

**A COLLECTION OF ABSTRACTS OF SHELAH'S PAPERS
E40**

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ABSTRACT. There are here abstracts of most of the papers up to 143 (and 217), mostly written in 1980/81 with Grossberg. Also more details than in the originals were added in B5, B8, B217 and C2, C3 were added (the Cxx are representations of the authors works).

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The author thanks Alice Leonhardt for the beautiful typing. References like [She, Th0.2=Ly5] means the label of Th.0.2 is y5. The reader should note that the version in my website is usually more updated than the one in the mathematical archive. First typed May 1987.

§ 1. NOTATION

- (A) (a) $\alpha, \beta, \gamma, \delta, \zeta, \xi, i, j$ denote ordinals; usually δ denotes a limit ordinal, ω the first infinite ordinal
 (b) $\lambda, \chi, \mu, \kappa$ stands for infinite cardinals
 (c) n, m, k, l natural numbers
 (d) λ^κ is cardinal exponentiation, $\lambda^{<\kappa} = \sum_{\mu < \kappa} \lambda^\mu$, $\beth(\lambda, \alpha) = \sum_{\beta < \alpha} 2^{\beth(\lambda, \beta)} + \lambda$,
 $\beth_\alpha = \beth(\aleph_0, \alpha)$
 (e) ${}^\alpha \lambda = \{f : i f : \alpha \rightarrow \lambda\}$ sometimes we call the member sequences
 (f) ${}^{\alpha > \lambda} = \bigcup_{\beta < \alpha} {}^\beta \lambda$
 (g) η, ν stands for sequences (or ordinals, usually)
 (h) GCH is the Generalized Continuum Hypothesis
- (B) (a) M, N (perhaps with index) are models
 (b) $\text{Th}(M)$ - the collection of first order sentences which are true in M . All languages are with equality.
 (c) a model is λ -like if it has cardinality λ and one of the relations of the model is an order $<$ such that every initial segment has cardinality less than λ , but not the whole.
 (d) vocabularies are denoted by τ .

§ 2. ABSTRACTS

(B1) Stable Theories, IJM 7(1969), 186-202.

The stability spectrum (i.e. the set of cardinals in which a theory T is stable) is characterized for countable theories and investigated in the general case; ranks for one formula are investigated, as well as existence of indiscernibles and prime among e.g. $|T|^+$ -saturated models. Also categoricity of elementary and of pseudo-elementary classes is investigated and lower bound on the number of non-isomorphic models of power \aleph_α is given for unstable and unsuperstable T (they are $|\alpha - \beta|$, $|(\alpha - \beta)/\omega|$ respectively, where $|T| = \aleph_\beta$).

(B2) Note on a Min-Max Problem of Leo Moser, J. Comb. Th. A 6(1969), 187-202

Moser asks how a pair of $(n+1)$ -sided dice should be loaded (identically) so that on throwing the dice the frequency of the most frequently occurring sum is as small as possible. G.F. Clements finds a relative minimum, conjecturing that it is always the solution. This conjecture is disproved for $n = 3$.

(B3) Finite diagrams stable in power, AML 2(1970), 69-118.

Let T be a complete first order theory, $D_m(T)$ the set of complete m -type (consistent with T), $D(T) = \bigcup_m D_m(T)$ and let $D \subseteq D(T)$. Let

$$K_D = \{M : M \text{ is a model of } T \text{ such that if } m < \omega \text{ and } \bar{a} \in {}^m M, \\ \text{then the type of } \bar{a} \text{ (in } M \text{ of course) belongs to } D\}.$$

Assume \mathfrak{C}_D is a D -monster, that is: it is a model of T , in K_D and is a $\bar{\kappa}$ -sequence homogeneous. We develop stability theory for it.

Anotated content

§1

[Let T be a complete first order theory, \mathfrak{C} its monster models M, N are $\prec \mathfrak{C}$, A, B are $\subseteq \mathfrak{C}$. Let $D(T)$ be the set of complete $L(T)$ -types realized in models of T and let D be a subset of $D(T)$. We call A a D -set if every finite sequence from A realizes (in \mathfrak{C}) a type from D , $D(A)$ the minimal such D . For a D -set let $\mathbf{S}_D^n(A)$ be the set of $p \in \mathbf{S}^n(A)$ such that if $\bar{a} = \langle a_\ell : \ell < n \rangle \in {}^n\mathfrak{C}$ realizes p then $A \cup \bar{a}$ is a D -set. M is called a D -model if $(M \prec \mathfrak{C})$ and $D(M) \subseteq D$; if $n = 1$ we may omit. We say M is (D, λ) -homogeneous if (it is a D -model and) $A \subseteq M$ and $|A| < \lambda$ and $p \in \mathbf{S}_D(A) \Rightarrow p$ realized in M (if $D = D(T)$ this means M is λ -saturated). Let D be fixed, all sets are D -sets, all models are D -models.]

§2

[We call D λ -good if for every λ there is (D, λ) -homogeneous model of cardinality $\geq \lambda$. We call D λ -stable if it is λ^+ -good and for every D -set A of cardinality $\leq \lambda$, $|A| \leq \lambda \Rightarrow |\mathbf{S}_D(A)| \leq \lambda$. We say D is stable if it is stable in some λ . We say $p \in \mathbf{S}_D^n(A)$ split over $B \subseteq A$ if 0 for $\bar{a}, \bar{b} \subseteq A$ realizing the same type over B and $\varphi, \varphi(\bar{x}, \bar{a}) \equiv \varphi(\bar{x}, \bar{b}) \in p$. Basic properties of splitting are proved. E.g. if $|\mathbf{S}_D(A)| > \mu_0 = |A|^{<\lambda} + \sum_{\mu < \lambda} 2^{|D|^\mu}$ then we can find $p \in \mathbf{S}_D(A)$ and an increasing sequence $\langle A_i : i < \lambda \rangle$, $|A_i| < \aleph_0 + |i|^+$ such that $p \upharpoonright A_{i+1}$ split over A_i , this in turn implies unstability in every $\mu < 2^\lambda$. Hence, if D is unstable in every $\lambda < \beth((2^{|T|})^+)$ then D is unstable (in fact have an indiscernible sequence (which forms a D -set and) which ordered by some formula).]

§3

[If D is stable, then D is good, and if $B \subseteq A$, $p \in \mathbf{S}_D^n(B)$ then p can be extended to some $q \in \mathbf{S}_D^n(A)$. We also prove that if D is stable in λ , $A \cup \bigcup\{\bar{a}_i : i < \lambda^+\}$ is a D -set, each \bar{a}_i a finite sequence then for some $X \subseteq \lambda^+$, $|X| = \lambda^+$, we have $\langle \bar{a}_i : i \in X \rangle$ is an indiscernible set over A .]

§4

[We say $p \in \mathbf{S}_D(A)$ splits strongly over $B \subseteq A$ if for some C , $\langle \bar{a}_i : i < \omega \rangle$ we have $A \cup \bigcup_{i < \omega} \bar{a}_i \subseteq C$ (the point is that C is a D -set) $\langle \bar{a}_i : i < \omega \rangle$ is an indiscernible sequence (equivalently set by §3) over A and for some ψ , $\psi(\bar{x}, \bar{a}_0), \neg\psi(\bar{x}, \bar{a}_i) \in p$. We investigate it and using it characterize the stability spectrum. I.e. if D is stable, then for some cardinal $\lambda = \lambda(D)$, $\kappa = \kappa(D)$ which are $< \beth((2^{|T|})^+)$ we have: D is μ -stable iff $\mu \geq \lambda(D)$ and $\mu = \mu^{<\kappa(D)}$. Also, a formula cannot divide an indiscernible set to two large pieces (i.e. of size $\geq \kappa(D)$).]

§5

[Some variants of λ -isolation are defined and so λ -prime models over (D) -sets are proved to exist using what is called in [She78, Ch.IV] primary models are defined; in fact, the framework of it applies here (including uniqueness of primary).]

§6

[We prove that there is a (D, λ) -homogeneous non- (D, λ^+) -homogeneous models in all but “degenerate” cases (parallel to unidimensional T).]

§7

[We prove the parallel of “characterizing the categoricity spectrum replacing categorical by all models are homogeneous, for class of the form $\{M : M \text{ a model of } T \text{ omitting every type in a set } \Gamma\}$.]

We prove: e.g. if T is first order complete countable not categorical in \aleph_2 then it has $\geq |\alpha + 1|$ non-isomorphic models in \aleph_0 .

See [She75e] on the spectrum of $\{\lambda : \text{there is a } (D, \lambda)\text{-homogeneous model of cardinality } \lambda\}$.

(B4) On Theories T categorical in $|T|$, JSL 35(1970),73-82.

We prove that, if T is categorical in $|T|$, $|T| = |T|^{\aleph_0}$, then T has a model of power $< |T|$; moreover, T is a definitional extension of a theory of smaller power. We first note that T has a \aleph_1 -compact model, and if T is unsuperstable also a non \aleph_1 -compact model of power \aleph_1 . So T is superstable. We construct a model which is “almost prime” over a countable indiscernible set, and prove it is maximal in it, but we can easily build a model without such set.

Superseded by [She78, Ch.IX].

(B5) On languages¹ with non-homogeneous strings of quantifiers, IJM 8(1970),75-79.

In infinitary logic we can introduce a string of quantifiers of the form

$$(\dots Q_i x_i \dots Q_j x_j \dots)_{i \in I} \varphi(\bar{x})$$

I a linear order, Q_i is \exists or \forall . For I well ordered the interpretation is by a game, generally by Skolem functions. The theorem is that we can express such quantifiers by a well ordered sequence of quantifiers (not so short; and propositional connectives). In fact, only the case when I an inverse of a well ordered set is presented in [She70]; the full proof appears in the author M. Sc. thesis.

