^\delta, \mathbf f^\delta)$ is a canonical limit of $\langle (\bar M^\alpha, \bar{\mathbf J}^\alpha, \mathbf f^\alpha) : \alpha < \delta\rangle$ <u>when</u>:

- (a) $\delta < \partial^+$
- (b) $\alpha < \beta < \delta \Rightarrow (\bar M^\alpha, \bar{\mathbf J}^\alpha, \mathbf f^\alpha) \leq^{\rm qt}_{\mathfrak u} (\bar M^\beta, \bar{\mathbf J}^\beta, \mathbf f^\beta)$
- (c) for some increasing continuous sequence $\langle \alpha_\varepsilon : \varepsilon < {\rm cf}(\delta)\rangle$ of ordinals with limit δ we have:

Case 1: $\mathrm{cf}(\delta) < \partial$.

For some club E of ∂ we have:

(α) $\zeta \in E \Rightarrow \mathbf{f}^\delta(\zeta) = \sup\{\mathbf{f}^{\alpha_\varepsilon}(\zeta) : \varepsilon < \mathrm{cf}(\delta)\}$;

(β) $\zeta \in E$ & $\xi < \mathrm{cf}(\delta)$ & $i \le \mathbf{f}^{\alpha_\xi}(\zeta)$ implies that $M^\delta_{\zeta+i} = \cup\{M^{\alpha_\varepsilon}_{\zeta+i} : \varepsilon < \mathrm{cf}(\delta)$ satisfies $\varepsilon \ge \xi\}$;

(γ) $\zeta \in E$ & $\xi < \mathrm{cf}(\delta)$ & $i < \mathbf{f}^\xi(\zeta)$ implies that $\mathbf{J}^\delta_{\zeta+i} = \cup\{\mathbf{J}^{\alpha_\varepsilon}_{\zeta+i} : \varepsilon < \mathrm{cf}(\delta)$ satisfies $\varepsilon \ge \xi\}$

(δ) if $\zeta \in E$ and $j = \mathbf{f}^\delta(\zeta) > \mathbf{f}^{\alpha_\varepsilon}(\zeta)$ for every $\varepsilon < \mathrm{cf}(\delta)$ then $M^\delta_{\zeta+j} = \cup\{M^\delta_{\zeta+i} : i < j\}$.

Case 2: $\mathrm{cf}(\delta) = \partial$.

Similarly, using diagonal unions.

4A) We say $\langle(\bar M^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha < \alpha(*)\rangle$ is $\le^{\mathrm{qt}}_{\mathfrak{u}}$-increasing continuous when it is $\le^{\mathrm{qt}}_{\mathfrak{u}}$-increasing and for every limit ordinal $\delta < \alpha(*)$, the triple $(\bar M^\delta, \bar{\mathbf{J}}^\delta, \mathbf{f}^\delta)$ is a canonical limit of $\langle(\bar M^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha < \delta\rangle$.

5) We define the relation $\le_{\mathrm{qs}} = \le^{\mathrm{qs}}_{\mathfrak{u}}$ on $K^{\mathrm{qt}}_{\mathfrak{u}}$ by:
$(\bar M', \bar{\mathbf{J}}', \mathbf{f}') \le_{\mathrm{qs}} (\bar M'', \bar{\mathbf{J}}'', \mathbf{f}'')$ if there is a $\le^{\mathrm{at}}_{\mathfrak{u}}$-tower $\langle(\bar M^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha \le \alpha(*)\rangle$ witnessing it, meaning that is is a sequence such that:

(a) the sequence is $\le^{\mathrm{qt}}_{\mathfrak{u}}$-increasing of length $\alpha(*) + 1 < \partial^+$

(b) $(\bar M^0, \bar{\mathbf{J}}^0, \mathbf{f}^0) = (\bar M', \bar{\mathbf{J}}', \mathbf{f}')$

(c) $(\bar M^{\alpha(*)}, \bar{\mathbf{J}}^{\alpha(*)}, \mathbf{f}^{\alpha(*)}) = (\bar M'', \bar{\mathbf{J}}'', \mathbf{f}'')$

(d) $(\bar M^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) \le^{\mathrm{at}}_{\mathfrak{u}} (\bar M^{\alpha+1}, \bar{\mathbf{J}}^{\alpha+1}, \mathbf{f}^{\alpha+1})$ for $\alpha < \alpha(*)$

(e) if $\delta \le \alpha(*)$ is a limit ordinal then $(\bar M^\delta, \bar{\mathbf{J}}^\delta, \mathbf{f}^\delta)$ is a canonical limit of $\langle(\bar M^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha < \delta\rangle$.

5A) Let $<_{\mathrm{qs}} = <^{\mathrm{qs}}_{\mathfrak{u}}$ be defined similarly but for at least one $\alpha < \alpha(*)$ we have $(\bar M^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) <^{\mathrm{at}}_{\mathfrak{u}} (\bar M^{\alpha+1}, \bar{\mathbf{J}}^{\alpha+1}, \mathbf{f}^{\alpha+1})$.

5B) Let $\le_{\mathrm{qr}} = \le^{\mathrm{qr}}_{\mathfrak{u}}$ be defined as in part (5) but in clause (d) we use $<^{\mathrm{at}}_{\mathfrak{u}}$. Similarly for $<_{\mathrm{qr}} = <^{\mathrm{qr}}_{\mathfrak{u}}$, i.e. when $\alpha(*) > 0$.

6) We say that $\langle(\bar M^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha < \alpha(*)\rangle$ is $\le^{\mathrm{qs}}_{\mathfrak{u}}$-increasing continuous when it is $\le^{\mathrm{qs}}_{\mathfrak{u}}$-increasing and clause (e) of part (5) holds. Similarly for $\le^{\mathrm{qr}}_{\mathfrak{u}}$.

Some obvious properties are (see more in Observation 1.19).

1.16 Observation. 1) $(\bar{M}^1, \bar{\mathbf{J}}^1, \mathbf{f}^1) \le^{\rm at}_{\mathfrak{u}} (\bar{M}^2, \bar{\mathbf{J}}^2, \mathbf{f}^2)$ iff for some club E of ∂ and sequence $\bar{\mathbf{I}} = \langle \mathbf{I}_\alpha : \alpha \in \cup\{[\delta, \delta + \mathbf{f}^1(\delta)] : \alpha \in E\}\rangle$ we have clause (a),(b),(c) of Definition 1.15(2) and

- $(f)'$ $(\langle M^1_{\delta+i} : i \le \mathbf{f}^1(\delta)\rangle, \langle M^2_{\delta+i} : i \le \mathbf{f}^1(\delta)\rangle, \langle \mathbf{J}^1_{\delta+i} : i < \mathbf{f}^1(\delta)\rangle, \langle \mathbf{J}^2_{\delta+i} : i < \mathbf{f}^1(\delta)\rangle, \langle \mathbf{I}_{\delta+i} : i \le \mathbf{f}^1(\delta)\rangle)$ is a $\mathfrak{u}$-free $(\mathbf{f}^1(\delta), 1)$-rectangle
- $(f)'_1$ if $\delta_1 < \delta_2$ are from E then $(M^1_{\delta_1+\mathbf{f}^1(\delta_1)}, M^2_{\delta_1+\mathbf{f}^1(\delta_1)}, \mathbf{I}_{\delta_1+\mathbf{f}^1(\delta_1)}) \le_2 (M^1_{\delta_2}, M^2_{\delta_2}, \mathbf{I}_{\delta_2})$.

2) The relation $\le^{\rm qt}_{\mathfrak{u}}, \le^{\rm at}_{\mathfrak{u}}, \le^{\rm qs}_{\mathfrak{u}}, \le^{\rm qr}_{\mathfrak{u}}$ and $\le^{\rm at}_{\mathfrak{u}}$ are preserved by equivalence, see Definition 1.15(3A) (and equivalence is an equivalence relation) and so are $<_{\rm at}, <_{\rm qt}, <_{\rm qs}, <_{\rm qr}$.

Proof. Straightforward. $\square_{1.16}$

1.17 Remark. 1) In some of our applications it is natural to redefine the partial order $\le^{\rm qs}_{\mathfrak{u}}$ we use on $K^{\rm qt}_{\mathfrak{u}}$ as the closure of a more demanding relation.
2) If we demand $\mathrm{FR}_1 = \mathrm{FR}^+_1$ hence we omit clause $(D)_1(e), (E)_1(d)$ of Definition 1.2, really $<^{\rm at}_{\mathfrak{u}}$ is the same as $\le^{\rm at}_{\mathfrak{u}}$. In Definition 1.15(3) we can choose $\mathbf{I}_\alpha = \emptyset$, then we get $\le^{\rm qt}_{\mathfrak{u}}$. But even so we would like to be able to say "repeat §1 with the following modifications". If in Definition 1.15(5) clause (d) we use $<^{\rm at}_{\mathfrak{u}}$, i.e. use $<^{\rm qr}_{\mathfrak{u}}$, the difference below is small.
3) Note that below $K^{\mathfrak{u},*}_\partial \subseteq K^{\rm up}_{\mathfrak{u},\partial}$ and $K^{\mathfrak{u},*}_{\partial^+} \subseteq K^{\rm up}_{\mathfrak{u},\partial^+}$, see the definition below.
4) Should we use $\le^{\rm qs}_{\mathfrak{u}}$ or $\le^{\rm qr}_{\mathfrak{u}}$ (see Definition 1.15(5A),(5B))? So far it does not matter.

1.18 Definition. 1) $K^{\mathfrak{u},*}_\partial = \{M : M = \cup\{M_\alpha : \alpha < \partial\}$ for some non-trivial $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}\}$, recalling Definition 1.15(1A).
2) $K^{\mathfrak{u},*}_{\partial^+} = \{\bigcup\{M^\gamma : \gamma < \partial^+\} : \langle(\bar{M}^\gamma, \bar{\mathbf{J}}^\gamma, \mathbf{f}^\gamma) : \gamma < \partial^+\rangle$ is $\le^{\rm qs}_{\mathfrak{u}}$-increasing continuous and for unboundedly many $\gamma < \partial^+$ we have $(\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) <^{\rm qs}_{\mathfrak{u}} (\bar{M}^{\alpha+1}, \bar{\mathbf{J}}^{\alpha+1}, \mathbf{f}^{\alpha+1})$ and as usual $M^\gamma = \cup\{M^\gamma_\alpha : \alpha < \partial\}\}$.

1.19 Observation. 1) $K^{\rm qt}_{\mathfrak u} \neq \emptyset$; moreover it has non-trivial members.
2) The two-place relations $\le^{\rm qt}_{\mathfrak u}$ and $\le^{\rm qs}_{\mathfrak u}, \le^{\rm qr}_{\mathfrak u}$ are quasi orders and so are $<_{\rm qt}, <_{\rm qs}, <_{\rm qr}$ <u>but</u> not necessarily $\le_{\rm at}, <_{\rm at}$.
3) Assume $(\bar M^1, \bar{\mathbf J}^1, \mathbf f^1) \in K^{\rm qt}_{\mathfrak u}$ and $\alpha < \partial, (M^1_\alpha, M^2_\alpha, \mathbf I^*) \in$ FR$_1$ and $\mathbf f^1 \le_{\mathscr D_\partial} \mathbf f^2 \in {}^\partial\partial$ and $M^2_\alpha \cap M^1 = M^1_\alpha$. <u>Then</u> we can find $\bar M^2, \bar{\mathbf J}^2, E$ and $\bar{\mathbf I} = \langle \mathbf I_\alpha : \alpha \in \cup\{[\delta, \mathbf f^1(\delta)] : \delta \in E\}\rangle$ such that

- (a) $(\bar M^1, \mathbf J^1, \mathbf f^1) \le^{\rm at}_{\mathfrak u} (\bar M^2, \bar{\mathbf J}^2, \mathbf f^2)$ as witnessed by $E, \bar{\mathbf I}$
- (b) if $\beta \in \cup\{[\delta, \mathbf f^2(\delta)] : \delta \in E\}$ then $(M^1_\alpha, M^2_\alpha, \mathbf I^*) \le_1 (M^1_\beta, M^2_\beta, \mathbf I_\beta)$
- (c) if $(M^1_\alpha, M^2_\alpha, \mathbf I^*) \in$ FR$^+_1$ then $(\bar M^1, \bar{\mathbf J}^1, \mathbf f^1) <^{\rm qs}_{\mathfrak u} (\bar M^2, \bar{\mathbf J}^2, \mathbf f^2)$ moreover $(\bar M^1, \bar{\mathbf J}^1, \mathbf f^1) <^{\rm at}_{\mathfrak u} (\bar M^2, \bar{\mathbf J}^2, \mathbf f^2)$.

4) If $\langle(\bar M^\alpha, \bar{\mathbf J}^\alpha, \mathbf f^\alpha) : \alpha < \delta\rangle$ is $\le^{\rm qt}_{\mathfrak u}$-increasing continuous (i.e. we use canonical limits) and δ is a limit ordinal $< \partial^+$, <u>then</u> it has a canonical limit $(\bar M^\delta, \bar{\mathbf J}^\delta, \mathbf f^\delta)$ which is unique up to equivalence (see 1.15(3A)). Similarly for $\le^{\rm qs}_{\mathfrak u}$ and $\le^{\rm qr}_{\mathfrak u}$.
5) $K^{\mathfrak u,*}_\partial, K^{\mathfrak u,*}_{\partial^+}$ are non-empty and included in $K^{\mathfrak u}$, in fact in $K^{\mathfrak u}_\partial, K^{\mathfrak u}_{\partial^+}$, respectively. Also if τ is a strong $\mathfrak u$-sub-vocabulary and $M \in K^{\mathfrak u,*}_\partial$ or $M \in K^{\mathfrak u,*}_{\partial^+}$ then $M^{[\tau]}$ has cardinality ∂ or ∂^+ respectively. If τ is a weak $\mathfrak u$-subvocabulary we get only $\le \partial, \le \partial^+$ respectively.

Proof. 1) We choose $M_i \in K_{\mathfrak u} = \mathfrak K_{<\partial}, \le_{\mathfrak u}$-increasing continuous with i as follows. For $i = 0$ use $\mathfrak K_{<\partial} \neq \emptyset$ by clause (C) of Definition 1.2. For i limit note that $\langle M_j : j < i\rangle$ is $\le_{\mathfrak K}$-increasing continuous and $j < i \Rightarrow M_j \in \mathfrak K_{\mathfrak u}$ hence $j < i \Rightarrow \|M_j\| < \partial$ but $i < \partial$ and ∂ is regular (by Definition 1.2, clause (B)) so $M_i := \cup\{M_j : j < i\}$ has cardinality $< \partial$ hence (by clause (C) of Definition 1.2) $M_i \in \mathfrak K_{\mathfrak u}$ and $j < i \Rightarrow M_j \le_{\mathfrak u} M_i$. For $i = j + 1$ by clause $(D)_2(d)$ of Definition 1.2 there are $M_i, \mathbf J_j$ such that $(M_j, M_i, \mathbf J_j) \in$ FR$^+_2$. Choose $\mathbf f \in {}^\partial\partial$, e.g., $\mathbf f(\alpha) = 1$.
Now $(\langle M_i : i < \partial\rangle, \langle \mathbf J_i : i < \partial\rangle, \mathbf f) \in K^{\rm qt}_{\mathfrak u}$ is as required; moreover is non-trivial, see Definition 1.15(1A).
2) We first deal with $\le^{\rm qt}_{\mathfrak u}$.

Trivially $(\bar M, \bar{\mathbf J}, \mathbf f) \le^{\rm qt}_{\mathfrak u} (\bar M, \bar{\mathbf J}, \mathbf f)$ for $(M, \bar{\mathbf J}, \mathbf f) \in K^{\rm qt}_{\mathfrak u}$.
[Why? It is witnessed by $E = \partial$ (as $(M_i, M_i, \emptyset) \in$ FR$_1$ by clause $(D)_1(e)$ of Definition 1.2 and $i < j < \partial \Rightarrow (M_i, M_i, \emptyset) \le_1 (M_j, M_j, \emptyset)$ by clause $(E)_1(d)$ of Definition 1.2.]

So assume $(\bar{M}^\ell, \bar{\mathbf{J}}^\ell, \mathbf{f}) \le^{\rm qt}_{\mathfrak{u}} (\bar{M}^{\ell+1}, \bar{\mathbf{J}}^{\ell+1}, \mathbf{f}^\ell)$ and let it be witnessed by E_ℓ for $\ell = 1, 2$. Let $E = E_1 \cap E_2$, it is a club of ∂. For every $\delta \in E$ by clause (b) of Definition 1.15(2) we have $\mathbf{f}^1(\delta) \le \mathbf{f}^2(\delta) \le \mathbf{f}^3(\delta)$ hence $\mathbf{f}^1(\delta) \le \mathbf{f}^3(\delta)$ and for $i \le \mathbf{f}^1(\delta)$ by clause (c) of Definition 1.15(2), clearly $M^1_{\mathbf{f}^1(\delta)+i} \le_{\mathfrak{u}} M^2_{\mathbf{f}^1(\delta)+i} \le_{\mathfrak{u}} M^3_{\mathbf{f}^1(\delta)+i}$ so as $\mathfrak{K}_{\mathfrak{u}}$ is a $(< \partial)$-a.c.e. we have $M^1_{\delta+i} \le_{\mathfrak{u}} M^3_{\delta+i}$. Similarly as $\le^2_{\mathfrak{u}}$ is a quasi order by clause $(\mathrm{E})_2$(a) of Definition 1.2 we have $\delta \in E \ \& \ i < \mathbf{f}^1(\delta) \Rightarrow (M^1_{\delta+i}, M^1_{\delta+i+1}, \mathbf{J}^1_{\delta+i}) \le^2_{\mathfrak{u}} (M^3_{\delta+i}, M^3_{\delta+i+1}, \mathbf{J}^3_{\delta+i})$ so clause (d) of Definition 1.15(2) holds.

Also if $\delta \in E$ and $i \le \mathbf{f}^1(\delta)$ then $M^2_{\delta+i} \cap (\cup\{M^1_\gamma : \gamma < \partial\}) = M^1_{\delta+i}$ and $M^3_{\delta+i} \cap (\cup\{M^2_\gamma : \gamma < \partial\}) = M^2_{\delta+i}$ hence $M^3_{\delta+i} \cap (\cup\{M^1_\gamma : \gamma < \partial\}) = M^1_{\delta+i}$, i.e. clause (e) there holds and clause (a) is trivial.

Together really $(\bar{M}^1, \bar{\mathbf{J}}^1, \mathbf{f}^1) \le^{\rm qt}_{\mathfrak{u}} (\bar{M}^3, \bar{\mathbf{J}}^3, \mathbf{f}^3)$.
So $\le^{\rm qt}_{\mathfrak{u}}$ is actually a quasi order. As for $\le^{\rm qs}_{\mathfrak{u}}$ and $\le^{\rm qr}_{\mathfrak{u}}$, this follows by the result on $\le^{\rm qt}_{\mathfrak{u}}$ and the definitions. Similarly for $<^{\rm qt}_{\mathfrak{u}}, <^{\rm qs}_{\mathfrak{u}}, <^{\rm qr}_{\mathfrak{u}}$.
3) By induction on $\beta \in [\alpha, \partial)$ we choose $(g_\beta, M^2_\beta, \mathbf{I}_\beta)$ and $\mathbf{J}_\beta$ (but $\mathbf{J}_\beta$ is chosen in the $(\beta+1)$-th step) such that

- (a) $M^2_\beta \in \mathfrak{K}_{\mathfrak{u}}$ is $\le_{\mathfrak{u}}$-increasing continuous
- (b) g_β is a $\le_{\mathfrak{u}}$-embedding of M^1_β into M^2_β, increasing and continuous with β
- (c) $(g_\beta(M^1_\beta), M^2_\beta, \mathbf{I}_\beta) \in \mathrm{FR}^{\mathfrak{u}}_1$ is $\le^1_{\mathfrak{u}}$-increasing and continuous
- (d) if $\beta = \gamma+1$ then $(g_\gamma(M^1_\gamma), g_\beta(M^1_\beta), g_\beta(\mathbf{J}^1_\beta)) \le^2_{\mathfrak{u}} (M^2_\gamma, M^2_\beta, \mathbf{J}^2_\beta)$.

For $\beta = \alpha$ let $g_\beta = \mathrm{id}_{M^1_\alpha}, M^2_\beta$ as given, $\mathbf{I}_\beta = \mathbf{I}^*$, so by the assumptions all is O.K.

For β limit use clause $(\mathrm{E})_1$(c) of Definition 1.2.

For $\beta = \gamma + 1$ use clause (F) of Definition 1.2. Having carried the induction, by renaming without loss of generality $g_\beta = \mathrm{id}_{M^1_\beta}$ for $\beta < \partial$. So clearly we are done.
4),5) Easy, too. $\square_{1.19}$

1.20 Definition. 1) $K^{\rm qt}_\alpha = K^{\rm qt}_{\mathfrak{u},\alpha}$ is defined as in Definition 1.15(1) above but $\alpha = \mathrm{Dom}(\mathbf{f}) = \ell g(\bar{\mathbf{J}}) = \ell g(\bar{M}) - 1$, where $\alpha \le \partial$.
2) $K^{\rm qt}_{<\alpha} = K^{\rm qt}_{\mathfrak{u},<\alpha}, K^{\rm qt}_{\le\alpha} = K^{\rm qt}_{\mathfrak{u},\le\alpha}$ are defined similarly.

* * *

1.21 Discussion: 1) Central here in Chapter are "τ-coding properties" meaning that they will help us in building $M \in K^{\mathfrak{u}}_{\partial^+}$, moreover in $K^{\mathfrak{u},*}_{\partial^+}$ (or $K^{\mathfrak{u},\mathfrak{h}}_{\partial^+}$, see below) such that we can code some subset of ∂^+ by the isomorphism type of $M^{[\tau]}$; that is during the construction, choosing $(\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) \in K^{\rm qt}_{\mathfrak{u}}$ which are $\le^{\rm qs}_{\mathfrak{u}}$-increasing with $\alpha < \partial^+$, we shall have enough free decisions. This means that, arriving to the α-th triple we have continuations which are incompatible in some sense. This will be done in §2,§3.
2) The following definition will help phrase coding properties which holds just for "almost all" triples from $K^{\rm qt}_{\mathfrak{u}}$. Note that in the weak version of coding we have to preserve $\mathbf{f}(\delta) = 0$ for enough δ's.
3) In the applications we have in mind, $\partial = \lambda^+$, the set of $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ for which $M_{\lambda^+} = \cup\{M_i : i < \lambda^+\}$ is saturated above λ, is dense enough which for our purpose means that for almost every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f})$ this holds.
4) Central in our proof will be having "for almost all $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ in some sense, satisfies ....". The first version (almost$_3$, in 1.22(0)), is related to Definition 1.24.
5) The version of Definition 1.22 we shall use mostly in 1.22(3C), "almost$_2$...", which means that for some stationary $S \subseteq \partial$, we demand the sequences to "strictly S-obey $\mathfrak{g}$"; and from Definition 1.24 is 1.24(7), "$\{0,2\}$-almost".

1.22 Definition. 0) We say that "almost$_3$ every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ satisfies Pr" when there is a function $\mathfrak{h}$ witnessing it which means:

- $(*)_1$ $(\bar{M}^\delta, \bar{\mathbf{J}}^\delta, \mathbf{f}^\delta)$ satisfies Pr when: the sequence $\mathbf{x} = \langle(\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha \le \delta\rangle$ is $\le^{\rm qs}_{\mathfrak{u}}$-increasing continuous ($\delta < \partial^+$ a limit ordinal, of course) and obeys $\mathfrak{h}$ which means that for some unbounded subset u of δ for every $\alpha \in u$ the sequence $\mathbf{x}$ does obey$_3$ or 3-obeys $\mathfrak{h}$ which means $(\bar{M}^{\alpha+1}, \bar{\mathbf{J}}^{\alpha+1}, \mathbf{f}^{\alpha+1}) = \mathfrak{h}((\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha)$ (and for notational simplicity the universes of $M^\alpha_\partial, M^{\alpha+1}_\partial$ are sets of ordinals[14]); we may write obeys instead 3-obeys when this is clear from the context; also below

[14] we may alternatively restrict yourself to models with universe $\subseteq \partial^+$ or use a universal choice function. Also if we use $\mathfrak{h}(\langle \bar{M}^\beta, \bar{\mathbf{J}}^\beta, \mathbf{f}^\beta) : \beta \le \alpha\rangle)$ the difference is minor: make the statement a little cumbersome and the checking a little easier. Presently we do not distinguish the two versions.

$(*)_2$ above $\mathbf{f}^\alpha(i) = 0 \Rightarrow \mathbf{f}^{\alpha+1}(i) = 0$ for a club of $i < \partial$ and $\{i : \mathbf{f}^\alpha(i) > 0\}$ is stationary for every $\alpha \le \delta$

$(*)_3$ $\mathfrak{h}$ has the domain and range implicit in $(*)_1 + (*)_2$

$(*)_4$ we shall restrict ourselves to a case where each of the models M^α_i above have universe $\subseteq \partial^+$, (or just be a set of ordinals) thus avoiding the problem of global choice; similarly below (e.g. in part (3)).

1) We say that the pair $((\bar M^1, \bar{\mathbf{J}}^1, \mathbf{f}^1), (\bar M^2, \bar{\mathbf{J}}^2, \mathbf{f}^2))$ does S-obey or S-obeys$_1$ the function $\mathfrak{g}$ (or $(\bar M^2, \bar{\mathbf{J}}^2, \mathbf{f}^2)$ does S-obeys or S-obeys$_1$ $\mathfrak{g}$ above $(\bar M^1, \bar{\mathbf{J}}, \mathbf{f}^1)$), <u>when</u> for some $\bar{\mathbf{I}}$ and E we have

(a) S is a stationary subset of ∂ and E is a club of ∂

(b) $(\bar M^1, \bar{\mathbf{J}}^1, \mathbf{f}^1) <^{\mathrm{at}}_{\mathfrak{u}} (\bar M^2, \bar{\mathbf{J}}^2, \mathbf{f}^2)$ as witnessed by E and $\bar{\mathbf{I}}$

(c) for stationarily many $\delta \in S$

- $\odot$ the triple $(\bar M^2 \restriction (\delta + \mathbf{f}^2(\delta) + 1), \bar{\mathbf{J}}^2 \restriction (\delta + \mathbf{f}^2(\delta)), \bar{\mathbf{I}} \restriction (\delta + \mathbf{f}^2(\delta)) + 1)$ is equal to, (in particular[15] $\mathfrak{g}$ is well defined in this case) $\mathfrak{g}(\bar M^1, \bar{\mathbf{J}}^1, \mathbf{f}^1, \bar M^2 \restriction (\delta + \mathbf{f}^1(\delta) + 1), \bar{\mathbf{J}}^2 \restriction (\delta + \mathbf{f}^1(\delta), \bar{\mathbf{I}} \restriction (\delta + \mathbf{f}^1(\delta) + 1), S)$
  or at least
- $\odot'$ for some $\gamma_1 \le \gamma_2$ from the interval $[\mathbf{f}^1(\delta), \mathbf{f}^2(\delta)]$, the triple $(\bar M^2 \restriction (\gamma_2 + 1), \bar{\mathbf{J}}_2 \restriction \gamma_2, \bar{\mathbf{I}} \restriction (\gamma_2 + 1))$ is equal to $\mathfrak{g}(\bar M^1, \bar{\mathbf{J}}^1, \mathbf{f}^1, \bar M^2 \restriction (\gamma_1 + 1)), \bar{\mathbf{J}}^2 \restriction \gamma_1, \bar{\mathbf{I}} \restriction (\gamma_1 + 1), S)$.

1A) Saying "strictly S-obeys$_1$" mean that in clause (1)(c) we replace "stationarily many $\delta \in S$" by "every $\delta \in E \cap S$ (we can add the "strictly" in other places, too). Omitting S means for some stationary $S \subseteq \partial$; we may assume $\mathfrak{g}$ codes S and in this case we write $S = S_{\mathfrak{g}}$ and can omit S. In the end of clause (1)(c), if the resulting value does not depend on some of the objects written as arguments we may omit them. We may use $\bar{\mathfrak{g}} = \langle \mathfrak{g}_S : S \subseteq \partial \text{ stationary}\rangle$ and obeying $\bar{\mathfrak{g}}$ means obeying $\mathfrak{g}_S$ for some S (where $\mathfrak{g} = \mathfrak{g}_S \Rightarrow S_{\mathfrak{g}} = S$).
2) A $\le^{\mathrm{qs}}_{\mathfrak{u}}$-increasing continuous sequence $\langle (\bar M^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha \le \delta \rangle$

[15] alternatively we can demand (as in §9,§10) that: the universe of M^1_∂ and of M^2_∂ is an ordinal $< \partial^+$

obeys$_1$ or 1-obeys $\bar{\mathfrak{g}}$ when δ is a limit ordinal $< \partial^+$ and for some unbounded $u \subseteq \delta$ there is a sequence $\langle S_\alpha : \alpha \in u \cup \{\delta\}\rangle$ of stationary subsets of ∂ decreasing modulo $\mathscr{D}_\partial$ such that for each $\alpha \in u$, the pair $((\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha), (\bar{M}^{\alpha+1}, \bar{\mathbf{J}}^{\alpha+1}, \mathbf{f}^{\alpha+1}))$ strictly S_α-obeys $\mathfrak{g}$.
2A) In part (2) we say S-obeys$_1$ when $S_\alpha \subseteq S$ mod $\mathscr{D}_\partial$ for $\alpha \in u \cup \{\delta\}$. Similarly for $\bar{S}'$-obey$_1$ when $\bar{S}' = \langle S'_\alpha : \alpha \in u'\rangle$ and $\alpha \in u \cup \{\delta\} \Rightarrow S^*_\alpha \subseteq S_\alpha$ and $u \cup \{\delta\} \subseteq u'$.
2B) In part (2) we say strictly S-obeys$_1$ when this holds in each case.
3) We say "almost$_1$ every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ satisfies Pr" <u>when</u> there is a function $\mathfrak{g}$ witnessing it, which means (note: the use of "obey" guarantees $\mathfrak{g}$ is as in part (2) and not as implicitly required on $\mathfrak{h}$ in part (0)):

- (c) if $\langle(\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha \leq \delta\rangle$ is $\leq^{\mathrm{qs}}_{\mathfrak{u}}$-increasing continuous obeying $\mathfrak{g}$ and $\delta < \partial^+$ a limit ordinal <u>then</u> $(\bar{M}^\delta, \bar{\mathbf{J}}^\delta, \mathbf{f}^\delta)$ satisfies the property Pr.

3A) We add "above $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}')$" <u>when</u> we demand in clause (c) that $(\bar{M}^0, \bar{\mathbf{J}}^0, \mathbf{f}^0) = (\bar{M}', \bar{\mathbf{J}}', \mathbf{f}')$.
3B) We replace[16] almost$_1$ by S-almost$_2$ <u>when</u> we require that the sequence "strictly S-obeys $\mathfrak{g}$".
3C) We replace[17] almost$_1$ by almost$_2$ <u>when</u> for every stationary $S \subseteq \partial, S$-almost$_1$ every triple $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ satisfies Pr; and "S-almost$_2$" we ?.

1.23 Definition. 1) For $\mathfrak{h}$ as[18] in Definition 1.22(0) we define $K^{\mathfrak{u},\mathfrak{h}}_{\partial^+}$ as the class of models M such that for some $\leq^{\mathrm{qs}}_{\mathfrak{u}}$-increasing continuous sequence $\mathbf{x} = \langle(\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha < \partial^+\rangle$ of members of $K^{\mathrm{qt}}_{\mathfrak{u}}$ such that a club of $\delta < \partial^+, \mathbf{x} \restriction (\delta + 1)$ obeys $\mathfrak{h}$ is the sense of part (0) of Definition 1.22 respectively, we have $M = \cup\{M^\alpha_\partial : \alpha < \partial^+\}$.
2) For $\mathfrak{g}$ as in Definition 1.22(1),(2) we define $K^{\mathfrak{u},\mathfrak{g}}_{\partial^+}$ similarly.
3) We call $\mathfrak{h}$ as in 1.22(0) appropriate$_3$ or 3-appropriate and $\mathfrak{g}$ as in Definition 1.22(1),(2) we call appropriate$_\ell$ or ℓ-appropriate for

[16]again assume that all elements are ordinals $< \partial^+$

[17]if we replaced it by "for a set of δ's which belongs to $\mathscr{D}$", $\mathscr{D}$ a normal filter on ∂, the difference is minor.

[18]we shall assume that no $\mathfrak{h}$ is both as required in Definition 1.22 and as required in Definition 1.23(0).

$\ell = 1, 2$; we may add "$\mathfrak{u}$-"if not clear from the context.
4) As in parts (1),(2) for $\mathfrak{h}$ as in (any relevant part of) Definition 1.24 below.
5) Also $K^{\mathfrak{u},\bar{\mathfrak{h}}}_{\partial^+} = \cap\{K^{\mathfrak{u},\mathfrak{h}_\varepsilon}_{\partial^+} : \varepsilon < \ell g(\bar{\mathfrak{h}})\}$ where each $K^{\mathfrak{u},\mathfrak{h}_\varepsilon}_{\partial^+}$ is well defined.

1.24 Definition. 1) We say $\langle(\bar{M}^\zeta, \bar{\mathbf{J}}^\zeta, \mathbf{f}^\zeta) : \alpha < \alpha_*\rangle$ does obey$_0$ (or 0-obey) the function $\mathfrak{h}$ in ζ <u>when</u> $\xi + 1 < \alpha_*$ and (if $\alpha_* = 2$ we can omit ζ):

- (a) $(\bar{M}^\varepsilon, \bar{\mathbf{J}}^\varepsilon, \mathbf{f}^\varepsilon) \in K^{\rm qt}_{\mathfrak{u}}$ is $\le^{\rm qt}_{\mathfrak{u}}$-increasing continuous (with ε)
- (b) M^ζ_∂ and even $M^{\zeta+1}_\partial$ has universe an ordinal $< \partial^+$
- (c) there is a club E of ∂ and sequence $\langle \mathbf{I}_\alpha : \alpha < \partial\rangle$ witnessing $(\bar{M}^\zeta, \bar{\mathbf{J}}^\zeta, \mathbf{f}^\zeta) \le^{\rm at}_{\mathfrak{u}} (\bar{M}^{\zeta+1}, \bar{\mathbf{J}}^{\zeta+1}, \bar{\mathbf{J}}, \mathbf{f}^{\zeta+1})$ such that $(M^\zeta_{\min(E)}, M^{\zeta+1}_{\min(E)}, \mathbf{I}_{\min(E)}) = \mathfrak{h}(\langle(\bar{M}^\xi, \bar{\mathbf{J}}^\xi, \mathbf{f}^\xi) : \xi \le \zeta\rangle) \in \text{ FR}^+_1$.

2) We say that $\mathfrak{h}$ is $\mathfrak{u}$-appropriate$_0$ or $\mathfrak{u} - 0$-appropriate <u>when</u>: $\mathfrak{h}$ has domain and range as required in part (1), particularly clause (c). We may say 0-appropriate or appropriate$_0$ when $\mathfrak{u}$ is clear from the context and we say "$(\bar{M}^\zeta, \dots), (\bar{M}^{\zeta+1}, \dots)$ does 0-obeys $\mathfrak{h}$".
2A) We say the function $\mathfrak{h}$ is $\mathfrak{u}$-1-appropriate <u>when</u> its domain and range are as required in Definition 1.22(3); in this case $S_{\mathfrak{h}} = S$.
2B) We say the function $\mathfrak{h}$ is $\mathfrak{u}$-2-appropriate for S <u>when</u> $S \subseteq \partial$ is stationary and its domain and range are as required in Definition 1.22(3B), i.e. 1.22(3).
2C) If in (2B) we omit S this means that $\bar{\mathfrak{h}} = \langle \mathfrak{h}_S : S \subseteq \partial$ is stationary$\rangle$, each $\mathfrak{h}_S$ as above.
3) For 0-appropriate $\mathfrak{h}$ we define $\mathfrak{K}^{\mathfrak{u},\mathfrak{h}}_{\partial^+}$ to be the family of models M, with universe ∂^+ for simplicity, as the set of models of the form $\cup\{M^\zeta_{\partial^+} : \zeta < \partial^+\}$ where $\langle(\bar{M}^\zeta, \bar{\mathbf{J}}^\zeta, \mathbf{f}^\zeta) : \zeta < \partial^+\rangle$ is $\le_{\rm qt}$-increasing continuous and 0-obeys $\mathfrak{h}$ in ζ for unboundedly many $\zeta < \partial^+$. Similarly for the other $\mathfrak{h}$, see below.
4) We say $\mathfrak{h}$ is $\mathfrak{u} - \{0, 2\}$-appropriate or $\mathfrak{u}$-appropriate for $\{0, 2\}$ if $\mathfrak{h} = \mathfrak{h}_0 \cup \mathfrak{h}_2$ and $\mathfrak{h}_\ell$ is ℓ-appropriate for $\ell = 0, 2$; we may omit $\mathfrak{h}$ when clear from the context.
5) For a $\{0, 1\}$-appropriate $\mathfrak{h}$ letting $\mathfrak{h}_0, \mathfrak{h}_1$ be as above we say

$\langle(\bar M^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha < \alpha(*)\rangle$ does $\{0,1\}$-obeys $\mathfrak{h}$ in $\zeta < \alpha(*)$ when $((\bar M^\zeta, \bar{\mathbf{J}}^\zeta, \mathbf{f}^\zeta), (\bar M^{\zeta+1}, \bar{\mathbf{J}}^{\zeta+1}, \mathbf{f}^\zeta))$ does ℓ-obey $\mathfrak{h}_\ell$ for $\ell = 0, 2$. We say strictly $\{0,2\} - S$-obeys $\mathfrak{h}$ in ζ when for stationary $S \subseteq \partial$, for unboundedly many $\zeta < \alpha(*)$ the pair 0-obeys $\mathfrak{h}_0$ and strictly $1-S$-obeys $\mathfrak{h}_1$.
6) For a $\{0,2\}$-appropriate $\mathfrak{h}$, we say $\langle(\bar M, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha < \delta \le \partial^+\rangle$ does $\{0,2\}$-obey $\mathfrak{h}$ when this holds for some stationary $S \subseteq \partial$ for unboundedly many $\zeta < \delta$ the sequence strictly $\{0,1\} - S$-obey $\mathfrak{h}$. Similarly we define "the sequence $\{0,2\} - S$-obeys $\mathfrak{h}$".
7) "$\{0,2\}$-almost every $(\bar M, \bar{\mathbf{J}}, \mathbf{f})$ (or every $(\bar M, \bar{\mathbf{J}}, \mathbf{f})$ above $(\bar M^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$)" is defined similarly to Definition 1.22.

1.25 Observation. 1) For any $\varepsilon^* < \partial^+$ and sequence $\langle \mathfrak{h}_\varepsilon : \varepsilon < \varepsilon^*\rangle$ of 3-appropriate $\mathfrak{h}$, there is an 3-appropriate $\mathfrak{h}$ such that $\mathfrak{K}^{\mathfrak{u},\mathfrak{h}}_{\partial^+} \subseteq \cap\{K^{\mathfrak{u},\mathfrak{h}_\varepsilon}_{\partial^+} : \varepsilon < \varepsilon^*\}$ and similarly for $<^{\mathrm{qs}}_{\mathfrak{u}}$-increasing sequences of $K^{\mathrm{qt}}_{\mathfrak{u}}$ length $< \partial^+$.
2) $K^{\mathfrak{u},\mathfrak{h}}_{\partial^+} \subseteq K^{\mathfrak{u},*}_{\partial^+}$ for any 3-appropriate function $\mathfrak{h}$.
3) Similarly to parts (1)+(2) for $\mathfrak{g}$ as in Definition 1.22(2).
4) Similarly to parts (1) + (2) for $\{0,2\}$-appropriate $\mathfrak{h}$, see Definition 1.24(4),(5),(6).

1.26 Remark. 1) Concerning 1.25, if in Definition 1.22(1)(c) we do not allow $\odot'$, then we better[19] in 1.22(2) add $\bar S = \langle S^\varepsilon : \varepsilon < \partial\rangle$ such that: $S^\varepsilon \subseteq \partial$ is stationary, $\varepsilon < \zeta < \lambda \Rightarrow S^\varepsilon \cap S^\zeta = \emptyset$ and $\varepsilon < \partial \wedge \alpha \in u \cup \{\delta\} \Rightarrow S^\varepsilon \cap S_\alpha$ is stationary.
2) A priori "almost$_3$" look the most natural, but we shall use as our main case "$\{0,2\}$-almost". We try to explain below.
3) Note that

- (a) in the proof of e.g. 10.10 we use $K^{\mathrm{rt}}_{\mathfrak{u}}$ not $K^{\mathrm{qt}}_{\mathfrak{u}}$, i.e. carry $\bar{\mathbb{F}}$; this does not allow us the freedom which "almost$_3$" require
- (b) model theoretically here usually there is a special model in $K^{\mathfrak{u}}_\partial$, normally the superlimit or saturated one, and we try to take care building the tree $\langle(\bar M_\eta, \bar{\mathbf{J}}_\eta, \mathbf{f}_\eta, (\bar{\mathbb{F}}_\eta)) : \eta \in {}^{\partial>}(2^\partial)\rangle$ that, e.g. $\eta \in {}^\gamma(2^\partial) \wedge \partial|\gamma \Rightarrow M^\eta_\partial$ is saturated.

[19] in the cases we would like to apply 1.25 there is no additional price for this.

In the 'almost$_3$" case this looks straight; in successor of successor cases we can take care.
4) We like to guarantee that for "almost" all $(\bar{M}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta)$ the model $M^\eta_\partial \in K^{\mathfrak{u},*}_\partial$ is saturated so that we have essentially one case. If we allow in the "almost", for, e.g. $\gamma + 2$, to choose some initial segment in $(\bar{M}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta)$ for η of length $\gamma + 1$, this guarantees saturation of M^η_∂ if $\mathrm{cf}(\ell g(\eta)) = \partial$, but

- (c) set theoretically we do not know that $S^{\partial^+}_\partial = \{\delta < \partial^+ : \mathrm{cf}(\delta) = \partial\}$ is not in the relevant ideal (in fact, even under GCH, $\diamondsuit_{S^{\partial^+}_\partial}$ may fail)
- (d) if K_∂ is categorical, there is no problem. However, if we know less, e.g. that there is a superlimit one, or approximation, using the almost$_2$, in $\gamma = \gamma' + 2$, we can guarantee that M_η for $\eta \in {}^\gamma(2^\partial)$ is up to isomorphism the superlimit one
- (e) we may conclude that it is better to work with $K^{\mathrm{rt}}_{\mathfrak{u}}$ rather than $K^{\mathrm{qt}}_{\mathfrak{u}}$, see Definition 10.3(1). This is true from the point of view of the construction but it is model theoretically less natural.

5) We may in Definition 1.24 demand on $\langle(\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha < \delta_*\rangle$ satisfies several $\mathfrak{h}$'s of different kinds say of $\{0, 2\}$ and of 3; make little difference.
6) In the usual application here for $\mathfrak{u} = \mathfrak{u}^\ell_{\mathfrak{s}}$ for some $\mathfrak{g}$, if $\langle(M^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha \leq \delta\rangle$ is $\leq^{\mathrm{qt}}_{\mathfrak{u}}$-increasing continuous and $\delta = u, u := \{\alpha < \delta : ((\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha), (\bar{M}^{\alpha+1}, \bar{\mathbf{J}}^{\alpha+1}, \mathbf{f}^{\alpha_1}))$ does strictly S-obey $\mathfrak{g}\}$, <u>then</u> M^δ_∂ is saturated. But without this extra knowledge, the fact that for $\alpha \in u$ we may have S_α disjoint to other may be hurdle. But using "strictly obey$_1$" seems more general and the definition of "almost$_2$" fits this feeling.

§2 Coding properties and non-structure

We now come to the definition of the properties we shall use as sufficient conditions for non-structure starting with Definition 2.2; in this section and §3 we shall define also some relatives needed for sharper results, those properties have parallel cases as in Definition 2.2.

2.1 Hypothesis. We assume $\mathfrak{u}$ be a nice construction framework and, τ a weak $\mathfrak{u}$-sub-vocabulary, see Definition 1.8(1).

Remark. The default value is $\tau_{\mathfrak{u}} = \tau(\mathfrak{K}_{\mathfrak{u}})$ or better the pair $(\tau, \tau_{\mathfrak{u}})$ such that $\tau_{\mathfrak{u}} = \tau'$, as in Definition 1.8(1) and 2.8(1),(2); see also $\mathfrak{u}$ has faked equality, see 3.17 later.

Among the variants of weak τ-coding in Definition 2.2 the one we shall use most is 2.2(5), "$\mathfrak{u}$ has the weak τ-coding$_1$ above $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$".

2.2 Definition. 1) We say that $M \in K_{\mathfrak{u}}$ has the weak τ-coding$_0$-property (in $\mathfrak{u}$) <u>when</u>:

- (A) if $N, \mathbf{I}$ are such that $(M, N, \mathbf{I}) \in \mathrm{FR}_1^+$ <u>then</u> $(M, N, \mathbf{I})$ has the weak τ-coding$_0$ property,
  where:
- (B) $(M, N, \mathbf{I})$ has the weak τ-coding$_0$ property <u>when</u> we can find $(M_*, N_\ell, \mathbf{I}_\ell) \in \mathrm{FR}_1$ for $\ell = 1, 2$ satisfying
  - (a) $(M, N, \mathbf{I}) \leq^1_{\mathfrak{u}} (M_*, N_\ell, \mathbf{I}_\ell)$ for $\ell = 1, 2$
  - (b) $M_* \cap N = M$ (follows)
  - (c) N_1, N_2 are τ-incompatible amalgamations of M_*, N over M in $K_{\mathfrak{u}}$, (see Definition 1.8(4)).

1A) We say that $(M, N, \mathbf{I}) \in \mathrm{FR}_1^+$ has the true weak τ-coding$_0$ <u>when</u>: if $(M, N, \mathbf{I}) \leq^1_{\mathfrak{u}} (M', N', \mathbf{I})$ then $(M', N', \mathbf{I}')$ has the weak τ-coding$_0$ property, i.e. satisfies the requirement in clause (B) of part (1).
1B) $\mathfrak{u}$ has the explicit weak τ-coding$_0$ property <u>when</u> every $(M, N, \mathbf{I}) \in \mathrm{FR}^+_{\mathfrak{u}}$ has the weak τ-coding property.
2) $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ has the weak τ-coding$_0$ property <u>when</u>: for a club of $\delta < \partial$, not only $M = M^*_\delta$ has the true weak τ-coding$_0$-property <u>but</u> in clause (B) of part (1) above we demand $M_* \leq_{\mathfrak{u}} M^*_\gamma$ for any $\gamma < \partial$ large enough.

3) We say that $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ has the weak τ-coding$_1$ property[20] <u>when</u>: (we may omit the superscript 1): recalling $M_\partial = \cup\{M_\alpha : \alpha < \partial\}$, there are $\alpha(0) < \partial$ and $N_0, \mathbf{I}_0$ such that $(M_{\alpha(0)}, N_0, \mathbf{I}_0) \in {\rm FR}_1, N_0 \cap M_\partial = M_{\alpha(0)}$ and for a club of $\alpha(1) < \partial$, if $(M_{\alpha(0)}, N_0, \mathbf{I}_1) \le^1_{\mathfrak{u}} (M_{\alpha(1)}, N_1, \mathbf{I}_1)$ satisfies $N_1 \cap M_\partial = M_{\alpha(1)}$ <u>then</u> there are $\alpha(2) \in (\alpha(1), \partial)$ and $N^\ell_2, \mathbf{I}^\ell_2$ for $\ell = 1, 2$ such that $(M_{\alpha(1)}, N_1, \mathbf{I}_1) \le^1_{\mathfrak{u}} (M_{\alpha(2)}, N^\ell_2, \mathbf{I}^\ell_2)$ for $\ell = 1, 2$ and N^1_2, N^2_2 are τ-incompatible amalgamations of $M_{\alpha(2)}, N_1$ over $M_{\alpha(1)}$ recalling Definition 1.8(4).
4) We say that $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f})$ has the S-weak τ-coding$_1$ property <u>when</u>: S is a stationary subset of ∂ and for some club E of ∂ the demand in (3) holds restricting ourselves to $\alpha(1) \in S \cap E$.
5) We say that $\mathfrak{u}$ has the weak τ-coding$_k$ property <u>when</u>: $\{0, 2\}$-almost every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_u$ has the weak τ-coding$_k$ property; omitting k means $k = 1$. Similarly for "above $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$". Similarly for "$S$-weak".

The following theorem uses a weak model theoretic assumption, but the price is a very weak but still undesirable, additional set theoretic assumption (i.e. clause (c)), recall that $\mu_{\rm unif}(\partial^+, 2^\partial)$ is defined in 0.4(7), see 9.4.

2.3 Theorem. *We have $\dot{I}_\tau(\partial^+, K^{\mathfrak{u}}_{\partial^+}) \ge \mu_{\rm unif}(\partial^+, 2^\partial)$, moreover for any $\mathfrak{u}$-0-appropriate $\mathfrak{h}$ (see Definition 1.24) and even $\{0, 2\}$-appropriate $\mathfrak{h}$ (see Definition 1.24(3),(7) and Definition 1.23) we have $\dot{I}(K^{\mathfrak{u},\mathfrak{h}}_{\partial^+}) \ge \mu_{\rm unif}(\partial^+, 2^\partial)$, <u>when</u>:*

- (a) $2^\theta = 2^{<\partial} < 2^\partial$
- (b) $2^\partial < 2^{\partial^+}$
- (c) *the ideal* ${\rm WDmId}(\partial)$ *is not* ∂^+*-saturated*
- (d) $\mathfrak{u}$ *has the weak τ-coding (or just the S-weak τ-coding property above some triple $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ with* ${\rm WDmId}(\partial) \restriction S$ *not ∂^+-saturated and $S \subseteq \mathbf{f}^{-1}\{0\}$).*

[20] the difference between coding$_0$ and coding$_1$ may seem negligible but it is crucial, e.g. in 4.1

Proof. This is proved in 10.10. $\square_{2.3}$

2.4 Remark. 1) Theorem 2.3 is used in 4.1, 4.3(1), 4.20, 4.16 and 4.28, for 4.3(2) we use the variant 2.5. We could use Theorem 2.7 below to get a somewhat stronger result.
In other words, e.g. it is used for "the minimal types are not dense in $\mathscr{S}(M)$ for $M \in \mathfrak{K}_\lambda$" for suitable $\mathfrak{K}$, see 4.1 (and Chapter VI or the older [Sh 576], [Sh 603]).
2) We may think that here at a minor set theoretic price (clause (c)), we get the strongest model theoretic version.
3) We can in 2.3 replace $\mathfrak{h}$ by $\bar{\mathfrak{h}}$, a sequence of $\{0,2\}$-appropriate $\mathfrak{h}$'s of length $\leq \partial^+$.
4) In part (3), we can fix a stationary $S \subseteq \partial$ such that $\mathrm{WDmId}(\partial)+S$ is not ∂^+-saturated (so ∂ is not in it) and restrict ourselves to strict S-obeying.
5) We can replace assumption (d) of 2.3 by

$(d)'$ for some $\mathscr{D}$

(i) $\mathscr{D}$ is a normal filter on ∂ disjoint to $\mathrm{WDmId}(\partial)$; moreover

$$(\forall A \in \mathscr{D})(\exists B)[B \subseteq A \wedge \partial \backslash B \in D \wedge B \in (\mathrm{WDmId}(\partial))^+]$$

(ii) almost$_2$ every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathscr{D}}$ has the weak τ-coding property (even just above some member of $K^{\mathrm{qt}}_{\mathscr{D}}$).

A variant is

2.5 Claim. *In Theorem 2.3 we can weaken the assumption to "$\mathfrak{u}$ has a weak τ-coding$_2$", see below.*

Proof. As in 10.10. $\square_{2.5}$

2.6 Definition. 1) We say that $\mathfrak{u}$ has the S-weak τ-coding$_2$ property (or the S-weak game τ-coding property) [above $(\bar{M}^*, \mathbf{J}^*, \mathbf{f}^*) \in K^{\rm qt}_{\mathfrak{u}}$] <u>when</u> $\{0,2\}$-almost every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{s}}$ [above $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$] has it.
2) We say $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ has the S-weak game τ-coding$_2$ property <u>or</u> S-weak τ-coding$_2$ property for a stationary set $S \subseteq \partial$ (omitting S means for every such S) <u>when</u>, recalling $M_\partial = \cup\{M_\alpha : \alpha < \partial\}$, in the following game $\eth_{\mathfrak{u},S}(\bar{M}, \bar{\mathbf{J}}, \mathbf{f})$, the Coder player has a winning strategy where:

$(*)_1$ a play of $\eth_{\mathfrak{u},S}$ last ∂ moves after the ε-th move a tuple $(\alpha_\varepsilon, e_\varepsilon, \bar{N}^\varepsilon, \bar{\mathbf{J}}^\varepsilon, \mathbf{f}^\varepsilon, \bar{I}^\varepsilon)$ is chosen such that:

- (a) $\alpha_\varepsilon < \partial$ is increasing continuous
- (b) e_ε is a closed subset of α_ε such that $\zeta < \varepsilon \Rightarrow \alpha_\zeta \in e \wedge e_\zeta = e_\varepsilon \cap \alpha_\zeta$
- (c) $\mathbf{f}^\varepsilon$ is a function with domain e_ε such that $\alpha + \mathbf{f}^\varepsilon(\alpha) < \min(e_\varepsilon \cup \{\alpha_\varepsilon\} \backslash \alpha)$ and $\mathbf{f}^\varepsilon(\alpha) \geq \mathbf{f}(\alpha)$
- (d) $u_\varepsilon = \cup\{[\alpha, \alpha + \mathbf{f}^\varepsilon(\alpha)] : \alpha \in e_\varepsilon\} \cup \{\alpha_\varepsilon\}$ and $u^-_\varepsilon = \cup\{[\alpha, \alpha + \mathbf{f}^\varepsilon(\alpha))) : \alpha \in e_\varepsilon\}$
- (e) $\bar{N}^\varepsilon = \langle N_\alpha : \alpha \in u_\varepsilon \rangle$ and $\mathbf{J}^\varepsilon = \langle \mathbf{J}_\alpha : \alpha \in u^-_\varepsilon \rangle$ and $\bar{\mathbf{I}}^\varepsilon = \langle \mathbf{I}_\alpha : \alpha \in u_\varepsilon \rangle$
- (f) $\langle (M_\alpha, N_\alpha, \mathbf{I}_\alpha) : \alpha \in u_\varepsilon \rangle$ is $\leq^1_{\mathfrak{u}}$-increasing
- (g) $N^\varepsilon_\alpha \cap M_\partial = M_\alpha$ for $\alpha \in u_\varepsilon$
- (h) $\bar{\mathbf{J}}^\varepsilon = \langle \mathbf{J}^*_\alpha : \alpha \in u^-_\varepsilon \rangle$
- (i) $(M_\alpha, M_{\alpha+1}, \mathbf{J}_\alpha) \leq^2_{\mathfrak{u}} (N_\alpha, N_{\alpha+1}, \mathbf{J}^*_\alpha)$ for $\alpha \in u^-_\varepsilon$
- (j) the coder chooses $(\alpha_\varepsilon, e_\varepsilon, \bar{N}^\varepsilon, \bar{\mathbf{J}}^\varepsilon, \bar{\mathbf{I}}^\varepsilon, \mathbf{f}^\varepsilon)$ if $\varepsilon = 0$ or $\varepsilon = \zeta + 1, \zeta$ a limit ordinal $\notin S$, and otherwise the anti-coder chooses

$(*)_2$ in the end the Coder wins the play <u>when</u> for a club of $\varepsilon < \partial$, if $\varepsilon \in S$, then the triple $(M_{\alpha_\varepsilon}, N_\varepsilon, \mathbf{I}_\varepsilon)$ has the weak τ-coding$_0$ property, i.e. satisfies clause (B) of Definition 2.2(1), moreover such that $M \leq_{\mathfrak{K}} M_\partial$.

We can also get "no universal" over $M_\partial \in \mathfrak{K}^{\mathfrak{u}}_\partial$ (suitable for applying 9.2).

2.7 Claim. *If* $(\bar{M}, \mathbf{J}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}, M = \cup\{M_\alpha : \alpha < \partial\}$ *and* $M \le_{\mathfrak{u}} N_\varepsilon \in \mathfrak{K}_{\le\mu}$ *for* $\varepsilon < \varepsilon^* < \mu^+$ *then there is* $(\bar{M}', \bar{\mathbf{J}}', \mathbf{f})$ *satisfying* $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \le^{\mathfrak{u}}_{\rm at} (\bar{M}', \bar{\mathbf{J}}', \mathbf{f}')$ *such that* $M'_\partial = \cup\{M_\alpha : \alpha < \partial\} \in K_\partial$ *cannot be* $\le_{\mathfrak{K}[\mathfrak{u}]}$*-embedded into* N_ε *for* $\varepsilon < \partial$ *over* M_∂ *provided that:*

(a),(b),(d) *as in 2.3*

(e) ${}^\partial 2$ *is not the union of* $\mathrm{cov}(\mu, \partial^+, \partial^+, 2)$ *sets from* $\mathrm{WDmTId}(\partial, 2^{<\partial})$.

Proof. As in the proof of 10.10, anyhow not used. $\square_{2.7}$

2.8 Exercise: 1) [Definition] Call $\mathfrak{u}$ a semi-nice construction framework when in Definition 1.2 we omit clause $(\mathrm{D})_\ell(\mathrm{d})$ and the disjointness demands $(\mathrm{E})_\ell(\mathrm{b})(\beta)$
2) For $\mathfrak{u}$ as above we define $\mathfrak{u}'$ as follows:

- (a) $\mathfrak{K}_{\mathfrak{u}'}$ is as in Definition 1.10(1)
- (b) $\mathrm{FR}^\ell_{\mathfrak{u}'} = \{(M, N, \mathbf{J}) : M \le_{\mathfrak{K}'_{\mathfrak{u}}} N$ so both of cardinality $< \partial$ and $\mathbf{J} \subseteq N \backslash M$ and letting $M^* = M/=_\tau, N^* = N/=_\tau$ and $\mathbf{J}^* = \{c/=^N_\tau: c \in \mathbf{J}$ and $(c/=_\tau) \notin M/=_\tau\}$ we have $(M^*, N^*, \mathbf{J}^*) \in \mathrm{FR}^\ell_{\mathfrak{u}}\}$
  pedantically $=_\tau$ means $=^N_\tau$ (even $M/=^N_\tau$)
- (c) $(M_1, N_1, \mathbf{J}_1) \le^\ell_{\mathfrak{u}'} (M_2, N_2, \mathbf{J}_1)$ iff $M_1 \le_{\mathfrak{u}'} M_2 \le_{\mathfrak{u}'} N_2, M_1 \le_{\mathfrak{u}'} N_1 \le_{\mathfrak{u}'} N_2, \mathbf{J}_1 \subseteq \mathbf{J}_2, M_2 \cap N_1 = M_1$ and $(M^*_1, N^*_1, \mathbf{J}_1) \le^\ell_{\mathfrak{u}} (M^*_2, N^*_2, \mathbf{J}^*)$ when we define them as in clause (b).

3) (Claim) If $\mathfrak{u}$ is a semi-nice construction framework then $\mathfrak{u}'$ is a nice construction framework.
4) [Definition] For $\mathfrak{u}$ a semi-nice construction framework we define $\mathfrak{u}''$ as in part (2) except that in clause (b) we demand $c \in \mathbf{J} \wedge a \in M \Rightarrow N \models \neg(a =_\tau c)$.
5) [Claim] If $\mathfrak{u}$ is a nice, [semi-nice], [semi-nice satisfying (D)(d)] construction framework then $\mathfrak{u}''$ is a nice, [semi-nice], [nice] construction framework.

Discussion: We now phrase further properties which are enough for the desired conclusions under weaker set theoretic conditions. The main case is vertical coding (part (4) but it relies on part (1) in Definition 2.9). On additional such properties, see later.

In the "vertical coding" version (see Definition 2.9 below), we strengthen the "density of τ-incompatibility" such that during the proof we do not need to preserve "$\mathbf{f}_\eta^{-1}\{0\}$ is large" even allowing $\mathbf{f}_\eta^{-1}\{0\} = \emptyset$.

We may say that "vertically" means that given $(\bar{M}^1, \bar{\mathbf{J}}^1, \mathbf{f}^1) \in K^{\rm qt}_{\mathfrak{u}}$ building $M^2_\alpha, \mathbf{J}^2_\alpha, \mathbf{I}^2_\alpha$ by induction on $\alpha < \partial$, arriving to some limit δ, we are committed to $M^2_{\alpha+i}$ for $i \le \mathbf{f}^1(\delta)$, but still like to have freedom in determining the type of M^2_δ over $\cup\{M^1_\beta : \beta < \partial\}$ (see more in the proof of Theorem 8.6 and Definition 8.7 on delayed uniqueness, which express failure of this freedom). In other words the property we have is a delayed version of the weak coding.
As usual, always $\mathfrak{u}$ is a nice construction framework.

2.9 Definition. 1) We say that $(\bar{M}^1, \bar{\mathbf{J}}^1) = (\langle M^1_i : i \le \beta\rangle, \langle \mathbf{J}^1_i : i < \beta\rangle)$ has the vertical τ-coding$_0$ property (in $\mathfrak{u}$) <u>when</u>:

$(A)(a)$ $\beta < \partial$

$(b)_1$ M^1_i is $\le_{\mathfrak{u}}$-increasing continuous for $i \le \beta$

$(c)_1$ $(M^1_i, M^1_{i+1}, \mathbf{J}^1_i) \in \mathrm{FR}_2$ for $i < \beta$

(B) if $(\langle M^2_i : i \le \beta\rangle, \langle \mathbf{J}^2_i : i < \beta\rangle, \langle \mathbf{I}_i : i \le \beta\rangle)$ satisfies $\circledast_1$ below, <u>then</u> we can find $\gamma_\ell, M^1_*, \mathbf{I}^\ell_*$ and $M^{2,\ell}_i$ (for $i \in (\beta, \gamma_\ell]$) and $\mathbf{J}^{2,\ell}_i$ (for $i \in [\beta, \gamma_\ell)$), for $\ell = 1,2$ satisfying $\circledast_2$ below <u>where</u>, letting $M^{2,\ell}_i = M^2_i$ for $i \le \beta$ and $\mathbf{J}^{2,\ell}_i = \mathbf{J}^2_i$ for $i < \beta$, we have:

$\circledast_1$(d) M^2_i $(i \le \beta)$ is $\le_{\mathfrak{u}}$-increasing continuous

(e) $M^2_i \cap M^1_\beta = M^1_i$

(f) $\langle (M^1_i, M^2_i, \mathbf{I}_i) : i \le \beta\rangle$ is $\le^1_{\mathfrak{u}}$-increasing continuous and $(M^1_i, M^2_i, \mathbf{I}_i) \in \mathrm{FR}^+_1$

(g) $(M^1_i, M^1_{i+1}, \mathbf{J}^1_i) \le^2_{\mathfrak{u}} (M^2_i, M^2_{i+1}, \mathbf{J}^2_i) \in \mathrm{FR}_2$ for $i < \beta$

(h) $(M^1_0, M^2_0, \mathbf{I}_0) = (M, N, \mathbf{I})$

$\circledast_2$ $M^{2,1}_{\gamma_1}, M^{2,2}_{\gamma_2}$ are τ-incompatible amalgamation of M^1_*, M^2_0 over M^1_0 in $\mathfrak{K}_{<\partial}$ and for $\ell = 1,2$ we have

$\circledast_{2,\ell}$(a)' $\beta < \gamma_\ell < \partial$

(b)$'_1$ $M^1_\beta \le_{\mathfrak{u}} M^1_*$

(c)$'_1$ $(M^{2,\ell}_i, M^{2,\ell}_{i+1}, \mathbf{J}^{2,\ell}_i) \in$ FR$_2$ for $i < \gamma_\ell$

(d)$'$ $M^{2,\ell}_i$ (for $i \le \gamma_\ell$) is $\le_{\mathfrak{u}}$-increasing continuous

(e)$'$ $M^1_* \le_{\mathfrak{u}} M^{2,\ell}_{\gamma_\ell}$

(f)$'$ $(M^1_\beta, M^2_\beta, \mathbf{I}_\beta) \le^1_{\mathfrak{u}} (M^1_*, M^{2,\ell}_{\gamma_\ell}, \mathbf{I}^\ell_*)$.

1A) We say that $(M, N, \mathbf{I})$ has the vertical τ-coding$_0$ property when: <u>if</u> $(\bar{M}^1, \bar{\mathbf{J}}^1) = (\langle M^1_i : i \le \beta\rangle, \langle \mathbf{J}^1_i : i < \beta\rangle)$ satisfies clause (A) of part (1) and $(\langle M^2_i : i \le \beta\rangle, \langle \mathbf{J}^2_i : i < \beta\rangle, \langle \mathbf{I}_i : i \le \beta\rangle)$ satisfies $\circledast_1$ of clause (B) of part (1) and $(M^1_0, M^2_0, \mathbf{I}_0) = (M, N, \mathbf{I})$ <u>then</u> we can find objects satisfying $\circledast_2$ of clause (B) of part (1).
1B) We say that $(M, N, \mathbf{I})$ has the true vertical τ-coding$_0$ property <u>when</u> it belongs to FR$^+_1$ and every $(M', N', \mathbf{J}')$ satisfying $(M, N, \mathbf{I}) \le^1_{\mathfrak{u}} (M', N', \mathbf{I}')$ has the vertical τ-coding$_0$ property.
1C) We say that $\mathfrak{u}$ has the explicit vertical τ-coding$_0$ property <u>when</u> for every M for some $N, \mathbf{I}$ the triple $(M, N, \mathbf{I}) \in$ FR$^+_1$ has the true vertical τ-coding$_0$ property.
2) $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*) \in K^{\rm qt}_{\mathfrak{u}}$ has the vertical τ-coding$_0$ property <u>when</u> for a club of $\delta < \partial$, the pair $(\bar{M}^1, \bar{\mathbf{J}}^1) = (\langle M^*_{\delta+i} : i \le \mathbf{f}^*(\delta)\rangle, \langle \mathbf{J}^*_{\delta+1} : i < \mathbf{f}^*(\delta)\rangle)$ satisfies part (1) even demanding $M^1_* \le_{\mathfrak{K}} M^*_\partial$.
3) We say that $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ has the vertical τ-coding$_1$ property <u>when</u> (we may omit the subscript 1) we can find $\alpha(0) < \partial$ and $(M_{\alpha(0)}, N_*, \mathbf{I}_*) \in$ FR$_1$ satisfying $N_* \cap M_\partial = M_{\alpha(0)}$ such that: for a club of $\delta < \partial$ the pair $(\bar{M}^1, \bar{\mathbf{J}}^1) = (\langle M_{\delta+i} : i \le \mathbf{f}(\delta)\rangle, \langle \mathbf{J}_{\delta+i} : i < \mathbf{f}(\delta)\rangle)$ satisfies part (1) when in clause (B) where we

(i) restrict ourselves to the case $(M_{\alpha(0)}, N_*, \mathbf{I}_*) \le^1_{\mathfrak{u}} (M^1_0, M^2_0, \mathbf{I}_0)$

(ii) demand that $M^1_* <_{\mathfrak{K}[\mathfrak{u}]} M_\partial$.

4) We say that $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ has the S-vertical τ-coding$_1$ property <u>when</u>: S is a stationary subset of ∂ and for club E of ∂ the requirement in part (3) holds when we restrict ourselves to $\delta \in S \cap E$.
4A) We say that $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ has the S-vertical τ-coding$_2$ property as in Definition 2.6.
5) For $k = 0, 1, 2$ we say $\mathfrak{u}$ has the vertical τ-coding$_k$ property <u>when</u>

$\{0,2\}$-almost every $(\bar{M},\bar{\mathbf{J}},\mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ has it. If $k=1$ we may omit it. Similarly adding "above $(\bar{M}^*,\bar{\mathbf{J}}^*,\mathbf{f}^*)$" and/or S-vertical, for stationary $S \subseteq \partial$.

The following observation is easy but very useful.

2.10 Observation. 1) Assume that some $(M,N,\mathbf{I}) \in \mathrm{FR}^+_1$ has the true vertical τ-coding$_0$ property (from Definition 2.9(1B)). If $(\bar{M},\bar{\mathbf{J}},\mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ and $M_\partial := \cup\{M_\alpha : \alpha < \partial\}$ is saturated (above λ, for $\mathfrak{K} = \mathfrak{K}^{\mathfrak{u}}$) <u>then</u> $(\bar{M},\bar{\mathbf{J}},\mathbf{f})$ has the vertical τ-coding property. See 2.9(2),2.9(2A) used in 4.14(5).
2) If $(\bar{M},\bar{\mathbf{J}},\mathbf{f})$ has the vertical τ-coding$_0$ property <u>then</u> it has the vertical τ-coding$_1$ property.
3) Similarly (to part (2)) for weak τ-coding.
4) Recalling $\mathfrak{K}_{\mathfrak{u}}$ has amalgamation (by claim 1.3(1))

- (a) if $M \in \mathfrak{K}_{\mathfrak{u}} \Rightarrow |\mathscr{S}_{\mathfrak{K}}(M)| \le \partial$ <u>then</u> there is a saturated $M \in K^{\mathfrak{u}}_\partial$
- (b) if every $M \in K^{\mathfrak{u},*}_\partial$ is saturated and every $(M,N,\mathbf{I}) \in \mathrm{FR}^+_1$ has weak τ-coding$_0$, <u>then</u> every $(\bar{M},\bar{\mathbf{J}},\mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ has weak τ-coding
- (c) similarly for vertical τ-coding
- (d) similarly replacing "every $M \in K^{\mathfrak{u},*}_\partial$" by "$M_\partial$ is saturated for $\{0,2\}$-almost every $(\bar{M},\bar{\mathbf{J}},\mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$" [or just above $(\bar{M}^*,\bar{\mathbf{J}}^*,\mathbf{f}^*) \in K^{\rm qt}_{\mathfrak{u}}$.]

Proof. Should be clear. $\square_{2.8}$

2.11 Theorem. *We have $\dot{I}_\tau(\partial^+,K^{\mathfrak{u}}_{\partial^+}) \ge \mu_{\rm unif}(\partial^+,2^\partial)$; moreover $\dot{I}(K^{\mathfrak{u},\mathfrak{h}}_{\partial^+}) \ge \mu_{\rm unif}(\partial^+,2^\partial)$ for any $\{0,2\}$-appropriate $\mathfrak{h}$ (see Definitions 1.24(2),(7), 1.23) <u>when</u>:*

- (a) $2^\theta = 2^{<\partial} < 2^\partial$
- (b) $2^\partial < 2^{\partial^+}$
- (c) *$\mathfrak{u}$ has the vertical τ-coding$_1$ property (or at least $\mathfrak{u}$ has the S-vertical τ-coding$_\tau$ property above some triple from $K^{\rm qt}_{\mathfrak{u}}$ for stationary $S \in (\mathrm{WDmId}(\partial))^+$ (recall τ is a weak $\mathfrak{u}$-sub-vocabulary, of course, by 2.1)).*

Remark. Theorem 2.11 is used in 4.10 and in 8.6.

Proof. Proved in 10.12. $\square_{2.11}$

* * *

2.12 Discussion: 1) In a sense the following property "horizontal τ-coding" is dual to the previous one "vertical τ-coding", it is "horizontal", i.e. in the ∂^+-direction. This will result in building $(\bar{M}^\eta, \bar{\mathbf{J}}^n, \bar{\mathbf{f}}^\eta)$ for $\eta \in {}^{\partial^+>}(2^{<\partial})$ such that letting $M^\eta_\partial = \cup\{M^{\eta\restriction\alpha}_\partial : \alpha < \partial^+\}$ we have $\eta \neq \nu \in {}^{\partial^+}(2^\partial) \Rightarrow M^\eta_\partial, M^\nu_\partial$ are not isomorphic over $M^{<>}_\partial$, so the set theory is simpler.
2) Note that in 2.13(4) below we could ask less than "for a club", e.g. having a winning strategy is the natural game; similarly in other definitions of coding properties, as in Exercise 2.5.

2.13 Definition. 1) We say that $(M_0, M_1, \mathbf{J}_2) \in \mathrm{FR}_2$ has the horizontal τ-coding$_0$ property when: if $\circledast_1$ holds then we can find $N^{+,\ell}_1, \mathbf{I}^\ell_5, \mathbf{J}^\ell_5$ for $\ell = 1, 2$ such that $\circledast_2$ holds when:

- $\circledast_1$ (a) $M_0 \le_{\mathfrak{u}} N_0 \le_{\mathfrak{u}} N_1, M_0 \le_{\mathfrak{u}} M_1 \le_{\mathfrak{u}} N_1$
  - (b) $(M_0, M_1, \mathbf{J}_2) \le^2_{\mathfrak{u}} (N_0, N_1, \mathbf{J}_3)$ so both are from FR_2
  - (c) $(N_0, N^+_0, \mathbf{I}_4) \in \mathrm{FR}^+_1$ and $N^+ \cap N_1 = N_0$
- $\circledast_2$ (α) $(N_0, N^+_0, \mathbf{I}_4) \le^1_{\mathfrak{u}} (N_1, N^{+,\ell}_1, \mathbf{I}^\ell_5)$
  - (β) $(N_0, N_1, \mathbf{J}_3) \le^2_{\mathfrak{u}} (N^+_0, N^{+,\ell}_1, \mathbf{J}^\ell_5)$
  - (γ) $N^{+,1}_1, N^{+,2}_1$ are τ-incompatible amalgamations of M_1, N^+_0 over M_0 in $K_{\mathfrak{u}}$.

2) We say that $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ has the S-horizontal τ-coding$_0$ property when S is a stationary subset of ∂ and for a club of $\delta \in S$, the triple $(M_\delta, M_{\delta+1}, \mathbf{J}_\delta)$ has it and $\mathbf{f}(\delta) > 0$. If $S = \partial$ we may omit it.
3) We say that $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ has the horizontal τ-coding$_1$ property when (we may omit the 1):

- $\circledast$ for $\{0, 2\}$-almost every $(\bar{M}', \bar{\mathbf{J}}', \mathbf{f}') \in K^{\mathrm{qt}}_{\mathfrak{u}}$ satisfying $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \le^{\mathrm{qs}}_{\mathfrak{u}} (\bar{M}', \bar{\mathbf{J}}', \mathbf{f}')$ we can find $\alpha < \partial$ and $N, \mathbf{I}^*$ satisfying $(M'_\alpha, N, \mathbf{I}^*) \in \mathrm{FR}^+_2$ and $N \cap M'_\partial = M'_\alpha$ such that

$\boxdot$ for a club of $\delta < \partial$ if $(M'_\alpha, N, \mathbf{I}^*) \le_1 (M'_\delta, N', \mathbf{I}') \in \mathrm{FR}_1^+$ and $N' \cap M'_\partial = M'_\alpha$, <u>then</u> the conclusion in 2.13(1) above holds with $M_{\alpha,i(*)}, M'_{i(*)}, M_\delta, M'_\delta, N', \mathbf{I}', \mathbf{J}_\delta, \mathbf{J}'_\delta$ here standing for $M_0, N_0, M_1, N_1, N_0^+, \mathbf{I}_4, \mathbf{J}_2, \mathbf{J}_3$ there.

4) We replace coding$_1$ by coding$_2$ when in $\boxdot$ we use the game version, as in 2.6.
5) We say $\mathfrak{u}$ has the horizontal τ-coding$_k$ property <u>when</u> some $(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ has it.

2.14 Claim. *The coding$_0$ implies the coding$_1$ versions in Definition 2.13 for $(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ in Definition 2.13(4) and for $\mathfrak{u}$ in Definition 2.13(5).*

Proof. Should be clear. $\square_{2.14}$

2.15 Theorem. *We have $\dot I_\tau(\partial^+, K^{\mathfrak{u}}_{\partial^+}) \ge 2^{\partial^+}$; moreover $\dot I(K^{\mathfrak{u},\mathfrak{h}}_{\partial^+}) \ge 2^{\partial^+}$ for any $\{0,2\}$-appropriate $\mathfrak{h}$ (see Definition 1.24(4),(5),(6)), <u>when</u>:*

(a) $2^\theta = 2^{<\partial} < 2^\partial$

(b) $2^\partial < 2^{\partial^+}$

(c) *the ideal* WDmId(∂) *is not ∂^+-saturated*

(d) *$\mathfrak{u}$ has the horizontal τ-coding property (or just the S-horizontal τ-coding$_2$ property for some stationary $S \subseteq \partial$).*

2.16 Remark. 1) Actually not used here.
2) What does this add compared to 2.3? getting $\ge 2^{\partial^+}$ rather than $\ge \mu_{\rm unif}(\partial^+, 2^\partial)$.

Proof. Proved in 10.13. $\square_{2.15}$

§3 Invariant codings

The major notion of this section is (variants of) uq-invariant coding properties. In our context, the point of coding properties is in esssence that their failure gives that there are many of uniqueness triples, $(M, N, \mathbf{J})$ ones, i.e. such that: if $(M, N, \mathbf{I}) \leq_1 (M', N'_\ell, \mathbf{I}'')$ for $\ell = 1, 2$ then N'_1, N'_2 are compatible over $N \cup M'$. For uq-invariant we ask for less: if $(M, N, \mathbf{I}) \leq (M', N', \mathbf{I}')$ and $\mathbf{d}$ is a $\mathfrak{u}$-free $(\xi, 0)$-rectanle with $(M^{\mathbf{d}}_{0,0}, M^{\mathbf{d}}_{\alpha(\mathbf{d}),0}) = (M, M')$, then we can "lift" $\mathbf{d}''$, i.e. find a $\mathfrak{u}$-free $(\xi + 1, 1)$-rectangle $\mathbf{d}^+$ such that $\mathbf{d}^+ \restriction (\xi, 0) = \mathbf{d}, (M^{\mathbf{d}}_{0,0}, M^{\mathbf{d}}_{0,1}, \mathbf{I}^{\mathbf{d}}_{0,0}) = (M, N, \mathbf{I})$ and $N' \leq_{\mathfrak{u}} M^{\mathbf{d}}_{\xi+1,1}$.

So we look at the simplest version, the weak ξ-uq-invariant coding, Definition 3.2, we can consider a "candidate" $(M, N, \mathbf{I}) \in \mathrm{FR}^1_{\mathfrak{u}}$ and challenge $\mathbf{d}$ (so $M^{\mathbf{d}}_{0,0} = M$) and looks for a pair of amalgamation which are incompatile in a specific way, but unlike in §2, they are not symmetric. One is really not an amalgamation but a family of those exhibiting $\mathbf{d}$ is "liftable", and "promise to continue to do so in the future", in the ∂^+-direction. The real one just has to contradict it.

Another feature is that instead of considering isomorphisms over $M^{\mathbf{d}}_{\alpha(\mathbf{d}),0} \cup N$ we consider isomorphisms over $M^{\mathbf{d}}_{\alpha(\mathbf{d}),0}$ with some remnants of preserving N; more specifically we consider two $\mathfrak{u}$-free rectangles $\mathfrak{d}_1, \mathfrak{d}_2$ which continues the construction in those two ways and demands $M^{\mathbf{d}_1}_{\alpha(\mathbf{d}_1)}$ is mapped onto $M^{\mathbf{d}_2}_{0,\alpha(\mathbf{d}_2)}$.

There are more complicating factors: we have for a candidate $(M, N, \mathbf{I})$, for every $M', M \leq_{\mathfrak{u}} M'$ to find a $\mathfrak{u}$-free $(\xi, 0)$-rectangle $\mathbf{d}$ with $M' = M^{\mathbf{d}}_{0,0}$ such that it will serve against $(M', N', \mathbf{I}')$ whenever $(M, N, \mathbf{I}) \leq_1 (M', N', \mathbf{I}')$, rather than choosing $\mathbf{d}$ after $(N', \mathbf{I}')$ is chosen, i.e. this stronger version is needed. The case $\xi < \partial$ should be clear but still we allow $\xi = \partial$, however then given $(N', \mathbf{I}')$ we take $\mathbf{d} \restriction (\xi', 0)$ for some $\xi' < \xi'$.

We can use only Definition 3.2, Claim 3.3, and Conclusion 3.5, for which "$2^\theta = 2^{<\partial} < 2^\partial < 2^{\partial^+}$+ the extra WdmId$(\partial)$ is not ∂^+-saturated" is needed, (if $\xi < \partial$ less is needed; however $\xi < \partial$ shall not be enough. But to get the sharp results (with the extra assumption) for almost good λ-frames we need a more elaborate approach - using vertical ξ-uq-invariant coding, see Definition 3.10.

Actually we shall use an apparently weaker version, the so called semi ξ-uq-invariant. However, we can derive from it the vertical version under reasonable demands on $\mathfrak{u}$; this last proof is of purely model theoretic characters. We also consider other variants.

In this section we usually do not use the τ from 3.1, i.e. use $\tau = \tau_{\mathfrak{u}}$ as it is not required presently.

3.1 Hypothesis. We assume $\mathfrak{u}$ is a nice construction framework and τ is a weak $\mathfrak{u}$-sub-vocabulary.

Remark. In Definition 3.2(1) below "$\mathbf{d}_*$ witnesses not being able to lift $\mathbf{d}$", of course we can ensure it can be lifted.

3.2 Definition. Let $\xi \leq \partial + 1$, if we omit it we mean $\xi = \partial + 1$.
1) We say that $(M, N, \mathbf{I}) \in \mathrm{FR}^1_{\mathfrak{u}}$ has weak ξ-uq-invariant coding$_0$ <u>when</u>:

- $\circledast$ if $M \leq_{\mathfrak{u}} M'$ and $M' \cap N = M$ <u>then</u> there are an ordinal $\alpha < \xi$ and a $\mathfrak{u}$-free $(\alpha, 0)$-rectangle $\mathbf{d}$ so $\alpha = \partial$ is O.K., such that:
  - (a) $M^{\mathbf{d}}_{0,0} = M'$ and $M^{\mathbf{d}}_{\alpha,0} \cap N = M$
  - (b) for every $N', \mathbf{I}'$ such that $(M', N', \mathbf{I}')$ is $\leq^1_{\mathfrak{u}}$-above $(M, N, \mathbf{I})$ and $N' \cap M^{\mathbf{d}}_{\alpha,0} = M'$ we can find $\alpha', \mathbf{I}^1, \alpha_*$ and $\mathbf{d}_*$ such that $\alpha' \leq \alpha, \alpha' < \partial$ (no harm[21] in $\alpha < \partial \Rightarrow \alpha' = \alpha$) and:

- (α) $\mathbf{d}_*$ is a $\mathfrak{u}$-free $(\alpha_*, 0)$-rectangle and $\alpha_* < \partial$
- (β) $M^{\mathbf{d}_*}_{0,0} = N'$ and $M^{\mathbf{d}}_{\alpha',0} \leq_{\mathfrak{u}} M^{\mathbf{d}_*}_{\alpha_*,0}$
- (γ) $(M', N', \mathbf{I}') \leq^1_{\mathfrak{u}} (M^{\mathbf{d}}_{\alpha,0}, M^{\mathbf{d}_*}_{\alpha_*,0}, \mathbf{I}^1)$
- (δ) there are no $\mathbf{d}_1, \mathbf{d}_2$ such that
  - $\bullet_1$ $\mathbf{d}_\ell$ is a $\mathfrak{u}$-free rectangle for $\ell = 1, 2$
  - $\bullet_2$ $\alpha(\mathbf{d}_1) = \alpha_*$ and $\mathbf{d}_1 \restriction (\alpha_*, 0) = \mathbf{d}_*$
  - $\bullet_3$ $\alpha(\mathbf{d}_2) \geq \alpha'$ and $\mathbf{d}_2 \restriction (\alpha', 0) = \mathbf{d} \restriction (\alpha', 0)$

[21]by natural monotonicity, similarly in 3.7, 3.10, 3.14

- $\bullet_4$ $(M^{\mathbf{d}_2}_{0,0}, M^{\mathbf{d}_2}_{0,1}, \mathbf{I}^{\mathbf{d}_2}_{0,0}) = (M', N', \mathbf{I}')$ or just $(M, N, \mathbf{I}) \leq^1_{\mathfrak{u}} (M^{\mathbf{d}_2}_{0,0}, M^{\mathbf{d}_2}_{0,1}, I^{\mathbf{d}_2}_{0,0})$
- $\bullet_5$ there are $N \in \mathfrak{K}_{\mathfrak{u}}$ and $\leq_{\mathfrak{u}}$-embeddings f_ℓ of $N^{\mathbf{d}_\ell}_{\alpha(\mathbf{d}_\ell),\beta(\mathbf{d}_\ell)}$ into N for $\ell = 1, 2$ such that $f_1 \restriction M^{\mathbf{d}}_{\alpha',0} = f_2 \restriction M^{\mathbf{d}}_{\alpha',0}$ and f_1, f_2 maps $M^{\mathbf{d}_1}_{0,\beta(\mathbf{d}_1)}$ onto $M^{\mathbf{d}_2}_{0,\beta(\mathbf{d}_2)}$.

2) We say that $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*) \in K^{\rm qt}_{\mathfrak{u}}$ has the weak ξ-uq-invariant coding$_0$ property <u>when</u>: if $\alpha(0) < \partial$ and $(M_*, N_0, \mathbf{I}_0) \in \mathrm{FR}^+_1$, $M_* \leq_{\mathfrak{u}} M^*_{\alpha(0)}$ and $N_0 \cap M^*_\partial = \emptyset$ <u>then</u> for some club E of ∂, for every $\delta \in E$ the statement $\circledast$ of part (1) holds, with $(M_*, N_0, \mathbf{I}_0), M_\delta$ here standing for $(M, N, \mathbf{I}), M'$ there but with some changes:

- $(*)_1$ $\mathbf{d}$ is such that $M^{\mathbf{d}}_{\alpha',0} \leq_{\mathfrak{u}} M^*_\beta$ for each $\alpha' < \alpha$ for any $\beta < \partial$ large enough and
- $(*)_2$ in clause (b) we demand $N' \cap M^*_\partial = M_\delta$ and $M^{\mathbf{d}_*}_{\alpha_*} \leq_{\mathfrak{u}} M^*_\beta$ for every $\beta < \partial$ large enough.

3) We say that $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ has the weak ξ-uq-invariant coding$_1$ property as in part (2) but require only that there are such $\alpha(0)$, $(M_*, N_0, \mathbf{I}_0)$ so without loss of generality $M_* = M^*_{\alpha(0)}$.
3A) We say that $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f})$ has the S-weak ξ-uq-invariant coding$_2$ property <u>when</u> we combine the above with Definition 2.6.
4) For $k = 0, 1$ we say that $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f})$ has the S-weak ξ-uq-invariant coding$_k$ property <u>when</u>: S is a stationary subset of ∂ and for some club E of ∂ the demand in part (2) if $k = 0$, part (3) if $k = 1$ holds restricting ourselves to $\delta \in S \cap E$.
5) We say that $\mathfrak{u}$ has the S-weak ξ-uq-invariant coding$_k$ property when: $\{0, 2\}$-almost every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ has the S-weak ξ-uq-invariant coding$_k$ property. Similarly for "above $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*) \in K^{\rm qt}_{\mathfrak{u}}$". If $S = \partial$ we may omit it; if $k = 1$ we may omit it.

3.3 Claim. *Assume ($\xi \leq \partial + 1$ and):*

- (a) *$S \subseteq \partial$ is stationary*
- (b) *$(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ has the S-weak ξ-uq-invariant coding property*

(*c*) $\mathbf{f} \restriction S$ *is constantly zero*

(*d*) $\xi \leq \partial + 1$ *and if* $\xi = \partial$ *then the ideal* $\mathrm{WDmId}(\partial) + (\lambda \backslash S)$ *is not* ∂*-saturated.*

Then we can find $\langle (\bar{N}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta) : \eta \in {}^\partial 2 \rangle$ *such that*

(α) $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \leq^{\mathrm{qs}}_{\mathfrak{u}} (\bar{N}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta)$

(β) $\mathbf{f}^\eta(\partial \backslash S) = \mathbf{f} \restriction (\partial \backslash S)$

(γ) *if* $\eta^1 \neq \eta^2 \in {}^\partial 2$ *and* $(\bar{N}^{\eta^\ell}, \bar{\mathbf{J}}^{\eta^\ell}, \mathbf{f}^{\eta^\ell}) \leq^{\mathrm{qt}}_{\mathfrak{u}} (\bar{N}^\ell, \bar{\mathbf{J}}^\ell, \mathbf{f}^\ell)$ *for* $\ell = 1, 2$ *then* $N^1_\partial, N^2_\partial$ *are not isomorphic over* M_∂.

3.4 Remark. Note that in Definition 3.2(1) we choose the $\mathfrak{u}$-free $(\alpha_\delta, 0)$-rectangle $\mathbf{d}_\delta$ for every $\delta \in S$ before we have arrived to choosing N^η_δ. This will be a burden in applying this.

Proof. For simplicity we first assume $\xi \leq \partial$. Let $\langle S_\varepsilon : \varepsilon \leq \partial \rangle$ be a sequence of pairwise disjoint stationary subsets of ∂ with union $S \backslash \{0\}$ such that $\varepsilon < \min(S_\varepsilon)$ (exists; if $\xi = \partial$ by assumption (d), otherwise if ∂ successor by applying Ulam matrixes, in general by a theorem of Solovay). Without loss of generality $0 \notin S$.

By assumption (b) we can find $\alpha(0) < \partial$ and $N_0, \mathbf{I}_0$ such that $N_0 \cap M_\partial = M_{\alpha(0)}$ and $(M_{\alpha(0)}, N_0, \mathbf{I}) \in \mathrm{FR}_1$ and a club E_0 of ∂ such that and for every $\delta \in S \cap E_0$ we can choose $\alpha_\delta, \mathbf{d}_\delta$ as in $\circledast$ of Definition 3.2(1) with $M_{\alpha(0)}, N_0, \mathbf{I}_0, M_\delta$ here standing for $M, N, \mathbf{I}, M'$ there but demanding $M^{\mathbf{d}_\delta}_{\alpha_\delta, 0} \leq_{\mathfrak{u}} M_{\beta_\delta}$ for some $\beta_\delta \in (\delta, \partial)$, see $(*)_1$ of Definition 3.2(2). Without loss of generality $\alpha(0)$ is a successor ordinal.

Let

$$\begin{aligned} E_1 = \{\delta \in E_0 :& \delta > \alpha(0) \text{ and if } \delta(1) \in \delta \cap S \cap E_0 \\ & \text{then } \beta_{\delta(1)} \leq \delta, \text{ i.e. } M^{\mathbf{d}_{\delta(1)}}_{\alpha_{\delta(1)}, 0} \leq_{\mathfrak{u}} M_\delta \\ & \text{and } (\forall \beta < \delta)(\beta \times \beta < \delta \wedge \mathbf{f}(\beta) < \delta)\}. \end{aligned}$$

Clearly E_1 is a club of ∂.

We now choose $\langle(\delta^\rho, u^\rho, \bar{N}^\rho, \bar{M}^\rho, \bar{\mathbf{J}}^{0,\rho}, \bar{\mathbf{J}}^{1,\rho}, \mathbf{f}^\rho, \mathbf{I}^\rho, e^\rho) : \rho \in {}^i 2\rangle$ by induction on $i < \partial$ such that

$\circledast$ (a) (α) $\delta^\rho < \partial$ belongs to $E_1 \cup \{\alpha(0)\}$

(β) e^ρ is a closed subset of $E_1 \cap \delta^\rho$

(γ) $\min(e^\rho) = \alpha(0)$, also $N^\rho_{\alpha(0)} = N_0, \mathbf{I}^\rho_{\alpha(0)} = \mathbf{I}$

(b) $\mathbf{f}^\rho : e^\rho \to \delta^\rho$

(c) $\mathbf{f}^\rho(\alpha) < \beta$ if $\alpha < \beta$ are from e^ρ

(d) if $\alpha \in e^\rho \backslash S$ then $\mathbf{f}^\rho(\alpha) = \mathbf{f}(\alpha)$

(e) (α) $\bar{N}^\rho = \langle N^\rho_i : i \leq \delta^\rho\rangle$ and $\bar{M}^\rho = \langle M^\rho_i : i \leq \delta^\rho\rangle$ are $\leq_{\mathfrak{u}}$-increasing continuous

(β) $\langle(M^\rho_i, N^\rho_i, \mathbf{I}^{\rho\restriction i}) : i \leq \delta\rangle$ is $\leq^1_{\mathfrak{u}}$-increasing continuous

(f) (α) $\bar{\mathbf{J}}^{\ell,\rho} = \langle \mathbf{J}^{\ell,\rho}_i : i < \delta^\rho\rangle$

(β) $(M^\rho_i, M^\rho_{i+1}, \mathbf{J}^{0,\rho}_i) \in \mathrm{FR}_2$ for $i < \delta^\rho$

(γ) $(M^\rho_i, M^\rho_{i+1,\rho}, \mathbf{J}^{0,\rho}_i) \leq^2_{\mathfrak{u}} (N^\rho_i, N^\rho_{i+1}, \mathbf{J}^{1,\rho}_i)$ when $i < \delta^\rho$ & $(\exists\alpha)(\alpha \in e^\rho \cap S \ \&\ \alpha \leq i \leq \alpha + \mathbf{f}^\rho(\alpha))$

(g) if $\alpha \in e^\rho$ <u>then</u> $M^\rho_\alpha = M_\alpha$ and $N_\alpha \cap M_\partial = M_\alpha$,

(h) if $\alpha \in e^\rho \backslash S$ then $(\mathbf{f}^\rho(\alpha) = \mathbf{f}(\alpha)$ and)
$i \leq \mathbf{f}(\alpha) \Rightarrow M^\rho_{\alpha+i} = M_{\alpha+i} \wedge N^\rho_{\alpha+i} \cap M_\partial = M_{\alpha+i} = M^\rho_{\alpha+i}$ and
$i < \mathbf{f}(\alpha) \Rightarrow \mathbf{J}^{0,\rho}_{\alpha+i} = \mathbf{J}_{\alpha+i}$ and
$i < \mathbf{f}(\alpha) \Rightarrow (M^\rho_{\alpha+i}, M^\rho_{\alpha+i+1}, \mathbf{J}^{0,\rho}_{\alpha+i}) \leq^2_{\mathfrak{u}} (N^\rho_{\alpha+i}, N^\rho_{\alpha+i+1}, \mathbf{J}^{1,\rho}_{\alpha+i})$

(i) if $\varrho \triangleleft \rho$ then $\delta_\varrho < \delta_\rho$, $e^\varrho = \delta_\varrho \cap e^\rho$, $\bar{M}^\varrho \triangleleft \bar{M}^\rho$, $\bar{N}^\varrho \triangleleft \bar{N}^\rho$, $\bar{\mathbf{J}}^{\ell,\varrho} \triangleleft \bar{\mathbf{J}}^{\ell,\rho}$ for $\ell = 0, 1$ and $\bar{\mathbf{I}}^\varrho \triangleleft \bar{\mathbf{I}}^\rho$

(j) if $\varepsilon < \partial$ and $\alpha \in e^\rho \cap S$ and $\rho(\alpha) = 1$ then $(\langle M^\rho_{\alpha+i} : i \leq \mathbf{f}^\rho(\alpha)\rangle$, $\langle \mathbf{J}^{0,\rho}_{\alpha+i} : i < \mathbf{f}^\rho(\alpha)\rangle)$ is equal to $\mathbf{d}_\alpha$

(k) if $\varepsilon < \partial$, $\alpha \in e^\rho \cap S$ and $\rho(\alpha) = 0$ <u>then</u> there is $\mathbf{d}_*$ as in clause (b) of $\circledast$ of Definition 3.2(1), for transparency of successor length β with $(M_{\alpha(0)}, N_0, \mathbf{I}_0)$, $(M^\rho_\alpha, N^\rho_\alpha, \mathbf{I}^\rho_\alpha)$, $\mathbf{d}_\alpha$, $\mathbf{d}_*$ here standing for $(M, N, \mathbf{I})$, $(M', N', \mathbf{I}')$, $\mathbf{d}$, $\mathbf{d}_*$ there, such that $\mathbf{f}^\rho(\alpha) = \alpha'$, $N^\rho_{\alpha+i,0} = M^{\mathbf{d}_*}_i$ for $i \leq \mathbf{f}^\rho(\alpha)$, $\mathbf{J}^{1,\rho}_{\alpha+i} = \mathbf{J}^{\mathbf{d}_*}_{i,0}$ for $i < \mathbf{f}^\rho(\alpha)$, and (for transparency) $M_{\alpha+i} = M_\alpha$, $\mathbf{J}^{0,\rho}_{\alpha+i} = \emptyset$ for $i \leq \mathbf{f}^\rho(\alpha)$.

Why can we construct?

Case 1: $i = 0$

For $\rho \in {}^i2$ let $\delta_\rho = \alpha(0)$ and by clause $(a)(\gamma)$ we define the rest (well for $i < \alpha(0)$ and $\ell = 0, 1$ let $M_i^\ell = M_{\alpha(0)}, N_i^\rho = N_0, \mathbf{J}_i^{\ell,\rho} = \emptyset, \mathbf{I}_i^\rho = \mathbf{I}$).

Case 2: i is a limit ordinal

For $\rho \in {}^i2$, let $\delta_\rho = \cup\{\delta_{\rho\restriction j} : j < i\}$ so $\delta_\rho \in E_1, \delta_\rho = \sup(E_1 \cap \alpha)$. By continuity we can define also the others.

Case 3: $i = j + 1$

Let $\rho \in {}^j2$ and we define for $\rho\hat{}\langle\ell\rangle$ for $\ell = 0, 1$ and first we deal only with $i \le \delta + \mathbf{f}^{\rho\hat{}<\ell>}(\delta)$.

Subcase 3A: $\delta_\rho \notin S$

We use clause (h) of $\circledast$ and 1.5(5).

Subcase 3B: $\delta \in S$

If $\ell = 1$ then we use $\mathbf{d}_\alpha$ for $\rho\hat{}\langle\ell\rangle$ as in clause (j) of $\circledast$ so the proof is as in subcase 3A. If $\ell = 0$ clause (b) of $\circledast$ of Definition 3.2 can be applied with $(M, N, \mathbf{I}), (M', N', \mathbf{I}'), \mathbf{d}$ there standing for $(M_{\alpha(0)}, N_0, \mathbf{I}_0), (M^\rho_{\delta_\rho}, N^\rho_{\delta_\rho}, \mathbf{I}^\rho_{\delta_\rho}), \mathbf{d}_{\delta_\rho}$ here; so we can find $\alpha'_\rho, \mathbf{I}^1_\rho, \alpha^\rho_*, \mathbf{d}^\rho_*$ as there (presently α'_ρ there can be α_δ); and without loss of generality $\ell g(\mathbf{d}^\rho_*)$ is a successor ordinal.

Now we choose:

$(*)$ (a) $\mathbf{f}^{\rho\hat{}<0>}(\delta_\rho) = \ell g(\mathbf{d}^\rho_*)$

(b) $M^{\rho\hat{}<0>}_{\delta_\rho+i} = M^\rho_{\delta_\rho}$ for $i < \mathbf{f}^{\rho\hat{}<0>}(\delta)$

(c) $M^{\rho\hat{}<0>}_{\delta_\rho+i} = M^{\mathbf{d}_\delta}_{\alpha'_\rho,0}$ for $i = \mathbf{f}^{\rho\hat{}<0>}(\delta_\rho)$

(d) $N^{\rho\hat{}<0>}_{\delta_\rho+i} = N^{\mathbf{d}^\rho_*}_{i,0}$ for $i \le \mathbf{f}^{\rho\hat{}<0>}(\delta_\rho)$

(e) $\mathbf{J}^{0,\rho\hat{}<\ell>}_{\delta_\rho+i} = \emptyset, \mathbf{J}^{1,\rho\hat{}<\ell>}_{\delta_\rho+i} = \mathbf{J}^{\mathbf{d}^\rho_*}_{i,0}$ for $i < \mathbf{f}^{\rho\hat{}<0>}(\delta_\rho)$.

Clearly clause (k) holds. This ends the division to cases 3A,3B.

Lastly, choose $\delta_{\rho\hat{}<\ell>} \in E_1$ large enough; we still have to choose $(M_i^{\rho\hat{}<\ell>}, N_i^{\rho\hat{}<\ell>}, \mathbf{I}^{\rho\hat{}<\ell>})$ for $i \in (\delta_\rho + \mathbf{f}^\rho(\delta_\rho), \delta_{\rho\hat{}<\ell>}]$; we choose them all equal, $M_i^{\rho\hat{}<\ell>} = M_{\delta_{\rho\hat{}<\ell>}}$ and use 1.3(2) to choose $N_i^{\rho\hat{}<\ell>}$,

$\mathbf{I}_i^{\rho\char`\^<\ell>}$. Then let $\mathbf{J}_i^{m,\rho\char`\^<\ell>} = \emptyset$ when $m < 2$ & $i \in [\delta_\rho + \mathbf{f}^\rho(\delta_\rho), \delta_{\rho\char`\^<\ell>})$.

So we have carried the induction. For $\rho \in {}^\partial 2$ we define $(\bar{M}^\rho, \bar{\mathbf{J}}^\rho, \mathbf{f}^\rho) \in K_{\mathfrak{u}}^{\rm qt}$ by $M_\alpha^\rho = N_\alpha^{\rho\restriction i}, \mathbf{J}_\alpha^\rho = \bar{\mathbf{J}}_\alpha^{1,\rho\restriction i}, \mathbf{f}^\rho(\alpha) = \mathbf{f}^{\rho\restriction i}(\alpha)$ for every $i < \partial$ large enough. Easily

$\odot_1$ $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \le_{\mathfrak{u}}^{\rm qt} (\bar{M}^\rho, \bar{\mathbf{J}}^\rho, \mathbf{f}^\rho)$.

Next

$\odot_2$ for $\nu \in {}^\partial 2$ let $\rho_\nu = \rho[\nu] \in {}^\partial 2$ be defined by $\rho_\nu(i) = \nu(\varepsilon)$ if $i \in S_\varepsilon \wedge \varepsilon < \partial$ and zero otherwise.

So it is enough to prove

$\odot_3$ if $\nu_1 \ne \nu_2 \in {}^\partial 2$ and $(\bar{M}^{\rho[\nu_\ell]}, \bar{\mathbf{J}}^{\rho[\nu_\ell]}, \mathbf{f}^{\rho[\nu_\ell]}) \le_{\mathfrak{u}}^{\rm qt} (\bar{M}^\ell, \bar{\mathbf{J}}^\ell, \mathbf{f}^\ell)$ for $\ell = 1, 2$, <u>then</u> $M_\partial^1, M_\partial^2$ are not isomorphic over M_∂.

Why this holds? As $\nu_1 \ne \nu_2$, by symmetry without loss of generality for some $\varepsilon < \partial$ we have $\nu_1(\varepsilon) = 1, \nu_2(\varepsilon) = 0$, and let f be an isomorphism from M_∂^1 onto M_∂^2 over M_∂ and let E^ℓ witness $(\bar{M}^{\rho[\nu_\ell]}, \bar{\mathbf{J}}^{\rho[\nu_\ell]}, \mathbf{f}^{\rho[\nu_\ell]}) \le_{\mathfrak{u}}^{\rm qt} (\bar{M}^\ell, \bar{\mathbf{J}}^\ell, \mathbf{f}^\ell)$ for $\ell = 1, 2$. Let $E := E^1 \cap E^2 \cap \{\delta < \partial : \delta \in \cup\{e^{\rho[\nu_\ell]\restriction i} : i < \partial\}$ and $\delta = \operatorname{otp}(\delta \cap e_{\rho[\nu_\ell]\restriction\delta})$ for $\ell = 1, 2$ and f maps M_δ^1 onto $M_\delta^2\}$, clearly it is a club of ∂.

Hence there is $\delta \in S_\varepsilon \cap E$, so $\rho[\nu_1](\delta) = 1, \rho[\nu_2](\delta) = 0$, now the contradiction is easy, recalling:

$(*)_1$ clause (δ) of Defintion 3.2(1)

$(*)_2$ $\mathbf{d}_\delta$ does not depend on $\rho[\nu_\ell] \restriction \delta$.

We still owe the proof in the case $\xi = \partial + 1$, it is similar with two changes. The first is in the choice of $\mathbf{d}_\delta$, as now $\alpha_{\mathbf{d}_\delta}$ may be ∂, so β_δ may be ∂, hence we have to omit "$\beta_{\delta(1)} \le \delta$" in the definition of E_1. In clause $\circledast(j), (k)$ we should replace $\mathbf{d}_\alpha$ by $\mathbf{d}_\alpha \restriction (0, \gamma_{\rho\restriction\alpha})$ so $\alpha + \gamma_{\rho\restriction\delta} < \min(e_\rho \backslash (\alpha+1))$ where $\gamma_{\rho\restriction\alpha} < \min\{\ell g(\mathbf{d}_\alpha) + 1, \partial\}$ which is the minimal α when we apply α in Definition 3.2.

Second, the choice of $\langle \rho_\nu : \nu \in {}^\partial 2\rangle$ with $\rho_\nu \in {}^\partial 2$ is more involved. For each $\varepsilon < \partial$ we choose $\varrho_\varepsilon \in {}^{(S_\varepsilon)}2$ such that:

$\boxdot$ if $\rho_1, \rho_2 \in {}^\partial 2$ then for stationarily many $\delta \in S_\varepsilon$ we have: $\mathbf{f}^{\rho_1}(\delta) \le \mathbf{f}^{\rho_2}(\delta) \Leftrightarrow \varrho_\varepsilon(\delta) = 1$

(noting that $\mathbf{f}^{\rho_\alpha}(\delta)$ depends only on $\rho \restriction \delta$).

[Why possible? As S_ε is not in the weak diamond ideal.]

Then we replace $\odot_2$ by

$\odot'_2$ for $\nu \in {}^\partial 2$ let $\rho_\nu = \rho[\nu] \in {}^\partial 2$ be defined by $\rho_\nu(i) = \varrho_\varepsilon(i) + \nu(\varepsilon)$ mod 2 when $i \in S_\varepsilon \wedge \varepsilon < \partial$ and $\rho_\nu(i)$ is zero otherwise.

Why this is O.K.? I.e. we have to prove $\odot_3$ in this case. Why this holds? As $\nu_1 \neq \nu_2$ by symmetry without loss of generality $\nu_1(\varepsilon) = 1, \nu_2(\varepsilon) = 0$ and let f, E be as before.

Let $\rho_\ell := \rho[\nu_\ell] \in {}^\partial 2$, so by the choice of ϱ_ε there is $\delta \in E \cap S_\varepsilon$ such that

$$\mathbf{f}^{\rho_1}(\delta) \le \mathbf{f}^{\rho_2}(\delta) \Leftrightarrow \varrho_\varepsilon(\delta) = 1.$$

First assume $\varrho_\varepsilon(\delta) = 1$ hence $\mathbf{f}^{\rho_1}(\delta) \le \mathbf{f}^{\rho_2}(\delta)$, so $\rho_1(\delta) = \varrho_\varepsilon(\delta) + \nu_1(\varepsilon) = 1 + 1 = 0$ mod 2 so $\rho_1(\delta) = 0$ and $\rho_2(\delta) = \varrho_\varepsilon(\delta) + \nu_2(\varepsilon) = 1 + 0 = 1$ mod 2, so $\nu_2(\delta) = 1$.

Now we continue as before, because what we need there for $(\bar{M}^{\rho[\nu_2]}, \bar{\mathbf{J}}^{\rho[\nu_2]}, \mathbf{f}^{\rho[\nu_2]})$ in δ is satisfied for $\delta + \mathbf{f}^{\nu[\rho_2]}(\delta)$ hence also for $\delta + \mathbf{f}^{\nu[\rho_1]}$.

The other case, $\varrho_\varepsilon(\delta) = 0$, is similar; exchanging the roles. $\square_{3.3}$

The following conclusion will be used in 6.15, 6.16.

3.5 Conclusion. We have $\dot{I}(\partial^+, \mathfrak{K}) \ge \mu_{\text{unif}}(\partial^+, 2^\partial)$ and moreover $\dot{I}(K^{\mathfrak{u},\mathfrak{h}}_{\partial^+}) \ge \mu_{\text{unif}}(\partial^+, 2^\partial)$ for any $\{0,2\}$-appropriate $\mathfrak{h}$ <u>when</u> ($\xi \le \partial + 1$ and):

- (a) $2^\partial < 2^{\partial^+}$ and $\partial > \aleph_0$
- (b) (α) $\mathscr{D}_\partial$, the club filter on ∂, is not ∂^+-saturated
  (β) if $\xi = \partial + 1$ then WDmId(∂) + $(\partial \backslash S)$ is not ∂^+-saturated
- (c) $\{0,2\} - S$-almost every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f})$ has the weak ξ-uq-invariant coding property even just above $(\bar{M}^*, \bar{\mathbf{J}}, \mathbf{f})$, so $S \subseteq \partial$ is stationary.

Proof. By 3.3 we can apply 3.6 below using

$\odot$ if $\mathscr{D}$ is a normal filter on the regular uncountable $\partial, \mathscr{D}$ not ∂^+-saturated then also the normal ideal generated by $\mathscr{A}$ is not ∂^+-saturated where $\mathscr{A} = \{A \subseteq \partial : A \in \mathscr{D}$ or $\partial \backslash A \in \mathscr{A}^+$ but $\mathscr{D} + (\partial \backslash A)$ is ∂-saturated$\}$.

$\square_{3.5}$

3.6 Theorem. *1) We have $\dot{I}(\partial^+, K^{\mathfrak{u}}_{\partial^+}) \geq \mu_{\text{unif}}(\partial^+, 2^\partial)$; moreover $\dot{I}(K^{\mathfrak{u},\mathfrak{h}}_{\partial^+}) \geq \mu_{\text{unif}}(\partial^+, 2^\partial)$ for any $\mathfrak{u} - \{0,2\}$-appropriate $\mathfrak{h}$ (see Definition 1.24) <u>when</u>:*

- (a) $2^\partial < 2^{\partial^+}$
- (b) *$\mathscr{D}$ is a non-∂^+-saturated normal filter on ∂*
- (c) *for $\{0,2\}$-almost every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\text{qt}}_{\mathfrak{u}}$ (maybe above some such triple $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$ satisfying $\mathscr{D}_\partial + \mathbf{f}^{-1}\{0\}$ is not ∂^+-saturated), if $S \subseteq \partial$ belongs to $\mathscr{D}^+$ and $\mathbf{f} \restriction S$ is constantly zero then we can find a sequence $\langle (\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha < 2^\partial \rangle$ such that*
  - (α) $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \leq^{\text{qt}}_{\mathfrak{u}} (\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha)$ *and* $\mathbf{f}^\alpha \restriction (\partial \backslash S) = \mathbf{f} \restriction (\partial \backslash S)$
  - (β) *if $\alpha(1) \neq \alpha(2) < 2^\partial$ and $(\bar{M}^{\alpha(i)}, \bar{\mathbf{J}}^{\alpha(i)}, \mathbf{f}^{\alpha(i)}) \leq^{\text{qt}}_{\mathfrak{u}} (\bar{M}^{\ell,*}, \bar{\mathbf{J}}^{\ell,*}, \mathbf{f}^{\ell,*})$ for $\ell = 1,2$ then $M^{1,*}_\partial, M^{2,*}_\partial$ are not isomorphic over M_∂.*

2) Similarly omitting the "∂^+-saturation" demands in clauses (b),(c) and omitting $\mathbf{f} \restriction S$ is constantly zero in clause (c).

Proof. 1) By Observation 1.25(4) without loss of generality $\mathfrak{h} = \mathfrak{h}_0 \cup \mathfrak{h}_2$ witness clause (c) of the assumption; we shall use $\mathfrak{h}_2$ for S^*_0 so without loss of generality $\mathfrak{h}_2$ is a $2 - S^*_0$-appropriate. By clause (b) of the assumption let $\bar{S}^*$ be such that

$\odot$ (a) $\bar{S}^* = \langle S^*_\alpha : \alpha < \partial^+ \rangle$
(b) $S^*_\alpha \subseteq \partial$ for $\alpha < \partial^+$
(c) $S^*_\alpha \backslash S^*_\beta \in [\partial]^{<\partial}$ for $\alpha < \beta < \partial^+$

(*d*) $S^*_{\alpha+1} \backslash S^*_\alpha$ is a stationary subset of ∂; moreover $\in \mathscr{D}^+$

(*e*) $S = S^*_0$ is stationary; it includes $\{\delta < \partial : \mathbf{f}^*(\delta) > 0\}$ when $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$ is given.

Let $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$ be as in clause (c) of the assumption; so $\{0,2\}$-almost every $(\bar{M}^{**}, \bar{\mathbf{J}}^{**}, \mathbf{f}^{**}) \in K^{\rm qt}_{\mathfrak{u}}$ which is $\leq_{\rm qs}$-above $(\bar{M}^*, \mathbf{J}^*, \mathbf{f}^*)$ is as there witnessed by $\mathfrak{h}$.

Now we choose $\langle (\bar{M}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta) : \eta \in {}^\alpha(2^\partial)\rangle$ by induction on $\alpha < \partial^+$ such that

$\circledast$ (a) $(\bar{M}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta) \in K^{\rm qt}_{\mathfrak{u}}$, and is equal to $(\bar{M}^*, \mathbf{J}^*, \mathbf{f}^*)$ if $\eta = <>$

(b) $\langle (\bar{M}^{\eta \restriction \beta}, \bar{\mathbf{J}}^{\eta \restriction \beta}, \mathbf{f}^{\eta \restriction \beta}) : \beta \leq \ell g(\eta)\rangle$ is $\leq^{\rm qs}_{\mathfrak{u}}$-increasing continuous

(c) $\mathbf{f}^\eta \restriction (\partial \backslash S^*_{\ell g(\eta)+1})$ is constantly zero

(d) if $\ell g(\eta) = \beta + 2 \leq \alpha$ and $\nu = \eta \restriction (\beta + 1))$ then $(\bar{M}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta) \leq^{\rm at}_{\mathfrak{u}} (\bar{M}^\nu, \bar{\mathbf{J}}^\nu, \mathbf{f}^\nu)$ and this pair strictly S-obeys $\mathfrak{h}$

(e) if $\ell g(\eta) = \delta < \alpha, \delta$ limit or zero, $\varepsilon^0 < \varepsilon^1 < 2^\partial$ and $(\bar{M}^{\eta\hat{}<\varepsilon^\ell>}, \bar{\mathbf{J}}^{\eta\hat{}<\varepsilon^\ell>}, \mathbf{f}^{\eta\hat{}<\varepsilon^\ell>}) \leq^{\rm qt}_{\mathfrak{u}} (\bar{M}^\ell, \bar{\mathbf{J}}^\ell, \mathbf{f}^\ell)$ for $\ell = 0, 1$ <u>then</u> $M^1_\partial, M^2_\partial$ are not isomorphic over M^η_∂.

The inductive construction is straightforward:

if $\alpha = 0$ let $(\bar{M}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta) = (\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$

if α is limit use claim 1.19(4)

if $\alpha = \beta + 2$ use clause $\circledast(d)$

if $\alpha = \delta + 1, \delta$ limit or zero use clause (c) of the assumption to satisfy clause $\circledast(e)$.

Having carried the induction, let $M_\eta = \cup\{M^{\eta \restriction \alpha} : \alpha < \partial^+\}$ for $\eta \in {}^{\partial^+}(2^\partial)$. By 9.1 we get that $|\{M^\eta/\cong : \eta \in {}^{\partial^+}(2^\partial)\}| \geq \mu_{\rm unif}(\partial^+, 2^\partial)$ so we are done.

2) Similarly. $\square_{3.5}$

* * *

We now note how we can replace the ξ-uq-invariant by ξ-up-invariant, a relative, not used.

3.7 Definition. Let $\xi \le \partial + 1$.
1) We say that $(M, N, \mathbf{I}) \in \mathrm{FR}^1_{\mathfrak{u}}$ has the weak ξ-up-invariant coding property <u>when</u>:

- $\circledast$ if $M \le_{\mathfrak{u}} M'$ and $M' \cap N = \emptyset$ <u>then</u> there are $\alpha_\ell < \xi$ and $\mathfrak{u}$-free $(\alpha_\ell, 0)$-rectangle $\mathbf{d}_\ell$ for $\ell = 1, 2$ such that:
  - (a) $M^{\mathbf{d}_1}_{0,0} = M' = M^{\mathbf{d}_2}_{0,0}$
  - (b) $M^{\mathbf{d}_1}_{\alpha_1,0} = M^{\mathbf{d}_2}_{\alpha_2,0}$
  - (c) $M^{\mathbf{d}_\ell}_{\alpha_\ell,0} \cap N = M$
  - (d) if $(M, N, \mathbf{I}) \le_1 (M', N', \mathbf{I}')$ and $M^{\mathbf{d}_\ell}_{\alpha_\ell,0} \cap N' = M'$ <u>then</u> there are no $\alpha'_\ell \le \alpha_\ell, \alpha'_\ell < \partial, \beta_\ell < \partial$ and $\mathfrak{u}$-free $(\alpha'_\ell, \beta_\ell)$-rectangles $\mathbf{d}^\ell$ for $\ell = 1, 2$ such that
    - $\bullet_1$ $\mathbf{d}^\ell \restriction (\alpha_\ell, 0) = \mathbf{d}_\ell \restriction (\alpha'_\ell, 0)$
    - $\bullet_2$ $(M', N', \mathbf{I}') \le_1 (M^{\mathbf{d}^\ell}_{0,0}, M^{\mathbf{d}^\ell}_{0,1}, \mathbf{I}^{\mathbf{d}^\ell}_{\alpha,0})$
    - $\bullet_3$ there are N'', f such that $M^{\mathbf{d}^2}_{\alpha'_2,\beta_2} \le_{\mathfrak{u}} N$ and f is a $\le_{\mathfrak{u}}$-embedding of $N^{\mathbf{d}^1}_{\alpha'_1,\beta_1}$ into N over $M^{\mathbf{d}_1}_{\alpha'_1,0} = M^{\mathbf{d}_2}_{\alpha_2,0}$ mapping $M^{\mathbf{d}^1}_{0,\beta_1}$ onto $M^{\mathbf{d}^2}_{0,\beta_2}$.

2)-5) As in Definition 3.2 replacing uq by up.

3.8 Claim. *Like 3.3 replacing uq-invariant by up-invariant.*

Proof. Similar. $\square_{3.8}$

3.9 Conclusion. Like 3.5 replacing uq-invariant by up-invariant (in clause (c)).

Proof. Similar. $\square_{3.11}$

* * *

Another relative is the vertical one.

3.10 Definition. Let $\xi \leq \partial + 1$, omitting ξ means $\partial + 1$. We say that $(M, N, \mathbf{I}) \in \mathrm{FR}^1_{\mathfrak{u}}$ has the vertical ξ-uq-invariant coding$_1$ property <u>when</u>:

$\otimes$ if $\alpha_0 < \partial$ and $\mathbf{d}_0$ is an $\mathfrak{u}$-free $(\alpha_0, 0)$-rectangle satisfying $M \leq_{\mathfrak{u}} M^{\mathbf{d}_0}_{0,0}$ and $M^{\mathbf{d}_0}_{\alpha_0,0} \cap N = M$ <u>then</u> there are $\alpha, \mathbf{d}$ such that:

(a) $\alpha_0 < \alpha < \xi$

(b) $\mathbf{d}$ is a $\mathfrak{u}$-free $(\alpha, 0)$-rectangle, though α is possibly ∂ this is O.K.

(c) $\mathbf{d} \restriction (\alpha_0, 0) = \mathbf{d}_0$

(d) $M^{\mathbf{d}}_{\alpha,0} \cap N = M$

(e) for every $N', \mathbf{I}'$ such that $(M^{\mathbf{d}}_{0,0}, N', \mathbf{I}') \in \mathrm{FR}_1$ is $\leq^1_{\mathfrak{u}}$-above $(M, N, \mathbf{I})$ and $N' \cap M^{\mathbf{d}}_{\alpha,0} = M^{\mathbf{d}}_{0,0}$ we can find $\alpha', \alpha_*, \mathbf{d}_*, \mathbf{I}'', M''$ such that

(α) $\alpha' \leq \alpha, \alpha' < \partial, \alpha_0 \leq \alpha_*$

(β) $\mathbf{d}_*$ is a $\mathfrak{u}$-free $(\alpha_*, 0)$-rectangle

(γ) $M^{\mathbf{d}_*}_{0,0} = N'$

(δ) there is an $\mathfrak{u}$-free $(\alpha_0, 1)$-rectangle $\mathbf{d}'$ such that $\mathbf{d}' \restriction (\alpha_0, 0) = \mathbf{d}_0, \mathbf{d}' \restriction ([0, \alpha_0), [1, 1]) = \mathbf{d}_* \restriction (\alpha_0, 0)$ and $\mathbf{I}^{\mathbf{d}'}_{0,0} = \mathbf{I}'$ and $(M^{\mathbf{d}'}_{\alpha_0,0}, M^{\mathbf{d}'}_{\alpha_0,1}, \mathbf{I}') \leq^1_{\mathfrak{u}} (M'', M^{\mathbf{d}_*}_{\alpha_*,0}, \mathbf{I}'')$

(ε) there are no $\mathbf{d}_1, \mathbf{d}_2$ such that

$\bullet_1$ $\mathbf{d}_\ell$ is a $\mathfrak{u}$-free rectangle for $\ell = 1, 2$

$\bullet_2$ $\alpha(\mathbf{d}_1) = \alpha_*$ and $\mathbf{d}_1 \restriction (\alpha_*, 0) = \mathbf{d}_*$

$\bullet_3$ $\alpha(\mathbf{d}_2) \geq \alpha'$ and $\mathbf{d}_2 \restriction (\alpha', 0) = \mathbf{d} \restriction (\alpha', 0)$

$\bullet_4$ $(M^{\mathbf{d}_2}_{0,0}, M^{\mathbf{d}_2}_{0,1}, \mathbf{I}^{\mathbf{d}_2}_{0,0}) = (M^{\mathbf{d}}_{0,0}, N', \mathbf{I}')$ or just $(M^{\mathbf{d}}_{0,0}, N', \mathbf{I}') \leq^1_{\mathfrak{u}} (M^{\mathbf{d}_2}_{0,0}, M^{\mathbf{d}_2}_{0,1}, \mathbf{I}^{\mathbf{d}_2}_{0,0})$

$\bullet_5$ there are N_1, N_2, f such that $M^{\mathbf{d}_\ell}_{0,\beta(\mathbf{d}_\ell)} \leq_{\mathfrak{u}} N_\ell$ for $\ell = 1, 2$
and f is an isomorphism from N_1 onto N_2 over $M^{\mathbf{d}}_{\alpha,0}$ mapping $M^{\mathbf{d}_1}_{0,\beta(\mathbf{d}_1)}$ onto $M^{\mathbf{d}_2}_{0,\beta(\mathbf{d}_2)}$.

2) We say that $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ has the vertical uq-invariant coding$_1$ propety as in Definition 3.2(2) only $(\langle M_{\delta+i} : i \le \mathbf{f}(\delta)\rangle, \langle \mathbf{J}_{\delta+i} : i < \mathbf{f}(\delta)\rangle)$ play the role of $\mathbf{d}_0$ in part (1). In all parts coding means coding$_1$.
3),4),5) Parallely to Definition 3.2.

3.11 Theorem. *Like 3.3 using vertical ξ-uq-invariant coding in clause (b) and omitting clause (c) of the assumption and omit clause (β) in the conclusion.* $\square_{3.11}$

Proof. Similarly.

3.12 Conclusion. Like 3.5 replacing clause (b)(β) of the assumption (by $\xi = \partial + 1 \Rightarrow (\exists\theta)2^\theta = 2^{<\partial} < 2^\partial$) and with vertical ξ-uq-invariant coding instead of the ξ-uq-invariant one (in clause (c), can use $S = \partial$).

Proof. Similar to the proof of 3.5. $\square_{3.12}$

* * *

3.13 Discussion: The intention below is to help in §6 to eliminate the assumption "WDmId(λ^+) is not λ^{++}-saturated" when $\mathfrak{s}$ fails existence for $K^{3,\mathrm{up}}_{\mathfrak{s},\lambda^+}$. We do using the following relatives, semi and vertical, from Definition 3.14, 3.10 are interesting because

- (a) under reasonable conditions (see Definition 3.17) the first implies the second
- (b) the second, as in Theorem 2.11 is enough for non-structure without the demand on saturation of WDmId(∂)
- (c) the first needs a weak version of a model theoretic assumption (in the application)
- (d) (not used) the semi-version implies the weak version (from 3.2).

3.14 Definition. Let $\xi \le \partial + 1$.
1) We say that $\mathfrak{u}$ has the semi ξ-uq-invariant coding$_1$ property [above some $(\bar M^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$] <u>when</u> for $\{0,2\}$-almost every $(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ [above $(\bar M^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$] for some $(\alpha, N, \mathbf{I})$ we have $\alpha < \partial, M \cap N = M_\alpha$ and $(M_\alpha, N, \mathbf{I}) \in \mathrm{FR}^+_1$ has the semi ξ-uq-invariant coding$_1$ property, see below but restricting ourselves to $M', M^{\mathbf{d}}_{\alpha_{\mathbf{d}},0} \in \{M_\beta : \beta \in (\alpha, \partial)\}$. Here and in part (2) we may write coding instead of coding$_1$.
2) We say that $(M, N, \mathbf{I}) \in \mathrm{FR}^1_{\mathfrak{u}}$ has the semi ξ-uq-invariant coding$_1$ property (we may omit the 1) <u>when</u>: if $M \le_{\mathfrak{u}} M'$ and $M' \cap N = M$ <u>then</u> we can find $\mathbf{d}$ such that:

⊛ (a) $\mathbf{d}$ is a $\mathfrak{u}$-free $(\alpha_{\mathbf{d}}, 0)$-rectangle with $\alpha_{\mathbf{d}} < \xi$ so $\alpha_{\mathbf{d}} \le \partial$

(b) $M^{\mathbf{d}}_{0,0} = M'$ and $M^{\mathbf{d}}_{\alpha(\mathbf{d}),0} \cap N = M$

(c) for any $N', \mathbf{I}'$:
if $(M, N, \mathbf{I}) \le_1 (M', N', \mathbf{I}')$ and $M^{\mathbf{d}}_{\alpha(\mathbf{d}),0} \cap N' = M'$
<u>then</u> we can find $\alpha', N'', \mathbf{I}''$ satisfying $\alpha' \le \alpha, \alpha' < \partial$ and $(M', N', \mathbf{I}') \le_1 (M^{\mathbf{d}}_{\alpha',0}, N'', \mathbf{I}'') \in \mathrm{FR}^1_{\mathfrak{u}}$ such that for no triple $(\mathbf{e}, f, N_*)$ do we have:

(α) $\mathbf{e}$ is a $\mathfrak{u}$-free rectangle

(β) $\alpha_{\mathbf{e}} = \alpha_{\mathbf{d}}$ and $\mathbf{e} \restriction (\alpha', 0) = \mathbf{d} \restriction (\alpha', 0)$

(γ) $(M^{\mathbf{e}}_{0,0}, M^{\mathbf{e}}_{0,1}, \mathbf{I}^{\mathbf{e}}_{0,0}) = (M^{\mathbf{d}}_{0,0}, N'', \mathbf{I}'')$

(δ) $M^{\mathbf{e}}_{\alpha',\beta(\mathbf{e})} \le_{\mathfrak{u}} N_*$

(ε) f is a $\le_{\mathfrak{u}}$-embedding of N'' into N_*

(ζ) $f \restriction M^{\mathbf{d}}_{\alpha',0}$ is the identity

(η) f maps N' into $M^{\mathbf{e}}_{0,\beta(\mathbf{e})}$.

3.15 Remark. 1) This is close to Definition 3.2 but simpler, cover the applications here and fit Claim 3.20.
2) We could have phrased the other coding properties similarly.

3.16 Claim. *If $(M,N,\mathbf{I}) \in \mathrm{FR}^1_{\mathfrak{u}}$ has the semi ξ-uq-invariant coding property* then *it has the weak ξ-uq-invariant coding property, see Definition 3.2.*

Proof. Should be clear. $\square_{3.16}$

The following holds in our natural examples when we add the fake, i.e. artificial equality and it is natural to demand $\mathfrak{u} = \mathfrak{u}^{[*]}$, see Definition 3.18.

3.17 Definition. 1) We say that $\mathfrak{u}$ has the fake equality $=_*$ when:

- (a) $\tau_{\mathfrak{K}}$ has only predicates and some two-place relation $=_* \in \tau_{\mathfrak{K}}$ is, for every $M \in K$, interpreted as an equivalence relation which is a congruence relation on M
- (b) $M \in K$ iff $M/=^M_*$ belongs to K
- (c) for $M \subseteq N$ both from K we have $M \le_{\mathfrak{u}} N$ iff $(M/=^N_*) \le_{\mathfrak{s}} (N/=^N_*)$
- (d) assume $M \le_{\mathfrak{u}} N$ and $\mathbf{I} \subseteq N \backslash M$ and $\mathbf{I}' = \{d \in \mathbf{I} : (\forall c \in M)(\neg c =^N d)\}, \ell \in \{1,2\}$. If $(M,N,\mathbf{I}) \in \mathrm{FR}_\ell$ then $(M/=^N, N/=^N, \mathbf{I}'/=^N) \in \mathrm{FR}_\ell$ which implies $(M,N,\mathbf{I}') \in \mathrm{FR}_\ell$
- (e) if $M \subseteq N$ are from K and $\mathbf{I} \subseteq \{d \in N : (\forall c \in M)(\neg c =^N d)\}$ and $\ell \in \{1,2\}$ then $(M,N,\mathbf{I}) \in \mathrm{FR}_\ell$ iff $(M/=^N, \mathbf{I}/=^N) \in \mathrm{FR}_\ell$.

1A) In part (1) we may say that $\mathfrak{u}$ has the fake equality $=_*$ or $=_*$ is a fake equality for $\mathfrak{u}$.
2) We say $\mathfrak{u}$ is hereditary when every $(M,N,\mathbf{I}) \in \mathrm{FR}^+_1$ is hereditary, see below.
3) We say $(M,N,\mathbf{I}) \in \mathrm{FR}^1_{\mathfrak{u}}$ is hereditary when:

- (a) if $\mathbf{d}$ is $\mathfrak{u}$-free $(1,2)$-rectangle, $M \le_{\mathfrak{u}} M^{\mathbf{d}}_{0,0}$ and $(M,N,\mathbf{I}) \le^1_{\mathfrak{u}} (M^{\mathbf{d}}_{0,1}, M^{\mathbf{d}}_{0,2}, \mathbf{I}^{\mathbf{d}}_{0,1})$
  then $(M,N,\mathbf{I}) \le^1_{\mathfrak{u}} (M^{\mathbf{d}}_{0,0}, M^{\mathbf{d}}_{0,2}, \mathbf{I}^{\mathbf{d}}_{0,1}) \le^1_{\mathfrak{u}} (M^{\mathbf{d}}_{1,0}, M^{\mathbf{d}}_{1,2}, \mathbf{I}^{\mathbf{d}}_{1,1})$.

4) We say $\mathfrak{u}$ is hereditary for the fake equality $=_*$ when every $(M,N,\mathbf{I}) \in \mathrm{FR}^+_1$ is hereditary for $=_*$ which means that clause (a) of part (3) above holds, $=_*$ is a fake equality for $\mathfrak{u}$ and:

(*b*) if $\mathbf{d}$ is a $\mathfrak{u}$-free $(0,2)$-rectangle, $M \le_{\mathfrak{u}} M^{\mathbf{d}}_{0,0}$ and $(M,N,\mathbf{I}) \le^1_{\mathfrak{u}} (M^{\mathbf{d}}_{0,1}, M^{\mathbf{d}}_{0,2}, \mathbf{I}^{\mathbf{d}}_{0,1})$ <u>then</u> we can find M_1, f, M_2 such that:

- (α) f is an isomorphism from $M^{\mathbf{d}}_{0,1}$ onto M_1 over $M^{\mathbf{d}}_{0,0}$
- (β) $M^{\mathbf{d}}_{0,0} \le_{\mathfrak{u}} M_1 \le_{\mathfrak{u}} M_2$ and $M_{0,2} \le_{\mathfrak{u}} M_2$
- (γ) $|M_2| = |M_1| \cup |M^{\mathbf{d}}_{0,2}|$
- (δ) $M_2 \models$ "$c =_* f(c)$" if $c \in M_{0,1}$
- (ε) $(M_{0,0}, M_{0,2}, \mathbf{I}^{\mathbf{d}}_{0,1}) \le^1_{\mathfrak{u}} (M_1, M_2, \mathbf{I}^{\mathbf{d}}_{0,1})$.

5) In parts (2),(3),(4) we can replace hereditary by weakly hereditary <u>when</u>: in clause (a) we assume $\mathbf{I} = \mathbf{I}^{\mathbf{d}}_{0,1} = \mathbf{I}^{\mathbf{d}}_{1,1}$ and in clause (b) we assume $\mathbf{I} = \mathbf{I}^{\mathbf{d}}_{0,1}$.

3.18 Definition. For $\mathfrak{u}$ is a nice construction framework we define $\mathfrak{u}^{[*]} = \mathfrak{u}^{[*]}$ like $\mathfrak{u}$ except that, for $\ell = 1,2$ we have $(M_1, N_1, \mathbf{I}_1) \le^{\ell}_{\mathfrak{u}[*]} (M_2, N_2, \mathbf{I}_2)$ <u>iff</u> $(M_1, N_1, \mathbf{I}_1) \le^{\ell}_{\mathfrak{u}} (M_2, N_2, \mathbf{I}_2)$ and $\mathbf{I}_1 \ne \emptyset \Rightarrow \mathbf{I}_1 = \mathbf{I}_2$.

3.19 Observation. 1) $\mathfrak{u}$ has the fake equality $=$ (i.e. the standard equality is also a fake equality).
2) $\mathfrak{u}'$ as defined in 2.8(2) has the fake equality $=_\tau$ (and is a nice construction framework, see 2.8(3)).
3) If $\mathfrak{u}$ is hereditary <u>then</u> $\mathfrak{u}''$ as defined in 2.8(4) is hereditary and even hereditary for the fake equality $=_\tau$ and is a nice construction framework, see 2.8(5).
4) $\mathfrak{u}^{[*]}$ is a nice construction framework, and if $\mathfrak{u}$ is weakly hereditary [for the fake equality $=_*$] <u>then</u> $\mathfrak{u}^{[*]}$ is hereditary [for the fake equality $=_*$].
5) If $\mathfrak{u}$ is hereditarily (for the fake equality $=_*$) <u>then</u> $\mathfrak{u}^{[*]}$ is hereditarily (for the fake equality $=_*$).

Proof. Check (really 3.1). $\square_{3.19}$

3.20 Claim. *Let $\xi \in \{\partial, \partial+1\}$ or just $\xi \le \partial+1$. Assume $\mathfrak{u} =$ dual$(\mathfrak{u})$ has fake equality $=_*$ and is hereditary for $=_*$.*

If $(M, N, \mathbf{I})$ has the semi ξ-uq-invariant coding property, then $(M, N, \mathbf{I})$ has the vertical ξ-uq-invariant coding property.

Remark. So no "$(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\text{qt}}_{\mathfrak{u}}$" here, but applying it we use $\mathfrak{K}^{\mathfrak{u}}$-universal homogeneous M_∂.

Proof. So let $\mathbf{d}_0$ be a $\mathfrak{u}$-free $(\alpha_{\mathbf{d}}, 0)$-rectangle satisfying $M \le_{\mathfrak{u}} M^{\mathbf{d}_0}_{0,\delta}$ and $M^{\mathbf{d}_0}_{\alpha(\mathbf{d}_0),0} \cap N = M$ so $\alpha_{\mathbf{d}_0} < \partial$ and we should find $\mathbf{d}_1$ satisfying the demand in Definition 3.10(1), this suffice. As we are assuming that "$(M, N, \mathbf{I})$ has the semi uq-invariant coding property", there is $\mathbf{e}_0$ satisfying the demands on $\mathbf{d}$ in 3.14(1)⊛ with $(M, N, \mathbf{I}, M^{\mathbf{d}_0}_{0,0})$ here standing for $(M, N, \mathbf{I}, M')$ there.

Without loss of generality

$(*)_1$ $M^{\mathbf{e}_0}_{\alpha(\mathbf{e}_0),0} \cap M^{\mathbf{d}_0}_{\alpha(\mathbf{d}),0} = M^{\mathbf{d}_0}_{0,0}$.

Let

$(*)_2$ $\mathbf{e}_1 =$ dual$(\mathbf{e}_0)$ so as $\mathfrak{u} =$ dual$(\mathfrak{u})$ clearly $\mathbf{e}_1$ is a $\mathfrak{u}$-free $(0, \alpha(\mathbf{e}_0))$-rectangle, so $\beta(\mathbf{e}_1) = \alpha(\mathbf{e}_0)$.

Now by 1.5(5) for some $\mathbf{e}_2$ (note: even the case $\beta(\mathbf{e}_1) = \alpha(\mathbf{e}_0) = \partial$ is O.K.)

$(*)_3$ (*a*) $\mathbf{e}_2$ is a $\mathfrak{u}$-free $(\alpha(\mathbf{d}_0), \beta(\mathbf{e}_1))$-rectangle

(*b*) $\mathbf{e}_2 \restriction (\alpha(\mathbf{d}_0), 0) = \mathbf{d}_0$

(*c*) $\mathbf{e}_2 \restriction (0, \beta(\mathbf{e}_1))) = \mathbf{e}_1$

and without loss of generality

(*d*) $M^{\mathbf{e}_2}_{\alpha(\mathbf{e}_2),\beta(\mathbf{e}_2)} \cap N = M$.

Now we choose $\mathbf{d}$ by

$(*)_4$ $\mathbf{d}$ is the $\mathfrak{u}$-free $(\alpha(\mathbf{d}_0) + \alpha(\mathbf{e}_0), 0)$-rectangle such that:

(*a*) $M^{\mathbf{d}}_{\alpha,0}$ is $M^{\mathbf{d}_0}_{\alpha,0}$ if $\alpha \le \alpha(\mathbf{d}_0)$

(*b*) $M^{\mathbf{d}}_{\alpha,0}$ is $M^{\mathbf{e}_2}_{\alpha(\mathbf{d}_0),\alpha-\alpha(\mathbf{d}_0)}$ for $\alpha \in [\alpha(\mathbf{d}_0), \alpha(\mathbf{d}_0)+\alpha(\mathbf{e}_0)]$

(*c*) $\mathbf{J}^{\mathbf{d}}_{\alpha,0} = \mathbf{J}^{\mathbf{d}_0}_{\alpha,0}$ if $\alpha < \alpha(\mathbf{d}_0)$

(*d*) $\mathbf{J}^{\mathbf{d}}_{\alpha,0} = \mathbf{I}^{\mathbf{e}_2}_{\alpha(\mathbf{d}_0),\alpha-\alpha(\mathbf{d}_0)}$ if $\alpha = [\alpha(\mathbf{d}_0), \alpha(\mathbf{d}_0)+\alpha(\mathbf{e}_0))$.

[Why is this O.K.? Check; the point is that $\mathfrak{u} = \text{dual}(\mathfrak{u})$.]

And we choose $\mathbf{d}_{2,\gamma}$ for $\gamma <\ \min\{\alpha(\mathbf{d}_0)+1, \partial\}$

$(*)_5$ $\mathbf{d}_{2,\gamma}$ is the $\mathfrak{u}$-free $(\gamma+1,0)$-rectangle such that:

(*a*) $M^{\mathbf{d}_{2,\gamma}}_{\alpha,0}$ is $M^{\mathbf{d}_0}_{\alpha,0}$ if $\alpha \le \gamma$

(*b*) $M^{\mathbf{d}_{2,\gamma}}_{\alpha,0}$ is $M^{\mathbf{e}_2}_{\alpha(\mathbf{d}),\alpha(\mathbf{e}_0)}$ if $\alpha = \gamma+1$

(*c*) $\mathbf{J}^{\mathbf{d}_{2,\gamma}}_{\alpha,0} = \mathbf{J}^{\mathbf{d}_0}_{\alpha,0}$ if $\alpha < \gamma$

(*d*) $\mathbf{J}^{\mathbf{d}_{2,\gamma}}_{\alpha,0}$ is $\emptyset$ if $\alpha = \gamma$.

[Why is this O.K.? Check.]

So it is enough to show

$(*)_6$ $\mathbf{d}$ is as required on $\mathbf{d}$ in 3.10 for our given $\mathbf{d}_0$ and $(M, N, \mathbf{I})$.

But first note that

$(*)_7$ $\mathbf{d} = \mathbf{d}_1$ is as required in clauses (b),(c),(d) of Definition 3.10(1).

Now, modulo $(*)_7$, clearly $(*)_6$ means that we have to show that

$\boxtimes$ if $(M, N, \mathbf{I}) \le_1 (M^{\mathbf{d}}_{0,0}, N', \mathbf{I}')$ and $M^{\mathbf{d}_1}_{\alpha(\mathbf{d}_1),0} \cap N = M$, i.e. $N', \mathbf{I}'$ are as in $\otimes(e)$ of 3.10(1), <u>then</u> we shall find $\alpha', \alpha_*, \mathbf{d}_*$, $\mathbf{I}'', M''$ as required in clauses $(\alpha) - (\varepsilon)$ of (e) from Definition 3.10(1).

By the choice of $\mathbf{e}_0$ to be as in Definition 3.14 before $(*)_1$, for $(N', \mathbf{J}')$ from $\boxtimes$ there are $\alpha', \alpha_*, \mathbf{e}_*, N'', \mathbf{I}''_*$ such that

$\odot_1(\alpha)$ $\alpha' \le \alpha(\mathbf{e}_0)$ and $\alpha' < \partial$

(β) $\mathbf{e}_*$ is a $\mathfrak{u}$-free $(\alpha_*, 0)$-rectangle

(γ) $M^{\mathbf{e}_*}_{0,0} = N'$

(δ) there is a $\mathfrak{u}$-free $(\alpha', 1)$-rectangle $\mathbf{e}'$ such that $\mathbf{e}' \restriction (\alpha', 0) = \mathbf{e}_0 \restriction (\alpha', 0)$ and $\mathbf{e}' \restriction (\alpha', [1,1]) = \mathbf{e}_* \restriction (\alpha', 0)$

(ε) there are no $\mathbf{d}_1, \mathbf{d}_2$ such that

- $\bullet_1$ $\mathbf{d}_\ell$ is a $\mathfrak{u}$-free rectangle for $\ell = 1, 2$
- $\bullet_2$ $\alpha(\mathbf{d}_1) = \alpha_*, \mathbf{d}_2 \restriction (\alpha_*, 0) = \mathbf{e}_*$
- $\bullet_3$ $\alpha(\mathbf{d}_2, 0) \geq \alpha'$ and $\mathbf{d}_2 \restriction (\alpha', 0) = \mathbf{d} \restriction (\alpha', 0)$
- $\bullet_4$ $(M^{\mathbf{d}_2}_{0,0}, M^{\mathbf{d}_2}_{0,1}, \mathbf{I}^{\mathbf{d}_2}_{0,0})$ is $(M', N', \mathbf{I})$ or just $\leq_1$-above it
- $\bullet_5$ there are N^*_1, N^*_2, f such that $M^{\mathbf{d}_\ell}_{\alpha(\mathbf{d}_\ell),\beta(\mathbf{d}_\ell)} \leq_{\mathfrak{u}} N^*_\ell$ for $\ell = 1, 2$, and f is an isomorphism from N^*_1 onto N^*_2 over $M^{\mathbf{d}}_{\alpha(\mathbf{d}),0}$ which maps $M^{\mathbf{d}_1}_{0,\beta(\mathbf{d}_1)}$ onto $M^{\mathbf{d}_2}_{0,\beta(\mathbf{d}_2)}$.

Without loss of generality

$\odot_2$ $N'' \cap M^{\mathbf{e}_2}_{\alpha',\beta(\mathbf{e}_2)} = M^{\mathbf{e}_0}_{\alpha',0}$.

Now by induction on $\alpha \leq \alpha(\mathbf{d})$ we choose $(M^*_\alpha, N^*_\alpha, \mathbf{I}^*_\alpha), \mathbf{J}^*_\alpha$ such that

$\odot_3$ (a) $(M^*_\alpha, N^*_\alpha, \mathbf{I}^*_\alpha) \in \mathrm{FR}^1_{\mathfrak{u}}$

(b) $\langle (M^*_\gamma, N^*_\gamma, \mathbf{I}^*_\gamma) : \gamma \leq \alpha \rangle$ is $\leq^1_{\mathfrak{u}}$-increasing continuous

(c) $(M^*_\alpha, N^*_\alpha, \mathbf{I}^*_\alpha) = (M^{\mathbf{e}_0}_{\alpha',0}, N'', \mathbf{I}'')$ for $\alpha = 0$

(d) $M^*_\alpha = M^{\mathbf{e}_2}_{\alpha,\alpha'}$

(e) $N^*_\alpha \cap M^{\mathbf{e}_2}_{\alpha(\mathbf{e}_2),\alpha'} = M^*_\alpha$

(f) if $\alpha = \alpha_1 + 1$ then $(M^{\mathbf{e}_2}_{\alpha_1,\alpha'}, M^{\mathbf{e}_2}_{\alpha_1+1,\alpha'}, \mathbf{J}^{\mathbf{e}_2}_{\alpha_1,\alpha(\mathbf{e}_0)}) \leq_2 (N^*_{\alpha_1}, N^*_{\alpha_1+1}, \mathbf{J}^*_{\alpha_1})$.

Now we shall use the assumption "$\mathfrak{u}$ is hereditary for $=_*$" to finish.
Choose

$\odot_4$ f is an isomorphism from $M^*_{\alpha(\mathbf{d})}$ onto a model M^* such that $M^{\mathbf{e}_2}_{\alpha',\beta(\mathbf{e}_2)} = M^* \cap M^*_{\alpha(\mathbf{d})}$ and $f \restriction M^{\mathbf{e}_2}_{\alpha',\beta(\mathbf{e}_2)}$ is the identity as well as $f \restriction N''$

$\odot_5$ M^{**} is the unique model $\in K_{\mathfrak{u}}$ such that $M^*_{\alpha(\mathbf{d})} \subseteq M^{**}$ and $M^* \subseteq M^{**}$ and $c \in M^*_{\alpha(\mathbf{d})} \Rightarrow M^{**} \models$ "$c = f(c)$".

Lastly

$\odot_6$ $\mathbf{d}_*$ is the following $\mathfrak{u}$-free $(\alpha(\mathbf{d})+1,1)$-rectangle:

- (a) $\mathbf{d}_* \restriction (\alpha(\mathbf{d}),0) = \mathbf{d}$
- (b) $M^{\mathbf{d}_*}_{\alpha(\mathbf{d})+1} = M^{\mathbf{e}_2}_{\alpha(\mathbf{e}_2),\alpha}$ and $\mathbf{J}^{\mathbf{d}_*}_{\alpha(\mathbf{d}),0} = \emptyset$
- (c) $M^{\mathbf{d}_*}_{\alpha(\mathbf{d})+1,1}$ is M^{**}
- (d) if $\alpha \le \alpha(\mathbf{d})$ the $M^{\mathbf{d}_4}_{\alpha,1}$ is $M^{\mathbf{d}}_{\alpha,0} \cup f(M^{\mathbf{e}_2}_{\alpha,\alpha'})$, i.e. the submodel of M^{**} with this universe
- (e) if $\alpha \le \alpha(\mathbf{d})$ then $\mathbf{I}^{\mathbf{d}_*}_{\alpha,0} = f(\mathbf{I}^*_\alpha)$ and $\mathbf{I}^{\mathbf{d}_*}_{\alpha(\mathbf{d})} = f(\mathbf{I}^*_{\alpha(\mathbf{d})})$
- (f) if $\alpha < \alpha(\mathbf{d})$ then $\mathbf{J}^{\mathbf{d}_*}_{\alpha,1} = f(\mathbf{J}^{\mathbf{e}_2}_{\alpha,\alpha'})$.

$\square_{3.20}$

To phrase a relative of 3.20, we need:

3.21 Definition. 1) We say $\mathfrak{u}$ satisfies (E)$_\ell$(f), is interpolative for ℓ or has interpolation for ℓ <u>when</u>:
if $(M_1,N_1,\mathbf{I}_1) \le^\ell_{\mathfrak{u}} (M_2,N_2,\mathbf{I}_2)$ then $(M_1,N_1,\mathbf{I}_1) \le^\ell_{\mathfrak{u}} (M_2,N_2,\mathbf{I}_1) \le^\ell_{\mathfrak{u}} (M_2,N_2,\mathbf{I}_2)$.
2) (E)(f) means (E)$_1$(f) + (E)$_2$(f).

Remark. This is related to but is different from monotonicity, see 1.13(1).

3.22 Claim. *1) Assume that for $\ell \in \{1,2\}, \mathfrak{u}$ satisfies (E)$_\ell$(f). For every $\mathfrak{u}$-free (α,β)-rectangle $\mathbf{d}$, also $\mathbf{e}$ is a $\mathfrak{u}$-free (α,β)-rectangle where $M^{\mathbf{e}}_{i,j} = M^{\mathbf{d}}_{i,j}, \ell = 1 \Rightarrow \mathbf{J}^{\mathbf{e}}_{i,j} = \mathbf{J}^{\mathbf{d}}_{i,j}, \ell = 1 \Rightarrow \mathbf{I}^{\mathbf{e}}_{i,j} = \mathbf{I}^{\mathbf{d}}_{0,j}$ and $\ell = 2 \Rightarrow \mathbf{J}^{\mathbf{e}}_{i,j} = \mathbf{J}^{\mathbf{d}}_{i,0}, \ell = 2 \Rightarrow \mathbf{I}^{\mathbf{e}}_{i,j} = \mathbf{I}^{\mathbf{d}}_{i,j}$.*
2) Similarly for $\mathfrak{u}$-free triangle.
3) If $\mathfrak{s}$ satisfies (E)(f) then in part (1) we can let $\mathbf{J}^{\mathbf{e}}_{i,j} = \mathbf{J}^{\mathbf{d}}_{i,0}, \mathbf{I}^{\mathbf{e}}_{i,j} = \mathbf{I}^{\mathbf{d}}_{0,j}$.

Proof. Easy. $\square_{3.22}$

Now we can state the variant of 3.20.

3.23 Claim. *Assume $\xi \le \partial + 1$ and $\mathfrak{u} = \mathrm{dual}(\mathfrak{u})$ has fake equality $=_*$, is weakly hereditary for $=_*$ and has interpolation.*

If $(M, N, \mathbf{I}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ has the semi ξ-uq-invariant coding property *then* *$(M, N, \mathbf{I}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ has the vertical ξ-uq-invariant coding property.*

Proof. Similar to the proof of 3.20, except that

- (A) in $\odot_3$ we add $\mathbf{I}'' = \mathbf{I}'$, justified by monotonicity; note that not only is it a legal choice but still exemplify 3.10
- (B) in $\odot_4$ we add $\mathbf{I}^*_\alpha = \mathbf{I}^*_0 = \mathbf{I}'' = \mathbf{I}'$.

3.24 Theorem. *We have $\dot{I}(K^{\mathfrak{u},\mathfrak{h}}_{\partial^+}) \ge \mu_{\mathrm{unif}}(\partial^+, 2^\partial)$* *when:*

- (a) $2^\partial < 2^{\partial^+}$
- (b) *some $(M, N, \mathbf{I}) \in \mathrm{FR}_2$ has the vertical ξ-uq-invariant coding property, see Definition 3.10*
- (c) *$\mathfrak{h}$ is a $\{0,2\}$-appropriate witness that for $\{0,2\}$-almost every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$, the model M_∂ is $K_{\mathfrak{u}}$-model homogeneous*
- (d) *$\xi = \partial$* *or* *$\xi = \partial^+$ and $2^\theta = 2^{<\partial} < 2^\partial$.*

3.25 Remark. 1) We can phrase other theorems in this way.
2) So if we change (b) to semi uq-invariant by 3.20 it suffices to add, e.g.

- (d) $\mathfrak{u}$ is hereditary for the faked equality $=_*$.

Proof. Easily by clause (b) we know $\mathfrak{u}$ has the vertical ξ-uq-invariant coding property. Now we apply 3.11, i.e. as in the proof of theorems 3.5 using 3.6(2) rather than 3.6(1) and imitating the proof of 3.3. $\square_{3.24}$

3.26 Exercise: Prove the parallel of the first sentence of the proof of 3.24 to other coding properties.

* * *

3.27 Discussion: We can repeat 3.13 - 3.24 with a game version. That is we replace Definition 3.14 by 3.28 and Definition 3.10 and then can imitate 3.20, 3.24 in 3.29, 3.30.

3.28 Definition. Let $\xi \le \partial + 1$.
1) We say that $\mathfrak{u}$ has the S-semi ξ-uq-invariant coding$_2$ property, [above $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*) \in K^{\rm qt}_{\mathfrak{u}}$] when $\{0,2\}$-almost every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ [above $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$] has it, see below; if $S = \partial$ we may omit it.
2) We say that $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ has the S-semi ξ-invariant coding$_2$ property when we can choose $\langle \mathbf{d}_\delta : \delta \in S \rangle$ such that

⊛ (a) $\mathbf{d}_\delta$ is a $\mathfrak{u}$-free $(\alpha(\mathbf{d}_\delta), 0)$-rectangle

(b) $M^{\mathbf{d}_\delta}_{0,0} = M_\delta$

(c) $M^{\mathbf{d}_\delta}_{\alpha(\mathbf{d}_\delta),0} \le_{\mathfrak{u}} M_\partial$

(d) in the following game the player Coder has a winning strategy; the game is defined as in Definition 2.6(2) except that the deciding who wins a play, i.e. we replace $(*)_2$ by

$(*)''_2$ in the end of the play the player Coder wins the play when:
for a club of $\delta \in \partial$ if $\delta \in S$ then there are $N'', \mathbf{I}''$ such that $(M_\delta, N_\delta, \mathbf{I}_\alpha) \le_{\mathfrak{u}} (M^{\mathbf{d}_\delta}_{\alpha(\mathbf{d}_\delta),0}, N'', \mathbf{I}'')$ and for no $(\mathbf{e}, f, N_*)$ do we have the parallel of $(\alpha) - (\eta)$ of clause (c) of Definition 3.14.

3) We define when $\mathfrak{u}$ or $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f})$ has the S-vertical ξ-uq-invariant coding$_2$ property as in parts (1),(2) replacing 3.14 by 3.10.

3.29 Claim. *Like 3.20 using Definition 3.28.*

3.30 Theorem. *Like 3.24 using 3.28.*

* * *

We now point out some variants of the construction framework, here amalgamation may fail (unlike 1.3(1) used in 2.10(4) but not usually). This relates to semi a.e.c.

3.31 Definition. We define when $\mathfrak{u}$ is a weak nice construction framework as in Definition 1.2 but we considerably weaken the demands of $\mathfrak{K}_\mathfrak{u}$ being an a.e.c.

- (A) $\mathfrak{u}$ consists of $\partial, \tau_\mathfrak{u}, \mathfrak{K}_\mathfrak{u} = (K_\mathfrak{u}, \leq_\mathfrak{u})$, FR_1, $\mathrm{FR}_2, \leq_1, \leq_2$ (also denoted by $\mathrm{FR}_1^\mathfrak{u}$, $\mathrm{FR}_2^\mathfrak{u}, \leq^1_\mathfrak{u}, \leq^2_\mathfrak{u}$)
- (B) ∂ is regular uncountable
- (C) (a) $\tau_\mathfrak{u}$ is a vocabulary
  - (b) $K_\mathfrak{u}$ is a non-empty class of $\tau_\mathfrak{u}$-models of cardinality $< \partial$ closed under isomorphisms (but $K^{\mathfrak{u},*}_\partial, K^{\mathfrak{u},*}_{\partial^+}$ are defined below)
  - (c) (restricted union) if $\ell \in \{1, 2\}$ then
    - (α) $\leq_\mathfrak{u}$ is a partial order on $K_\mathfrak{u}$,
    - (β) $\leq_\mathfrak{u}$ is closed under isomorphism
    - (γ) restricted union existence: if $\ell = 1, 2$ and $\langle M_\alpha : \alpha \leq \delta\rangle$ is $\leq_\mathfrak{K}$-increasing continuous, δ a limit ordinal $< \partial$ and $(M_\alpha, M_{\alpha+1}, \mathbf{I}_\alpha) \in \mathrm{FR}^\ell_\mathfrak{u}$ for $\alpha < \delta$ and $\delta = \sup\{\alpha < \delta$: $(M_\alpha, M_{\alpha+1}, \mathbf{I}_\alpha) \in \mathrm{FR}^{\ell,+}_\mathfrak{u}$ <u>then</u> $M_\delta := \{M_\alpha : \alpha < \delta\}$ belongs to $K_\mathfrak{u}$ and $\alpha < \delta \Rightarrow M_\alpha \leq_\mathfrak{u} M_\delta$
  - (d) restricted smoothness: in clause (c)(γ) if $\alpha < \delta \Rightarrow M_\alpha \leq_\mathfrak{u} N$ <u>then</u> $M_\delta \leq_\mathfrak{u} N$
- $(D)_\ell$ as in Definition 1.2
- $(E)_\ell$ as in Definition 1.2 but we replace clause (c) by
  - $(c)'$ $((M_\delta, N_\delta, \mathbf{J}_\delta) \in \mathrm{FR}^\ell_\mathfrak{u}$ and $\alpha < \delta \Rightarrow (M_\alpha, N_\alpha, \mathbf{J}_\alpha) \leq^\ell_\mathfrak{u} (M_\delta, N_\delta, \mathbf{J}_\delta)$, <u>when</u>:
    - (α) $\delta < \partial$ is a limit ordinal
    - (β) $\langle (M_\alpha, N_\alpha, \mathbf{J}_\alpha) : \alpha < \delta\rangle$ is $\leq^\ell_\mathfrak{u}$-increasing continuous
    - (γ) $(M_\delta, N_\delta, \mathbf{J}_\delta) = (\cup\{M_\alpha : \alpha < \delta\}, \cup\{N_\alpha : \alpha < \delta\}, \cup\{\mathbf{J}_\alpha : \alpha < \delta\})$
    - (δ) M_δ is a $\leq_\mathfrak{u}$-upper bound of $\langle M_\alpha : \alpha \geq \delta\rangle$

(ε) N_δ is a $\leq_{\mathfrak{u}}$-upper bound of $\langle N_\alpha : \alpha < \delta \rangle$

(ζ) $M_\delta \leq_{\mathfrak{u}} N_\delta$

(F) As in Definition 1.2.

3.32 Remark. 1) We may in condition (E)$_\ell$(c) use essentially a $\mathfrak{u}$-free $(\delta, 2)$-rectangle (or $(2, \delta)$-rectangle).
2) A stronger version of (E)$_\ell$(c) is: (E)$_2$(c)$^+$ as in (E)$_1$(c)$'$ adding:

(η) $(M_\alpha, M_{\alpha+1}, \mathbf{I}_\alpha) \leq^1_{\mathfrak{u}} (N_\alpha, N_{\alpha+1}, \mathbf{I}^1_\alpha)$

(θ) $(M_\alpha, N_\alpha, \mathbf{I}^m_\alpha) \in \mathrm{FR}^{1,+}_{\mathfrak{u}}$ for $m = 0, 1$ for unboundedly many $\alpha < \delta$ (so clause (ζ) follows).