(In somewhat revised formulation this was sent to a student of Takeuli and included in his thesis, Urbana 1973).

More formally

Definition 2.1. (Mostowski). The meaning of

$$M \models (\dots, (Q_t \bar{x}_t), \dots)_{t \in I} \varphi(\dots, \bar{x}_t, \dots)$$

where I is a linear order, $\bar{x}_t = \langle x_{t,\alpha} : \alpha < \alpha_t \rangle$, $Q_t \in \{\forall, \exists\}$, letting

$$I_\exists = \{t : Q_t = \exists\}, \quad I_\forall = \{t \in I : Q_t = \forall\}$$

is that there are functions $F_{t,\alpha} = F_{t,\alpha}(\dots, x_{s,\beta}, \dots)_{s \in I_\forall, s < t, \beta < \alpha_s}$ for $t \in I_\exists$, $\alpha < \alpha_t$ such that for every $a_{s,\beta} \in M$ (for $s \in I_\forall, \beta < \alpha_s$); letting $a_{t,\alpha} = F_{t,\alpha}(\dots, a_{s,\beta}, \dots)_{s \in I_\forall, s < t, \beta < \alpha_s}$ we have $M \models \varphi[\dots, \langle a_{t,\gamma} : \gamma < \alpha \rangle, \dots]_{t \in I}$.

Theorem 2.2. 1) The formula $(\dots, (Q_t \bar{x}_t) \dots)_{t \in I} \varphi(\dots, \bar{x}_t, \dots)$ is equivalent to the formula ψ^* defined below.

Let

- (A) $\mathcal{T} = \{\bar{J} : \bar{J} = \langle J_i : i \leq j \rangle \text{ is a strictly } \subseteq\text{-increasing continuous sequence of proper initial segments of } I, J_0 = \emptyset\}$
- (B) For $\bar{J} = \langle J_i : i \leq j \rangle$ let $t \in \bar{J}$ mean $t \in J_j \setminus \bigcup_{i < j} J_i$ and let $\bar{J}_1 \leq \bar{J}_2$ mean \bar{J}_1 is an initial segment of \bar{J}_2
- (C) For $\bar{J} = \langle J_i : i \leq j \rangle \in \mathcal{T}$ let

$$\bar{y}_{\bar{J}} = \langle \dots, x_{s,\beta}, \dots \rangle_{s \in \bar{J}, \beta < \alpha_s, Q_s = \forall}$$

$$\bar{z}_{\bar{J}} = \langle \dots, x_{s,\beta}, \dots \rangle_{s \in \bar{J}, \beta < \alpha_s, Q_s = \exists}$$

- (D) Lastly, define the formula $\psi^* = (\dots, (\forall \bar{y}_{\bar{J}})(\exists \bar{z}_{\bar{J}}), \dots)_{\bar{J} \in \mathcal{T}} \langle \varphi(\dots, \bar{y}_{\bar{J}_1}, \bar{z}_{\bar{J}_1}, \dots)_{\bar{J}_1 \leq \bar{J}_2} : \bar{J} \in \mathcal{T} \rangle$ which means: as explained below

- (E) *in the following game between the player \exists and \forall , the player \exists has a winning strategy;*
- (a) *in stage i of a play an initial segment J_i of I , strictly increasing with i is constructed and a sequence $\langle \dots, a_{s,\beta}^i, \dots \rangle_{s \in I, \beta < \alpha_s}$ are constructed such that*
- (*) $s \in J_{i_0}, i_0 < i_1 \Rightarrow a_{s,\beta}^{i_0} = a_{s,\beta}^{i_1}$
- (b) *in stage i , the player \forall chooses $a_{s,\beta}^i (s \in I, \beta < \alpha_s, Q_s = \forall)$ but satisfying (*), then player \exists chooses $a_{s,\beta}^i (s \in I, \beta < \alpha_s, Q_s = \exists)$ but satisfying (*), and then player \forall chooses J_j as required. If in stage β $M \models \neg \varphi(\dots, a_{s,\beta}^i, \dots)$, player \exists loses immediately. If \exists never loses, he wins.*

2) *Formulas as in (1) (with tree quantifiers indexed by \mathcal{T} a well founded tree) can be translated to one of the form $(\forall \bar{y}^0)(\exists \bar{z}^0) \dots (\forall \bar{y}^\alpha)(\exists \bar{z}^\alpha) \dots \bigwedge_{\Upsilon \in \mathcal{T}} \varphi_\Upsilon(\bar{x}^\Upsilon)$ where γ is the number of levels of the tree (so in our case, every increasing sequence of initial segments has length $< \gamma$) and φ_Υ the formula “sitting” in the node Υ (we need \bar{x}^Υ not to repeat variables appearing in incomparable nodes of the tree, in our case all are the same φ). In fact $\gamma^* \geq \text{otp}(X)$ for every branch of the tree \mathcal{T} , then γ^* can serve and if $\alpha < \gamma^*$, $\ell g(\bar{y}^\alpha) = \Sigma\{|\ell g(\bar{y}_t)| : t \in \mathcal{T} \text{ of level } \alpha\}$, $\ell g(\bar{z}^\alpha) = \Sigma\{|\ell g(\bar{z}_t)| : t \in \mathcal{T} \text{ of level } \alpha\}$ are O.K.*

Proof. Straightforward, we may wonder: in the new formulas the “ $(\exists \bar{z}^\alpha)$ may depend on additional variables but because of the structure of $\bigwedge_{\Upsilon \in \mathcal{T}} \varphi_\Upsilon(\bar{x}^\Upsilon)$ this is not a problem. □_{2.2}

Another result from the M.Sc. thesis is:

Theorem 2.3. *[Hanf number of $\mathbb{L}_{\lambda^+, \lambda^+}$ with unary function only.]*

Let the vocabulary τ be $\{F\} \cup \{P_i : i < \lambda\}$ where F is a unary function and the P_i are unary predicates.

- (A) *if $\psi \in \mathbb{L}_{\lambda^+, \lambda^+}$ has models in arbitrarily large cardinality $< \beth_{\lambda^+}$ (in fact $> \beth_{\text{q.d.}(\psi)} + 1(\lambda)$ suffice), then it has models in arbitrarily large cardinalities (in fact, any one above $\beth_{\text{q.d.}(\psi)} + 1(\lambda)$)*
- (B) *if $T \subseteq \mathbb{L}_{\lambda^+, \lambda^+}$ has a model of cardinality $> \beth_{\lambda^+}$, then it has models in arbitrarily large cardinalities.*

Proof. For an L -model M and $x \in M$ let

$$M^{[x]} = (M \upharpoonright \{y : \text{for some } n < \omega, F^n(y) = x\}, x)$$

(so $F^{M^{[x]}}$ may be partial and x serve as an individual constant) where $F^0(x) = x$, $F^{n+1}(x) = F(F^n(x))$ and let $M^{(x)} = M \upharpoonright \{y : y \in \bigcup_{n < \omega} M^{F^n(x)}\}$. Let $\text{Pre}_M(x) = \{y : F^M(y) = x\}$.

Why (A)? Assume for simplicity that $\text{q.d.}(\psi)$ is a limit ordinal. If M is a model of T of cardinality $> \beth_{\text{q.d.}(\psi)}(\lambda)$ then one of the following occurs:

- (α) the number of components (i.e. $M^{(x)}, x \in M$) is $> \beth_{\text{q.d.}(\psi)+1}$. So for some x , for each $\beta < \text{q.d.}(\psi)$ the set $\{M^{(y)} : M^{(y)}, M^{(x)}\}$ satisfies the same

L_{∞, λ^+} -formulas of quantifier depth $\leq \beta$ has cardinality $> \lambda$. So we can add to M any number of isomorphic copies of $M^{(x)}$

- (β) for some $x \in M$, $\beth_{\text{q.d.}(\psi)+1} < |\text{Pre}_M(x)|$ so for some $y \in \text{Pre}_M(x)$, for every $\beta < \text{q.d.}(\psi)$, we have $\{z \in \text{Pre}_M(x) : M^{[z]}, M^{[x]}$ satisfies the same L_{∞, λ^+} formulas of quantifier depth $\leq \text{q.d.}(\psi)\}$ has cardinality $> \beth_{\text{q.d.}(\psi)}$. So we can just “increase” $\text{Pre}_M(x)$, giving more copies of $M^{(x)}$.

Why (B)? Similarly.

□_{2.3}

- (B6) A note on Hanf numbers, PJM, 34(1970),539-543.

We show that for every $\xi < (2^\kappa)^+$, there is a theory T and set of type Γ in a language of power κ , such that there is a model of T which omits every $p \in \Gamma$ of power λ if and only if $\lambda \leq \beth_\xi$. This completes the computation of an appropriate Hanf number. We also disprove a conjecture of Morley on the existence of algebraic elements in such examples.

- (B7) On the cardinality of ultraproduct of finite sets, JSL, 35(1970), 83-84.

We prove that if D is an ultrafilter and $\aleph_0 \leq \lambda = \prod n_i/D$, then $\lambda^{\aleph_0} = \lambda$. The method is to use the amount of number theory $N = \prod n_i/D$ inherit and \aleph_1 -saturation of N .

- (B8) Two cardinal compactness, IJM 9(1971), 103-198.

Let K be the class of (λ, μ) -models, i.e. models such that $\|M\| = \lambda$, $|P^M| = \mu$. We prove that when $\mu^{\aleph_0} = \mu$, K is μ -compact (i.e. a theory T of cardinality $\leq \mu$ has a (λ, μ) -model iff every finite subtheory has a (λ, μ) -model).

In fact \aleph_0 -compactness implies μ -compactness, see the abstract in the Notices of the AMS, [She71b]. We also prove that then any T , $|T| \leq \mu$ which has a (λ, μ) -model, has a (λ, μ) -model in which only $\leq 2^{\aleph_0} + |T|$ types are realized (e.g. if $|T| = \aleph_0$, and the class of (λ, μ, \aleph_0) -models is compact, we can get \aleph_0 types). Also we can get models with many automorphisms (by a sequence of μ indiscernibles).