(E)$_1$(c)$^+$ means clause (E)$_2$(c)$^+$ is satisfied by dual($\mathfrak{u}$).
3) We may demand for $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ that for a club of δ, if $\mathbf{f}(\delta) > 0$ then:

(a) $\mathbf{f}(\delta)$ is a limit ordinal

(b) $\mathbf{f}(\delta) = \sup\{i < \mathbf{f}(\delta) : (M_{\delta+i}, M_{\delta+i+1}, \mathbf{J}_{\delta+i}) \in \mathrm{FR}^{2,+}_{\mathfrak{u}}\}$

(c) in the examples coming for an almost good λ-frame $\mathfrak{s}$, see §5: if $i < \delta$ and $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_{\delta+i})$ then $\delta = \sup\{j : j \in (i, \delta)$ and $\mathbf{J}_{\delta+j} = \{a_{\delta+j}\}$ and $\mathbf{tp}_{\mathfrak{s}}(a_{\delta+j}, M_{\delta+j}, M_{\delta+j+1})$ is a non-forking extension of p; see more in §5 on this.

4) We may in part (3)(b) and in 3.33(A)(a) below restrict ourselves to sucessor i.

3.33 Lemma. *We can repeat §1 + §2 (and §3) with Definition 3.31 instead of Definition 1.2 with the following changes:*

(A) *from Definition 1.15:*

(a) *in the Definition of* $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ *we demand* $S = \{\delta < \partial : \mathbf{f}(\delta) > 0\}$ *is stationary and for a club of* $\delta \in S, \mathbf{f}(\delta)$ *is a limit ordinal and*
$i < \mathbf{f}(\delta) \Rightarrow (M_{\delta+i}, M_{\delta+i+1}, \mathbf{J}_{\delta+i}) \in \mathrm{FR}^{2,+}_{\mathfrak{u}}$

(b) *we redefine* $\leq^{\mathrm{qr}}_{\mathfrak{u}}, \leq^{\mathrm{qs}}_{\mathfrak{u}}$ *as* $\leq^{\mathrm{qt}}_{\mathfrak{u}}$ *was redefined*

(B) *proving 1.19:*

- (a) *in part (1), we should be given a stationary* $S \subseteq \partial$ *and for* $\alpha < \partial$ *let* $\mathbf{f}(\alpha) = \omega$
- (b) *in part (2), we use the restricted version of union existence and smoothness*
- (c) *in part (3), we demand* $(M^1_\alpha, M^2_\alpha, \mathbf{I}^*) \in \mathrm{FR}^+_1$ *and let* $E \subseteq \lambda\backslash\alpha$ *witness* $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ *and* $S = \{\delta \in E : \mathbf{f}(\delta) > 0\}$ *and inthe induction we use just* $\langle M_\beta : \beta \in \{\alpha\} \cup \bigcup\{[\delta, \delta, \mathbf{f}(\delta)] : \delta \in S\}\rangle$.

3.34 Exercise: Rephrase this section with τ-coding$_k$ instead coding$_k$.
[Hint: 1) Of course, we replace "coding" by "τ-coding" and isomorphic by τ-isomorphic.
2) We replace $\bullet_5$ of 3.2(1)(δ) by:

- $\bullet'_5$ there are N_1, N_2 such that $N^{\mathbf{d}_\ell}_{\alpha(\mathbf{d}_\ell),\beta(\mathbf{d}_\ell)} \leq_{\mathfrak{u}} N_\ell$ for $\ell = 1, 2$ and there is a τ-isomorphism f from N_1 onto N_2 extending $\mathrm{id}_{M^{\mathbf{d}}_{\alpha',0}}$ and mapping $M^{\mathbf{d}_1}_{0,\beta(\mathbf{d}_1)}$ onto $M^{\mathbf{d}_2}_{0,\beta(\mathbf{d}_2)}$.

3) In Claim 3.3 in the end of clause (γ) "$N^1_\partial, N^2_\partial$ are not τ-isomorphic over M_γ", of course.
4) In 3.5 replace $\dot{I}$ by $\dot{I}_\tau$.
5) In 3.6, in the conclusion replace $\dot{I}$ by $\dot{I}_\tau$, in clause (c)(β) use "not τ-isomorphic".
6) In Definition 3.7 like (2), in Definition 3.10(e)(δ) as in (2).
7) Change 3.14(c)(ε) as in (2), i.e.

- $(\varepsilon)'$ f is a τ-isomorphism from N^* onto N_* for some N^* such that $N'' \leq_{\mathfrak{u}} N^*$.]

§4 Straight Applications of (weak) coding

Here, to try to exemplify the usefulness of Theorem 2.3, the "lean" version, i.e. using weak coding, we revisit older non-structure results.

First, recall that the aim of [Sh 603] or better VI§3,§4 is to show that the set of minimal types in $\mathscr{S}^{\rm na}_{\mathfrak{K}}(M), M \in \mathfrak{K}_\lambda$ is dense, <u>when</u>:

$\boxtimes^\lambda_{\mathfrak{K}}$ $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}, \mathfrak{K}$ is an a.e.c. with $\mathrm{LS}(\mathfrak{K}) \le \lambda$, categorical in λ, λ^+ and have a medium number of models in λ^{++} (hence $\mathfrak{K}$ has amalgamation in λ and in λ^+).

More specifically we have to justify Claim VI.3.13 when the weak diamond ideal on λ^+ is not λ^{++}-saturated and we have to justify claim VI.4.12 when some $M \in K_{\lambda^+}$ is saturated; in both cases inside the proof there we quote results from here.

We interpret medium as $1 \le \dot{I}(\lambda^{++}, K) < \mu_{\rm unif}(\lambda^{++}, 2^{\lambda^+})$ (where the latter is usually $2^{\lambda^{++}}$). This is done in 4.1-4.15, i.e., where we prove the non-structure parts relying on the one hand on the pure model theoretic part done in Chapter VI and on the other hand on coding theorems from §2. More elaborately, as we are relying on Theorem 2.3, in 4.1 - 4.9, i.e. §4(A) we assume that the normal ideal $\mathrm{WDmId}(\lambda^+)$ is not λ^{++}-saturated and prove for appropriate $\mathfrak{u}$ that (it is a nice construction framework and) it has the weak coding property. Then in 4.10 - 4.15, i.e. §4(B) relying on Theorem 2.11, we assume more model theory and (for the appropriate $\mathfrak{u}$) prove the vertical coding property, hence eliminate the extra set theoretic assumption (but retaining the relevant cases of the WGCH, i.e. $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$).

Second, we relook at the results in VI§6, i.e. [Sh 576, §6] which were originally proved relying on [Sh 576, §3]. That is, our aim is to prove the density of uniqueness triples (M, N, a) in $K^{3,\rm na}_\lambda$, assuming medium number of models in λ^{++}, and set theoretically $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ and in addition assume (for now) the non-λ^{++}-saturation of the weak diamond ideal on λ^+. So we use the "weak coding" from Definition 2.2, Theorem 2.3 (see 4.20, i.e. §4(D)). The elimination of the extra assumption is delayed as it is more involved (similarly for §4(E)).

Third, we fulfill the ("lean" version of the) promise from II§5, proving density of uniqueness triples in $K^{3,\rm bs}_{\mathfrak{s}}$, for $\mathfrak{s}$ a good λ-frame, also originally relying on [Sh 576, §3], see 4.28, i.e. §4(E).

Fourth, we deal with the promises from I§5 by Theorem 2.3 in

4.16 - 4.19, i.e. §4(C).

But still we owe the "full version", this is §4(F) in which we eliminate the extra set theoretic result relying on the model theory from §5-§8.

* * *

(A) Density of the minimal types for $\mathfrak{K}_\lambda$

4.1 Theorem. *We have* $\dot{I}(\lambda^{++}, K) \geq \mu_{\text{unif}}(\lambda^{++}, 2^{\lambda^+})$ *when:*

$\odot$ (a) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$

(b) *the ideal* $\text{WDmId}(\lambda^+)$*, a normal ideal on* λ^+*, is not* λ^{++}*-saturated*

(c) $\mathfrak{K}$*, an a.e.c. with* $\text{LS}(\mathfrak{K}) \leq \lambda$*, has amalgamation in* λ*, the* JEP *in* λ*, for simplicity and* $K_{\lambda^+} \neq \emptyset$*;*

(d) *for every* $M \in K_{\lambda^+}$ *and* $\leq_{\mathfrak{K}}$*-representation* $\bar{M} = \langle M_\alpha : \alpha < \lambda^+ \rangle$ *of* M *we can find* (α_0, N_0, a)*, i.e., a triple* (M_{α_0}, N_0, a) *such that:*

(α) $M_{\alpha_0} \leq_{\mathfrak{K}_\lambda} N_0$

(β) $a \in N_0 \backslash M_{\alpha_0}$ *and* $\mathbf{tp}_{\mathfrak{K}}(a, M_{\alpha_0}, N_0)$ *is not realized in* M

(γ) *if* $\alpha_0 < \alpha_1 < \lambda^+, M_{\alpha_1} \leq_{\mathfrak{K}} N_1$ *and* f *is a* $\leq_{\mathfrak{K}}$*-embedding of* N_0 *into* N_1 *over* M_{α_0} *then we can find* $\alpha_2 \in (\alpha_1, \lambda^+)$ *such that* M_{α_2}, N_1 *are not uniquely amalgamated over* M_{α_1} *(in* $\mathfrak{K}_\lambda$*), i.e.* $\text{NUQ}_\lambda(M_{\alpha_1}, M_{\alpha_2}, N_1)$*, see VI.3.3(2).*

4.2 Remark. 1) Used in Claim VI.3.13, more exactly the relative 4.3 is used.

2) A further question, mentioned in VI.2.17(3) concern $\dot{I}\dot{E}(\lambda^{++}, K)$ but we do not deal with it here.

3) Recall that for $M \in K_\lambda$ the type $p \in \mathscr{S}^{\text{na}}_{\mathfrak{K}_\lambda}(M)$ is minimal when there is no $M_2 \in K_\lambda$ which $\leq_{\mathfrak{K}}$-extends M and p has at least 2 extensions in $\mathscr{S}^{\text{na}}_{\mathfrak{K}_\lambda}(M_2)$; see Definition VI.1.11.

3A) We say that in $\mathfrak{K}_\lambda$ the minimal types are dense when: for any $M \in K_\lambda$ and $p \in \mathscr{S}^{\text{na}}_{\mathfrak{K}_\lambda}(M)$ there is a pair (N, q) such that $M \leq_{\mathfrak{K}_\lambda} N$ and $q \in \mathscr{S}^{\text{na}}_{\mathfrak{K}_\lambda}(N)$ is minimal and extend p (see VI.1.11(1A)).

4) A weaker version of Clause (d) of 4.1 holds when any $M \in K_{\lambda^+}$ is saturated (above λ) and the minimal types are not dense (i.e. omit subclause (β) and in subclause (γ) add $f(a) \notin M_{\alpha_1}$; the proof is similar (but using 4.11). Actually, 4.1 as phrased is useful normally only when $2^\lambda > \lambda^+$, but otherwise we use 4.3.
5) In VI.3.13 we work more to justify a weaker version (d)$''$ of Lemma 4.3 below which suffice.

Similarly

4.3 Lemma. *1) Like 4.1 but we replace clause (d) by:*

- (d)$'$ *there is a superlimit $M \in K_{\lambda^+}$ and for it clause (d) of 4.1 holds.*

2) For τ a K-sub-vocabulary, see 1.8(5), we have $\dot{I}_\tau(\lambda^{++}, K) \geq \mu_{\text{unif}}(\lambda^{++}, 2^{\lambda^+})$ when (a),(b),(c) of 4.1 holds and

- (d)$''$ *there[22] is K''_{λ^+} such that*
  - (α) *$K''_{\lambda^+} \subseteq K_{\lambda^+}, K''_{\lambda^+} \neq \emptyset$ and K''_{λ^+} is closed under unions of $\leq_{\mathfrak{K}}$-increasing continuous sequences of length $\leq \lambda^+$*
  - (β) *there is $M_* \in K''_{\lambda^+}$ such that for any $\leq_{\mathfrak{K}_\lambda}$-increasing continuous sequence $\langle M^1_\alpha : \alpha < \lambda^+ \rangle$ with union $M^1 \in K''_{\lambda^+}$ satisfying $M_* \leq_{\mathfrak{K}} M^1$, in the following game the even player has a winning strategy. In the α-th move a triple $(\beta_\alpha, M_\alpha, f_\alpha)$ is chosen such that $\beta_\alpha < \lambda^+, M_\alpha \in \mathfrak{K}_\lambda$ has universe $\subseteq \lambda^+$ and f_α is a $\leq_{\mathfrak{K}}$-embedding of $M^1_{\beta_\alpha}$ into M_α, all three are increasing continuous with α. Of course, the α-th move is done by the even/odd iff α is even/odd. Lastly, in the end the even player wins iff $\cup\{M_\alpha : \alpha < \lambda^+\}$ belongs to K''_{λ^+} and for a club of $\alpha < \lambda^+$ for some $\gamma \in (\beta_\alpha, \lambda^+)$ and some $N \in K_\lambda$ such that there is an isomorphism from N onto M_α extending f_α we have $\text{NUQ}_\tau(M^1_{\beta_\alpha}, N, M^1_\gamma)$, i.e. N, M_γ can be amalgamated over M_{β_α} in $\mathfrak{K}_\lambda$ in at least two τ-incompatible ways.*

[22]Why not K'_λ? Just because we use this notation in 1.10.

4.4 Remark. 0) We may replace β_α in clause (d)$''(\beta)$ above by $\beta = \alpha+1$, many times it does not matter.
1) We may weaken the model theoretic assumption $(d)''$ of 4.3(2) so weaken (d) in 4.1 and $(d)'$ in 4.3 if we strengthen the set theoretic assumptions, e.g.

- $(*)_1$ for some stationary $S \subseteq S^{\lambda^{++}}_{\lambda^+}$ we have $S \in \check{I}[\lambda^{++}]$ but $S \notin$ WDmId(λ^{++})
- $(*)_2$ in 4.3, in subclause (α) of clause (d)$''$ we weaken the closure under union of K''_{λ^+} to: for a $\leq_{\mathfrak{K}}$-increasing sequence in K''_{λ^+} of length λ^+, its union belongs to K''_{λ^+}.

2) If, e.g. $\lambda = \lambda^{<\lambda}$ and $\mathbf{V} = \mathbf{V}^{\mathbb{Q}}$ where $\mathbb{Q}$ is the forcing notion of adding λ^+-Cohen subsets of λ and the minimal types are not dense <u>then</u> $\dot{I}(\lambda^+, K) = 2^{\lambda^+}$, (hopefully see more in [Sh:E45]).
3) If in part (2) of 4.3, we may consider omitting the amalgamation, but demand "no maximal model in $\mathfrak{K}_\lambda$". However, the minimality may hold for uninteresting reasons.
4) This is used in VI.3.13.
5) We may assume $2^\lambda \neq \lambda^+$ as essentially the case $2^\lambda = \lambda^+$ is covered[23] by Lemma 4.10 below.
6) In clause (d)$''(\beta)$ of Lemma 4.3, we can let the even player choose also for $\alpha = \delta+1$ for $\delta \in S$ <u>when</u> $S \subseteq \lambda^+$ but WDmId$(\lambda^+) + S$ is not λ^{++}-saturated.

Proof of 4.1. We shall apply Theorem 2.3. So (model theoretically) we have an a.e.c. $\mathfrak{K}$ with LS$(\mathfrak{K}) \leq \lambda$, and we are interested in proving $\dot{I}(\lambda^{++}, \mathfrak{K}) \geq \mu_{\rm unif}(\lambda^{++}, 2^{\lambda^+})$.

We shall define (in Definition 4.5 below) a nice construction framework $\mathfrak{u}$ such that $\partial_{\mathfrak{u}} = \lambda^+$; the set theoretic assumptions of 2.3 hold; i.e.

[23] by the assumption $\mathfrak{K}_\lambda$ has amalgamation so togther with $2^\lambda = \lambda^+$, if $|\tau_K| \leq \lambda$ or just $M \in \mathfrak{K}_\lambda \Rightarrow |\mathscr{S}_{\mathfrak{K}_\lambda}(M)| \leq \lambda^+$ <u>then</u> there is a model $M^* \in \mathfrak{K}_{\lambda^+}$ which is saturated above λ; in general this is how it is used in VI§4; but if there is $M \in \mathfrak{K}_\lambda$ with $|\mathscr{S}_{\mathfrak{K}_\lambda}(M)| > \lambda^+$; we can use 4.1.

(a) $\lambda < \partial$ and $2^\lambda = 2^{<\partial} < 2^\partial$; i.e. we choose $\theta := \lambda$ and this holds by clause $\odot(a)$ of Theorem 4.1

(b) $2^\partial < 2^{\partial^+}$; holds by clause $\odot(a)$ of 4.1

(c) the ideal WDmID(∂) is not ∂^+-saturated; holds by clause $\odot(b)$ of 4.1

We still have to find $\mathfrak{u}$ (and τ) as required in clause (d) of Theorem 2.3. We define it in Definition 4.5 below, in particular we let $\mathfrak{K}_\mathfrak{u} = \mathfrak{K}'_\lambda, \tau = \tau_\mathfrak{K}$, see Definition 1.10 where $\mathfrak{K}$ is the a.e.c. from 4.3 hence the conclusion "$\dot{I}_\tau(\partial^+, K^{\mathfrak{u},\mathfrak{h}}_{\partial^+}) \ge \mu_{\text{unif}}(\partial^+, 2^\partial) > 2^\partial$ for any $\{0,2\}$-appropriate function $\mathfrak{h}$" of Theorem 2.3 implies that $\dot{I}(\lambda^{++}, K^\mathfrak{u}) \ge \mu_{\text{unif}}(\lambda^{++}, 2^{\lambda^+})$ as required using 4.7(3); we can use Exercise 1.12. So what we should actually prove is that we can find such nice construction frameworks $\mathfrak{u}$ with the weak coding property which follows from $\mathfrak{u}$ having the weak coding property (by 2.10(2),(3)) and $\mathfrak{h}$ such that every $M \in K^{\mathfrak{u},\mathfrak{h}}_{\lambda^{++}}$ is τ-fuller. This is done in 4.6, 4.7 below. $\square_{4.1}$

Proof of 4.3. 1) By part (2), in particular letting $K''_{\lambda^+} = \{M \in K_{\lambda^+} : M$ is superlimit$\}$.

But why does subclause (β) of clause (d)$''$ hold? Let $M_* \in K''_{\lambda^+}$ be superlimit, let $\langle M^1_\alpha : \alpha < \lambda^+\rangle$ be $\le_{\mathfrak{K}_\lambda}$-increasing continuous with union $M^1 \in K''_{\lambda^+}$ and assume $M_* \le_\mathfrak{K} M^1$. Without loss of generality M^1 has universe $\{\beta < \lambda^+ : \beta$ odd$\}$.

We shall prove that the even player has a winning strategy. We describe it as follows: the even player in the α-th move also choose (N_α, g_α) for α even and also for α odd (after the odd's move) such that

$(*)$ (a) $N_\alpha \in K_{\lambda^+}$ is superlimit and $M^1 \le_\mathfrak{K} N_0$

(b) the universe of N_α is $\{\gamma < \lambda^+ : \gamma$ is not divisible by $\lambda^\alpha\}$

(c) $N_\beta \le_\mathfrak{K} N_\alpha$ for $\beta < \alpha$

(d) g_α is a $\le_\mathfrak{K}$-embedding of M_α into N_α

(e) $g_\beta \subseteq g_\alpha$ for $\beta < \alpha$

(f) if α is odd then the universe of $g_{\alpha+1}(M_{\alpha+1})$ includes $\alpha = N_\alpha \cap \alpha$.

It should be clear that the even player can do this. Also for any such play $\langle (M_\alpha, f_\alpha, N_\alpha, g_\alpha) : \alpha < \lambda^+ \rangle$ we have $\lambda^+ = \cup\{N_\alpha \cap \alpha : \alpha < \lambda^+\} \subseteq \cup\{g_\alpha(M_\alpha) : \alpha < \lambda^+\} \subseteq \cup\{N_\alpha : \alpha < \lambda^+\} \subseteq \lambda^+$, so $g = \cup\{g_\alpha : \alpha < \lambda^+\}$ is an isomorphism from $\cup\{M_\alpha : \alpha < \lambda^+\}$ onto $\cup\{N_\alpha : \alpha < \lambda^+\}$. As the latter is superlimit we are done (so for this part being (λ^+, λ^+)-superlimit suffice).
2) The proof is like the proof of 4.1 but we have to use a variant of 2.3, i.e. we use the variant of weak coding where we use a game, see 2.5. $\square_{4.3}$

4.5 Definition. [Assume clause (c) of 4.1.]

We define $\mathfrak{u} = \mathfrak{u}^1_{\mathfrak{K}_\lambda}$ as follows (with $\tau(\mathfrak{u}) = (\tau_{\mathfrak{K}})'$ so $=_\tau$ is a congruence relation in $\tau(\mathfrak{u})$, O.K. by 1.10; this is a fake equality 3.17(1), 3.19)

- (a) $\partial_{\mathfrak{u}} = \lambda^+$
- (b) essentially $\mathfrak{K}_{\mathfrak{u}} = \mathfrak{K}_\lambda$; really $\mathfrak{K}'_\lambda$ (i.e. $=_\tau$ is a congruence relation!)
- (c) $\mathrm{FR}_1 = \{(M, N, \mathbf{I}) : M \leq_{\mathfrak{K}_{\mathfrak{u}}} N, \mathbf{I} \subseteq N \backslash M$ empty or a singleton $\{a\}\}$
- (d) $\mathrm{FR}_2 = \mathrm{FR}_1$
- (e) $(M_1, N_1, \mathbf{J}_1) \leq_\ell (M_2, N_2, \mathbf{J}_2)$ <u>when</u>
  - (i) both triples are from FR_ℓ
  - (ii) $M_1 \leq_{\mathfrak{K}} M_2, N_1 \leq_{\mathfrak{K}} N_2$ and $\mathbf{J}_1 \subseteq \mathbf{J}_2$
  - (iii) $M_2 \cap N_1 = M_1$.

4.6 Observation. $\mathfrak{u}$ is a nice construction framework which is self-dual.

Proof. Easy and the proof of 4.13 can serve when we note that $(\mathrm{D})_1$(d) and (F) are obvious in our context, recalling we have fake equality. $\square_{4.6}$

4.7 Observation. [Assume $\lambda, \mathfrak{K}$ are as in 4.1 or 4.3(1) or 4.3(2).]
1) $\mathfrak{u}$ has the weak coding property (see Definition 2.2).
2) If $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ and so $M_\partial \in K_{\lambda^+}$ but for 4.3(2) the model M_∂ is superlimit then $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f})$ has weak coding property.
3) For some $\mathfrak{u}-0$-appropriate function $\mathfrak{h}$, every $M \in K^{\mathfrak{u},\mathfrak{h}}_{\lambda^{++}}$ is $\tau_{\mathfrak{s}}$-fuller, i.e. the model $M/=^M$ has cardinality λ^{++} and the set $a/=^M$ has cardinality λ^{++} for every $a \in M$.

Proof. Easy.
1) By part (2) and (3).
2) For proving 4.1 by clause (d) there we choose $(\alpha(0), N_0, \mathbf{I}_0)$, it is as required in Definition 2.2(3), noting that we can get the necessary disjointness because $\mathfrak{K}'_\lambda$ has fake equality. Similarly for 4.3(1).

For proving 4.3(2) we fix a winning strategy **st** for the even player in the game from clause (d) of 4.3. Again by the fake equality during the game we can demand $\alpha_1 < \alpha \Rightarrow f_\alpha(M_{\beta_\alpha}) \cap M_\alpha = f_{\alpha_1}(M_{\beta_1})$.
3) By 1.25(3) is suffice to deal separately with each aspect of being $\tau_{\mathfrak{s}}$-fuller.

First, we choose a $\mathfrak{u}$-0-appropriate function $\mathfrak{h}_0$ such that if $((\bar{M}^1, \bar{\mathbf{J}}^1, \mathbf{f}^1), (\bar{M}^2, \bar{\mathbf{J}}^2, \mathbf{f}^2))$ does 0-obeys $\mathfrak{h}_0$ as witnessed by $(E, \bar{\mathbf{I}})$ then for any $\delta \in E, (M^1_\delta, M^2_\delta, \mathbf{I}_\delta) \in \ \mathrm{FR}^+_1$ and there is $a \in \mathbf{I}_\delta$ such that $c \in M^1_\partial \Rightarrow M^2_\partial \models \neg(a =_\tau c)$; this is possible as in every $(\bar{M}^1, \bar{\mathbf{J}}^1, \mathbf{f}^1) \in K^{\rm qt}_{\mathfrak{u}}$ there are $\alpha < \lambda^+$ and $p \in \mathscr{S}^{\rm na}(M^1_\alpha)$ not realized in $M^1 = \cup\{M^1_\beta : \beta < \alpha\}$. Why? For 4.1 by 4.1(d)(β), similarly for 4.3(1) and for 4.3(2), if it fails, then the even player cannot win, because

- $(*)$ if $M_0 \le_{\mathfrak{u}} M_\ell$ for $\ell = 1,2$ and $(\forall b \in M_1)(\exists b \in M_1)(\exists a \in M_0)(M_1 \models a =_\tau b)$ then M_1, M_2 can be uniquely disjointly amalgamated in $\mathfrak{K}_{\mathfrak{u}}$.

Second, we choose a $\mathfrak{u} - 0$-suitable $\mathfrak{h}_2$ such that if $\langle(\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha < \partial^+\rangle$ does 0-obeys $\mathfrak{h}_2$, then for every $\alpha < \partial^+$ and $a \in M_\alpha$ for λ^{++} many $\delta \in (\alpha, \lambda^{++})$, we have $(a/=^{M_\delta}_\tau) \subset (a/=^{M_{\delta+1}}_\tau)$. $\square_{4.7}$

4.8 Example For $\mathfrak{K}, \mathfrak{u}$, **st** as in the proof of 4.7(2).
1) For some initial segment $\mathbf{x} = \langle(\beta_\alpha, M_\alpha, f_\alpha) : \alpha \le \alpha_*\rangle$, of a play of the game of length $\alpha_* < \lambda^+$ in which the even player uses the

strategy **st**, for any longer such initial segment $\langle(\beta_\alpha, M_\alpha, f_\alpha) : \alpha \leq \alpha_{**}\rangle$ of such a play we have $M_{\alpha_*} \cap f_{\alpha_{**}}(M^1_{\beta_{\alpha_{**}}}) = f_{\alpha_*}(M^1_{\beta_{\alpha_*}})$ and $f_{\alpha_*}(M^1_{\beta_{\alpha_*}}) <_{\mathfrak{K}} M_\alpha$.
[Why? As in the proof of the density of reduced triples; just think.]
2) Moreover if $c \in M_\partial \backslash M_{\beta_{\alpha_*}}$ then $f_{\alpha_*}(\mathbf{tp}_{\mathfrak{K}}(c, M_{\beta_{\alpha_*}}, M_\partial))$ is not realized in M_{α_*} (recall the definition of NUQ). $\square_{4.1}$

4.9 Remark. So we have finished proving 4.1, 4.3.

* * *

<u>(B) Density of minimal types: without λ^{++}-saturation of the ideal</u>

The following takes care of VI.4.12, of its assumptions, (a)-(g) are listed in VI.4.10.

4.10 Theorem. *We have $\dot{I}(\lambda^{++}, K) \geq \mu_{\text{unif}}(\lambda^{++}, 2^{\lambda^+})$ <u>when</u>:*

- (a) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{+2}}$
- (b) *$\mathfrak{K}$ is an abstract elementary class,* $\text{LS}(\mathfrak{K}) \leq \lambda$
- (c) $K_{\lambda^{++}} \neq \emptyset$,
- (d) *$\mathfrak{K}$ has amalgamation in λ*
- (e) *the minimal types, for $\mathfrak{K}_\lambda$ are not dense, see 4.2(3A)*
- (f) *$\mathfrak{K}$ is categorical in λ^+ or at least has a superlimit model in λ^+*
- (g) *there is $M \in K_{\lambda^+}$ which is saturated (in $\mathfrak{K}$) above λ.*

Proof. The proof is broken as in the other cases.

4.11 Definition. We define $\mathfrak{u} = \mathfrak{u}^2_{\mathfrak{K}_\lambda}$ as in 4.5 so $\mathfrak{K}_{\mathfrak{u}} = \mathfrak{K}'_\lambda$ but replacing clauses (c),(e) by (we shall use the fake equality only for having disjoint amalgamation):

- $(c)'$ $\text{FR}_1 = \{(M, N, \mathbf{I}) : M \leq_{\mathfrak{K}_{\mathfrak{u}}} N, \mathbf{I} \subseteq N \backslash M$ empty or a singleton $\{a\}$ and if $(a/=^M) \notin M/=^M$ <u>then</u> $\mathbf{tp}_{\mathfrak{K}_\lambda}(a, M, N)$

has no minimal extension, i.e. and $\mathbf{tp}_{\mathfrak{K}_\lambda}((a/=^M_\tau,(M/=^M_\tau),(N/=^N_\tau))$ has no minimal extension$\}$

(*e*) $(M_1,N_1,\mathbf{J}_1)\le_\ell(M_2,N_2,\mathbf{J}_2)$ <u>when</u>: clauses (i),(ii),(iii) there and

(*iv*) if $b\in\mathbf{J}_1$ and $(\forall a\in M_1)(\neg a=^{N_1}_\tau b)$ <u>then</u> $(\forall a\in M_2)(\neg a=^{N_2}_\tau b)$.

4.12 Observation. Without loss of generality $\mathfrak{K}$ has $(\text{jep})_\lambda$ and for every $M\in K_\lambda$ there is $p\in\mathscr{S}^{\text{na}}_{\mathfrak{K}_\lambda}(M)$ with no minimal extension.

Proof. Why?

Because if (p_*,M_*) are as required we can replace $\mathfrak{K}$ by $\mathfrak{K}^*=\mathfrak{K}\restriction\{M\in K$: there is a $\le_{\mathfrak{K}}$-embedding of M_* into $M\}$. Clearly $\mathfrak{K}^*$ satisfies the older requirements and if h is a $\le_{\mathfrak{K}}$-embedding of M_* into $M\in\mathfrak{K}^*_\lambda$ then $h(p_*)$ can be extended to some $p\in\mathscr{S}_{\mathfrak{K}^*_\lambda}(M)=\mathscr{S}_{\mathfrak{K}_\lambda}(M)$ as required. Why it can be extended? As any triples $(M,N,a)\in K^{3,\text{na}}_\lambda$ with no minimal extension has the extension property, see VI.2.5(1). $\square_{4.12}$

The first step is to prove that (and its proof includes a proof of 4.6).

4.13 Claim. $\mathfrak{u}$ *is a nice construction framework (if it is as in 4.12).*

Proof. Clauses (A),(B),(C) of Definition 1.2 are obvious. Also $(D)_1=(D)_2$ and $(E)_1=(E)_2$ as $\text{FR}_1=\text{FR}_1$ and $\le_1=\le_2$. Now $(D)_1$(a), (b), (c), (e) and $(E)_1$(a), (b)(α), (c), (d) holds by the definition of $\mathfrak{u}$. Concerning $(D)_1$(d), by assumption (e) of Lemma 4.10 clearly $\text{FR}^+_1\ne\emptyset$ and by Observation 4.12 we have $[M\in K_\lambda\Rightarrow(M,N,a)\in\text{FR}_1$ for some pair $(N,\mathbf{I})]$, i.e. $(D)_1$(d) holds. As $\mathfrak{K}_\lambda$ has amalgamation and $(D)_1$(d) holds, clearly

(∗) if $M\le_{\mathfrak{K}_\lambda}N$ then for some pair $(N',\mathbf{J})$ we have $N\le_{\mathfrak{K}_\lambda}N'$ and $(M,N',\mathbf{J})\in\text{FR}^+_1$.

Concerning $(E)_1(b)(\beta)$, just remember that $=_\tau$ is a fake equality, recalling subclause (iii) of clause (e) of Definition 4.5. Now the main point is amalgamation = clause (F) of Definition 1.2. We first ignore the $=_\tau$ and disjointness, that is, we work in $\mathfrak{K}_\lambda$ not $\mathfrak{K}'_\lambda$; easily this suffices. So we assume that $(M_0, M_\ell, \mathbf{J}_\ell) \in \mathrm{FR}_\ell$ for $\ell = 1,2$ and by $(*)$ above without loss of generality $\ell = 1,2 \Rightarrow \mathbf{J}_\ell \neq \emptyset$ so let $\mathbf{J}_\ell = \{a_\ell\}$. Let $p_\ell = \mathbf{tp}_{\mathfrak{K}_\lambda}(a_\ell, M_0, M_\ell)$ and by VI.1.14(1) we can find a reduced $(M'_0, M'_1, a_1) \in K^{3,\mathrm{na}}_\lambda$ which is $\leq_{\mathrm{na}}$-above (M_0, M_1, a_1). We can apply Claim VI.2.5(1) because: first Hypothesis VI.2.1 holds (as $\mathfrak{K}$ is an a.e.c., $\mathrm{LS}(\mathfrak{K}) \leq \lambda$ and $K_\lambda \neq \emptyset$) and, second, $(\mathrm{amg})_\lambda$ holds by assumption (d) of 4.10. So by VI.2.5(1), the extension property for such types (i.e. ones with no minimal extensions) holds, so there are M'_2 such that $M'_0 \leq_{\mathfrak{K}_\lambda} M'_2$ and $\leq_{\mathfrak{K}}$-embedding g of M_2 into M'_2 over M_0 such that $g(a_2) \notin M'_0$.

Again by VI.2.5(1) we can find $M''_1 \in K_\lambda$ such that $M'_2 \leq_{\mathfrak{K}} M''_1$ and f which is a $\leq_{\mathfrak{K}}$-embedding of M'_1 into M''_1 such that $f(a_1) \notin M'_2$.

By the definition of "(M'_0, M'_1, a) is reduced", see Definition VI.1.11 it follows that $f(M'_1) \cap M'_2 = M'_0$, so $M'_1 \cap M'_2 = M'_0$. In particular $f(a_1) \in M'_2 \wedge g(a_2) \notin f(M'_1)$ so we are done. Now the result with disjointness follows because $=_\tau$ is a fake equality. $\square_{4.13}$

4.14 Claim. *1) $\mathfrak{u}$ has the vertical coding property, see Definition 2.9(5).*
2) If $(M, N, \mathbf{I}) \in \mathrm{FR}^+_2$ and $a \in \mathbf{I}$ & $b \in M \Rightarrow \neg(a =^{N_1}_\tau b)$ <u>then</u> this triple has the true vertical coding$_0$ property (see Definition 2.9(1B).
3) $\mathfrak{K}_{\lambda^+}$ has a superlimit model which is saturated.
4) For almost$_2$ every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ the model M_∂ is saturated.
5) Every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ has the vertical coding property (see Definition 2.9(3)) <u>when</u> $M_{\lambda^+} \in \mathfrak{K}_{\lambda^+}$ is saturated.
6) For some $\mathfrak{u} - 0$-appropriate function $\mathfrak{h}$, for every $M \in K^{\mathfrak{u},\mathfrak{h}}_{\lambda^{++}}$ the model $M/=^M$ have cardinality λ^{++} and the set $a/=^M_\tau$ has cardinality λ^{++} for every $a \in M$.

Proof. 1) Follows by part (3),(4),(5).
2) By the choice of $\mathrm{FR}^{\mathfrak{u}}_1$ for some $a, \mathbf{I} = \{a\}$ and the type $\mathbf{tp}_{\mathfrak{K}_\lambda}(a, M, N)$ has no minimal extension. To prove the true vertical coding property

assume that $(\langle M^\ell_i : i \le \beta\rangle, \langle \mathbf{J}^\ell_i : i < \beta\rangle)$ for $\ell = 1,2$ and $\langle \mathbf{I}_i : i \le \beta\rangle$ are as in Definition 2.9(1), i.e., they form a $\mathfrak{u}$-free $(\beta,1)$-rectangle with $(M, N, \{a\}) \le^1_{\mathfrak{u}} (M^1_0, M^2_0, \mathbf{I}_0)$; i.e. there is $\mathbf{d}$, a $\mathfrak{u}$-free $(\beta,1)$-rectangle such that $M^{\mathbf{d}}_{i,0} = M^1_i, M^{\mathbf{d}}_{i,1} = M^2_i, \mathbf{J}^{\mathbf{d}}_{i,\ell} = \mathbf{J}^\ell_i, \mathbf{I}^{\mathbf{d}}_{i,\ell} = \mathbf{I}_i$).

So $M \le_{\mathfrak{K}_\lambda} M^1_\beta, N \le_{\mathfrak{K}_\lambda} M^2_\beta, a \in N \backslash M^1_\beta$ and $\mathbf{I}_i = \{a\}$. As $\mathbf{tp}_{\mathfrak{K}_\lambda}(a, M, N)$ has no minimal extension we can find $M^1_{\beta+1}$ such that $M^1_\beta \le_{\mathfrak{K}_\lambda} M^1_{\beta+1}$ and $\mathbf{tp}_{\mathfrak{K}_\lambda}(a, M^1_\beta, M^2_\beta)$ has at least two non-algebraic extensions in $\mathscr{S}_{\mathfrak{K}_\lambda}(M^1_\beta)$, hence we can choose $p_1 \ne p_2 \in \mathscr{S}^{\mathrm{na}}_{\mathfrak{K}_\lambda}(M^1_{\beta+1})$ extending $\mathbf{tp}_{\mathfrak{K}_\lambda}(a, M^1_\beta, M^2_\beta)$. Now treating equality as congruence without loss of generality $M^1_{\beta+1} \cap M^2_\beta = M^1_\beta$ and there are $N^1, N^2 \in K_\lambda$ such that $M^1_{\beta+1} \le_{\mathfrak{K}} N^\ell, M^2_\beta \le_{\mathfrak{K}} N^\ell$ and $\mathbf{tp}_{\mathfrak{K}_\lambda}(a, M^1_{\beta+1}, N_\ell) = p_\ell$ for $\ell = 1,2$.

Letting $M^{2,\ell}_{\beta+1} := N_\ell$ we are done.

3) If $\mathfrak{K}$ is categorical in λ^+ then the desired conclusion holds as every $M \in \mathfrak{K}_{\lambda^+}$ is saturated above λ by clause (g) of the assumption of 4.10. If $\mathfrak{K}$ only has a superlimit model in $\mathfrak{K}_{\lambda^+}$ as there is a $M' \in \mathfrak{K}_{\lambda^+}$ saturated above λ, necessarily the superlimit $M' \in \mathfrak{K}_{\lambda^+}$ is saturated above λ by VI.2.8(4).

4) We prove the existence of $\mathfrak{g}$ for the "almost$_2$" (or use the proof of 4.3(1)). Now recalling $(*)_4$ of 1.22(1) for each $M \in K_\lambda$ with universe $\in [\lambda^{++}]^\lambda$, we can choose a sequence $\langle p_{M,\alpha} : \alpha < \lambda^+\rangle$ listing $\mathscr{S}_{\mathfrak{K}_\lambda}(M)$. When defining the value $\mathfrak{g}(\bar M^1, \bar{\mathbf{J}}^1, \mathbf{f}^1, \bar M^2 \restriction (\delta + \mathbf{f}^1(\delta) + 1)), \bar{\mathbf{J}}^2 \restriction (\delta + \mathbf{f}^1(\delta), \bar{\mathbf{I}} \restriction (\delta + \mathbf{f}^1(\delta) + 1), S)$, see Definition 1.22(1)(c) we just realize all $p_{M^2_i, j}$ with $i, j < \delta$. Recalling that by part (3) the union of a $\le_{\mathfrak{K}}$-increasing sequence of length $< \lambda^{++}$ of saturated members of K_{λ^+} is saturated, we are done.

5) Holds by Observation 2.10(1).

6) Easy, as in 4.7(3). $\square_{4.14}$

Continuation of the Proof of 4.10. By the 1.19, 4.13, 4.14(1) we can apply Theorem 2.11. $\square_{4.10}$

4.15 Remark. So we have finished proving 4.10.

* * *

(C) The symmetry property of $\mathrm{PC}_{\aleph_0}$ classes

Here we pay a debt from Theorem I.5.34(1), so naturally we assume knowledge of I§5; of course later results supercede this. Also we can avoid this subsection altogether, dealing with the derived good $\aleph_0$-frame in II.3.4.

4.16 Theorem. $\dot{I}(\aleph_2, \mathfrak{K}) \geq \mu_{\mathrm{unif}}(\aleph_2, 2^{\aleph_1})$,
moreover $\dot{I}(\aleph_2, \mathfrak{K}(\aleph_1\text{-saturated})) \geq \mu_{\mathrm{unif}}(\aleph_2, 2^{\aleph_1})$ *when:*

⊛ (a) *(set theory)*
  (α) $2^{\aleph_0} < 2^{\aleph_1} < 2^{\aleph_0}$ *and*
  (β) $\mathrm{WDmId}(\aleph_1)$ *is not* $\aleph_2$*-saturated*
 (b) $\mathfrak{K}$*, an a.e.c., is* $\aleph_0$*-representable, i.e., is* $\mathrm{PC}_{\aleph_0}$*-a.e.c., see Definition I.1.4(4),(5)*
 (c) $\mathfrak{K}$ *is as in I.4.8, I.5.1*
 (d) $\mathfrak{K}$ *fails the symmetry property or the uniqueness of two sided stable amalgamation, see Definition I.5.31, equivalently*
 (d) $\mathfrak{K}$ *fails the uniqueness of one-sided amalgamation*
 (e) $\mathbf{D}$ *is countable, see Definition I.5.2, I.5.10 and II§3(B).*

Remark. 1) On omitting "$\mathrm{WDmId}(\aleph_1)$ is not $\aleph_2$-saturated", see Conclusion 4.35.
2) Clause (e) of 4.16 is reasonable as we can without loss of generality assume it by Observation I.5.36.

Proof. Let $\lambda = \aleph_0$, without loss of generality

 (f) for $M \in K$, any finite sequence is coded by an element.

Now by II§3 (B), i.e. II.3.4 we have $\mathfrak{s}_{\aleph_0}$, which we call $\mathfrak{s} = \mathfrak{s}_{\mathfrak{K}} = \mathfrak{s}^1_{\mathfrak{K}}$, is a good $\aleph_0$-frame as defined (and proved) there, moreover $\mathfrak{s}_\lambda$ is type-full and $\mathfrak{K}^{\mathfrak{s}} = \mathfrak{K}$.

The proof is broken, as in other cases, i.e., we prove it by Theorem 2.3 which is O.K. as by 4.18 + 4.19 below its assumptions holds.

Remark. Note in the following definition $\mathrm{FR}_1, \mathrm{FR}_2$ are quite different even though $\le^1_{\mathfrak{u}}, \le^2_{\mathfrak{u}}$ are the same, except the domain.

4.17 Definition. We define $\mathfrak{u} = \mathfrak{u}^3_{\mathfrak{K}_{\aleph_0}}$ by

(a) $\partial = \partial_{\mathfrak{u}} = \aleph_1$

(b) $\mathfrak{K}_{\mathfrak{u}} = \mathfrak{K}_{\aleph_0}$ so $K^{\mathrm{up}}_{\mathfrak{u}} = \mathfrak{K}$

(c) FR_2 is the family of triples $(M, N, \mathbf{J})$ such that:
- (α) $M \le_{\mathfrak{K}} N \in \mathfrak{K}_{\aleph_0}$
- (β) $\mathbf{J} \subseteq N \backslash M$ and $|\mathbf{J}| \le 1$

(d) $(M_1, N_1, \mathbf{J}_1) \le_2 (M_2, N_2, \mathbf{J}_2)$ <u>iff</u>
- (α) both are from FR_1
- (β) $M_1 \le_{\mathfrak{K}} M_2$ and $N_1 \le_{\mathfrak{K}} N_2$
- (γ) $\mathbf{J}_1 \subseteq \mathbf{J}_2$
- (δ) if $\bar{c} \in \mathbf{J}$ then $\mathrm{gtp}(\bar{c}, M_2, N_2)$ is the stationarization of $\mathrm{gtp}(\bar{c}, M_1, N_1)$

(e) FR_1 is the class of triples $(M, N, \mathbf{J})$ such that
- (α) $M \le_{\mathfrak{K}} N$ are countable
- (β) $\mathbf{J} \subseteq N \backslash M$ or, less pedantically, $\mathbf{J} \subseteq {}^{\omega >}N \backslash {}^{\omega >}M$
- (γ) if $|\mathbf{J}| > 1$ then $\mathbf{J} = {}^{\omega >}N \backslash {}^{\omega >}M$ and N is $(\mathbf{D}(M), \aleph_0)^*$-homogeneous

(f) $\le_1$ is defined as in clause (d) <u>but</u> on FR_2.

4.18 Claim. *1)* $\mathfrak{u}$ *is a nice construction framework.*
2) For almost$_2$ *all triples* $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ *the model* M *is saturated (for* $\mathfrak{K}$*).*

Proof. 1) The main points are:

$\boxtimes_1$ $(M, N, \mathbf{J}) \in \mathrm{FR}_2$ and $n < \omega \Rightarrow (M_n, N_n, \mathbf{J}_n) \le_2 (M, N, \mathbf{J})$ when:

- (a) $(M_n, N_n, \mathbf{J}_n) \in \mathrm{FR}_2$
- (b) $(M_n, N_n, \mathbf{J}_n) \le_2 (M_{n+1}, N_{n+1}, \mathbf{J}_{n+1})$ for $n < \omega$
- (c) $M = \cup\{M_n : n < \omega\}$
- (d) $N = \cup\{N_n : n < \omega\}$
- (e) $\mathbf{J} = \cup\{\mathbf{J}_n : n < \omega\}$.

[Why $\boxtimes_1$ holds? See I.5.24(9).]

$\boxtimes_2$ $(M, N, \mathbf{J}) \in \mathrm{FR}_1$ and $n < \omega \Rightarrow (M_n, N_n, \mathbf{J}_n) \le_1 (M, N, \mathbf{J})$ when

- (a) $(M_n, N_n, \mathbf{J}_n) \in \mathrm{FR}_1$
- (b) $(M_n, N_n, \mathbf{J}_n) \le_1 (M_{n+1}, N_{n+1}, \mathbf{J}_{n+1})$ for $n < \omega$
- (c) $M = \cup\{M_n : n < \omega\}$
- (d) $N = \cup\{N_n : n < \omega\}$
- (e) $\mathbf{J} = \cup\{\mathbf{J}_n : n < \omega\}$.

[Why does $\boxtimes_2$ holds? If $|\mathbf{J}| \le 1$ then the proof is similar to the one in $\boxtimes_1$, so assume that $|\mathbf{J}| > 1$, so as $n < \omega \Rightarrow \mathbf{J}_n \subseteq \mathbf{J}_{n+1}$ by clause (b) of $\boxtimes_2$ and $\mathbf{J} = \cup\{\mathbf{J}_n : n < \omega\}$ by clause (e) of the $\boxtimes_2$ necessarily for some $n, |\mathbf{J}_n| > 1$, so without loss of generality $|\mathbf{J}_n| > 1$ for every $n < \omega$. So by the definition of FR_1, we have:

$(*)_1$ $\mathbf{J}_n = {}^{\omega>}(N_n) \backslash {}^{\omega>}(M_n)$

$(*)_2$ N_n is $(\mathbf{D}(M_n), \aleph_0)^*$-homogeneous.

Hence easily

$(*)_3$ $\mathbf{J} = {}^{\omega>}N \backslash {}^{\omega>}M$

and as in the proof of $\boxtimes_1$ clearly

$(*)_4$ if $n < \omega$ and $\bar{c} \in \mathbf{J}_n$ then $\mathrm{gtp}(\bar{c}, M, N)$ is the stationarization of $\mathrm{gtp}(\bar{c}, M_n, N_n)$.

So the demands for "$(M_n, N_n, \mathbf{J}_n) \le_1 (M, N, \mathbf{J})$" holds except that we have to verify

$(*)_5$ N is $(\mathbf{D}(M), \aleph_0)^*$-homogeneous.

[Why this holds? Assume $N \le_{\mathfrak{K}_{\aleph_0}} N^+$, $\bar{a} \in {}^{\omega>}N$, $\bar{b} \in {}^{\omega>}(N^+)$. So $\mathrm{gtp}(\bar{a}\hat{\ }\bar{b}, M, N^+) \in \mathbf{D}(M)$, hence by I.5.24(9) it is the stationarization of $\mathrm{gtp}(\bar{a}\hat{\ }\bar{b}, M_{n_0}, N^+)$ for some $n_0 < \omega$. Also for some $n_1 < \omega$, we have $\bar{a} \in {}^{\omega>}(N_{n_1})$. Now some $n < \omega$ is $\ge n_0, n_1$, so by $(*)_1 + (*)_2$ for some $\bar{b}' \in {}^{\omega>}(N_n)$ the type $\mathrm{gtp}(\bar{a}\hat{\ }\bar{b}', M_n, N_n)$ is equal to $\mathrm{gtp}(\bar{a}\hat{\ }\bar{b}, M_n, N^+)$ so $\bar{b}' \in \mathbf{J}_n$. But by $(*)_4$, also the type $\mathrm{gtp}(\bar{a}\hat{\ }\bar{b}', M, N)$ is a stationarization of $\mathrm{gtp}(\bar{a}\hat{\ }\bar{b}', M_n, N) = \mathrm{gtp}(\bar{a}\hat{\ }\bar{b}, M_n, N^+)$ hence $\mathrm{gtp}(\bar{a}\hat{\ }\bar{b}', M, N) = \mathrm{gtp}(\bar{a}\hat{\ }\bar{b}, M, N^+)$ so we are done.]
Thus we have finished proving $\boxtimes_2$]

$\boxtimes_3$ clause (F) of Definition 1.2 holds.

[Why $\boxtimes_3$? By I.5.30.]
Together we have finished proving $\mathfrak{u}$ is a nice construction framework.
2) Note that

$\boxtimes_4$ $M_{\omega_1} \in K_{\aleph_1}$ is saturated <u>when</u>:

- (a) $\langle M_\alpha : \alpha \le \omega_1 \rangle$ is $\le_{\mathfrak{K}}$-increasing continuous
- (b) $\alpha < \omega_1 \Rightarrow M_\alpha \in K_{\aleph_0}$
- (c) $M_{\alpha+1}$ is $(\mathbf{D}(M_\alpha), \aleph_0)^*$-homogeneous for $\alpha < \omega_1$ or just
- $(c)'$ if $\alpha < \omega_1$ and $p \in \mathbf{D}(M_\alpha)$ is a 1-type then for some $\beta \in [\alpha, \omega_1)$ and some $c \in M_{\beta+1}$, $\mathrm{gtp}(c, M_\beta, M_{\beta+1})$ is the stationarization of p.

[Why? Obvious.]
Let $S \subseteq \omega_1$ be stationary, so clearly it suffices to prove:

$\boxtimes_5$ there is $\mathfrak{g}$ as in Definition 1.22(1), 1.23(3) for S and our $\mathfrak{u}$ such that:
if $\langle (\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha \le \delta \rangle$ is $\le^{\mathrm{qs}}_{\mathfrak{u}}$-increasing continuous 1-obeying $\mathfrak{g}$
(and $\delta < \partial^+$ is a limit ordinal) <u>then</u> $M^\delta \in \mathfrak{K}_{\aleph_1}$ is saturated.

[Why? Choose $\mathfrak{g}$ such that if the pair $((\bar{M}', \bar{\mathbf{J}}', \mathbf{f}'), (\bar{M}'', \bar{\mathbf{J}}'', \mathbf{f}''))$ does 1-obey $\mathfrak{g}$ then for every $\alpha < \partial$ and $p \in \mathbf{D}(M''_\alpha)$ we have

- $(*)$ for stationarily many $\delta \in S$ for some $i < \mathbf{f}''(\delta)$ the type $\mathrm{gtp}(a, M''_\alpha, M''_{\delta+i+1})$ is the stationarization of p where $\mathbf{J}''_{\delta+i} = \{a\}$.

Now assume that $\langle(\bar{M}^\zeta, \bar{\mathbf{J}}^\zeta, \mathbf{f}^\zeta) : \zeta < \delta\rangle$ is $\leq_{\mathrm{qs}}$-increasing (for our present $\mathfrak{u}$) and $\delta = \sup(u)$ where $u = \{\zeta : ((\bar{M}^\zeta, \mathbf{J}^\zeta, \mathbf{f}^\zeta), (\bar{M}^{\zeta+1}, \bar{\mathbf{J}}^{\zeta+1}, \mathbf{f}^{\zeta+1}))$ does 1-obeys $\mathfrak{g}\}$, and we should prove that $M^\delta := \cup\{M^\zeta_\partial : \zeta < \partial\}$ is saturated. Without loss of generality u contains all odd ordinals $< \delta$ and $\delta = \mathrm{cf}(\delta)$. If $\delta = \aleph_1$ this is obvious, and if $\delta = \aleph_0$ just use non-forking of types, and the criterion in $\boxtimes_4$ using $(*)$. So $\boxtimes_5$ is proved.] $\square_{4.18}$

4.19 Claim. $\mathfrak{u}$ *has the weak coding property.*

Proof. Clearly by clause $(d)'$ of the assumption, i.e. by Definition I.5.31(2),(3) there are $N_\ell\,(\ell \leq 2)), N'_3, N''_3$ such that:

$(*)_1$
- (a) $N_0 \leq_{\mathfrak{u}} N_1$ and $N_0 \leq_{\mathfrak{u}} N_2$
- (b) $N_\ell \leq_{\mathfrak{u}} N'_3$ and $N_\ell \leq_{\mathfrak{u}} N''_3$ for $\ell = 1, 2$
- (c) N_0, N_1, N_2, N'_3 is in one-sided amalgamation, i.e. $\bar{a} \in {}^{\omega>}(N_1) \Rightarrow (N_0, N_1, \{\bar{a}\}) \leq^2_{\mathfrak{u}} (N_2, N'_3, \{\bar{a}\})$ (hence $N_1 \cap N_2 = N_0$)
- (d) N_0, N_1, N_2, N''_3 is in one sided amalgamation
- (e) there are no (N_3, f) such that $N''_3 \leq_{\mathfrak{u}} N_3$ and f is a $\leq_{\mathfrak{u}}$-embedding of N'_3 into N_3 over $N_1 \cup N_2$.

Now without loss of generality

- $(*)_2$ $(N_0, N_1, \mathbf{J}) \in \mathrm{FR}_1$ where $\mathbf{J} \in {}^{\omega>}(N_1)\backslash{}^{\omega>}(N_0)$ such that $|\mathbf{J}| > 1$.

[Why? We can find $N_1^+ \in K_{\mathfrak{u}}$ which is $(\mathbf{D}(N_1), \aleph_0)^*$-homogeneous over N_1 and without loss of generality $N_1^+ \cap N'_3 = N_1 = N_1^+ \cap N''_3$. Now we can find N_3^* and $N_3^{**} \in K_{\mathfrak{u}}$ such that $(N_1, N_1^+, N'_3, N_3^*)$ as well as $(N_1, N_1^+, N_3'', N_3^{**})$ is in one sided stable amalgamation. It follows that $(N_0, N_1^+, N_2, N_3^*, N_3^{**})$ satisfies all the requirements on $(N_0, N_1, N_2, N_3', N_3'')$ and in addition the demand in $(*)_2$ so we are done.]

Also without loss of generality

$(*)_3$ $(N_0, N_2, \mathbf{I}) \in \mathrm{FR}_2$ and $|\mathbf{I}| = 1$ and N_2 is $(\mathbf{D}(N_0), \aleph_0)^*$-homogeneous.

[Why? Similarly to $(*)_1$.]

To prove $\mathfrak{u}$ has the weak coding we can assume (the saturation is justified by 4.18(2))

$(*)_4$ $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K_{\mathfrak{u}}^{\mathrm{qt}}$ and $M := \cup\{M_\alpha : \alpha < \omega_1\}$ is saturated for $\mathfrak{K}$.

Now by renaming without loss of generality

$(*)_5$ $N_0 \leq_{\mathfrak{K}} M_{\alpha(0)}$ and $N_1 \cap M = N_0$ and $(N_0, N_1, \mathbf{J}) \leq_1 (M_{\alpha(0)}, N_1', \mathbf{J}')$ and $N_1' \cap M = M_{\alpha(0)}$.

It suffices to prove that $(\alpha(0), N_1', \mathbf{J}')$ is as required in 2.2(3). Next by the definition of "having the weak coding property", for our purpose we can assume we are given $(N'', \mathbf{J}'')$ such that

$(*)_6$ $\alpha(0) \leq \delta < \omega_1$ and $(N_0, N_1', \mathbf{J}') \leq_1 (M_\delta, N'', \mathbf{J}'')$.

By the definition of $\leq_1$ we know that

$(*)_7$ N'' is $(\mathbf{D}(M_\delta), \aleph_0)^*$-homogeneous over M_δ.

As $\cup\{M_\alpha : \alpha < \omega_1\}$ is saturated (for $K^{\mathfrak{u}}$) we can find $\beta \in (\delta, \omega_1)$ such that M_β is $(\mathbf{D}(M_\delta), \aleph_0)^*$-homogeneous over M_δ.

As $\mathfrak{K}$ is categorical in $\aleph_0$

$(*)_8$ there is an isomorphism f_0 from N_0 onto M_δ.

Similarly using the uniqueness over N_0 of a countable $(\mathbf{D}(M_0), \aleph_0)^*$-homogeneous model over N_0

$(*)_9$ there are isomorphisms f_1, f_2 from N_1, N_2 onto N'', M_β respectively extending f_0.

Lastly, M_β, N can be amalgamated over M_δ in the following two ways:

- $\odot_1$ there are $f', M' \in K_{\mathfrak{u}}$ such that f' is an isomorphism from N'_3 onto M' extending $f_1 \cup f_2$
- $\odot_2$ there are $f'', M'' \in K_{\mathfrak{u}}$ such that f'' is an isomorphism from N^1_3 onto M'' extedning $f_1 \cup f_2$.

This is clearly enough. The rest should be clear.] $\square_{4.19}$

Proof of 4.16. By the claims above. $\square_{4.16}$

* * *

<u>(D) Density of $K^{3,\mathrm{uq}}_\lambda$ when minimal triples are dense</u>

Having taken care of VI§3,§4 and of I§5, we now deal with proving the non-structure results of VI§6, i.e. [Sh 576, §6], relying on 2.3 instead of [Sh 576, §3]. Of course, later we prove stronger results but have to work harder, both model theoretically (including "$\mathfrak{s}$ is almost a good λ-frame") and set theoretically (using (vertical coding in) Theorem 2.11 and §3 rather than (weak coding in) Theorem 2.3).

This is used in VI.6.11.

4.20 Theorem. *The non-structure results of VI.6.11, Case 1 holds. It details:* $\dot{I}(\lambda^{++}, \mathfrak{K}) \geq \mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$ <u>*when*</u> *we are assuming:*

- (A) *set theoretically:*
  - (a) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ *and*
  - (b) *the weak diamond ideal on* $\partial := \lambda^+$ *is not* ∂^+*-saturated*
- (B) *model theoretically:*
  - (a) $\mathfrak{K}$ *is an a.e.c.,* $\lambda \geq \mathrm{LS}(\mathfrak{K})$
  - (b) (α) $\mathfrak{K}$ *is categorical in* λ

    (β) $\mathfrak{K}$ *is categorical in* λ^+ *or just has a superlimit model in* λ^+

(c) (α) *$\mathfrak{K}$ has amalgamation in λ*

(β) *$\mathfrak{K}$ is stable in λ <u>or</u> just $M \in K_\lambda \Rightarrow |\mathscr{S}^{\min}_{\mathfrak{K}}(M)| \le \lambda$*

(d) (α) *the minimal types are dense (for $M \in \mathfrak{K}_\lambda$)*

(β) *for $M \in K_\lambda$ the set $\mathscr{S}^{\min}_{\mathfrak{K}_\lambda}(M) = \{p \in \mathscr{S}_{\mathfrak{K}_\lambda}(M) : p$ minimal$\}$ is inevitable*

(γ) *the $M \in K^{\rm slm}_{\lambda^+}$ is saturated above λ*

(e) *above (by $\le_{\rm na}$) some $(M^*, N^*, a) \in K^{3,{\rm na}}_\lambda$ there is no triple with the uniqueness property, i.e. from $K^{3,{\rm uq}}_\lambda$, see VI.6.3.*

4.21 Remark. 1) Note: every $M \in K_{\lambda^+}$ is saturated above λ when the first, stronger version of (B)(b)(β) holds noting (B)(c)(β) + (B)(d)(β).
2) When we use the weaker version of clause (b)(β), i.e. "there is superlimit $M \in K_{\lambda^+}$" then we have to prove that for almost$_2$ every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f})$, the model M_{λ^+} is saturated above λ which, as in earlier cases, can be done; see VI.2.8(4).
3) Concerning clause (B)(d): "the minimal types are dense", it follows from $(\text{amg})_\lambda+$ $(\text{stb})_\lambda$, i.e. from clause (c) recalling VI.2.3(4).
4) Note that 4.23, 4.24 does not depend on clause (A)(b) of 4.20.

4.22 Definition. We define $\mathfrak{u} = \mathfrak{u}_4 = \mathfrak{u}^4_{\mathfrak{K}_\lambda}$ as follows:

(a) $\partial_{\mathfrak{u}} = \lambda^+$

(b) $\mathfrak{K}_{\mathfrak{u}} = \mathfrak{K}_\lambda$ or pedantically $\mathfrak{K}'_\lambda$, see Definition 1.10

$(c)_1$ $\text{FR}^{\mathfrak{u}}_1$ is the set of triples $(M, N, \mathbf{I})$ satisfying $M \le_{\mathfrak{K}} N \in K_\lambda, \mathbf{I} = \emptyset$ or $\mathbf{I} = \{a\}$ and the type $\mathbf{tp}_{\mathfrak{K}_\lambda}(a, M, N)$ is minimal, pedantically, if $a/=^N \notin M/=^N$ then $\mathbf{tp}(a/=^N, M/=^N, N/=^N)$ is minimal

$(c)_2$ $(M_1, N_1, \mathbf{I}_1) \le_1 (M_2, N_2, \mathbf{I}_2)$ iff (both are $\text{FR}^{\mathfrak{u}}_1$ and) $M_1 \le_{\mathfrak{K}} M_2, N_1 \le_{\mathfrak{K}} N_2$ and $\mathbf{I}_1 \subseteq \mathbf{I}_2$ (in the non-trivial cases, equivalently, $\mathbf{I}_1 = \mathbf{I}_2$), pedantically, if $(a/=^N) \notin M/=^N$ then $\mathbf{tp}(a/=^N, M/=^N, N/=^N)$ is minimal

(d) $\text{FR}^{\mathfrak{u}}_2 = \text{FR}^{\mathfrak{u}}_1$ and $\le^2_{\mathfrak{u}} = \le^1_{\mathfrak{u}}$.

4.23 Claim. $\mathfrak{u}$ *is a nice construction framework which is self-dual.*

Proof. Easy (amalgamation, i.e. clause (F) of Definition 1.2 holds by the proof of symmetry in Axiom (E)(f) in proof of Theorem VI.8.1). $\square_{4.23}$

4.24 Claim. *1) Every* $(M, N, \mathbf{I}) \in \mathrm{FR}_1$ *such that* $[a \in \mathbf{I} \;\;\&\;\; b \in M \Rightarrow \neg a =^N_\tau b]$ *has the true weak coding property (see Definition 2.2(1A)).*
2) $\mathfrak{u}$ *has the weak coding property.*
3) For almost$_2$ *every* $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ *the model* $M_\partial \in K_{\lambda^+}$ *is saturated above* λ.
4) For some $\mathfrak{u} - \{0, 2\}$*-appropriate function* $\mathfrak{h}$*, for every* $M \in K^{\mathfrak{u},\mathfrak{h}}_{\lambda^{++}}$ *the model* $M/=^M$ *has cardinality* λ^{++} *and is saturated above* λ.

Proof. 1) Straight.
2) By part (1) above and part (3) below.
3) By clauses (B)(c)(α), (β) of 4.20, clearly there is a $M \in K_{\lambda^+}$ which is saturated above λ. If in (B)(b)(β) we assume categoricity in λ^+ then every $M \in K_{\lambda^+}$ is saturated above λ, but then it is obvious that part (1) implies part (2) by 2.10(4)(b). For any stationary $S \subseteq \partial$, we choose $\mathfrak{h}$ such that

(∗) if $((\bar{M}^1, \bar{\mathbf{J}}^1, \mathbf{f}^1), (\bar{M}^2, \bar{\mathbf{J}}^2, \mathbf{f}^2))$ does 2-obeys $\mathfrak{h}$ then: for stationarily many $\delta \in S$ there is successor ordinal $i < \mathbf{f}^2(\delta)$ such that $M^2_{\delta+i+1}$ is $<_{\mathfrak{K}_\lambda}$-universal over $M^2_{\delta+i}$ (hence M^2_∂ is saturated above λ and is the superlimit model in $\mathfrak{K}_{\lambda^+}$).

Alternatively do as in 4.14(4), using VI.8.1.
4) As in 4.7(3).

Proof of 4.20. By 4.23, 4.24 and Theorem 2.3. $\square_{4.20}$

* * *

<u>(E) Density of $K^{3,\mathrm{uq}}_{\mathfrak{s}}$ for good λ-frames</u>

We now deal with the non-structure proof in II.5.9, that is justifying why the density of $K^{3,\mathrm{uq}}_{\mathfrak{s}}$ holds.

Before we state the theorem, in order to get rid of the problem of disjoint amalgamation, one of the ways is to note:

4.25 Definition. Assume that $\mathfrak{s}$ is a good λ-frame (or just an almost good λ-frame see Definition in 5.2 below or just a pre-λ-frame, see VI.8.2).
1) We say that $\mathfrak{s}$ has fake equality $=_*$ when $\mathfrak{K}_{\mathfrak{s}}$ has the fake equality $=_*$, see Definition 3.17(1) and $\mathbf{tp}_{\mathfrak{s}}(a, M_1, M_2)$ does not fork over M_0 iff $M_0 \le_{\mathfrak{s}} M_1 \le_{\mathfrak{s}} M_2, a \in M_2$ and letting $M'_\ell = M_\ell / =^{M_2}_*$ we have $(\forall b \in M_1)(\neg(a =^{M_2}_* b)) \Rightarrow \mathbf{tp}_{\mathfrak{s}}(a/ =^{M_2}_*, M'_1, M'_2)$ does not fork over M'_0.
2) We define $\mathfrak{s}' = (K_{\mathfrak{s}'}, \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}'}, \bigcup\limits_{\mathfrak{s}'})$ as follows:

- (a) $\mathfrak{K}_{\mathfrak{s}'} = K'_{\mathfrak{s}}$, see 1.10 as in 4.5
so $\tau_{\mathfrak{s}'} = \tau'_{\mathfrak{s}} = \tau_{\mathfrak{s}} \cup \{=_\tau\}$ and a $\tau'_{\mathfrak{s}}$-model M belongs to $K_{\mathfrak{s}'}$ iff $=^M_*$ is a congruence relation and the model $M/ =^{M'}_\tau$ belongs to $\mathfrak{K}_{\mathfrak{s}}$
- (b) for $M' \in K_{\mathfrak{s}'}$ we let $\mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}'}(M') = \{\mathbf{tp}_{\mathfrak{K}_{\mathfrak{s}'}}(a, M', N') : M' \le_{\mathfrak{s}'} N'$ and $\mathbf{tp}_{\mathfrak{s}}(a/ =^{N'}_\tau, M'/ =^{M'}_\tau, N'/ =^{N'}_\tau) \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M'/ =^{M'}_\tau)$ or $a \in N' \backslash M'$ but $(\exists b \in M')(a =_\tau b)\}$
- (c) $\mathbf{tp}_{\mathfrak{s}'}(a, M'_1, M'_2)$ does not fork over M'_0 when $M'_0 \le_{\mathfrak{s}'} M'_1 \le_{\mathfrak{s}'} M'_2$ and either $\mathbf{tp}_{\mathfrak{s}}(a/ =^{M'_2}_\tau, M'_1 =^{M'_2}_\tau, M'_2/ =^{M'_2}_\tau)$ does not fork over $M'_0/ =^{M'_2}$ or for some $b \in M'_0$ we have $M'_2 \models$ "$a =_\tau b$" but $a \notin M'_1$.

4.26 Claim. *Let $\mathfrak{s}, \mathfrak{s}'$ be as in 4.25(2).*
1) If $\mathfrak{s}$ is a good λ-frame <u>*then*</u> *$\mathfrak{s}'$ is a good λ-frame and if $\mathfrak{s}$ is an almost good λ-frame then $\mathfrak{s}'$ is an almost good λ-frame; and if $\mathfrak{s}$ is a pre-λ-frame then $\mathfrak{s}'$ is a pre-λ-frame. In all cases $\mathfrak{s}'$ has the fake equality $=_\tau$.*
2) For $\mu \ge \lambda, \dot I(\mu, K^{\mathfrak{s}}) = |\{M'/ \cong: M' \in K^{\mathfrak{s}'}_\mu$ and is $=_\tau$-fuller, that is $a \in M' \Rightarrow \|M'/ =^{M'}_\tau \| = \mu = |\{b \in M' : a =^{M'}_\tau b\}|\}$.

3) If $M' \in K^{\mathfrak{s}'}$ <u>then</u> M' is λ^+-saturated above λ for $\mathfrak{s}'$ iff $M'/=^{M'}_\tau$ is λ^+-saturated above λ for $\mathfrak{s}$ and M' is $(\lambda^+, =_\tau)$-full (recalling 1.10(5A)).

4.27 Remark. 1) By 4.26(2), the proof of "$\dot{I}(\mu, K^{\mathfrak{s}'})$ is $\geq \chi$" here usually gives "$\dot{I}(\mu, K^{\mathfrak{s}})$ is $\geq \chi$".
2) We define $\mathfrak{s}'$ such that for some 0-appropriate $\mathfrak{h}$, if $\langle(\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha < \partial^+\rangle$ is $\leq_{\rm qt}$-increasing continuous 0-obeying $\mathfrak{h}$, then $M = \cup\{M^\alpha_\partial : \alpha < \partial^+\}$ satisfies the condition in 4.26(2); it does not really matter if we need $\{0,2\}$-appropriate $\mathfrak{h}$.
3) Recall Example 1.12 as an alternative to 4.26(2).
4) Another way to deal with disjointness is by 5.22, 5.23 below.

Proof. Easy and see 1.11. $\square_{4.26}$

4.28 Theorem. *Like 4.20 but dealing with $\mathfrak{s}$, i.e. replacing clause (B) by clause (B)′ stated below; that is, $\dot{I}(\lambda^{++}, K^{\mathfrak{s}}) \geq \mu_{\rm unif}(\lambda^{++}, 2^\lambda)$ <u>when</u>:*

- (A) *set theoretically:*
  - (a) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ *and*
  - (b) *the weak diamond ideal on $\partial := \lambda^+$ is not ∂^+-saturated*
- $(B)'$ *model theoretic*
  - (a) *$\mathfrak{s}$ is a good λ-frame (or just an almost good λ-frame, see 5.2) with $\mathfrak{K}_{\mathfrak{s}} = \mathfrak{K}_\lambda$*
  - (b) *density of $K^{3,\rm uq}_{\mathfrak{s}}$ fail, i.e. for some $(M,N,a) \in K^{3,\rm bs}_{\mathfrak{s}}$ we have $(M,N,a) \leq_{\rm bs} (M',N',a) \Rightarrow (M',N',a) \notin K^{3,\rm uq}_{\mathfrak{s}}$, see Definition 5.3.*

Proof. We apply 2.3, its assumption holds by Definition 4.29 and Claim 4.30 below applied to $\mathfrak{s}'$ from 4.25 by using 4.26.

4.29 Definition. Let $\mathfrak{s}$ be as in 4.28 (or just a pre-λ-frame). We let $\mathfrak{u} = \mathfrak{u}_{\mathfrak{s}} = \mathfrak{u}^1_{\mathfrak{s}}$ be defined as

(a) $\partial_{\mathfrak{u}} = \lambda^+_{\mathfrak{s}}$

(b) $\mathfrak{K}_{\mathfrak{u}} = \mathfrak{K}_{\mathfrak{s}}$

(c) $\mathrm{FR}^{\mathfrak{u}}_1 = \{(M, N, \mathbf{I}) : M \le_{\mathfrak{K}_{\mathfrak{u}}} N, \mathbf{I} = \emptyset$ or $\mathbf{I} = \{a\}$ where $a \in N$ and $\mathbf{tp}_{\mathfrak{s}}(a, M, N) \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)\}$

(d) $\le^1_{\mathfrak{u}}$ is defined by $(M_1, N_1, \mathbf{I}_1) \le^1_{\mathfrak{u}} (M_2, N_2, \mathbf{I}_2)$ <u>when</u> both are from $\mathrm{FR}^{\mathfrak{u}}_1, M_1 \le_{\mathfrak{s}} M_2, N_1 \le_{\mathfrak{s}} N_2, \mathbf{I}_1 \subseteq \mathbf{I}_2, M_1 = M_2 \cap N_1$ and if $\mathbf{I}_1 = \{a\}$ then $\mathbf{tp}_{\mathfrak{s}}(a, M_2, N_2)$ does not fork (for $\mathfrak{s}$) over M_1 (so if $\mathbf{I}_2 = \{a_\ell\}$ for $\ell = 1, 2$ this means $(M_1, N_1, a_1) \le_{\mathrm{bs}} (M_2, N_2, a_1)$)

(e) $\mathrm{FR}^{\mathfrak{u}}_2 = \mathrm{FR}^{\mathfrak{u}}_1$ and $\le^2_{\mathfrak{u}} = \le^1_{\mathfrak{u}}$.

4.30 Claim. *Let $\mathfrak{u} = \mathfrak{u}_{\mathfrak{s}'}$ where $\mathfrak{s}'$ is from Definition 4.25 or $\mathfrak{u} = \mathfrak{s}$ except when we mention equality (or $=_\tau$-fuller).*
1) $\mathfrak{u}$ is a nice construction framework which is self dual.
2) For almost$_2$ every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ the model $M := \cup\{M_\alpha : \alpha < \lambda^+\}$ is saturated, see Definition 1.22(3C), see 4.3.
3) $\mathfrak{u}$ has the weak coding property.
4) There is a $\mathfrak{u}$-0-appropriate function $\mathfrak{h}$ such that every $M \in K^{\mathfrak{u},\mathfrak{h}}_{\lambda^{++}}$ is λ^+-saturated above λ and is $=_\tau$-fuller (hence $M/=^M_\tau$ has cardinality λ^{++}).
5) Moreover, there is a $\mathfrak{u} - \{0, 2\}$-appropriate function $\mathfrak{h}$ such that if $\langle(\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha < \lambda^{++}\rangle$ obeys $\mathfrak{h}$ <u>then</u> for some club E of λ^{++} the model M^δ_∂ is saturated above λ for $\delta \in E$ and $\cup\{M^\zeta_\partial : \zeta < \lambda^{++}\}$ is $=_\tau$-fuller.
6) Also $\mathfrak{u}$

(α) *satisfies $(E)_\ell(e)$, monotonicity (see 1.13(1))*

(β) *is hereditary (see Definition 3.17(2),(3))*

(γ) *if $\mathfrak{u} = \mathfrak{u}_{\mathfrak{s}}, \mathfrak{s}$ from 4.25(2) then $=_\tau$ is a fake equality for $\mathfrak{u}$, (see Definition 3.16(1))*

(δ) *$\mathfrak{u}$ is hereditary for the fake equality $=_\tau$, (see Definition 3.17(4))*

(ε) *$\mathfrak{u}$ is interpolative, see Definition 3.21.*

4.31 Remark. 1) In claim 5.11 we shall deal with the almost good case, (see Definition 5.2), the proof below serves there too.
2) In 4.30, only clause $(B)'(a)$ from the assumptions of Theorem 4.28 is used except in part (3) which uses also clause (B)′(b).
3) Part (6) of 4.30 is used only in 6.19.
4) Most parts of 4.30 holds also for $\mathfrak{u} = \mathfrak{u}_{\mathfrak{s}}$, i.e. we have to omit the statements on $=_\tau$-fuller, fake equality.

Proof. 1) Note

Clause $(D)_\ell(d)$: Given $M \in K_{\mathfrak{s}}$, it is not $<_{\mathfrak{s}}$-maximal hence there is N such that $M <_{\mathfrak{s}} N$ hence by density (Ax(D)(c) of (almost) good λ-frames) there is $c \in N$ such that $\mathbf{tp}_{\mathfrak{s}}(c, M, N) \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$, so $(M, N, a) \in \mathrm{FR}^+_1$, as required.

Clause $(E)_\ell(c)$: Preservation under increasing union.

Holds by axiom $(E)(h)$ of Definition of II.2.1 of $\mathfrak{s}$ being a good λ-frame (and similarly for being an almost good λ-frame).

Clause (F), amalgamation:

This holds by symmetry axiom $(E)(i)$ of Definition II.2.1 of $\mathfrak{s}$ being a good λ-frame (and similarly for $\mathfrak{s}$ being an almost good λ-frame). The disjointness is not problematic in proving clause (F) of Definition 1.2 because

- $(*)_1$ for $\mathfrak{u} = \mathfrak{u}_{\mathfrak{s}}$ we can prove it (when $\mathfrak{K}$ is categorical in λ, see 5.23, 5.12 below) and it follows by our allowing the use of $=_\tau$ or use $\mathfrak{s}'$ (see 4.30).

2) We just use

- $(*)$ M is saturated ($\in \mathfrak{K}_{\lambda^+}$) when
  - (a) $M = \cup\{M_\alpha : \alpha < \lambda^+\}$
  - (b) $M_\alpha \in K_{\mathfrak{s}}$ is $\le_{\mathfrak{s}}$-increasing continuous
  - (c) if $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_\alpha)$ then for some $\beta \in [\alpha, \lambda^+)$ the non-forking extension $q \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_\beta)$ of p is realized in $M_{\beta+1}$ (or just in some $M_\gamma, \gamma \in (\beta, \lambda^+)$).

See II§4; more fully see the proof of part (5).
3) Let $(M,N,a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ be such that there is no triple $(M',N',a) \in K^{3,\mathrm{uq}}_{\mathfrak{s}}$ which is $<_{\mathrm{bs}}$-above it, exists by clause (B)$'$(b) from Theorem 4.28. Let $\mathbf{I} = \{a\}$, so if $(M,N,\mathbf{I}) \le^1_{\mathfrak{u}} (M',N',\mathbf{I}')$ then ($\mathbf{I}' = \mathbf{I} = \{a\}$ and) $(M',N',a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}} \backslash K^{3,\mathrm{uq}}_{\mathfrak{s}}$ hence there are M'', N_1, N_2 such that $(M',N',\mathbf{I}) \le^1_{\mathfrak{u}} (M'',N_\ell,\mathbf{I})$ and N_1, N_2 are $\le_{\mathfrak{s}}$-incompatible amalgamations of M'', N' over M'. This shows that $(M',N',\mathbf{I})$ has the true weak coding property. As for almost$_2$ every triple $(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{s}}, M_\partial = M_{\lambda^+}$ is saturated, by 2.10(4) and part (5) we get that $\mathfrak{u}$ has the weak coding property.
4) Easy to check by 1.11 or as in (5).
5) We choose $\mathfrak{h}$ such that:

⊠ if $\mathbf{x} = \langle (\bar M^\zeta, \bar{\mathbf{J}}^\zeta, \mathbf{f}^\zeta) : \zeta \le \zeta(*)\rangle$ is $\le_{\mathrm{qt}}$-increasing continuous and obey $\mathfrak{h}$ is $\xi < \zeta(*)$ <u>then</u>
- (α) $(\bar M^\xi, \bar{\mathbf{J}}^\xi, \mathbf{f}^\xi) <^{\mathrm{at}}_{\mathfrak{u}} (\bar M^{\xi+1}, \bar{\mathbf{J}}^{\xi+1}, \mathbf{f}^{\alpha+1})$ and let it be witnessed by $E, \bar{\mathbf{I}}$
- (β) $M^{\xi+1}_{\delta+1}$ is brimmed over M^ξ_δ for a club of $\delta < \lambda^+$
- (γ) if $\zeta \le \xi$ is minimal such that one of the cases occurs, then the demand in the first of the cases below holds:

<u>Case A</u>: There is $a \in M^\xi_\partial$ such that $a/=^{M^\xi}_\tau$ is $\subseteq M^\zeta_\partial$ and $\zeta < \xi$.

<u>Then</u> for some such $a', M^{\xi+1}_\partial \models$ "$a' =_\tau b$" (but $b \notin M^\zeta_\partial$), in fact $b' \in \mathbf{I}_\alpha$ for some $\alpha < \partial$ large enough.

<u>Case B</u>: $\zeta < \xi$, not Case A (for ζ) but for some $\alpha < \partial$ and $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M^\zeta_\alpha)$ for no $\varepsilon \in [\zeta,\xi)$ are there $a \in M^{\varepsilon+1}_\partial$ such that $\mathbf{tp}_{\mathfrak{s}}(a, M^\varepsilon_\beta, M^{\varepsilon+1}_\beta)$ is a non-forking extension of p for every $\beta < \partial$ large enough.

<u>Then</u> for some such (p,α) we have $\mathbf{tp}_{\mathfrak{s}}(b, M^\zeta_\beta, M^{\zeta+1}_\partial)$ is a non-forking extension of p for every $\beta < \partial$ large enough.

<u>Case C</u>: $\zeta < \xi$, Cases A,B fail for ζ and there is a pair (α,p) such that $\alpha < \partial, p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M^\zeta_\alpha)$ such that for no $\varepsilon \in [\zeta,\xi)$ is the set $S := \{\delta < \partial$: there is $i < \mathbf{f}^{\varepsilon+1}(\delta)$ such that $\mathbf{tp}_{\mathfrak{s}}(a_{\mathbf{J}^{\varepsilon+1}_{\delta+i}}, M^{\varepsilon+1}_{\delta+i}, M^{\varepsilon+1}_{\delta+i+1})$ is a non-forking extension of $p\}$ stationary.

Then for some such pair (α, p), the condition above holds for ξ.

Case D: $\zeta = \xi$.
Does not matter.
6) Easy, too. $\square_{4.30}$

* * *

(F) The better versions of the results:

Here we prove the better versions of the results, i.e. without using on "WDmId(λ^+) is λ^{++}-saturated" but relying on later sections.

Of course, the major point is reproving the results of §4(E), i.e. "non-structure for a good λ-frame $\mathfrak{s}$ failing the density of $K^{3,\mathrm{uq}}_{\mathfrak{s}}$", we have to rely on §5-§8.

We also deal with §4(D); here we rely on VI§8, so we get an almost good λ-frame $\mathfrak{s}$ (rather than good λ-frames). But in §5-§8 we deal also with this more general case (and in §7, when we discard a non-structure case, we prove that $\mathfrak{s}$ is really a good λ-frame).

Lastly, we revisit §4(C).

4.32 Theorem. *1) In Theorem 4.28 we can omit the assumption (A)(b).*
2) $\dot{I}(\lambda^{++}, \mathfrak{K}) \geq \mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$ *and moreover* $\dot{I}(\lambda^{++}, K^{\mathfrak{u},\mathfrak{h}}) \geq \mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$ *for* $\mathfrak{u} = \mathfrak{u}^1_{\mathfrak{s}}$ *from Definition 4.29 or* $\mathfrak{u} = \mathfrak{u}^3_{\mathfrak{s}}$ *from Definition 8.3 and any* $\mathfrak{u} - \{0,2\}$*-appropriate function* $\mathfrak{h}$*, when:*

- (A) *(set theoretic assumption),* $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$
- (B) *(model theoretic assumptions),*
  - (a) $\mathfrak{s}$ *is an almost good* λ*-frame*
  - (b) $\mathfrak{s}$ *is categorical in* λ
  - (c) $\mathfrak{s}$ *is not a good* λ*-frame or* $\mathfrak{s}$ *(is a good* λ*-frame which) fail density for* $K^{3,\mathrm{uq}}_{\mathfrak{s}}$.

4.33 Remark. 1) This proves VI.0.2 from VI.8.4, proving the main theorem VI.0.2.

2) We can phrase the theorem also as: if (A),(B)(a),(B)(b) holds and the $\dot{I}(K^{\mathfrak{u},\mathfrak{h}}_{\lambda^{++}}) < \mu_{\text{unif}}(\lambda^{++}, 2^\lambda)$ <u>then</u> $\mathfrak{s}$ is a good λ-frame for which $K^{3,\text{uq}}_{\mathfrak{s}}$ is dense in $(K^{3,\text{bs}}_{\mathfrak{s}}, \leq_{\text{bs}})$ (and so has existence for $K^{3,\text{uq}}_{\mathfrak{s}}$).

Proof. 1) This is a special case of part (2).
2) Toward contradiction assume that the desired conclusion fail.

First, the Hypothesis 5.1 of §5 holds for $\mathfrak{s}$ hence its results. Second, the Hypothesis 6.1 of §6 apply hence its results. So consider conclusion 6.17(2); its assumption "$2^{\lambda^+} < 2^{\lambda^{++}}$" holds by assumption (A) here, and its assumption "$\dot{I}(K^{\mathfrak{s},\mathfrak{h}}_{\lambda^{++}}) < \mu_{\text{unif}}(\lambda^{++}, 2^{\lambda^+})$ for $\mathfrak{u} = \mathfrak{u}^1_{\mathfrak{s}}$ for some $\mathfrak{u} - \{0,2\}$-appropriate $\mathfrak{h}$" holds by our present assumption toward contradiction and its assumption "$\mathfrak{K}_{\mathfrak{s}}$ is categorical" holds by clause (B)(b) of the assumption of 4.32.

Hence the conclusion of 6.17 holds which says that

(∗) $\mathfrak{s}$ has existence for $K^{3,\text{up}}_{\mathfrak{s},\xi}$ for every $\xi \leq \lambda^+$, see Definition 6.4.

Now consider Hypothesis 7.1; now part (1) there ($\mathfrak{s}$ is an almost good λ-frame) holds by the present assumption (B)(a), part (2) there was just proven; part (3) there ($\mathfrak{s}$ is categorical in λ) holds by the present assumption (B)(b), and lastly, part (4) there (disjointness) is proved in 5.23. So Hypothesis 7.1 of §7 holds hence the results of that section up to 7.20 apply.

In particular, $\text{WNF}_{\mathfrak{s}}$ defined in 7.3(1),(2) is well defined and by 7.17(1) is a weak non-forking relation on ${}^4(K_{\mathfrak{s}})$ respecting $\mathfrak{s}$. Also $\mathfrak{s}$ is a good λ-frame by Lemma 7.19(1) so the first possibility in clause (B)(c) of 4.32 does not hold. By inspection all parts of Hypothesis 8.1 of §8 holds hence the results of that section apply.

Now in Claim 8.19, its conclusion fails as this means our assumption toward contradiction and among its assumptions, clause (a), saying "$2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$" holds by clause (A) of 4.32, clause (c) saying "K is categorical in λ" holds by clause (B)(b) of 4.32 and clause (d) saying "$\mathfrak{u} = \mathfrak{u}^3_{\mathfrak{s}}$ from 8.3 has existence for $K^{3,\text{up}}_{\mathfrak{s},\lambda^+}$" was proved above. So clause (b) of 8.19 fails, i.e. $\mathfrak{s}$ fails the non-uniqueness for $\text{WNF}_{\mathfrak{s}}$, but by 8.12(1) this implies that we have uniqueness for WNF.

Lastly, we apply Observation 8.12(2), it has two assumptions, the first "$\mathfrak{s}$ has existence for $K^{3,\mathrm{up}}_{\mathfrak{s},\lambda^+}$", was proved above, and second "$\mathfrak{s}$ has uniqueness for WNF" has just been proved; so the conclusion of 8.12 holds. This means "$\mathfrak{s}$ has existence for $K^{3,\mathrm{uq}}_{\mathfrak{s}}$", so also the second possibility of clause (B)(c) of 4.32 fails; a contradiction. $\square_{4.32}$

4.34 Theorem. *1) In Theorem 4.20 we can omit the assumption (A)(b) at least if K is categorical in λ^+.*
2) $\dot{I}(\lambda^{++}, K^{\mathfrak{u},\mathfrak{h}}) \geq \mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$ <u>when</u>:

- (A) *(set theoretic)* $2^\lambda < 2^{\lambda^+} < 2^{\lambda^+}$
- (B) *(model theoretic) as in 4.20, but $\mathfrak{K}$ categorical in λ^+*
- (C) $\mathfrak{u} = u^4_{\mathfrak{K}_\lambda}$, *see Definition 4.22, $\mathfrak{h}$ is a $\mathfrak{u} - \{0,2\}$-appropriate function.*

Remark. This theorem is funny as VI§6 and in particular VI.6.11 is a shortcut, but we prove this by a detour (using VI§8) so in a sense 4.34 is less natural than 4.20; but no harm done.

Proof. 1) By part (2) and 4.24(4) recalling 4.21(4).
2) Toward contradiction, assume that the desired conclusion fails. By VI.8.1 there is an almost good λ-frame $\mathfrak{s}$ such that $\mathfrak{K}_{\mathfrak{s}} = \mathfrak{K}_\lambda$ and $\mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ is the set of minimal $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$.

Note that categoricity in λ^+ is used in Chapter VI to deduce the stability in λ for minimal types and the set of minimal types in $\mathscr{S}_{\mathfrak{K}}(M)$ being inevitable, but this is assumed in clause (B)(d) of the assumption of 4.20, so natural to conjecture that it is not needed, see Chapter VI.

Now using the meaning of the assumption (B)(e) of Theorem 4.20 is that "$K^{3,\mathrm{uq}}_{\mathfrak{s}}$ is not dense in $(K^{3,\mathrm{bs}}_{\mathfrak{s}}, \leq_{\mathrm{bs}})$" so we can apply Theorem 4.32 to get the desired result.

$\square_{4.34}$

4.35 Theorem. *1) In Theorem 4.16 we can weaken the set theoretic assumption, omitting the extra assumption (a)(β).*
2) $\dot{I}(\aleph_2, K^{\mathfrak{u},\mathfrak{h}}) \ge \mu_{\rm unif}(\aleph_2, 2^{\aleph_1})$ *when:*

(a) *(set theory)* $2^{\aleph_0} < 2^{\aleph_1} < 2^{\aleph_2}$

(b) − (e) *as in 4.20*

(f) $\mathfrak{u} = \mathfrak{u}^3_{\mathfrak{K}_{\aleph_0}}$ *from Definition 4.17 and* $\mathfrak{h}$ *is a* $\mathfrak{u}-\{0,2\}$*-appropriate function.*

4.36 Discussion: 1) This completes a promise from I§5. You may say that once we prove in II§3(B) that $\mathfrak{s} = \mathfrak{s}^1_{\mathfrak{K}}$ is a good $\aleph_0$-frame we do not need to deal with $\mathfrak{K}$ any more, so no need of 4.35. In addition to keeping promises this is only partially true because of the following.
2) First, arriving to $\mathfrak{s}^+$, see III§1, we do not know that $\le_{\mathfrak{s}(+)} = \le_{\mathfrak{K}} \restriction \mathfrak{K}_{\mathfrak{s}(+)}$, because this is proved only if $\mathfrak{s}$ is good$^+$ (see III§1). Now by looking at the definitions (and II.3.4, equality of the various types), we know that $\mathfrak{s}$ being good$^+$ is equivalent to the symmetry property, i.e. every one sided stable amalgamation. We prove that its failure implies non-structure in 4.38, 4.39, 4.40 below.
3) Another point is that even if $\mathfrak{s}$ is weakly successful (i.e. we have existence for $K^{3,{\rm uq}}_{\mathfrak{s}}$), we can define NF = $\mathrm{NF}_{\mathfrak{s}}$ and so we have unique non-forking amalgamation, it is not clear that this is equal to the one/two sided stable amalgamation from Chapter I.
4) Also defining $\mathfrak{s}^{+n}$ as in Chapter II we may hope not to shrink $K^{\mathfrak{s}(+n)}$, i.e. to get all the $(\aleph_0, n)$-properties (as in [Sh 87b]). If we start with $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ as in [Sh 48] this seems straight, in general, this is a priori not clear, hopefully see [Sh:F888].
5) Concerning (2) above, we like to use 3.14 - 3.20 in the proof as in the proof of 6.14. If we have used $\mathfrak{u} = \mathfrak{u}^3_{\mathfrak{K}}$ from Definition 4.17, this fails, e.g. it is not self dual. We can change $(\mathrm{FR}_2, \le_2)$ to make it symmetric but still it will fail "hereditary", so it is natural to use $\mathfrak{u}_2$ defined in 4.38 below, but then we still need $(M_\delta, N_\delta, \mathbf{I}_\delta) \in \mathrm{FR}^1_{\mathfrak{u}_2}$ to ensure N_δ is $(\mathbf{D}(M_\delta), \aleph_0)^*$-homogeneous over M_δ. This can be done by using the game version of the coding property. This is fine but was not our "main road" so rather we use the theorem on $\mathfrak{u}_3$ but use $\mathfrak{u}_5$ to apply §3.

A price of using §3 is having to use fake equality. Also together with symmetry, we deal with lifting free $(\alpha, 0)$-rectangles.
6) To complete the proof of 4.35, by 4.40 it suffices to prove the uniqueness of two-sided stable amalgamation. We use §8 and toward this we define WNF_*, prove that it is a weak non-forking relation of $\mathfrak{K}_{\aleph_0}$ respecting $\mathfrak{s}$, using the "lifting" from §5. Then we can apply §8.
7) A drawback of 4.35 as well as 4.16 and II§3(B) is that we restrict ourselves to a countable $\mathbf{D}$. Now in Chapter I this is justified as it is proved that for some increasing continuous sequence $\langle \mathbf{D}_\alpha : \alpha < \omega_1 \rangle$ with each $\mathbf{D}_\alpha$ countable, $\mathbf{D} = \cup\{\mathbf{D}_\alpha : \alpha < \omega_1\}$, i.e. for every $M \in K_{\aleph_0}$, the sequence $(\mathbf{D}_\alpha(M) : \alpha < \omega_1\rangle$ is an increasing sequence of sets of types with union $\mathbf{D}(M)$. However, from the positive results on every $\mathbf{D}_\alpha$ we can deduce positive results on $\mathbf{D}$. See, hopefully, in [Sh:F888].

4.37 Remark. 1) Assumption (d) of 4.16 gives: usually $(M, N, \mathbf{I}) \in \mathrm{FR}^1_{\mathfrak{u}}$ has non-uniqueness, i.e. when $\mathbf{I} = ({}^{\omega>}N)\backslash({}^{\omega>}M)$. We like to work as in 4.32.
2) So as indirectly there we would like to use 3.24; for this we need the vertical uq-invariant whereas we naturally get failure of the semi uq-invariant coding property. So we would like to quote 3.20 but this requires $\mathfrak{u}$ to be self dual.
3) Hence use also a relative of $\mathfrak{u}$ from 4.41, for it we prove the implication and from this deduce what we need for the old.
4) Our problem is to prove that $\mathfrak{s} = \mathfrak{s}_{\aleph_0}$ is good$^+$, equivalently prove the symmetry property, this is done in Claim 4.40. It is natural to apply 3.13 - 3.24.
5) The proof of 4.35 will come later.

4.38 Definition. In 4.35 we define $\mathfrak{u}_5 = \mathfrak{u}^5_{\mathfrak{K}_{\aleph_0}}$ as follows (ℓ is $1, 2$)

- (a) $\partial = \partial_{\mathfrak{u}} = \aleph_1$
- (b) $\mathfrak{K}_{\mathfrak{u}} = \mathfrak{K}_{\aleph_0}$ more pedantically $\mathfrak{K}_{\mathfrak{u}} = \mathfrak{K}'_{\aleph_0}$
- $(c)_1$ $\mathrm{FR}^{\mathfrak{u}}_1$ is the class of triples $(M, N, \mathbf{I})$ such that $\mathbf{I} \subseteq {}^{\omega>}N \backslash {}^{\omega>}M$
- $(c)_2$ $(M_1, N_1, \mathbf{I}_1) \le_1 (M_1, N_2, \mathbf{I}_2)$ iff both are from $\mathrm{FR}^{\mathfrak{u}}_1, M_1 \le_{\mathfrak{K}} M_2, N_1 \le_{\mathfrak{K}} N_2$ and $\bar{c} \in \mathbf{I} \Rightarrow \mathrm{gtp}(\bar{c}, M_2, N_2)$ is the stationarization of $\mathrm{gtp}(\bar{c}, M_1, N_1)$

(*d*) $\mathrm{FR}_2 = \mathrm{FR}_1$ and $\leq^2_{\mathfrak{u}} = \leq^1_{\mathfrak{u}}$.

Now we have to repeat various things.

4.39 Claim. *1)* $\mathfrak{u}_5$ *is a nice construction framework.*
2) For almost$_2$ *every triples* $(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}_5}$ *the model* $M_\partial = M_{\lambda^+}$ *belongs to* K_{λ^+} *and is saturated.*
3) $\mathfrak{u}_5$ *has fake equality* $=_\tau$ *and is monotonic, see Definition (1.13(1)), and weakly hereditary for the fake equality* $=_\tau$*, see Definition 3.17(5) and interpolative (see Definition 3.21).*

Proof. Should be clear (e.g. part (2) as in 6.2). $\square_{4.39}$

4.40 Claim. $\dot I(\lambda^{++}, \mathfrak{K}) \geq \mu_{\mathrm{unif}}(\aleph_2, 2^{\aleph_0})$ *and moreover* $\dot I(\aleph_2, \mathfrak{K}(\aleph_1 -$ *saturated*$)) \geq \mu_{\mathrm{unif}}(\aleph_2, 2^{\aleph_2})$ *<u>when</u>:*

$\circledast$ $(a)(\alpha), (b), (c), (e)$ *from 4.16 and*
$(d)''(\alpha)$ $\mathfrak{K}$ *fails the symmetry property <u>or</u>*
(β) $\mathfrak{K}$ *fails the lifting property, see Definition 4.41 below.*

4.41 Definition. We define a 4-place relation WNF_* on $K_{\aleph_0}$ as follows:
$\mathrm{WNF}_*(M_0, M_1, M_2, M_3)$ <u>when</u>

- (*a*) $M_\ell \in K_{\aleph_0}$ for $\ell \leq 3$
- (*b*) $M_0 \leq_{\mathfrak{K}} M_\ell \leq_{\mathfrak{K}} M_3$ for $\ell = 1, 2$
- (*c*) for $\ell = 1, 2$ if $\bar a \in {}^{\omega>}(M_\ell)$ then $\mathrm{gtp}(\bar a, M_{3-\ell}, M_3)$ is the stationarization of $\mathrm{gtp}(\bar a, M_0, M_3) = \mathrm{gtp}(\bar a, M_0, M_\ell)$.

4.42 Definition. We say that $(\mathfrak{K}, \mathrm{WNF}_*)$ has the lifting property when WNF_* satisfies clause (g) of Definition 7.18, see the proof of 7.18, i.e. if $\mathrm{WNF}_*(M_0, N_0, M_1, N_1)$ and $\alpha < \lambda^+$ and $\langle M_{0,i} : i \leq \alpha \rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous, $M_{0,0} = M_0$ and $N_0 \leq_{\mathfrak{K}_\lambda}, M_{0,\alpha}$ <u>then</u> we can find a $\leq_{\mathfrak{K}_\lambda}$-increasing continuous sequence $\langle M_{1,i} : i \leq \alpha + 1 \rangle$

such that $M_{1,0} = M_1, N_1 \leq_{\mathfrak{K}} M_{1,\alpha+1}$ and for each $i < \alpha$ we have $\mathrm{WNF}_*(M_{0,i}, M_{0,i+1}, M_{1,i}, M_{1,i+1})$ for $i < \alpha$.

Proof of 4.40. We start as in the proof of 4.16, choosing the good $\aleph_0$-frame $\mathfrak{s} = \mathfrak{s}_{\aleph_0}$ and define $\mathfrak{u} = \mathfrak{u}^3_{\mathfrak{K}}$ as there, (except having the fake inequality which causes no problem), so it is a nice construction framework by 4.18(1) and for almost$_2$ all triples $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ the model $M \in \mathfrak{K}_{\aleph_1}$ is saturated (by 4.18(2)).

Now Theorem 3.24 gives the right conclusion, so to suffice to verify its assumptions. Of course, $\mathfrak{u}$ is as required in Hypothesis 3.1.

Clause (a) there means $2^{\aleph_0} < 2^{\aleph_0} < 2^{\aleph_2}$ (as $\partial_{\mathfrak{u}} = \aleph_1$ and we choose $\theta = \aleph_0$), which holds by clause $(a)(\alpha)$ of the present claim.

Clause (c) there says that for $\{0,2\}$-almost every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ the model $M_\partial \in K_{\aleph_2}$ is $K_{\mathfrak{u}}$-model homogeneous; this holds and can be proved as in 6.2.

We are left with clause (b), i.e. we have to prove that some $(M, N, \mathbf{I}) \in \mathrm{FR}^{\mathfrak{u}}_1$ has the vertical uq-invariant coding property, see Definition 3.10. Choose $(M, N, \mathbf{I}) \in \mathrm{FR}^{\mathfrak{u}}_1$ such that $|\mathbf{I}| > 1$, hence N is $(\mathbf{D}(M), N)^*$-homogeneous and $\mathbf{I} = ({}^{\omega>}N) \backslash ({}^{\omega>}M)$ and we shall prove that it has the vertical uq-invariant coding, so assume

- $(*)$ $\mathbf{d}_0$ is a $\mathfrak{u}$-free $(\alpha_{\mathbf{d}}, 0)$-rectangle satisfying $M \leq_{\mathfrak{s}} M' = M^{\mathbf{d}}_{0,0}$ and $M^{\mathbf{d}}_{\alpha(\mathbf{d}),0} \cap N = M$.

We have to find $\mathbf{d}$ as required in Definition 3.10.

Note that by the choice of $(M, N, \mathbf{I})$, and the assumption "$\mathfrak{K}$ fails the symmetry property" we can find (M_*, N_*) and then $\bar{c}$

$(*)_2$
- (a) $M \leq_{\mathfrak{s}} M_* \leq_{\mathfrak{s}} N_*$ and $N \leq_{\mathfrak{s}} N_*$
- (b) M, N, M_*, N_* is in one-sided stable amalgamation, i.e. if $\bar{b} \in {}^{\omega>}N$ then $\mathrm{gtp}(\bar{b}, M_*, N_*)$ is the stationarization of $\mathrm{gtp}(\bar{b}, M, N)$
- (c) M, M_*, N, N_* is not in one sided stable amalgamation, so
- (c)$^+$ $\bar{c} \in {}^{\omega>}(M_*)$ and $\mathrm{gtp}(\bar{c}, N, N_*)$ is not the stationarization of $\mathrm{gtp}(\bar{c}, M, M_*)$.

We like to apply the semi version, i.e. Definition 3.14 and Claim 3.20. There are technical difficulties so we apply it to $\mathfrak{u}_5$, see 4.38, 4.39 above and in the end increase the models to have the triples in $\mathrm{FR}^1_{\mathfrak{u}}$ and use 3.23 instead of 3.20, so all should be clear.

Alternatively, works only with $\mathfrak{u}_5$ but use the game version of the coding theorem.

$\square_{4.40}$

4.43 Claim. *1) If* ($\mathfrak{K}$,WNF_*) *has lifting, see Definition 4.41, then* WNF_* *is a weak non-forking relationg of* $\mathfrak{K}_{\aleph_0}$ *respecting* $\mathfrak{s}$ *with disjointness (7.18(3)).*
2) WNF_* *is a pseudo non-forking relation of* $\mathfrak{K}_{\aleph_0}$ *respecting* $\mathfrak{s}$ *meaning clauses (a)-(f) with disjointness, see the proof or see Definition 7.18(4),(3).*
3) If $\mathfrak{K}_{\aleph_0}$ *satisfies symmetry then in clause (c) of Definition 4.41, it is enough if it holds for one* ℓ.

Proof. 1) We should check all the clauses of Definition 7.18, so see II.6.1 or the proof of 7.17(1).

Clause (a): WNF_* is a 4-place relation on $\mathfrak{K}_{\aleph_0}$.
[Why? By Definition 4.41, in particular clause (a).]

Clause (b): $\mathrm{WNF}_*(M_0, M_1, M_2, M_3)$ implies $M_0 \leq_{\mathfrak{K}} M_\ell \leq_{\mathfrak{K}} M_3$ for $\ell = 1, 2$ and is preserved by isomorphisms.
[Why? By Definition 4.41, in particular clause (c).]

Clause (c): Monotonicity
[Why? By properties of gtp, see I.5.24(1).]

Clause (d): Symmetry
[Why? Read Definition 4.41.]

Clause (e): Long Transitivity
As in the proof of I.5.29.

Clause (f): Existence
This is proved in I.5.32.

Clause (g): Lifting, see Definition 4.41.
This holds by an assumption.
WNF_* respects $\mathfrak{s}$ and has disjointness.

Clear by the definition (in particular of gtp).
2) The proof included in the proof of part (1).
3) Should be clear. $\square_{4.43}$

Proof of 4.35. Let $\lambda = \aleph_0$ and toward contradiction assume that $\dot{I}(\lambda^{++}, \mathfrak{K}(\lambda^+\text{-saturated})) < \mu_{\text{unif}}(\lambda^{++}, 2^\lambda)$.

As in the proof of 4.16, by II.3.4 $\mathfrak{s} := \mathfrak{s}_{\aleph_0}$ is a good λ-frame categorical in λ. By Theorem 4.32, recalling our assumption toward contradiction, $K^{3,\text{uq}}_{\mathfrak{s}}$ is dense in $K^{3,\text{bs}}_{\mathfrak{s}}$ hence $\mathfrak{s}$ has existence for $K^{3,\text{uq}}_{\mathfrak{s}}$; i.e. is weakly successful, but we shall not use this.

By 4.40 we know that $\mathfrak{K}$ has the symmetry property, hence the two-sided stable amalgamation fails uniqueness. Also by 4.40 we know that it has the lifting property, so by 4.43, 4.40 we know that WNF_* is a weak non-forking relation on $\mathfrak{K}_{\aleph_0}$ which respects $\mathfrak{s}$, so Hypothesis 8.1 holds.

Let $\mathfrak{u}$ be defined as in 8.3 (for our given $\mathfrak{s}$ and WNF_*). Now we try to apply Theorem 8.19. Its conclusion fails by our assumption toward contradiction and clause (a),(b),(c) there holds. So clause (b) there fails so by 8.12(1). So we can conclude that we have uniqueness for WNF_* by 8.12(2) clearly $\mathfrak{s}$ has existence for $K^{3,\text{bs}}_{\mathfrak{s}}$, i.e. is weakly successful.

So $\mathfrak{s}^+$ is a well defined good λ^+-frame, see Chapter III. By II§8, Chapter III and our assumption toward contradiction, we know that $\mathfrak{s}$ is successful. Now if $\mathfrak{s}$ is not good$^+$ then $\mathfrak{K}$ fails the symmetry property hence by 4.40 we get contradiction, so ncessarily $\mathfrak{s}$ is good$^+$ hence we have $\leq_{\mathfrak{s}(+)} = \leq_{\mathfrak{s}} \restriction K_{\mathfrak{s}(+)}$. This proves that the saturated $M \in \mathfrak{K}_{\lambda^+}$ is super limit (see Chapter III also this is I.5.39). $\square_{4.35}$

§5 On almost good λ-frames

Accepting "WDmId(∂) is not ∂^+-saturated" where $\partial = \lambda^+$ we have accomplished in §4 the applications we promised. Otherwise for II§5 we have to prove for a good λ-frame $\mathfrak{s}$ that among the triples $(M, N, a) \in K^{3,\text{bs}}_{\mathfrak{s}}$, the ones with uniqueness are not dense, as otherwise non-structure in λ^{++} follows. Toward this (in this section) we have to do some (positive structure side) work which may be of self interest. Now in the case we get $\mathfrak{s}$ in II§3 starting from [Sh 576] or better from VI§8, but with $\lambda_{\mathfrak{s}} = \lambda$ rather than $\lambda_{\mathfrak{s}} = \lambda^+$, we have to start with an almost good λ-frames $\mathfrak{s}$ rather than with good λ-frames. However, there is a price: for eliminating the non-∂^+-saturation of the weak diamond ideal and for using the "almost λ-good" version, we have to work more.

First, we shall not directly try to prove density of uniqueness triples $(M, N, \mathbf{J})$ but just the density of poor relatives like $K^{3,\text{up}}_{\mathfrak{s}}$.

Second, we have to prove some positive results, particularly in the almost good λ-frame case. This is done here in §5 and more is done in §7 assuming existence for $K^{3,\text{up}}_{\mathfrak{s},\lambda^+}$ justified by the non-structure result in §6 and the complimentary full non-structure result is proved in §8.

5.1 Hypothesis. $\mathfrak{s}$ is an almost good λ-frame (usually categorical in λ) and for transparency $\mathfrak{s}$ has disjointness, see Definitions 5.2, 5.5 below; the disjointness is justified in the Discussion 5.6 and not used in 5.21 - 5.25 which in fact prove it and let $\partial = \lambda^+$.

5.2 Definition. "$\mathfrak{s}$ is an almost good λ-frame" is defined as in II.2.1 except that we weaken (E)(c) to (E)(c)$^-$ and strengthen (D)(d) to (D)(d)$^+$ where (recall $\mathbf{tp}_{\mathfrak{s}} = \mathbf{tp}_{\mathfrak{K}_{\mathfrak{s}}}$):

Ax(E)(c)$^-$: the local character

if $\langle M_i : i \leq \delta + 1\rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous and the set $\{i < \delta : N_i <^*_{\mathfrak{s}} N_{i+1}$, i.e. N_{i+1} is universal over $N_i\}$ is unbounded in δ then for some $a \in M_{\delta+1}$ the type $\mathbf{tp}_{\mathfrak{s}}(a, M_\delta, M_{\delta+1})$ belongs to $\mathscr{S}^{\text{bs}}_{\mathfrak{s}}(M_\delta)$ and does not fork over M_i for some $i < \delta$

Ax$(D)(d)^+$ if $M \in K_{\mathfrak{s}}$ then $\mathscr{S}_{\mathfrak{K}_{\mathfrak{s}}}(M)$ has cardinality $\leq \lambda$ (for good λ-frame this holds by II.4.2).

As in Chapter II

5.3 Definition. 1) $K^{3,\text{bs}}_{\mathfrak{s}}$ is the class of triples (M, N, a) such that $M \leq_{\mathfrak{s}} N$ and $a \in N \backslash M$.
2) $\leq_{\text{bs}} = \leq^{\text{bs}}_{\mathfrak{s}}$ is the following two-place relation (really partial order)

on $K^{3,\text{bs}}_{\mathfrak{s}}$. We let $(M_1, N_1, a_1) \le_{\text{bs}} (M_2, N_2, a_2)$ iff $a_1 = a_2, M_1 \le_{\mathfrak{s}} M_2, N_1 \le_{\mathfrak{s}} N_2$ and $\mathbf{tp}_{\mathfrak{s}}(a_1, N_1, N_2)$ does not fork over M_1.

5.4 Claim. *1) $K^{3,\text{bs}}_{\mathfrak{s}}$ and $\le_{\text{bs}}$ are preserved by isomorphisms.*
2) $\le_{\text{bs}}$ is a partial order on $K^{3,\text{bs}}_{\mathfrak{s}}$.
3) If $\langle (M_\alpha, N_\alpha, a) : \alpha < \delta \rangle$ is $\le_{\text{bs}}$-increasing and $\delta < \lambda^+$ is a limit ordinal and $M_\delta := \cup\{M_\alpha : \alpha < \delta\}, N_\delta := \cup\{N_\alpha : \alpha < \delta\}$ then $\alpha < \delta \Rightarrow (M_\alpha, N_\alpha, a) \le_{\text{bs}} (M_\delta, N_\delta, a) \in K^{3,\text{bs}}_{\mathfrak{s}}$ (using Ax(E)(h)).

Proof. Easy. $\square_{5.4}$

5.5 Definition. We say $\mathfrak{s}$ has disjointness or is disjoint when:

- (a) strengthen Ax(C), i.e. $\mathfrak{K}_{\mathfrak{s}}$ has disjoint amalgamation which means that: if $M_0 \le_{\mathfrak{s}} M_\ell$ for $\ell = 1, 2$ and $M_1 \cap M_2 = M_0$ then for some $M_{\mathfrak{s}} \in K_{\mathfrak{s}}$ we have $M_\ell \le_{\mathfrak{s}} M_{\mathfrak{s}}$ for $\ell = 0, 1, 2$
- (b) strengthen Ax(E)(i) by disjointness: if above we assume in addition that $(M_0, M_\ell, a_\ell) \in K^{3,\text{bs}}_{\mathfrak{s}}$ for $\ell = 1, 2$ then we can add $(M_0, M_\ell, a_\ell) \le_{\text{bs}} (M_{3-\ell}, M_{\mathfrak{s}}, a_\ell)$ for $\ell = 1, 2$.

5.6 Discussion: How "expensive" is the (assumption of) disjoint amalgamation (in Ax(C) and Ax(E)(i))?
1) We can "get it for free" by using $\mathfrak{K}'$ and $\mathfrak{s}'$, see Definition 1.10 and 4.25(2) so we assume it.
2) Alternatively we can prove it assuming categoricity in λ (see 5.23 which relies on 5.22).
3) So usually we shall ignore this point.

5.7 Exercise. There is a good λ-frame without disjoint amalgamation.

[Hint: Let

$\circledast_1$ (a) $\tau = \{F\}, F$ a unary function

(b) ψ the first order sentence
$(\forall x, y)[F(x) \neq x \wedge F(y) \neq y \rightarrow F(x) = F(y)] \wedge$
$((\forall x)[F(F(x)) = F(x)]$

(c) $K = \{M : M$ is a τ-model of $\psi\}$, so $M \in K \Rightarrow |\varphi(M)| \leq 1$ where we let $\varphi(x) = (\exists y)(F(y) = x \wedge y \neq x)$

(d) $M \leq_{\mathfrak{K}} N'$ iff $M \subseteq N$ are from K and $\varphi(M) = \varphi(N) \cap M$.

Now note

$(*)_1$ $\mathfrak{K} := (K, \leq_{\mathfrak{K}})$ is an a.e.c. with $\text{LS}(\mathfrak{K}) = \aleph_0$

$(*)_2$ $\mathfrak{K}$ has amalgamation.

[Why? If $M_0 \leq_{\mathfrak{K}} M_\ell$ for $\ell = 1, 2$, then separate the proof to three cases: the first when $\varphi(M_0) = \varphi(M_1) = \varphi(M_2) = \emptyset$ the second when $|\varphi(M_1)| + |\varphi(M_2)| = 1$ and $\varphi(M_0) = \emptyset$; in the third case $\varphi(M_0) = \varphi(M_1) = \varphi(M_2)$ is a singleton.]

So

$(*)_3$ $\mathfrak{K}_\lambda = (K_\lambda, \leq_{\mathfrak{K}} \restriction K_\lambda)$.

Now we define $\mathfrak{s}$ by letting

$(*)_4$ (a) $\mathfrak{K}_{\mathfrak{s}} = \mathfrak{K}_\lambda$

(b) $K^{3,\text{bs}}_{\mathfrak{s}} := \{(M, N, a) : M \leq_{\mathfrak{K}_\lambda} N, a \in N \backslash M \backslash \varphi(N)\}$

(c) for $M_1 \leq_{\mathfrak{K}} M_2 \leq_{\mathfrak{K}} M_3$ and $a \in M_3$ we say $\mathbf{tp}_{\mathfrak{K}}(a, M_2, M_3)$ does not fork over M_1 iff $a \notin M_2$ & $F^{M_3}(a) \notin M_2 \backslash M_1$.

Lastly

$(*)_5$ $\mathfrak{s}$ is a good λ-frame.

[Why? Check. E.g.

<u>Ax(D)(c)</u>: Density

So assume $M <_{\mathfrak{s}} N$ now if there is $a \in N \backslash M \backslash \varphi(M)$ then a $\mathbf{tp}(a, M, N) \in \mathscr{S}^{\text{bs}}_{\mathfrak{s}}(M)$ so a is as required. Otherwise, as $M \neq M$ necessary $\varphi(N)$ is non-empty and $\subseteq N \backslash M$, let it be $\{b\}$. By the definition of φ there is $a \in N$ such that $F^N(a) = b \wedge a \neq b$ so necessarily $a \notin M$ and is as required.]

Ax(E)(e): Uniqueness

The point is that:

$(*)_6$ if $\varphi(M) = \emptyset$ then $\mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ contains just two types p_1, p_2 such that if $p_\ell = \mathbf{tp}(a, M, N) \Rightarrow$ then $\ell = 1 \Rightarrow F^N(a) = a \in N \backslash M$ and $\ell = 2 \Rightarrow F^N(a) \in N \backslash M \backslash \{a\}$

$(*)_7$ if $\varphi(M) = \{b\}$ then $\mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ contains just two types p_1, p_2 such that p_1 is as above and $p_2 = \mathbf{tp}(a, M, N) \Rightarrow F^N(a) = b$

$(*)_8$ there are $M_0 \leq_{\mathfrak{K}} M_1 = M_2$ such that $\varphi(M_1) \neq \emptyset = \varphi(M_0)$ and let $\varphi(M_\ell) = \{b\}$ for $\ell = 1, 2$ so $b \in M_\ell \backslash M_0$. So we cannot disjointly amalgamate M_1, M_2 over M_0.

[Why? Think.]
So we are done with Example 5.7.]

Recalling II.1.15, II.1.16:

5.8 Claim. *1) For $\kappa = \text{cf}(\kappa) \leq \lambda$*

- (a) *there is a (λ, κ)-brimmed $M \in \mathfrak{K}_{\mathfrak{s}}$, in fact (λ, κ)-brimmed over M_0 for any pregiven $M_0 \in \mathfrak{K}_{\mathfrak{s}}$*
- (b) *M is unique up to isomorphism over M_0 (but we fix κ)*
- (c) *if $M \in \mathfrak{K}_{\mathfrak{s}}$ is (λ, κ)-brimmed over M_0 then it is $\leq_{\mathfrak{s}}$-universal over M_0.*

2) So the superlimit $M \in \mathfrak{K}_{\mathfrak{s}}$ is (λ, κ)-brimmed for every $\kappa \leq \text{cf}(\kappa) \leq \lambda$ hence is brimmed.
3) If $\kappa = \text{cf}(\kappa) \leq \lambda$ and $M_1 \leq_{\mathfrak{s}} M_2$ are both (λ, κ)-brimmed and $\Gamma \subseteq \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_2)$ has cardinality $< \kappa$ and every $p \in \Gamma$ does not fork over M_1 then there is an isomorphism f from M_2 onto M_1 such that $p \in \Gamma \Rightarrow f(p) = p \restriction M_1$.

Proof. 1) By Definition II.1.15 and Claim II.1.16 because $\mathfrak{K}_\lambda$ has amalgamation, the JEP recalling II.2.1 and has no $<_{\mathfrak{K}_\lambda}$-maximal member (having a superlimit model).
2) By the definition of being brimmed and "superlimit in $\mathfrak{K}_{\mathfrak{s}}$" which exists by "$\mathfrak{s}$ is almost good λ-frame".
3) Exactly as in the proof of III.1.21. $\square_{5.8}$

5.9 Remark. 1) It seems that there is no great harm in weakening (E)(h) to (E)(h)$^-$ as in (E)(c)$^-$, but also no urgent need, where:.

Ax(E)(h)$^-$: assume $\langle M_i : i \le \delta\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous and $\delta = \sup\{i : M_{i+1}$ is $\le_{\mathfrak{s}}$-universal over $M_i\}$. If $p \in \mathscr{S}_{\mathfrak{K}_{\mathfrak{s}}}(M_\delta)$ and $i < \delta \Rightarrow p \restriction M_i \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_i)$ <u>then</u> $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_\delta)$.

2) That is, if we weaken Ax(E)(h) then we are drawn to further problems. After defining $\mathfrak{u}$, does $\le_\ell$-increasing sequence $\langle (M_i, N_i, \mathbf{J}_i) : i < \delta\rangle$ has the union as a $\le_\ell$-upper bound? If M_{i+1} is universal over M_i for $i < \delta$ this is O.K., but using triangles in the limit we have a problem; see part (4) below.

3) Why "no urgent need"? The case which draws us to consider Ax(E)(c)$^-$ is VI§8, i.e. by the $\mathfrak{s}$ derived there satisfies Ax(E)(h). So we may deal with it elsewhere, [Sh:F841].

4) When we deal with $\mathfrak{u}$ derived from such $\mathfrak{s}$, i.e. as in part (1) we may demand:

- (A) First, dealing with $\mathfrak{u}$-free rectangles and triangles we add
  - (a) $\mathbf{I}^{\mathbf{d}}_{i,j} = \emptyset$ when j is a limit ordinal
  - (b) $\mathbf{J}^{\mathbf{d}}_{i,j} = \emptyset$ when i is a limit ordinal
  - (c) it is everywhere universal (each $M_{i+1,j+j}$ is $\le_{\mathfrak{s}}$-universal over $M^{\mathbf{d}}_{i+1,j} \cup M^{\mathbf{d}}_{i,j+1}$ (or at least each)

- (B) similarly with $K^{\rm qt}_{\mathfrak{s}}$, i.e. defining $(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{s}}$ we add the demands
  - (a) $\mathbf{J}_\delta = \emptyset$ for limit δ
  - (b) if $\delta \in S \cap E$, then $\mathbf{d}$, the $(\mathfrak{u}_{\mathfrak{s}})$-free $(\mathbf{f}(\delta), 0)$-rectangle $(\langle M_{\delta+i} : i \le \mathbf{f}(\delta)\rangle, \langle \mathbf{J}_{\delta+i} : i < \mathbf{f}(\delta)\rangle)$ is strongly full (defined as below) and so for any $\theta \le \lambda$, if $\lambda | j$ and $i < j \le \mathbf{f}(\delta)$ and $\mathrm{cf}(\delta) = \theta$ then $M_{\delta,j}$ is (λ,θ)-brimmed over $M_{\delta+i}$.

- (C) similarly for $\le^{\rm at}_{\mathfrak{u}_{\mathfrak{s}}}, \le^{\rm qr}_{\mathfrak{u}_{\mathfrak{s}}}$.

5.10 Definition. 1) We define $\mathfrak{u} = \mathfrak{u}_{\mathfrak{s}} = \mathfrak{u}^1_{\mathfrak{s}}$ as in Definition 4.29, it is denoted by $\mathfrak{u}$ for this section so may be omitted and we may write

$\mathfrak{s}$-free instead of $\mathfrak{u}_{\mathfrak{s}}$-free.
2) We say $(M, N, \mathbf{I}) \in \mathrm{FR}_\ell$ realizes p <u>when</u> $\mathbf{tp}_{\mathfrak{s}}(a_{\mathbf{I}}, M, N) = p$, recalling $\mathbf{I} = \{a_{\mathbf{I}}\}$.

5.11 Claim. *1) $\mathfrak{u}_{\mathfrak{s}}$ is a nice construction framework which is self-dual.*
2) Also $\mathfrak{u}_{\mathfrak{s}}$ is monotonic and hereditary and interpolative.

5.12 Remark. 1) Here we use "$\mathfrak{s}$ has disjointness" proved in 5.23.
2) Even without 5.23, if $\mathfrak{s} = \mathfrak{s}'_1$ for some almost good λ-frame $\mathfrak{s}_1$ then $\mathfrak{s}$ has disjointness.
3) Mostly it does not matter if we use $\mathfrak{u}^1_{\mathfrak{s}}, \mathfrak{s}'$ from 4.25 (see 6.18) but in proving 6.13(1), the use of $\mathfrak{u}^1_{\mathfrak{s}'}$ is preferable; alternatively in defining nice construction framework we waive the disjointness, which is a cumbersome but not serious change.

Proof. As in 4.30 except disjointness which holds by Hypothesis 5.2 and is justified by 5.6 above, (or see 5.23 below). $\square_{5.11}$

5.13 Remark. Because we assume on $\mathfrak{s}$ only that it is an almost good λ-frame we have to be more careful as (E)(c) may fail, in particular in proving brimmness in triangles of the right kind. I.e. for a $\mathfrak{u}$-free $(\bar{\alpha}, \beta)$- triangle $\mathbf{d}$, we need that in the "vertical sequence", $\langle M^{\mathbf{d}}_{i,\beta} : i \le \alpha_\beta\rangle$ the highest model $M^{\mathbf{d}}_{\alpha_\beta,\beta}$ is brimmed over the lowest $M^{\mathbf{d}}_{0,\beta}$. This motivates the following.

5.14 Definition. 1) We say $\bar{M}$ is a (Γ, δ)-correct sequence <u>when</u>: $\Gamma \subseteq \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_\delta)$, the sequence $\bar{M} = \langle M_\alpha : \alpha \le \alpha(*)\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous, $\delta \le \alpha(*)$ and: <u>if</u> $N \in K_{\mathfrak{s}}$ satisfies $M_\delta <_{\mathfrak{s}} N$ <u>then</u> for some $c \in N \backslash M_\delta$ and $\alpha < \delta$ the type $\mathbf{tp}_{\mathfrak{s}}(c, M_\delta, N)$ does not fork over M_α and belongs to Γ.
2) We omit δ when this holds for every limit $\delta \le \alpha(*)$.
3) We say Γ is M-inevitable when $\Gamma \subseteq \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ and: if $M <_{\mathfrak{s}} N$ <u>then</u> some $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M) \cap \Gamma$ is realized in N.

4) Using a function $\mathscr{S}^*$ instead of Γ we mean we use $\Gamma = \mathscr{S}^*(M_\delta)$.
5) We may omit Γ (and write δ-correct) above <u>when</u> Γ is $\mathscr{S}^{\rm bs}_{\mathfrak{s}}$.
6) For $\bar M = \langle M_\alpha : \alpha \le \alpha(*)\rangle$ let $\text{correct}_\Gamma(\bar M) = \{\delta \le \alpha(*) : \bar M$ is (δ,Γ)-correct so δ is a limit ordinal$\}$ and we may omit Γ if $\Gamma = \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_{\alpha(*)})$.
7) If $\mathbf{d}$ is a $\mathfrak{u}_{\mathfrak{s}}$-free $(\bar\alpha,\beta)$-triangle let $\Gamma_{\mathbf{d}} = \{p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_{\alpha_\beta,\beta}) : p$ does not fork over $M_{i,j}$ for some $j < \beta, i < \alpha_j\}$.

5.15 Definition. 1) We say that $\mathbf{d}$ is a brimmed (or universal) $\mathfrak{u}_{\mathfrak{s}}$-free or $\mathfrak{s}$-free triangle <u>when</u>:

- (a) $\mathbf{d}$ is a $\mathfrak{u}_{\mathfrak{s}}$-free triangle
- (b) if $i < \alpha_j(\mathbf{d})$ and $j < \beta(\mathbf{d})$ <u>then</u> $M^{\mathbf{d}}_{i+1,j+1}$ is brimmed (or universal) over $M^{\mathbf{d}}_{i+1,j}$.

2) We say strictly brimmed (universal) <u>when</u> also $M^{\mathbf{d}}_{i+1,j+1}$ is brimmed (universal) over $M^{\mathbf{d}}_{i+1,j} \cup M^{\mathbf{d}}_{i,j+1}$ when $j < \beta$, $i < \alpha_j(\mathbf{d})$.
2A) We say that $\mathbf{d}$ is a weakly brimmed (or weakly universal) $\mathfrak{u}_{\mathfrak{s}}$-free or $\mathfrak{s}$-free triangle <u>when</u>:

- (a) $\mathbf{d}$ is a $\mathfrak{u}_{\mathfrak{s}}$-free triangle
- (b) if $j_1 < \beta_{\mathbf{d}}, i_1 < \alpha_j(\mathbf{d})$ then we can find a pair (i_2, j_2) such that $j_1 \le j_2 < \beta_{\mathbf{d}}, i_1 \le i_2 < \alpha_j(\mathbf{d})$ and $M^{\mathbf{d}}_{i_2+1,j_2+1}$ is brimmed (or is $\le_{\mathfrak{s}}$-universal) over $M^{\mathbf{d}}_{i_2,j_2}$ or just over $M^{\mathbf{d}}_{i_1,j_1}$.

2B) We say that $\mathbf{d}$ is a weakly brimmed (weakly universal) $\mathfrak{u}_{\mathfrak{s}}$-free rectangle <u>when</u> it and its dual (see Definition 1.7(3)) are weakly brimmed $\mathfrak{u}_{\mathfrak{s}}$-free triangles. Similarly for brimmed, strictly brimmed, universal, strictly universal.
3) We say that a $\mathfrak{u}_{\mathfrak{s}}$-free triangle $\mathbf{d}$ is full <u>when</u>:

- (a) $\lambda_{\mathfrak{s}}$ divides $\alpha_{\beta(\mathbf{d})}(\mathbf{d})$ and $\beta(\mathbf{d})$ is a limit ordinal and $\bar\alpha$ is continuous or just $\alpha_{\beta(\mathbf{d})}(\mathbf{d}) = \cup\{\alpha_j(\mathbf{d}) : j < \beta\}$
- (b) if $i < \alpha_j(\mathbf{d}), j < \beta := \beta(\mathbf{d})$ and $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_{i,j})$ <u>then</u>:
  - $(*)$ the following subset $S^{\mathbf{d}}_p$ of $\alpha_\beta(\mathbf{d})$ has order type $\ge \lambda_{\mathfrak{s}}$ where $S^{\mathbf{d}}_p := \{i_1 < \alpha_\beta(\mathbf{d})$: for some $j_1 \in (j,\beta)$ we have $i \le i_1 < \alpha_{j_1}(\mathbf{d})$ and for some $c \in \mathbf{J}_{i_1,j_1}$ the type $\mathbf{tp}_{\mathfrak{s}}(c, M_{i_1,j_1}, M_{i_1+1,j_1})$ is a non-forking extension of $p\}$.

3A) We say that the $\mathfrak{u}_{\mathfrak{s}}$-free triangle is strongly full <u>when</u>:

(a) as in part (3)

(b) if $i < \alpha_j(\mathbf{d})$ and $j < \beta(\mathbf{d})$ and $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M^{\mathbf{d}}_{i,j})$ then for λ ordinals $i_1 \in [i, i+\lambda)$ for some $j_1 \in [j_2, \beta)$ the type $\mathbf{tp}_{\mathfrak{s}}(b^{\mathbf{d}}_{i_1,j_1}, M^{\mathbf{d}}_{i_1,j_1} M^{\mathbf{d}}_{i_1+1,j_1})$ is a non-forking extension of p and $i_1 < \alpha_{j_1}(\mathbf{d})$, of course.

3B) We say a $\mathfrak{u}_{\mathfrak{s}}$-free rectangle $\mathbf{d}$ is full [strongly full] <u>when</u> both $\mathbf{d}$ and its dual are full [strongly full] $\mathfrak{u}_{\mathfrak{s}}$-free triangles.

5.16 Observation. 1) If $\bar{M} = \langle M_\alpha : \alpha \leq \delta\rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous, δ a limit ordinal and $\delta = \sup\{\alpha < \delta : M_{\alpha+1}$ or just M_β for some $\beta \in (\alpha, \delta)$, is $\leq_{\mathfrak{s}}$-universal over $M_\alpha\}$ <u>then</u> $\delta \in$ correct$(\bar{M})$.
2) If $\text{cf}(\delta_\ell) = \kappa$ and $\bar{M}^\ell = \langle M^\ell_\alpha : \alpha \leq \delta_\ell\rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous and $h_\ell : \kappa \rightarrow \delta_\ell$ is increasing with $\delta_\ell = \sup(\text{Rang}(h_\ell))$ for $\ell = 0, 1$ and $\varepsilon < \kappa \Rightarrow M^1_{h_1(\varepsilon)} = M^2_{h_2(\varepsilon)}$ <u>then</u> $\bar{M}^1$ is δ_1-correct iff $\bar{M}^2$ is δ_2-correct.
3) Instead of the h_1, h_2 is part (2) it suffices that $(\forall \alpha < \delta_\ell)(\exists \beta < \delta_{3-\ell})(M^\ell_\alpha \leq_{\mathfrak{s}} M^{3-\ell}_\beta)$ for $\ell = 1, 2$. Also instead of "δ_ℓ-correct" we can use (Γ, δ_ℓ)-correct.
4) If $\bar{M} = \langle M_\alpha : \alpha \leq \delta\rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous and $\Gamma_{\bar{M}} = \{p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_\delta) : p$ does not fork over M_α for some $\alpha < \delta\}$ <u>then</u> $\bar{M}$ is δ-correct iff $\bar{M}$ is (δ, Γ)-correct.

Proof. 1) By Ax(E)(c)$^-$ and the definition of correct$(\bar{M})$.
2),3),4) Read the definitions. $\square_{5.16}$

5.17 Observation. Assume $\mathbf{d}^{\rm ver}$ is a $\mathfrak{u}_{\mathfrak{s}}$-free $(\alpha, 0)$-rectangle, $\mathbf{d}_{\rm hor}$ is a $\mathfrak{u}_{\mathfrak{s}}$-free $(0, \beta)$-rectangle and $M^{\mathbf{d}^{\rm ver}}_{0,0} = M^{\mathbf{d}_{\rm hor}}_{0,0}$. <u>Then</u> there is a pair $(\mathbf{d}, f)$ such that:

(a) $\mathbf{d}$ is a strictly brimmed $\mathfrak{u}$-free (α, β)-rectangle

(b) $\mathbf{d} \restriction (0, \beta) = \mathbf{d}_{\rm hor}$

(c) f is an isomorphism from $\mathbf{d}^{\rm ver}$ onto $\mathbf{d} \restriction (\alpha, 0)$ over $M^{\mathbf{d}}_{0,0}$

(d) if α is divisible by λ and $S^{\rm ver} = \{\alpha' < \alpha : \mathbf{J}^{\mathbf{d}^{\rm ver}}_{\alpha'} = \emptyset\}$ is an unbounded subset of α of order-type divisible by λ (so in particular α is a limit ordinal) <u>then</u> we can add $\mathbf{d}$ as a triangle is full; we can add strongly full if $\alpha' < \alpha \Rightarrow \lambda = |S^{\rm ver} \cap [\alpha', \alpha' + \lambda]|$

(e) if $S_{\rm hor} := \{\beta' < \beta : \mathbf{I}^{\rm hor}_{\beta} = 0\}$ is an unbounded subset of β of order-type divisible by λ <u>then</u> we can add "dual($\mathbf{d}$) as a triangle is full"; and we can add strongly full if $\beta' < \beta \Rightarrow \lambda = |S_{\rm hor} \cap [\beta', \beta' + \lambda]|$.

Proof. Easy. $\square_{5.17}$

<u>5.18 Exercise</u>: Show the obvious implication concerning the notions from Definition 5.15. Let $\mathbf{d}$ be a $\mathfrak{u}$-free $(\bar{\alpha}, \beta)$-triangle and $\mathbf{e}$ be a $\mathfrak{u}$-free (α, β)-rectangle.
1) $\mathbf{d}$ strictly brimmed implies $\mathbf{d}$ is brimmed which implies $\mathbf{d}$ is weakly brimmed.
2) Like (1) replacing brimmed by universal.
3) If $\mathbf{d}$ is strictly brimmed/brimmed/weakly brimmed <u>then</u> $\mathbf{d}$ is strictly universal/universal/weakly universal.
4) If $\mathbf{d}$ is strongly full <u>then</u> $\mathbf{d}$ is full.
5) Similarly for the rectangle $\mathbf{e}$.
6) If $\mathbf{e}$ is strictly brimmed/brimmed/weakly brimmed <u>then</u> so is dual($\mathbf{e}$).
7) If $\mathbf{e}$ is strictly universal/universal/weakly universal <u>then</u> so is dual($\mathbf{e}$).

5.19 The Correctness Claim. *1) Assume $\delta < \lambda^+$ is a limit ordinal, $\bar{M}^\ell = \langle M^\ell_\alpha : \alpha \le \delta \rangle$ is $\le_{\mathfrak{s}}$-increasing continuous sequence for $\ell = 1, 2$ and $\alpha < \delta \Rightarrow M^1_\alpha \le_{\mathfrak{s}} M^2_\alpha$ and $M^1_\delta = M^2_\delta$. If $\bar{M}^1$ is δ-correct <u>then</u> $\bar{M}^2$ is δ-correct.*
2) M_δ is $(\lambda, \mathrm{cf}(\delta))$-brimmed over M_0; moreover over M_i for any $i < \delta$ <u>when</u>:

(a) *δ is a limit ordinal divisible by λ (the divisibility follows by clause (d))*

(b) $\bar{M} = \langle M_\alpha : \alpha \le \delta \rangle$ *is* $\le_{\mathfrak{s}}$*-increasing continuous*

(c) $\bar{M}$ *is* (δ, Γ)*-correct, so* $\Gamma \subseteq \{p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_\delta) : p$ *does not fork over* M_α *for some* $\alpha < \delta\}$*, if* Γ *is equal this means* $\delta \in \mathrm{correct}(\bar{M})$*, recalling Definition 5.14(1),(6)*

(d) *if* $\alpha < \delta$ *and* $p \in \{q \restriction M_\alpha : q \in \Gamma\} \subseteq \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_\alpha)$ *<u>then</u>: for* $\ge \lambda$ *ordinals* $\beta \in (\alpha, \delta)$ *<u>there</u> is* $c \in M_{\beta+1}$ *such that* $\mathbf{tp}_{\mathfrak{s}}(c, M_\beta, M_{\beta+1})$ *is a non-forking extension of* p*.*

2A) M_δ *is* $(\lambda, \mathrm{cf}(\delta))$*-brimmed over* M_0 *<u>when</u> clauses (a),(b) of part (2) holds and* $\delta = \sup(S)$ *where* $S = \{\delta' : \delta' < \delta$ *and* $\bar{M} \restriction (\delta'+1)$ *satisfies clauses (a)-(d) from part (2)*$\}$*.*

3) Assume $\mathbf{d}$ *is a* $\mathfrak{u}_{\mathfrak{s}}$*-free* $(\bar{\alpha}, \beta)$*-triangle,* β *is a limit ordinal,* $\bar{\alpha}$ *is continuous (or just* $\alpha_\beta = \alpha_\beta(\mathbf{d}) = \cup\{\alpha_{j+1} : j < \beta\}$*) and* $\bar{\alpha} \restriction \beta$ *is not eventually constant,*

(a) *if* $\mathbf{d}$ *is brimmed or just weakly universal <u>then</u>* $\langle M^{\mathbf{d}}_{\alpha,\beta} : \alpha \le \alpha_\beta \rangle$ *is* α_β*-correct; moreover is correct for* $(\alpha_\beta, \Gamma_{\mathbf{d}})$ *recalling* $\Gamma_{\mathbf{d}} = \{p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M^{\mathbf{d}}_{\alpha_\beta,\beta}) : p$ *does not fork over* $M^{\mathbf{d}}_{i,j}$ *for some* $j < \beta, i < \alpha_j\}$

(b) *if* $\mathbf{d}$ *is weakly universal and full <u>then</u>* $M_{\alpha_\beta,\beta}$ *is brimmed over* $M_{i,\beta}$ *for every* $i < \alpha_\beta$*.*

3A) Assume $\mathbf{d}$ *is an* $\mathfrak{u}_{\mathfrak{s}}$*-free* (α, β)*-rectangle*

(a) *if* $\mathbf{d}$ *is brimmed or just weakly universal (see Definition 5.15(2A)) and* $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta) \ge \aleph_0$ *<u>then</u>* $\langle M^{\mathbf{d}}_{i,\beta} : i \le \alpha \rangle$ *is* α*-correct and even* $(\alpha, \Gamma_{\mathbf{d}})$*-correct*

(b) *if in clause (a),* $\mathbf{d}$ *is full <u>then</u>* $M^{\mathbf{d}}_{\alpha,\beta}$ *is* $(\lambda, \mathrm{cf}(\alpha))$*-brimmed over* $M_{i,\beta}$ *for* $i < \alpha$

(c) *if* $\mathbf{d}$ *is strictly brimmed (or just strictly universal, see Definition 5.15(2)) and strongly full, (see Definition 5.15(3A),(3B)) and* $\lambda^2\omega$ *divides* α *(but no requirement on the cofinalities) <u>then</u>* $M^{\mathbf{d}}_{\alpha,\beta}$ *is* $(\lambda, \mathrm{cf}(\alpha))$*-brimmed over* $M^{\mathbf{d}}_{i,\beta}$ *for every* $i < \alpha$*.*

4) For $M \in K_{\mathfrak{s}}$ *there is* $N \in K_{\mathfrak{s}}$ *which is brimmed over* M *and is unique up to isomorphism over* M *(so in other words, if* M_ℓ *is* (λ, κ_ℓ)*-brimmed over* M *for* $\ell = 1, 2$ *<u>then</u>* N_1, N_2 *are isomorphic over* M*).*

Proof. 1) Assume $M^2_\delta <_{\mathfrak{s}} N$, hence $M^1_\delta <_{\mathfrak{s}} N$ so as $\delta \in$ correct$(\bar M^1)$ necessarily for some pair (p,α) we have: $\alpha < \delta$ and $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M^1_\delta)$ is realized in N and does not fork over M^1_α. As $M^1_\alpha \le_{\mathfrak{s}} M^2_\alpha \le_{\mathfrak{s}} M^2_\delta = M^1_\delta$ and montonicity of non-forking it follows that "$p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M^2_\delta)$ does not fork over M^2_α", and, of course, p is realized in N. So (p,α) are as required in the definition of "$\delta \in$ correct$(\bar M^2)$".
2) Similar to II§4 but we give a full self-contained proof. The "moreover" can be proved by renaming.

Let $\langle \mathscr{U}_\alpha : \alpha < \delta\rangle$ be an increasing continuous sequence of subsets of λ such that $|\mathscr{U}_0| = \lambda, |\mathscr{U}_{\alpha+1} \backslash \mathscr{U}_\alpha| = \lambda$. We choose a triple $(\bar{\mathbf{a}}^\alpha, N_\alpha, f_\alpha)$ by induction on $\alpha \le \delta$ such that:

⊕ (a) $N_\alpha \in \mathfrak{K}_{\mathfrak{s}}$ and $N_0 = M_0$
(b) f_α is a $\le_{\mathfrak{s}}$-embedding of M_α into N_α
(c) $\langle N_\beta : \beta \le \alpha\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous
(d) $\langle f_\beta : \beta \le \alpha\rangle$ is $\subseteq$-increasing continuous and $f_0 = {\rm id}_{M_0}$
(e) $\bar{\mathbf{a}}^\alpha = \langle a_i : i \in \mathscr{U}_\alpha\rangle$, so $\beta < \alpha \Rightarrow \bar{\mathbf{a}}^\beta = \bar{\mathbf{a}}^\alpha \restriction \mathscr{U}_\beta$
(f) $\bar{\mathbf{a}}^\alpha$ lists the elements of N_α each appearing λ times
(g) if $\alpha = \beta + 1$ then N_α is $\le_{\mathfrak{s}}$-universal over N_β
(h) if $\alpha = \beta + 1$ and $\mathscr{W}_\beta := \{i \in \mathscr{U}_\beta$: for some $c \in M_{\alpha+1} \backslash M_\alpha$ we have $f_\beta(\mathbf{tp}_{\mathfrak{s}}(c, M_\beta, M_\alpha)) = \mathbf{tp}_{\mathfrak{s}}(a_i, f_\beta(M_\beta), N_\beta)\}$ is not empty and $i_\beta :=$ min$(\mathscr{W}_\beta)$ then $a_{i_\beta} \in$ Rang(f_α).

There is no problem to carry the definition; and by clauses (c),(g) of ⊕ obviously

⊙ N_δ is $(\lambda, {\rm cf}(\delta))$-brimmed over N_0 (hence over $f_0(M_0)$).

Also by renaming without loss of generality $f_\alpha = {\rm id}_{M_\alpha}$ for $\alpha \le \delta$ hence $M_\delta \le_{\mathfrak{s}} N_\delta$.

Now if $M_\delta = N_\delta$ then by ⊙ we are done. Otherwise by clause (c) of the assumption $\bar M$ is (δ, Γ)-correct hence by Definition 5.14(1), for some $c \in N_\delta \backslash f_\delta(M_\delta)$ and $\alpha_0 < \delta$ the type $\mathbf{tp}_{\mathfrak{s}}(c, M_\delta, N_\delta)$ belongs to $\Gamma \subseteq \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_\delta)$ and does not fork over M_{α_0}. As $\langle N_\beta : \beta \le \delta\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous, for some $\alpha_1 < \delta$ we have $c \in N_{\alpha_1}$, hence for some $i_* \in \mathscr{U}_{\alpha_1}$ we have $c = a_{i_*}$. So $\alpha_2 :=$ max$\{\alpha_0, \alpha_1\} < \delta$ and by clause (d) of the assumption the set $\mathscr{W} := \{\alpha < \delta : \alpha \ge$

α_2 and for some $c' \in M_{\alpha+1}$ the type $\mathbf{tp}_{\mathfrak{s}}(c', M_\alpha, M_{\alpha+1})$ is a non-forking extension of $\mathbf{tp}_{\mathfrak{s}}(c, M_{\alpha_0}, N_\delta)\}$ has at least λ members and is $\subseteq [\alpha_2, \delta)$ and by the monotonicity and uniqueness properties of non-forking we have $\mathscr{W} = \{\alpha < \delta : \alpha \geq \alpha_2$ and some $c' \in M_{\alpha+1}$ realizes $\mathbf{tp}_{\mathfrak{s}}(c, M_\alpha, N_\delta)$ in $M_{\alpha+1}\}$. Now for every $\alpha \in \mathscr{W} \subseteq [\alpha_2, \delta)$ the set $\mathscr{W}_\alpha$ defined in clause (h) of $\oplus$ above is not empty, in fact, $i_* \in \mathscr{W}_\alpha$ hence $\beta \in \mathscr{W} \subseteq [\alpha_2, \delta) \Rightarrow i_\beta = \min(\mathscr{W}_\beta) \leq i_*$ but $|\mathscr{W}| = \lambda$, so by cardinality consideration for some $\beta_1 < \beta_2$ from $\mathscr{W}$ we have $i_{\beta_1} = i_{\beta_2}$ but $a_{i_{\beta_1}} \in \mathrm{Rang}(f_{\beta_1+1}) \subseteq \mathrm{Rang}(f_{\beta_2})$ whereas $a_{i_{\beta_2}} \notin \mathrm{Rang}(f_{\beta_2})$, contradiction.

2A) If $\alpha < \beta \in S$ then by part (2) applied to the sequence $\langle M_{\alpha+\gamma} : \gamma \leq \beta - \alpha \rangle$, the model M_β is $(\lambda, \mathrm{cf}(\beta))$-brimmed over M_α hence M_β is $\leq_{\mathfrak{s}}$-universal over M_α by 5.8(1)(c). Choose an increasing continuous sequence $\langle \alpha_\varepsilon : \varepsilon < \mathrm{cf}(\delta) \rangle$ with limit δ such that $\varepsilon < \mathrm{cf}(\delta) \Rightarrow \alpha_{\varepsilon+1} \in S$ and $\alpha_0 = 0$, so clearly $\langle M_{\alpha_\varepsilon} : \varepsilon < \mathrm{cf}(\delta) \rangle$ exemplifies that M_δ is $(\lambda, \mathrm{cf}(\delta))$-brimmed over M_0.

3) Clause (a):

Note that necessarily $\mathrm{cf}(\alpha_\beta) = \mathrm{cf}(\beta)$ as $\bar{\alpha} = \langle \alpha_j : j \leq \beta \rangle$ is non-decreasing and $\bar{\alpha} \restriction \beta$ is not eventually constant.

Let $\langle \beta_\varepsilon : \varepsilon < \mathrm{cf}(\alpha_\beta) \rangle, \langle \gamma_\varepsilon : \varepsilon < \mathrm{cf}(\alpha_\beta) \rangle$ be increasing continuous sequences of ordinals with limit β, α_β respectively such that $\gamma_\varepsilon \leq \alpha_{\beta_\varepsilon}$ for every $\varepsilon < \mathrm{cf}(\alpha_\beta)$.

We now choose a pair $(i_\varepsilon, j_\varepsilon)$ by induction on $\varepsilon < \mathrm{cf}(\alpha_\beta)$ such that:

$\odot$ (a) $j_\varepsilon < \beta$ is increasing continuous with ε

(b) $i_\varepsilon \leq \alpha_{j_\varepsilon}$ is increasing continuous with ε

(c) if $\varepsilon = \zeta + 1$ then $M^{\mathbf{d}}_{i_\varepsilon, j_\varepsilon}$ is $\leq_{\mathfrak{s}}$-universal over $M^{\mathbf{d}}_{i_\zeta, j_\zeta}$

(d) if $\varepsilon = \zeta + 1$ then $i_\varepsilon > \gamma_\varepsilon, j_\varepsilon > \beta_\varepsilon$.

There is no problem to carry the definition as $\mathbf{d}$ is weakly universal, see Definition 5.15(2A) and $\bar{\alpha}$ not eventually constant. Now the sequence $\langle i_\varepsilon : \varepsilon < \mathrm{cf}(\alpha_\beta) \rangle$ is increasing with limit α_β (by clause $\odot$(d)), and $\langle j_\varepsilon : \varepsilon < \beta \rangle$ is an increasing continuous sequence and has limit β (as $\langle \alpha_j : j < \beta \rangle$ is not eventually constant), hence

$(*)$ $\langle M^{\mathbf{d}}_{i_\varepsilon, j_\varepsilon} : \varepsilon < \mathrm{cf}(\beta) \rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous with union $M^{\mathbf{d}}_{\alpha_\beta, \beta}$.

Let $(i_{\mathrm{cf}(\beta)}, j_{\mathrm{cf}(\beta)}) := (\alpha_\beta, \beta)$.

So by $\odot(c) + (*)$ it follows that $\langle M^{\mathbf{d}}_{i_\varepsilon, j_\varepsilon} : \varepsilon \le \mathrm{cf}(\beta)\rangle$ satisfies the assumptions of claim 5.16(1), hence its conclusion, i.e. the sequence $(\langle M^{\mathbf{d}}_{i_\varepsilon, j_\varepsilon} : \varepsilon \le \mathrm{cf}(\beta)\rangle$ is cf(β)-correct.

We shall apply part (1) of the present claim 5.19. Now the pair $(\langle M^{\mathbf{d}}_{i_\varepsilon, j_\varepsilon} : \varepsilon \le \mathrm{cf}(\beta)\rangle, \langle M^{\mathbf{d}}_{i_\varepsilon, \beta} : \varepsilon \le \mathrm{cf}(\beta)\rangle)$ satisfies its assumptions hence its conclusion holds and it says that $\langle M^{\mathbf{d}}_{i_\varepsilon, \beta} : \varepsilon \le \mathrm{cf}(\beta)\rangle$ is cf(β)-correct. As $\langle i_\varepsilon : \varepsilon \le \mathrm{cf}(\beta)\rangle$ is increasing continuous with last element $i_\varepsilon = \alpha_\beta$ and also $\langle M^{\mathbf{d}}_{i,\beta} : i \le \alpha_\beta\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous also $\langle M^{\mathbf{d}}_{i,\beta} : i \le \alpha_\beta\rangle$ is α_β-correct, by Observation 5.16(2), as required.

<u>Clause (b)</u>:

We shall apply part (2) of the present claim on the sequence $\langle M^{\mathbf{d}}_{i,\beta} : i \le \alpha_\beta\rangle$ and $\Gamma = \Gamma_{\mathbf{d}} := \{p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M^{\mathbf{d}}_{\alpha_\beta,\beta}) : p$ does not fork over $M^{\mathbf{d}}_{i,j}$ for some $i < \alpha_j, j < \beta\}$. By the definition of an $\mathfrak{s}$-free triangle it is $\le_{\mathfrak{s}}$-increasing continuous hence clause (b) of part (2) is satisfied. As $\mathbf{d}$ is full by clause (a) of Definition 5.15(3) the ordinal $\alpha_\beta = \alpha_\beta(d)$ is divisible by λ, i.e. clause (a) of part (2) holds. Clause (c) of the assumption of part (2) is satisfied because we have proved clause (a) here.

As for clause (d) of part (2) let $i_1 < \alpha_\beta$ and $p_1 \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M^{\mathbf{d}}_{i_1,\beta})$ be given; let $p_2 \in \Gamma_{\mathbf{d}} = \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M^{\mathbf{d}}_{\alpha_\beta,\beta})$ be a non-forking extension of p_1. By the definition of $\Gamma_{\mathbf{d}}$, see part (2) we can find $j_2 < \beta$ and $i_2 \le \alpha_{j_2}$ such that p_2 does not fork over $M^{\mathbf{d}}_{i_2,j_2}$. By monotonicity, without loss of generality $i_2 \ge i_1$ and $i_2 < \alpha_{j_2}$. As $\mathbf{d}$ is full (see clause (b) of Definition 5.15(3)) we can find $S \subseteq [i_2, \alpha_\beta)$ of order type $\ge \lambda_{\mathfrak{s}}$ such that for each $i \in S$ an ordinal $j_*(i) < \beta$ satisfying $j_*(i) > j_2$ and an element $c \in \mathbf{J}^{\mathbf{d}}_{i,j_*(i)}$ such that $i < \alpha_{j_*(i)}$ and $\mathbf{tp}_{\mathfrak{s}}(c, M^{\mathbf{d}}_{i,j_*(i)}, M^{\mathbf{d}}_{i+1,j_*(i)})$ is a non-forking extension of $p_2 \restriction M^{\mathbf{d}}_{i_2,j_2}$. So by the definition of $\mathfrak{u}_{\mathfrak{s}}$ we have $\mathbf{J}^{\mathbf{d}}_{i,j_*(i)} = \{c\}$ and so by the definition of "$\mathbf{d}$ is a $\mathfrak{u}_{\mathfrak{s}}$-free triangle" we have $(M^{\mathbf{d}}_{i,j_*(i)}, M^{\mathbf{d}}_{i+1,j_*(i)}, c) \le^1_{\mathfrak{u}} (M^{\mathbf{d}}_{i,\beta}, M^{\mathbf{d}}_{i+1,\beta}, c)$. Hence $(M^{\mathbf{d}}_{i,\beta}, M^{\mathbf{d}}_{i+1,\beta}, c)$ realizes a non-forking extension of $p_2 \restriction M_{i_2,j_2}$, hence, by the uniqueness of non-forking extensions the element c realizes $p_2 \restriction M^{\mathbf{d}}_{i,\beta}$. So S is as required in clause (d) of the assumption of part (2).

So all the assumptions of part (2) applied to the sequence $\langle M^{\mathbf{d}}_{\alpha,\beta} : \alpha \le \alpha_\beta\rangle$ and the set $\Gamma_{\mathbf{d}}$ are satisfied hence its conclusion which says that $M^{\mathbf{d}}_{\alpha_\beta,\beta}$ is $(\lambda, \mathrm{cf}(\alpha_\beta))$-brimmed over $M_{i,\beta}$ for every $i < \alpha_\beta$. So we are done proving clause (b) hence part (3).
3A) We prove each clause.

Clause (a):

So $\theta := \mathrm{cf}(\alpha) = \mathrm{cf}(\beta)$. Let $\langle \alpha_\varepsilon : \varepsilon < \theta\rangle$ be an increasing sequence of ordinals with limit α and $\langle \beta_\varepsilon : \varepsilon < \theta\rangle$ be an increasing sequence of ordinals with limit β. Now for each $\varepsilon < \theta$ we can find $i \in (\alpha_\varepsilon, \alpha)$ and $j \in (\beta_\varepsilon, \beta)$ such that $M^{\mathbf{d}}_{i,j}$ is $\le_{\mathfrak{s}}$-universal over $M^{\mathbf{d}}_{\alpha_\varepsilon,\beta_\varepsilon}$; this holds as we are assuming $\mathbf{d}$ is weakly universal, see Definition 5.15(2A).

By monotonicity without loss of generality $i \in \{\alpha_\zeta : \zeta \in (\varepsilon, \theta)\}$ and $j \in \{\beta_\zeta : \zeta \in (\varepsilon, \theta)\}$. So without loss of generality $M^{\mathbf{d}}_{\alpha_{\varepsilon+1},\beta_{\varepsilon+1}}$ is $\le_{\mathfrak{s}}$-universal over $M^{\mathbf{d}}_{\alpha_\varepsilon,\beta_\varepsilon}$ for $\varepsilon < \theta$. Let $\alpha_\theta := \alpha, \beta_\theta = \beta$.

Hence by Observation 5.16(1) we have $\theta \in \mathrm{correct}(\langle M^{\mathbf{d}}_{\alpha_\varepsilon,\beta_\varepsilon} : \varepsilon \le \theta\rangle)$ which means $\theta \in \mathrm{correct}_{\Gamma_{\mathbf{d}}}(\langle M^{\mathbf{d}}_{\alpha_\varepsilon,\beta_\varepsilon} : \varepsilon \le \theta\rangle)$, see 5.16(4). So by part (1) also $\theta \in \mathrm{correct}_{\Gamma_{\mathbf{d}}}(\langle M^{\mathbf{d}}_{\alpha_\varepsilon,\beta} : \varepsilon \le \theta\rangle)$, hence by Observation 5.16(2) also $\alpha \in \mathrm{correct}_{\Gamma_{\mathbf{d}}}(\langle M^{\mathbf{d}}_{i,\beta} : i \le \alpha\rangle)$, as required.

Clause (b):

We can apply part (2) of the present claim to the sequence $\langle M^{\mathbf{d}}_{i,\beta} : i \le \alpha\rangle$. This is similar to the proof of clause (b) of part (3), alternatively letting $\alpha'_j = \sup\{\alpha_\varepsilon : \varepsilon < \theta \text{ and } \beta_\varepsilon \le j\}$ for $j < \beta$ and $\bar\alpha' = \langle \alpha'_j : j \le \beta\rangle$ use part (3) for the $\mathfrak{u}$-free triangle $\mathbf{d} \restriction (\bar\alpha, \beta)$, i.e. $\bar M^{\mathbf{d}} = \langle M_{i,j} : j \le \beta \text{ and } i \le \alpha'_j\rangle$, etc.; this applies to clause (a), too.

Clause (c):

Note that "$\mathrm{cf}(\alpha) = \mathrm{cf}(\beta)$" is not assumed.

We use part (2A) of the present claim, but we elaborate. Let $S := \{\alpha' < \alpha : \alpha' \text{ is divisible by } \lambda \text{ and has cofinality } \mathrm{cf}(\beta)\}$.

Now S is a subset of α, unbounded (as for every $i < \alpha$ we have $i + \lambda(\mathrm{cf}(\beta)) \in S \cup \{\alpha\}$ and $i + \lambda(\mathrm{cf}(\beta)) \le i + \lambda^2 < i + \lambda^2\omega \le \alpha$), hence it is enough to show that $i_1 < i_2 \in S \Rightarrow M^{\mathbf{d}}_{i_2,\beta}$ is brimmed over $M^{\mathbf{d}}_{i_1,\beta}$.