Similar theorems hold for μ -like models, hence cardinality quantifiers, and even for several such demands.

In the cases that \aleph_0 -compactness holds, transfer theorems are reduced to partition theorems with finite conclusions.

The models are like incomplete Ehrenfeucht-Mostowski ones, but the indiscernibility is not “full”, just as much as corresponding to the appropriate partition theorems.

- (B9) Remark to “local definability theory of Reyes”, AML 2(1971), 441-448.

Let $\tau \subseteq \tau_1$, $P \in \tau_1 \setminus \tau$ and T a theory in τ_1 . For a model M let

$$\text{df}(M) = |\{P : (M, P) \text{ is a reduct of a model of } T\}|$$

and let $\text{Df}(\lambda) := \sup\{\text{df}(M)^+ : \|M\| = \lambda\}$.

Then the following are equivalent:

- (i) for no formula $\theta(\bar{x}, \bar{y}) \in \tau$, $T \vdash (\exists \bar{y})(\forall \bar{x}) [P(\bar{x}) \equiv \theta(\bar{x}, \bar{y})]$
- (ii) for some $\lambda \geq |\tau_1|$, $\text{Df}(\lambda) > \lambda^+$,
- (iii) for every $\lambda \geq |\tau_1|$, $\text{Df}(\lambda) \geq \text{Ded}^*(\lambda)$ (which is the first μ such that any tree of power λ , has $< \mu$ branches of a fixed height).

Conjecture 2.4. If for some $\lambda \geq |\tau_1|$, $\text{Df}(\lambda) > \text{Ded}(\lambda)$ then for every λ , $\text{Df}(\lambda) = (2^\lambda)^+$.

(B10) Stability, the f.c.p. and superstability, AML 3(1971), 271-362.

We investigate in detail stable formulas, ranks of types and their definability, the f.c.p., some syntactical properties of unstable formulas, indiscernible sets and degrees of types in superstable theories. There is a list of all results connected with those properties, or whose proof use them.

This list may still be of value. Superseded by [She78, Ch.II].

(B11) On the number of non-almost isomorphic models of T in a power, PJM, 36(1971), 811-818.

Let T be a first order theory. Two models are almost isomorphic if they are elementarily equivalent in the language $\mathbb{L}_{\infty, \omega}$. We investigate the number of non almost-isomorphic models of T of power λ as a function of λ , $I(T, \lambda)$. We prove $\mu > \lambda \geq |T|$ and $I(T, \lambda) \leq \lambda$ implies $I(T, \mu) \leq I(T, \lambda)$.

We also get downward Lowenheim Skolem theorem for the corresponding variant of a rigid model.

(B12) The number of non-isomorphic models of an unstable first-order theory, IJM 4(1971), 473-487.

It is proved that if T is an unstable (first-order) theory, $\lambda > |T| + \aleph_0$, then T has exactly 2^λ non-isomorphic models of cardinality λ . In fact we have stronger results: this is true for pseudo-elementary classes, and for almost every $\lambda \geq |T| + \aleph_1$.

The method is contradictory orders. Not too many Ehrenfeucht Mostowski models built on pairwise contradictory order are isomorphic. Many pairwise contradictory orders are constructed, by using stationary sets.

In passing we deal with the existence of a family of 2^λ subsets of λ , each of power λ , the intersection of any two finite, if it does not exist, our proof works for $\lambda = |T| + \aleph_1$, if it does exist, for some $\aleph_\alpha < \lambda$, $2^\lambda = 2^{|\alpha|} + 2^{\aleph_0}$ (in fact it follows that $\lambda < 2^{\aleph_0} \vee (\exists \mu, \alpha), \mu < \alpha = \aleph_\alpha \leq \lambda < \mu^{\aleph_0}$) and then we use a different proof.

Superseded by [She87c, Ch.III, §3], even better [Shear, Ch.III, §3].

(B13) Every two elementary equivalent models have isomorphic ultrapowers, IJM, 10(1971), 224-233.

We prove that every two elementarily equivalent models have isomorphic ultrapower. In fact if M, N are of cardinality $\leq \lambda$, then there is such an ultrafilter on 2^λ .

Later the author notes that (see Stern, [Ste76]) the parallel theorem for Banach spaces, and so get Craig interpolation theorem for the suitable logic.

(B14) Saturation of Ultrapowers and Keisler's Order

Superseded by [She78, Ch.VI], except §1, which gives examples of uncountable theories.

(B15) Uniqueness and characterization of prime models over sets for totally transcendental first-order theories, JSL (1972), 107-113.

If T is a complete first-order totally transcendental theory then, over every T -structure A , the prime model is unique up to isomorphism over A . Moreover, M is a prime model over A iff:

- (a) every finite sequence for M realizes an isolated type over A , and
- (b) there is no uncountable indiscernible set of A in M . Hence the uniqueness of the differential closure of a differential field of characteristic zero follows.

The proof is by a suitable induction on rank.

(B16) A combinatorial problem: stability and order for models and theories in infinitary languages PJM 41(1972).

Some infinite combinatorial problems are solved. Their model-theoretic representation is: if the model M is unstable in λ , $\lambda = \lambda^{<\mu} + \sum_{\kappa < \mu} 2^{2^\kappa}$, then in M there is a set of sequences from M , of fixed finite length ordered by one formula. We then prove that if M is stable in λ and has no “large” ordered set, every set of $> \lambda$ elements contains an indiscernible subset. We also prove that if $\psi \in \mathbb{L}_{\lambda^+, \omega}$, $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\infty, \omega}$ and ψ has a model in which ψ orders a set of power μ for arbitrarily large μ (or just $\mu < \beth_{(2^\kappa)^+}$ where $\psi, \varphi \in \mathbb{L}_{\kappa^+, \omega}$) then ψ has 2^μ non-isomorphic models of power μ for each μ .

Most model theory is superseded by [She87c, Ch.I] and Theorem 2.5 by [GS86].

(B17) For what filters is every reduced product saturated IJM 12(1972), 23-31.

In this paper we characterize the filters D such that for every sequence $\langle M_i : i < I \rangle$ we have $\prod_{i \in I} M_i / D$ is λ -saturated (where $\lambda > \aleph_0$). The characterization is: D is λ -good, D is \aleph_0 -incomplete and $\mathcal{P}(I)/D$ is a λ -saturated Boolean Algebra. (Note that when the Boolean algebra $\mathcal{P}(I)/D$ is λ -saturated it can be characterized directly; we can also restrict ourselves e.g. to Π_n -formulas only).

(B18) On Models with power-like orderings JSL 37(1972), 247-267.

We prove here theorems of the form: if T has a model M in which $P_1(M)$ is κ_1 -like ordered, $P_2(M)$ is κ_2 -like ordered..., and $Q_1(M)$ is of power λ_1, \dots , then T has a model N in which $P_1(M)$ is κ'_1 -like ordered..., $Q_1(N)$ is of power λ'_1, \dots (In this article κ is a strong limit singular cardinal, and κ' is a singular cardinal). If the language has power χ , this is written $\chi : \langle \kappa_1, \kappa_2, \dots, |\lambda_1, \dots \rangle \rightarrow \langle \kappa'_1, \kappa'_2, \dots, |\lambda'_1, \dots \rangle$.

We also sometimes add the condition that M, N omits some types. The results are seemingly the best possible, i.e. according to our knowledge about n -cardinal problems (or, more precisely, a certain variant of them, speaking on cofinality, too).

Our method will enable us to reduce such problems to transfer problems which do not mention κ -like orderings. In many cases we can translate problems of the form $\chi : \langle \kappa_1, \dots, |\lambda_1, \dots \rangle \rightarrow \langle \kappa'_1, \dots, |\lambda'_1, \dots \rangle$ to problems of the form

$$\chi : \langle \text{cf}(\kappa_1), \dots, \lambda_1, \dots \rangle \rightarrow \langle \text{cf}(\kappa'_1), \dots, \lambda'_1, \dots \rangle.$$

As an example of our results, concerning transfer between pairs of cardinals we have: if $\chi \leq \lambda' < \kappa', \chi \leq \lambda < \kappa$, then $\chi : \langle \kappa | \lambda \rangle \rightarrow \langle \kappa' | \lambda' \rangle$ when at least one of the following conditions is satisfied:

- (A) $\text{cf}(\kappa') \leq \lambda'$,
- (B) $\text{cf}(\kappa) > \beth_\omega(\lambda)$,
- (C) $\text{cf}(\kappa) > \lambda, \text{cf}(\kappa') = (\lambda')^+, \lambda' = \sum_{\mu < \lambda} (\lambda')^\mu$.

In general, the transfers are affected by using incomplete Ehrenfeucht-Mostowski-like types obtained from certain polarized partition theorems.

In addition we prove that:

Theorem 2.5. *There is a nice logic which is (fully) compact; stronger than first order. We got it adding the quantifier “the cofinality of an order is κ ” to first order logic gives a compact logic (provably in ZFC of course; subsequently more such logics were found).*

We also suggest transfer theorems starting with finite cardinals, some problems on them and the easily answered cases. (Those corresponding to the gaps $(\aleph_0), (\beth_\omega, \aleph_0)$), see p.250 and see more in [She75b].

In proving the theorems or omitting types we use passing to non-well ordered models of fragments of set theory (§5).

(B19) Separability properties of almost disjoint families of sets, IJM 12(1972), 207-214 (with P. Erdős).

We deal with families of subsets of ω . It is proved (in ZFC) that: there is a strongly n -separable non $(n+1)$ -separable maximal almost disjoint family (of infinite subset of ω), any for $n = 2$, but (under MA) not necessarily for $n = 3$. Also we have some similar results, also independent families and related finitary problems are discussed.

Saharon: read.

(B20) On power-like models for hyperinaccessible cardinals JSL 37(1972), 531-537 (with J.H. Schmerl).

The main result of this paper is the following transfer theorem: if T is an elementary theory which has κ -like model where κ is Mahlo of order ω , then T has a λ -like model for each $\lambda > \text{card}(T)$. This can be expressed as a transfer theorem for generalized quantifiers (we get compactness and omitting-type results, too).

The partition theorem originally used by the second author is:

Theorem 2.6. *If f_ℓ is an ℓ -place function on κ , where κ is $(m+n)$ -Mahlo then there is an m -Mahlo $\lambda < \kappa$ and $A \subseteq \lambda$ unbounded, such that if $\alpha_1 < \dots < \alpha_\ell \in A$, $\ell \leq n$, $f(\alpha_1, \dots, \alpha_\ell) < \alpha_i$ then the value of f does not depend on $\alpha_{i+1}, \dots, \alpha_n$.*

What is sufficient (and proved there) is the version with having $A = \{\alpha_1 < \alpha_2 < \dots < \alpha_n\}$ with $m = 0$.

(B21) On problems of Moser and Hanson Proc. Symp. in Graph Theory, Springer Lecture, No.303, 75-80 (with P. Erdős).

The following problem is due to L. Moser. Let A_1, \dots, A_n be any n sets. Take the largest subfamily A_{i_1}, \dots, A_{i_r} which is union-free; i.e. $A_{i_{j_1}} \cup A_{i_{j_2}} \neq A_{i_{j_3}}$, $1 \leq j_i \leq r$, $1 \leq j_2 \leq r$, $1 \leq j_3 \leq r$, for every triple of distinct sets $A_{j_1}, A_{j_2}, A_{j_3}$. Put $f(n) = \min(r)$, where the minimum is taken over all families of n distinct sets.

Determine or estimate $f(n)$. Improving previous bounds we show

$$\sqrt{2n} - 1 < f(n) < 2\sqrt{n+1}.$$

(B22) A note on model complete models and generic models, Proc. AMS 34(1972), 509-514.

We prove that there are many maximal model complete (= generic) models, and that there exists an (uncountable) theory with no generic models.

(B23) Some counterexamples in the partition calculus, J. Comb. Th. A 15(1973), 167-174. (with F. Galvin)

We show that $\aleph_1 \rightarrow [\aleph_1]_4^2$, i.e. the pairs (2-element subsets; edges of the complete graph) of a set of cardinality \aleph_1 can be colored with 4 colors so that every uncountable subset contains pairs of every color, and $2^{\aleph_0} \rightarrow [2^{\aleph_0}]_{\aleph_0}^2$, i.e. the pairs of real numbers can be colored with \aleph_0 colors so that every set of reals of cardinality 2^{\aleph_0} contains pairs of every color. These results are counterexamples to certain transfinite analogs of Ramsey's theorem. Results of this kind were obtained previously by Sierpinski and by Erdős, Hajnal and Rado. The Erdős-Hajnal-Rado result is much stronger than ours, but they used the continuum hypothesis and we do not. As by-products, we get an uncountable tournament with no uncountable transitive subtournament, and an uncountable partially ordered set such that every uncountable subset contains an infinite antichain and a chain isomorphic to the rationals.

(B24) First order theory of permutation groups, IJM 14(1973), 149-162.

(B25) Errata to first order, theory of permutation groups, IJM 15(1973), 437-441.

We solve the problem of the elementary equivalence (definability) of the permutation groups over cardinals \aleph_α , by getting bi-interpretability results. We show that it suffices to solve the problem of elementary equivalence (definability) for the ordinals α in certain second order logic, and this is reduced to the case of $\alpha < (2^{\aleph_0})^+$. E.g. if the ordinal $((2^{\aleph_0})^+)^{\omega}$ (ordinal exponentiation) divides α_1, α_2 and $\text{cf}(\alpha_1), \text{cf}(\alpha_2) > 2^{\aleph_0}$ or $\text{cf}(\alpha_1) = \text{cf}(\alpha_2)$, then the permutation groups of $\aleph_{\alpha_1}, \aleph_{\alpha_2}$ are elementarily equivalent. We also solve a problem of Mycielski and McKenzie on embedding of free groups in permutation groups, and discuss some weak second-order quantifiers.

(B26) Notes on combinatorial set theory, IJM 14(1973), 262-277.

We shall prove some unconnected theorems:

- 1) (GCH) $\omega_{\alpha+1} \rightarrow (\omega_\alpha + \xi)_2^2$ when \aleph_α is regular and $|\xi|^+ < \aleph_\alpha$.
- 2) There is a Jónsson algebra in $\aleph_{\alpha+n}$, and $\aleph_{\alpha+n} \rightarrow [\aleph_{\alpha+n}]_{\aleph_{\alpha+n}}^{n+1}$ if $2^{\aleph_\alpha} \leq \aleph_{\alpha+n}$.
- 3) [Universal graph] If $\lambda > \aleph_0$ is a strong limit cardinal, then among the graphs with $\leq \lambda$ vertices each of valence $< \lambda$ there is a universal one.
- 4) (GCH) If f is a set mapping on $\omega_{\alpha+1}$ (\aleph_α regular) satisfying $\alpha < \beta < \lambda \Rightarrow |f(\alpha) \cap f(\beta)| < \aleph_\alpha$, then there is a free subset of order-type ξ for every $\xi < \omega_{\alpha+1}$.

(B27) Size direction games over the real line. III., IJM, 14(1973), 442-449 (with G. Moran).

(B28) There are just four second-order quantifiers, IJM, 14(1973), 262-277.

Among the second-order quantifiers ranging over relations satisfying a first-order sentence, there are four for which any other one is bi-interpretable with one of them: the trivial, monadic, permutational and a full second-order. (We deal with infinite, one sorted models). The interpretation is in fact first-order (second-order parameters but not quantifiers are used).

See representation in Handbook of model theoretic logics, Baldwin's article and [She86].

(B29) A substitute for Hall's theorem for families with infinite sets, J. Comb. Th. A. 16(1974), 199-208.

A sufficient condition for the existence of a system of distinct representatives for a family S is that $x \in A \in S$ implies the number of elements of A is not smaller than the number of sets in S to which x belongs.

The method is: by Hall's theorem we reduce the problem for countable S to the divergence of an infinite sum, which is then proved to diverge. For S uncountable see [MS74].

(B30) The cardinals of simple models for universal theories, Proc. of the Symp. in honor of Tarski's 70th birthday, Proc. Symp. in Pure Math 25(1974), 53-74 (with R. McKenzie)

Some results about spectra of cardinals of simple algebras in varieties and, more generally, about the cardinals of T -simple models where T is a universal theory are obtained and applications discussed. M is T -simple if every homomorphism from M to a model N of T is an embedding or a constant. It is shown that if the language of T has power κ and if there exists a T -simple model whose power exceeds 2^κ , then T -simple models exist in all powers $\lambda \geq \kappa$ (the point is that 2-indiscernibility is enough, even a sequence of approximations). It is further shown that if the language of T is countable, and if there exists an uncountable T -simple model, then there exists a T -simple model with the power 2^{\aleph_0} . Also counterexamples are given showing the results are essentially best possible.

There are generalizations.

(B33) The Hanf number of omitting complete types, PJM 50(1974).

It is proved in this paper that the Hanf number m^c of omitting complete types by models of complete countable theories is the same as that of omitting not necessarily complete type by models of a countable theory. I.e.

Theorem 2.7. *For every ordinal $\alpha < \omega_1$ there is a countable first-order vocabulary τ_α and a complete theory T_α in $\mathbb{L}(\tau_\alpha)$ such that:*

- (i) $p = \{P(\chi_0)\} \cup \{\chi_0 \neq c_n : n < \omega\}$ is a complete type for T_α
- (ii) T_α has a model of cardinality \beth_α omitting p
- (iii) T_α has no model of cardinality $> \beth_\alpha$ omitting p .

(B34) Weak definability in infinitary languages, JSL 38(1973), 339-404.

We prove that if a model of cardinality κ can be expanded to a model of a sentence ψ from $\mathbb{L}_{\lambda^+, \omega}$ by adding a suitable predicate in more than κ ways, then it has a submodel of power μ which can be expanded to a model of ψ in $> \mu$ ways provided that λ, κ, μ satisfy suitable conditions.

The Main Theorem 2.8. *1) Let ψ be a sentence in $\mathbb{L}_{\lambda^+, \omega}(\tau + P)$, P a unary predicate not in τ , $|\tau| \leq \lambda$, M a τ -model of cardinality $\aleph_{\alpha+\beta}$ such that:*

$$|\{P : P \subseteq |M|, (M, P) \models \psi\}| > \aleph_{\alpha+\beta}.$$

Assume further that $\beta < \omega_1$, \aleph_α has cofinality \aleph_0 , $\mu_n \geq \lambda$, $\mu = \sum_{n < \omega} \mu_n$, $\mu_n < \mu_{n+1}$ and $\kappa < \aleph_\alpha \Rightarrow \kappa^{\mu_n} < \aleph_\alpha$ for $n < \omega$.

Then M has an elementary submodel N of cardinality μ such that

$$|\{P : P \subseteq |N|, (N, P) \models \psi\}| \geq \mu^{\aleph_0}.$$

Another theorem which we shall not prove, as its proof is simpler is

Theorem 2.9. *Let $\psi \in \mathbb{L}_{\lambda^+, \omega}(\tau + P)$, M a τ -model of cardinality κ such that $|\{P : P \subseteq |M|, (M, P) \models \psi\}| > \kappa$. Assume further that $\mu \geq \lambda$, $\kappa^\mu = \kappa$. Then M has an elementary submodel N of cardinality μ such that*

$$|\{P : P \subseteq |N|, (N, P) \models \psi\}| > \mu^{\aleph_0}.$$

Also, (see middle of page 400 there)

Theorem 2.10. *If T is a complete (first order) theory, $|T| = \lambda^+$, λ regular (for simplicity) and every n -type of cardinality $< \lambda$ can be extended to complete n -type of cardinality $< \lambda$ (holds if $|D(T)| < 2^\lambda$), then T has a model in which every finite sequence realizes a complete type of cardinality $< \lambda$.*

(B35) Sufficiency conditions for the existence of transversals *Cand. J. Math* 26 (1974), 948-961 (with E.C. Milner).