Now this follows by clause (b) of part (3A) which we have just proved applied to $\mathbf{d}' = \mathbf{d} \restriction (i_2, \beta)$, it is a $\mathfrak{u}$-free (i_2, β)-rectangle, it is strongly full hence full and $\mathrm{cf}(i_2) = \mathrm{cf}(\beta) \geq \aleph_0$. So the assumptions of part (2A) holds hence its conclusion so we are done.
4) Let κ_1, κ_2 be regular $\leq \lambda$ and choose $\alpha_\ell = \lambda^2 \times \kappa_\ell$, for $\ell = 1, 2$. Let $M \in K_{\mathfrak{s}}$ and define a $\mathfrak{u}$-free $(\alpha_1, 0)$-rectangle by $M^{\mathbf{d}_0^{\mathrm{ver}}}_{(i,0)} = M$ for $i \leq \alpha_1$ and $\mathbf{J}^{\mathbf{d}_0^{\mathrm{ver}}}_{(i,0)} = \emptyset$ for $i < \alpha_1$.

Define a $\mathfrak{u}$-free $(0, \alpha_2)$-rectangle $\mathbf{d}_{\mathrm{hor}}$ by $M^{\mathbf{d}_{\mathrm{hor}}}_{(0,j)} = M$ for $j \leq \alpha_2$ and $\mathbf{I}^{\mathbf{d}}_{(0,j)} = \emptyset$ for $j < \alpha_2$. By Observation 5.17 there is a strongly full strictly brimmed $\mathfrak{u}$-free (α_1, α_2)-rectangle $\mathbf{d}$ such that its dual is strongly full too (and automatically strictly brimmed recalling 5.18(6)).

We can apply clause (c) of part (3A) with $(\mathbf{d}, \alpha_1, \alpha_2)$ here standing for $(\mathbf{d}, \alpha, \beta)$ there; so we can conclude in particular that M_{α_1,α_2} is $(\lambda, \mathrm{cf}(\alpha_1))$-brimmed over $M^{\mathbf{d}}_{0,\alpha_2}$ hence over $M_{0,0} = M$. But $\mathfrak{u}_{\mathfrak{s}}$ is self-dual so dual($\mathbf{d}$) is a $\mathfrak{u}_{\mathfrak{s}}$-free (α_2, α_1)-rectangle, and by the choice of $\mathbf{d}$ (recalling 5.17(d)) it is still strongly full and, e.g. by 5.18(7) is universal. So applying clause (c) of part (3A) we get that $M^{\mathrm{dual}(\mathbf{d})}_{\alpha_2,\alpha_1}$ is $(\lambda, \mathrm{cf}(\alpha_2))$-brimmed over $M^{\mathrm{dual}(\mathbf{d})}_{0,\alpha_1}$ hence over $M^{\mathrm{dual}(\mathbf{d})}_{0,0} = M^{\mathbf{d}}_{0,0} = M$. However $M^{\mathrm{dual}(\mathbf{d})}_{\alpha_2,\alpha_1} = M^{\mathbf{d}}_{\alpha_1,\alpha_2}$ so this model $\leq_{\mathfrak{s}}$-extend M and is $(\lambda, \mathrm{cf}(\alpha_\ell))$-brimmed over it for $\ell = 1, 2$; this means (λ, κ_ℓ)-brimmed over M. So as for each regular $\kappa \leq \lambda$, the "(λ, κ)-brimmed model over M for some regular $\kappa \leq \lambda$" is unique up to isomorphism over M we conclude that the brimmed model over M is unique, so we are done.

$\square_{5.19}$

5.20 Claim. *If $M_1 \leq_{\mathfrak{s}} M_2$ are brimmed and $p_i \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_2)$ does not fork over M_1 for $i < i_* < \lambda_{\mathfrak{s}}$ <u>then</u> for some isomorphism π from M_2 onto M_1 we have $i < i_* \Rightarrow \pi(p_i) = p_i \restriction M_1$.*

Proof. Easy, by 5.19(4) and 5.8(3), i.e. as in III.1.21. $\square_{5.20}$

* * *

Another way to deal with disjointness is through reduced triples (earlier we have used 1.10, 1.11, 4.25 (with some repetitions).

5.21 Definition. 1) $K^{3,\mathrm{rd}}_{\mathfrak{s}}$ is the class of triples $(M,N,a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ which are reduced[24] which means: if $(M,N,a) \le_{\mathrm{bs}} (M_1,N_1,a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ then $N \cap M_1 = M$.
2) We say that $\mathfrak{s}$ has existence for $K^{3,\mathrm{rd}}_{\mathfrak{s}}$ <u>when</u> for every $M \in K_{\mathfrak{s}}$ and $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ there is a pair (N,a) such that the triple $(M,N,a) \in K^{3,\mathrm{rd}}_{\mathfrak{s}}$ realizes p, i.e. $p = \mathbf{tp}_{\mathfrak{s}}(a,M,N)$.
3) Let $\xi^{\mathrm{rd}}_{\mathfrak{s}}$ be the minimal ξ from Claim 5.22(4) below for $M \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ which is superlimit.
3A) For $M \in K_{\mathfrak{s}}$, let $\xi^{\mathrm{rd}}_{\mathfrak{s},M} = \xi^{\mathrm{rd}}_M$ be the minimal $\xi < \lambda^+_{\mathfrak{s}}$ in 5.22(4) below for M <u>when</u> it exists, ∞ otherwise (well defined, i.e. $< \infty$ if $\mathfrak{s}$ has existence for $K^{3,\mathrm{rd}}_{\mathfrak{s}}$).

5.22 Claim. *1) For every $(M,N,a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ there is $(M_1,N_1,a) \in K^{3,\mathrm{rd}}_{\mathfrak{s}}$ such that $(M,N,a) \le_{\mathrm{bs}} (M_1,N_1,a)$ and morever M_1,N_1 are brimmed over M,N respectively.*
2) $K^{3,\mathrm{rd}}_{\mathfrak{s}}$ is closed under increasing unions of length $< \lambda^+$, i.e. if $\delta < \lambda^+$ is a limit ordinal and $(M_\alpha,N_\alpha,a) \in K^{3,\mathrm{rd}}_{\mathfrak{s}}$ is $\le^{\mathrm{bs}}_{\mathfrak{s}}$-increasing with $\alpha(<\delta)$ and $M_\delta := \cup\{M_\alpha : \alpha < \delta\}$ and $N_\delta := \cup\{N_\alpha : \alpha < \delta\}$ <u>then</u> $(M_\delta,N_\delta,a) \in K^{3,\mathrm{rd}}_{\mathfrak{s}}$ and $\alpha < \delta \Rightarrow (M_\alpha,N_\alpha,a) \le^{\mathrm{bs}}_{\mathfrak{s}} (M_\delta,N_\delta,a)$.
3) If $\mathfrak{K}_{\mathfrak{s}}$ is categorical (in λ) <u>then</u> $\mathfrak{s}$ has existence for $K^{3,\mathrm{rd}}_{\mathfrak{s}}$.
4) For every $M \in K_{\mathfrak{s}}$ and $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ there are $\xi < \lambda^+$, a $\le_{\mathfrak{s}}$-increasing continuous $\bar{M} = \langle M_\alpha : \alpha \le \xi\rangle$ and $\bar{a} = \langle a_\alpha : \alpha < \xi\rangle$ such that $M_0 = M, M_\xi$ is brimmed over M_0 and each $(M_\alpha,M_{\alpha+1},a_\alpha)$ is a reduced member of $K^{3,\mathrm{bs}}_{\mathfrak{s}}$ and a_0 realizes p in M_1, <u>provided that</u>[25] $\mathfrak{s}$ is categorical or M is brimmed (equivalently superlimit) or $\mathfrak{s}$ has existence for $K^{3,\mathrm{rd}}_{\mathfrak{s}}$.
5) If $(M_1,N_1,a) \in K^{3,\mathrm{rd}}_{\mathfrak{s}}$ and $(M_1,N_1,a) \le_{\mathrm{bs}} (M_2,N_2,a)$ then $M_2 \cap N_1 = M_1$.

[24] This is different from our choice in Definition VI.1.11(2), <u>but</u> here this is for a given almost good λ-frame, there it is for a λ-a.e.c. $\mathfrak{K}$.

[25] Why not $\xi \le \lambda$? The bookkeeping is O.K. but then we have to use Ax(E)(c) in the end, but see Exercise 5.26.

Proof. 1),2),3) Easy, or see details in [Sh:F841].
4) As in the proof of 7.10(2B) using Fodor lemma; for the "M brimmed" case, use the moreover from part (1).
5) By the definitions. $\square_{5.22}$

5.23 Conclusion. (Disjoint amalgamation) Assume $\mathfrak{K}_{\mathfrak{s}}$ is categorical or just has existence for $K^{3,\mathrm{rd}}_{\mathfrak{s}}$ recalling 5.22(3). If $(M, N_\ell, a_\ell) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ for $\ell = 1, 2$ and $N_1 \cap N_2 = M$ <u>then</u> there is $N_3 \in K_{\mathfrak{s}}$ such that $(M, N_\ell, a_3) \leq_{\mathrm{bs}} (N_{3-\ell}, N_3, a_\ell)$ for $\ell = 1, 2$.

Hence $\mathfrak{s}$ has disjointness, see definition 5.5.

Proof. Straight by 5.22(4) similarly to Observation 5.17 using 5.22(5), of course. $\square_{5.23}$

<u>5.24 Question</u>: Is 5.23 true without categoricity (and without assuming existence for $K^{3,\mathrm{rd}}_{\mathfrak{s}}$)?

5.25 Remark. So we can redefine $\mathfrak{u}$ such that the amalgamation is disjoint by restricting ourselves to $\mathfrak{s}_{[M]}, M \in K_{\mathfrak{s}}$ superlimit or assuming $\mathfrak{s}$ has existence for $K^{3,\mathrm{rd}}_{\mathfrak{s}}$.

We can work with $\mathfrak{u}$ which includes disjointness so this enters the definition of $K^{3,\mathrm{up}}_{\mathfrak{s}}$ defined in 6.4, so this is a somewhat different property or, as we prefer, we ignore this using $=_\tau$ as in §1.

<u>5.26 Exercise</u>: If M is superlimit and $(M, N, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ <u>then</u> for some $\leq_{\mathfrak{s}}$-increasing continuous sequence $\bar{M} = \langle M_\alpha : \alpha < \lambda \rangle$ and $\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle, \mathbf{d}$ we have $M = M_0 \leq_{\mathfrak{s}} N \leq_{\mathfrak{s}} M_\lambda$ and each $(M_\alpha, M_{\alpha+1}, a_\alpha) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ is reduced and $a = a_0$ and M_λ is brimmed over M_0.

[<u>Hint</u>: Let $\bar{\mathscr{U}} = \langle \mathscr{U}_\alpha : \alpha \leq \lambda \rangle$ be an increasing continuous sequence of subsets of λ such that $|\mathscr{U}_0| = \lambda = |\mathscr{U}_{\alpha+1} \backslash \mathscr{U}_0|$ and $\min(\mathscr{U}_\alpha) \geq \alpha$ for $\alpha < \lambda$ and $\mathscr{U}_\lambda = \lambda$.

Let $\alpha_i = i$ for $i \leq \lambda$ and $\bar{\alpha}^i = \langle \alpha_j : j \leq i \rangle$. We choose pairs $(\mathbf{d}_\beta, \bar{p}^\beta)$ by induction on $\beta \leq \lambda$ such that

$\circledast$ (a) $\mathbf{d}_\beta$ is a $\mathfrak{u}$-free $(\bar{\alpha}^\beta, \beta)$-triangle

(b) $M^{\mathbf{d}_\beta}_{i+1,j+1}$ is $\le_{\mathfrak{s}}$-universal over $M^{\mathbf{d}_\beta}_{i+1,j} \cup M^{\mathbf{d}_\beta}_{i,j+1}$ when $j < \beta$ and $i < \alpha_j$ and $M^{\mathbf{d}_\beta}_{0,j+1}$ is $\le_{\mathfrak{s}}$-universal over $M^{\mathbf{d}_\beta}_{0,j}$ when $j < \beta$

(c) $\bar{p}^\beta = \langle p_i : i \in \mathscr{U}_\alpha \rangle$ list $\cup\{\mathscr{S}^{\rm bs}_{\mathfrak{s}}(M^{\mathbf{d}_\beta}_{i,j}) : j \le \beta, i \le \alpha_j\}$ each appearing λ times

(d) if $\beta = 2\alpha + 1, \alpha \in \mathscr{U}_\varepsilon$ <u>then</u> $\mathbf{J}^{\mathbf{d}_\beta}_{2\alpha,\beta} \ne \emptyset$ and letting a be the unique member of $\mathbf{J}^{\mathbf{d}_\beta}_{2\alpha,\beta}$, the type $\mathbf{tp}_{\mathfrak{s}}(a, M^{\mathbf{d}_\beta}_{2\alpha,\beta}, M^{\mathbf{d}_\beta}_{p,\beta})$ is a non-forking extension of p_α

(e) if $\beta = 2\alpha + 2$ and $\alpha \in \mathscr{U}_\varepsilon$ then there is $(M'_\beta, N'_\beta, \mathbf{J}_\beta) \in K^{3,\rm rd}_{\mathfrak{s}}$ such that $(M^{\mathbf{d}_\beta}_{\varepsilon,2\alpha+1}, M^{\mathbf{d}_\beta}_{\varepsilon+1,2\alpha+1}, \mathbf{J}^{\mathbf{d}_\beta}_{\varepsilon,2\alpha+1}) \le^{\rm bs}_{\mathfrak{s}} (M'_\beta, N'_\beta, \mathbf{J}_\beta) \le^{\rm bs}_{\mathfrak{s}} (M^{\mathbf{d}_\beta}_{\varepsilon,\beta}, M^{\mathbf{d}_\beta}_{\varepsilon+1,\beta}, \mathbf{J}^{\mathbf{d}_\beta}_{\varepsilon,\beta})$]

(f) $M^{\mathbf{d}_0}_{0,0} = M$ and $(M^{\mathbf{d}_1}_{0,1}, M^{\mathbf{d}_1}_{1,1}, \mathbf{J}^{\mathbf{d}_1}_{0,1}) = (M, N, \{a\})$.

Now $\langle M^{\mathbf{d}_\lambda}_{i,\lambda} : i \le \lambda \rangle$ is as required except "$M = M_{0,\lambda} \le_{\mathfrak{s}} N \le_{\mathfrak{s}} M_{\lambda,\lambda}$". But $(M, N, \{a\}) = (M^{\mathbf{d}_1}_{0,1}, M^{\mathbf{d}_1}_{1,1}, \{a\}) \le^1_{\mathfrak{u}} (M^{\mathbf{d}_\lambda}_{0,\lambda}, M^{\mathbf{d}_\lambda}_{1,\lambda}, \mathbf{J}^{\mathbf{d}_\lambda}_{0,\lambda})$, that is $(M, N, a) \le_{\rm bs} (M^{\mathbf{d}_\lambda}_{0,\lambda}, a)$ and both M and $M^{\mathbf{d}_\lambda}_{0,\lambda}$ are brimmed equivalently superlimit, hence by 5.8(3) there is an isomorphism π from M onto M onto $M^{\mathbf{d}_\lambda}_{0,\lambda}$ mapping $\mathbf{tp}(a, M, N)$ to $\mathbf{tp}(a, M^{\mathbf{d}_\lambda}_{0,\lambda}, M^{\mathbf{d}_\lambda}_{1,\lambda})$. Recalling $M^{\mathbf{d}_\lambda}_{\lambda,\lambda}$is brimmed over $M^{\mathbf{d}_\lambda}_{0,\lambda}$ we can extend π to a $\le_{\mathfrak{s}}$-embedding π^+ of N into $M^{\mathbf{d}_\lambda}_{\lambda,\lambda}$ mapping a to itself, so renaming we are done.]

§6 Density of weak version of uniqueness

We would like to return to "density of $K^{3,\rm uq}_{\mathfrak{s}}$", where $\mathfrak{s}$ is a good λ-frame or just almost good λ-frames, i.e. to eliminate the (weak) extra set theoretic assumption in the non-structure results from failure of density of $K^{3,\rm uq}_{\mathfrak{s}}$. But we start with a notion $K^{3,\rm up}_{\mathfrak{s}}$, weaker then $K^{3,\rm uq}_{\mathfrak{s}}$ related to weak non-forking relations defined later in Definition 7.18. Defining weak non-forking we shall waive uniqueness, but still can "lift $\mathfrak{u}_{\mathfrak{s}}$-free $(\alpha, 0)$-rectangles". We now look for a dichotomy - either a non-structure results applying the theorems of §2 or actually

§3, or density of $K^{3,\mathrm{up}}_{\mathfrak{s}}$. Using the last possibility in the subsequent sections §7, §8 we get a similar dichotomy with $K^{3,\mathrm{uq}}_{\mathfrak{s}}$.

It turns out here that what we prove is somewhat weaker than density for $K^{3,\mathrm{up}}_{\mathfrak{s}}$ in some ways. Mainly we prove it for the $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ version for each $\xi \le \lambda^+$. Actually for every $M \in K_{\mathfrak{s}}$ and $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ what we find is a triple $(M_1, N_1, a) \in K^{3,\mathrm{up}}_{\mathfrak{s}}$ such that $M \le_{\mathfrak{s}} M_2$ and the type $\mathbf{tp}_{\mathfrak{s}}(a, M_1, N_1)$ is a non-forking extension of p; not a serious difference when $\mathfrak{s}$ is categorical which is reasonable for our purposes. Eventually in this section we have to use $\mathfrak{K}_{\mathfrak{s}}$ with fake equality (to apply 3.16), this is justified in 6.18.

Discussion: Why do we deal with $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ for $\xi \le \lambda^+$ rather than with $K^{3,\mathrm{up}}_{\mathfrak{s}}$? The point is that, e.g. in the weak/semi/vertical uq-invariant coding property in §3 (see Definitions 3.2, 3.14, 3.10), given $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ and $(M_{\alpha(0)}, N_0, \mathbf{I}) \in \mathrm{FR}^1_{\mathfrak{u}}$, for a club of $\delta < \partial$ we promise the existence of a $\mathfrak{u}$-free $(\alpha, 0)$-rectangle $\mathbf{d}_\delta$, which is O.K. for every N_δ such that $(M_{\alpha(*)}, N_0, \mathbf{I}) \le^1_{\mathfrak{u}} (M_\delta, N_\delta, \mathbf{I})$. So the failure gives (not much more than) that for every $\mathfrak{u}$-free $(\alpha, 0)$-rectangle $\mathbf{d}$ and $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M^{\mathbf{d}}_{\alpha,0})$ there is a pair $(N, \mathbf{I})$ such that $(M, N, \mathbf{I})$ realizes p and $\mathbf{d}$ is what we call uq-orthogonal to $(M, N, \mathbf{I})$. We like to invert the quantifiers, i.e. "for $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ there is $(N, \mathbf{I})$ such that for every $\mathbf{d}$....". Of course, we assume categoricity (of $K_{\mathfrak{s}}$, that is in λ), but we need to use a "universal $\mathbf{d}$". This is guaranteed by 6.12 for $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ for any $\xi \le \lambda^+$ (but we have to work more for $\xi = \lambda^+$, i.e. for all $\xi < \lambda^+$ at once, i.e. for $K^{3,\mathrm{up}}_{\mathfrak{s}}$).

6.1 Hypothesis. 1) As in 5.1 and for transparency $\mathfrak{s}$ has disjointness.
2) $\mathfrak{u} = \mathfrak{u}_{\mathfrak{s}}$, see Definition 5.10, Claim 5.11, so $\partial = \lambda^+$.

6.2 Claim. *1) For almost$_2$ all $(M, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{u}}$ the model M_∂ belongs to $K^{\mathfrak{s}}_{\lambda^+}$ and is saturated (above λ).*
2) If $\mathfrak{s}$ has the fake equality $=_\tau$ (e.g. $\mathfrak{s} = \mathfrak{t}'$ where $\mathfrak{t}$ is an almost good λ-frame and $\mathfrak{t}'$ is defined as in 4.25(1), $\mathfrak{u} = \mathfrak{u}^1_{\mathfrak{s}}$, see 5.10,5.11, then for some $\mathfrak{u} - 0$-appropriate $\mathfrak{h}$, if $\langle(\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha < \partial^+_{\mathfrak{u}}\rangle$ is $\le^{\mathrm{qt}}_{\mathfrak{u}}$-increasing continuous and obeys $\mathfrak{h}$, then $M = \cup\{M^\alpha_j : \alpha < \partial^+_{\mathfrak{u}}\rangle$ is $=_\tau$-fuller.

Remark. 1) See Definition 1.22.
2) Compare with 4.30.

Proof. 1) We choose $\mathfrak{g}$ as in Definition 1.22(2) such that:

$(*)_1$ if $S \subseteq \partial$ is a stationary subset of ∂ and the pair $((\bar{M}^1, \bar{\mathbf{J}}^1, \mathbf{f}^1), (\bar{M}^2, \bar{\mathbf{J}}^2, \mathbf{f}^2))$ strictly S-obeys $\mathfrak{g}$ <u>then</u>:
$\odot$ for some club E of ∂ for every $\delta \in S \cap E$, we have
(a) if $i < \mathbf{f}^1(\delta)$ then $M^2_{\delta+i+1}$ is $\leq_{\mathfrak{s}}$-universal over $M^1_{\delta+i+1} \cup M^2_{\delta+i}$
(b) $\mathbf{f}^2(\delta)$ is $> \mathbf{f}^1(\delta)$ and is divisible by λ^2
(c) if $i \in [\mathbf{f}^1(\delta), \mathbf{f}^2(\delta))$ then
(α) $M^2_{\delta+i+1}$ is $\leq_{\mathfrak{s}}$-universal over $M^2_{\delta+i}$
(β) if $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M^2_{\delta+i})$,
<u>then</u> for λ ordinals $i_1 \in [i, i+\lambda)$
the type $\mathbf{tp}_{\mathfrak{s}}(a_{\mathbf{J}^2_{\delta+i}}, M^2_{\delta+i}, M^2_{\delta+i+1})$
is a non-forking extension of p, where
where $b_{\mathbf{J}^2_{\delta+i}}$ is the unique member of $\mathbf{J}^2_{\delta+i}$.

We can find such $\mathfrak{g}$. Now

$(*)_2$ assume $\langle (\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha \leq \delta \rangle$ is $\leq^{\mathrm{qt}}_{\mathfrak{u}}$-increasing continuous (see Definition 1.15(4A)) and obey $\mathfrak{g}$ (i.e. for some[26] stationary $S \subseteq \partial$ for unboundedly many $\alpha < \delta$). <u>Then</u> M^δ_∂ is saturated above λ.

[Why? Let $\kappa =$ cf(δ), of course $\aleph_0 \leq \kappa \leq \lambda^+$.

We can find an increasing continuous sequence $\langle \alpha_\varepsilon : \varepsilon <$ cf$(\delta) = \kappa \rangle$ of ordinals with limit δ such that:

$(*)_3$ if $\varepsilon = \zeta + 1 < \kappa$ and ε is an even ordinal then $\alpha_{\varepsilon+1} = \alpha_\varepsilon + 1$ and letting $\alpha_{\mathrm{cf}(\delta)} = \delta$ the pair $\langle (\bar{M}^{\alpha_\varepsilon}, \bar{\mathbf{J}}^{\alpha_\varepsilon}, \mathbf{f}^{\alpha_\varepsilon}), (\bar{M}^{\alpha_{\varepsilon+1}}, \bar{\mathbf{J}}^{\alpha_{\varepsilon+1}}, \mathbf{f}^{\alpha_{\varepsilon+1}}) \rangle$ does S-obey $\mathfrak{g}$.

[26] we can use a decreasing sequence of S's but then we really use the last one only, the point is that $\mathfrak{g}$ treat each $\delta \in S$ in the same way (rather than dividing it according to tasks, a reasonable approach, but not needed here

Now clearly

$(*)_4$ for every $\varepsilon < \lambda^+$ satisfying $\varepsilon \le \kappa$ for some club $E = E_\varepsilon$ of λ^+, for every $\delta \in S \cap E$ we have:

(a) $\langle \mathbf{f}^{\alpha_\zeta}(\delta) : \zeta \le \varepsilon \rangle$ is non-decreasing and is continuous

(b) $\mathbf{d}_\delta$ is a $\mathfrak{u}$-free $(\bar\alpha^\delta, \beta)$-triangle where $\bar\alpha^\delta = \langle \mathbf{f}^{\alpha_\zeta}(\delta) : \zeta \le \varepsilon \rangle, \beta = \varepsilon$ such that $M^{\mathbf{d}_\delta}_{i,\zeta} = M^{\alpha_\zeta}_{\delta+i}$ for $\zeta \le \varepsilon, i \le \mathbf{f}^{\alpha_\zeta}(\delta)$ and $\mathbf{J}^{\mathbf{d}_\delta}_{i,\zeta} = \mathbf{J}^{\alpha_\zeta}_{\delta+i}$ for $\zeta \le \varepsilon, i < \mathbf{f}^{\alpha_\zeta}(\delta)$ and $\mathbf{I}^{\mathbf{d}_\delta}_{i,\zeta} = \emptyset$ for $\zeta < \varepsilon, i \le \mathbf{f}^{\alpha_\zeta}(\delta)$.

Sorting out the definition by 5.14(3A)(a) and the correctness claim 5.19(3)(b), clearly:

$(*)$ for a club of $\gamma < \partial$, if $\gamma \in S$ <u>then</u> $M^{\alpha_\kappa}_{\gamma+\mathbf{f}^{\alpha_\kappa}(\gamma)}$ is brimmed over $M^{\alpha_\kappa}_\gamma$ (and even $M^{\alpha_\kappa}_{\gamma+i}$ for $i < \mathbf{f}^{\alpha_\kappa}(\gamma)$)

which means that $M^\delta_{\gamma+\mathbf{f}^\delta(\gamma)}$ is brimmed over M^δ_γ.

This is clearly enough.

2) Should be clear. $\square_{6.2}$

6.3 Conclusion. For any $(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ and for stationary $S \subseteq \partial$ there is $\mathfrak{u} - 2$-appropriate $\mathfrak{g}$ with $S_{\mathfrak{g}} = S$ (see 1.22 and 1.23) such that:

$\circledast$ if $\langle (\bar M^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha \le \partial \rangle$ is $\le^{\rm qs}_{\mathfrak{u}}$-increasing continuous and 2-obeys $\mathfrak{g}$ such that $(\bar M^0, \bar{\mathbf{J}}^0, \mathbf{f}^0) = (\bar M, \bar{\mathbf{J}}, \mathbf{f})$ (so $\mathbf{f}^\partial(\delta) = \sup\{\mathbf{f}^\alpha(\delta) : \alpha < \delta\}$ for a club of $\delta < \partial$) <u>then</u> for a club of $\delta < \partial$ the model $M^\partial_\delta = \cup\{M^\alpha_\delta : \alpha < \delta\} \in K_{\mathfrak{s}}$ is brimmed over $M_\delta = M^0_\delta$.

Proof. Similar to the proof of 6.2; only the $\mathfrak{u}$-free triangle is flipped, i.e. it is a dual($\mathfrak{u}$)-free triangle but dual($\mathfrak{u}$) = $\mathfrak{u}$. $\square_{6.3}$

6.4 Definition. Let $1 \le \xi \le \lambda^+$, if we omit it we mean $\xi = \lambda^+$.
1) $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ is the class of $(M,N,a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ which has up-ξ-uniqueness, which means:

- $\circledast$ if $(M,N,a) \le^1_{\mathfrak{u}} (M',N',a)$ and $\mathbf{d}$ is a $\mathfrak{u}$-free $(\alpha,0)$-rectangle with $\alpha \le \xi, \alpha < \lambda^+$ satisfying $(M^{\mathbf{d}}_{0,0}, M^{\mathbf{d}}_{\alpha,0}) = (M,M')$ <u>then</u> $\mathbf{d}$ can be lifted for $((M,N,a),N')$ which means:
  - $\boxdot$ we can find a $\mathfrak{u}$-free $(\alpha+1,1)$-rectangle $\mathbf{d}^*$ and f such that $\mathbf{d}^* \restriction (\alpha,0) = \mathbf{d}, f(N) \le_{\mathfrak{s}} M^{\mathbf{d}^*}_{0,1}, f(a) = a^{\mathbf{d}^*}_{0,0}$, i.e. $\mathbf{I}^{\mathbf{d}^*}_{0,0} = \{f(a)\}$ and f is a $\le_{\mathfrak{s}}$-embedding of N' into $M^{\mathbf{d}^*}_{\alpha+1,1}$ over M' hence also $f \restriction M = \mathrm{id}_M$.

2) We say that $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ is dense (in $K^{3,\mathrm{bs}}_{\mathfrak{s}}$) or $\mathfrak{s}$ has density for $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ <u>when</u> for every $(M_0,N_0,a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ there is $(M_1,N_1,a) \in K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ such that $(M_0,N_0,a) \le_{\mathrm{bs}} (M_1,N_1,a)$.
3) We say that $\mathfrak{s}$ has (or satisfies) existence for $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ or $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ has (or satisfies) existence <u>when</u>: if $M \in K_{\mathfrak{s}}$ and $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ <u>then</u> for some pair (N,a) we have $(M,N,a) \in K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ and $p = \mathbf{tp}_{\mathfrak{s}}(a,M,N)$.
3A) We say that $\mathfrak{s}$ has existence for $K^{3,\mathrm{up}}_{\mathfrak{s},<\xi}$ <u>when</u> $\mathfrak{s}$ has existence for $K^{3,\mathrm{up}}_{\mathfrak{s},\zeta}$ for every $\zeta \in [1,\xi)$; similarly in the cases below.
4) Let $K^{3,\mathrm{up}}_{\mathfrak{s},\xi} = \cap\{K^{3,\mathrm{up}}_{\mathfrak{s},\zeta} : \zeta < \xi\}$.
5) Let $K^{3,\mathrm{up+rd}}_{\mathfrak{s},\xi}$ be defined as $K^{3,\mathrm{up}}_{\mathfrak{s},\xi} \cap K^{3,\mathrm{rd}}_{\mathfrak{s}}$ recalling Definition 5.21, and we repeat parts (2),(3),(4) for it.

6.5 Observation. 1) $K^{3,\mathrm{up}}_{\mathfrak{s},\xi} \subseteq K^{3,\mathrm{up}}_{\mathfrak{s},\zeta} \subseteq K^{3,\mathrm{rd}}_{\mathfrak{s}}$ recalling Definition 5.21 when $1 \le \zeta \le \xi \le \lambda^+$.
2) $\xi = \lambda^+ \Rightarrow K^{3,\mathrm{up}}_{\mathfrak{s},\xi} = \cap\{K^{3,\mathrm{up}}_{\mathfrak{s},\zeta} : \zeta \in [1,\xi)\}$.
3) The triple $(M,N,a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ does not belong to $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ <u>iff</u> we can find $\mathbf{d}_1, \mathbf{d}_2$ such that for $\ell = 1,2$

$\square$ (a) $\mathbf{d}_\ell$ is a $\mathfrak{u}$-free $(\alpha_\ell,1)$-rectangle and $\alpha_\ell <\ \min\{\xi+1, \lambda^+_{\mathfrak{s}}\}$
(b) $(M^{\mathbf{d}_\ell}_{0,0}, M^{\mathbf{d}_\ell}_{0,1}, \mathbf{I}^{\mathbf{d}_\ell}_{0,0}) = (M,N,\{a\})$
(c) $M^{\mathbf{d}_1}_{\alpha_1,0} = M^{\mathbf{d}_2}_{\alpha_2,0}$ and $M^{\mathbf{d}_1}_{0,0} = M = M^{\mathbf{d}_2}_{0,0}$

(d) there is no triple $(\mathbf{d}, f)$ such that

(α) $\mathbf{d}$ is a $\mathfrak{u}$-free $(\alpha_{\mathbf{d}}, \beta_{\mathbf{d}})$-rectangle

(β) $\mathbf{d} \restriction (\alpha_1, 1) = \mathbf{d}_1$ so $\beta_{\mathbf{d}} \geq 1, \alpha_{\mathbf{d}} \geq \alpha_1$

(γ) f is a $\leq_{\mathfrak{s}}$-embedding of $M^{\mathbf{d}_2}_{\alpha_2,1}$ into $M^{\mathbf{d}}_{\alpha_{\mathbf{d}},\beta_{\mathbf{d}}}$ over $M^{\mathbf{d}}_{\alpha_1,0} = M^{\mathbf{d}_1}_{\alpha(\mathbf{d}_1),0}$ mapping $M^{\mathbf{d}_2}_{0,1}$ into $M^{\mathbf{d}}_{0,\beta(\mathbf{d})}$ and $a^{\mathbf{d}_2}_{0,0}$ to itself.

Proof. 0) By the definition; as for a $\mathfrak{u}$-free (α, β)-rectangle or $(\bar{\alpha}, \beta)$-triangle $\mathbf{d}$ we have: if $i = \min\{i_1, i_2\}, j = \min\{j_1, j_2\}$ and $M^{\mathbf{d}}_{i_1,j_1}$, $M^{\mathbf{d}}_{i_2,j_2}$ are well defined then $M^{\mathbf{d}}_{i,j} = M^{\mathbf{d}}_{i_1,j_1} \cap M^{\mathbf{d}}_{i_2,j_2}$.
1) By the definition (no need of categoricity).
2) By 5.8(3).
3) Straight, recalling that $\mathfrak{u}$ satisfies monotonicity, (E)(e), see 1.13 but we elaborate.

<u>The Direction $\Rightarrow$</u>:

So (M, N, a) belongs to $K^{3,\mathrm{bs}}_{\mathfrak{s}}$ but not to $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$. So by Definition 6.4(1) there is $(M', N', \mathbf{d})$ exemplifying the failure of ⊛ from 6.4(1), which means

⊙ (a) $(M, N, a) \leq^1_{\mathfrak{u}} (M', N', a)$, i.e. see Definition 5.10, i.e. 4.29, i.e. $M = N \cap M'$ and $(M, N, a) \leq_{\mathrm{bs}} (M', N', a)$

(b) $\mathbf{d}$ is a $\mathfrak{u}$-free $(\alpha, 0)$-rectangle with $\alpha \leq \xi, \alpha < \lambda^+$

(c) $(M^{\mathbf{d}}_{0,0}, M^{\mathbf{d}}_{\alpha,0}) = (M, M')$

(d) $\mathbf{d}$ cannot be lifted for (M, N, a, N'), i.e. there is no pair $(\mathbf{d}^*, f)$ such that

(α) $\mathbf{d}^*$ is a $\mathfrak{u}$-free $(\alpha + 1, 1)$-rectangle

(β) $\mathbf{d}^* \restriction (\alpha, 0) = \mathbf{d}$

(γ) f is a $\leq_{\mathfrak{s}}$-embedding of N' into $M^{\mathbf{d}^*}_{\alpha+1,1}$ over $M' = M^{\mathbf{d}}_{\alpha,0}$

(δ) $f(N) \leq_{\mathfrak{s}} M^{\mathbf{d}^*}_{0,1}$ and $f(a) = a^{\mathbf{d}^*}_{0,0}$.

We define $\mathbf{d}_2$ by

$\odot_2$ $\mathbf{d}_2$ is the $\mathfrak{u}$-free $(1,1)$-rectangle if

(a) $(M^{\mathbf{d}_2}_{0,0}, M^{\mathbf{d}_2}_{1,0}, \mathbf{J}^{\mathbf{d}_2}_{0,0}) = (M, M', \emptyset)$

(b) $(M^{\mathbf{d}_2}_{0,0}, M^{\mathbf{d}_2}_{1,0}, \mathbf{I}^{\mathbf{d}_2}_{0,0}) = (M, N, \{a\})$

(c) $M^{\mathbf{d}_2}_{1,1} = N, \mathbf{J}^{\mathbf{d}_2}_{0,1} = \emptyset, \mathbf{I}^{\mathbf{d}_2}_{1,0} = \{a\}$.

We choose $\mathbf{d}_1$ such that

$\odot_2$ (a) $\mathbf{d}_2$ is a $\mathfrak{u}$-free $(\alpha, 1)$-rectangle

(b) $\mathbf{d}_2 \restriction (\alpha, 0) = \mathbf{d}^*$

(c) $(M^{\mathbf{d}_2}_{0,0}, M^{\mathbf{d}_2}_{0,1}, \mathbf{I}^{\mathbf{d}_2}_{0,0}) = (M, N, \{a\})$.

The Direction $\Leftarrow$:
Choose $\mathbf{d} = \mathbf{d}_1$ and use Exercise 1.13. $\square_{6.5}$

6.6 Definition. We say (M, N, a) is a up-orthogonal to $\mathbf{d}$ when:

$\circledast$ (a) $(M, N, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$

(b) $\mathbf{d}$ is a $\mathfrak{u}$-free $(\alpha_{\mathbf{d}}, 0)$-rectangle

(c) $M^{\mathbf{d}}_{0,0} = M$

(d) Case 1: $N \cap M^{\mathbf{d}}_{\alpha(\mathbf{d}),0} = M$.
If N_1 satisfies $(M, N, a) \leq^1_{\mathfrak{u}} (M^{\mathbf{d}}_{\alpha(\mathbf{d}),0}, N_1, a)$, so N_1 does $\leq_{\mathfrak{s}}$-extends $M^{\mathbf{d}}_{\alpha(\mathbf{d}),0}$ and N,
then the rectangle $\mathbf{d}$ can be lifted for $((M, N, a), N_1)$;
Case 2: possibly $N \cap M^{\mathbf{d}}_{(\alpha(\mathbf{d}),0)} \neq M$.
We replace (N, a) by (N', a') such that $N' \cap M^{\mathbf{d}}_{\alpha(\mathbf{d}),0} = M$ and there is an isomorphism from N onto N' over M mapping a to a'.

We now consider a relative of Definition 6.4.

6.7 Definition. Let $\xi \le \lambda^+$ but $\xi \ge 1$, if $\xi = \lambda^+$ we may omit it.
1) We say that $\mathfrak{s}$ has almost-existence for $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ <u>when</u>: if $M \in K_{\mathfrak{s}}$ and $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ we have

$\odot_{M,p}$ if $\alpha \le \xi$ and $\mathbf{d}$ is a $\mathfrak{u}$-free $(\alpha, 0)$-rectangle with $M^{\mathbf{d}}_{0,0} = M$ (yes, we allow $\alpha = \xi = \lambda^+$) <u>then</u> there is a triple $(M, N, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ such that $p = \mathbf{tp}_{\mathfrak{s}}(a, M, N)$ and (M, N, a) is up-orthogonal to $\mathbf{d}$.

2) We say that $\mathfrak{s}$ has the weak density for $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ <u>when</u>: if $M \in K_{\mathfrak{s}}$ and $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ <u>then</u> for some (M_1, p_1) the demand $\odot_{M_1,p_1}$ in part (1) holds and $M \le_{\mathfrak{s}} M_1$ and $p_1 \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_1)$ is a non-forking extension of p.
3) We write "almost-existence/weak density for $K^{3,\mathrm{up}}_{\mathfrak{s},<\xi}$" when this holds for every $\xi' < \xi$.

6.8 Observation. Assume $\mathfrak{s}$ is categorical (in $\lambda_{\mathfrak{s}}$) and $\xi \le \lambda^+$.
1) Then $\mathfrak{s}$ has weak density for $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ <u>iff</u> $\mathfrak{s}$ has almost existence for $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$.
2) If $\mathfrak{s}$ has existence for $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ <u>then</u> $\mathfrak{s}$ has almost existence for $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$.

Proof. 1) The weak density version implies the existence version, i.e. the first implies the second because if $M \le_{\mathfrak{s}} M_1$ and $p_1 \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_1)$ does not fork over M then there is an isomorphism f from M_1 onto M mapping p to $p \restriction M$, see 5.20. The inverse is obvious.
2) Read the definitions. $\square_{6.8}$

<u>6.9 Discussion</u>: Below we fix $p_* \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_*)$ and look only at stationarization of p_*. We shall use the failure of almost-existence for $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ to get non-structure.

We first present a proof in the case $\mathscr{D}_\partial$ is not ∂^+-saturated (see 6.15) but by a more complicated proof this is not necessary, see 6.14. As it happens, we do not assume $2^\lambda < 2^{\lambda^+}$, but still assume $2^{\lambda^+} < 2^{\lambda^{++}}$ using the failure of weak density for $K^{3,\mathrm{up}}_{\mathfrak{s},<\lambda^+}$ to get an up-invariant coding property (avoiding the problem we encounter when we try to use $\mathbf{d}_\delta$ depending on N^η_δ).

So in 6.15 for each $\alpha < \lambda^{++} = \partial^+$ we "give" a stationary $S_\alpha \subseteq \partial$ almost disjoint to S_β for $\beta < \alpha^+$.

Well, we have for the time being decided to deal only with up-uniqueness arguing that it will help to deal with "true" uniqueness. Also, in the non-structure we use failure of weak density for $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$, whereas for the positive side, in §7, we use existence. The difference is that for existence we have (N, a) for given (M, p, ξ) whereas for almost existence we are given $(M, p, \mathbf{d})$. However, we now prove their equivalence. To get the full theorem 6.14 we use 3.10 - 3.24.

6.10 Claim. *Assume $\mathfrak{s}$ is categorical; if $\xi \le \lambda^+$ and $\mathfrak{s}$ has almost-existence for $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ <u>then</u> $\mathfrak{s}$ has existence for $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$.*

Proof. This follows from the following two subclaims, 6.11, 6.12.

6.11 Subclaim. *If $(M, N, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ is up-orthogonal to $\mathbf{d}_2$, <u>then</u> it is up-orthogonal to $\mathbf{d}_1$ <u>when</u>:*

⊛ (*a*) $\mathbf{d}_\ell$ *is a $\mathfrak{u}$-free $(\alpha_\ell, 0)$-rectangle for $\ell = 1, 2$ where $\alpha_\ell \le \lambda^+$*

(*b*) $M^{\mathbf{d}_1}_{0,0} = M = M^{\mathbf{d}_2}_{0,0}$

(*c*) *h is an increasing function from α_1 to α_2*

(*d*) *f is an $\le_{\mathfrak{s}}$-embedding of $M^{\mathbf{d}_1}_{\alpha_1,0}$ into $M^{\mathbf{d}_2}_{\alpha_2,0}$*

(*e*) $f \restriction M^{\mathbf{d}_1}_{0,0} = \mathrm{id}_M$

(*f*) *if $\beta < \alpha_1$ then*

(*α*) $f(b^{\mathbf{d}_1}_{\beta,0}) = b^{\mathbf{d}_2}_{h(\beta),0}$

(*β*) *f maps $M^{\mathbf{d}_1}_{\beta,0}$ into $M^{\mathbf{d}_2}_{h(\beta),0}$*

(*γ*) $\mathbf{tp}_{\mathfrak{s}}(b^{\mathbf{d}_2}_{h(\beta),0}, M^{\mathbf{d}_2}_{h(\beta),0}, M^{\mathbf{d}_2}_{h(\beta)+1,0})$ *does not fork over $f(M^{\mathbf{d}_1}_{\beta,0})$.*

Proof. Without loss of generality f is the identity and $M^{\mathbf{d}_2}_{\alpha_2,0} \cap N = M$.

So assume $N_1 \in K_{\mathfrak{s}}$ is a $\le_{\mathfrak{s}}$-extension of N and of $M^{\mathbf{d}_1}_{\alpha_1,0}$ such that $(M,N,a) \le_{\rm bs} (M^{\mathbf{d}_1}_{\alpha_1,0},N_1,a)$ and we should prove the existence of a suitable lifting. Without loss of generality $N_1 \cap M^{\mathbf{d}_2}_{\alpha_2,0} = M^{\mathbf{d}_1}_{\alpha_1,0}$. Hence there is N_2 which does $\le_{\mathfrak{s}}$-extend $M^{\mathbf{d}_2}_{\alpha_2,0}$ and N_1 and $(M^{\mathbf{d}_1}_{\alpha_1,0},N_1,a) \le^1_{\mathfrak{u}} (M^{\mathbf{d}_2}_{\alpha_2,0},N_2,a)$; but $\le^1_{\mathfrak{u}}$ is a partial order hence $(M,N,a) \le^1_{\mathfrak{u}} (M^{\mathbf{d}_2}_{\alpha_{2,0}},N_2,a)$.

Recall that we are assuming (M,N,a) is up-orthogonal to $\mathbf{d}_2$ hence we can find $\mathbf{d}^2, f$ as in Definition 6.6, i.e. as in $\odot$ inside Definition 6.4(1), so $\mathbf{d}^2$ is a $\mathfrak{u}$-free $(\alpha_2+1,1)$-rectangle, $\mathbf{d}^2 \restriction (\alpha_2,0) = \mathbf{d}_2$, f is a $\le_{\mathfrak{s}}$-embedding of N_2 into $M^{\mathbf{d}_2}_{\alpha_2+1,1}$ over $M^{\mathbf{d}_2}_{\alpha_2,0}$ mapping N into $M^{\mathbf{d}}_{0,1}$ satisfying $f(a) = a^{\mathbf{d}^2}_{0,0}$, note: $\mathbf{I}^{\mathbf{d}}_{\alpha,0} = \{a\}$ for $\alpha \le \alpha_2+1$. Now we define $\mathbf{d}^1$, a $\mathfrak{u}$-free $(\alpha_1+1,1)$-rectangle by

$\boxtimes$ (a) $\mathbf{d}^1 \restriction (\alpha_1,0) = \mathbf{d}_1$

(b) $M^{\mathbf{d}^1}_{\alpha,1}$ is $M^{\mathbf{d}^2}_{h(\alpha),1}$ if $\alpha \le \alpha_1$ is a non-limit ordinal
and is $\cup\{M^{\mathbf{d}^2}_{\beta,1} : \beta < h(\alpha)\}$ if $\alpha \le \alpha_1$ is a limit ordinal
and is $M^{\mathbf{d}^2}_{\alpha_1+1,1}$ if $\alpha = \alpha_1+1$

(c) $M^{\mathbf{d}^1}_{\alpha_1+1,0} = M^{\mathbf{d}^2}_{\alpha_1,0}$

(d) $\mathbf{I}^{\mathbf{d}_1}_{\alpha,0} = \mathbf{I}^{\mathbf{d}_2}_{h(\alpha),0}$ for $\alpha \le \alpha_1$

(e) $\mathbf{J}^{\mathbf{d}_1}_{\alpha,1} = \mathbf{J}^{\mathbf{d}_2}_{h(\alpha),0}$ for $\alpha < \alpha_1$

(f) $\mathbf{I}^{\mathbf{d}^1}_{\alpha_1+1,0} = \mathbf{I}^{\mathbf{d}^2}_{\alpha_2+1,0}$

(g) $\mathbf{J}^{\mathbf{d}^1}_{\alpha_1,0} = \emptyset = \mathbf{J}^{\mathbf{d}^1}_{\alpha_1,1}$.

Now check. $\square_{6.11}$

6.12 Claim. *For every $\alpha_1 \le \lambda^+$ there is $\alpha_2 \le \lambda^+$ (in fact $\alpha_2 = \lambda\alpha_1$ is O.K.) such that: for every $M \in K_{\mathfrak{s}}$ there is a $\mathfrak{u}$-free $(\alpha_2,0)$-rectangle $\mathbf{d}_2$ with $M^{\mathbf{d}_2}_{\alpha_2,0} = M$ such that*

(∗) *if $\mathbf{d}_1$ is a $\mathfrak{u}$-free $(\alpha_1,0)$-rectangle with $M^{\mathbf{d}_1}_{0,0} = M$ then there are h,f as in $\circledast$ of 6.11.*

Proof. Let $\langle \mathscr{U}_i : i \le \lambda\rangle$ be a $\subseteq$-increasing continuous sequence of subsets of λ such that $\mathscr{U}_\lambda = \lambda$, $\min(\mathscr{U}_i) \ge i, \lambda = |\mathscr{U}_0|$ and $\lambda =$

$|\mathscr{U}_{\alpha+1}\backslash\mathscr{U}_\alpha|$ for $\alpha < \lambda$. We now choose $(M_i, \bar{p}_i, a_i)$ by induction on $i \le \lambda\alpha_1$ such that

$\oplus$ (a) $\langle M_i : j \le i\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous

(b) $M_0 = M$

(c) if $i = j+1$ then M_i is $\le_{\mathfrak{s}}$-universal over M_j

(d) if $i = \lambda i_1 + i_2$ and $i_1 < \alpha_1, i_2 < \lambda$ then $\bar{p}_i = \langle p^{i_1}_\varepsilon : \varepsilon \in \mathscr{U}_{i_2}\rangle$ where $p^{i_1}_\varepsilon \in \cup\mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_{\lambda i_1+i}) : i \le i_2\}$

(e) if $i = \lambda i_1 + i_2, i_1 < \alpha_1, j \le i_2 < \lambda$ and $q \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_{\lambda i_1+i_2})$ then $(\exists^\lambda \zeta \in \mathscr{U}_{i_2})(p^i_\zeta = q)$

(f) if $i = \lambda i_1 + i_2, i_2 = j_2 + 1 < \lambda$, and $j_2 \in \mathscr{U}_\varepsilon$ then $a_{i-1} \in M_i$ and the type $\mathbf{tp}_{\mathfrak{s}}(a_{i-1}, M_{i-1}, M_i)$ is a non-forking extension of $p^{i_1}_{j_2}$.

Now choose $\mathbf{d}_2 = (\langle M_i : i \le \lambda\alpha\rangle, \langle a_i : i < \lambda\alpha\rangle)$. So assume $\mathbf{d}_1$ is a $\mathfrak{u}$-free $(\alpha_1, 0)$-rectangle with $M^{\mathbf{d}_1}_{0,0} = M = M^{\mathbf{d}_2}_{0,0}$. We now choose a pair (f_i, h_i) by induction on $i \le \alpha_1$ such that: f_i is a $\le_{\mathfrak{s}}$-embedding of $M^{\mathbf{d}_1}_i$ into $M^{\mathbf{d}_2}_{\lambda,i}, h_i : i \rightarrow \lambda i$ such that $h(j) \in (\lambda j, \lambda + \lambda), f_i$ is $\subseteq$-increasing, h_i is $\subseteq$-increasing, $\mathbf{tp}_{\mathfrak{s}}(a^{\mathbf{d}_2}_{h(j),0}, M^{\mathbf{d}_2}_{h(j)+1,0})$ is a non-forking extension of $f_{j+1}(\mathbf{tp}_{\mathfrak{s}}(a^{\mathbf{d}_1}_j, M^{\mathbf{d}_1}_j, M^{\mathbf{d}_1}_{j+1}))$ and check; in fact $\oplus$ gives more than necessary. $\square_{6.12}$ $\square_{6.10}$

6.13 Claim. *Assume $\xi \le \lambda^+$ and $M_* \in K_{\mathfrak{s}}$ and $p_* \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_*)$ witness that $\mathfrak{s}$ fails the weak density for $K^{3,\rm up}_{\mathfrak{s},\xi}$, see Definition 6.7.*

1) If $(M, N, \{a\}) \in K^{3,\rm bs}_{\mathfrak{s}}$ and $M_* \le_{\mathfrak{s}} M$ and $\mathbf{tp}_{\mathfrak{s}}(a, M, N)$ is a non-forking extension of p_* then $(M, N, \{a\})$ has the weak ξ-uq-invariant coding property for $\mathfrak{u}$; pedantically assuming $\mathfrak{s}$ has fake equality see Definition 4.25, see 6.18, similarly in part (2); on this coding property, (see Definition 3.2(1).

2) Moreover the triple $(M, N, \{a\}) \in {\rm FR}^1_{\mathfrak{u}}$ has the semi ξ-uq-invariant coding property, (see Definition 3.14).

Remark. If below 6.15 suffices for us then part (2) of 6.13 is irrelevant.

Proof. 1) Read the definitions, i.e. Definition 6.7(2) on the one hand and Definition 3.2(1) on the other hand. Pedantically one may worry

that in 6.7(2) we use $\leq^{\rm bs}_{\mathfrak{s}}$, where disjointness is not required whereas in $\leq^{\ell}_{\mathfrak{u}}$ it is, however as we allow using fake equality in $\mathfrak{K}_{\mathfrak{u}}$ this is not problematic.
2) Similar. $\square_{6.13}$

Now we arrive to the main result of the section.

6.14 Theorem. *$\dot I(\lambda^{++},\mathfrak{K}^{\mathfrak{s}}) \ge \mu_{\rm unif}(\lambda^{++},2^{\lambda^+})$, and even $\dot I(\lambda^{++}, K^{\mathfrak{s}}(\lambda^+$-saturated above $\lambda)) \ge \mu_{\rm unif}(\lambda^{++},2^{\lambda^+})$, and even $\dot I(K^{\mathfrak{s},\mathfrak{h}}_{\lambda^{++}}) \ge \mu_{\rm unif}(\lambda^{++},2^{\lambda^+})$ for any $\mathfrak{u}_{\mathfrak{s}} - \{0,2\}$-appropriate $\mathfrak{h}$ <u>when</u>:*

- *(a) $2^{\lambda^+} < 2^{\lambda^{++}}$*
- *(b) (α) $\mathfrak{s}$ fail the weak density for $K^{3,{\rm up}}_{\mathfrak{s},\xi}$ where $\xi \le \lambda^+$ or just for $\xi = \lambda^+$*
  - *(β) if $\xi = \lambda^+$ then $2^\lambda < 2^{\lambda^+}$.*

Before we prove 6.14 we prove a weaker variant when we strengthen the set theoretic assumption.

6.15 Theorem. *Like 6.14 but we add to the assumption*

- *(c) $\mathscr{D}_{\lambda^+}$ is not λ^{++}-saturated (see 6.16(1) below).*

6.16 Remark. 0) In the section main conclusion, 6.17, if we add clause (c) of 6.15 to the assumptions then we can rely there on 6.15 instead of on 6.14.
1) Recall that $\lambda > \aleph_0 \Rightarrow \mathscr{D}_{\lambda^+}$ is not λ^+-saturated by Gitik-Shelah [GiSh 577], hence this extra set theoretic assumption is quite weak.
2) We use 3.5 in proving 6.15.
3) We can add the version with $\mathfrak{h}$ to the other such theorems.

Proof of 6.15. We can choose a stationary $S \subseteq \partial = \lambda^+$ such that $\mathscr{D}_{\lambda^+} + (\lambda^+ \backslash S)$ is not λ^{++}-saturated. We shall apply Theorem 3.5 for the S we just chose for ξ' which is $\lambda + 1$ if $\xi = \lambda$ and is ξ if $\xi < \lambda^+$.

We have to verify 3.5's assumption (recalling $\partial = \lambda^+$): clauses (a) + (b) of 3.5 holds by clauses (a) + (c) of the assumption of 6.15 if

$\xi < \lambda^+$ and clauses (a) + (b)(β) of the assumptions of 6.15 if $\xi = \lambda^+$. Clause (c) of 3.5 holds by 6.2, 6.13(1), whose assumption holds by clause (b) of the assumption of 6.14. Really we have to use 6.13(1), 1.8(6). $\square_{6.14}$

6.17 Conclusion. Assume $2^{\lambda^+} < 2^{\lambda^{++}}$ and $\xi \le \lambda^+$ but $\xi = \lambda^+ \Rightarrow 2^\lambda < 2^{\lambda^+}$.
1) If $\dot{I}(\lambda^{++}, K^{\mathfrak{s}}(\lambda^+\text{-saturated}))$ is $< \mu_{\text{unif}}(\lambda^{++}, 2^{\lambda^+})$ and $\mathfrak{K}_{\mathfrak{s}}$ is categorical then $\mathfrak{s}$ has existence for $K^{3,\text{up}}_{\mathfrak{s},\xi}$ for every $\xi \le \lambda^+$.
2) Similarly for $\dot{I}(\mathfrak{K}^{\mathfrak{s},\mathfrak{h}}_{\lambda^{++}})$ for any $\mathfrak{u}_{\mathfrak{s}} - \{0,2\}$-appropriate $\mathfrak{h}$.

Proof. Let $\xi \le \lambda^+$. We first try to apply Theorem 6.14. Its conclusion fails, but among its assumptions clauses (a) and (b)(β) hold by our present assumptions. So necessarily the demand in clause (b)(α) of 6.14 fails. So we have deduced that $\mathfrak{s}$ has weak density for $K^{3,\text{up}}_{\mathfrak{s},\xi}$. By Observation 6.8(1), recalling we are assuming that $\mathfrak{K}_{\mathfrak{s}}$ is categorical, it follows that $\mathfrak{s}$ has almost existence for $K^{3,\text{up}}_{\mathfrak{s},\xi}$. By Claim 6.10 again recalling we are assuming $\mathfrak{K}_{\mathfrak{s}}$ is categorical we can deduce that $\mathfrak{s}$ has existence for $K^{3,\text{up}}_{\mathfrak{s},\xi}$.

So we have gotten the desired conclusion. But we still have to prove Theorem 6.14 in the general case. $\square_{6.17}$

6.18 Claim. *Assume $\mathfrak{t}'$ is an almost good λ-frame derived from $\mathfrak{t}$ as in Definition 4.25 and $\mathfrak{u}' = \mathfrak{u}^1_{\mathfrak{t}'}$ is as defined in Definition 5.10, i.e. Definition 4.29 (and see Claims 4.30, 5.11).*
Then

$\boxtimes$ (a) $\mathfrak{t}'$ *satisfies all the assumptions on* $\mathfrak{t}$ *in 6.1*
(b) *so all that we have proved on* $(\mathfrak{t}, \mathfrak{u})$ *in this section apply to* $(\mathfrak{t}', \mathfrak{u}')$*, too*
(c) $\mathfrak{u}'$ *has fake equality* $=_\tau$ *(see Definition 3.17(1)*
(d) $\mathfrak{u}'$ *is hereditary for the fake equality* $=_\tau$ *(see Definition 3.17(4)).*

Proof. Clause (a) holds by Claim 4.26. Clause (b) holds by Claim 4.30, 5.11. Clause (c) holds by direct inspection on 4.29(6)(γ).

In clause (d), "$\mathfrak{u}'$ is hereditary for the fake equality $=_\tau$", i.e. satisfies clause (a) of Definition 3.17(3) by Claim 4.30(6)(β), 5.11 applied to $\mathfrak{t}'$.

Lastly, to prove "$\mathfrak{u}'$ is hereditary for the fake equality $=_\tau$" we have to show that it satisfies clause (b) of 3.17(4), which holds by 4.30(6)(δ). $\square_{6.18}$

6.19 Proof of 6.14: We shall use Claim 6.18 to derive $(\mathfrak{s}', \mathfrak{u}')$, so we can use the results of this section to $(\mathfrak{s}, \mathfrak{u})$ and to $(\mathfrak{s}', \mathfrak{u}')$. Now by 6.2 for some $\mathfrak{u}_{\mathfrak{s}'} - 2$-appropriate $\mathfrak{h}$, every $M \in K^{\mathfrak{u}',\mathfrak{h}}_{\partial^+}$ is τ-fuller, see Definition 1.8(6), so by 1.8(6) it is enough to prove Theorem 6.14 for $(\mathfrak{s}', \mathfrak{u}')$. Now by Claim 6.13(1) and clause (b)(α) of the assumption we know that some $(M, N, \{a\}) \in \mathrm{FR}^1_{\mathfrak{u}'}$ has the semi uq-invariant coding property (for $\mathfrak{u}'$). Also $\mathfrak{u}'$ has the fake equality $=_\tau$ and is hereditary for it by 6.18 and is self dual by 5.11(1).

Hence in Claim 3.20 all the assumptions hold for $\mathfrak{u}', (M, N, a)$, hence its conclusion holds, i.e. $(M, N, \{a\})$ has the weak vertical ξ-uq-invariant coding property. This means that clause (b) from the assumptions of Theorem 3.24 holds. Clause (a) there means $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ (choosing $\theta := \lambda, \partial = \lambda^+$) and clause (c) holds by 6.2. So we are done. Having shown that the assumptions of Theorem 3.24 hold, we get its conclusion, which is the conclusion of the present theorem (reclaling we show that it suffices to prove it for $\mathfrak{s}'$, so we are done. $\square_{6.14}$

§7 Pseudo uniqueness

Our explicit main aim is to help in §8 to show that under the assumptions of Chapter VI, i.e. [Sh 576] we can get a good λ-frame not just a good λ^+-frame as done in II§3(D),3.7. For this we deal with almost good frames (see 7.1 and Definition 5.2) and assume existence for $K^{3,\mathrm{up}}_{\mathfrak{s},\lambda^+}$ (see Definition 6.4(3A) justified by 6.17) and get enough of the results of II§6 and few from Chapter III. This means that $\mathrm{WNF}_{\mathfrak{s}}$ is defined in 7.3 and proved to be so called "a weak non-forking relation on $\mathfrak{K}_{\mathfrak{s}}$ respecting $\mathfrak{s}$"; we also look at almost good λ-frames with such relations and then prove that they are good λ-

frames in 7.19(2). Those results are used in the proof of 4.32 (and in §8).

But this has also interest in itself as in general we like to understand pre-λ-frames which are not as good as the ones considered in Chapter III, i.e. weakly successful good λ-frames. We will try to comment on this, too. Note that below even if $\mathfrak{s}$ is a good λ-frame satisfying Hypothesis 7.1 which is weakly successful (i.e. we have existence for $K^{3,\mathrm{uq}}_{\mathfrak{s}}$, still $\mathrm{WNF}_{\mathfrak{s}}$ defined below is not in general equal to $\mathrm{NF}_{\mathfrak{s}}$). We may wonder, is the assumption (3) of 7.1 necessary? The problem is in 7.14, 7.11.

Till 7.20 we use:

7.1 Hypothesis. 1) $\mathfrak{s}$ is an almost good λ-frame (see Definition 5.2).
2) $\mathfrak{s}$ has existence[27] for $K^{3,\mathrm{up}}_{\mathfrak{s},\lambda^+}$, see Definition 6.4(3A) and sufficient condition in 6.17.
3) $\mathfrak{s}$ is categorical in λ (used only from 7.14 on).
4) $\mathfrak{s}$ has disjointness (see Definition 5.5; used only from 7.10 on, just for transparency, in fact follows from parts (1) + (3) by 5.23).

7.2 Definition. 1) Let $\mathfrak{u} = \mathfrak{u}_{\mathfrak{s}}$ be as in Definition 4.29 and 5.10.
2) Let $\mathrm{FR}_{\mathfrak{s}}$ be $\mathrm{FR}^{\ell}_{\mathfrak{u}_{\mathfrak{s}}}$ for $\ell = 1, 2$ (they are equal).

7.3 Definition. 1) Assume $\xi \leq \lambda^+$. Let $\mathrm{WNF}^{\xi}_{\mathfrak{s}}(M_0, N_0, M_1, N_1)$ mean that: $M_0 \leq_{\mathfrak{s}} N_0 \leq_{\mathfrak{s}} N_1, M_0 \leq_{\mathfrak{s}} M_1 \leq_{\mathfrak{s}} N_1$ and if $\alpha < \xi$ so $\alpha < \lambda^+$ and $\mathbf{d}$ is an $\mathfrak{u}$-free $(0, \alpha)$-rectangle and f is a $\leq_{\mathfrak{s}}$-embedding of N_0 into $M^{\mathbf{d}}_{0,\alpha}$ such that $f(M_0) = M^{\mathbf{d}}_{0,0}$ <u>then</u> we can find a model N^* and a $\mathfrak{u}$-free $(1, \alpha)$-rectangle $\mathbf{d}^+$ satisfying $\mathbf{d}^+ \restriction (0, \alpha) = \mathbf{d}$ and $M^{\mathbf{d}^+}_{1,\alpha} \leq_{\mathfrak{s}} N^*$ and $\leq_{\mathfrak{s}}$-embedding $g \supseteq f$ of N_1 into N^* such that $M^{\mathbf{d}^+}_{1,0} = g(M_1)$.
2) If $\xi = \lambda^+$ we may omit it. So $\mathrm{WNF}^{\xi}_{\mathfrak{s}}$ is also considered as the class of such quadruples of models.

[27] if $\mathfrak{s}$ is a good λ-frame, then actually $K^{3,\mathrm{up}}_{\mathfrak{s},<\lambda^+}$ is enough, see 6.4(3A); the main difference is in the proof of 7.14

7.4 Remark. 0) Definition 7.3(1) is dull for $\xi = 0$.
1) So this definition is not obviously symmetric but later we shall prove it is.
2) Similarly, it seemed that the value of ξ is important, but we shall show that for $\xi < \lambda^+$ large enough it is not <u>when</u> $\mathfrak{s}$ is a good λ-frame; see e.g. 7.6.
3) In Definition 7.3 we may ignore $\xi = 0$ as it essentially says nothing.

7.5 Observation. 1) If $1 \le \xi \le \lambda^+$ and $\mathrm{WNF}^\xi_{\mathfrak{s}}(M_0, N_0, M_1, N_1)$ and $(M_0, N_0, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ <u>then</u> $(M_0, N_0, a) \le_{\mathrm{bs}} (M_1, N_1, a)$ and in particular $(M_1, N_1, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$.
2) $\mathrm{WNF}^\xi_{\mathfrak{s}}$ is $\subseteq$-decreasing with ξ.
3) $\mathrm{WNF}_{\mathfrak{s}} = \mathrm{WNF}^{\lambda^+}_{\mathfrak{s}} = \cap\{\mathrm{WNF}^\xi_{\mathfrak{s}} : \xi < \lambda^+\}$.
4) In Definition 7.3(1) in the end we can weaken "$M^{\mathbf{d}^+}_{1,0} = g(M_1)$" to "$g(M_1) \le_{\mathfrak{s}} M^{\mathbf{d}^+}_{1,0}$".

Proof. 1) Straight, use $\mathbf{d}$, the $\mathfrak{u}$-free $(0,1)$-rectangle such that $M^{\mathbf{d}}_{0,0} = M_0$ and $M^{\mathbf{d}}_{0,1} = M$ and $a^{\mathbf{d}}_{0,0} = a$, i.e. $\mathbf{I}^{\mathbf{d}}_{0,0} = \{a\}$.
2),3) Trivial.
4) Given $(N^*, \mathbf{d}^1, g)$ as in Definition 7.5(4). We define $\mathbf{d}'$, a $\mathfrak{u}$-free $(1,\alpha)$-rectangle by $M^{\mathbf{d}'}_{i,j}$ is $g(M_1)$ if $(i,j) = (1,0)$ and is $M^{\mathbf{d}^+}_{i,j}$ when $i \le 1$ & $j \le \alpha$ & $(i,j) \ne (1,0)$ and $\mathbf{I}^{\mathbf{d}'}_{0,j} = \mathbf{I}^{\mathbf{d}'}_{1,j} = \mathbf{I}^{\mathbf{d}}_{0,j}$ for $j < \alpha$ and $\mathbf{J}^{\mathbf{d}'}_{0,j} = \emptyset$ for $j \le \alpha$. The only non-obvious point is why $(M^{\mathbf{d}'}_{0,0}, M^{\mathbf{d}'}_{0,1}, \mathbf{I}^{\mathbf{d}'}_{0,0}) \le^1_{\mathfrak{u}} (M^{\mathbf{d}'}_{1,0}, M^{\mathbf{d}'}_{1,1}, \mathbf{I}^{\mathbf{d}'}_{0,0})$ which means $(M^{\mathbf{d}}_{0,0}, M^{\mathbf{d}}_{0,1}, \mathbf{I}^{\mathbf{d}}_{0,0}) \le^1_{\mathfrak{u}} (g(M_1), M^{\mathbf{d}^+}_{1,1}, \mathbf{I}^{\mathbf{d}^+}_{0,0})$. This is because $\mathfrak{u}$ is interpolative by 4.30(6)(ε), see Definition 3.21. $\square_{7.5}$

7.6 Claim. *[Monotonicity] Assume* $1 \le \xi \le \lambda^+$.
If $\mathrm{WNF}^\xi_{\mathfrak{s}}(M_0, N_0, M_1, N_1)$ *and* $M_0 \le_{\mathfrak{s}} N'_0 \le_{\mathfrak{s}} N_0$ *and* $M_0 \le_{\mathfrak{s}} M'_1 \le M_1$ *and* $N_1 \le_{\mathfrak{s}} N'_1, N'_0 \cup M'_1 \subseteq N''_1 \le_{\mathfrak{s}} N'_1$ <u>*then*</u> $\mathrm{WNF}^\xi_{\mathfrak{s}}(M_0, N'_0, M'_1, N''_1)$ *holds.*

Proof. It is enough to prove that for the case three of the equalities $N'_0 = N_0, M'_1 = M_1, N''_1 = N'_1, N'_1 = N_1$ hold. Each follows: in the case $N'_0 \neq N_0$ by the Definition 7.3, in the case $M'_1 \neq M_1$ by 7.5(4), and in the cases $N''_1 = N'_1 \vee N'_1 = N_1$ by amalgamation (in $\mathfrak{K}_{\mathfrak{s}}$) and the definition of $\mathrm{WNF}^{\xi}_{\mathfrak{s}}$ and 7.5(4).

$\square_{7.6}$

7.7 Observation. [$\mathfrak{s}$ categorical in λ or M_0 is brimmed or $\mathfrak{s}$ has existence for $K^{3,\mathrm{rd}}_{\mathfrak{s}}$, see Definition 5.21.]

If $\xi > \xi^{\mathrm{rd}}_{M_0}$, see Definition 5.21 and $\mathrm{WNF}^{\xi}_{\mathfrak{s}}(M_0, N_0, M_1, N_1)$ then $M_1 \cap N_0 = M_0$.

Remark. 1) Recall that if $\mathfrak{s}$ is an almost good λ-frame then it has density of $K^{3,\mathrm{rd}}_{\mathfrak{s}}$ hence if $\mathfrak{s}$ is categorical then it also has existence for $K^{3,\mathrm{rd}}_{\mathfrak{s}}$.

It is convenient to assume this but not essential. Proving density for $K^{3,\mathrm{up}}_{\mathfrak{s}}$ we actualy prove density for $K^{3,\mathrm{up}}_{\mathfrak{s}} \cap K^{3,\mathrm{rd}}_{\mathfrak{s}}$; moreover $K^{3,\mathrm{up}}_{\mathfrak{s}} \subseteq K^{3,\mathrm{rd}}_{\mathfrak{s}}$.
2) Recall $\xi^{\mathrm{rd}}_{M_0} = \xi^{\mathrm{rd}}_{\mathfrak{s}}$ if M_0 is superlimit, e.g. when $K_{\mathfrak{s}}$ is categorical.

Proof. Recall that letting $\alpha = \xi^{\mathrm{rd}}_{M_0}$ by 5.22(3) or 5.22(4) there is a $\mathfrak{u}$-free $(0,\alpha)$-rectangle $\mathbf{d}$ such that $M^{\mathbf{d}}_{0,0} = M_0$, $N_0 \leq_{\mathfrak{s}} M^{\mathbf{d}}_{0,\alpha}$ and each $(M^{\mathbf{d}}_{0,\alpha}, M^{\mathbf{d}}_{0,\alpha+1}, a^{\mathbf{d}}_{0,\alpha})$ is reduced (see Definition 5.21). Now apply Definition 7.3 to this $\mathbf{d}$. Alternatively, recall for $\mathfrak{u}$-free (α,β)-rectangle (or $(\bar{\alpha},\beta)$-triangle) $\mathbf{d}$ we have $M^{\mathbf{d}}_{i_1,j_1} \cap M^{\mathbf{d}}_{i_2,j_2} = M^{\mathbf{d}}_{\min\{i_1,i_2\},\min\{j_1,j_2\}}$, or we can use 6.5(0). $\square_{7.7}$

7.8 Claim. *[Long transitivity]*

Assume $1 \leq \xi \leq \lambda^+$. *We have* $\mathrm{WNF}^{\xi}_{\mathfrak{s}}(M_0, N_0, M_{\alpha(*)}, N_{\alpha(*)})$ *when:*

(a) $\langle M_\alpha : \alpha \leq \alpha(*)\rangle$ *is* $\leq_{\mathfrak{s}}$*-increasing continuous*

(b) $\langle N_\alpha : \alpha \leq \alpha(*)\rangle$ *is* $\leq_{\mathfrak{s}}$*-increasing continuous*

(c) $\mathrm{WNF}^{\xi}_{\mathfrak{s}}(M_\alpha, N_\alpha, M_{\alpha+1}, N_{\alpha+1})$ *for every* $\alpha < \alpha(*)$.

Remark. 1) Recall that we do not know symmetry for $\mathrm{WNF}_{\mathfrak{s}}$ and while this claim is easy its dual is not clear at this point.

Proof. By chasing arrows. $\square_{7.8}$

7.9 Claim. *[weak existence] Assume* $1 \le \xi \le \lambda^+$.
If $(M_0, M_1, a) \le_{\mathrm{bs}} (N_0, N_1, a)$ *and* $(M_0, M_1, a) \in K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$
then $\mathrm{WNF}^{\xi}_{\mathfrak{s}}(M_0, N_0, M_1, N_1)$.

Proof. Let $\alpha \le \xi$ be such that $\alpha < \lambda^+$ and $\mathbf{d}$ be a $\mathfrak{u}$-free $(0,\alpha)$-rectangle and let f be a $\le_{\mathfrak{s}}$-embedding of N_0 into $M^{\mathbf{d}}_{0,\alpha}$ such that $f(M_0) = M^{\mathbf{d}}_{0,0}$. Let $N'_1 \in K_{\mathfrak{s}}$ be $\le_{\mathfrak{s}}$-universal over $M^{\mathbf{d}}_{0,\alpha}$, exist by 5.8, and let $N'_0 = M^{\mathbf{d}}_{0,\alpha}$.

As $(N_0, N_1, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ we can find a $\le_{\mathfrak{s}}$-embedding g of N_1 into N'_1 extending f such that $(g(N_0), g(N_1), g(a)) \le_{\mathrm{bs}} (N'_0, N'_1, g(a))$ so as $\le_{\mathrm{bs}}$ is a partial order preserved by isomorphisms, clearly $(g(M_0), g(M_1), g(a)) \le_{\mathrm{bs}} (N'_0, N'_1, g(a))$. Now as $(M_0, M_1, a) \in K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ it follows that $(g(M_0), g(M_1), g(a)) \in K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$. Applying the definition of $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$, see Definition 6.4(1) with $g(M_0), g(M_1), g(a), N'_0, N'_1$, $\mathrm{dual}(\mathbf{d})$ here standing for $M, N, a, M', N', \mathbf{d}$ there, recalling that $\mathfrak{u}$ is self-dual we can find a $\mathfrak{u}$-free $(\alpha+1,1)$-rectangle $\mathbf{d}^*$ and h such that: $\mathbf{d}^* \restriction (\alpha, 0) = \mathrm{dual}(\mathbf{d}), h(g(M_1) \le_{\mathfrak{s}} M^{\mathbf{d}^*}_{0,1}, h(g(a)) = a^{\mathbf{d}^*}_{0,0}$ and h is a $\le_{\mathfrak{s}}$-embedding of N'_1 into $M^{\mathbf{d}^*}_{\alpha+1,1}$ over $N'_0 = M^{\mathbf{d}}_{0,\alpha}$ hence $h \restriction g(M_0) = \mathrm{id}_{g(M_0)}$.

Now $N'_1, \mathbf{d}^+ := \mathrm{dual}(\mathbf{d}^*) \restriction (1, \alpha)$ and $h \circ g$ are as required in Definition 7.3 (standing for $N^*, \mathbf{d}^+, g$ there) recalling 7.5. $\square_{7.9}$

7.10 Lemma. *[Amalgamation existence] Let* $\xi < \lambda^+$ *or just* $\xi \le \lambda^+$ *and* $\xi \ge 1$.
1) If $M_0 \le_{\mathfrak{s}} M_\ell$ *for* $\ell = 1, 2$ *and* $M_1 \cap M_2 = M_0$ *then for some* M_3 *we have* $\mathrm{WNF}^{\xi}_{\mathfrak{s}}(M_0, M_1, M_2, M_3)$.
2) If $M_0 \le_{\mathfrak{s}} M_2$ *then we can find an* $\mathfrak{u}$*-free rectangle* $\mathbf{d}$ *satisfying* $\beta_{\mathbf{d}} = 0$ *such that*

$\boxtimes$ (a) $M^{\mathbf{d}}_{0,0} = M_0$

(*b*) $M_2 \le_{\mathfrak{s}} M^{\mathbf{d}}_{\alpha(\mathbf{d}),0}$

(*c*) $(M^{\mathbf{d}}_{\alpha,0}, M^{\mathbf{d}}_{\alpha+1,0}, b^{\mathbf{d}}_{\alpha,0})$ *belongs to* $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ *for* $\alpha < \alpha_{\mathbf{d}}$

(*d*) *if* M_2 *is* $(\lambda,*)$*-brimmed over* M_0 *then* $M^{\mathbf{d}}_2 = M^{\mathbf{d}}_{\alpha_{\mathbf{d}},0}$.

2A) If $(M_0, M_2, b) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ *we can add* $b^{\mathbf{d}}_{0,0} = b$.
2B) Assume $K^{3,*}_{\mathfrak{s}} \subseteq K^{3,\mathrm{bs}}_{\mathfrak{s}}$ *is such that* $\mathfrak{s}$ *has existence for* $K^{3,*}_{\mathfrak{s}}$ *then in parts 2), 2A) we can replace* $K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ *by* $K^{3,*}_{\mathfrak{s}}$.
3) In part (1) if $(M_0, M_\ell, b_\ell) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ *for* $\ell = 1,2$ *then we can add* $(M_0, M_\ell, b_\ell) \le_{\mathrm{bs}} (M_{3-\ell}, M_3, b_\ell)$ *for* $\ell = 1,2$.

Proof. 1) Follows by part (3).
2) By Ax(D)(c), density, of almost good λ-frames there is $b \in M_2 \backslash M_0$ such that $\mathbf{tp}_{\mathfrak{s}}(b, M_0, M_2) \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_0)$, hence $(M_0, M_2, b) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$, by the definition of $K^{3,\mathrm{bs}}_{\mathfrak{s}}$ it follows that so we can apply part (2A).
2A) By part (2B).
2B) So let $(M_0, M_2, b) \in K^{3,*}_{\mathfrak{s}}$ be given. We try to choose $(M_{0,\alpha}, M_{2,\alpha})$ and if $\alpha = \beta+1$ also a_β by induction on $\alpha < \lambda^+$ such that:

⊛ (*a*) $M_{\ell,\alpha} \in K_{\mathfrak{s}}$ is $\le_{\mathfrak{s}}$-increasing continuous for $\ell = 0,2$

(*b*) $M_{0,\alpha} \le_{\mathfrak{s}} M_{2,\alpha}$

(*c*) $(M_{0,\alpha}, M_{2,\alpha}) = (M_0, M_2)$ for $\alpha = 0$

(*d*) if $\alpha = \beta+1$ then $(M_{0,\beta}, M_{0,\alpha}, a_\beta) \in K^{3,*}_{\mathfrak{s}}$ and $a_\beta \in M_{2,\alpha} \backslash M_{0,\alpha}$

(*e*) if $\alpha = \beta+1$ then $M_{2,\alpha}$ is $\le_{\mathfrak{s}}$-brimmed over $M_{2,\beta}$

(*f*) if $\alpha = 0$ then $a_\alpha = b$.

By Fodor lemma we cannot choose for every $\alpha < \lambda^+$. For $\alpha = 0$ and α limit there are no problems, hence for some $\alpha = \beta+1$, we have defined up to β but cannot define for α clearly $\beta < \lambda^+$. First assume

$(*)$ $M_{0,\beta} \neq M_{2,\beta}$.