The main theorem has an interesting formulation in terms of bipartite graphs. A bipartite graph is a triple $\Gamma = \langle X, \Delta, Y \rangle$ with vertex set $X \cup Y$ (X, Y disjoint sets) and edge set $\Delta \subseteq \{\{x, y\} : x \in X, y \in Y\}$. Let $v(z) = |\{u \in X \cup Y : \{u, z\} \in \Delta\}|$ (for $z \in X \cup Y$) be the valency function of Γ . Then the main theorem is equivalent to the following statement: if $\Gamma = \langle X, \Delta, Y \rangle$ is a bipartite graph such that $v(x) > 0$ for $x \in X$ and $v(x) \geq v(y)$ whenever $x \in X, y \in Y$ and $\{x, y\} \in \Delta$, then there is a matching from X into Y , i.e. there is a 1-1 function $\varphi : X \rightarrow Y$ such that $\{x, \varphi(x)\} \in \Delta$ (for $x \in X$).

In §7 we prove even stronger results.

(B36) Remarks on Cardinal invariants in Topology, General Topology and its Applications 7(1977), 251-259.

In §2 we investigate to what degree singular calibres are preserved by products (for regular calibres Sanin solved the problem). We also generalize the concept to pairs of cardinals.

We prove that

Theorem 2.11. *If λ is a calibre of X , $\text{cf}(\lambda) > \aleph_0$, then λ is a calibre of X^I .*

In §1 we construct a T_2 space X with $d(X) = 2^{\aleph_0}$ and $s(x) = \aleph_0$; i.e. there is no discrete subspace of power \aleph_1 . We also show that the upper bound in the definition of $h(X) = \lambda$ is realized if X is Hausdorff and $\text{cf}(\lambda) = \aleph_0$. The theorem on $h(X) = \lambda$, for X Hausdorff was proved independently by Hajnal and Juhasz and the author.

In §3 we show that the class of calibres of a T_2 space is just about arbitrary.

(B37) A two-cardinal theorem, *Proc. AMS* 481(1975), 207-213.

We prove the following theorem and deal with some related questions: if for all $n < \omega$, T has a model M such that $n + |Q^M|^n \leq |P^M| < \aleph_0$ then for all λ, μ such that $|T| \leq \mu \leq \lambda < \text{Ded}^*(\mu)$ (e.g. $\mu = \aleph_0, \lambda = 2^{\aleph_0}$), T has a model of type (λ, μ) , i.e. $|Q^M| = \mu, |P^M| = \lambda$. We use the existence of free subsets.

Question: Is our result the best possible? That is, does there exist a sentence which for every n has a model M , $\aleph_0 > |P^M| > |Q^M|^n, |Q^M| \geq n$, but does not have a $(2^\mu, \mu)$ -model for some μ , and even: has a (λ, μ) -model iff $\mu \leq \lambda < \text{Ded}^*(\mu)$ (assuming for some $\mu, \text{Ded}^*(\mu) \leq 2^\mu$).

Conjecture: $\{(m_i, n_i, k_i) : i < \omega\} \rightarrow (\lambda, \mu, k)$ when $m_i \geq n_i^i, n_i \geq k_i^i, k_i \geq i, k \leq \mu \leq \lambda < \text{Ded}^*k$.

Conjecture: $\{(2^{n_i}, n_i) : i < \omega\} \rightarrow (2^\mu, \mu)[n_i \geq i]$. (Some information is suggested).

(B38) Graphs with prescribed asymmetry and minimal number of edges, Erdős Symp. (infinite and finite sets), Vol.III(1975), 1241-1256.

We shall deal with non-directed graphs, without loops and double edges, and having a finite number of vertices.

A graph is symmetric if it has a non-trivial automorphism = a permutation of its vertices, such that a pair of vertices is connected iff their images are connected. The asymmetry of a graph is the minimal number of changes (i.e. adding and deleting of edges) which is necessary to make the graph symmetric. Erdős and Renyi [ER63] defined and investigated this notion, and defined, $F(n, k) [C(n, k)]$ for $k \geq 1, n > 1$ as the minimal number of edges in a [connected] graph with n vertices which is asymmetric is k ; if there is no such graph the value of the function will be ∞ (if n is too small, this happens).

Theorem 2.12. *For n sufficiently large*

$$F(n, 2) = n + 1, C(n, 2) = n + 2.$$

Theorem 2.13. *For odd $k > 2$ and n sufficiently larger than k*

$$F(n, k) = C(n, k) = [(k + 3)n/4] - 0.5[2n/(k + 3) + 1/2].$$

Theorem 2.14. *For even $k > 2$ and n sufficiently larger than k*

$$F(n, k) = C(n, k) = [(k + 2)n/4 + 1/2].$$

The proof is given only for $k \geq 41$, and uses a detailed analysis of examples.

(B39) Differentially closed fields, IJM 16(1973), 314-328.

We prove that even the prime, differentially closed field of characteristic zero, is not minimal; that over every differential radical field of characteristic p , there is a closed prime one, and that the theory of closed differential radical fields of characteristic $p(> 0)$ is stable (hence the prime model above is unique).

(B40) Notes on Partition Calculus, Erdős Symp. Vol.III, 1257-1276.

We deal here with some separate problems which appeared in the problem list of Erdős and Hajnal [EH71].

In §1 we solve problem 3 of [EH71] affirmatively. Thus, if $\aleph_\omega < 2^{\aleph_{n(0)}} < \dots < 2^{\aleph_{n(k)}} < \dots$, then $\sum_{n < \omega} 2^{\aleph_n} \rightarrow (\aleph_\omega, \aleph_\omega)^2$. We prove a canonization lemma for it.

In §2 we deal with problem 32 of [EH71] which asks whether there is a graph G with \aleph_1 vertices and with no $[[\aleph_0, \aleph_1]]$ subgraph for which $\aleph_1 \rightarrow (G, G)^2$. We provide a wide class of such graphs, assuming CH. If $\mathbf{V} = \mathbf{L}$ is assumed we show that $\aleph_1 \rightarrow (G, G)^2$ iff G has coloring number $\leq \aleph_0$.

In §3 we deal with problems 48, 50 of [EH71] (asked by Erdős and Hajnal): partition relations concerning coloring numbers.

In §4 we deal with problem 42 of [EH71]. We get compactness and incompactness results on the existence of transversals and on property B .

We also find sufficient and necessary conditions for the existence of transversals for a family of sets.

(B41) Some theorems on transversals Erdős Symp., Vol.III, 115-1126 (with E.C. Milner).

We prove e.g. $\text{NPT}(\lambda, \aleph_0) \rightarrow \text{NPT}(\lambda^+, \aleph_0)$ where $P(\lambda)$: there is a family F of λ countable sets such that F has no transversals but every $F' \subseteq F$, $|F'| < |F|$ has a transversal.

(B42) The monadic theory of order, Annals of Math 102(1975), 379-419.

First, a decidability method is developed and used to decide in a uniform and simplified way all known decidable fragments of the monadic theory of order: the monadic theory of ω , the monadic theory of ω_1 , the monadic theory of countable orders and so on. The method is used to reduce the monadic theory of ordinals $\alpha < \lambda^+$ to that of λ and to reduce the monadic theory of ω_2 to the first-order theory of the Boolean algebra of subsets of ω_2 module closed unbounded sets with an operation $F(X) = \{\alpha : \text{cf}(\alpha) = \omega_1 \text{ and } X \cap \alpha \text{ is stationary in } \alpha\}$ and similarly for any cardinal. The new proof of decidability of the monadic theory of countable orders helps to prove that countable orders cannot be characterized in the monadic logic. To develop the method a useful Feferman-Vaught type theorem for the monadic theory of order is proven and also two Ramsey type theorems are proven.

E.g.

Theorem 2.15. 1) *If c is an additive two-place function from a dense linear order (i.e. if $x < y < z$, $f(x, z)$ is determined by $f(x, y), f(y, z)$) and f has a finite range, then there is a c -homogeneous set which is dense in some interval (this is made to help prove decidability).*

2) *If λ is regular, c and additive two-place function from λ with range of cardinality $< \lambda$, then there is a c -homogeneous set which is unbounded in λ .*

Second, the true first-order arithmetic is interpreted under the Continuum Hypothesis in the monadic theory of the real line and in the monadic theory of order.

There are more results and many conjectures. The introduction surveys preceding results.

(B43) Generalized quantifiers and compact logic, Trans. AMS 204(1975), 342-364.

We show the existence of a logic stronger than first-order logic even for countable models, but still satisfying the general compactness theorem, assuming e.g. the existence of a weakly compact cardinal. We also discuss several kinds of generalized quantifiers.

We prove the compactness of $\mathbb{L}(Q_{C_1}^{\text{cf}}, \dots, Q_{C_n}^{\text{cf}})$ when C_n is a convex class of cardinals, Q_C^{cf} say the cofinality of an order belongs to C , and axiomatize $L(Q_{C_1}^{\text{cf}})$. We also prove that \aleph_0 -compactness of $(Q_{\aleph_1}^{\text{stat}}x)$ (being stationary for order of cofinality \aleph_1) and transfer theorems.

We define generalized second-order quantifiers: allowing in forming formulas the use of second-order variables, and second-order quantifiers $(QP)\psi(P)$, such that its satisfaction depends only on the isomorphism type of $\{P : \models \psi[P]\}$. In particular, stationary logic (that is, the quantifier $(\text{aa}X)$) is introduced (developed subsequently by Barwise, Kaufman and Makkai, [BKM78]) and we note we could have used diamonds instead of a weakly compact cardinal in the main theorem.

(B44) Infinite abelian groups, Whitehead problem and some constructions, IJM, 18(1974), 243-256.

We deal with three problems: the existence of indecomposable and even rigid (abelian) groups, the number of reduced separable p -groups and Whitehead problem.

We prove that for any cardinal $\lambda > \aleph_0$ there are 2^λ (abelian) groups G_i ($i < 2^\lambda$) such that any homomorphism $h : G_i \rightarrow G_j$ is zero or $i = j$, $h(x) = nx$. We do it by coding deepness of well-founded trees and stationary sets.

For regular $\lambda > \aleph_0$ we build a family of 2^λ reduced separable p -groups, each of power λ , no one embeddable into another, by imitating the proof for non-superstable theories (see [1], Ch.VIII and [Shear] for singular λ). The proof indicates that reduced separable p -groups cannot be characterized by reasonable invariants.

An abelian group G is W (= Whitehead) if $\text{Ext}(G, \mathbb{Z}) = 0$, or equivalently, when: if $\mathbb{Z} \subseteq H$ (\mathbb{Z} the integers), $h : H \rightarrow G$ a homomorphism onto G with kernel \mathbb{Z} , then for some homomorphism $g : G \rightarrow H$ we have $hg = 1_G$. It is proved that “every Whitehead group of power \aleph_1 is free”, is independent of ZFC: it follows from $\mathbf{V} = \mathbf{L}$ (or even \diamond_S for every stationary $S \subseteq \omega_1$) and is contradicted by $\text{MA} + 2^{\aleph_0} > \aleph_1$. We partition the \aleph_1 -free groups of power \aleph_1 to three classes: I (“unstable in \aleph_0 ”), II (intermediate) and III (the free ones). Class II is similar to “strongly \aleph_1 -free but not free” but somewhat wider.

For latter use let $\Gamma(G) = \{i : G/G_i \text{ is not } |G|\text{-free}\}$, where

$$G = \bigcup_{i < |G|} G_i,$$

G_i increasing continuums, $|G_i| < G$, and G is $|G|$ -free ($\Gamma(G)$ is determined modulo the filter of closed unbounded subsets of $|G|$).

(B45) Existence of rigid-like families of abelian p -groups, model theory and algebra, lecture notes in Math. 498, Springer-Verlag, 1975, 385-402.

We prove that for arbitrarily large λ there is a family of 2^λ (abelian) separable p -groups, each of power λ with only the necessary homomorphism (called simple) between them. A homomorphism $h : G \rightarrow H$ is zero-like if there are no $m < \omega$ and $a_n \in G$ of order p^{m+n} , $h(a)$ having order $\geq p^n$; we call h simple if $h = h_1 + h_2$, h_1 zero-like, h_2 a multiplication by a p -adic integer (if $H \neq G$, $h_2 = 0$).

We prove this for λ strong limit of uncountable cofinality; the method is that we can for each strong limit $\mu < \lambda$, $\text{cf}(\mu) = \aleph_0$, diagonalize over all approximation of a homomorphism; we can do it as $2^\mu = \mu^{\aleph_0}$.

For $\lambda^{\aleph_0} = 2^\lambda > 2^{\aleph_0}$ we prove a weaker theorem

(*) i.e. $\forall x \left[\bigvee_{n < \omega} p^n x = 0 \right]$, and the group has no divisible subgroup $\neq 0$.

(B46) Colouring without triangles and partition relation, IJM 20(1975), 1-12.

We say the colouring h of λ proves $\lambda \rightarrow [\mu]_\chi^2$ if h is a function from the edges of the complete graph on λ into χ such that whenever $S \subseteq \lambda$, $|S| = \mu$, then the range of h restricted to the edges in S is the whole of χ .

Erdős and Hajnal (Problem 68) asked if it is true (assuming CH) that for every colouring which witness $\aleph_1 \rightarrow [\aleph_1]_3^2$ there is a triangle whose edges have three colours. We answer this negatively and prove the stronger result (Theorem 1.1) that, if $2^{\aleph_\alpha} = \aleph_{\alpha+1}$, then there is a colouring which witness $\aleph_{\alpha+1} \rightarrow [\aleph_{\alpha+1}]_{\aleph_\alpha}^2$ and for which there is no triangle with three colours. We also show that, if $\mathbf{V} = \mathbf{L}$, and \aleph_α is regular then there is even such a colouring which witness $\aleph_{\alpha+1} \rightarrow [\aleph_{\alpha+1}]_{\aleph_{\alpha+1}}^2$.

In §2, we deal with the corresponding finite problem and confirm a conjecture of Erdős. We prove, for example, that if the edges of the complete graph on n points are colored with three colours so that no triangle has three colours, then there is a set $A \subseteq n$, such that $|A| \geq n^c$ (c some constant) and which contains only two colours. This should be compared with the result $n \not\rightarrow [c_1 \log(n), n]_3^2$ proved by Erdős by a probabilistic method.

In §3, we show that examples for 1.1 should contain quite homogeneous sets; and then introduce new partition relations, (of interest in the finite and infinite case), give some results and suggest some problems.

It was proved by Erdős and Hajnal that any example of $\aleph_1 \rightarrow [[\aleph_1, \aleph_0]]_3^2$ is \aleph_0 -universal. Erdős and Hajnal asked whether every example of $\aleph_2 \rightarrow [[\aleph_2, \aleph_1]]_3^2$ is \aleph_1 -universal. We prove the consistency of the negative answer and the method is applicable in many other cases.

(B47) Δ -Logics and generalized quantifiers, AML 10(1976), 155-192 (with J.A. Makowsky and Y. Stavi).

We study the δ -closure of an abstract logic, which is its smallest extension satisfying the Δ -interpolation theorem or, equivalently, its largest extension having the same PC -classes.

We study properties which are preserved by the Δ -closure such as compactness, and various Lowenheim and Hauf numbers.

We apply these to exhibit new logics with various properties, and we study Δ -closed fragments of $\mathbb{L}_{\omega_1, \omega}$ in great detail.

(B48) Categoricity in \aleph_1 of sentences in $\mathbb{L}_{\omega_1, \omega}(Q)$, IJM 20(1975), 127-148.

We investigate the possible numbers of models of a sentence $\psi \in \mathbb{L}_{\omega_1, \omega}(Q)$ of cardinality \aleph_1 .

Assume $2^{\aleph_0} < 2^{\aleph_1}$ and $1 \leq I(\aleph_1, \psi) < 2^{\aleph_1}$, it is proved that there exists a class K of models which are the atomic models of some first-order countable theory T_ψ such that $\forall \lambda [I(\lambda, K) \leq I(\lambda, \psi)]$ and $I(\aleph_1, K) \geq 1$.

Under the same assumptions it is proved that K is \aleph_0 -stable.

The Main Theorem is: assuming \diamond_{\aleph_1} if $1 \leq I(\aleph_1, \psi) < 2^{\aleph_1}$, then there exists a model of ψ of cardinality \aleph_2 . This is proved by showing:

- if the \aleph_0 -amalgamation property fails for K , then using \diamond_{\aleph_1} we can construct 2^{\aleph_1} non-isomorphic models in K of cardinality \aleph_1 .
- The \aleph_0 -amalgamation property is equivalent to a suitable rank being always $< \infty$ (and some other properties).
- Assuming the rank is $< \infty$, we prove that the order property (= there are \aleph_1 sequences in a model of ψ , ordered by some formula) is equivalent to two variants of the symmetry property (one of them: $R(\text{tp}(\bar{a}, M \cup \bar{b})) = R(\text{tp}(\bar{a}, M))$ iff $R(\text{tp}(\bar{b}, M, \bar{a})) = R(\text{tp}(\bar{b}, M))$ essentially). The second asymmetry property, enables us to prove $I(\aleph_1, K) = 2^{\aleph_1}$ by coding stationary subsets of ω_1 . But the holding of the symmetry property enable us to properly extend every model of power \aleph_1 . Thus by iteration, we get a model of power \aleph_2 .

Remark 2.16. 1) We continue this work on $\mathbb{L}_{\omega_1, \omega}$ sentences in [Sh:87a], [Sh:87b] where we improve some of the theorems from this paper and some of the claims here are preparation for them and also in [She87a].

2) The main difficulty is in 3).

(B49) A two-cardinal theorem and a combinatorial theorem, Proc. AMS 62(1977), 134-136.

We prove a new two-cardinal theorem, e.g. $(\aleph_\omega, \aleph_0) \rightarrow (2^{\aleph_0}, \aleph_0)$.

For this we prove a new partition theorem, with “tree indiscernibility”.

The problem seemed to have been the right induction. Also compactness is gotten.

Conjecture 2.17. A) $(\aleph_{\alpha+\omega+\omega}, \aleph_{\alpha+\omega}, \aleph_\alpha) \rightarrow (\lambda, \mu, \chi)$ whenever $\chi < \mu < \lambda < \text{Ded}^* \chi$.

B) If a countable theory T has a λ -like model, λ a limit cardinal, and $|T| \leq \mu < \lambda_1 \text{Ded}^*(\mu)$, λ_1 a singular cardinal, then T has a λ_1 -like model. If λ is ω -Mahlo weakly inaccessible cardinal, we can remove the cardinal, we can remove the singularity of λ_1 .

C) If $\psi \in L_{\omega_1, \omega}$ has a model of cardinality \aleph_{ω_1} , then ψ has a model of cardinality 2^{\aleph_0} (but see [Sh:522]).