So by Ax(D)(c) of Definition 5.2 of almost good λ-frame we can choose $a_\beta \in M_{2,\beta} \backslash M_{0,\beta}$ such that $\mathbf{tp}_{\mathfrak{s}}(b, M_{0,\beta}, M_{2,\beta}) \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_{0,\beta})$ and $a_\beta = b$ if $\beta = 0$. By the assumption on $K^{3,*}_{\mathfrak{s}}$ there is $N_\beta \in K_{\mathfrak{s}}$

such that $(M_{0,\alpha}, N_\beta, a_\beta) \in K^{3,*}_{\mathfrak{s}}$ and $\mathbf{tp}_{\mathfrak{s}}(a_\beta, M_{0,\beta}, N_\beta) = \mathbf{tp}_{\mathfrak{s}}(a_\beta, M_{0,\beta}, M_{2,\beta})$.

By the definition of orbital type (and amalgamation of $\mathfrak{K}_{\mathfrak{s}}$) without loss of generality for some $M'_{2,\beta}$ we have $N_\beta \le_{\mathfrak{s}} M'_{2,\beta}$ and $M_{2,\beta} \le_{\mathfrak{s}} M'_{2,\beta}$.

Let $M_{2,\alpha} \in K_{\mathfrak{s}}$ be brimmed over $M'_{2,\beta}$. So we can choose for α, contradiction.

Hence $(*)$ cannot hold so $M_{0,\beta} = M_{2,\beta}$, easily $\beta \ge 1$ (as $M_2 \ne M_0$) and by clause (e) of $\circledast$ $M_{2,\beta}$ is brimmed over $M_{2,0} = M_2$ hence over M_0. What about clause (d) of the conclusion? It follows because any two brimmed extensions of M_0 are isomorphic over it by 5.19(4) and with a little more work even over $M_0 \cup \{b\}$.

3) So let $K^{3,*}_{\mathfrak{s}} = K^{3,\mathrm{up}}_{\mathfrak{s}}$ or just $K^{3,*}_{\mathfrak{s}} \subseteq K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ and $\mathfrak{s}$ has existence for $K^{3,*}_{\mathfrak{s}}$.

Let $\mathbf{d}$ be as guaranteed in parts (2),(2A) so $a^{\mathbf{d}}_{0,0} = b_2$ and $M_0 = M^{\mathbf{d}}_{0,0}, M_2 \le_{\mathfrak{s}} M^{\mathbf{d}}_{\alpha_{\mathbf{d}},0}$. Without loss of generality $M^{\mathbf{d}}_{\alpha_{\mathbf{d}},0} \cap M_1 = M_0$ and now we choose N_α by induction on $\alpha \le \alpha_{\mathbf{d}}$ such that

$\boxplus$ (a) $N_\alpha \in K_{\mathfrak{s}}$ is $\le_{\mathfrak{s}}$-increasing continuous

(b) $M^{\mathbf{d}}_{\alpha_{\mathbf{d}},0} \cap N_\alpha = M^{\mathbf{d}}_{\alpha,0}$

(c) $M^{\mathbf{d}}_{\alpha,0} \le_{\mathfrak{s}} N_\alpha$

(d) $(M^{\mathbf{d}}_{\alpha,0}, M^{\mathbf{d}}_{\alpha+1,0}, a^{\mathbf{d}}_{\alpha,0}) \le_{\mathrm{bs}} (N_\alpha, N_{\alpha+1}, a^{\mathbf{d}}_{\alpha,0})$

(e) $N_\alpha = M_1$ for $\alpha = 0$.

There is no problem to carry the choice by Hypothesis 7.1(4) and Definition 5.5.

Now for each $\alpha < \alpha_{\mathbf{d}}$ by clause (c) of $\boxtimes$ of part (2) or (2B) we have

$$(M^{\mathbf{d}}_{0,\alpha}, M^{\mathbf{d}}_{0,\alpha+1}, a^{\mathbf{d}}_{\alpha,0}) \in K^{3,*}_{\mathfrak{s}} \subseteq K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$$

recalling the choice of $\mathbf{d}$ and by clause (d) of $\boxplus$ we have

$$(M^{\mathbf{d}}_{0,\alpha}, M^{\mathbf{d}}_{0,\alpha+1}, a^{\mathbf{d}}_{\alpha,0}) \le_{\mathrm{bs}} (N_\alpha, N_{\alpha+1}, a^{\mathbf{d}}_{\alpha,0}),$$

hence by the weak existence Claim 7.9 we have $\mathrm{WNF}^{\xi}_{\mathfrak{s}}(M^{\mathbf{d}}_{0,\alpha}, N_\alpha, M^{\mathbf{d}}_{0,\alpha+1}, N_{\alpha+1})$.

As $\langle M^{\mathbf{d}}_{0,\alpha} : \alpha \le \alpha_{\mathbf{d}}\rangle$ and $\langle N_\alpha : \alpha \le \alpha_{\mathbf{d}}\rangle$ are $\le_{\mathfrak{s}}$-increasing continuous, it follows the long transitivity claim 7.8 that $\mathrm{WNF}^{\xi}_{\mathfrak{s}}(M^{\mathbf{d}}_{0,0}, N_0, M^{\mathbf{d}}_{\alpha_{\mathbf{d}},0}, N_{\alpha(\mathbf{d})})$ which means that $\mathrm{WNF}^{\xi}_{\mathfrak{s}}(M_0, M_1, M^{\mathbf{d}}_{\alpha_{\mathbf{d}},0}, N_{\alpha(\mathbf{d})})$. Let $M_3 := N_{\alpha(\mathbf{d})}$, but now $M_0 \le_{\mathfrak{s}} M_2 \le_{\mathfrak{s}} M^{\mathbf{d}}_{\alpha(\mathbf{d}),0}$ hence by the monotonicity Claim 7.6 we have $\mathrm{WNF}^{\xi}_{\mathfrak{s}}(M_0, M_1, M_2, M_3)$.

This proves the desired conclusion of part (1), but there are more demands in part (3). One is $(M_0, M_1, b_1) \le_{\mathrm{bs}} (M_2, M_3, b_1)$, but $M_1 = N_0$ and $M_3 = N_{\alpha(\mathbf{d})}$, so this means $(M_0, N_0, b_1) \le_{\mathrm{bs}} (M_2, N_{\alpha(\mathbf{d})}, b_1)$ and by monotonicity of non-forking it suffices to show $(M_0, N_0, b_1) \le_{\mathrm{bs}} (M^{\mathbf{d}}_{\alpha(\mathbf{d}),0}, N_{\alpha(\mathbf{d})})$.
But recall $\mathrm{WNF}^{\xi}_{\mathfrak{s}}(M_0, N_0, M^{\mathbf{d}}_{\alpha(\mathbf{d}),0}, N_{\alpha(\mathbf{d})})$; this implies $(M_0, N_0, b_1) \le_{\mathrm{bs}} (M^{\mathbf{d}}_{\alpha(\mathbf{d}),0}, N_{\alpha(\mathbf{d})}, b_1)$ by Observation 7.5(1) which as said above suffices.

Now also we have chosen $a^{\mathbf{d}}_{0,0}$ as b_2, so by clause (d) of $\boxplus$ for $\alpha = 0$ we have easily $(M_0, M^{\mathbf{d}}_{1,0}, b_2) = (M^{\mathbf{d}}_{0,0}, M^{\mathbf{d}}_{1,0}, a^{\mathbf{d}}_{0,0}) \le_{\mathrm{bs}} (N_0, N_1, a^{\mathbf{d}}_{0,0}) = (M_1, N_1, b_2) \le_{\mathrm{bs}} (M_1, N_{\alpha(\mathbf{d})}, b_2) = (M_1, M_3, b_2)$; but $M_0 \le_{\mathfrak{s}} M_1 \le_{\mathfrak{s}} M^{\mathbf{d}}_{\alpha(\mathbf{d}),0}$ so by easy monotonicity we have $(M_0, M_2, b_2) \le_{\mathrm{bs}} (M_1, M_3, b_2)$, as desired in part (3); so we are done.

$\square_{7.10}$

7.11 Remark. In the proof of 7.10(2B), if $\mathfrak{s}$ is a good λ-frame, in fact, λ steps in the induction suffice by a careful choice of a_β using bookkeeping as in the proof of 5.19(1), so we get $\alpha_{\mathbf{d}} = \lambda$. Without this extra hypothesis on $\mathfrak{s}$, this is not clear.

7.12 Claim. *Assume that* $1 \le \xi \le \lambda^+, \alpha < \lambda^+, \mathbf{d}$ *is a* $\mathfrak{u}$*-free* $(0,\alpha)$*-rectangle and* $M_0 \le_{\mathfrak{s}} M^{\mathbf{d}}_{0,0} \le_{\mathfrak{s}} M^{\mathbf{d}}_{0,\alpha} \le_{\mathfrak{s}} N_1$. *Then we can find* $\alpha', \mathbf{d}', h$ *such that*

$\oplus$ (a) $\alpha' \in [\alpha, \lambda^+)$
(b) $\mathbf{d}'$ *is a* $\mathfrak{u}$*-free* $(0,\alpha')$*-rectangle*
(c) h *is an increasing function,* $h : \alpha + 1 \to \alpha' + 1$
(d) $M^{\mathbf{d}'}_{0,0} = M_0$
(e) $N_1 \le_{\mathfrak{s}} M^{\mathbf{d}'}_{0,\alpha'}$

(f) $i \le \alpha \Rightarrow M^{\mathbf{d}}_{0,i} \le_{\mathfrak{s}} M^{\mathbf{d}'}_{0,h(i)}$

(g) *for* $i < \alpha, a^{\mathbf{d}}_{0,i} = a^{\mathbf{d}'}_{0,h(i)}$ *and* $\mathbf{tp}_{\mathfrak{s}}(a^{\mathbf{d}'}_{0,h(i)}, M^{\mathbf{d}'}_{0,h(i)}, M^{\mathbf{d}'}_{0,h(i)+1})$ *does not fork over* $M^{\mathbf{d}}_{0,i}$

(h) $(M^{\mathbf{d}'}_{0,\beta}, M^{\mathbf{d}'}_{0,\beta+1}, a^{\mathbf{d}'}_{0,\beta}) \in K^{3,\mathrm{up}}_{\mathfrak{s},\xi}$ *for* $\beta < \alpha'$.

Proof. We can choose M'_i by induction on $i \le 1 + \alpha + 1$ such that

(α) $\langle M'_j : j \le i\rangle$ is increasing continuous

(β) $M'_0 = M_0$

(γ) M'_i is brimmed over M'_j if $i = j + 1 \le 1 + \alpha + 1$

(δ) $M_\alpha \cap M'_{1+i} = M^{\mathbf{d}}_{0,i}$ if $i \le \alpha$

(ε) $M^{\mathbf{d}}_{0,i} \le_{\mathfrak{s}} M'_{1+i}$ if $i \le \alpha$

(ζ) $\mathbf{tp}_{\mathfrak{s}}(a^{\mathbf{d}}_{0,i}, M'_i, M'_{i+1})$ does not fork over $M^{\mathbf{d}}_{0,i}$ if $i < \alpha$

(η) $N_1 \le_{\mathfrak{s}} M'_{1+\alpha+1}$.

This is possible because we have disjoint amalgamation (see 5.23). Now for each $i \le 1+\alpha$ use 7.10(2A) with $M'_i, M'_{i+1}, a^{\mathbf{d}}_{0,i}$ here standing for M_0, M_2, b there (so clause (d) there apply). $\square_{7.12}$

7.13 Remark. Recall that from now on we are assuming that $\mathfrak{K}_{\mathfrak{s}}$ is categorical.

7.14 Claim. *[Symmetry] There is* $\xi = \xi_{\mathfrak{s}} < \lambda^+$ *such that* ($\xi \ge \xi^{\mathrm{rd}}_{\mathfrak{s}}$ *for simplicity, see 5.21 and) for every* $\zeta < \lambda^+$, *if* $\mathrm{WNF}^{\xi}_{\mathfrak{s}}(M_0, N_0, M_1, N_1)$ *<u>then</u>* $\mathrm{WNF}^{\zeta}_{\mathfrak{s}}(M_0, M_1, N_0, N_1)$ *holds.*

Remark. 1) Yes, the models N_0, M_1 exchange places.
2) Without categoricity, $\xi = \xi_{\mathfrak{s},M_0}$ is O.K.

Proof. By 7.10(2) there are $\xi = \xi(*) < \lambda^+$ and $\mathbf{d}$, a $\mathfrak{u}$-free $(0,\xi)$-rectangle with each $(M^{\mathbf{d}}_{0,\alpha}, M^{\mathbf{d}}_{0,\alpha+1}, a^{\mathbf{d}}_{0,\alpha})$ belonging to $K^{3,\mathrm{up}}_{\mathfrak{s},\zeta}$ for every

$\zeta < \lambda^+$ such that $M^{\mathbf{d}}_{0,\xi}$ is brimmed over $M^{\mathbf{d}}_{0,0}$ and $M_0 = M^{\mathbf{d}}_{0,0}$. Note that the choice of ξ does not depend on $\zeta, \langle M_0, N_0, M_1, N_1\rangle$, just on M_0 by 7.10 and it does not depend on M_0 recalling $K_{\mathfrak{s}}$ is categorical.

As $M^{\mathbf{d}}_{0,\xi}$ is $\le_{\mathfrak{s}}$-universal over $M^{\mathbf{d}}_{0,0}$ without loss of generality $N_0 \le_{\mathfrak{s}} M^{\mathbf{d}}_{0,\xi}$.

Now let $\zeta < \lambda^+$ and recall that we assume $\mathrm{WNF}^{\xi}_{\mathfrak{s}}(M_0, N_0, M_1, N_1)$. Let $\mathbf{d}^+, N^+, f$ be as guaranteed by Definition 7.3(1) and by renaming without loss of generality the function f is the identity. Now for each $\alpha < \xi$, we shall apply the weak existence claim 7.9, with $M^{\mathbf{d}^+}_{0,\alpha}, M^{\mathbf{d}^+}_{1,\alpha}, M^{\mathbf{d}^+}_{0,\alpha+1}, M^{\mathbf{d}^+}_{1,\alpha+1}, a^{\mathbf{d}^+}_{0,\alpha}$ here standing for M_0, N_0, M_1, N_1, a as there; this is O.K. as its assumptions mean $(M^{\mathbf{d}^+}_{0,\alpha}, M^{\mathbf{d}^+}_{0,\alpha+1}, a^{\mathbf{d}^+}_{0,\alpha}) \le_{\mathrm{bs}} (M^{\mathbf{d}^+}_{1,\alpha}, M^{\mathbf{d}^+}_{1,\alpha+1}, a^{\mathbf{d}^+}_{0,\alpha})$ and $(M^{\mathbf{d}^+}_{0,\alpha}, M^{\mathbf{d}^+}_{0,\alpha+1}, a^{\mathbf{d}^+}_{0,\alpha}) \in K^{3,\mathrm{up}}_{\mathfrak{s},\zeta}$ which hold by clause $\boxtimes(c)$ of Claim 7.10(2), i.e. by the choice of $\mathbf{d}$ as $\mathbf{d}^+ \restriction (0, \alpha(\mathbf{d})) = \mathbf{d}$. Hence the conclusion of 7.9 applies, which gives that we have $\mathrm{WNF}^{\zeta}_{\mathfrak{s}}(M^{\mathbf{d}^+}_{0,\alpha}, M^{\mathbf{d}^+}_{1,\alpha}, M^{\mathbf{d}^+}_{0,\alpha+1}, M^{\mathbf{d}^+}_{1,\alpha+1})$. Of course $\langle M^{\mathbf{d}^+}_{0,\alpha} : \alpha \le \xi\rangle$ and $\langle M^{\mathbf{d}^+}_{1,\alpha} : \alpha \le \xi\rangle$ are $\le_{\mathfrak{s}}$-increasing continuous. Together by the long transitivity, claim 7.8 we have $\mathrm{WNF}^{\zeta}_{\mathfrak{s}}(M^{\mathbf{d}^+}_{0,0}, M^{\mathbf{d}^+}_{1,0}, M^{\mathbf{d}^+}_{0,\xi}, M^{\mathbf{d}^+}_{1,\xi})$. But $M^{\mathbf{d}^+}_{0,0} = M_0, M^{\mathbf{d}^+}_{1,0} = M_1$ and $N_0 \le_{\mathfrak{s}} M^{\mathbf{d}}_{0,\xi}$ and $N_1 \le_{\mathfrak{s}} N^+, M^{\mathbf{d}^+}_{1,\xi} \le_{\mathfrak{s}} N^+$ so by the monotonicity claim, 7.6, we have $\mathrm{WNF}^{\zeta}_{\mathfrak{s}}(M_0, M_1, N_0, N_1)$ as required. $\square_{7.14}$

7.15 Conclusion. If $\mathrm{WNF}^{\xi}_{\mathfrak{s}}(M_0, N_0, M_1, N_1)$ and $\xi \ge \xi_{\mathfrak{s}}$, see 7.14 then $\zeta < \lambda^+ \Rightarrow \mathrm{WNF}^{\zeta}_{\mathfrak{s}}(M_0, N_0, M_1, N_1)$ that is $\mathrm{WNF}_{\mathfrak{s}}(M_0, N_0, M_1, N_1)$.

Proof. Applying 7.14 twice recalling 7.5(3) in the end. $\square_{7.15}$

7.16 Claim. *(*$\mathrm{WNF}_{\mathfrak{s}}$ *lifting or weak uniqueness)*

If $\mathrm{WNF}_{\mathfrak{s}}(M_0, N_0, M_1, N_1)$ *and* $\alpha < \lambda^+$ *and* $\langle M_{0,i} : i \le \alpha\rangle$ *is* $\le_{\mathfrak{s}}$*-increasing continuous,* $M_{0,0} = M_0$ *and* $M_{0,\alpha} = N_0$ *then we can find a* $\le_{\mathfrak{s}}$*-increasing continuous sequence* $\langle M_{1,i} : i \le \alpha + 1\rangle$ *such that* $M_{1,0} = M_1, N_1 \le_{\mathfrak{s}} M_{1,\alpha+1}$ *and for each* $i < \alpha$ *we have* $\mathrm{WNF}_{\mathfrak{s}}(M_{0,i}, M_{0,i+1}, M_{1,i}, M_{1,i+1})$ *for* $i < \alpha$.

Proof. We shall use 7.12.

By induction on $i \le \alpha$ we can find M'_i which is $\le_{\mathfrak{s}}$-increasing continuous such that $M'_i \cap N_1 = M_{0,i}, M_{0,i} \le_{\mathfrak{s}} M'_i$ and if $i = j+1$ then M'_i is brimmed over M'_j and $\mathrm{WNF}_{\mathfrak{s}}(M_{0,i}, M_{0,i+1}, M'_i, M'_{i+1})$.

So by 7.10(2) and see 7.14 we can find a $\mathfrak{u}_{\mathfrak{s}}$-free $(0, \xi_{\mathfrak{s}}\alpha)$-rectangle $\mathbf{d}$ such that $M^{\mathbf{d}}_{\xi_{\mathfrak{s}} i} = M'_i$ for $i \le \xi_{\mathfrak{s}}\alpha$ and $(M^{\mathbf{d}}_{\varepsilon,0}, M^{\mathbf{d}}_{\varepsilon+1,0}, a^{\mathbf{d}}_{\varepsilon,0}) \in K^{3,\mathrm{up}}_{\mathfrak{s}}$ for $\varepsilon < \xi_{\mathfrak{s}}\alpha$. Recalling $M_{0,\alpha} = N_0$, without loss of generality $M^{\mathbf{d}}_{0,\xi_{\mathfrak{s}}\alpha} \cap N_1 = N_0$ so by 7.10(1) we can find N^*_1 such that $\mathrm{WNF}_{\mathfrak{s}}(N_0, N_1, M^{\mathbf{d}}_{0,\xi_{\mathfrak{s}}\alpha}, N^*_1)$. Recalling that we are assuming $\mathrm{WNF}_{\mathfrak{s}}(M_0, N_0, M_1, N_1)$, by symmetry, i.e. 7.14 we have $\mathrm{WNF}_{\mathfrak{s}}(M_0, M_1, N_0, N_1)$ hence by transitivity, i.e. Claim 7.8 we can deduce that $\mathrm{WNF}_{\mathfrak{s}}(M_0, M_1, M^{\mathbf{d}}_{0,\xi_{\mathfrak{s}}\alpha}, N^*_1)$ hence by 7.14, i.e. symmetry $\mathrm{WNF}_{\mathfrak{s}}(M_0, M^{\mathbf{d}}_{0,\xi_{\mathfrak{s}}\alpha}, M_1, N^*_1)$. By the definition of $\mathrm{WNF}_{\mathfrak{s}}$, we can find a $\mathfrak{u}_{\mathfrak{s}}$-free $(1, \xi_{\mathfrak{s}}\alpha+1)$-rectangle $\mathbf{d}^+$ such that $\mathbf{d}^+ \restriction (0, \xi_{\mathfrak{s}}\alpha) = \mathbf{d}$ and $N^*_1 \le_{\mathfrak{s}} M^{\mathbf{d}^+}_{1,\xi_{\mathfrak{s}}\alpha+1}$ and $M_1 = M^{\mathbf{d}^+}_{1,0}$.

By the weak existence claim 7.9, we have $\mathrm{WNF}_{\mathfrak{s}}(M^{\mathbf{d}^+}_{0,\varepsilon}, M^{\mathbf{d}^+}_{1,\varepsilon}, M^{\mathbf{d}^+}_{0,\varepsilon+1}, M^{\mathbf{d}^+}_{1,\varepsilon+1})$ for each $\varepsilon < \xi_{\mathfrak{s}}\alpha$.

Let $M_{1,\alpha+1} = M^{\mathbf{d}^+}_{1,\xi_{\mathfrak{s}}\alpha+1}$ and for $i \le \alpha$ let $M_{1,i} := M^{\mathbf{d}^+}_{1,\xi_{\mathfrak{s}} i}$. So clearly $\langle M_{1,i} : i \le \alpha+1 \rangle$ is $\le_{\mathfrak{s}}$-increasing continuous, $M_{1,0} = M_1, N_1 \le_{\mathfrak{s}} M_{1,\alpha+1}$. Now to finish the proof we need to show, for $i < \alpha$ that $\mathrm{WNF}_{\mathfrak{s}}(M_{0,i}, M_{0,i+1}, M_{1,i}, M_{1,i+1})$.

For each $i < \alpha$ by the long transitivity claim, i.e. 7.8 applied to $\langle M^{\mathbf{d}^+}_{0,\xi_{\mathfrak{s}} i+\varepsilon} : \varepsilon \le \xi_{\mathfrak{s}} \rangle$ and $\langle M^{\mathbf{d}^+}_{1,\xi_{\mathfrak{s}},i+\varepsilon} : \varepsilon \le \xi_{\mathfrak{s}} \rangle$ we have $\mathrm{WNF}_{\mathfrak{s}}(M^{\mathbf{d}^+}_{0,\xi_{\mathfrak{s}} i}, M^{\mathbf{d}^+}_{1,\xi_{\mathfrak{s}} i}, M^{\mathbf{d}^+}_{0,\xi_{\mathfrak{s}}(i+1)}, M^{\mathbf{d}^+}_{1,\xi_{\mathfrak{s}}(i+1)})$, by symmetry we have $\mathrm{WNF}_{\mathfrak{s}}(M^{\mathbf{d}^+}_{0,\xi_{\mathfrak{s}} i}, M^{\mathbf{d}^+}_{0,\xi_{\mathfrak{s}}(i+1)}, M^{\mathbf{d}^+}_{1,\xi_{\mathfrak{s}} i}, M^{\mathbf{d}^+}_{1,\xi_{\mathfrak{s}}(i+1)})$ which means $\mathrm{WNF}_{\mathfrak{s}}(M'_i, M'_{i+1}, M_{1,i}, M_{i,1+i})$.

Recall that for each $i < \alpha$ we have $\mathrm{WNF}_{\mathfrak{s}}(M_{0,i}, M_{0,i+1}, M'_i, M'_{i+1})$. By the transitivity claim 7.8, the two previous sentences imply $\mathrm{WNF}_{\mathfrak{s}}(M_{0,i}, M_{0,i+1}, M_{1,i}, M_{1,i+1})$ as required. $\square_{7.16}$

7.17 Theorem. *1)* $\mathrm{WNF}_{\mathfrak{s}}$ *is a weak non-forking relation on* $\mathfrak{K}_\lambda$ *respecting* $\mathfrak{s}$ *and having disjointness (see Definition 7.18 below).*
2) If NF *is a weak non-forking relation of* $\mathfrak{K}_\lambda$ *respecting* $\mathfrak{s}$, <u>then</u> $\mathrm{NF} \subseteq \mathrm{WNF}_{\mathfrak{s}}$.

A relative of Definition II.6.1 is:

7.18 Definition. Let $\mathfrak{K}_\lambda$ be a λ-a.e.c.
1) We say that NF is a weak non-forking relation on $\mathfrak{K}_\lambda$ when: it satisfied the requirements in Definition II.6.1(1); except that we replace uniqueness (clause (g) there) by weak uniqueness meaning that the conclusion of Claim 7.16 holds (replacing $\mathrm{WNF}_\mathfrak{s}$ by NF); or see the proof of 7.17 below for a list or use Definition II.6.37(1).
2) Let $\mathfrak{t}$ be an almost good λ-frame and $\mathfrak{K}_\lambda = \mathfrak{K}_\mathfrak{t}$ and NF be a weak non-forking relation on $\mathfrak{K}_\lambda$. We say that NF respects $\mathfrak{t}$ when: if $\mathrm{NF}(M_0, N_0, M_1, N_1)$ and $(M_0, N_0, a) \in K^{3,\mathrm{bs}}_\mathfrak{t}$ then $\mathbf{tp}_\mathfrak{t}(a, M_1, N_1)$ does not fork over M_0. We say NF is a weak $\mathfrak{t}$-non-forking relation when it is a weak $\mathfrak{t}$-non-forking relation respecting $\mathfrak{t}$.
3) In part (1) we say NF has disjointness when
$\mathrm{WNF}(M_0, N_0, M_1, N_1) \Rightarrow M_0 \cap M_1 = M_0$.
4) We say NF is a pseudo non-forking relation on $\mathfrak{K}_\lambda$ when we have clauses (a)-(f) of Definition II.6.1 or see the proof below. Also here parts (2),(3) are meaningful.

Proof of 7.17. 1) Let us list the conditions on NF := $\mathrm{WNF}_\mathfrak{s}$ being a weak non-forking relation let $\mathfrak{K}_\lambda = \mathfrak{K}_\mathfrak{s}$. We shall use 7.15 freely.

Condition (a): NF is a 4-place relation on $\mathfrak{K}_\lambda$.
[Why? This holds by Definition 7.3(1),(2).]

Condition (b): $\mathrm{NF}(M_0, M_1, M_2, M_3)$ implies $M_0 \le_{\mathfrak{K}_\lambda} M_\ell \le_{\mathfrak{K}_\lambda} M_3$ for $\ell = 1, 0$ and NF is preserved by isomorphisms.
[Why? The preservation by isomorphisms holds by the definition, and also the order demands.]

Condition (c)$_1$: [Monotonicity] If $\mathrm{NF}(M_0, M_1, M_2, M_3)$ and $M_0 \le_{\mathfrak{K}_\lambda} M'_\ell \le_{\mathfrak{K}_\lambda} M_\ell$ for $\ell = 1, 2$ then $\mathrm{NF}(M_0, M'_1, M'_2, M_3)$.
[Why? By 7.6.]

Condition (c)$_2$: [Monotonicity] If $\mathrm{NF}(M_0, M_1, M_2, M_3)$ and $M_3 \le_{\mathfrak{K}_\lambda} M'_3$ and $M_1 \cup M_2 \subseteq M''_3 \le_{\mathfrak{K}_\lambda} M_3$ then $\mathrm{NF}(M_0, M_1, M_2, M''_3)$.
[Why? By Claim 7.6.]

Condition (d): [Symmetry] If $\mathrm{NF}(M_0, M_1, M_2, M_3)$ then $\mathrm{NF}(M_0, M_2, M_1, M_3)$.
[Why? By Claim 7.14.]

Condition (e): [Long Transitivity] If $\alpha < \lambda^+$, $\mathrm{NF}(M_i, N_i, M_{i+1}, N_{i+1})$ for each $i < \alpha$ and $\langle M_i : i \le \alpha\rangle, \langle N_i : i \le \lambda\rangle$ are $\le_{\mathfrak{K}_\lambda}$-increasing continuous sequences then $\mathrm{NF}(M_0, N_0, M_\alpha, N_\alpha)$.
[Why? By Claim 7.8.]

Condition (f): [Existence] Assume $M_0 \le_{\mathfrak{K}_\lambda} M_\ell$ for $\ell = 1, 2$. Then for some M_3, f_1, f_2 we have $M_0 \le_{\mathfrak{K}_\lambda} M_3 \in \mathfrak{K}_\lambda$, f_ℓ is a $\le_{\mathfrak{K}_\lambda}$-embedding for M_ℓ into M_3 over M_0 for $\ell = 1, 2$ and $\mathrm{NF}(M_0, f_1(M_1), f_2(M_2), M_3)$. Here we have the disjoint version, i.e. $f_2(M_1) \cap f_2(M_2) = M_2$.
[Why? By Lemma 7.10(1).]

Condition (g): Lifting or weak uniqueness [a replacement for uniqueness]

This is the content of 7.16.

Thus we have finished presenting the definition of "NF is a weak non-forking relation on $\mathfrak{K}_{\mathfrak{s}}$" and proving that $\mathrm{WNF}_{\mathfrak{s}}$ satisfies those demands.

But we still owe "$\mathrm{WNF}_{\mathfrak{s}}$ respect $\mathfrak{s}$" where NF respect $\mathfrak{s}$ means that if $\mathrm{NF}(M_0, N_0, M_1, N_1)$ and $(M_0, N_0, a) \in K^{3,\mathfrak{s}}_{\mathfrak{s}}$ then $\mathbf{tp}_{\mathfrak{s}}(a, M_1, N_1)$ does not fork over M_0, i.e. $(M_0, N_0, a) \le_{\mathrm{bs}} (M_1, N_1, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$.
[Why? This holds by Observation 7.5(1).]

Also the disjointness of WNF is easy; use 7.7 and categoricity.

Proof of 7.17(2).

So assume $\mathrm{NF}(M_0, N_0, M_1, N_1)$ and we should prove $\mathrm{WNF}_{\mathfrak{s}}(M_0, N_0, M_1, N_1)$ so let $\mathbf{d}$ be as in Definition 7.3(1). As NF satisfies existence, transitivity and monotonicity without loss of generality it suffices to deal with the case $M^{\mathbf{d}}_{0,\alpha(\mathbf{d})} = N_0$.

This holds by the definition of $\mathrm{WNF}_{\mathfrak{s}}$ in 7.3 and clause (g) in the definition of being weak non-forking relation and "respecting $\mathfrak{s}$". $\square_{7.17}$

At last we can get rid of the "almost" in "almost good λ-frame", of course, this is under the Hypothesis 7.1, otherwise we do not know.

7.19 Lemma. *1) If* $\mathfrak{t}$ *is an almost good* λ*-frame and* WNF *is a weak non-forking relation on* $\mathfrak{K}_\lambda$ *respecting* $\mathfrak{t}$ *then* $\mathfrak{t}$ *is a good* λ*-frame.*
2) In 7.1, $\mathfrak{s}$ *is a good* λ*-frame.*

Proof. Part (2) follows from part (1) and 7.17(1) above. So recalling Definition 5.2 we should just prove that $\mathfrak{t}$ satisfies Ax(E)(c). Note that the Hypothesis 7.20 below holds hence we are allowed to use 7.25.

Let $\langle M_i : i \leq \delta \rangle$ be $\leq_{\mathfrak{s}}$-increasing continuous and $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_\delta)$ and we should prove that p does not fork over M_i for some $i < \delta$. By renaming without loss of generality $\delta < \lambda^+$ and δ is divisible by $\lambda^2\omega$ and $\varepsilon \leq \lambda \wedge i < \delta \Rightarrow M_{i+1} = M_{i+1+\varepsilon}$. Let $\mathfrak{u}$ be as in Definition 7.2, so $\mathfrak{u}$ is a nice construction framework. Let $\alpha = \delta, \beta = \delta$.

Now

$(*)_1$ there is $\mathbf{d}$ such that
- (α) $\mathbf{d}$ is a $\mathfrak{u}$-free (α, β)-rectangle
- (β) $\mathbf{d}$ is a strictly brimmed, see Definition 5.15(2)
- (γ) if $i < \alpha$ and $j < \delta$ then WNF$(M^{\mathbf{d}}_{i,j}, M^{\mathbf{d}}_{i,j+1}, M^{\mathbf{d}}_{i+1,j}, M^{\mathbf{d}}_{i+1,j+1})$
- (δ) $M^{\mathbf{d}}_{i,0} = M_i$ for $i \leq \delta$
- (ε) $\mathbf{J}^{\mathbf{d}}_{i,j} = \emptyset$ when $i < \delta, j \leq \delta$ and $\mathbf{I}^{\mathbf{d}}_{i,j} = \emptyset$ when $i \leq \delta, j < \delta$.

This will be done as in the proof of Observation 5.17.

By the properties of WNF (i.e. using twice long transitivity and symmetry)

$(*)_2$ if $i_1 < i_2 \leq \alpha$ and $j_1 < j_2 \leq \beta$ <u>then</u> WNF$(M^{\mathbf{d}}_{i_1,j_1}, M^{\mathbf{d}}_{i_1,j_2}, M^{\mathbf{d}}_{i_2,j_1}, M^{\mathbf{d}}_{i_2,j_2})$ and WNF$(M^{\mathbf{d}}_{i_1,j_1}, M^{\mathbf{d}}_{i_2,j_1}, M^{\mathbf{d}}_{i_1,j_2}, M^{\mathbf{d}}_{i_2,j_2})$.

Now we can choose a $\mathfrak{u}$-free $(\alpha_{\mathbf{d}}, \beta_{\mathbf{d}})$-rectangle $\mathbf{e}$ such that

$(*)_3$ (a) $M^{\mathbf{e}}_{i,j} = M^{\mathbf{d}}_{i,j}$ for $i \leq \alpha_{\mathbf{d}}, j \leq \beta_{\mathbf{d}}$
- (b) if $i < \alpha, j < \beta$ and $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M^{\mathbf{d}}_{i,j})$ then for λ ordinals $\varepsilon < \lambda$ we have:
  - (α) $\mathbf{tp}_{\mathfrak{t}}(b^{\mathbf{d}}_{i+\varepsilon,j}, M^{\mathbf{d}}_{i+\varepsilon,j+1}, M^{\mathbf{d}}_{i+\varepsilon+1,j+1})$ is a non-forking extension of p, recalling $\mathbf{J}^{\mathbf{e}}_{i+\varepsilon,j} = \{b^{\mathbf{e}}_{i+\varepsilon,j}\}$,
  - (β) $\mathbf{tp}_{\mathfrak{t}}(a^{\mathbf{e}}_{i,j+\varepsilon}, M^{\mathbf{d}}_{i+1,j+\varepsilon}, M^{\mathbf{d}}_{i+1,j+\varepsilon+1})$ is a non-forking extension of p, recalling $\mathbf{I}^{\mathbf{e}}_{i,j+\varepsilon} = \{a^{\mathbf{e}}_{i+1,j+\varepsilon}\}$.

[Why? We can choose $\mathbf{e} \restriction (\alpha, \alpha)$ by induction on $\alpha \leq \delta$. The non-forking condition in the definition of $\mathfrak{u}$-free holds because WNF respects $\mathbf{t}$ and $(*)_2$.]

So $\mathbf{d}$ is full (see Definition 5.15(3),(3A), even strongly full) hence by Claim 5.19(3A)(c)

$(*)_4$ $M^{\mathbf{d}}_{\alpha,\beta}$ is brimmed over $M^{\mathbf{d}}_{\alpha,0} = M_\delta$.

Hence $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_\delta) = \mathscr{S}^{\rm bs}_{\mathfrak{t}}(M^{\mathbf{d}}_{\alpha,0})$ is realized in $M_{\alpha,\beta}$ say by $c \in M_{\alpha,\beta}$, so for some $i < \alpha$ we have $c \in M_{i,\beta}$.
As $\mathrm{WNF}(M^{\mathbf{d}}_{i,0}, M^{\mathbf{d}}_{i,\beta}, M^{\mathbf{d}}_{\alpha,0}, M^{\mathbf{d}}_{\alpha,\beta})$ holds by $(*)_2$ above by Claim 7.25 below, it follows that $\mathbf{tp}_{\mathfrak{t}}(c, M^{\mathbf{d}}_{\alpha,0}, M^{\mathbf{d}}_{\alpha,\beta})$ does not fork over $M^{\mathbf{d}}_{i,0}$ which means $i < \delta$ and p does not fork over M_i as required. $\square_{7.19}$

Now for the rest of the section we replace Hypothesis 7.1 by

7.20 Hypothesis. Assume $\mathfrak{s}$ is an almost good λ-frame categorical in λ and WNF is a weak non-forking relation on $\mathfrak{K}_{\mathfrak{s}}$ respecting $\mathfrak{s}$.

The following is related to the proof of 7.19.

7.21 Definition. 1) We say $\mathbf{d} = \langle M_{\alpha,\beta} : \alpha \leq \alpha_{\mathbf{d}}, \beta \leq \beta_{\mathbf{d}} \rangle$ is a WNF-free rectangle (or $(\alpha_{\mathbf{d}}, \beta_{\mathbf{d}})$-rectangle) when:

- (a) $\langle M_{\alpha,j} : j \leq \beta_{\mathbf{d}} \rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous for each $\alpha \leq \alpha_{\mathbf{d}}$
- (b) $\langle M_{i,\beta} : i \leq \alpha_{\mathbf{d}} \rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous for each $\beta \leq \beta_{\mathbf{d}}$
- (c) $\mathrm{WNF}(M_{i,j}, M_{i+1,j}, M_{i,j+1}, M_{i+1,j+1})$ for $i < \alpha, j < \beta$.

2) Let $\bar{\alpha} = \langle \alpha_j : j \leq \beta \rangle$ be $\leq$-increasing.
We say $\mathbf{d} = \langle M_{i,j} : i \leq \alpha_j \text{ and } j \leq \beta \rangle$ is a WNF-free $(\langle \alpha_j : j \leq \beta \rangle, \beta)$-triangle <u>when</u>:

- (a) $\langle M_{i,j} : i \leq \alpha_j \rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous for each $j \leq \beta$
- (b) $\langle M_{i,j} : j \leq \beta$ satisfies $i \leq \alpha_j \rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous for each $i < \alpha_\beta$
- (c) $\mathrm{WNF}(M_{i,j}, M_{i+1,j}, M_{i,j+1}, M_{i+1,j+1})$ for $j < \beta, i < \alpha_j$.

Now we may note that some facts proved in Chapter III for weakly successful good λ-frame can be proved under Hypothesis 7.1 or just 7.20. Systematically see [Sh 842].

7.22 Claim. *$M^{\mathbf{d}}_{\alpha_\beta,\beta}$ is brimmed over $M_{0,\beta}$ when:*

- *(a) $\bar\alpha = \langle \alpha_j : \alpha \le \beta\rangle$ is increasing continuous*
- *(b) $\mathbf{d}$ is a* WNF*-free $(\bar\alpha,\beta)$-triangle*
- *(c) $M_{i+1,j+1}$ is $\le_{\mathfrak{s}}$-universal over $M_{i,j}$ when $i < \alpha_j, j < \beta$.*

Proof. If β and each α_j is divisible by λ we can repeat (part of the) proof of 7.19 and this suffices for proving 7.25 hence for proving 7.19 (no vicious circle!)

In general we have to repeat the proof of 5.19 or first find a WNF-free $(\langle \lambda\alpha_j : j \le \beta\rangle, \beta)$-rectangle $\mathbf{d}'$ which is brimmed and full and $M^{\mathbf{d}'}_{\lambda,j} = M^{\mathbf{d}^+}_{i,j}$ for $i \le \alpha_j, j \le \beta$ and then use the first sentence. $\square_{7.22}$

It is natural to replace $\le_{\rm bs}$ by the stronger $\le_{\rm wnf}$ defined below (and used later).

7.23 Definition. 1) Let $\le_{\rm wnf}$ be the following two-place relation on $K^{2,{\rm bs}}_{\mathfrak{s}} := \{(M,N) : M \le_{\mathfrak{s}} N \text{ are from } \mathfrak{K}_{\mathfrak{s}}\}$, we have $(M_0,N_0) \le_{\rm wnf} (M_1,N_1)$ iff:

- (a) $(M_\ell,N_\ell) \in K^{2,{\rm bs}}_{\mathfrak{s}}$ for $\ell = 0,1$
- (b) WNF(M_0,N_0,M_1,N_1).

2) Let $(M_1,N_1,a) \le_{\rm wnf} (M_2,N_2,a)$ means $(M_\ell,N_\ell,a) \in K^{3,{\rm bs}}_{\mathfrak{s}}$ for $\ell = 1,2$ and $(M_1,N_1) \le_{\rm wnf} (M_2,N_2)$.

7.24 Claim. *1) $\le_{\rm wnf}$ is a partial order on $K^{2,{\rm bs}}_{\mathfrak{s}}$.*
2) If $\langle (M_\alpha,N_\alpha) : \alpha < \delta\rangle$ is $\le_{\rm wnf}$-increasing continuous then $\alpha < \delta \Rightarrow (M_\alpha,N_\alpha) \le_{\rm wnf} (\bigcup\limits_{\beta<\delta} M_\beta, \bigcup\limits_{\beta<\delta} N_\beta) \in K^{3,{\rm bs}}_{\mathfrak{s}}$.
3) If $(M_1,N_1) \le_{\rm wnf} (M_2,N_2)$ and $(M_1,N_1,a) \in \mathfrak{K}^{3,{\rm bs}}_{\mathfrak{s}}$ then (M_1,N_1,a) $\le_{\rm bs} (M_2,N_2,a) \in K^{3,{\rm bs}}_{\mathfrak{s}}$.

4) If $M \le_{\mathfrak{s}} N' \le_{\mathfrak{s}} N$ and $(M,N) \in K^{2,\mathrm{bs}}_{\mathfrak{s}}$ then $(M,N') \le_{\mathrm{wnf}} (M,N)$.
5) If $(M_1,N_1) \le_{\mathrm{wnf}} (M_2,N_2)$ and $M_1 \le_{\mathfrak{s}} N'_1 \le_{\mathfrak{s}} N_1, M_1 \le_{\mathfrak{s}} M'_2 \le_{\mathfrak{s}} M_2$ and $N'_1 \cup M'_2 \subseteq N'_2 \le_{\mathfrak{s}} N''_2$ and $N_2 \le_{\mathfrak{s}} N''_2$ then $(M_1,N'_1) \le_{\mathrm{wnf}} (M'_2,N'_2)$.
6) Similarly to (1),(2),(4),(5) for $(K^{3,\mathrm{bs}}_{\mathfrak{s}}, \le_{\mathrm{wnf}})$.

Proof. Easy by now. $\square_{7.24}$

The following is a "downward" version of "WNF respect $\mathfrak{s}$" which was used in the proof of 7.19.

7.25 Claim. *If* WNF(M_0,N_0,M_1,N_1) *and* $c \in N_0$ *and* $(M_1,N_1,a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ *then* $(M_0,N_0,c) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ *and* $\mathbf{tp}_{\mathfrak{s}}(c,M_1,N_1)$ *does not fork over* M_0.

Proof. The second phrase in the conclusion (about "$\mathbf{tp}_{\mathfrak{s}}(a,M_1,N_1)$ does not fork over M_0") follows from the first by "WNF respects $\mathfrak{t}$"; so if $\mathfrak{s}$ is type full (i.e. $\mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M) = \mathscr{S}^{\mathrm{na}}_{\mathfrak{s}}(M)$) this is easy. (So if $\mathfrak{s}$ is type-full the result is easy.)

Toward contradiction assume

$(*)_0$ (M_0,N_0,M_1,N_1,c) form a counterexample.

Also by monotonicity properties without loss of generality :

$(*)_1$ (a) N_0 is brimmed over M_0
(b) M_1 is brimmed over M_0
(c) N_1 is brimmed over $N_0 \cup M_1$.

Let $\langle \mathscr{U}_\varepsilon : \varepsilon < \lambda_{\mathfrak{s}}\rangle$ be an increasing sequence of subsets of $\lambda_{\mathfrak{s}}$ such that $|\mathscr{U}_0| = |\mathscr{U}_{\varepsilon+1} \backslash \mathscr{U}_\varepsilon| = \lambda_{\mathfrak{s}}$. We now by induction on $\varepsilon \le \lambda$ choose $\mathbf{d}_\varepsilon, \bar{\mathbf{a}}_\varepsilon$ such that $\mathbf{d}_\varepsilon = \langle M_{i,j} : i \le \lambda j, j \le \varepsilon\rangle$ and:

⊛ (a) $\mathbf{d}_\varepsilon$ is a WNF-free triangle
(b) $M_{i+1,j+1}$ is brimmed over $M_{i+1,j} \cup M_{i,j+1}$ when $i < \lambda \wedge j < \varepsilon$
(c) $M_{0,j+1}$ is brimmed over $M_{0,j}$

(d) $\bar{\mathbf{a}}_\varepsilon = \langle a_\alpha : \alpha \in \mathscr{U}_\varepsilon \rangle$ list the elements of $M_{\varepsilon\lambda,\varepsilon}$

(e) if $\varepsilon = j+1$ and $\mathscr{W}_j \subseteq \mathscr{U}_j$ defined below is $\neq \emptyset$ and $\gamma_j = \min(\mathscr{W}_j)$, then $\mathbf{tp}_{\mathfrak{s}}(a_{\gamma_j}, M_{0,\varepsilon}, M_{\lambda j,\varepsilon}) \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_{0,\varepsilon})$ and $[\mathbf{tp}_{\mathfrak{s}}(a_{\gamma_j}, M_{0,j}, M_{\lambda j,j}) \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_{0,j}) \Rightarrow \mathbf{tp}_{\mathfrak{s}}(a_{\gamma_j}, M_{0,\varepsilon}, M_{\lambda j,\varepsilon})$ forks over $M_{0,j}]$ where $\mathscr{W}_j = \{\alpha \in \mathscr{U}_i$: we can find M', N' such that $\mathrm{WNF}(M_{0,j}, M_{\lambda j,j}, M', N')$ and $\mathbf{tp}_{\mathfrak{s}}(a_\alpha, M', N') \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M')$ but is not a non-forking extension of $\mathbf{tp}_{\mathfrak{s}}(a, M_{0,j}, M_{\lambda j,j})$, e.g. $\mathbf{tp}_{\mathfrak{s}}(a, M_{0,j}, M_{\lambda j,j}) \notin \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_{0,j})\}$.

There is no problem to carry the definition.

Now by 7.22

$(*)_2$ $M_{\lambda\lambda,\lambda}$ is brimmed over $M_{0,\lambda}$.

So by $(*)_1 + (*)_2$ and $\mathfrak{s}$ being categorical without loss of generality

$(*)_3$ $(M_0, N_0) = (M_{0,\lambda}, M_{\lambda\lambda,\lambda})$.

Clearly $M_{\lambda\lambda,\lambda}$ is the union of the $\leq_{\mathfrak{s}}$-increasing continuous chain $\langle M_{\lambda j,j} : j \leq \lambda \rangle$ hence $j_1(*)$ is well defined where:

$(*)_4$ $j_1(*) = \min\{j < \lambda : c \in M_{\lambda j,j}\}$.

By clause (d) of $\circledast$ for some $j(*)$ we have

$(*)_5$ $j(*) \in [j_1(*), \lambda)$ and $c \in \{a_\alpha : \alpha \in \mathscr{U}_{j(*)}\}$.

So by the choice of $j(*), \gamma(*)$ is well defined where

$(*)_6$ $\gamma(*) = \min\{\gamma \in \mathscr{U}_{j(*)} : a_\gamma = c\} < \lambda$.

Note that

$(*)_7$ $(M_{0,j}, M_{\lambda j,j}) \leq_{\rm wnf} (M_{0,\lambda}, M_{\lambda\lambda,\lambda}) = (M_0, N_0) \leq_{\rm wnf} (M_1, N_1)$ for $j \in [j(*), \lambda)$.

Also

$(*)_8$ if $j \in [j(*), \lambda)$ then $\mathbf{tp}_{\mathfrak{s}}(c, M_{0,j}, M_{\lambda j,j}) \notin \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_{0,j})$.

[Why? By $(*)_7$, we have $\mathrm{WNF}_{\mathfrak{s}}(M_{0,j}, M_{\lambda j,j}, M_{0,\lambda}, M_{\lambda\lambda,\lambda})$ hence if $(\mathbf{tp}_{\mathfrak{s}}(c, M_{0,j}, M_{\lambda j,j}) \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_{0,j})$ then $(M_{0,j}, M_{\lambda j,j}, c) \in K^{3,\rm bs}_{\mathfrak{s}}$ but by $(*)_7$ we have $\mathrm{WNF}(M_{0,j}, M_{\lambda j,j}, M_1, N_1)$ and WNF respects $\mathfrak{s}$ hence $\mathbf{tp}_{\mathfrak{s}}(c, M_1, N_1)$ does not fork over $M_{0,j}$, so by monotonicity it does not fork over $M_{0,\lambda} = M_0$ and this contradicts $(*)_0$.]

$(*)_9$ if $j \in [j(*), \lambda)$ then $\min(\mathscr{W}_j) \le \gamma(*)$.

[Why? As $\gamma(*) \in \mathscr{W}_j$, i.e. satisfies the requirement which appear in clause (e) of $\circledast$ that is (M_1, N_1) here can stand for (M', N') there.]

So by cardinality considerations for some $j_1 < j_2$ from $[j(*), \lambda)$ we have $\min(\mathscr{W}_{j_1}) = \min(\mathscr{W}_{j_2})$ but this gives a contradiction as in the proof of $(*)_8$. $\square_{7.25}$

<u>7.26 Exercise</u>: Show that in Hypothesis 7.20 we can omit "$\mathfrak{s}$ is categorical (in λ)".

[<u>Hint</u>: The only place it is used is in showing $(*)_3$ during the proof of 7.25. To avoid it in $\circledast$ there waive clause (c) and add $j \le \lambda \Rightarrow M_{0,j} = M_0$.]

<u>7.27 Exercise</u>: Show that in this section we can replace clause (c) of Hypothesis 7.1, i.e. "$\mathfrak{s}$ is categorical in λ" by

$(c)'$ $\dot{I}(K_{\mathfrak{s}}) \le \lambda$
or just

$(c)''$ $\xi_{\mathfrak{s}} = \sup\{\xi_M : M \in K_{\mathfrak{s}}\} < \lambda^+$ where for $M \in K_{\mathfrak{s}}$ we let $\xi_N = \min\{\alpha_{\mathbf{d}}$: there is a $\mathfrak{u}$-free $(\alpha_{\mathbf{d}}, 0)$-rectangle such that $M^{\mathbf{d}}_{0,0} = M, (M^{\mathbf{d}}_{i,0}, M^{\mathbf{d}}_{i,0}, a^{\mathbf{d}}_{i,0}) \in K^{3,\mathrm{up}}_{\mathfrak{s}}$ and $M^{\mathbf{d}}_{\alpha(\mathbf{d}),0}$ is universal over $M^{\mathbf{d}}_{0,0} = M\}$.

[Hint: The only place we use "$\mathfrak{s}$ is categorical in λ" is in claim 7.14 more fully in 7.10(2) we would like to have a bound $< \lambda^+$ on $\alpha_{\mathbf{d}}$ not depending on M_0, see 7.11 (and then quoting it). It is used to define $\xi_{\mathfrak{s}}$. As $(c)' \Rightarrow (c)''$ without loss of generality we assume $(c)''$.]

<u>7.28 Exercise</u>: (Brimmed lifting, compare with III.1.17(4).)
1) For any $\le_{\mathfrak{s}}$-increasing continuous sequence $\langle M_\alpha : \alpha \le \alpha(*)\rangle$ with $\alpha(*) < \lambda^+_{\mathfrak{s}}$ we can find $\bar{N}$ such that

$\circledast$ (a) $\bar{N} = \langle N_\alpha : \alpha \le \alpha(*)\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous (*b*) WNF$(M_\alpha, N_\alpha, M_\beta, N_\beta)$ for $\alpha < \beta \leq \alpha(*)$

(*c*) N_α is brimmed over M_α for $\alpha = 0$ and moreover for every $\alpha \leq \alpha(*)$

(*d*) $N_{\alpha+1}$ is brimmed over $M_{\alpha+1} \cup N_\alpha$ for $\alpha < \alpha(*)$.

2) In fact the moreover in clause (c) follows from (a),(b),(d); an addition

$(c)^+$ N_β is brimmed over $N_\alpha \cup M_\beta$ for $\alpha < \beta \leq \alpha(*)$.

[Hint: Similar to 5.19 or 5.20 or note that by 7.19 we can quote Chapter III.]

§8 Density of $K^{3,\mathrm{uq}}_{\mathfrak{s}}$ for good λ-frames

We shall prove non-structure from failure of density for $K^{3,\mathrm{uq}}_{\mathfrak{s}}$ in two rounds. First, in 8.6 - 8.10 we prove the wnf-delayed version. Second, in 8.14 - 8.17 - we use its conclusion to prove the general case. Of course, by §6, we can assume as in §7:

8.1 Hypothesis. *We assume (after 8.2)*

(*a*) $\mathfrak{s}$ *is an almost*[28] *good λ-frame*

(*b*) WNF *is a weak non-forking relation on $K_{\mathfrak{s}}$ respecting $\mathfrak{s}$ with disjointness*
(not necessarily the one from Definition 7.3, but Hypothesis 7.20 holds).

We can justify Hypothesis 8.1 by

8.2 Observation. Instead clause (b) of 8.1 we can assume

$(b)'$ $\mathfrak{s}$ has existence for $K^{3,\mathrm{up}}_{\mathfrak{s},\lambda^+}$ (so we may use the consequences of Conclusion 7.17)

(*c*) $\mathfrak{s}$ is categorical in λ.

[28]by 7.17 clauses (b) + (c) implies that $\mathfrak{s}$ is actually a good λ-frame but we may ignore this

Proof. Why? First note that Hypothesis 7.1 holds:

<u>Part (1)</u> there, $\mathfrak{s}$ is an almost good λ-frame, is clause (a) of Hypothesis 8.1.

<u>Part (2)</u> there is clause (b)$'$ assumed above.

<u>Part (3)</u>, categoricity in λ, there is clause (c) above.

<u>Part (4)</u> there, disjointness of $\mathfrak{s}$, holds by 5.23.

So the results of §7 holds, in particular the relation WNF := $\mathrm{WNF}_{\mathfrak{s}}$ defined in Definition 7.3 is a weak non-forking relation (on $K_{\mathfrak{s}}$) respecting $\mathfrak{s}$, by Claim 7.17.

$\square_{8.2}$

We now use a relative of $\mathfrak{u}^1_{\mathfrak{s}}$ from Definition 4.29; this will be the default value of $\mathfrak{u}$ in this section so ∂ will be $\partial_{\mathfrak{u}} = \lambda^+$.

8.3 Definition. For $\mathfrak{s}$ as in 8.1 we define $\mathfrak{u} = \mathfrak{u}^3_{\mathfrak{s}}$ as follows:

(*a*) $\partial_{\mathfrak{u}} = \lambda^+ (= \lambda^+_{\mathfrak{s}})$

(*b*) $\mathfrak{K}_{\mathfrak{u}} = \mathfrak{K}_{\mathfrak{s}}$ (or $\mathfrak{K}'_{\mathfrak{s}}$ see 4.25, 4.26; but not necessary by (c) of 8.1)

(*c*) $\mathrm{FR}_\ell = \{(M,N,\mathbf{J}) : M \le_{\mathfrak{s}} N$ and $\mathbf{J} = \emptyset$ or $\mathbf{J} = \{a\}$ and $(M,N,a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}\}$

(*d*) $\le^\ell_{\mathfrak{u}}$ is defined by $(M_0,N_0,\mathbf{J}_0) \le^\ell_{\mathfrak{u}} (M_1,N_1,\mathbf{J}_1)$ <u>when</u>

- (α) $\mathrm{WNF}(M_0,N_0,M_1,N_1)$
- (β) $\mathbf{J}_1 \subseteq \mathbf{J}_0$
- (γ) if $\mathbf{J}_0 = \{a\}$ then $\mathbf{J}_1 = \{a\}$ hence $(M_0,N_0,a) \le^{\mathrm{bs}}_{\mathfrak{s}} (M_1,N_1,a)$ by "WNF respects $\mathfrak{s}$", see Hypothesis 8.1 and Definition 7.18; if we use $\mathrm{WNF}_{\mathfrak{s}}$ then we can quote 7.17(2), also by 7.5(1).

8.4 Remark. 0) The choice in 8.3 gives us symmetry, etc., i.e. $\mathfrak{u}$ is self-dual, this sometimes helps.
1) We could define $\mathrm{FR}_1, \le_1$ as above but

(*e*) $\mathrm{FR}_2 = \{(M, N, \mathbf{J}) : M \le_{\mathfrak{s}} N$ and $\mathbf{J} = \emptyset$ or $\mathbf{J} = \{a\}, (M, N, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}\}$

(*f*) $\le_2$ is defined by:
$(M_0, N_0, \mathbf{J}_0) \le_1 (M_1, N_1, \mathbf{J}_1)$ <u>when</u> (both are from FR_1 and)

(α) $M_0 \le_{\mathfrak{s}} M_1$ and $N_0 \le_{\mathfrak{s}} N_1$

$(\beta)'$ $\mathbf{J}_0 \subseteq \mathbf{J}_1$

$(\beta)''$ if $\mathbf{J}_0 = \{a\}$ then $\mathbf{J}_1 = \{a\}$ and $(M_0, N_0, a) \le^{\mathrm{bs}}_{\mathfrak{s}} (M_1, N_1, a)$.

2) We call it $\mathfrak{u}^{3,*}_{\mathfrak{s}}$. However, then for proving 8.14 we have to use $\mathfrak{u}^{*,3}_{\mathfrak{s}}$ which is defined similarly interchanging $(\mathrm{FR}_1, \le_1)$ with $(\mathrm{FR}_2, \le_2)$. Thus we lose "self-dual".

8.5 Claim. *1) $\mathfrak{u}$ is a nice construction framework which is self-dual.*
2) For almost$_2$ every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\mathrm{qt}}_{\mathfrak{s}}$ the model M_∂ is saturated above λ.
3) $\mathfrak{u}$ is monotonic (see 1.13(1)), hereditary (see 3.17(12)), hereditary for $=_*$ if $=_*$ is a fake equality for $\mathfrak{s}$ (see 4.25) and has interpolation (see 3.21).

Proof. 1) As in earlier cases (see 4.30(1)), 5.11(1)).
2) As in 4.30(2) or 5.11(2).
3) check. $\square_{8.5}$

8.6 Theorem. *We have $\dot{I}(\lambda^{++}, K^{\mathfrak{s}}(\lambda^+\text{-saturated})) \ge \mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$ and even $\dot{I}(K^{\mathfrak{u},\mathfrak{h}}_{\lambda^{++}}) \ge \mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$ for any $\mathfrak{u} - \{0,2\}$-appropriate function $\mathfrak{h}$ <u>when</u>:*

(*a*) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$

(*b*) $\mathfrak{u}$ *fails wnf-delayed uniqueness for* WNF *see Definition 8.7 below.*

Remark. Note that we have some versions of delayed uniqueness: the straight one, the one with WNF and the one in §5 and more.

Before we prove Theorem 8.6

8.7 Definition. We say that (the almost good λ-frame) $\mathfrak{s}$ has wnf-delayed uniqueness for WNF <u>when</u>: (if WNF is clear from the context we may omit it)

⊠ for every $(M_0, N_0, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ we can find (M_1, N_1) such that

(a) $(M_0, N_0, a) \leq^1_{\mathfrak{u}} (M_1, N_1, a)$, i.e. $(M_0, N_0, a) \leq_{\mathrm{bs}} (M_1, N_1, a)$ and $\mathrm{WNF}(M_0, N_0, M_1, N_1)$, see clause (b) of Hypothesis 8.1 and

(b) if $(M_1, N_1, a) \leq^1_{\mathfrak{u}} (M_\ell, N_\ell, a)$ hence $\mathrm{WNF}(M_1, N_1, M_\ell, N_\ell)$ for $\ell = 2, 3$ and $M_2 = M_3$ <u>then</u> N_2, N_3 are $\leq_{\mathfrak{s}}$-compatible over $M_2 \cup N_0$, that is we can find a pair (f, N') such that

  (α) $N_3 \leq_{\mathfrak{s}} N'$

  (β) f is a $\leq_{\mathfrak{s}}$-embedding of N_2 ito N'

  (γ) f is the identity on N_0 (not necessarily N_1!) and on $M_2 = M_3$.

Remark. A point of 8.7 is that we look for uniqueness among $\leq^1_{\mathfrak{u}}$-extensions (if 8.2 apply then $\leq_{\mathrm{wnf}}$-extensions) and, of course, it is "delayed", i.e. possibly $M_0 \neq M_1$.

8.8 Observation. In Definition 8.7 we can without loss of generality demand that M_1 is brimmed over M_0. Hence M_1 can be any pregiven $\leq_{\mathfrak{s}}$-extension of M_0 brimmed over it such that $M_1 \cap N_0 = M_0$.

Proof. Read the definition. $\square_{8.8}$

8.9 Claim. *If $\mathfrak{s}$ (satisfies 8.1) and fails wnf-delayed uniqueness for* WNF *(i.e. satisfies 8.6(b))* <u>*then*</u> $\mathfrak{u} = \mathfrak{u}^3_{\mathfrak{s}}$ *has vertical coding, see Definition 2.9.*

Proof. Straight. $\square_{8.9}$

Proof of 8.6. Straight by the above and Theorem 2.11. $\square_{8.9}$

8.10 Remark. Note that the assumption of 8.9, failure of wnf-delayed uniqueness may suffice for a stronger version of 8.6 because given $\eta \in {}^{\partial^+}2$ and $(\bar{M}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta) \in K^{\rm qt}_{\mathfrak{s}}$, we can find 2^∂ extensions $\langle(\bar{M}^{\eta,\rho}, \bar{\mathbf{J}}^{\eta,\rho}, \mathbf{f}^{\eta,\rho}) : \rho \in {}^\omega 2\rangle$ and $\alpha(*) < \partial$ such that $M^{\eta,\rho}_{\alpha(0)} = N_*$ and $\langle M^{\eta,\rho}_\partial : \rho \in {}^\partial 2\rangle$ are pairwise non-isomorphic over $M^\eta_\partial \cup N_*$. Does this help to omit the assumption $2^\lambda < 2^{\lambda^+}$?

8.11 Definition. 1) We say $\mathfrak{s}$ has uniqueness for WNF <u>when</u>:

if $\mathrm{WNF}(M^k_0, M^k_1, M^k_2, M^k_3)$ for $k = 1, 2$ and f_ℓ is an isomorphism from M^1_ℓ onto M^2_ℓ for $\ell = 0, 1, 2$ and $f_0 \subseteq f_1, f_0 \subseteq f_2$ <u>then</u> there is a pair (N, f) such that $M^2_3 \le_{\mathfrak{s}} N$ and f is a $\le_{\mathfrak{s}}$-embedding of M^1_3 into N extending $f_1 \cup f_2$.

2) We say $(M_0, M_1, a) \in K^{3,\rm bs}_{\mathfrak{s}}$ has non-uniqueness for WNF <u>when</u>: if $(M_0, M_1, a) \le^1_{\mathfrak{u}} (M'_0, M'_1, a)$ <u>then</u> we can find $\langle M^k_\ell : \ell \le 3, k = 1, 2\rangle, \langle f_\ell : \ell \le 2\rangle$ such that

⊛ (a) $(M^k_0, M^k_1, a) \le^1_{\mathfrak{u}} (M^k_2, M^k_3, a)$ for $k = 1, 2$

(b) $M^k_0 = M'_0, M^k_1 = M'_1$ for $k = 1, 2$

(c) $M^1_2 = M^2_2$ and f_2 is the identity on M^2_1

(d) there is no pair (N, f) such that $M^2_3 \le_{\mathfrak{s}} N$ and f is a $\le_{\mathfrak{s}}$-embedding of M^1_3 into N extending $\mathrm{id}_{M^k_1} \cup \mathrm{id}_{M^k_2}$ (which does not depend on k).

3) We say that $\mathfrak{s}$ has non-uniqueness for WNF when some triple $(M_0, M_1, a) \in K^{3,\rm bs}_{\mathfrak{s}}$ has it.

8.12 Observation. 1) Assume $\mathfrak{s}$ is categorical (in λ). <u>Then</u> $\mathfrak{s}$ has non-uniqueness for WNF <u>iff</u> it does not have uniqueness for WNF <u>iff</u> $\mathfrak{s}$ fails existence for $K^{3,\rm ur}_{\mathfrak{s},\mathfrak{u}}$, see below.

2) If $\mathfrak{s}$ is categorical (in λ), has existence for $K^{3,\rm up}_{\mathfrak{s}} = K^{3,\rm up}_{\mathfrak{s},\lambda^+}$ and has uniqueness for WNF and 7.9 holds for WNF (i.e. if $(M_1, N_1, a) \in K^{3,\rm up}_{\mathfrak{s}}$ and $(M_1, N_1, a) \le_{\rm bs} (M_2, N_2, a)$ then[29] $\mathrm{WNF}(M_1, M_1, N_1, N_2)$)

[29]so really WNF = $\mathrm{WNF}_{\mathfrak{s}}$

then $\mathfrak{s}$ has existence for $K^{3,\mathrm{uq}}_{\mathfrak{s}}$ and $K^{3,\mathrm{up}}_{\mathfrak{s}} \subseteq K^{3,\mathrm{uq}}_{\mathfrak{s}}$.
3) $K^{3,\mathrm{uq}}_{\mathfrak{s}} \subseteq K^{3,\mathrm{up}}_{\mathfrak{s}}$.
4) If $\mathfrak{s}$ has uniqueness for WNF then WNF is a non-forking relation on $\mathfrak{K}_\lambda$ respecting $\mathfrak{s}$.

Remark. Note that 8.12(3) is used in §4(F).

8.13 Definition. 1) Let $K^{3,\mathrm{ur}}_{\mathfrak{s},\mathfrak{u}}$ be the class of triples $(M,N,a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ such that: if $(M,N,a) \le^1_{\mathfrak{u}} (M',N'_\ell,a)$ for $\ell = 1,2$ then we can find a pair (N^*,f) such that $N'_2 \le_{\mathfrak{s}} N^*$ and f is a $\le_{\mathfrak{s}}$-embedding of N'_1 into N^* extending $\mathrm{id}_N \cup \mathrm{id}_{M'}$.
2) $\mathfrak{s}$ has existence for $K^{3,\mathrm{ur}}_{\mathfrak{s}}$ when: if $M \in K_{\mathfrak{s}}$ and $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ then for some pair (N,a) the triple $(M,N,a) \in K^{3,\mathrm{ur}}_{\mathfrak{s}}$ realizes p.
3) If WNF is $\mathrm{WNF}_{\mathfrak{s}}$ and $\mathfrak{u}$ is defined as in 8.7 above then we may omit $\mathfrak{u}$.

Proof. 1) By the definition

- $(*)$ if $\mathfrak{s}$ has non-uniqueness for WNF then $\mathfrak{s}$ does not have uniqueness for $\mathfrak{s}$.

Now

Case 1: $\mathfrak{s}$ fails existence for $K^{3,\mathrm{ur}}_{\mathfrak{s}}$.

We shall show that $\mathfrak{s}$ has the non-uniqueness property; this suffices by $(*)$. Let (M,p) exemplify it and let (N,a) be such that $(M,N,a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ realizes p and we shall prove that (M,N,a) is as required in Definition 8.11(2).

Let $(M',N',a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ be $\le^1_{\mathfrak{u}}$-above (M,N,a). We can find (f_*,N_*,a_*) such that $(M,N_*,a_*) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ realizes p and f_* is an isomorphism from N' onto N_* which maps M' onto M and a to a_*. [Why it exists? See 5.20 recalling $\mathfrak{s}$ is categorical.] So $(M,N_*,a_*) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ and $\mathbf{tp}(a^*,M,N_*) = p$ hence by the choice of M and p clearly $(M,N_*,a_*) \notin K^{3,\mathrm{ur}}_{\mathfrak{s},\mathfrak{u}}$ so by Definition 8.13(1) we can find M',N_1,N_2 as there such that there are no (N^*,f) as there. But this means that for (M,N_*,a_*) we can find $\langle M^k_\ell : \ell \le 3, k = 1,2\rangle$ as required in 8.11(2). By chasing maps this holds also for (M', N', a) so we are done.

Case 2: $\mathfrak{s}$ has existence for $K^{3,\mathrm{ur}}_{\mathfrak{s}}$.

We shall show that $\mathfrak{s}$ has uniqueness for WNF. This suffices by $(*)$.

First note

⊠ if $(*)_1$ and $(*)_2$ below then M^1_3, M^2_3 are isomorphic for $M_2 \cup M_1$ when:

$(*)_1$ (a) $\bar{M} = \langle M_{0,\alpha} : \alpha \leq \alpha(*)\rangle$ is $<_{\mathfrak{s}}$-increasing continuous

(b) $(M_{0,\alpha}, M_{0,\alpha+1}, a_\alpha) \in K^{3,\mathrm{ur}}_{\mathfrak{s},\mathfrak{u}}$ for $\alpha < \alpha(*)$

(c) $M_{0,\alpha(*)}$ is brimmed over $M_{0,0}$

$(*)_2$ (α) $\mathrm{WNF}(M_0, M_1, M_2, M^k_3)$ for $k = 1, 2$

(β) $M_0 = M_{0,0}$ and $M_1 = M_{0,\delta}$

(γ) M^k_3 is brimmed over $M_1 \cup M_2$.

[Why? As in previous arguments in §7, we lift $\bar{M}$ by $(*)_2(\alpha)$ and clause (g) of Definition 7.18 of "WNF is a weak non-forking relation on $\mathfrak{K}_{\mathfrak{s}}$", i.e. being as in the proof of 7.17 and then use $(*)_1(b)$.]

Next

⊞ we can weaken $(*)_2(\beta)$ to $M_{0,0} = M_0 \leq_{\mathfrak{s}} M_1 \leq_{\mathfrak{s}} M_{0,\alpha(*)}$.

[By the properties of WNF.]

Checking the definitions we are done recalling 7.10(2B), or pedantically repeating its proof to get $\langle M_{0,i} : i \leq \alpha(*)\rangle$ as in $(*)_1$.

2) We are assuming that $\mathfrak{s}$ has existence for $K^{3,\mathrm{up}}_{\mathfrak{s}}$ so it suffices to prove $K^{3,\mathrm{up}}_{\mathfrak{s}} \subseteq K^{3,\mathrm{uq}}_{\mathfrak{s}}$. So assume $(M_0, N_0, a) \in K^{3,\mathrm{up}}_{\mathfrak{s}}$ and $(M_0, N_0, a) \leq_{\mathrm{bs}} (M_\ell, N_\ell, a)$ for $\ell = 2, 3$ and $M_2 = M_3$. By 7.9 it follows that $\mathrm{WNF}(M_0, N_0, M_\ell, N_\ell)$ so by Definition 8.3 we have $(M_0, N_0, a) \leq^1_{\mathfrak{u}} (M_\ell, N_\ell, a)$. Applying Definition 8.11(1) we are done.

3),4) Clear by the definitions. $\square_{8.12}$

* * *

8.14 Theorem.
$\dot I(\lambda^{++},K^{\mathfrak{s}}) \ge \dot I(\lambda^{++},K^{\mathfrak{s}}(\lambda^+\text{-saturated})) \ge \mu_{\mathrm{unif}}(\lambda^{++},2^{\lambda^+})$ and even $\dot I(K^{\mathfrak{u},\mathfrak{h}}_{\lambda^{++}}) \ge \mu_{\mathrm{unif}}(\lambda^{++},2^{\lambda^+})$ for any $\mathfrak{u}_{\mathfrak{s}} - \{0,2\}$-appropriate $\mathfrak{h}$ (so we can restrict ourselves to models λ^+-saturated above λ and if $\mathfrak{s} = \mathfrak{s}'$ also to $\tau_{\mathfrak{s}}$-fuller ones) when:

(a) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^+}$

(b) *$\mathfrak{s}$ has non-uniqueness for* WNF *(for every $M \in K_{\mathfrak{s}}$)*

(c) *$\mathfrak{s}$ has* wnf-*delayed uniqueness for* WNF.

Proof. We first prove claim 8.16.

Note that proving them we can use freely 8.7, 8.3, 8.8 and that wnf-delayed uniqueness replaces the use of 10.7.

8.15 Explanation: As $\mathrm{FR}^{\mathfrak{u}}_1 = \mathrm{FR}^{\mathfrak{u}}_2$ there is symmetry, i.e. $\mathfrak{u}$ is self-dual. The wnf-delayed uniqueness was gotten vertically, i.e. from its failure we got a non-structure result (8.6) relying on vertical coding, i.e. 2.11. But now we shall use it horizontally; we shall construct over $(\bar M,\bar{\mathbf{J}},\mathbf{f})$ with $M_\partial \in K^{\mathfrak{s}}_{\lambda^+}$ saturated above λ, a tree $\langle(M^\rho,\bar{\mathbf{J}}^\rho,\mathbf{f}^\ell) : \rho \in {}^{\partial\ge}2\rangle$ as in weak coding but each is not as usual but a sequence of length $\ell g(\rho)$ such extensions. In fact we use the λ-wide case of §10, i.e. 10.14, 10.15 without quoting. So the "non-structure" is done in the "immediate successor" of $(\bar M,\bar{\mathbf{J}},\mathbf{f})$. The rest of the section is intended to make the rest of the construction, in the ∂^+-direction, irrelevant (well, mod $\mathscr{D}_\partial$, etc) using the wnf-delayed uniqueness assumed in clause (c) of 8.14, justified by 8.6. The net result is that we can find $\langle(\bar M^\rho,\bar{\mathbf{J}}^\rho,\mathbf{f}^\rho) : \rho \in {}^\partial 2\rangle$ which are $\le^{\mathrm{qt}}_{\mathfrak{u}}$-above $(\bar M,\bar{\mathbf{J}},\mathbf{f})$ and for $\rho \ne \nu \in {}^\partial 2$, there is no $\le_{\mathfrak{K}}$-embedding of M^ρ_∂ into M'_∂ if $(\bar M^\nu,\mathbf{J}^\nu,\mathbf{f}^\nu) \le^{\mathrm{qs}}_{\mathfrak{u}} (\bar M',\bar{\mathbf{J}}',\mathbf{f}')$. That comes instead of using $\bar{\mathbb{F}}$, the amalgamation choice functions in §10.

For constructing $\langle(\bar M^\rho,\bar{\mathbf{J}}^\rho,\mathbf{f}^\rho) : \rho \in {}^\partial 2\rangle$ as above, again we use $\langle(\bar M^{\rho,\alpha},\bar{\mathbf{J}}^{\rho,\alpha},\mathbf{f}^{\rho,\alpha}) : \rho \in {}^i 2, \alpha < \lambda\rangle$ for $i \le \partial$ such that for a club of $\delta < \partial$ the model $\cup\{M^{\rho,\alpha}_\delta : \alpha < \delta\}$ is brimmed over $M^{\rho,\beta}$ for $\beta < \gamma$.

8.16 Claim. *[Under the assumptions of 8.14] If* $\boxtimes$ *then* $\circledast$, *where*

$\boxtimes$ (a) $(M, N, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ *has non-uniqueness for* $\mathrm{WNF}_{\mathfrak{s}}$

(b) $\delta < \lambda^+$ *is divisible by* λ^3

(c) $\mathbf{d}$ *is a* $\mathfrak{u}$*-free* $(0,\delta)$*-rectangle, let* $M_\alpha = M^{\mathbf{d}}_{0,\alpha}$ *for* $\alpha \le \delta, a_\alpha = a^{\mathbf{d}}_{0,\alpha}$ *for* $\alpha < \delta$ *and* $a = a_0$

(d) $(M, N, a) \le_{\mathrm{wnf}} (M_0, M_1, a)$ *equivalently* $(M, N, \{a\}) \le^1_{\mathfrak{u}} (M_0, M_1, \mathbf{I}^{\mathbf{d}}_{0,0})$

(e) M_δ *is brimmed over* M_α *for* $\alpha < \delta$

(f) $\delta \in \mathrm{correct}(\langle M_\alpha : \alpha \le \delta \rangle)$, *i.e. if* $M_\delta <_{\mathfrak{s}} N$ *then some* $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_\delta)$ *is realized in* N *and does not fork over* M_β *for some* $\beta < \delta$ *(on correctness, see Definition 5.14)*

(g) $(M_0, M', b) \in \mathrm{FR}_2$ *and* $M' \cap M_\delta = M_0$ *and* M' *is brimmed over* M_0

$\circledast$ *there are* $\mathbf{d}_1, \mathbf{d}_2$ *such that*

(α) $\mathbf{d}_\ell$ *is a* $\mathfrak{u}$*-free* $(1, \delta+1)$*-rectangle for* $\ell = 1, 2$

(β) $b^{\mathbf{d}}_{0,0} = b$

(γ) $\mathbf{d}_\ell \restriction (0, \delta) = \mathbf{d}$ *for* $\ell = 1, 2$

(δ) $\mathbf{d}_1 \restriction (1, 0) = \mathbf{d}_2 \restriction (1, 0)$

(ε) $M^{\mathbf{d}_1}_{1,0} = M^{\mathbf{d}_2}_{1,0} = M'$ *and* $b^{\mathbf{d}_1}_{0,0} = b = b^{\mathbf{d}_2}_{0,0}$

(ζ) $M^{\mathbf{d}_1}_{\alpha,\delta}, M^{\mathbf{d}_2}_{\alpha,\delta}$ *are* $\tau_{\mathfrak{s}}$*-incompatible over* $(M^{\mathbf{d}_1}_{1,0} = M^{\mathbf{d}_2}_{1,0}) + M_1$

(η) *if* $k, \mathbf{d}_1, \mathbf{d}_2, f$ *satisfies* $\bullet_1 - \bullet_4$ *below, then we can find a triple* (g, N_1, N_2) *such that* $\ell = 1, 2 \Rightarrow M^{\mathbf{d}_\ell}_{1,\alpha(\mathbf{d}_\ell)} \le_{\mathfrak{s}} N_\ell$ *and* g *is an isomorphism from* N_1 *onto* N_2 *extending* $id_{M'} \cup f$ *where (for* $\ell = 1, 2$*):*

- $\bullet_1$ $\mathbf{d}_\ell$ *is a* $\mathfrak{u}$*-free rectangle*
- $\bullet_2$ $\beta(\mathbf{d}_\ell) = 1$
- $\bullet_3$ $\alpha(\mathbf{d}_\ell) \ge \delta$ *and* $\mathbf{d}_\ell \restriction (0, \delta) = \mathbf{d}$
- $\bullet_4$ f *is an isomorphism from* $M^{\mathbf{d}_1}_{0,\alpha(\mathbf{d}_1)}$ *onto* $M^{\mathbf{d}_2}_{0,\alpha(\mathbf{d}_2)}$ *over* $M^{\mathbf{d}}_{0,\delta}$.

Remark. 1) In the proof we use wnf-delayed uniqueness.
2) This claim helps.

Proof. First, letting $M_* = M'$, we can choose M^1_*, M^2_* such that (for $\ell = 1,2$)

- $\circledast_1$ $M_* \leq_{\mathfrak{s}} M^\ell_*$
- $\circledast_2$ $M_1 \leq_{\mathfrak{s}} M^\ell_*$
- $\circledast_3$ $M^\ell_* \cap M_\delta = M_1$
- $\circledast_4$ $\mathrm{WNF}(M_0, M_1, M_*, M^\ell_*)$ hence $(M_0, M_*, b) \leq_{\mathrm{bs}} (M_1, M^\ell_*, b)$
- $\circledast_5$ M^1_*, M^2_* are τ-incompatible over $M_* + M_1$.

[Why? By $\boxtimes(a)$ and $\boxtimes(g)$ recalling Definition 8.11(2).]

Second, we choose N^ℓ_* for $\ell = 1,2$ such that

- $\circledast_6$ $\mathrm{WNF}(M_1, M_\delta, M^\ell_*, N^\ell_*)$ for $\ell = 1,2$
- $\circledast_7$ wnf-delayed uniqueness: if $\ell \in \{1,2\}$ and $M_{\delta+1}, N^1_{\delta+1}, N^2_{\delta+1}$ satisfies $\mathrm{WNF}(M_\delta, M_{\delta+1}, N^\ell_*, N^k_{\delta+1})$ for $k = 1,2$ <u>then</u> we can find (f, N) such that $N^2_{\delta+1} \leq_{\mathfrak{s}} N$ and f is a $\leq_{\mathfrak{s}}$-embedding of $N^1_{\delta+1}$ into N over M^1_* (hence over M_*) and over $M_{\delta+1}$.

[Why is this possible? As M_δ is brimmed over M_1 by cluase (e) of $\boxtimes$ we are assuming, and $\mathfrak{s}$ has wnf-delayed uniqueness by clause (c) of Theorem 8.14 and we apply it $(M_1, M^\ell_*, b) \leq_2 (M_\delta, N^\ell_*, b)$ recalling $\mathfrak{u}$ is self-dual and 8.8.]

Note that in $\circledast_6$ we can replace N^ℓ_* by N^ℓ_{**} if $N^\ell_* \leq_{\mathfrak{s}} N^\ell_{**}$ or $M^\ell_* \cup M_\delta \subseteq N^\ell_{**} \leq_{\mathfrak{s}} N^\ell_*$.

Third, by the properties of WNF, for $\ell = 1,2$ we can choose N^ℓ_{**} and a $\mathfrak{u}$-free $(1,\delta)$-rectangle $\mathbf{d}'_\ell$ with $M^{\mathbf{d}'_\ell}_{1,0} = M^\ell_*, b^{\mathbf{d}_\ell}_{0,0} = b, \mathbf{d}'_\ell \restriction (0,\delta) = \mathbf{d} \restriction ([0,0],[1,\delta])$ and $M^{\mathbf{d}'_\delta}_{1,\delta} \leq_{\mathfrak{s}} N^\ell_{**}, N^\ell_* \leq_{\mathfrak{s}} N^\ell_{**}$.

Now $\mathbf{d}'_1, \mathbf{d}'_2$ are as required. $\square_{8.16}$

8.17 Proof of 8.14. In this case, for variety, instead of using a theorem on $\mathfrak{u}$ from §2 or §3, we do it directly (except quoting 9.1). We fix a stationary $S \subseteq \partial$ such that $\partial \backslash S \notin$ WDmId(∂) and S is a set of limit ordinals.

We choose $\mathfrak{g}$ witnessing 8.5(2) for S so without loss of generality $S_{\mathfrak{g}} = S$ so $\mathfrak{g}$ is $\mathfrak{u}$-2-appropriate. Let $\mathfrak{h}$ be any $\mathfrak{u} - \{0,2\}$-appropriate function. We restrict ourselves to $K^{\text{qt},*}_{\mathfrak{u}} := \{(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\text{qt}}_{\mathfrak{u}} : M_\partial \in K^{\mathfrak{s}}_{\lambda^+}$ is saturated (above λ), M_∂ has universe an ordinal $< \partial^+$ and $\mathbf{f} \restriction (\partial \backslash S)$ is constantly 1 and $(\alpha \in \partial \backslash S \Rightarrow M_{\alpha+1}$ is brimmed over $M_\alpha\}$. We now choose $\langle (\bar{M}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta) : \eta \in {}^\alpha(2^\partial)\rangle$ by induction on $\alpha < \partial^+$ such that

$\oplus_\alpha$ (a) $(\bar{M}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta) \in K^{\text{qt},*}_{\mathfrak{u}}$ for $\eta \in {}^\alpha(2^\partial)$

(b) $\langle (\bar{M}^{\eta \restriction \beta}, \bar{\mathbf{J}}^{\eta \restriction \beta}, \mathbf{f}^{\eta \restriction \beta}) : \beta \leq \alpha\rangle$ is $\leq^{\text{qt}}_{\mathfrak{u}}$-increasing continuous

(c) if $\alpha = \beta + 1$ and β is non-limit and $\eta \in {}^\alpha(2^\partial)$ then the pair $((\bar{M}^{\eta \restriction \beta}, \bar{\mathbf{J}}^{\eta \restriction \beta}, \mathbf{f}^{\eta \restriction \beta}), (\bar{M}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta))$ obeys[30] $\mathfrak{g}$ if α is even and obeys $\mathfrak{h}$ if α is odd

(d) if $\alpha = \beta + 1$, β is a limit ordinal, $\nu \in {}^\beta(2^\partial)$ and $\varepsilon_1 \neq \varepsilon_2 < 2^\partial$ and so $\eta^\ell = \nu\hat{\ }\langle \varepsilon_\ell \rangle$ is from ${}^\alpha(2^\partial)$ for $\ell = 1, 2$ <u>then</u> not only $M^{\eta^1}_\partial, M^{\eta^2}_\partial$ are not isomorphic over M^ν_∂, but if $(\bar{M}^{\eta^\ell}, \bar{\mathbf{J}}^{\eta^\ell}, \mathbf{f}^{\eta^\ell}) \leq^{\text{qt}}_{\mathfrak{u}} (\bar{M}^\ell, \bar{\mathbf{J}}^\ell, \mathbf{f}^\ell)$ for $\ell = 1, 2$ then $M^1_\partial, M^2_\partial$ are not isomorphic over M^ν_∂.

By 9.1 this suffices. For $\alpha = 0$ and α limit there are no problems (well we have to show that the limit exists which hold by 1.19(4), and belongs to $K^{\text{qt},*}_{\mathfrak{u}}$, but this is easy by 7.28(2)).

So assume $\alpha = \beta + 1, \eta \in {}^\beta 2$ and we should choose $\langle (\bar{M}^{\eta\hat{\ }<\varepsilon>}, \bar{\mathbf{J}}^{\eta\hat{\ }<\varepsilon>}, \mathbf{f}^{\eta\hat{\ }<\varepsilon>}) : \varepsilon < 2^\partial\rangle$, let γ_* be the universe of M^η_∂.

Let E_1 be a club of $\partial = \lambda^+$ such that if $\alpha < \delta \in E_1$ then $\mathbf{f}(\alpha) < \delta$ and M^η_δ is brimmed over M^η_α. Let $E_2 = E_1 \cup \{[\delta, \delta + \mathbf{f}(\delta)] : \delta \in S \cap E_1\}$, and without loss of generality M^η_0 is brimmed and if $\delta \in S \cap E_1$ then $M_{\delta+1}$ is brimmed over M_δ (can use $\mathfrak{g}$ to guarantee this, or increase it inside M^η_γ with no harm). Let h be the increasing continuous function from λ^+ onto E_2 and $E = \{\delta < \lambda^+ : \delta$ a limit ordinal and $h(\delta) = \delta\}$ a club of $\lambda^+ = \partial$.

So

$\boxplus$ (a) $(M^\eta_\alpha, M^\eta_{\alpha+1}, \mathbf{J}^\eta_\alpha) \in \text{FR}^2_{\mathfrak{u}} = \text{FR}^1_{\mathfrak{u}}$

[30] we may combine

(b) M^η_θ is brimmed

(c) $M^\eta_{h(\alpha+1)}$ is brimmed over $M^\eta_{h(\alpha)}$ if $\alpha \in E_1 \cap S$

(d) if $h(\alpha) \in S$ then $h(\alpha+1) = h(\alpha)+1$ (used?).

Now $\mathfrak{s}$ has non-uniqueness for WNF hence we can find (N, a) such that the triple (M^η_0, N, a) has the non-uniqueness property for WNF; without loss of generality $N \backslash M^\eta_0$ is $[\gamma_*, \gamma_* + i_{<>})$ for some ordinal $i_{<>} \leq \lambda$.

Now we choose $\mathbf{d}_\rho$ for $\rho \in {}^\varepsilon 2$ by induction on $\varepsilon < \lambda^+$ such that (recalling $\mathfrak{u}$ is self-dual; note that $\mathbf{d}$ looks inverted letting $\bar\alpha^\varepsilon = \langle \lambda(1+\zeta) : \zeta \leq \varepsilon\rangle$

$\odot$ for $\rho \in {}^\varepsilon 2$

(a) $\mathbf{d}_\rho$ is an $\mathfrak{u}$-free $(\bar\alpha^\varepsilon, \varepsilon)$-triangle

(b) $(M^\eta_0, N, \{a\}) = (M^{\mathbf{d}_\rho}_{0,0}, M^{\mathbf{d}_\rho}_{1,0}, \mathbf{J}^{\mathbf{d}_\rho}_{0,0})$

(c) if $\zeta < \varepsilon$ then $\mathbf{d}_{\rho \restriction \zeta} = \mathbf{d}_\rho \restriction (\bar\alpha^\zeta, \zeta)$

(d) $M^{\mathbf{d}_\rho}_{0,\zeta} = M^\eta_{h(\zeta)}$ for $\zeta \leq \varepsilon$ and $\mathbf{I}^{\mathbf{d}_\rho}_{0,\zeta} = \mathbf{J}^\eta_\zeta$ for $\zeta < \varepsilon$

(e) $M^{\mathbf{d}_\rho}_{i+1,\zeta+1}$ is brimmed over $M^{\mathbf{d}_\rho}_{i,\zeta+1} \cup M^{\mathbf{d}_\rho}_{i,\zeta+1}$ when $\zeta < \varepsilon, i < \lambda(1+\zeta)$

(f) if $\varepsilon = \zeta + 1$ and $i < \lambda$ then[31] $M^{\mathbf{d}_\varepsilon}_{\lambda\varepsilon+i+1,\varepsilon}$ is brimmed over $M^{\mathbf{d}_\rho}_{\lambda\varepsilon+i,\varepsilon}$

(g) if $\varepsilon = \zeta + 1$ and $i < \lambda$ and $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M^{\mathbf{d}_\varepsilon}_{\lambda\varepsilon+i,\varepsilon})$ <u>then</u> for λ ordinals $j \in [i, \lambda)$, the non-forking extension of p in $\mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M^{\mathbf{d}_\varepsilon}_{\lambda\varepsilon+j,\varepsilon})$ is realized by the $b \in \mathbf{J}^{\mathbf{d}_\varepsilon}_{\lambda\varepsilon+1,\varepsilon}$ in $M^{\mathbf{d}_\varepsilon}_{\lambda\varepsilon+j+1,\varepsilon}$

(h) if $\varepsilon \in E_! \cap S$ so $1+\varepsilon = \varepsilon$ then clause (η) of $\circledast$ of 8.16 holds with dual$(\mathbf{d}_\rho \restriction (\lambda\varepsilon, 0))$, $\lambda\varepsilon$, (M^η_0, N, a), $(M^\eta_\varepsilon, M^\eta_{\varepsilon+1}, \mathbf{J}^\eta_\varepsilon)$, dual$(\mathbf{d}_{\rho\hat{}<0>} \restriction [0, \lambda\varepsilon], [\varepsilon, \varepsilon+1])$, dual$(\mathbf{d}_{\rho\hat{}<1>} \restriction [0, \lambda\varepsilon], [\varepsilon, \varepsilon+1))$ here standing for $\mathbf{d}$, δ, (M, N, a), $(M_0, M', \{b\})$, $\mathbf{d}_1$, $\mathbf{d}_2$ there

(i) the set $M^{\mathbf{d}_\rho}_{\lambda(1+\varepsilon),\varepsilon} \backslash M^\eta_\varepsilon$ is $[\gamma_*, \gamma_* + \lambda(1+\varepsilon))$.

[31] actually can waive clause (f),(g)

There is no problem to carry the definition.

Lastly for $\rho \in {}^{\partial}2$ we define $(\bar{M}^{\eta,\rho}, \bar{\mathbf{J}}^{\eta,\rho}, \mathbf{f}^{\eta,\rho})$ by: (let $E \subseteq \partial = \lambda^+$ be a thin enough club):

$\boxtimes$ (a) $M^{\eta,\rho}_\varepsilon = M^{\mathbf{d}_{\rho\restriction\varepsilon}}_{\lambda(1+\varepsilon),\varepsilon}$ for $\varepsilon < \lambda$

(b) $\mathbf{f}^{\eta,\rho} = \mathbf{f}^\eta$

(c) $\mathbf{J}^{\eta,\rho}_\varepsilon = \mathbf{J}^\eta_\varepsilon$ when $\varepsilon \in \cup\{[\delta, \delta + \mathbf{f}^\eta(\delta)) : \delta \in E\}$.

Now let $\langle S_\varepsilon : \varepsilon < \partial = \lambda^+\rangle$ be a partition of $\lambda^+ \backslash S$ to (pairwise disjoint) sets from $(\mathrm{WDmId}_{\lambda^+})^+$.

Now we define a function $\mathbf{c}$:

$(*)_1$ its domain is the set of $\mathbf{x} = (\rho_1, \rho_2, f, \mathbf{d})$ such that: for some $\varepsilon \in S \cap E \subseteq \lambda^+$

(a) $\rho_1, \rho_2 \in {}^{\varepsilon}2$

(b) $\mathbf{d}$ is a $\mathfrak{u}$-free $(\varepsilon, 1)$-rectangle with $\mathbf{I}^{\mathbf{d}}_{\zeta,0} = \emptyset$ for $\zeta \leq \varepsilon$

(c) $M^{\mathbf{d}}_{\zeta,0} = M^{\mathbf{d}_{\rho_2}}_{\lambda(1+\zeta),\zeta}$ for $\zeta \leq \varepsilon$

(d) $\mathbf{J}^{\mathbf{d}}_{\zeta,0} = \mathbf{I}^{\mathbf{d}_{\rho_2}}_{0,\zeta}$ for $\zeta \leq \varepsilon$

(e) $M^{\mathbf{d}}_{\varepsilon,1} \backslash M^{\mathbf{d}_{\rho_2}}_{\lambda(1+\varepsilon),\varepsilon} \subseteq [\gamma_* + \lambda^+, \gamma_* + \lambda^+ + \lambda^+)$

(f) f is a $\leq_{\mathfrak{s}}$-embedding of $M^{\mathbf{d}_{\rho_1}}_{\lambda(1+\varepsilon),\varepsilon}$ into $M^{\mathbf{d}}_{\varepsilon,1}$ over $M^\eta_{h(\varepsilon)} = M^{\mathbf{d}_{\rho_\ell}}_{0,\varepsilon}$ for $\ell = 1, 2$

$(*)_2$ for $\mathbf{x} = (\rho_1, \rho_2, f, \mathbf{d})$ as above, say with $\varepsilon = \ell g(\rho_1) = \ell g(\rho_2)$ we have $\mathbf{c}(\mathbf{x}) = 1$ <u>iff</u> there are $\mathbf{d}^+, f^+, N^*$ such that letting $\nu_\ell = \rho_\ell{}^\frown\langle 0\rangle$ for $\ell = 1, 2$:

(a) $\mathbf{d}^+$ is a $\mathfrak{u}$-free $(\varepsilon + 1, 1)$-rectangle

(b) $\mathbf{d}^+ \restriction (\varepsilon, 1) = \mathbf{d}$

(c) $M^{\mathbf{d}_{\nu_2}}_{\lambda(1+\varepsilon),\varepsilon+1} \leq_{\mathfrak{s}} N^*$ and $M^{\mathbf{d}^+}_{\varepsilon+1,1} \leq_{\mathfrak{s}} N^*$

(d) f^+ is a $\leq_{\mathfrak{s}}$-embedding of $M^{\mathbf{d}_{\nu_1}}_{\lambda(1+\varepsilon),\varepsilon+1}$ into N^* extending $f \cup \mathrm{id}_{M^\eta_{h(\varepsilon)+1}}$.

Note

⊛ if $(\bar{M}^{\eta,\rho}, \bar{\mathbf{J}}^{\eta,\ell}, \mathbf{f}^{\eta,\rho}) \le_{\mathrm{qt}} (\bar{M}^*, \mathbf{J}^*, \mathbf{f}^*)$ and the universe of M^*_∂ is an ordinal $< \lambda^{++}$ and π is a one-to-one mapping from M^*_∂ onto $\gamma_* + \partial + \partial$ over $\gamma_* + \partial$, then we can find $\langle \mathbf{e}_\varepsilon : \varepsilon < \lambda^+ \rangle$ such that for some club E of ∂

- (a) $\mathbf{e}_\varepsilon$ is a $\mathfrak{u}$-free $(\varepsilon, 1)$-rectangle
- (b) $M^{\mathbf{e}_\varepsilon}_{\zeta,0} = M^{\mathbf{d}_\rho}_{\lambda(1+\zeta),\zeta}$ for $\zeta \le \varepsilon$
- (c) $\mathbf{J}^{\mathbf{e}_\varepsilon}_{\zeta,0} = \mathbf{I}^{\mathbf{d}_\rho}_{\zeta,\zeta}$ for $\zeta < \varepsilon$
- (d) $M^{\mathbf{e}_\varepsilon}_{\varepsilon,1} \backslash M^{\mathbf{d}_\rho}_{\lambda(1+\varepsilon),\varepsilon} \subseteq [\gamma_* + \lambda^+, \gamma_* + \lambda^+ + \lambda^+)$
- (e) if $\varepsilon \in E$ then there is a $\mathfrak{u}$-free $(\varepsilon+1, 1)$-rectangle $\mathbf{e}^+_\varepsilon$ such that $\mathbf{e}^+_\varepsilon \restriction (\varepsilon+1) = \mathbf{e}_{\varepsilon+1}$
- (f) $\langle M^{\mathbf{e}_\varepsilon}_{\varepsilon,1} : \varepsilon < \partial \rangle$ is a $\le_{\mathfrak{s}}$-increasing continuous sequence with union $\pi(M^*_\partial)$ which has universe $\gamma_* + \lambda^+ + \lambda^+$.

For each $\zeta < \lambda^+$ as $S_\zeta \subseteq \lambda^+$ does not belong to the weak dimaond ideal, there is a sequence $\varrho_\zeta \in {}^{(S_\zeta)}2$ such that

$(*)_3$ for any $\rho_1, \rho_2 \in {}^\partial 2$ such that $\rho_1 \restriction S_\zeta = 0_{S_\zeta}$ and $N_* \in K^{\mathfrak{s}}_{\lambda^+}$, $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*) \in K^{\mathrm{qs}}_{\mathfrak{s}}$ which is $\le_{\mathrm{qt}}$-above $(\bar{M}^{\eta,\rho_2}, \bar{\mathbf{J}}^{\eta,\rho_2}, \mathbf{f}^{\eta,\rho_2})$ and $|M^*_\partial| = \gamma_{**} < \partial^+$ and $\le_{\mathfrak{K}[\mathfrak{s}]}$-embedding f of M^{η,ρ_1}_∂ into M^*_∂ over M^η_∂, letting $\langle \mathbf{e}_\delta : \delta < \partial \rangle, \pi$ be as in ⊛ for the pair $((\bar{M}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta), (\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$ the set $\{\delta \in S_\varepsilon : \mathbf{c}(\rho_1 \restriction \delta, \rho_2 \restriction \delta, f \restriction \delta, \mathbf{e}_\delta) = \varrho(\delta)\}$ is stationary.

Now for any $u \subseteq \partial$ we define $\rho_u \in {}^\lambda 2$ by

$(*)_4$ for $\zeta < \partial, \ell < 2$ let $\rho_u \restriction S_{2\zeta+\ell}$ be $0_{2\zeta+\ell}$ if $[\zeta \notin u \leftrightarrow \ell = 0]$ and ϱ_ζ otherwise

$(*)_5$ let $\rho_u \restriction S_0$ be constantly zero.

Let $\langle u(\alpha) : \alpha < 2^\partial \rangle$ list $\mathscr{P}(\gamma)$ and for $\alpha < 2^\partial$ let $(\bar{M}^{\eta\hat{\ }<\alpha>}, \mathbf{J}^{\eta\hat{\ }<\alpha>}, \mathbf{f}^{\eta\hat{\ }<\alpha>})$ be $(\bar{M}^{\eta,\rho_{u(\alpha)}}, \mathbf{J}^{\eta,\rho_{u(\alpha)}}, \mathbf{f}^{\eta,\rho_{u(\alpha)}})$.

Clearly they are as required. $\square_{8.14}$

* * *

8.18 Exercise: 1) Assume $\mathfrak{t}$ is an almost good λ-frame, $\mathfrak{u} = \mathfrak{u}^1_{\mathfrak{t}}$ from Definition 4.29 then for some $\mathfrak{u} - \{0,2\}$-appropriate $\mathfrak{h}$, for every $M \in K^{\mathfrak{t},\mathfrak{h}}_{\lambda^{++}}$ we have

(a) M is λ^+-saturated
(b) if $M_0 \in K_{\mathfrak{t}}$ is $\leq_{\mathfrak{K}[\mathfrak{t}]} M$ and $p \in \mathscr{S}^{\rm bs}_{\mathfrak{t}}(M_0)$

then $\dim(p, M) = \lambda^{++}$ that is, there is a sequence $\langle a_\alpha : \alpha < \lambda^{++}\rangle$ of members of M realizing p such that: if $M_0 \leq_{\mathfrak{t}} M_1 <_{\mathfrak{K}[\mathfrak{t}]} M$ then $\{\alpha < \lambda^{++} : \mathbf{tp}_{\mathfrak{t}}(a, M_1, M)$ does not fork over $M_0\}$ is a co-bounded subset of λ^{++}.
2) Similarly if $\mathfrak{t}$ has existence for $K^{3,\rm up}_{\mathfrak{s}}$ and $\mathfrak{u} = \mathfrak{u}^3_{\mathfrak{t}}$, see Definition 8.3.

8.19 Theorem.
$\dot I(\lambda^{++}, K^{\mathfrak{s}}) \geq \dot I(\lambda^{++}, K^{\mathfrak{s}}(\lambda^+\text{-saturated})) \geq \mu_{\rm unif}(\lambda^{++}, 2^{\lambda^+})$ and even $\dot I(K^{\mathfrak{u},\mathfrak{h}}_{\lambda^{++}}) \geq \mu_{\rm unif}(\lambda^{++}, 2^{\lambda^+})$ for any $\mathfrak{u}_{\mathfrak{s}} - \{0,2\}$-appropriate $\mathfrak{h}$ (so we can restrict ourselves to models λ^+-saturated above λ and if $\mathfrak{s} = \mathfrak{s}'$ also to $\tau_{\mathfrak{s}}$-fuller ones) when:

(a) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^+}$
(b) *$\mathfrak{s}$ has non-uniqueness for* WNF *(for every $M \in K_{\mathfrak{s}}$)*
(c) *K is categorical in λ*
(d) *$\mathfrak{u}$ has existence for $K^{3,\rm up}_{\mathfrak{s},\lambda^+}$.*

Proof. We shall use 8.6, 8.12, 8.14. So assume toward contradiction that the conclusion fails. We try to apply Theorem 8.6, now its conclusion fails by our assumption toward contradiction, and clause (a) there which says "$2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$" holds by clause (a) of the present theorem. So necessarily clause (b) of Theorem 8.6 fails which means that $\mathfrak{u}$ has wnf-delayed uniqueness, see Definition 8.7.

Next we try to apply Theorem 8.14, again it assumption fails by our assumption toward contradiction, and among its assumptions clause (a) which says that "$2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$" holds by clause (a) of the present theorem, and clause (c) which says "$\mathfrak{s}$ has wnf-delayed uniqueness" has just been proved. So necessarily clause (b) of 8.14

fails which means that $\mathfrak{s}$ fails non-uniqueness for WNF, i.e. for some M.

Now we apply Observation 8.12, noting that its assumption "$\mathfrak{s}$ is categorical in λ" holds by clause (c) of the present theorem, so by the previous sentence one of the equivalent phrases the first fails, hence all of them. In particular $\mathfrak{s}$ has uniqueness for WNF. $\square_{8.19}$

§9 The combinatorial part

We deal here with the "relatively" pure-combinatorial parts. We do just what is necessary. We can get results on $\dot{I}\dot{E}(\partial^+, \mathfrak{K}_\mathfrak{u})$, we can weaken the cardinal arithmetic assumptions to $\emptyset \notin \mathrm{DfWD}_\partial$, see [Sh:E45], we can weaken the demands on $\mathfrak{K}$; but not here.

Recall the obvious by the definitions:

9.1 Theorem. *If $2^\partial < 2^{\partial^+}$ then $\{M_\eta / \cong : \eta \in {}^{\partial^+}(2^\partial)$ and $\|M_\eta\| = \partial^+\}$ has cardinality $\geq \mu_{\mathrm{unif}}(\partial^+, 2^\partial)$ when the following conditions hold:*

$\circledast$ (a) $\bar{M} = \langle M_\eta : \eta \in {}^{\partial^+ >}(2^\partial)\rangle$

(b) *for $\eta \in {}^{\partial^+ >}(2^\partial)$ the model M_η has cardinality $\leq \partial$ and for notational simplicity has universe an ordinal $< \partial^+$*

(c) *$M_\eta \subseteq M_\nu$ if $\eta \triangleleft \nu \in {}^{\partial^+ >}(2^\partial)$, so no a.e.c. appear here!*

(d) *$\langle M_{\eta \restriction \alpha} : \alpha < \ell g(\eta)\rangle$ is $\subseteq$-increasing continuous for any $\eta \in {}^{\partial^+ >}(2^\partial)$*

(e) *$M_\eta := \cup\{M_{\eta\restriction\alpha} : \alpha < \partial^+\}$ for $\eta \in {}^{\partial^+}(2^\partial)$*

(f) *if $\eta \in {}^{\partial^+ >}(2^\partial)$ and $\alpha_1 < \alpha_2 < 2^\partial$ and $\eta\hat{\ }\langle\alpha_\ell\rangle \trianglelefteq \nu_\ell \in {}^\delta 2$ for $\ell = 1, 2$ and $\delta < \partial^+$ then M_{ν_2}, M_{ν_1} are not isomorphic over M_η or just*

$(\mathrm{f})_1^-$ *for $\eta \in {}^{\partial^+ >}2$, there is $\mathscr{U}_\eta \subseteq 2^\partial$ of cardinality 2^∂ such that: if $\alpha_0 \neq \alpha_1$ are from $\mathscr{U}_\eta$ and $\eta\hat{\ }\langle\alpha_\ell\rangle \trianglelefteq \nu_\ell \in {}^\delta 2$ for $\ell < 2$ and $\delta < \partial^+$ then M_{ν_0}, M_{ν_1} are not isomorphic over M_η.*

Proof. Concerning clause $(\mathrm{f})_1^-$ we can by renaming get clause (f), so in the rest of the proof of 9.1 we can ignore clause $(\mathrm{f})_1^-$.

Note that $\Xi_0 := \{\eta \in {}^{\partial^+}(2^\partial) : \|M_\eta\| < \partial^+\}$ has cardinality $\le 2^\partial$ (because for each $\eta \in \Xi_0$ there is $\alpha_\eta < \partial^+$ such that $M_\eta = M_{\eta \restriction \alpha_\eta}$; and note that by clause (f) we have $\eta_1 \in \Xi_0 \wedge \eta_2 \in \Xi_0 \wedge \alpha_{\eta_1} = \alpha_{\eta_2} \wedge \eta_1 \restriction \alpha_{\eta_1} = \eta_2 \restriction \alpha_{\eta_2} \Rightarrow \eta_1 = \eta_2$). So by clause (b) of $\circledast$ it follows that $\eta \in {}^{\partial^+}(2^\partial) \backslash \Xi_0 \Rightarrow |M_\eta| = \partial^+$.

It suffices to assume that $\Xi \subseteq {}^{\partial^+}(2^\partial)$ has cardinality $< \mu_{\rm unif}(\partial^+, 2^\partial)$ and find $\eta \in {}^{\partial^+}(2^\partial)$ such that $\nu \in \Xi \Rightarrow M_\eta \ncong M_\nu$, because without loss of generality $\Xi_0 \subseteq \Xi$.

Let $\langle \eta_\zeta : \zeta < |\Xi| \rangle$ list Ξ and let $N_\zeta := M_{\eta_\zeta}$ and toward contradiction for every $\nu \in {}^{\partial^+}(2^\partial)$ we can choose $\zeta_\nu = \zeta(\nu) < |\Xi|$ and an isomorphism f_ν from M_ν onto $N_{\zeta(\nu)}$, so f_ν is a function from M_ν onto $M_{\eta_{\zeta(\nu)}}$.

For $\zeta < |\Xi|$ let $W_\zeta = \{\nu \in {}^{\partial^+}(2^\partial) : \zeta_\nu = \zeta\}$, so clearly:

$(*)_1$ ${}^{\partial^+}(2^\partial)$ is equal to $\cup\{W_\zeta : \zeta < |\Xi|\}$.

[Why? Obvious by our assumption toward contradiction.]

$(*)_2$ if $i < \partial^+$ and $\rho \in {}^i(2^\partial)$, <u>then</u> there are no $\varepsilon_1 \neq \varepsilon_2 < 2^\partial$ such that $\rho\char94\langle \varepsilon_\ell \rangle \triangleleft \nu_\ell \in W_\zeta$ for $\ell = 1, 2$ and $f_{\nu_1} \restriction M_\eta = f_{\nu_2} \restriction M_\eta$.

[Why? By Clause (f) of the assumption.]

Together we get a contradiction to the definition of $\mu_{\rm unif}(\partial^+, 2^\partial)$, see Definition 0.4(7). $\square_{9.1}$

Similarly

9.2 Claim. *1) In 9.1 we can replace 2^∂ by $\langle \chi_i : i < \partial \rangle$ with $\chi_i \le 2^\partial$.*
2) Also we can weaken clause (f) or $(f)_1^-$ there by demanding $\delta = \partial^+$.
3) Assume $\mathfrak{K}$ is an a.e.c. and in 9.1 we demand $M_\nu \le_{\mathfrak{K}} M_\eta \in \mathfrak{K}$ for $\nu \triangleleft \eta \in {}^{\partial^+ >}(2^\partial)$. If we strengthen there clause $(f)_1^-$ by strengthening the conclusion to "if $\eta\char94\langle \ell \rangle \triangleleft \eta_\ell \in {}^{\partial^+}2$ for $\ell = 1, 2$ then M_{ν_0}, M_{ν_1} cannot be $\le_{\mathfrak{K}}$-amalgamated over M_η" <u>then</u>:

$(*)$ *for every $\Xi \subseteq {}^{\partial^+}(2^\partial)$ of cardinality $< \mu_{\rm unif}(\partial^+, 2^\partial)$ for some $\eta \in {}^{\partial^+}(2^\partial)$ the model M_η has cardinality ∂^+ and cannot be $\le_{\mathfrak{K}}$-embedded in M_ν for any $\nu \in \Xi$*

(**) *if* $2^{\partial^+} > (2^\partial)^+$ <u>then</u> *there is* $\Xi \subseteq {}^{\partial^+}2$ *of cardinality* 2^{∂^+} *such that if* $\eta \neq \nu \in \Xi$ *then* M_η *cannot be* $\leq_{\mathfrak{K}}$*-embedded into* M_ν*.*

Proof. Left to the reader (easier than 9.7 below and will not be used here). $\square_{9.2}$

Remark. Why do we prefer to state 9.1? As this is how it is used.

9.3 Lemma. *Assuming* $2^\theta = 2^{<\partial} < 2^\partial$ *(and naturally but not used* $2^\partial < 2^{\partial^+}$*) and* $\circledast(a) - (e)$ *of 9.1, a sufficient condition for clause* $(f)_1^-$ *of 9.1 is:*

$(a)^+$ $\bar{M} = \langle M_\eta : \eta \in {}^{\partial^+ >}(2^\partial)\rangle$ *and* $\langle M^*_{\eta,\zeta} : \zeta < \partial\rangle$ *is* $\subseteq$*-increasing with union* M_η *such that* $\zeta < \partial \Rightarrow \|M_{\eta,\zeta}\| < \partial$

$(f)_2^-$ *for each* $\eta \in {}^{\partial^+ >}2$ *we can find* $\langle M_{\eta,\rho} : \rho \in {}^{\partial \geq}2\rangle$ *such that*

- (α) $\langle M_{\eta,\rho} : \rho \in {}^\partial 2\rangle$ *is a subsequence of* $\langle M_{\eta\char`^<\alpha>} : \alpha < 2^\partial\rangle$ *with no repetitions so* $M_{\eta,\rho} = M_{\eta\char`^<\alpha(\rho)>}$ *for some one-to-one function* $\rho \mapsto \alpha(\rho)$ *from* ${}^\partial 2$ *to* 2^∂
- (β) *if* $\rho \in {}^{\partial >}2$ <u>then</u> $M_{\eta,\rho} \in K_{<\partial}$
- (γ) *if* $\rho \in {}^{\partial >}2$ <u>then</u> $\langle M_{\eta,\rho \restriction \alpha} : \alpha \leq \ell g(\rho)\rangle$ *is* $\subseteq$*-increasing continuous*
- (δ) $\cup\{M_{\eta,\rho\restriction\varepsilon} : \varepsilon < \partial\}$ *is equal to* $M_{\eta,\rho} = M_{\eta\char`^<\alpha(\rho)>}$ *for any* $\rho \in {}^\partial 2$
- (ε) ∂ *is regular uncountable and for some sequence* $\langle S_\varepsilon : \varepsilon < \partial\rangle$ *of pairwise disjoint non-small stationary subsets of* ∂ *(i.e.* $\varepsilon < \partial \Rightarrow S_\varepsilon \in (\mathrm{WDmId}_\partial)^+$*) we have*
  - $(*)$ *for every* $\varepsilon < \partial$*, there is a pair* $(\bar{g}, \mathbf{c}) = (\bar{g}^\varepsilon, \mathbf{c}^\varepsilon)$*, may not depend on* ε *such that:*
  - $\bullet_1$ $\bar{g} = \langle g_{\eta,\rho} : \rho \in {}^\partial 2\rangle$
  - $\bullet_2$ $g_{\eta,\rho}$ *is a function from* ∂ *to* $\mathscr{H}_{<\partial}(\partial^+)$
  - $\bullet_3$ *if* $2^\partial > \partial^+$ *and* $\rho_0, \rho_1 \in {}^\partial 2, \rho_1 \restriction S_\varepsilon$ *is constantly zero,* $\delta < \partial^+$*,* $\eta\char`^\langle\alpha(\rho_\ell)\rangle \trianglelefteq \nu_\ell \in {}^\delta(2^\partial)$ *for* $\ell = 0, 1$ *and* f *is an isomorphism from* M_{ν_0} *onto* M_{ν_1} <u>then</u>

for some club E of ∂, if $\zeta \in E \cap S_\varepsilon$ we have $\rho_0(\zeta) = \mathbf{c}^\varepsilon(\rho_0 \restriction \zeta,\ M_{\eta,\rho_0\restriction\zeta}, \rho_1 \restriction \zeta, M_{\eta,\rho_1\restriction\zeta}, g_{\eta,\rho_0} \restriction \zeta,\ g_{\eta,\rho_1} \restriction \zeta, M_{\nu_0,\zeta}, M_{\nu_\zeta}, f \restriction M_{\eta,\rho_0\restriction\zeta})$

- $\bullet_4$ *if $2^\partial = \partial^+$: as above but $\mathbf{c}$ is preserved by any partial order preserving function from ∂^+ to ∂^+ extending* $\mathrm{id}_{M^\eta_\partial}$.

Remark. 1) We can imitate 9.7.
2) If $2^\partial = \partial^+$ then it follows that $\partial = \partial^{<\partial}$, so they give us stronger ways to construct.

Proof. First

$\boxtimes$ for $\eta \in {}^{\partial^+>}(2^\partial)$ and $\varepsilon < \partial$ there is $\varrho_{\eta,\varepsilon} \in {}^{(S_\varepsilon)}2$ such that:

(∗) if $\rho_0 \neq \rho_1$ are from ${}^\partial 2, \eta\hat{\ }\langle\alpha(\rho_\ell)\rangle \triangleleft \nu_\ell \in {}^\delta 2, \delta < \partial^+$ and f is an isomorphism from M_{ν_0} onto M_{ν_1} <u>then</u> for stationary many $\zeta \in S_\varepsilon$ we have:

$$\begin{aligned}\varrho_\eta(\zeta) =& \mathbf{c}^\varepsilon(\rho_0 \restriction \zeta, M_{\eta,\rho_0\restriction\zeta}, \rho_1 \restriction \zeta, M_{\eta,\rho_1\restriction\zeta}, g_{\eta,\rho_0} \restriction \zeta, g_{\eta,\rho_1} \restriction \zeta,\\ & M^*_{\nu_0,\zeta}, M^*_{\nu_1,\zeta}, f \restriction M_{\eta,\rho_0\restriction\zeta})\end{aligned}$$

[Why? First if $2^\theta \geq \partial^+$, use the definition of $S_\varepsilon \notin$ WDmId(∂), (see more in the proof of 9.6). If $2^\theta = \partial \wedge 2^\partial = \partial^+$, the proof is similar using the invariance of $\mathbf{c}^\varepsilon$, i.e. $\bullet_4$.

Lastly, if $2^\theta = \partial \wedge 2^\partial > \partial^+$, use $\mu_{\mathrm{wd}}(\partial) > \partial^+$, see 0.5(1A).]

Let $\eta \in {}^{\partial^+>}(2^\partial)$. For any $w \subseteq \partial$ we define $\rho_{\eta,w} \in {}^\partial 2$ as follows: $\rho_{\eta,w}(i)$ is $\varrho_{\eta,\varepsilon}$ <u>if</u> for some $\varepsilon < \partial$ and $\ell < 2$ we have $i \in S_{2\varepsilon+\ell} \wedge [\varepsilon \in w \equiv \ell = 1]$ and is zero otherwise. So $\{\alpha_\eta(\rho_{\eta,w}) : w \subset \partial\}$ is as required in $(\mathrm{f})^-_1$. $\square_{9.3}$

9.4 Theorem. *If* $2^\partial < 2^{\partial^+}$ *and* $\mu = \mu_{\rm unif}(\partial^+, 2^\partial)$ *then:*

(*A*) $\emptyset \notin {\rm UnfTId}_\mu(\partial^+)$

(*B*) ${\rm UnfTId}_{\mu_1}(\partial^+)$ *is* μ_1*-complete when* $\aleph_0 \le \mu_1 = {\rm cf}(\mu_1) < \mu$; *see 0.4(4),(5)*

(*C*) $\mu = 2^{\partial^+}$ *except maybe when (all the conditions below hold):*

$\circledast$ *(a)* $\mu < \beth_\omega$

(b) $\mu^{\aleph_0} = 2^{\partial^+}$

(c) there is a family $\mathscr{A} \subseteq [\mu]^{\partial^+}$ *of cardinality* $\ge 2^{\partial^+}$ *such that the intersection of any two distinct members of* $\mathscr{A}$ *is finite.*

Remark. So in the aleph sequence μ is much larger than 2^∂, when $\mu \ne 2^{\partial^+}$.

Proof. By [Sh:f, AP,1.16] we have clauses (b) + (c) of $\circledast$ and they imply clause (a) by [Sh 460] (or see [Sh 829]). $\square_{9.4}$

9.5 Claim. *Assume* $\partial > \theta \ge \aleph_0$ *is regular and* $2^\theta = 2^{<\partial} < 2^\partial$. *Then* $\{M_\eta/\cong : \eta \in {}^\partial 2$ *and* M_η *has cardinality* $\partial\}$ *has cardinality* 2^∂ *when the following condition holds:*

$\circledast$ (a) $\bar M = \langle M_\eta : \eta \in {}^{\partial\ge}2\rangle$ *with* M_η *a* τ*-model*

(b) *for* $\eta \in {}^\partial 2, \langle M_{\eta\restriction\alpha} : \alpha \le \partial\rangle$ *is* $\subseteq$*-increasing continuous*

(c) *if* $\eta \in {}^{\partial>}2$ *and* $\eta\hat{\ }\langle\ell\rangle \trianglelefteq \nu_\ell \in {}^\alpha 2$ *for* $\ell = 0,1$ *and* $\alpha < \partial$ *then* M_{ν_0}, M_{ν_1} *are not isomorphic over* $M_{<>}$ *or just for* $\alpha = \partial$

(d) $M_{<>}$ *has cardinality* $< \partial$

(e) M_η *has cardinality* $\le \partial$ *for* $\eta \in {}^{\partial>}2$.

Proof. As in the proof of 9.1 we can ignore the $\eta \in {}^\partial 2$ for which M_η has cardinality $< \partial$.

As $\partial^{\|M_{<>}\|} \le 2^{<\partial} < 2^\partial$ this is obvious, see I.0.3 or see Case 1 in the proof of 9.7 below. $\square_{9.5}$

The following is used in VI.3.9 (and can be used in VI.2.18; but compare with 9.5!)

9.6 Claim. *The set $\{M_\eta/\cong : \eta \in {}^\partial 2$ and M_η has cardinality $\partial\}$ has cardinality $\ge \mu$ when*

$\boxtimes_1$ $\partial = \mathrm{cf}(\partial) > \aleph_0$ *and*

- *(a)* M_η *is a τ-model of cardinality $< \partial$ for $\eta \in {}^{\partial>}2$*
- *(b)* *for each $\eta \in {}^\partial 2$, $\langle M_{\eta\restriction\alpha} : \alpha < \partial\rangle$ is $\subseteq$-increasing continuous with union, called M_η*
- *(c)* *if $\eta \in {}^{\partial>}2, \eta\hat{}\langle \ell\rangle \triangleleft \rho_\ell \in {}^\partial 2$ for $\ell = 1,2$ <u>then</u> M_{ρ_1}, M_{ρ_2} are not isomorphic over M_η*

$\boxtimes_2$ $\partial \notin \mathrm{WDmId}_{<\mu}(\partial)$, *e.g.* $\mu = \mu_{\mathrm{wd}}(\partial)$.

Proof. Let $\Xi = \{\eta \in {}^\partial 2 : M_\eta$ has cardinality $< \partial\}$ and for $\eta \in \Xi$ let $\alpha_\eta = \min\{\alpha \le \partial : M_\eta = M_{\eta\restriction\alpha}\}$, clearly

$\boxdot_1$ (*a*) $\eta \in \Xi$ implies $\alpha_\eta < \partial$

(*b*) if $\eta \in \Xi$ and $\eta \restriction \alpha \triangleleft \nu \in \Xi\backslash\{\eta\}$ then $\alpha_\nu > \alpha_\eta$.

For each $\varrho \in {}^\partial 2$ we define $F_\varrho : {}^{\partial\ge}2 \to {}^{\partial\ge}2$ by: for $\eta \in {}^\alpha 2$ let $F_\varrho(\eta) \in {}^{2\ell g(\eta)}2$ be defined by $(F_\varrho(\eta))(2i) = \eta(i)$, $(F_\varrho(\eta))(2i+1) = \varrho(i)$ for $i < \alpha$. Easily $\langle \mathrm{Rang}(F_\varrho) : \varrho \in {}^\partial 2\rangle$ are pairwise disjoint, hence for some $\varrho \in {}^\gamma 2$, the sets $\mathrm{Rang}(F_\varrho)$ is disjoint to Ξ so without loss of generality (by renaming):

$\boxdot_2$ $\Xi = \emptyset$.

Let $\{N_\varepsilon : \varepsilon < \varepsilon_*\}$ be a maximal subset of $\{M_\rho : \rho \in {}^\partial 2\}$ consisting of pairwise non-isomorphic models.

Without loss of generality the universe of each $M_\eta, \eta \in {}^{\partial>}2$ is an ordinal $\gamma_\eta < \partial$ and so the universe of each $M_\eta, \eta \in {}^\partial 2$ is $\gamma_\eta := \cup\{\gamma_{\eta\restriction i} : i < \partial\} = \partial$, in particular the universe of N_ε is ∂ and

$\eta \in {}^{\partial}2 \Rightarrow \gamma_\eta = \partial$. For $\alpha < \partial$ and $\eta \in {}^{\alpha}2$ let the function h_η be $h_\eta(i) = M_{\eta\restriction(i+1)}$ for $i < \ell g(\eta)$. For each $\varepsilon < \varepsilon_*$ we define $\Xi_\varepsilon \subseteq {}^{\partial}2$ by $\Xi_\varepsilon = \{\eta \in {}^{\partial}2 : M_\eta$ is isomorphic to $N_\varepsilon\}$.

For $\eta \in \Xi_\varepsilon$ choose $f^\varepsilon_\eta : M_\eta \to N_\varepsilon$, an isomorphism, hence $f^\varepsilon_\eta \in {}^{\partial}\partial$. By the assumption

$\boxdot_3$ if $\varepsilon < \varepsilon_*$ and $\eta \in {}^{\partial>}2$ and $\eta\hat{\ }\langle \ell \rangle \triangleleft \nu_\ell \in \Xi_\varepsilon$ for $\ell = 0, 1$ then $f^\varepsilon_{\nu_0} \restriction \gamma_\eta \neq f^\varepsilon_{\nu_1} \restriction \gamma_\eta$.

We also for each $\varepsilon < \varepsilon_*$ define a function (= colouring) $\mathbf{c}_\varepsilon$ from $\bigcup\limits_{\alpha<\partial} ({}^{\alpha}2 \times {}^{\alpha}\partial)$ to $\{0, 1\}$ by:

$\boxdot_4$ $\mathbf{c}_\varepsilon(\eta, f)$ is : 0 <u>if</u> there is ν such that $\eta \triangleleft \nu \in \Xi_\varepsilon$ and $f \subseteq f^\varepsilon_\nu$ and $\nu(\ell g(\eta)) = 0$
$\mathbf{c}_\varepsilon(\eta, f)$ is: 1 <u>if</u> otherwise.

Now for any $\eta \in \Xi_\varepsilon$, the set

$$E^\varepsilon_\eta = \{\delta < \partial : \gamma_{\eta\restriction\delta} = \delta \text{ and } f^\varepsilon_\eta \restriction \delta \text{ is a function from } \delta \text{ to } \delta\}$$

is clearly a club of ∂.

Now

$\boxdot_5$ if $\varepsilon < \varepsilon_*, \eta \in \Xi_\varepsilon$ and $\delta \in E^\varepsilon_\eta$ then $\mathbf{c}_\varepsilon(\eta \restriction \delta, f^\varepsilon_\eta \restriction \delta) = \eta(\delta)$.

[Why? If $\eta(\delta) = 0$ then $\eta \restriction \delta$ witness that $\mathbf{c}_\varepsilon(\eta \restriction \delta, f^\varepsilon_\eta \restriction \delta) = 0$. If $\eta(\delta) = 1$ just recall $\boxdot_3$.]

Hence we have $\Xi_\varepsilon \in$ WDmTId(∂). To get a contradiction it is enough to prove $\cup\{\Xi_\varepsilon : \varepsilon < \varepsilon_*\} \neq {}^{\partial}2$, but as $\varepsilon_* < \mu$ clearly $\bigcup\limits_{\varepsilon<\varepsilon_*} \Xi_\varepsilon$ belongs to WDmId$_{<\mu}(\partial)$ hence is not ${}^{\partial}2$, so we are done. $\square_{9.6}$

The following is used in VI.2.18, VI.3.11, VI.3.9 which repeat the division to cases.

9.7 Claim. *The set $\{M_\eta/\cong : \eta \in {}^{\partial}2$ and $\|M_\eta\| = \partial\}$ has cardinality 2^∂ <u>when</u>:*

$\boxtimes_1$ *M_η is a τ-model of cardinality $< \partial$ for $\eta \in {}^{\partial>}2, \langle M_{\eta\restriction\alpha} : \alpha \leq \ell g(\eta)\rangle$ is $\subseteq$-increasing continuous, and: if $\delta < \delta(1) < \partial$*

are limit ordinals, $\eta_0, \eta_1 \in {}^{\delta}2$ *and* $\eta_0{}^\frown\langle \ell \rangle \triangleleft \nu_\ell \in {}^{\delta(1)}2$ *and* $\eta_1{}^\frown\langle 0 \rangle \triangleleft \nu'_\ell \in {}^{\delta(1)}2$ *for* $\ell = 0, 1$ *then there are no* f_0, f_1 *such that*

- (α) f_ℓ *is an isomorphism from* M_{ν_ℓ} *onto* $M_{\nu'_\ell}$ *for* $\ell = 0, 1$
- (β) $f_0 \restriction M_{\eta_0} = f_1 \restriction M_{\eta_0}$ *and* $M_{\eta_1} = f_0(M_{\eta_0})$
- (γ) *for some* $\rho_0, \rho_1 \in {}^{\partial}2$ *we have* $\nu'_\ell \triangleleft \rho_\ell$ *for* $\ell = 0, 1$ *and* M_{ρ_0}, M_{ρ_1} *are isomorphic over* M_{η_1}

$\boxtimes_2$ $\partial = \text{cf}(\partial) > \aleph_0$, $\partial \notin \text{WDmId}_{<\mu}(\partial)$ *(hence* $2^{<\partial} < 2^{\partial}$*) and moreover*

$\boxtimes_3$ ∂ *is a successor cardinal, or at least there is no* ∂*-saturated normal ideal on* ∂*, or at least* $\text{WDmId}(\partial)$ *is not* ∂*-saturated (which holds if for some* $\theta < \partial, \{\delta < \partial : \text{cf}(\delta) = \theta\} \notin \text{WDmId}(\partial)$ *because the ideal is normal).*

9.8 Remark. 1) Compare with I.3.8 - which is quite closed but speak on $\mathfrak{K}$ rather than on a specific $\langle M_\eta : \eta \in {}^{\partial>}2\rangle$. Can we get $\dot{I}\dot{E}_{\mathfrak{K}}(\partial, \mathfrak{K}) = 2^{\partial}$? Also λ^+ there corresponds to ∂ here, a minor change.
2) The parallel claim was inaccurate in the [Sh 576, §3].
3) Used in VI.2.18.

Proof of 9.7. Easily, as in the proof of 9.6 without loss of generality

$\boxdot_1$ $\eta \in {}^{\partial}2 \Rightarrow \|M_\eta\| = \partial$ while, of course, preserving $\boxtimes_1$.

We divide the proof into cases according to the answer to the following:

<u>Question</u>: Is there $\eta^* \in {}^{\partial>}2$ such that for every ν satisfying $\eta^* \trianglelefteq \nu \in {}^{\partial>}2$ there are $\rho_0, \rho_1 \in {}^{\partial>}2$ such that: $\nu \triangleleft \rho_0, \nu \trianglelefteq \rho_1$, and for any $\nu_0, \nu_1 \in {}^{\partial}2$ satisfying $\rho_\ell \triangleleft \nu_\ell$, (for $\ell = 0, 1$) the models M_{ν_0}, M_{ν_1} are not isomorphic over M_{η^*}?

But first we can find a function $h : {}^{\partial>}2 \to {}^{\partial>}2$, such that:

(∗) the function h is one-to-one, mapping ${}^{\partial>}2$ to ${}^{\partial>}2$, preserving $\triangleleft$, satisfying $(h(\nu))\hat{}\langle\ell\rangle \trianglelefteq h(\nu\hat{}\langle\ell\rangle)$ and h is continuous, for $\nu \in {}^{\partial}2$ we let $h(\nu) := \bigcup_{\alpha<\partial} h(\nu \restriction \alpha)$, so $\ell g(\eta) < \partial \Leftrightarrow \ell g(h(\eta)) < \partial$ and:

$(b)_{\rm yes}$ <u>when</u> the answer to the question above is yes, it is exemplified by $\eta^* = h(\langle\rangle)$ and $M_{h(\rho_0)}, M_{h(\rho_1)}$ are not isomorphic over $M_{h(<>)}$ whenever $\nu \in {}^{\partial>}2$ and $h(\nu\hat{} < \ell >) \triangleleft \rho_\ell \in {}^{\partial}2$ for $\ell = 0, 1$

$(b)_{\rm no}$ when the answer to the question above is no, $h(\langle\rangle) = \langle\rangle$ and if $\alpha + 1 < \beta < \partial, \eta \in {}^{\alpha+1}2$ and $h(\eta) \triangleleft \rho_\ell \in {}^{\beta}2$ for $\ell = 1, 2$ <u>then</u> we can find ν_1, ν_2 and g^* such that $\rho_\ell \triangleleft \nu_\ell \in {}^{\partial}2$ and g^* is an isomorphism from M_{ν_1} onto M_{ν_2} over $M_{h(\eta\restriction\alpha)}$.

[Why can we get $(b)_{\rm no}$? We choose $h(\eta)$ for $\eta \in {}^{\alpha}2$ by induction on α such that $h(\eta) = \eta$ for $\alpha = 0, h(\eta) = \cup\{h(\eta \restriction \beta) : \beta < \alpha\}$ when α is a limit ordinal, and if $\alpha = \beta + 1, \ell < 2$ apply the assumption ("the answer is no") with $h(\eta)\hat{} < \ell >$ standing for η^* and let $h(\eta\hat{} < \ell >)$ be a counterexample to "for every ν"; so we get even more then the promise; the isomorphism is over $M_{h(\eta\restriction\alpha)\hat{}\langle\ell\rangle}$ rather than $M_{h(\eta\restriction\alpha)}$, and note that $h(\eta)\hat{}\langle\ell\rangle \triangleleft h(\eta\hat{}\langle\ell\rangle)$.]

<u>Case 1</u>: The answer is yes.

We do not use the non-∂-saturation of WDmId(∂) in this case. Without loss of generality h is the identity, by renaming.
For any $\eta \in {}^{\partial}2$ and $\subseteq$-embedding g of $M_{\langle\rangle}$ into $M_\eta := \bigcup_{\alpha<\partial} M_{\eta\restriction\alpha}$, let

$$\Xi_{\eta,g} := \{\nu \in {}^{\partial}2 : \text{there is an iso. from } M_\nu \text{ onto } M_\eta \text{ extending } g\}$$

$$\Xi_\eta := \{\nu \in {}^{\partial}2 : \text{there is an isomorphism from } M_\nu \text{ onto } M_\eta\}.$$

So:

$\boxdot_2$ $|\Xi_{\eta,g}| \le 1$ for any g and $\eta \in {}^{\partial}2$.

[Why? As if $\nu_0, \nu_1 \in \Xi_{\eta,g}$ are distinct then for some ordinal $\alpha < \partial$ and $\nu \in {}^{\alpha}2$ we have $\nu := \nu_0 \restriction \alpha = \nu_1 \restriction \alpha, \nu_0(\alpha) \ne \nu_1(\alpha)$ and use the choice of $h(\nu\hat{}\langle\ell\rangle))$, see $(b)_{\rm yes}$ above.]

Since $\Xi_\eta = \cup\{\Xi_{\eta,g} : g$ is a $\leq_{\mathfrak{K}}$ -embedding of $M_{\langle\rangle}$ into $M_\eta\}$, we have

$\boxdot_3$ $|\Xi_\eta| \leq \partial^{\|M_{\eta^*}\|} \leq 2^{<\partial}$.

Hence we can by induction on $\zeta < 2^\partial$ choose $\eta_\zeta \in {}^\partial 2 \setminus \bigcup\limits_{\xi<\zeta} \Xi_{\eta_\xi}$, (exist by cardinality considerations as $2^{<\partial} < 2^\partial$). Then $\xi < \zeta \Rightarrow M_{\eta_\xi} \not\cong M_{\eta_\zeta}$ so we have proved the desired conclusion.

Case 2: The answer is no.

Without loss of generality M_η has as universe the ordinal $\gamma_\eta < \partial$ for $\eta \in {}^{\partial>}2$.
Let $\langle S_i : i < \partial\rangle$ be a partition of ∂ to sets, none of which is in WDmId(∂), possible by the assumption $\boxtimes_3$. For each $i < \partial$ we define a function $\mathbf{c}_i$ as follows:

$\boxdot_4$ if $\delta \in S_i$ and $\eta, \nu \in {}^\delta 2$ and $\gamma_\eta = \gamma_\nu = \delta = \gamma_{h(\eta)} = \gamma_{h(\nu)}$, and $f : \delta \to \delta$ then
- (a) $\mathbf{c}_i(\eta, \nu, f) = 0$ if we can find $\eta_1, \eta_2 \in {}^\partial 2$ satisfying $h(\eta)\hat{\ }\langle 0\rangle \triangleleft \eta_1$ and $h(\nu)\hat{\ }\langle 0\rangle \triangleleft \eta_2$ such that f can be extended to an isomorphism from M_{η_1} onto M_{η_2}
- (b) $\mathbf{c}_i(\eta, \nu, f) = 1$ otherwise.

So for $i < \partial$, as $S_i \notin$ WDmId(∂), for some $\varrho^*_i \in {}^\partial 2$ we have:

$(*)_i$ for every $\eta \in {}^\partial 2, \nu \in {}^\partial 2$ and $f \in {}^\partial\partial$ the following set of ordinals is stationary:

$$\{\delta \in S_i : \mathbf{c}_i(\eta \restriction \delta, \nu \restriction \delta, f \restriction \delta) = \varrho^*_i(\delta)\}.$$

Now for any $X \subseteq \partial$ let $\eta_X \in {}^\partial 2$ be defined by:

$\boxdot_5$ if $\alpha \in S_i$ then $i \in X \Rightarrow \eta_X(\alpha) = 1 - \varrho^*_i(\alpha)$ and $i \notin X \Rightarrow \eta_X(\alpha) = 0$.

For $X \subseteq \partial$ let $\rho_X := \eta_{\{2i:i\in X\}\cup\{2i+1:i\in\partial\setminus X\}} \in {}^\partial 2$. Now we shall show

$\oplus$ if $X, Y \subseteq \partial$, and $X \neq Y$ then $M_{h(\rho_X)}$ is not isomorphic to $M_{h(\rho_Y)}$.

Clearly $\oplus$ will suffice for finishing the proof.

Assume toward a contradiction that f is an isomorphism of $M_{h(\rho_X)}$ onto $M_{h(\rho_Y)}$; as $X \neq Y$ there is i such that $i \in X \Leftrightarrow i \notin Y$ so there is $j \in \{2i, 2i+1\}$ such that

$\boxdot_6$ $\rho_X \upharpoonright S_j = \langle 1 - \varrho^*_j(\alpha) : \alpha \in S_j \rangle$ and $\rho_Y \upharpoonright S_j$ is identically zero.

Clearly the set $E = \{\delta : f$ maps δ onto δ and $h(\rho_X \upharpoonright \delta), h(\rho_Y \upharpoonright \delta) \in {}^{\delta}2$ and the universes of $M_{h(\rho_X \upharpoonright \delta)}, M_{h(\rho_Y \upharpoonright \delta)}$ are $\delta\}$ is a club of ∂ and hence $S_j \cap E \neq \emptyset$.

So if $\delta \in S_j \cap E$ then f extends $f \upharpoonright M_{h(\rho_X) \upharpoonright \delta}$ and f is an isomorphism from $M_{h(\rho_X)}$ onto $M_{h(\rho_Y)}$; by the choice of ϱ^*_j we can choose $\delta \in S_j \cap E$ such that:

$\boxdot_7$ $\mathbf{c}_j(\rho_X \upharpoonright \delta, \rho_Y \upharpoonright \delta, f \upharpoonright \delta) = \varrho^*_j(\delta)$.

Also by the choice of j, i.e. $\boxdot_6$ we have

$\boxdot_8$ $\rho_X(\delta) = 1 - \varrho^*_i(\delta)$ and $\rho_Y(\delta) = 0$.

<u>Subcase 2A</u>: $\rho_X(\delta) = 0$.

Now $\rho_X \upharpoonright \delta \triangleleft (\rho_X \upharpoonright \delta)\hat{}\langle \rho_X(\delta) \rangle = (\rho_X \upharpoonright \delta)\hat{}<0> \triangleleft \rho_X \in {}^{\partial}2$ and $(\rho_Y \upharpoonright \delta) \triangleleft (\rho_Y \upharpoonright \delta)\hat{}\langle \rho_Y(\delta) \rangle = \rho_Y\hat{}<0> \triangleleft \rho_Y \in {}^{\partial}2$ (as $\rho_X(\delta) = 0$ by the case and $\rho_Y(\delta) = 0$ as $\delta \in S_j$ and the choice of j, i.e. by $\boxdot_6$). Hence f, ρ_X, ρ_Y witness that by the definition of $\mathbf{c}_j$ we get

$\otimes_1$ $\mathbf{c}_j(\rho_X \upharpoonright \delta, \rho_Y \upharpoonright \delta, f \upharpoonright \delta) = 0$.

Also, by $\boxdot_8$

$\otimes_2$ $0 = \rho_X(\delta) = 1 - \varrho^*_j(\delta)$ so $\varrho^*_j(\delta) = 1$.

But $\otimes_1 + \otimes_2$ contradict the choice of δ, (indirectly the choice of ϱ^*_j), i.e., contradicts $\boxdot_7$.

<u>Subcase 2B</u>: $\rho_X(\delta) = 1$.

By $\boxdot_7$ and $\boxdot_6$ and the case assumption we have $\mathbf{c}_j(\rho_X \upharpoonright \delta, \rho_Y \upharpoonright \delta, f \upharpoonright \delta) = \varrho^*_j(\delta) = 1 - \rho_X(\delta) = 0$ hence by the definition of $\mathbf{c}_j$ there are $\eta_1, \eta_2 \in {}^{\partial}2$ such that $h(\rho_X \upharpoonright \delta)\hat{}<0> \triangleleft \eta_1, h(\rho_Y \upharpoonright \delta)\hat{}<0> \triangleleft \eta_2$, and there is an isomorphism g from M_{η_1} onto M_{η_2} extending $f \restriction \delta$. There is $\delta_1 \in (\delta, \partial)$ such that: f maps $M_{h(\rho_X) \restriction \delta_1}$ onto $M_{h(\rho_Y) \restriction \delta_1}$ and g maps $M_{\eta_1 \restriction \delta_1}$ onto $M_{\eta_2 \restriction \delta_2}$. Now by the choice of h, i.e., clause $(b)_{\rm no}$ above, with $h(\rho_Y \restriction \delta)\hat{}<0>, \eta_2 \restriction \delta_1, h(\rho_Y) \restriction \delta_1$ here standing for ν, ρ_1, ρ_2 there and get ν_1, ν_2, g^* as there so $\eta_2 \restriction \delta_1 \triangleleft \nu_1 \in {}^{\partial}2, h(\rho_Y) \restriction \delta_1 \triangleleft \nu_2 \in {}^{\partial}2$ and g^* is an isomorphism form M_{ν_1} onto M_{ν_2} over $M_{h(\rho_Y) \restriction \delta\hat{}<0>}$. So this contradicts $\boxtimes_1$ in the assumption of the claim with $\delta, \delta_1, h(\rho_X \restriction \delta), h(\rho_Y \restriction \delta), \eta_1 \restriction \delta_1, h(\rho_X \restriction \delta_1), \eta_2 \restriction \delta_1, h(\rho_Y) \restriction \delta_1, f \restriction M_{\eta_1 \restriction \delta_1}, g \restriction M_{h(\rho_X \restriction \delta_1)}, \nu_1, \nu_2$ here standing for $\delta, \delta(1), \eta_0, \eta_1, \nu_0, \nu_1, \nu_0', \nu_1', f_0, f_1, \rho_0, \rho_1$ there. $\square_{9.7}$

§10 Proof of the non-structure theorems with choice functions

When we try to apply several of the coding properties, we have to use the weak diamond (as e.g. in 9.7), but in order to use it we have to fix some quite arbitrary choices; this is the role of the $\bar{\mathbb{F}}$'s here. Of course, we can weaken 10.1, but no need here.

10.1 Hypothesis. $\mathfrak{u}$ is a nice ∂-construction framework (so ∂ is regular uncountable) and τ is a $\mathfrak{u}$-sub-vocabulary.

10.2 Definition. We call a model $M \in \mathfrak{K}_{\mathfrak{u}}$ standard if $M \in K^{\circ}_{\mathfrak{u}} := \{M \in K_{\mathfrak{u}}$: every member of M is an ordinal $< \partial^+\}$ and $\mathfrak{K}^{\circ}_{\mathfrak{u}} = (K^{\circ}_{\mathfrak{u}}, \leq_{\mathfrak{K}} \restriction K^{o}_{\mathfrak{u}})$.

<u>Convention</u>: Models will be standard in this section if not said otherwise.

10.3 Definition. 1) Let $K^{\rm rt}_{\partial} = K^{\rm rt}_{\mathfrak{u}}$ be the class of quadruples $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}, \bar{\mathbb{F}})$ such that:

- (A) $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ recalling 1.15, and $M_{\partial} = \cup\{M_\alpha : \alpha < \partial\}$ has universe some ordinal $< \partial^+$ divisible by ∂ hence M_α is standard for $\alpha < \partial$

(B) $\bar{\mathbb{F}} = \langle \mathbb{F}_\alpha : \alpha < \partial \rangle$ where $\mathbb{F}_\alpha$ is a $\mathfrak{u}$-amalgamation choice function, see part (2) below and[32] if $2^\partial = \partial^+$ then each $\mathbb{F}_\alpha$ has strong uniqueness, see Definition 10.4(2) below.

2) We say that $\mathbb{F}$ is a $\mathfrak{u}$-amalgamation function <u>when</u>:

- (a) $\mathrm{Dom}(\mathbb{F}) \subseteq \{(M_0, M_1, M_2, \mathbf{J}_1, \mathbf{J}_2, A) : M_\ell \in \mathfrak{K}^\circ_\mathfrak{u}$ for $\ell \le 2, M_0 \le_\mathfrak{K} M_\ell, (M_0, M_\ell, \mathbf{J}_\ell) \in$ FR$_\ell$ for $\ell = 1, 2$ and $M_1 \cap M_2 = M_0$ and $M_1 \cup M_2 \subseteq A \subseteq \partial^+$, and $|A \backslash M_1 \backslash M_2| < \partial\}$
- (b) if $\mathbb{F}(M_0, M_1, M_2, \mathbf{J}_1, \mathbf{J}_2, A)$ is well defined <u>then</u> it has the form $(M_3, \mathbf{J}^+_1, \mathbf{J}^+_2)$ such that
  - (α) $M_\ell \le_\mathfrak{K} M_3 \in K^\circ_{<\partial}$ for $\ell = 1, 2$
  - (β) $|M_3| = A$
  - (γ) $(M_0, M_\ell, \mathbf{J}_\ell) \le^\ell_\mathfrak{u} (M_{3-\ell}, M_3, \mathbf{J}^+_\ell)$ for $\ell = 1, 2$.
- (c) if $(M_0, M_1, M_2, \mathbf{J}_1, \mathbf{J}_2)$ are as in clause (a) then for some A we have:
  $\mathbb{F}(M_0, M_1, M_2, \mathbf{J}_1, \mathbf{J}_2, A)$ is well defined and for any such A, the set $A \backslash M_1 \backslash M_2$ is disjoint to $\sup\{\gamma+1 : \gamma \in M_1$ or $\gamma \in M_2\}$
- (d) if[33] $\mathbb{F}(M_0, M_1, M_2, \mathbf{J}_1, \mathbf{J}_2, A^k)$ is well defined for $k = 1, 2$ then $|A^1 \backslash M_1 \backslash M_2| = |A^2 \backslash M_1 \backslash M_2|$
  moreover
- (e) if $\mathbb{F}(M_0, M_1, M_2, \mathbf{J}_1, \mathbf{J}_2, A^1)$ is well defined and $M_1 \cup M_2 \subseteq A^2 \subseteq \partial^+$ and $\mathrm{otp}(A^2 \backslash M_1 \backslash M_2) = \mathrm{otp}(A^1 \backslash M_1 \backslash M_2)$ then also $\mathbb{F}(M_0, M_1, M_2, \mathbf{J}_1, \mathbf{J}_2, A^2)$ is well defined.

3) Let $(\bar{M}^1, \bar{\mathbf{J}}^1, \mathbf{f}^1, \bar{\mathbb{F}}^1) <^{\mathrm{at}}_\mathfrak{u} (\bar{M}^2, \bar{\mathbf{J}}^2, \mathbf{f}^2, \bar{\mathbb{F}}^2)$ with at standing for atomic, hold <u>when</u> both quadruples are from $K^{\mathrm{rt}}_\mathfrak{u}$ and there are a club E of ∂ and sequence $\bar{\mathbf{I}} = \langle \mathbf{I}_\alpha : \alpha < \partial \rangle$ witnessing it which means that we have

- (a) $\delta \in E \Rightarrow \mathbf{f}^1(\delta) \le \mathbf{f}^2(\delta)$ & $\mathrm{Min}(E \backslash (\delta + 1)) > \mathbf{f}^2(\delta)$
- (b) for $\delta \in E$, if $i \le \mathbf{f}^1(\delta)$ then $M^1_{\delta+i} \le_{\mathfrak{K}_{<\partial}} M^2_{\delta+i}$ and if $i < \mathbf{f}^i(\delta)$ then $\mathbb{F}^1_{\delta+i} = \mathbb{F}^2_{\delta+i}$

[32]can demand this always

[33]Dropping clause (d) causes little change

(c) $\langle (M^0_\alpha, M^1_\alpha, \mathbf{I}_\alpha) : \alpha \in \cup\{[\delta, \delta + \mathbf{f}^1(\delta)] : \delta \in E\}\rangle$ is $\leq^1_{\mathfrak{u}}$-increasing continuous.

(d) if $\delta \in E$ and $i < \mathbf{f}^1(\delta)$ and A is the universe of $M^2_{\delta+i+1}$ <u>then</u> $(M^2_{\delta+i+1}, \mathbf{I}^2_{\delta+i+1}, \mathbf{J}^2_{\delta+i}) =$
$= \mathbb{F}^1_{\delta+i}(M^1_{\delta+i}, M^1_{\delta+i+1}, M^2_{\delta+i}, \mathbf{I}_{\delta+i}, \mathbf{J}^1_{\delta+i}, A)$.

4) We say that $(\bar M^\delta, \bar{\mathbf{J}}^\delta, \mathbf{f}^\delta, \bar{\mathbb{F}}^\delta)$ is a canonical upper bound of $\langle(\bar M^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha, \bar{\mathbb{F}}) : \alpha < \delta\rangle$ as in Definition 1.15(4) adding: in clause (c), case 1 subclause (γ) to the conclusion $\mathbb{F}^\delta_{\zeta+i} = \mathbb{F}^{\alpha_\xi}_{\zeta+i}$ (and similarly in case 2).
4A) We say $(\langle \bar M^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha, \bar{\mathbb{F}}^\alpha) : \alpha < \alpha(*)\rangle$ is a $<^{\rm at}_{\mathfrak{u}}$-tower <u>if</u>:

(a) $(\bar M^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha, \bar{\mathbb{F}}^\alpha) <^{\rm at}_{\mathfrak{u}} (\bar M^{\alpha+1}, \mathbf{J}^{\alpha+1}, \bar{\mathbb{F}}^{\alpha+1})$ for when $\alpha + 1 < \alpha(*)$

(b) if $\delta < \alpha(*)$ is a limit ordinal, <u>then</u> $(\bar M^\delta, \bar{\mathbf{J}}^\delta, \mathbf{f}^\delta, \bar{\mathbb{F}}^\delta) \in K^{\rm rt}_{\mathfrak{u}}$ is a canonical upper bound of the sequence $\langle(\bar M^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha, \mathbb{F}^\alpha) : \alpha < \delta\rangle$.

5) Let $(\bar M, \bar{\mathbf{J}}, \mathbf{f}, \bar{\mathbb{F}}) \leq^{\rm rs}_{\mathfrak{u}} (\bar M', \bar{\mathbf{J}}', \mathbf{f}', \bar{\mathbb{F}}')$ means that for some $<^{\rm at}_{\mathfrak{u}}$-tower $\langle(\bar M^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha, \bar{\mathbb{F}}^\alpha) : \alpha \leq \alpha(*)\rangle$ we have $(\bar M, \bar{\mathbf{J}}, \mathbf{f}, \bar{\mathbb{F}}) = (\bar M^0_{\alpha(*)}, \bar{\mathbf{J}}^0, \mathbf{f}^0, \bar{\mathbb{F}}^0)$ and $(\bar M', \bar{\mathbf{J}}', \mathbf{f}', \bar{\mathbb{F}}') = (\bar M^{\alpha(*)}, \bar{\mathbf{J}}^{\alpha(*)}, \mathbf{f}, \bar{\mathbb{F}}^{\alpha(*)})$.
6) We say that the sequence $\langle(\bar M^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha, \bar{\mathbb{F}}^\alpha) : \alpha < \alpha(*))\rangle$ is $\leq^{\rm rs}_{\mathfrak{u}}$-increasing continuous if it is $\leq^{\rm rs}_{\mathfrak{u}}$-increasing and for any limit $\delta < \alpha(*)$ the tuple $(\bar M^\delta, \bar{\mathbf{J}}^\delta, \mathbf{f}^\delta, \bar{\mathbb{F}}^\delta)$ is a canonical upper bound of the sequence of $\langle(\bar M^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha, \bar{\mathbb{F}}^\alpha) : \alpha < \delta\rangle$.

10.4 Definition. For $\mathbb{F}$ a $\mathfrak{u}$-amalgamation choice function, see Definition 10.3(2):
1) $\mathbb{F}$ has uniqueness <u>when</u>:

⊛ if $\mathbb{F}(M^\ell_0, M^\ell_1, M^\ell_2, \mathbf{J}^\ell_1, \mathbf{I}^\ell_1, A^\ell) = (M^\ell_3, \mathbf{J}^\ell_2, \mathbf{I}^\ell_2)$ for $\ell = 1, 2$ (all models standard) and f is a one to one function from A^1 onto A^2 preserving the order of the ordinals, f maps M^1_i to M^2_i for $i = 0, 1, 2$ (i.e. $f \restriction M^1_i$ is an isomorphism from M^1_i onto M^2_i) and $\mathbf{J}^1_1, \mathbf{I}^1_1$ onto $\mathbf{J}^2_1, \mathbf{I}^2_1$, respectively, <u>then</u> f is an isomorphism from M^1_3 onto M^2_3 mapping $\mathbf{J}^1_2, \mathbf{I}^1_2$ onto $\mathbf{J}^2_2, \mathbf{I}^2_2$ respectively.

2) $\mathbb{F}$ has strong uniqueness <u>when</u> $\mathbb{F}$ has uniqueness and

> ⊛ if $\mathbb{F}(M_0^\ell, M_1^\ell, M_2^\ell, \mathbf{J}_1^\ell, \mathbf{I}_1^\ell, A^\ell) = (M_3^\ell, \mathbf{J}_2^\ell, \mathbf{I}_2^\ell)$ for $\ell = 1, 2$ and f is a one to one mapping from $M_1^1 \cup M_2^1$ onto $M_1^2 \cup M_2^2$ such that: $f \restriction M_i^1$ is an isomorphism from M_i^1 onto M_i^2 for $i = 0, 1, 2$ and it maps $\mathbf{J}_1^1, \mathbf{I}_1^1$ onto $\mathbf{J}_1^2, \mathbf{I}_1^2$, respectively, <u>then</u> $|A^1 \backslash M_1^1 \backslash M_2^1| = |A^2 \backslash M_1^2 \backslash M_2^2|$, and there is an isomorphism g from M_3^1 onto M_3^2 extending f and mapping $\mathbf{J}_2^1, \mathbf{I}_2^1$ onto $\mathbf{J}_2^2, \mathbf{I}_2^2$ respectively; and moreover, $\text{otp}(A^1 \backslash M_1^1, M_2^1) = \text{otp}(A^2 \backslash M_1^2 \backslash M_2^2)$ and $f \restriction (A \backslash M_1^1 \backslash M_2^1)$ is order preserving.

10.5 Remark. 1) In Definition 10.3, in part (2) we can replace clause (e) by demanding A is an interval of the form $[\gamma_*, \gamma_* + \theta]$ where $i < \partial, \gamma_* = \cup\{\gamma + 1 : \gamma \in M_1$ or $\gamma \in M_2\}$. Then in part (3) we have M_∂^1 has universe δ for some $\delta < \partial^+$ and M_γ^2 has universe $\delta + \partial$. Also the results in 10.7, 10.8 becomes somewhat more explicit.
2) We can fix $\mathbb{F}_*$, i.e. demand $\mathbb{F}_\alpha = \mathbb{F}_*$ in Definition 10.3.

10.6 Claim. *1) There is a $\mathfrak{u}$-amalgamation function with strong uniqueness.*
2) $K_\mathfrak{u}^{\rm rt}$ is non-empty, moreover for any stationary $S \subseteq \partial$ and triple $(\bar M, \bar{\mathbf{J}}, \mathbf{f}) \in K_\mathfrak{u}^{\rm qt}$, there is $\bar{\mathbb{F}}$ such that $(\bar M, \bar{\mathbf{J}}, \mathbf{f}, \bar{\mathbb{F}}) \in K_\mathfrak{u}^{\rm rt}$ with $S = \{\delta < \partial : \mathbf{f}(\delta) > 0\}$.
3) If $(\bar M, \bar{\mathbf{J}}^1, \mathbf{f}^1, \bar{\mathbb{F}}^1) \in K_\mathfrak{u}^{\rm rt}$ <u>then</u> for some $(\bar M^2, \bar{\mathbf{J}}^2, \mathbf{f}^2, \bar{\mathbb{F}}^2) \in K_\mathfrak{u}^{\rm rt}$ we have $(\bar M^1, \bar{\mathbf{J}}^1, \mathbf{f}^1, \bar{\mathbb{F}}^1) <_\mathfrak{u}^{\rm at} (\bar M^2, \bar{\mathbf{J}}^2, \mathbf{f}^2, \bar{\mathbb{F}}^2)$; moreover, if δ is the universe of M_∂^1 then $\alpha < \partial \Rightarrow M_\alpha^2 \backslash \delta \in \{[\delta, \delta + i) : i < \partial\}$.
4) Canonical upper bound as in 10.3(4) exists.

Proof. 1) Let $\mathbf{X}$ be the set of quintuples $\mathbf{x} = (M_0^\mathbf{x}, M_1^\mathbf{x}, M_2^\mathbf{x}, \mathbf{J}_1^\mathbf{x}, \mathbf{J}_2^\mathbf{x})$ as in clause (a) of Definition 10.3(2). We define a two-place relation E on $\mathbf{X} : \mathbf{x} E \mathbf{y}$ iff $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and there is a one-to-one function f from $M_1^\mathbf{y} \cup M_2^\mathbf{y}$ onto $M_1^\mathbf{x} \cup M_2^\mathbf{x}$ such that $f \restriction M_\ell^\mathbf{x}$ is an isomorphism from $M_\ell^\mathbf{x}$ onto $M_\ell^\mathbf{y}$ for $\ell = 0, 1, 2$ and f maps $\mathbf{J}_\ell^\mathbf{y}$ onto $\mathbf{J}_\ell^\mathbf{x}$ for $\ell = 1, 2$. Clearly E is an equivalence relation, and let $\mathbf{Y} \subseteq \mathbf{X}$ be a set of representatives and for every $\mathbf{x} \in \mathbf{X}$ let $\mathbf{y}(\mathbf{x})$ be the unique $\mathbf{y} \in \mathbf{Y}$

which is E-equivalent to it and let $f = f_{\mathbf{x}}$ be a one-to-one function from $M_1^{\mathbf{y}} \cup M_2^{\mathbf{y}}$ onto $M_1^{\mathbf{x}} \cup M_2^{\mathbf{x}}$ witnessing the equivalence.