(B50) Decomposing uncountable squares to countably many chains, J. Comb. Th. A. 21(1976), 110-114.

We construct an ordered set I of cardinality \aleph_1 , such that its square is the union of \aleph_0 chains (in the natural partial order). This gives a complete order, isomorphic to any open interval of itself but not to its universe. This is proved by constructing a “very special” Aronszajn tree.

Conjecture 2.18. It is consistent that:

- any Specker order contains a suborder as above
- if I, J are as above, I, J or I, J^* have uncountable isomorphic suborders.

The consistency of 2) is proved in [AS85].

As for (1) it is equivalent to:

- (*) if T is an Aronszajn tree, $c : T \rightarrow 2$ then for some uncountable $X \subseteq T$ and $\ell \neq \{0, 1\}$ for every $x \neq y$ in X , the maximal $z \leq_T x, y$ satisfies $c(z) = \ell$.

(B51) Why there are many non-isomorphic models for unsuperstable theories, Proc. of the International Congress of Math. Vancouver (1974), 553-557.

We review the proof (see [She78]) that for unsuperstable T , $T \subseteq T_1$, $I(\lambda, T_1, T) = 2^\lambda$ when e.g. $\lambda > |T_1|$ or $\lambda = |T_1| + \aleph_1$, T unsuperstable. To exemplify the generality of the method, we use it to prove that in any uncountable cardinality λ there is a rigid Boolean algebra (for λ regular satisfying the countable chain condition), and a rigid order (with a rigid completion).

(B52) A compactness theorem in singular cardinals, free algebras, Whitehead problem and transversals, IJM 21(1975), 319-349.

We prove in an axiomatic way a compactness theorem for singular cardinals, concluding that if $\mathbf{V} = \mathbf{L}$ every Whitehead group is free. For this we use index sets of submodels $\langle M_u : u \in \mathcal{P}(n) \rangle$ and prove various assertion by induction on the cardinality for all n at once.

There is an application for transversals, i.e. if $\kappa < \lambda$, λ singular $\{A_i : i < \lambda\}$ a family of λ sets each of power $\leq \kappa$, and every subfamily of smaller cardinality has a transversal then the family has a transversal.

Similar conclusions hold for freeness of groups, freeness abelian groups and colouring numbers.

The general theorem:

- (*) “if V is a variety with $< \lambda$ functions, $A \in V$ an algebra of power λ , and every (or “many”) subalgebras of A of cardinality $< \lambda$ are free, then A is free”

and does not follow by the axiomatic treatment. However, we develop here some filters on $S_\kappa^\delta(A) = \{\langle A_i : i < \delta \rangle : A_i \text{ a subset of } A \text{ of power } \leq \kappa, A_i \text{ increasing continuous}\}$, using those we prove (*).

Further work on the situation in regular cardinals is in [She85a], and the author eliminates AxI^* , V from the axiomatic system (see [She19], Hodges [Hod81]).

Theorem 2.19. *For a λ singular, $\chi^* < \lambda$ assume \mathbf{F} is a set of pairs (A, B) , (\mathbf{F} for free) $A, B \subseteq \lambda$ satisfying the axioms (II-IV, VI, VII) below. Let $A^*, B^* \subseteq \lambda$, then $B^*/A^* \in \mathbf{F}$ if B^*/A^* is λ -free in a weak sense which means (see Definition below):*

(*)₀ for the $D_{\chi^*}(B^*)$ -majority of $B \in [B^*]^{<\lambda}$ we have $B/A^* \in \mathbf{F}$ or just

(*)₁ for a club of $\mu < \lambda$ we have

$$\{\mu < \lambda : \{B \in [B^*]^\mu : B/A^* \in \mathbf{F}\} \in E_\mu^{\mu^+}(B^*)\}$$

contains a club of λ all $B \in [B^*]^\mu$ (see below), $B/A^* \in \mathbf{F}$

or at least

(*)₂ for some set C of cardinals $< \lambda$, unbounded in λ and closed (meaningful only if $\text{cf}(\lambda) > \aleph_0$) for a $E_\mu^{\mu^+}(B^*)$ -positive set of $B \in [B^*]^\mu$, $B/A^* \in \mathbf{F}$.

The axioms are:

Ax II: $B/A \in \mathbf{F} \Leftrightarrow A \cup B/A \in \mathbf{F}$.

Ax III: if $A \subseteq B \subseteq C$, $B/A \in \mathbf{F}$, $C/B \in \mathbf{F}$ then $C/A \in \mathbf{F}$.

Ax IV: if A_i is increasingly continuous for $i \leq \theta = \text{cf}(\theta)$, $A_{i+1}/A_i \in \mathbf{F}$ then $A_\theta/A_0 \in \mathbf{F}$.

Ax VI: if $A/B \in \mathbf{F}$ then for the D_{χ^*} -majority of $A' \subseteq A$, $A'/B \in \mathbf{F}$.

Ax VII: if $A/B \in \mathbf{F}$ then for the D_{χ^*} -majority of $A' \subseteq A$, $A/B \cup A' \in \mathbf{F}$.

Definition 2.20. 1) Let for D a function giving for any set B^* a filter $D(B^*)$ on $\mathcal{P}(B^*)$ (or on $[B^*]^\mu$), then to say for the D -majority of $B \subseteq B^*$ (or $B \in Y^*$) we have $\varphi(B)$ means $\{B \subseteq B^* : \varphi(B)\} \in D(B^*)$ or $\{B : B \in Y^*, \neg\varphi(B)\} = \emptyset \text{ mod } D$.

2) Let $D_\mu(B^*)$ be $\{Y : Y \subseteq \mathcal{P}(B^*)\}$ such that for some algebra M with universe B^* and \leq_μ functions, $Y \supseteq \{B \subseteq B^* : B \neq \emptyset \text{ closed under the functions of } M\}$.

$$E_\mu^{\mu^+}(B^*) = \{Y \subseteq [B^*]^\mu : \text{for some } \chi, x \text{ such that } \{B^*, x\} \in \mathcal{H}(\chi) \\ \text{and if } \bar{M} = \langle M_i : i < \mu^+ \rangle \text{ is an increasingly} \\ \text{continuous sequence of elementary submodels of} \\ (\mathcal{H}(\chi), \in) \text{ such that } x \in M_0 \text{ and} \\ \bar{M} \upharpoonright (i+1) \in M_{i+1} \text{ then for some club } C \text{ of } \mu^+, \\ i \in C \Rightarrow M_i \cap B^* \in Y\}$$

(see [She75a], [BD78]). Now the proof in [She75a] of singular compactness used two axioms which the author eliminated later; see [She19], Hodges [Hod81].

There are some cases of incompleteness ([FS89], [She90]).

(B53) Models with few non-isomorphic expansions, IJM 28(1971), 331-338 (with A. Litman).

If M is a countable model (with countable language) the number of expansions of M by one one-place predicate is, up to isomorphism, \aleph_0 or 2^{\aleph_0} . Moreover, we characterize when the number is \aleph_0 : when M is definable in $N \times \omega$ for some finite model N .

(B54) The lazy model theorist's guide to stability, six days of model theory, Proc. of a conference in Louvain-la-Neuve, March 1975, edited by Paul Henramd, Pub. by Paul Castella, Switzerland, 1661 Albeuve (1978), 9-76.

This paper is an introduction to classification theory and an outline of the main results updated to 1975. The main goal is to convince the reader that stability is useful for algebraic theories, and though it was developed for elementary classes, it can be adapted to others, mainly:

- the existentially closed models of a universal theory with the joint embedding property and the amalgamation property
- the existentially closed models of a universal theory with the JEP.

We assume knowledge of only basic model theory (as in the book of Chang and Keisler); stability is defined and some of the harder combinatorial theorems are just quoted and appear with an outline of proof.

Morley's categoricity theorem is proved, uniqueness of prime model for stable countable theory, characterization of stability using rank, the basic properties of non-forking (sometimes with simpler proofs than in the book [She78]). We prove that for a ring R , if K is the class of R -modules then either every R -module M is the direct sum of modules of $\leq |R| + \aleph_0$, or in every $\lambda > \aleph_0$ there are 2^λ non-isomorphic R -modules; the first case occurs iff the theory is superstable. Application to group theory (chain condition) and field are discussed.

It is noted (with Macintyre) that by [She73], some theories of separably closed field of characteristic p are stable (ie. the reducts of the theory of differentially closed fields of characteristic p), and by a similar proof, all such theories are stable.

Since this paper is very condensed and is itself a summary of results, we could not mention in this summary all the theorems. We gave just a representative sample list.

(B55) Universal locally finite groups, J. of Alg. 43(1976), 168-175 (with Macintyre).

Call G a u.f.l. (universal locally finite) group if:

- (a) every finitely generated subgroup is finite
- (b) every finite group can be embedded into G
- (c) every isomorphism by one finitely generated subgroup to another is induced by an inner automorphism.

We prove that for $\lambda > \aleph_0$, there are 2^λ u.f.l. non-isomorphic groups, no one embeddable into another (using [She72a]).

(B56) Refuting Ehrenfeucht Conjecture on Rigid models, IJM 25(1976), 273-288.

We prove that the class of cardinalities in which a first-order sentence has a rigid model can be very complicated. In fact, we essentially characterized those classes: if GCH holds any Σ_2^1 class (of pure second order logic) occurs.

(B57) The complete finitely axiomatized theories of order are dense, IJM 23(1976), 200-208 (with A. Amit).

We prove that every sentence of the theory of linear order which has a model, has a model with finitely axiomatized theory.

(B58) Decidability of a portion of the predicate calculus, IJM 28(1977).