For each $\mathbf{y} \in \mathbf{Y}$ by clause (F) of Definitin 1.2 there is a triple $(M_3^{\mathbf{y}}, \mathbf{J}_1^{+,\mathbf{y}}, \mathbf{J}_2^{+,\mathbf{y}})$ such that

(∗) $(M_0^{\mathbf{x}}, M_\ell^{\mathbf{x}}, \mathbf{J}_\ell^{\mathbf{x}}) \le_{\mathfrak{u}}^{\ell} (M_{3-\ell}^{\mathbf{y}}, M_3^{\mathbf{y}}, \mathbf{J}_\ell^{+,\mathbf{y}})$ for $\ell = 1, 2$.

Without loss of generality $M_3^{\mathbf{y}}$ has universe $\subseteq \partial^+$, we can add has universe $M_1^{\mathbf{y}} \cup M_2^{\mathbf{y}} \cup [\gamma, \gamma + \theta)$ where $\gamma = \sup\{\alpha + 1 : \alpha \in M_1^{\mathbf{y}} \cup M_2^{\mathbf{y}}\}$ where θ is the cardinality of $M_3^{\mathbf{y}} \backslash M_1^{\mathbf{y}} \backslash M_2^{\mathbf{y}}$.

Lastly, let us define $\mathbb{F}$ as follows:

(a) $\zeta_{\mathbf{x}} = \mathrm{otp}(M_3^{\mathbf{y}(\mathbf{x})} \backslash M_1^{\mathbf{y}(\mathbf{x})} \backslash M_2^{\mathbf{y}(\mathbf{x})}) < \partial$
and

(b) $\mathrm{Dom}(\mathbb{F}) = \{ (\mathbf{x}, A) : \mathbf{x} \in \mathbf{X},\ M_1^{\mathbf{x}} \cup M_2^{\mathbf{x}} \subseteq A \subseteq \partial$ and $\mathrm{otp}(A \backslash M_1^{\mathbf{x}} \backslash M_2^{\mathbf{x}}) = \zeta_{\mathbf{x}}\}$,
where $(\mathbf{x}, A)$ means $(M_0^{\mathbf{x}}, M_1^{\mathbf{x}}, M_2^{\mathbf{x}}, \mathbf{J}_2^{\mathbf{x}}, \mathbf{J}_2^{\mathbf{x}}, A)$

(c) for $(\mathbf{x}, A) \in \mathrm{Dom}(\mathbb{F})$ let $f_{\mathbf{x},A}$ be the unique one to one function from $M_3^{\mathbf{y}(\mathbf{x})}$ onto A which extends $f_{\mathbf{x}}$ and is order preserving mapping from $M_3^{\mathbf{y}(\mathbf{x})} \backslash M_1^{\mathbf{y}(\mathbf{x})} \backslash M_2^{\mathbf{y}(\mathbf{x})}$ onto $A \backslash M_1^{\mathbf{x}} \backslash M_2^{\mathbf{x}}$

(d) for $(\mathbf{x}, A) \in \mathrm{Dom}(\mathbb{F})$ let $\mathbb{F}(\mathbf{x}, A)$ be the image under $f_{\mathbf{x},A}$ of $(M_3^{\mathbf{y}(\mathbf{x})}, \mathbf{J}_1^{+,\mathbf{y}(\mathbf{x})}, \mathbf{J}_2^{+,\mathbf{y}(\mathbf{x})})$.

Now check.
2) By part (1) and "$K_{\mathfrak{u}}^{\mathrm{qt}} \neq \emptyset$", see 1.19(1).
3) Put together the proof of 1.19(3) and part (2).
4) As in the proof of 1.19(4). $\square_{10.6}$.

10.7 Claim. *1) There is a function* $\mathbf{m}$ *satisfying:*

⊛$_1$ *if* $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}, \bar{\mathbb{F}}) \le_{\mathfrak{u}}^{\mathrm{rs}} (\bar{M}', \bar{\mathbf{J}}', \mathbf{f}', \bar{\mathbb{F}}')$, *recalling 10.3(5)* *<u>then</u> for some function* $h : \partial \to \mathscr{H}_{<\partial}(\partial^+)$, *but if* $2^\partial = \partial^+$ *then* $h : \partial \to \mathscr{H}_{<\partial}(\partial)$ *we have:*

⊙ *for a club of* $\delta < \partial$ *the object* $\mathbf{m}(h \restriction \delta, \bar{M}, \bar{\mathbf{J}}, \mathbf{f}, \bar{\mathbb{F}}, M'_\delta)$ *is a model* $N \in K_{\mathfrak{u}}$ *such that*

(a) $M_{\delta + \mathbf{f}(\delta)} \le_{\mathfrak{u}} N$

(b) $M'_\delta \le_{\mathfrak{u}} N$

(c) *there is* $N' \leq_u M'_\partial$ *isomorphic to* N *over* $M_{\delta+\mathbf{f}(\delta)} + M'_\delta$ *in fact* $N' = M'_{\delta+\mathbf{f}(\delta)}$

$\circledast_2$ $\mathbf{m}$ *is preserved by partial, order preserving functions from* ∂^+ *to* ∂^+ *compatible with* id_{M_∂}

$\circledast_3$ *in fact in* $\odot$ *above,* $\mathbf{m}(h \restriction \delta, \bar{M}, \bar{\mathbf{J}}, \mathbf{f}, \bar{\mathbb{F}}, M'_\delta)$ *is actually* $\mathbf{m}(h \restriction \delta, \bar{M} \restriction [\delta, \delta + \mathbf{f}(\delta) + 1, \bar{\mathbf{J}} \restriction [\delta, \delta + \mathbf{f}(\delta)), \bar{\mathbb{F}} \restriction [\delta, \delta + \mathbf{f}(\delta)], M'_\delta)$.

Proof. By 10.8 we can define $\mathbf{m}$ explicitly. $\square_{10.7}$

10.8 Claim. $(\bar{M}', \bar{\mathbf{J}}', \mathbf{f}', \bar{\mathbb{F}}') \leq^{\mathrm{rs}}_{\mathfrak{u}} (\bar{M}'', \bar{\mathbf{J}}'', \mathbf{f}'', \bar{\mathbb{F}}'')$ *iff both are from* $K^{\mathrm{rt}}_{\mathfrak{u}}$ *and we can find a* $E, \alpha(*) = \alpha_*, \bar{u}, \bar{\mathbf{d}}$ *such that:*

$\circledast$ (a) *E is a club of* ∂

(b) *α_* an ordinal* $< \partial^+$

(c) *$\bar{u}$ is a $\subseteq$-increasing continuous sequence* $\langle u_i : i < \partial \rangle$

(d) $i < \partial \Rightarrow |u_i| < \partial$ *and* $\alpha_* + 1 = \cup\{\alpha_i : i < \partial\}$ *and* $\alpha \in u_0, 0 \in u_0$ *and* $(\forall \beta < \alpha)(\beta \in u_i \equiv \beta + 1 \in u_i)$

(e) $\bar{\mathbf{d}} = \langle \mathbf{d}_\delta : \delta \in E \rangle$

(f) *$\mathbf{d}_\delta$ is a $\mathfrak{u}$-free* $(\mathbf{f}(\delta), \mathrm{otp}(u_\delta))$*-rectangle, see Definition 1.4*

(g) *there is a $\leq^{\mathrm{at}}_{\mathfrak{u}}$-tower* $\langle (\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha, \bar{\mathbb{F}}^\alpha) : \alpha \leq \alpha_* \rangle$ *as in 10.3(6) witnessing the assumption and* $\langle \bar{\mathbf{I}}^\alpha : \alpha < \alpha_* \rangle$ *with* $\mathbf{I}^\alpha$ *witnessing* $(\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha, \bar{\mathbb{F}}^\alpha) <^{\mathrm{at}}_{\mathfrak{u}_*} (\bar{M}^{\alpha+1}, \bar{\mathbf{J}}^{\alpha+1}, \mathbf{f}^{\alpha+1}, \bar{\mathbb{F}}^{\alpha+1})$ *(so* $(\bar{M}^0, \bar{\mathbf{J}}^0, \mathbf{f}^0, \bar{\mathbb{F}}^0) = (\bar{M}', \bar{\mathbf{J}}', \mathbf{f}', \bar{\mathbb{F}}'), (\bar{M}^{\alpha(*)}, \bar{\mathbf{J}}^{\alpha(*)}, \mathbf{f}^{\alpha(*)}, \bar{\mathbb{F}}^{\alpha(*)}) = (\bar{M}'', \bar{\mathbf{J}}'', \mathbf{f}'', \bar{\mathbb{F}}''))$ *such that*

$\boxdot$ *if* $\beta \in [0, \alpha_* + 1), \delta \in E$ *and* $\beta \in u_\delta$ *and* $j = \mathrm{otp}(\beta \cap u_\delta)$ *then for every* $i \leq \mathbf{f}(\delta)$

(α) $M^{\mathbf{d}_\delta}_{i,j} = M^\beta_{\delta+i}$

(β) $\mathbf{J}^{\mathbf{d}_\delta}_{i,0} = \mathbf{J}'_{\delta+i}$ *when* $i < \mathbf{f}(\delta)$

(γ) $\mathbf{I}^{\mathbf{d}_\delta}_{0,j} = \mathbf{I}^\beta_\delta$ *when* $\beta < \alpha_*$

(δ) *if* $i < \mathbf{f}(\delta)$ *and* $\beta < \alpha_*$ *(so* $\mathrm{otp}(\beta \cap u_\delta) < \mathrm{otp}(u_\delta)$*) then*

$$(M^{\mathbf{d}_\delta}_{i+1,j+1}, \mathbf{I}^{\mathbf{d}_\delta}_{i,j+1}, \mathbf{J}^{\mathbf{d}_\delta}_{i+1,j}) = \\ = \mathbb{F}_{\delta+i}(M^{\mathbf{d}_\delta}_{i,j}, M^{\mathbf{d}_\delta}_{i+1,j}, M^{\mathbf{d}_\delta}_{i,j+1}, \mathbf{I}^{\mathbf{d}_\delta}_{i,j}, \mathbf{J}^{\mathbf{d}_\delta}_{i,j}, |M^{\mathbf{d}_\delta}_{i+1,j+1}|).$$

Proof. Straight. $\square_{10.8}$

10.9 Remark. If we define the version of 10.3(3) with $|M^2| = |M^1| + \partial$ then $\mathbf{m}(-)$ is O.K. not only up to isomorphism but really given the value.

10.10 Claim. *Theorem 2.3 holds.*
That is, $\dot{I}_\tau(\partial^+, K^{\mathfrak{u},\mathfrak{h}}_{\partial^+}) \geq \mu_{\rm unif}(\partial^+, 2^\partial)$ *when:*

$\circledast$ (a) $2^\theta = 2^{<\partial} < 2^\partial$
(b) $2^\partial < 2^{\partial^+}$
(c) *the ideal* WDmId(∂) *is not* ∂^+*-saturated*
(d) $\mathfrak{u}$ *has the weak* τ*-coding, see Definition 2.2(5) (or just above* $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*) \in K^{\rm qt}_{\mathfrak{u}}$ *with* WDmId$(\partial) \restriction (\mathbf{f}^*)^{-1}\{0\}$ *not* ∂^+*-saturated)*
(e) $\mathfrak{h}$ *is* $\mathfrak{u} - \{0,2\}$*-appropriate.*

10.11 Remark. 1) Similarly 2.7 holds.
2) We can below (in $\boxtimes$) imitate the proof of 3.3.

Proof. Clearly when $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$ as in clause (d) is not given, by 10.6(2) we can choose it, even with $\mathbf{f}^*$ constantly zero, so without loss of generality such a triple is given. By 1.25(4) and clauses (d) + (e), without loss of generality :

$\otimes$ $\mathfrak{h} = (\mathfrak{h}_0, \mathfrak{h}_2)$ witness that $\{0,2\}$-almost every triple $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K^{\rm qt}_{\mathfrak{u}}$ above $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$ has the weak coding property.

Let $\bar{S}$ be such that

$\odot$ *(a)* $\bar{S} = \langle S^*_\zeta : \zeta < \partial^+ \rangle$
(b) $S^*_\zeta \subseteq \partial$
(c) S^*_ζ is increasing modulo $[\partial]^{<\partial}$
(d) S^*_0 and $S^*_{\zeta+1} \backslash S^*_\zeta \notin$ WDmId(∂)
(e) $\mathbf{f}^* \restriction (\partial \backslash S^*_0)$ is constantly zero.

Such sequence exists by clause (c) of the hypothesis. It suffices to deal with the case $\mathfrak{h}_2$ is a $\mathfrak{u} - 2 - S^*_0$-appropriate function, see Definition 1.24(2A).

We choose $\langle(\bar{M}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta, \bar{\mathbb{F}}^\eta) : \eta \in {}^\gamma(2^\partial)\rangle$ by induction on $\gamma < \partial^+$ such that:

⊞ (a) $(\bar{M}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta, \bar{\mathbb{F}}^\eta) \in K^{\rm rt}_{\mathfrak{u}}$ and if $\gamma = 0$ then $(\bar{M}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta) = (\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$

(b) if $i \notin S^*_{\ell g(\eta)}$ then $\mathbf{f}_\eta(i) = 0$

(c) $\langle(\bar{M}^{\eta\restriction\beta}, \bar{\mathbf{J}}^{\eta\restriction\beta}, \mathbf{f}^{\eta\restriction\beta}, \bar{\mathbb{F}}^{\eta\restriction\beta} : \beta \le \gamma\rangle$ is $\le^{\rm rs}_{\mathfrak{u}}$-increasing continuous,

(d) if $\eta \in {}^{\gamma+1}(2^\partial)$ and γ is a non-limit ordinal and $\alpha < 2^\partial$ <u>then</u> the pair $((\bar{M}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta), (\bar{M}^{\eta\hat{}<\alpha>}, \bar{\mathbf{J}}^{\eta\hat{}<\gamma>}, \mathbf{f}^{\eta\hat{}<\alpha>}))$ strictly $1 - S^*_0$-obey $\mathfrak{h}_2$ and 0-obey $\mathfrak{h}_0$ (see Definition 1.22(1), 1.24(2) so without loss of generality for all $\eta\hat{}\langle\alpha\rangle$ we choose the same value

(e) $\eta \in {}^\gamma(2^\partial)$ and $\alpha_1 \ne \alpha_2 < 2^\partial, \gamma$ a limit ordinal (even ∂ divides γ) and $(\bar{M}^{\eta\hat{}<\alpha_2>}, \bar{\mathbf{J}}^{\eta\hat{}<\alpha_2>}, \bar{\mathbb{F}}^{\eta\hat{}<\alpha_2>}) \le^{\rm rt}_{\mathfrak{u}} (M', \bar{\mathbf{J}}', \mathbf{f}', \bar{\mathbb{F}}')$ <u>then</u> $M^{\eta\hat{}<\alpha_1>}_\partial$ cannot be $\le_{\mathfrak{K}}$-embedded into M'_∂ over M_η.

Why this is enough? By 9.1, noting that

(∗) if $\gamma(*) < \partial^+$ and $\eta\hat{}\langle\alpha_i\rangle \trianglelefteq \nu_i \in {}^{\gamma(*)}(2^\partial)$ for $i = 0, 1$ and $\alpha_0 < \alpha_1 < 2^\partial$ and f is an isomorphism from $M^{\nu_0}_\partial$ onto $M^{\nu_1}_\partial$ over M^η_∂, <u>then</u> $\eta, f \restriction M^{\eta\hat{}<\alpha_0>}_\partial, (\bar{M}_{\nu_1}, \bar{\mathbf{J}}_{\nu_1}, \mathbf{f}_{\nu_2}, \bar{\mathbb{F}}_{\nu_1})$ form a counterexample to clause (e) of ⊞.

For $\gamma = 0$ clause (a) of ⊞, i.e. choose $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$ recallng our use of 10.6(2).

For γ limit use 10.6(4).

So assume $\eta \in {}^{\gamma(*)}(2^\partial)$ and $(\bar{M}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta, \bar{\mathbb{F}}^\eta)$ has been defined and we should deal with $\Xi_\eta := \{\eta\hat{}\langle\alpha\rangle : \alpha < 2^\partial\}$.

We choose $(\alpha(0), N_0, \mathbf{I}_0)$ such that

⊕ (a) $\alpha(0) < \partial$

(b) $(M^\eta_{\alpha(0)}, N_0, \mathbf{I}_0) \in$ FR$_1$

(c) $N_0 \cap M^\eta_\partial = M^\eta_{\alpha(0)}$

(d) if $\gamma(*)$ is a non-limit ordinal then $(\alpha(0), N, \mathbf{I})$ is as dictated by $\mathfrak{h}$, i.e. $\mathfrak{h}_0$, see Definition 1.24(1)(c)

(e) if $\gamma(*)$ is a limit ordinal and $(\bar{M}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta)$ has the weak coding$_1$ property (see Definition 2.2(3)) <u>then</u> for a club of $\alpha(1) \in (\alpha(0), \gamma)$ we have:

  (*) if $(M_{\alpha(0)}, N_0, \mathbf{I}_0) \leq^1_{\mathfrak{u}} (M_{\alpha(1)}, N^*_1, \mathbf{I}^*_1)$ and $M^\eta_\partial \cap N^*_1 = M^\eta_{\alpha(1)}$
  <u>then</u> there are $\alpha(2) \in (\alpha(1), \gamma)$ and $N^\ell_2, \mathbf{I}^\ell_2$ for $\ell = 1, 2$ such that: $(M^\eta_{\alpha(1)}, N_1, \mathbf{I}_1) \leq^1_{\mathfrak{u}} (M^\eta_{\alpha(2)}, N^\ell_2, \mathbf{I}^\ell_2)$ for $\ell = 1, 2$ and N^1_2, N^2_2 are τ-incompatible amalgamations of $M^\eta_{\alpha(2)}, N_1$ over $M^\eta_{\alpha(1)}$.

Without loss of generality

(f) $|N_0| \backslash M^\eta_{\alpha(0)}$ is an initial segment of $\partial^+ \backslash |M^\eta_\partial|$.

We shall use 9.3. Toward this we choose u_i and if $i \in u_i$ also $E^\eta_\rho, M^\eta_\rho, \mathbf{I}^\eta_\rho, \mathbf{J}^\eta_\rho$ (but $\mathbf{J}^\eta_\rho$ is chosen in the $(i+1)$-th step) for $\rho \in {}^i 2$ induction on $i \in [\alpha(0), \partial)$ such that:

$\boxtimes_1$ (a) $u_i \subseteq [\alpha(0), i]$ is closed

(b) if $j < i$ then $u_i \cap (j+1) = u_j$

(c) E^η_ρ is a closed subset of $i \cap u_i$

(d) $j < \ell g(\rho) = E^\eta_{\rho \restriction j} = E^\eta_\rho \cap j$

(e) if $j \in E^\eta_\rho$ then $\mathbf{f}_\eta(j) <$ $\min\{(E^\eta_\rho \backslash j)$ or $E^\eta_\rho \subseteq j\}$ and $(j, j + f^\eta(j)] \subseteq u_i$

(f) $M^\eta_\rho \in K^{\mathfrak{u}}_{<\partial}$ and $M^\eta_\rho \cap M^\eta = M^\eta_{\ell g(\rho)}$

(g) $\langle M^\eta_{\rho \restriction j} : j \in u_i \rangle$ is $\leq_{\mathfrak{K}_{<\partial}}$-increasing continuous

(h) $\langle (M^\eta_j, M^\eta_{\rho \restriction j}, \mathbf{I}^\eta_\rho) : j \in u_i \rangle$ is $\leq^1_{\mathfrak{u}}$-increasing continuous

(i) (α) $(M^\eta_j, M^\eta_{j+1}, \mathbf{J}^\eta_j) \leq^2_{\mathfrak{u}} (M^\eta_{\rho \restriction j}, M^\eta_{\rho \restriction (j+1)}, \mathbf{J}^\eta_{\rho \restriction j})$ when $j \in i \cap u_i$

(β) if $j \in \cup\{[\zeta, \zeta + \mathbf{f}^\eta(\zeta)) : \zeta \in E^\eta_\rho\}$ then moreover we get $(M^\eta_{\rho \restriction (j+1)}, \mathbf{I}^\eta_{\rho \restriction j}, \mathbf{J}^\eta_{\rho \restriction j})$ by applying $\mathbb{F}^\eta_j$

(j) if $i = \ell g(\rho) = j + 1$, j is limit $\in S^*_{\gamma(*)+1} \backslash S^*_{\gamma(*)}$ and $\cup\{E_{\rho \restriction j} : j < \ell g(\rho)\}$ is unbounded in $\ell g(\rho)$ and $\mathbf{f}_\eta(\ell g(\rho)) = 0$ and if we can <u>then</u> $u_i = u_j \cup \{i\}$ and $(M^\eta_{\rho\hat{\ }<\ell>}, \mathbf{I}^\eta_{\rho\hat{\ }<\ell>}, \mathbf{J}^\eta_{\rho\hat{\ }<\ell>})$ for $\ell =$

0, 1 are gotten as in $\oplus$ above so in particular $M^\eta_{\rho\hat{}<0>}, M^\eta_{\rho\hat{}<\ell>}$ are τ-incompatible amalgamations of M^η_i, M^η_ρ over M^η_j

(k) if $i = j+1, \delta = \max(E^\eta_{\rho\restriction j}) \le i, \delta \in S^*_0, j = \delta + \mathbf{f}^\eta(\delta)$, <u>then</u> we act as dictated by $\mathfrak{h}$, i.e. $\mathfrak{h}_2$; moreover this holds for all the interval $[\delta + \mathbf{f}^\eta(\delta), \delta + \mathbf{f}^\eta(\delta) + i']$ for an appropriate $i' < \partial$ by the "dictation" of $\mathfrak{h}_2$ (see Definition 1.22).

In clause (j) this is possible for enough times, if $(\bar{M}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta)$ has the weak coding$_1$ property, i.e. for $\rho \in {}^\partial 2$, for a club of $i \in \mathbf{f}^{-1}_\eta\{0\}$ by the choice of $\mathfrak{h}$. Trace the Definitions.
Also $(\bar{M}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta)$ has the weak coding property by the choice of $\mathfrak{h}$ and the induction hypothesis.

Clearly we can carry the induction on $i < \lambda$ and by 9.3 carrying the induction to $\gamma(*)+1$, so we have finished carrying the induction. So by 9.1 we are done.

$\square_{10.10}$

10.12 Claim. *Theorem 2.11 holds.*
That is, $\dot{I}_\tau(\partial^+, K^{\mathfrak{u}}_{\partial^+}) \ge \mu_{\text{unif}}(\partial^+, 2^\partial)$, *<u>when</u>:*

$\circledast$ (a) $2^\theta = 2^{<\partial} < 2^\partial$

(b) $2^\partial < 2^{\partial^+}$

(c) $\mathfrak{u}$ *has the vertical* τ*-coding*$_1$ *property above some triple from* $K^{\text{qt}}_{\mathfrak{u}}$.

Proof. Like the proof of 10.10 but:

<u>Change (A)</u>: We omit $\bar{S}$, i.e. $\odot$, the choice of $\langle S^*_\varepsilon : \varepsilon < \partial^+\rangle$, can use $S^*_0 = \partial$, or more transparently, use $S^*_\zeta = S^*_0$ stationary, $\partial \backslash S^*_\zeta \in (\text{WDmId}(\partial))^+$, on S^*_0 act as before on $\partial \backslash S^*_0$ and act as on $S^*_{\zeta+1} \backslash S^*_\zeta$ before.

<u>Change (B)</u>: We change clause $\boxtimes(j)$ to fit the present coding so any limit ordinal j. $\square_{10.12}$

10.13 Claim. *Theorem 2.15 holds.*
That is $\dot{I}_\tau(\partial^+, K^{\mathfrak{u}}_{\partial^+}) \ge 2^{\partial^+}$ *when:*

(a) $2^\theta = 2^{<\partial} < 2^\partial$

(b) $2^\partial < 2^{\partial^+}$

(c) $\mathfrak{u}$ *has horizontal* τ*-coding property, say just above* $(\bar{M}^*, \bar{\mathbf{J}}^*, \mathbf{f}^*)$

(d) *the ideal* WDmId(∂) *is not* ∂^+*-saturated.*

Proof. Similar to the proof of 10.10 but:

Change (A): We omit $\odot$, i.e. $\bar{S}^*$ and use $S^*_0 = \partial$

Change (B): In $\boxplus$ we use only $\eta \in {}^{\partial^+}2$ and clause (e) is changed to:

(e)″ if $(\bar{M}^{\eta\hat{\ }<1>}, \bar{\mathbf{J}}^{\eta\hat{\ }<1>}, \mathbf{f}^{\eta\hat{\ }<1>}, \bar{\mathbb{F}}^{\eta\hat{\ }<1>}) \le^{\text{rt}}_{\mathfrak{u}} (\bar{M}', \bar{\mathbf{J}}', \mathbf{f}', \bar{\mathbb{F}}')$ then $M^{\eta\hat{\ }<0>}_\partial$ cannot be $\le_{\mathfrak{K}}$-embedded into M'_∂ over $M^{<>}_\partial$.

Change (C): We change $\boxtimes$ clause (j) to deal with the present coding.

Change (D): We use 9.5 rather than 9.1. $\square_{10.13}$

* * *

10.14 Discussion: 1) Instead constructing $\le^{\text{at}}_{\mathfrak{u}}$-successors $(\bar{M}^{\eta\hat{\ }\langle\alpha\rangle}, \mathbf{J}^{\eta\hat{\ }\langle\alpha\rangle}, \mathbf{f}^{\eta\hat{\ }\langle\alpha\rangle})$ of $(\bar{M}^\eta, \bar{\mathbf{J}}^\eta, \mathbf{f}^\eta)$, we may like to build, for each $\alpha < 2^\partial$ an increasing sequence of length ζ, first with $\zeta < \partial$ then even $\zeta < \partial^+$ but a sequence of approximations of height ∂.

We would like to have in quite many limit $\delta < \partial$ a "real choice" as the various coding properties says. How does this help? If arriving to $\eta \in {}^\delta(2^\partial), \delta < \partial^+, \eta\hat{\ }\langle\alpha\rangle$, the model $M^{\eta\hat{\ }<\alpha>}_\delta$ is brimmed over M_δ; this is certainly beneficial and having a tower arriving to δ help toward this. But it has a price - we have to preserve it. In case we have existence for $K^{3,\text{up}}_{\mathfrak{s}}$ this occurs, see the proof of Theorem 8.14 (but was proved in an ad-hoc way).
2) So we have a function ι such that; so during the construction, for $\eta \in {}^{\partial^+}(2^\partial)$ letting $S_\eta := \{\xi < \ell g(\eta) : \langle(\bar{M}^{(\eta\restriction\xi)\hat{\ }<\alpha>}, \bar{\mathbf{J}}^{(\eta\restriction\xi)\hat{\ }<\alpha>}, \mathbf{f}^{(\eta\restriction\xi)\hat{\ }<\alpha>}) : \alpha < 2^\partial\rangle$ is not constant$\}$ we have:

(a) if $\xi = \ell g(\eta) = \sup(S_\eta) + 1$ and $\iota = \iota(\bar{M}^\eta \restriction (\xi+1), \bar{\mathbf{J}}^{\eta \restriction (\xi+1)}, f^{\eta\restriction(\xi+1)}), \bar{\mathbb{F}}^{\eta\restriction(\xi+1)})$ and $\eta \triangleleft \nu_\ell \in {}^{\ell g(\eta)+\iota}(2^\partial)$ for $\ell = 1,2$ then $(\bar{M}^{\nu_1}, \bar{\mathbf{J}}^{\nu_1}, \mathbf{f}^{\nu_1}, \bar{\mathbb{F}}^{\nu_1}) = (\bar{M}^{\nu_2}, \bar{\mathbf{J}}^{\nu_2}, \mathbf{f}^{\nu_2}, \mathbb{F}^{\nu_2})$.

To formalize this we can use (see a concrete example in the proof in 8.17).

10.15 Definition. 1) We say $\mathbf{i}$ is a ∂-parameter <u>when</u>:

(a) $\mathbf{i} = (\iota, \bar{u})$
(b) ι is an ordinal ≥ 1 but $< \partial^+$
(c) $\bar{u} = \langle u_\varepsilon : \varepsilon < \partial \rangle$ is a $\subseteq$-increasing sequence of subsets of ι of cardinality $< \partial$ with union ι
(d) if δ is a limit ordinal $< \partial$ then u_δ is the closure of $\cup\{u_\varepsilon : \varepsilon < \delta\}$.

2) We say $\mathbf{d}$ is a $\mathfrak{u}$-free $([\varepsilon_1, \varepsilon_2], \mathbf{i})$-rectangle <u>when</u>:

(a) $\varepsilon_1 \leq \varepsilon_2 < \partial$
(b) $\mathbf{d}$ is a $\mathfrak{u}$-free $([\varepsilon_1, \varepsilon_2], \iota)$-rectangle

(so ι may be $\geq \partial$, but then not serious; in fact, it is an $\mathfrak{u}$-free $([\varepsilon_1, \varepsilon_2], u_\delta)$-rectangle but we complete it in the obvious way.)
3) The short case is when $\mathbf{i}$ is short, i.e. $\iota = 1, u_\alpha = 1$.
The long case is $\iota = \partial, u_\varepsilon = \varepsilon + 1$.

<u>10.16 Discusssion</u>: 1) Above we have concentrated on what we may call the "short" case, the "long" case as described in 10.14, 10.15 allows more constructions by "consuming" more levels.
2) Above we can restrict ourselves to the case $\partial = \lambda^+$ so in 10.3 we then demand on $(A \backslash M_1 \backslash M_2)$ is just "equal to λ" and the possible variants of 10.3(2),(a) + (b)(B) are irrelevant.

§11 Remarks on pcf

This section will provide us two pcf claims we use. One is 11.1, a set-theoretic division into cases when $2^\lambda < 2^{\lambda^+}$ (it is from pcf theory; note that the definition of WDmId(λ) is recalled in 0.3(4)(b) = 0.3(4)(b) and of $\mu_{\rm wd}(\lambda)$ is recalled in 0.3(8) = 0.3(8)), we can replace λ^+ by regular λ such that $2^\theta = 2^{<\lambda} < 2^\lambda$ for some θ). The second deals with the existence of large independent subfamilies of sets, 11.4. This is a revised version of a part of [Sh 603]. See on history related to 11.1 in [Sh:g] particularly in [Sh:g, II,5.11] and [Sh 430].

Remark. Recall that

$$\mathrm{cov}(\chi,\mu,\theta,\sigma) = \chi + \mathrm{Min}\big\{|\mathscr{P}| : \mathscr{P} \subseteq [\chi]^{<\mu} \text{ and every member of } [\chi]^{<\theta} \text{ is included in the union of } < \sigma \text{ members of } \mathscr{P}\big\}.$$

11.1 Claim. *Assume* $2^\lambda < 2^{\lambda^+}$.

Then one of the following cases occurs: (clauses $(\alpha) - (\lambda)$ appear later)

- $(A)_\lambda$ $\chi^* = 2^{\lambda^+}$ *and for some* μ *clauses* $(\alpha) - (\varepsilon)$ *hold*
- $(B)_\lambda$ *for some* $\chi^* > 2^\lambda$ *and* μ *clauses* $(\alpha) - (\kappa)$ *hold (note:* μ *appear only in* $(\alpha) - (\varepsilon)$*)*
- $(C)_\lambda$ $\chi^* = 2^\lambda$ *and clauses* $(\eta) - (\mu)$ *hold*

  where

  - (α) $\lambda^+ < \mu \le 2^\lambda$ *and* $\mathrm{cf}(\mu) = \lambda^+$
  - (β) $\mathrm{pp}(\mu) = \chi^*$, *moreover* $\mathrm{pp}(\mu) =^+ \chi^*$
  - (γ) $(\forall \mu')(\mathrm{cf}(\mu') \le \lambda^+ < \mu' < \mu \Rightarrow \mathrm{pp}(\mu') < \mu)$ *hence* $\mathrm{cf}(\mu') \le \lambda^+ < \mu' < \mu \Rightarrow \mathrm{pp}_{\lambda^+}(\mu') < \mu$
  - (δ) *for every regular cardinal* χ *in the interval* $(\mu, \chi^*]$ *there is an increasing sequence* $\langle \lambda_i : i < \lambda^+ \rangle$ *of regular cardinals* $> \lambda^+$ *with limit* μ *such that* $\chi = \mathrm{tcf}\left(\prod_{i<\lambda^+} \lambda_i / J^{\rm bd}_{\lambda^+}\right)$, *and* $i < \lambda^+ \Rightarrow \max \mathrm{pcf}\{\lambda_j : j < i\} < \lambda_i < \mu$
  - (ε) *for some regular* $\kappa \le \lambda$, *for any* $\mu' < \mu$ *there is a tree* $\mathscr{T}$ *with* $\le \lambda$ *nodes,* κ *levels and* $|\lim_\kappa(\mathscr{T})| \ge \mu'$ *(in fact*

e.g. $\kappa = \mathrm{Min}\{\theta : 2^\theta \geq \mu\}$ *is appropriate; without loss of generality* $\mathscr{T} \subseteq {}^{\kappa>}\lambda$)

(ζ) *there is no normal* λ^{++}*-saturated ideal on* λ^+

(η) *there is* $\langle \mathscr{T}_\zeta : \zeta < \chi^* \rangle$ *such that:* $\mathscr{T}_\zeta \subseteq {}^{\lambda^+>}2$*, a subtree of cardinality* λ^+ *and* ${}^{\lambda^+}2 = \{\lim_{\lambda^+}(\mathscr{T}_\zeta) : \zeta < \chi^*\}$

(θ) $\chi^* < 2^{\lambda^+}$ *moreover* $\chi^* < \mu_{\mathrm{unif}}(\lambda^+, 2^\lambda)$*, but* $< \mu_{\mathrm{unif}}(\lambda^+, 2^\lambda)$ *is not used here,*

(ι) *for some* $\zeta < \chi^*$ *we have* $\lim_{\lambda^+}(\mathscr{T}_\zeta) \notin \mathrm{UnfmTId}_{(\chi^*)^+}(\lambda^+)$*, not used here*

(κ) $\mathrm{cov}(\chi^*, \lambda^{++}, \lambda^{++}, \aleph_1) = \chi^*$ *or* $\chi^* = \lambda^+$*, equivalently* $\chi^* = \sup[\{\mathrm{pp}(\chi) : \chi \leq 2^\lambda, \aleph_1 \leq \mathrm{cf}(\chi) \leq \lambda^+ < \chi\} \cup \{\lambda^+\}]$ *by [Sh:g, Ch.II,5.4]; <u>note</u> that clause* (κ) *trivially follows from* $\chi^* = 2^{\lambda^+}$

(λ) *for no* $\mu \in (\lambda^+, 2^\lambda]$ *do we have* $\mathrm{cf}(\mu) \leq \lambda^+$*,* $\mathrm{pp}(\mu) > 2^\lambda$*; equivalently* $2^\lambda > \lambda^+ \Rightarrow \mathrm{cf}([2^\lambda]^{\lambda^+}, \subseteq) = 2^\lambda$

(μ) *if there is a normal* λ^{++}*-saturated ideal on* λ^+*, moreover the ideal* $\mathrm{WDmId}(\lambda^+)$ *is, <u>then</u>* $2^{\lambda^+} = \lambda^{++}$ *(so as* $2^\lambda < 2^{\lambda^+}$ *clearly* $2^\lambda = \lambda^+$*).*

Proof. This is related to [Sh:g, II,5.11]; we assume basic knowledge of pcf (or a readiness to believe quotations). Note that by their definitions

$\circledast_1$ if $2^\lambda > \lambda^+$ then for any $\theta \in [\aleph_0, \lambda^+]$ we have $\mathrm{cf}([2^\lambda]^{\leq\lambda^+}, \subseteq) = 2^\lambda \Leftrightarrow \mathrm{cov}(2^\lambda, \lambda^{++}, \lambda^{++}, 2) = 2^\lambda \Leftrightarrow \mathrm{cov}(2^\lambda, \lambda^{++}, \lambda^{++}, \theta) = 2^\lambda$.

[Why? Because $(2^\lambda)^{<\lambda^+} = 2^\lambda$ and $\mathrm{cf}([2^\lambda]^{\leq\lambda^+}, \subseteq) = \mathrm{cov}(2^\lambda, \lambda^{++}, \lambda^{++}, 2) \geq \mathrm{cov}(2^\lambda, \lambda^{++}, \lambda^{++}, \aleph_0) \geq \mathrm{cov}(2^\lambda, \lambda^{++}, \lambda^{++}, \theta) \geq 2^\lambda$ for $\theta \in [\aleph_0, \lambda]$.]

Note also that

$\circledast_2$ $\lambda^+ \notin \mathrm{WDmId}(\lambda^+)$ and ${}^{\lambda^+}2 \notin \mathrm{WDmTId}(\lambda^+)$.

[Why? Theorem 0.5(2) with θ, ∂ there standing for λ, λ^+ here.]

Possibility 1: $2^\lambda > \lambda^+$ and $\mathrm{cov}(2^\lambda, \lambda^{++}, \lambda^{++}, \aleph_1) = 2^\lambda$ or $2^\lambda = \lambda^+$; and let $\chi^* := 2^\lambda$.

We shall show that case (C) holds (for the cardinal λ), the first assertion "$\chi^* = 2^\lambda$" holds by our choice.
Now clause (κ) is obvious. As for clause (η), we have $\chi^* = 2^\lambda < 2^{\lambda^+}$. Now if $2^\lambda = \lambda^+$ we let $\mathscr{T}_\zeta = {}^{\lambda^+ >}2$, for $\zeta < \chi^*$ so clause (η) holds, otherwise as ${}^{\lambda^+ >}2$ has cardinality 2^λ, by the definitions of $\mathrm{cov}(2^\lambda, \lambda^{++}, \lambda^{++}, \aleph_1)$ and the possibility assumption (and obvious equivalence) there is $\mathscr{P} \subseteq [{}^{\lambda^+ >}2]^{\lambda^+}$ of cardinality χ^* such that any $A \in [{}^{\lambda^+ >}2]^{\lambda^+}$ is included in the union of $\le \aleph_0$ of them. So $\eta \in {}^{\lambda^+}2 \Rightarrow (\exists A \in \mathscr{P})(\exists^{\lambda^+} \alpha < \lambda^+)(\eta \restriction \alpha \in A)$ so let $\langle A_\zeta : \zeta < \chi^* \rangle$ list $\mathscr{P}$ and let $\mathscr{T}_\zeta = \{\eta \restriction \alpha : \eta \in A_\zeta \text{ and } \alpha \le \ell g(\eta)\}$, now check that they are as required in clause (η).

As on the one hand by [Sh:f, AP,1.16 + 1.19] or see 9.4 we have $\left(\mu_{\mathrm{unif}}(\lambda^+, 2^\lambda)\right)^{\aleph_0} = 2^{\lambda^+} > 2^\lambda = \chi^*$ and on the other hand $(\chi^*)^{\aleph_0} = (2^\lambda)^{\aleph_0} = 2^\lambda = \chi^*$ necessarily $\chi^* < \mu_{\mathrm{unif}}(\lambda^+, 2^\lambda)$ so clause (θ) follows; next clause (ι) follows from clause (η) by the definition of $\mathrm{UnfTId}_{(\chi^*)^+}(\lambda^+)$. In fact in our possibility for some ζ, $\lim_{\lambda^+}(\mathscr{T}_\zeta) \notin \mathrm{WDmTId}(\lambda^+)$ because $\mathrm{WDmTId}(\lambda^+)$ is $(2^\lambda)^+$-complete by 0.5(2),(4) recalling $\circledast_2$ and having chosen $\chi^* = 2^\lambda$.

Now if $2^{\lambda^+} > \lambda^{++}$, (so $2^{\lambda^+} \ge \lambda^{+3}$), then for some $\zeta < \chi^*$, $\mathscr{T}_\zeta$ is (a tree with $\le \lambda^+$ nodes, λ^+ levels and) at least λ^{+3} λ^+-branches which is well known (see e.g. [J]) to imply "no normal ideal on λ^+ is λ^{++}-saturated"; so we got clause (μ). Also if $2^{\lambda^+} \le \lambda^{++}$ then $2^\lambda = \lambda^+, 2^{\lambda^+} = \lambda^{++}$.

As for clause (λ), by the definition of χ^* and the assumption $\chi^* = 2^\lambda$ we have the first two phrases. The "equivalently" holds as $(2^\lambda)^{\aleph_0} = 2^\lambda$.

Possibility 2: $\chi^* := \mathrm{cov}(2^\lambda, \lambda^{++}, \lambda^{++}, \aleph_1) > 2^\lambda > \lambda^+$.

So $(\mathrm{C})_\lambda$ fails, and we have to show that $(A)_\lambda$ or $(B)_\lambda$ holds.

Let

$(*)_0$ $\mu := \mathrm{Min}\{\mu : \mathrm{cf}(\mu) \le \lambda^+, \lambda^+ < \mu \le 2^\lambda \text{ and } \mathrm{pp}(\mu) = \chi^*\}$.

We know by [Sh:g, II,5.4] that μ exists and (by [Sh:g, II,2.3](2)) clause (γ) holds, also $2^\lambda < \text{pp}(\mu) \le \mu^{\text{cf}(\mu)} \le (2^\lambda)^{\text{cf}(\mu)} = 2^{\lambda+\text{cf}(\mu)}$ hence $\text{cf}(\mu) = \lambda^+$. So clauses $(\alpha), (\beta), (\gamma)$ hold (of course, for clause (β) use [Sh:g, Ch.II,5.4](2)), and by (γ) + [Sh:g, VIII,§1] also clause (δ) holds.

Toward trying to prove clause (ε) let

$(*)_1$ $\Upsilon := \text{Min}\{\theta : 2^\theta \ge \mu\}$,
clearly[34]

$(*)_2$ $\alpha < \Upsilon \Rightarrow 2^{|\alpha|} < \mu$ and $\Upsilon \le \lambda$ (as $2^\lambda \ge \mu$) hence $\text{cf}(\Upsilon) \le \Upsilon \le \lambda < \lambda^+ = \text{cf}(\mu)$ hence $2^{<\Upsilon} < \mu$.

Let

$(*)_3$ (a) u be a closed unbounded subset of Υ of order type $\text{cf}(\Upsilon)$
(b) $\mathscr{T}^* = (\bigcup_{\alpha \in u} {}^\alpha 2, \triangleleft)$ is a tree with $\text{cf}(\Upsilon)$ levels and $\le 2^{<\Upsilon}$ nodes.

Now we shall prove clause (ε), i.e.

$(*)_4$ there is a tree with λ nodes, $\text{cf}(\Upsilon)$ levels and $\ge \mu$ Υ-branches.

Case A: Υ has cofinality $\aleph_0$.

In the case $\Upsilon = \aleph_0$ or just $2^{<\Upsilon} \le \lambda$ clearly there is a tree as required, i.e. $\mathscr{T}^*$ is a tree having $\le 2^{<\Upsilon} \le \lambda$ nodes. So we can assume $2^{<\Upsilon} > \lambda$ and $\Upsilon > \text{cf}(\Upsilon) = \aleph_0$ hence $\langle 2^\theta : \theta < \Upsilon \rangle$ is not eventually constant.

So necessarily $(\exists \theta < \Upsilon)(2^\theta \ge \lambda)$hence $2^{<\Upsilon} > \lambda^+$ (and even $2^{<\Upsilon} \ge \lambda^{+\omega}$) and for some $\theta < \Upsilon$ we have $\lambda^{++} < 2^\theta < 2^{<\Upsilon} < \mu$. Let $\chi' = \text{cov}(2^{<\Upsilon}, \lambda^{++}, \lambda^{++}, \aleph_1)$, so $\chi' \ge 2^{<\Upsilon}$ and $\chi' < \mu$ by [Sh:g, II,5.4] and clause (γ) of 11.1 which have been proved (in our present possibility).

We try to apply claim 11.3 below with $\aleph_0, \lambda^+, 2^{<\Upsilon}, \chi'$ here standing for $\theta, \kappa, \mu, \chi$ there; we have to check the assumptions of 11.3 which means $2^{<\Upsilon} > \lambda^+ > \aleph_0$ and $\chi' = \text{cov}(2^{<\Upsilon}, \lambda^{++}, \lambda^{++}, \aleph_1)$,

[34]Below we show that $\Upsilon > \text{cf}(\Upsilon) \Rightarrow \text{cf}(\Upsilon) > \aleph_0$.

both clearly hold. So the conclusion of 11.3 holds which means that $(\chi')^{\aleph_0} \geq \mathrm{cov}((2^{<\Upsilon})^{\aleph_0}, (\lambda^{++})^{\aleph_0}, \lambda^{++}, 2)$, now $(\chi')^{\aleph_0} \leq \mu^{\aleph_0} \leq (2^\lambda)^{\aleph_0} = 2^\lambda$ and $(2^{<\Upsilon})^{\aleph_0} = 2^\Upsilon \geq \mu$ because presently $\mathrm{cf}(\Upsilon) = \aleph_0$ and the choice of Υ and $(\lambda^{++})^{\aleph_0} \leq (2^\theta)^{\aleph_0} = 2^\theta$. So by monotonicity $2^\lambda \geq \mathrm{cov}(\mu, 2^\theta, \lambda^{++}, \aleph_1)$. But $\mathrm{cov}(2^\theta, \lambda^{++}, \lambda^{++}, \aleph_1) \leq \chi' := \mathrm{cov}(2^{<\Upsilon}, \lambda^{++}, \lambda^{++}, \aleph_1) < \mu \leq 2^\lambda$ by clause (γ) which we have proved (and [Sh:g, ChII,5.4]) so by transitivity of cov, see 11.2(4), also $2^\lambda \geq \mathrm{cov}(\mu, \lambda^{++}, \lambda^{++}, \aleph_1)$ contradicting the present possibility.

Case B: $\mathrm{cf}(\Upsilon) > \aleph_0$.

Let $h : \mathscr{T}^* \to 2^{<\Upsilon}$ be one-to-one, see $(*)_3(b)$. Let $\mathscr{P} \subseteq [2^{<\Upsilon}]^{\leq\lambda}$ be such that every $X \in [2^{<\Upsilon}]^{\leq\lambda^+}$ is included in the union of countably members of $\mathscr{P}$, exists by clause (γ) of 11.1 by [Sh:g, II.5.4].

Now for every $\bar\nu = \langle \nu_\varepsilon : \varepsilon < \Upsilon\rangle \in \lim_{\mathrm{cf}(\Upsilon)}(\mathscr{T}^*)$, for some $A_{\bar\nu} \in \mathscr{P}$ we have $\Upsilon = \sup\{\varepsilon \in u : h(\nu_\varepsilon) \in A_{\bar\nu}\}$, so for every $\mu' \in (2^{<\Upsilon}, \mu)$ for some $A \in \mathscr{P}$ we have $\mu' \leq |\{\bar\nu : \bar\nu \in \lim_{\mathrm{cf}(\Upsilon)}(\mathscr{T}^*)$ and $A_{\bar\nu} = A\}|$.

Now let $\mathscr{T}'$ be the closure of A to initial segments of length $\in u$, easily $\mathscr{T}'$ is as required.

So we have proved $(*)_4$ so in possibility (2) the demand $(\alpha) - (\varepsilon)$ in $(A)_\lambda$ holds.

Sub-possibility 2α: $\chi^* < 2^{\lambda^+}$.

We shall prove $(B)_\lambda$, so by the above we are left with proving clauses $(\zeta) - (\kappa)$ when $\chi^* < 2^{\lambda^+}$. By the choice of χ^*, easily the demand in clause (ζ) (in Case B of 11.1) holds; that is let $\{u_\zeta : \zeta < \chi^*\}$ be a family of subsets of ${}^{\lambda^+>}2$, a set of cardinality 2^λ, each of cardinality λ^+ such that any other such subset is included in the union of $\leq \aleph_0 < \aleph_1$ of them, exist by the choice of χ^*.

Let $\mathscr{T}_\zeta = \{\nu \restriction i : \nu \in u_\zeta$ and $i \leq \ell g(\nu)\}$. Now $\langle \mathscr{T}^*_\zeta : \zeta < \chi^*\rangle$ is as required.

In clause (η), "$2^\lambda < \chi^*$" holds as we are in possibility 2α.

Also as $\mathrm{pp}(\mu) = \chi^*$ and $\mathrm{cf}(\mu) = \lambda^+$ by the choice of μ necessarily (by transitivity of pcf, i.e., [Sh:g, Ch.II,2.3](2)) we have $\mathrm{cf}(\chi^*) > \lambda^+$ but $\mu > \lambda^+$. Easily $\lambda^+ < \chi \leq \chi^* \wedge \mathrm{cf}(\chi) \leq \lambda^+ \Rightarrow \mathrm{pp}(\chi) \leq \chi^*$ hence $\mathrm{cov}(\chi^*, \lambda^{++}, \lambda^{++}, \aleph_1) = \chi^*$ by [Sh:g, Ch.II,5.4], which gives clause (λ). Now let $\mathscr{A} \subseteq [\chi^*]^{\lambda^+}$ exemplify $\mathrm{cov}(\chi^*, \lambda^{++}, \lambda^{++}, \aleph_1) = \chi^*$ and let $\mathscr{A}' = \{B : B$ is an infinite countable subset of some $A \in \mathscr{A}\}$. So $\mathscr{A}' \subseteq [\chi^*]^{\aleph_0}$ and easily $A \in [\chi^*]^{\lambda^+} \Rightarrow (\exists B \in \mathscr{A}')(B \subseteq A)$ and $|\mathscr{A}'| \le \chi^*$ as $(\lambda^+)^{\aleph_0} \le 2^\lambda < \chi^*$ certainly there is no family of $> \chi^*$ subsets of χ^* each of cardinality λ^+ with pairwise finite intersections. But by 9.4 there is $\mathscr{A}' \subseteq [\mu_{\text{unif}}(\lambda^{++}, 2^{\lambda^+})]^{\lambda^+}$ of cardinality $2^{\lambda^{++}}$ such that $A \ne B \in \mathscr{A}' \Rightarrow |A \cap B| < \aleph_0$, hence we have $\chi^* < \mu_{\text{unif}}(\lambda^+, 2^\lambda)$ thus completing the proof of (θ).

Now clause (ι) follows by clauses (η)+(θ)+(κ) as $\emptyset \notin$ $\text{UnfTId}_{(\chi^*)^+}$ $\in (\lambda^+)$ which is $(\chi^*)^+$-complete ideals, see 9.4. Note also that $\emptyset \in$ $\text{WDmTId}_{\chi^*}(\lambda^+)$ by 0.5(2) and it is a $(\chi^*)^+$ complete ideal by 0.5(4). Also by clause (α) which we have proved $2^{\lambda^+} \ne \lambda^{++}$ hence $2^{\lambda^+} \ge \lambda^{+3}$ so by clause (η) (as $\chi^* < 2^{\lambda^+}$), we have $|\lim_{\lambda^+}(\mathscr{T}_\zeta)| \ge \lambda^{+3}$ for some ζ which is well known (see [J]) to imply no normal ideal on λ^+ is λ^{++}-saturated; i.e., clause (μ). So we have proved clauses $(\alpha) - (\lambda)$ holds, i.e. that case $(B)_\lambda$ holds.

<u>Sub-possibility 2β</u>: $\chi^* = 2^{\lambda^+}$ (and $\chi^* > 2^\lambda > \lambda^+$).

We have proved that case $(A)_\lambda$ holds, as we already defined μ and χ^* and proved clauses (α), (β), (γ), (δ), (ε) so we are done. $\square_{11.1}$

It may be useful to recall (actually $\lambda^{<\kappa>_{\text{tr}}} = \lambda$ suffice)

11.2 Fact. 1) Assume $\lambda > \theta \ge \kappa = \text{cf}(\kappa) \ge \kappa_1$. Then $\lambda^{<\kappa>_{\text{tr}}} \le \text{cov}(\lambda, \theta^+, \kappa^+, \kappa_1)$ recalling $\mu^{<\kappa>_{\text{tr}}} = \sup\{\lim_\kappa(\mathscr{T}) : \mathscr{T}$ is a tree with $\le \mu$ nodes and κ levels, e.g. $\mathscr{T}$ a subtree of ${}^{\kappa>}\mu\}$.
2) If $\mu > \aleph_0$ is strong limit and $\lambda > \mu$ <u>then</u> for some $\kappa < \mu$ we have $\text{cov}(\lambda, \mu, \mu, \kappa)$.
3) If $\mathscr{T} \subseteq {}^{\lambda^+>}2$ is a tree, $|\mathscr{T}| \le \lambda^+$ and $\lambda \ge \beth_\omega$ <u>then</u> for every regular $\kappa < \beth_\omega$ large enough, we can find $\langle Y_\delta : \delta < \lambda^+, \text{cf}(\delta) = \kappa\rangle, |Y_\delta| \le \lambda$ such that:
for every $\eta \in \lim_{\lambda^+}(\mathscr{T})$ for a club of $\delta < \lambda^+$ we have $\text{cf}(\delta) = \kappa \Rightarrow \eta \restriction \delta \in Y_\delta$.
4) [Transitivity of cov] If $\mu_3 \ge \mu_2 \ge \mu_1 \ge \theta \ge \sigma = \text{cf}(\sigma)$ and $\lambda_2 = \text{cov}(\mu_3, \mu_2, \theta, \sigma)$ and $\mu < \mu_2 \Rightarrow \lambda_1 \ge \text{cov}(\mu, \mu_1, \theta, \sigma)$ <u>then</u> $\lambda_1 + \lambda_2 \ge \text{cov}(\mu_3, \mu_1, \theta, \sigma)$.

Proof. 1) E.g. proved inside 11.1.
2) By [Sh 460] or see [Sh 829].

3) Should be clear from part (2).
4) Let $\mathscr{P}_2 \subseteq [\mu_3]^{<\mu_2}$ exemplify $\lambda_2 = \text{cov}(\mu_3, \mu_2, \theta, \sigma)$ and for each $A \in P_2$ let f_A be one to one function from $|A|$ onto A and $\mathscr{P}_{1,A} \subseteq [|A|]^{<\mu_1}$ exemplify $\lambda_1 \geq \text{cov}(|A|, \mu_1, \theta, \sigma)$. Lastly, let $\mathbf{P} = \{f_A(\alpha) : \alpha \in u\} : u \in \mathbf{P}_{1,A}, A \in \mathbf{P}_2\}$ it exemplify $\lambda_1 + \lambda_2 \geq \text{cov}(\mu_3, \mu_1, \theta, \sigma)$ as required. $\square_{11.3}$

We have used in proving 11.1 also

11.3 Observation. Assume $\mu > \kappa > \theta$.
If $\chi = \text{cov}(\mu, \kappa^+, \kappa^+, \theta^+)$ <u>then</u> $\chi^\theta \geq \text{cov}(\mu^\theta, (\kappa^\theta)^+, \kappa^+, 2)$.

Proof. Let $\mathscr{P} \subseteq [\mu]^\kappa$ exemplify $\chi = \text{cov}(\mu, \kappa^+, \kappa^+, \theta^+)$.
Let $\langle \eta_\alpha : \alpha < \mu^\theta \rangle$ list ${}^\theta\mu$. Now for $\mathscr{U} \in [\mu]^\kappa$ define $\mathscr{U}^{[*]} = \{\alpha < \mu^\theta$: if $i < \theta$ then $\eta_\alpha(i) \in \mathscr{U}\}$, and let $\mathscr{P}_1 = \mathscr{P}, \mathscr{P}_2 = \{\bigcup_{i<\theta} A_i : A_i \in \mathscr{P}$ for $i < \theta\}$ and $\mathscr{P}_3 = \{\mathscr{U}^{[*]} : \mathscr{U} \in \mathscr{P}_2\}$.
So (of course $\chi \geq \mu$ as $\mu > \kappa$, hence $\chi > \kappa$)

$(*)_1$ (a) $\mathscr{P}_1 \subseteq [\mu]^\kappa$ has cardinality χ
(b) $\mathscr{P}_2 \subseteq [\mu]^{\kappa^\theta}$ has cardinality $\leq \chi^\theta$
(c) $\mathscr{P}_3 \subseteq [\mu^\theta]^{\kappa^\theta}$ has cardinality $\leq \chi^\theta + \kappa^\theta = \chi^\theta$

and

$(*)_2$ if $\mathscr{U} \in [\mu^\theta]^{\leq\kappa}$ then
(a) $\mathscr{U}' := \{\eta_\alpha(i) : \alpha \in \mathscr{U}$ and $i < \theta\} \in [\mu]^{\leq\kappa}$
(b) there are $A_i \in \mathscr{P}_1$ for $i < \theta$ such that $\mathscr{U}' \subseteq \bigcup_{i<\theta} A_i$
(c) $\bigcup_{i<\theta} A_i \in \mathscr{P}_2$
(d) $\mathscr{U} \subseteq (\bigcup_{i<\theta} A_i)^{[*]} \in \mathscr{P}_3 \subseteq [\mu^\theta]^{\kappa^\theta}$.

[Why? Clause (a) holds by cardinal arithmetic, clause (b) holds by the choice of $\mathscr{P} = \mathscr{P}_1$, clause (c) holds by the definition of $\mathscr{P}_2$ and clause (d) holds by the definition of $(-)^{[*]}$ and of $\mathscr{P}_3$.]

Together we are done. $\square_{11.3}$

The following is needed when we like to get in the model theory not just many models but many models no one $\le_{\mathfrak{K}}$-embeddable into another and even just for $\dot{I}$, (see VI.4.7).

11.4 Claim. *Assume:*

(a) $\mathrm{cf}(\mu) \le \kappa < \mu, \kappa^+ < \theta < \chi^*$ *and* $\mathrm{pp}_\kappa(\mu) = \chi^*$, *moreover* $\mathrm{pp}_\kappa(\mu) =^+ \chi^*$

(b) $\mathbf{F}$ *is a function, with domain* $[\mu]^\kappa$, *such that: for* $a \in [\mu]^\kappa, \mathbf{F}(a)$ *is a family of* $< \theta$ *members of* $[\mu]^\kappa$

(c) F *is a function with domain* $[\mu]^\kappa$ *such that*

$$a \in [\mu]^\kappa \Rightarrow a \subseteq F(a) \in \mathbf{F}(a).$$

Then we can find pairwise distinct $a_i \in [\mu]^\kappa$ *for* $i < \chi^*$ *such that* $\mathscr{I} = \{a_i : i < \chi^*\}$ *is* $(F, \mathbf{F})$*-independent which means*

$$(*)_{F,\mathbf{F},\mathscr{I}} \qquad a \ne b \ \ \&\ \ a \in \mathscr{I} \ \ \&\ \ b \in \mathscr{I} \ \ \&\ \ c \in \mathbf{F}(a) \Rightarrow \neg(F(b) \subseteq c).$$

11.5 Remark. 1) Clearly this is a relative to Hajnal's free subset theorem [Ha61].
2) Note that we can choose $F(a) = a$.
3) Also if $\mu_1 \le \mu$, $\mathrm{cf}(\mu_1) \le \kappa \le \kappa + \theta < \mu_1$ and $\mathrm{pp}_\kappa(\mu_1) \ge \mu$ then by [Sh:g, Ch.II,2.3] the Fact for μ_1 implies the one for μ.
4) Note that if $\lambda =\ \mathrm{cf}([\mu]^\kappa, \subseteq)$ then for some $\mathbf{F}, F$ as in the Fact we have

⊛ if $a_i \in [\mu]^\kappa$ for $i < \lambda^+$ are pairwise distinct then not every pair $\{a_i, a_j\}$ is $(\mathbf{F}, F)$-independent
[why? let $\mathscr{P} \subseteq [\mu]^\kappa$ be cofinal (under $\subseteq$) of cardinality λ, and let $F, \mathbf{F}$ be such that
$\mathbf{F}(a) \subseteq \{b \in [\mu]^\kappa : a \subseteq b$ and $b \in \mathscr{P}\}$ has a $\subseteq$-maximal member $F(a)$;
obviously there are such $F, \mathbf{F}$.

Now clearly

$(*)_1$ if $a \neq b$ are from $[\mu]^\kappa$ and $F(a) = F(b)$ then $\{a, b\}$ is not $(\mathbf{F}, F)$-independent.

[Why? Just look at the definition of $(\mathbf{F}, F)$-independent.]

$(*)_2$ if $\mathscr{I} \subseteq [\mu]^\kappa$ is of cardinality $> \lambda$ (e.g. λ^+) <u>then</u> $\mathscr{I}$ is not $(\mathbf{F}, F)$-independent.

[Why? As $\mathrm{Rang}(F \restriction \mathscr{I}) \subseteq \mathrm{Rang}(F) \subseteq \mathscr{P}$ and $\mathscr{P}$ has cardinality λ necessarily there are $a \neq b$ from $\mathscr{I}$ such that $F(a) = F(b)$ and use $(*)_1$.]

Proof.

$\boxtimes_1$ it suffices to prove the variant with $[\mu]^\kappa$ replaced by $[\mu]^{\leq\kappa}$.

[Why? So we are given $\mathbf{F}, F$ as in the claim. We define $g : [\mu]^{\leq\kappa} \to [\mu]^\kappa$ and functions $F', \mathbf{F}'$ with domain $[\mu]^{\leq\kappa}$ as follows:

$$g(a) = \{\kappa + \alpha : \alpha \in a\} \cup \{\alpha : \alpha < \kappa\}$$

$$\mathbf{F}'(a) = \{\{\alpha : \kappa + \alpha \in b\} : b \in \mathbf{F}(g(a))\}$$

$$F'(a) = \{\alpha : \kappa + \alpha \in F(g(a))\}.$$

Now $\mathbf{F}', F'$ are as in the claim only replacing everywhere $[\mu]^\kappa$ by $[\mu]^{\leq\kappa}$, and if $\mathscr{I}' = \{a_i : i < \chi\} \subseteq [\mu]^{\leq\kappa}$ with no repetitions satisfying $(*)_{F',\mathbf{F}',\mathscr{I}'}$ then we shall show that $\mathscr{I} := \{g(a_i) : i < \chi\}$ is with no repetitions and $(*)_{F,\mathbf{F},\mathscr{I}}$ holds.

This clearly suffices, but why it holds? Clearly g is a one-to-one function so $i \neq j < \chi \Rightarrow g(a_i) \neq g(a_j)$ and $\mathrm{Rang}(g) \subseteq [\mu]^\kappa$ so $g(a_i) \in [\mu]^\kappa$. Let $i \neq j$ and we should check that $[c' \in \mathbf{F}(g(a_i)) \Rightarrow F(g(a_j)) \nsubseteq c']$, so fix c' such that $c' \in \mathbf{F}(g(a_i))$.

By the definition of $\mathbf{F}'(a_i)$ clearly $c := \{\alpha : \kappa + \alpha \in c'\}$ belongs to $\mathbf{F}'(a_i)$. By the choice of $\mathscr{I}' = \{a_i : i < \chi\}$ we know that $c \in \mathbf{F}'(a_i) \Rightarrow F'(a_j) \nsubseteq c$, but by the previous sentence the antecedent hold hence $F'(a_j) \nsubseteq c$ hence we can choose $\alpha \in F'(a_j) \backslash c$. By the choice of $F'(a_j)$ we have $\kappa + \alpha \in F(g(a_j))$ and by the choice of c we have $\kappa + \alpha \notin c'$, so α witness $F(g(a_j)) \nsubseteq c'$ as required.]

So we conclude that we can replace $[\mu]^\kappa$ by $[\mu]^{\le\kappa}$. In fact we shall find the a_i in $[\mu]^{|\mathfrak{a}|}$ where $\mathfrak{a}$ chosen below.
As μ is a limit cardinal $\in (\kappa, \chi^*)$, if $\theta < \mu$ then we can replace θ by θ^+ but $\kappa^{++} < \mu$ so without loss of generality $\kappa^{++} < \theta$.

Now we prove

$\boxtimes_2$ for some unbounded subset w of χ we have $\langle \text{Rang}(f_\alpha) : \alpha \in w\rangle$ is $(\mathbf{F}, F)$-independent <u>when</u>:

$\oplus_{\chi,\mathfrak{a},\bar{f}}$ $\theta < \chi = \text{cf}(\Pi\mathfrak{a}/J)$ where $\mathfrak{a} \subseteq \mu \cap \text{Reg}\backslash\kappa^+, |\mathfrak{a}| \le \kappa, \sup(\mathfrak{a}) = \mu, J^{\text{bd}}_{\mathfrak{a}} \subseteq J$ and for simplicity $\chi = \max \text{pcf}(\mathfrak{a})$ and $\bar{f} = \langle f_\alpha : \alpha < \chi\rangle$ is a sequence of members of $\Pi\mathfrak{a}, <_J$-increasing, and cofinal in $(\Pi\mathfrak{a}, <_J)$, so, of course, $\chi \le \chi^*$.

Without loss of generality $f_\alpha(\lambda) > \sup(\mathfrak{a} \cap \lambda)$ for $\lambda \in \mathfrak{a}$.

Also for every $a \in [\mu]^\kappa$, define $\text{ch}_a \in \Pi\mathfrak{a}$ by $\text{ch}_a(\lambda) = \sup(a \cap \lambda)$ for $\lambda \in \mathfrak{a}$ so for some $\zeta(a) < \chi$ we have $\text{ch}_a <_J f_{\zeta(a)}$ (as $\langle f_\alpha : \alpha < \chi\rangle$ is cofinal in $(\Pi\mathfrak{a}, <_J)$). So for each $a \in [\mu]^\kappa$, as $|\mathbf{F}(a)| < \theta < \chi = \text{cf}(\chi)$ clearly $\xi(a) := \sup\{\zeta(b) : b \in \mathbf{F}(a)\}$ is $< \chi$, and clearly $(\forall b \in \mathbf{F}(a))[\text{ch}_b <_J f_{\xi(a)}]$. So $C := \{\gamma < \chi :$ for every $\beta < \gamma, \xi(\text{Rang}(f_\beta)) < \gamma\}$ is a club of χ.

For each $\alpha < \chi$, $\text{Rang}(f_\alpha) \in [\mu]^\kappa$, hence $\mathbf{F}(\text{Rang}(f_\alpha))$ has cardinality $< \theta$, but $\theta < \chi = \text{cf}(\chi)$ hence for some $\theta_1 < \theta$ we have $\theta_1 > \kappa^+$ and $\chi = \sup\{\alpha < \chi : |\mathbf{F}(\text{Rang}(f_\alpha)| \le \theta_1\}$, so without loss of generality $\alpha < \chi \Rightarrow \theta_1 \ge |\mathbf{F}(\text{Rang}(f_\alpha))|$.

As $\kappa^+ < \theta_1$, by [Sh 420, §1] there are $\bar{d}, S$ such that

$(*)_1$(a) $S \subseteq \theta_1^+$ is a stationary

(b) $S \subseteq \{\delta < \theta_1^+ : \text{cf}(\delta) = \kappa^+\}$

(c) S belongs to $\check{I}[\theta_1^+]$,

(d) $\langle d_i : i < \theta_1^+\rangle$ witness it, so $\text{otp}(d_i) \le \kappa^+, d_i \subseteq i, [j \in d_i \Rightarrow d_j = d_i \cap i]$ and $i \in S \Rightarrow i = \sup(d_i)$,
and for simplicity (see [Sh:g, III])

(e) for every club E of θ_1^+ for stationarily many $\delta \in S$ we have $(\forall \alpha \in d_\delta)[(\exists \beta \in E)(\sup(\alpha \cap d_\delta) < \beta < \alpha)]$.

Now try to choose by induction on $i < \theta_1^+$, a triple (g_i, α_i, w_i) such that:

$(*)_2$(a) $g_i \in \Pi\mathfrak{a}$

(b) if $j < i$ then[35] $g_j <_J g_i$

(c) $(\forall \lambda \in \mathfrak{a})(\sup\limits_{j \in d_i} g_j(\lambda) < g_i(\lambda))$

(d) $\alpha_i < \chi$ and $\alpha_i > \sup(\bigcup\limits_{j<i} w_j)$

(e) $j < i \Rightarrow \alpha_j < \alpha_i$

(f) $g_i <_J f_{\alpha_i}$

(g) $\beta \in \bigcup\limits_{j<i} w_j \Rightarrow \xi(\mathrm{Rang}(f_\beta)) < \alpha_i \ \& \ f_\beta <_J g_i$

(h) w_i is a maximal subset of (α_i, χ) satisfying

$(*)$ $\beta \in w_i \ \& \ \gamma \in w_i \ \& \ \beta \neq \gamma \ \& \ a \in \mathbf{F}(\mathrm{Rang}(f_\beta)) \Rightarrow \neg(F(\mathrm{Rang}(f_\gamma)) \subseteq a)$

and moreover

$(*)^+$ $\beta \in w_i \ \& \ \gamma \in w_i \ \& \ \beta \neq \gamma \ \& \ a \in \mathbf{F}(\mathrm{Rang}(f_\beta)) \Rightarrow \{\lambda \in \mathfrak{a} : f_\gamma(\lambda) \in a\} \in J$.

Note that really (as indicated by the notation)

$\otimes$ if $w \subseteq (\alpha_i, \chi)$ satisfies $(*)^+$ then it satisfies $(*)$.

[Why? let us check $(*)$, so let $\beta \in w$, $\gamma \in w$, $\beta \neq \gamma$ and $a \in \mathbf{F}(\mathrm{Rang}(f_\beta))$; by $(*)^+$ we know that $\mathfrak{a}' = \{\lambda \in \mathfrak{a} : f_\gamma(\lambda) \in a\} \in J$. Now as J is a proper ideal on $\mathfrak{a}$ clearly for some $\lambda \in \mathfrak{a}$ we have $\lambda \notin \mathfrak{a}'$, hence $f_\gamma(\lambda) \notin a$ but $f_\gamma(\lambda) \in \mathrm{Rang}(f_\gamma)$ and by the assumption on $(\mathbf{F}, F)$ we have $\mathrm{Rang}(f_\gamma) \subseteq F(\mathrm{Rang}(f_\gamma))$ hence $f_\gamma(\lambda) \in F(\mathrm{Rang}(f_\gamma)) \backslash a$ so $\neg(F(\mathrm{Rang}(f_\gamma)) \subseteq a)$, as required.]

We claim that we cannot carry the induction because if we succeed, then as $\mathrm{cf}(\chi) = \chi > \theta \geq \theta_1^+$ there is α such that $\bigcup\limits_{i<\theta_1^+} \alpha_i <$

[35] in fact, without loss of generality $\min(\mathfrak{a}) > \theta_1^+$, so we can demand $g_j < g_i$ so clause (c) is redundant

$\alpha < \chi$ and let $\mathbf{F}(\text{Rang}(f_\alpha)) = \{a^\alpha_\zeta : \zeta < \theta_1\}$ (possible as $1 \le |\mathbf{F}(\text{Rang}(f_\alpha))| \le \theta_1$). Now for each $i < \theta_1^+$, by the choice of w_i clearly $w_i \cup \{\alpha\}$ does not satisfy the demand in clause (h) and let it be exemplified by some pair (β_i, γ_i). Now $\{\beta_i, \gamma_i\} \subseteq w_i$ is impossible by the choice of w_i, i.e. as w_i satisfies clause (h). Also $\beta_i \in w_i \wedge \gamma_i = \alpha$ is impossible as $\beta_i \in w_i \Rightarrow \xi(\text{Rang}(f_{\beta_i})) < \alpha_{i+1} < \alpha$, so necessarily $\gamma_i \in w_i$ and $\beta_i = \alpha$, so for some $a' \in \mathbf{F}(\text{Rang}(f_{\beta_i})) = \mathbf{F}(\text{Rang}(f_\alpha))$ the conclusion of $(*)^+$ fails, so as $\langle a^\alpha_\zeta : \zeta < \theta_1\rangle$ list $\mathbf{F}(\text{Rang}(f_\alpha))$ it follows that for some $\zeta_i < \theta_1$ we have

$$\mathfrak{a}_i = \{\lambda \in \mathfrak{a} : f_{\gamma_i}(\lambda) \in a^\alpha_{\zeta_i}\} \notin J.$$

[why use the ideal? In order to show below that $\mathfrak{b}_\varepsilon \neq \emptyset$.] But $\text{cf}(\theta_1^+) = \theta_1^+ > \theta_1$, so for some $\zeta(*) < \theta_1^+$ we have $A := \{i : \zeta_i = \zeta(*)\}$ is unbounded in θ_1^+. Hence $E = \{\alpha < \theta_1^+ : \alpha$ a limit ordinal and $A \cap \alpha$ is unbounded in $\alpha\}$ is a club of θ_1^+. So for some $\delta \in S$ we have $\delta = \sup(A \cap \delta)$, moreover letting $\{\alpha_\varepsilon : \varepsilon < \kappa^+\}$ list d_δ in increasing order, we have $(\forall \varepsilon)[E \cap (\sup\limits_{\zeta<\varepsilon} \alpha_\zeta, \alpha_\varepsilon) \neq \emptyset]$ hence we can find $i(\delta,\varepsilon) \in (\sup\limits_{\zeta<\varepsilon} \alpha_\zeta, \alpha_\varepsilon) \cap A$ for each $\varepsilon < \kappa^+$.

Clearly for each $\varepsilon < \kappa^+$

$$\begin{aligned}\mathfrak{b}_\varepsilon = \big\{\lambda \in \mathfrak{a} : g_{i(\delta,\varepsilon)}(\lambda) < f_{\alpha_{i(\delta,\varepsilon)}}(\lambda) < f_{\gamma_{i(\delta,\varepsilon)}}(\lambda) \\ < g_{i(\delta,\varepsilon)+1}(\lambda) < f_{\alpha_{i(\delta,\varepsilon)+1}}(\lambda) < f_\alpha(\lambda)\} = \mathfrak{a} \bmod J\end{aligned}$$

hence $\mathfrak{b}_\varepsilon \cap \mathfrak{a}_{i(\delta,\varepsilon)} \notin \emptyset$. Moreover, $\mathfrak{b}_\varepsilon \cap \mathfrak{a}_{i(\delta,\varepsilon)} \notin J$. Now for each $\lambda \in \mathfrak{a}$ let $\varepsilon(\lambda)$ be $\sup\{\varepsilon < \kappa^+ : \lambda \in \mathfrak{b}_\varepsilon \cap \mathfrak{a}_{i(\delta,\varepsilon)}\}$ and let $\varepsilon(*) = \sup\{\varepsilon(\lambda) : \lambda \in \mathfrak{a}$ and $\varepsilon(\lambda) < \kappa^+\}$ so as $|\mathfrak{a}| \le \kappa$ clearly $\varepsilon(*) < \kappa^+$. Let $\lambda^* \in \mathfrak{b}_{\varepsilon(*)+1} \cap \mathfrak{a}_{i(\delta,\varepsilon(*)+1)}$, so $B := \{\varepsilon < \kappa^+ : \lambda^* \in \mathfrak{b}_\varepsilon \cap \mathfrak{a}_{i(\delta,\varepsilon)}\}$ is unbounded in κ^+, $\langle f_{\beta_{i(\delta,\varepsilon)}}(\lambda^*) : \varepsilon \in B\rangle$ is strictly increasing (see clause (c) above and the choice of $\mathfrak{b}_\varepsilon$) and $\varepsilon \in B \Rightarrow f_{\beta_{i(\delta,\varepsilon)}}(\lambda^*) \in a^\alpha_{\zeta(*)}$ (by the definition of $\mathfrak{a}_{i(\delta,\varepsilon)}$, and $\zeta(*)$ as $\zeta_{i(\delta,\varepsilon)} = \zeta(*)$). We get contradiction to $a \in \mathbf{F}(\text{Rang}(f_\alpha)) \Rightarrow |a| \le \kappa$.

So really we cannot carry the induction in $(*)_2$ so we are stuck at some $i < \theta_1^+$. If $i = 0$, or i limit, or $i = j+1$ & $\sup(w_j) < \chi$

we can find g_i and then α_i and then w_i as required. So necessarily $i = j+1, \sup(w_j) = \chi$. So we have finished proving $\boxtimes_2$.

$\boxtimes_3$ there is $\mathscr{I} \subseteq [\mu]^{\leq \kappa}$ as required.

Now if χ^* is regular, recalling that we assume $\mathrm{pp}_\kappa(\mu) =^+ \chi^*$ there are $\mathfrak{a}, J$ as required in $\oplus$ above for $\chi = \chi^*$, hence also such $\bar{f}$. Applying $\boxtimes_2$ to $(\chi, \mathfrak{a}, J, \bar{f})$ we get w as there. Now $\langle \mathrm{Rang}(f_\alpha) : \alpha \in w \rangle$ is as required in the fact. So the only case left is when χ^* is singular. Let $\chi^* = \sup\limits_{\varepsilon < \mathrm{cf}(\chi^*)} \chi_\varepsilon$ and $\chi_\varepsilon \in (\mu, \chi^*) \cap$ Reg is (strictly) increasing with ε. By [Sh:g, Ch.II,§3] we can find, for each $\varepsilon < \mathrm{cf}(\chi^*), \mathfrak{a}_\varepsilon, J_\varepsilon, \bar{f}^\varepsilon = \langle f^\varepsilon_\alpha : \alpha < \chi_\varepsilon \rangle$ satisfying the demands in $\oplus$ above, but in addition

- $\odot$ $\bar{f}^\varepsilon$ is μ^+-free i.e. for every $u \in [\chi_\varepsilon]^\mu$ there is a sequence $\langle \mathfrak{b}_\alpha : \alpha \in u \rangle$ such that $\mathfrak{b}_\alpha \in J_\varepsilon$ and for each $\lambda \in \mathfrak{a}_\varepsilon, \langle f^\varepsilon_\alpha(\lambda) :$ α satisfies $\lambda \notin \mathfrak{b}_\alpha \rangle$ is strictly increasing.

So for every $a \in [\mu]^{\leq \kappa}$ and $\varepsilon < \mathrm{cf}(\chi^*)$ we have

$$\big\{\alpha < \chi_\varepsilon : \{\lambda \in \mathfrak{a}_\varepsilon : f_\alpha(\lambda) \in a\} \notin J_\varepsilon\big\} \text{ has cardinality } \leq \kappa.$$

Hence for each $a \in [\mu]^{\leq \kappa}$

$$\big\{(\varepsilon, \alpha) : \varepsilon < \mathrm{cf}(\chi^*) \text{ and } \alpha < \chi_\varepsilon \text{ and } \{\lambda \in \mathfrak{a}_\varepsilon : f_\alpha(\lambda) \in a\} \notin J_\varepsilon\big\}$$

has cardinality $\leq \kappa + \mathrm{cf}(\chi^*) = \mathrm{cf}(\chi^*)$ as for singular $\mu > \kappa \geq \mathrm{cf}(\mu)$ we have $\mathrm{cf}(\mathrm{pp}_\kappa(\mu)) > \kappa$.

Define: $X = \{(\varepsilon, \alpha) : \varepsilon < \mathrm{cf}(\chi^*), \alpha < \chi_\varepsilon\}$

$$\begin{aligned} F'\big((\varepsilon, \alpha)\big) = \big\{(\varepsilon', \alpha') :& (\varepsilon', \alpha') \in X \backslash \{(\varepsilon, \alpha)\} \text{ and for some} \\ & d \in \mathbf{F}(\mathrm{Rang}(f^\varepsilon_\alpha)) \text{ we have} \\ & \{\lambda \in \mathfrak{a}_\varepsilon : f^{\varepsilon'}_{\alpha'}(\lambda) \in d\} \notin J_{\varepsilon'}\big\} \end{aligned}$$

so $F'\big((\varepsilon, \alpha)\big)$ is a subset of X of cardinality $< \mathrm{cf}(\chi^*)^+ + \theta < \chi^*$.

So by Hajnal's free subset theorem [Ha61] we finish proving $\boxtimes_3$ (we could alternatively, for χ^* singular, have imitated his proof).

Recalling $\boxtimes_1$ we are done. $\square_{11.4}$

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