During this century many fragments of first-order logic were proved to be decidable or undecidable. The most natural fragments are classes of prenex sentences defined by restrictions on prefix and numbers of monadic, dyadic, etc., predicate and function symbols. Such classes are called standard below. The decision problem for standard classes without equality is settled. See the complete picture in the article, "The decision problem for standard classes" by Y. Gurevich in JSL 41(1976). According to that article the only open case of the decision problem for standard classes with equality and at least one function symbol is the class (let's call it K) of $\exists \dots \exists \forall \exists \dots \exists$ sentences with arbitrary predicate symbols and exactly

one function symbol, which is unary. Here we settle the decision problem for standard classes with equality and at least one function symbol by proving decidability. Also the decision problem for satisfiability in finite models is decidable. Let us say a few words about the method. Given $\varphi \in K$ we study closure properties of the class of models of φ mainly under what operations it is preserved.

See the book by Berger Gurevich.

(B59) Singular Cohomology in \mathbf{L} , IJM 26(1977), 313-319 (with H.L. Hiller).

Theorem 2.21. ($\mathbf{V} = \mathbf{L}$). *If G is a κ -free non-free abelian group of power κ then $\text{Ext}(G, \mathbb{Z})$ has power κ^+ hence for no torsion free G , $\text{Ext}(G, \mathbb{Z})$ has power \aleph_0 .*

A consequence is that there does not exist a topological space X and integer n such that $H^n(X, \mathbb{Z}) = \mathbb{Q}$ (H^n - ordinary, singular cohomology).

In [She81d] we construct a universe of set theory where there exists a topological space X and an integer such that $H^n(X, \mathbb{Z}) = \mathbb{Q}$.

(B60) Possible orderings of an indiscernible sequence, Bull. London, Math. Soc. 9(1977), 212-215 (with W. Hodges and A. Lachlan).

Let n be a positive integer and X a set of cardinality $\geq \max(2n - 3), n + 2$. Suppose $(X, <)$ is indiscernible with respect to an n -relation R , i.e. $<$ linearly orders X , and whether Rx_1, \dots, x_n holds depends only on the relative order of x_1, \dots, x_n in X .

Then one of the following holds:

- for all linear orderings $<_1$ of X , $(X, <_1)$ is indiscernible w.r.t. R ;
- if $<_1$ linearly orders R than $(X, <_1)$ is indiscernible w.r.t. R iff $<_1$ comes from $<$ by moving an initial segment to the end and maybe reversing the order;
- if $<_1$ linearly orderings X then $(X, <_1)$ is indiscernible w.r.t. R iff $<_1$ is either $<$ or $>$, except that at each end of $(X, <)$ some elements may be permuted (but at most n in all). The theorem is equivalent to an earlier result of Frasnay not mentioning indiscernibles, but our bounds improve his.

(B61) Interpreting set theory in the endomorphism semi-group of a free algebra or in a category, Ann. Sci. Univ. Clermont Sec. Math. fase 13(1976), 1-29.

In the category of any variety we interpret set theory, essentially. Moreover, in the semi-group of endomorphism of F_λ , the free algebra with λ generators, we interpret $(\mathcal{H}(\lambda^+), \in)$, essentially. The essentially indicate that we assume some terms call beautiful does not exist, but even if they exist we get the best possible results.

The method is using stationary sets and deepness of trees.

A related problem is about the group of automorphism of F , $\text{Aut}(F_\lambda)$, and we believe

Conjecture 2.22. For varieties satisfying $(*)$ below, a similar answer holds, whereas if $(*)$ fails the group is like the group of permutations of λ (see [24,25]).

- $(*)$ Let F be freely generated by $\{x_i : i < \omega + \omega\}$. Then there is an automorphism h of F , $h(x_i) = x_i$ for $i < \omega$ and $h(x_\omega) \notin \langle x_i : \omega \leq i < \omega + \omega \rangle_F$.

(B62) The theorems of Beth and Craig in Abstract Model theory I, The abstract setting, Trans AMS 256(1979) (with J.A. Makowsky).

In the context of abstract model theory various definability properties, their interrelations and their relation to compactness are investigated.

Beth's theorem together with a Feferman-Vaught theorem for tree-like sums implies a weak form of Robinson's consistency lemma, hence e.g. assuming compactness, Craig theorem follows. So in many cases Craig and Beth's theorems are equivalent. Also, essentially, Robinson consistency lemma together with the Feferman-Vaught theorem for pairs implies full compactness.

(B63) The Hanf number of the first-order theory of Banach spaces, Trans AMS 244(1978) (with J. Stern).

In this paper, we discuss the possibility of developing a nice, i.e. first order theory for Banach spaces: the restrictions on the set of sentences for recent compactness arguments applied to Banach spaces as well as for other model-theoretic results are both natural and necessary; without them we rove here that we essentially get a second order logic with quantification over countable sets. Especially, the Hanf number for sets of sentences of the first-order theory of Banach spaces is exactly the Hanf number for the second order logic of binary relations (with the second order quantifiers ranging over countable sets).

(B64) Whitehead group may not be free even assuming Ch.I, IJM 29(1978), 239-247.

In spite of its name this is a paper in set theory. Its purpose is to prove that many consequences of MA are consistent with GCH. More specifically, it is possible that GCH holds, $S \subseteq \omega_1$ is a stationary, co-stationary set of limit ordinals, $\omega_1 \setminus S$ is not small in fact $\diamond_{\omega_1 \setminus S}^*$ holds, but e.g. every ladder system $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$ can be \aleph_0 -uniformized where " $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$ is a ladder system" means each η_δ is an increasing ω -sequence converging to δ and κ -uniformized means that for every $\bar{c} = \langle c_\delta : \delta \in S \rangle$, $c_\delta \in {}^\omega \kappa$ there is $h : \omega_i \rightarrow \kappa$ such that for every $\delta \in S$, for large enough n , $c(\eta_\delta(n)) = c_\delta(n)$. By [DS78], $\omega_1 \setminus S$ has to be stationary. So we prove that $\diamond_{\omega_1} \neq \diamond_S$ ($S \subseteq \omega_1$ stationary).

We use straightforwardly countable support iteration of length ω_2 ; the main problem is why S remains stationary. For this we take $N \prec (\mathcal{H}(\aleph_3), \in)$ and build a sequence of $\bar{q}^n = \langle q_\eta : \eta \in T_n \rangle$, T_n a finite tree, and in the end get by inverse limit, a tree of candidates for being a condition generic over N . Using $\omega_1 \setminus S$ we then prove at least one is contained in a condition.

(B65) A weak form of the diamond which follows from $2^{\aleph_0} < 2^{\aleph_1}$ IJM 29(1978), 239-247 (with K. Devlin).

We prove that weak CH (i.e. $2^{\aleph_0} < 2^{\aleph_1}$) implies a weak variant of the diamond, explicitly:

(Φ) for each function $F : {}^{\omega_1} 2 \rightarrow 2$ there is $g \in {}^{\omega_1} 2$, such that for any $f \in {}^{\omega_1} 2$

$$\{\alpha < \omega_1 : F(f \upharpoonright \alpha) = g(\alpha)\} \text{ is stationary.}$$

Note that the "guessing power" is being able to say "yes or no" (i.e. 0 or 1), i.e. the range of g is 2 (not even 3). Also the guesses are on ω_1 , not any stationary

subset S , so we can phrase $\Phi(S)$, and let us call S small if it fails. The small sets form a normal idea. There are generalization for higher cardinals.

A corollary is that if $\Gamma(G) \subseteq \omega_1$ (see [She74]) is not small then G is not Whitehead. Also if $S \subseteq \omega_1$ is not small, $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$ a ladder system then it is not even 2-uniformizable (see [She77]).

(B66) End extensions and numbers of countable models, JSL 43(1978), 550-562.

We prove that:

- every model of $T = \text{Th}(\omega, <, \dots)$ for T countable has a (proper) end extension (so addition and multiplication are not used, hence definition by induction is not available);
- every countable theory with an infinite order and Skolem functions has 2^{\aleph_0} non-isomorphic countable models; (instead of Skolem functions, existence of arbitrarily large finite intervals can be assumed);
- if every model of T has an end extension, then every $|T|$ -universal model of T has an end extension definable with parameters.

(B67) On the number of minimal models, JSL 43(1978), 475-480.

We prove that for every κ , $1 \leq \kappa \leq \aleph_0$, there is a (countable) complete theory T , with no prime model, and exactly κ minimal models (up to isomorphism). The examples are specifically built for this purpose.

It is clear that if the number is $> \aleph_1$ it is 2^{\aleph_0} .

(B68) Jónsson algebras in successor cardinals, IJM 30(1978), 57-64.

We shall show here that in many successor cardinals λ , there is a Jónsson algebra (in other words $\text{Jn}(\lambda)$ or λ is not a Jónsson cardinal). In connection with this we show that, e.g., for every ultrafilter D over ω , in $(\omega_\omega, <)^{\omega}/D$ there is no increasing sequence of length $\aleph_{(2^{\aleph_0})+}$ and a lemma on the existence of l.u.b. We prove e.g., $\text{Jn}(\aleph_{\omega+1})$ when $2^{\aleph_0} \leq \aleph_{\omega+1}$ and $\text{Jn}(2^{\aleph_0})$ when $2^{\aleph_0} = \aleph_{\alpha+1}$, $\alpha < \omega_1$ and similar results for higher cardinals.

(B69) On a problem of Kurosh, Jónsson groups and applications work problem, II, edited by Adjan, Boone and Higman North Holland Publ. Col., 1979

We prove some results in group theory in model theoretic spirit.

- 1) Construct Jónsson groups of cardinality \aleph_1 , and others as well (G is Jónsson group of cardinality λ ($> \aleph_0$) if it has no proper subgroup of cardinality λ); this solves a problem of Kurosh.
- 2) Our group is simple with no maximal subgroup; so it follows that taking Frathini subgroups does not commute with direct products.
- 3) Assuming $2^{\aleph_0} = \aleph_1$, our group has no non-trivial Hausdorff topology; this answers a question of Markov.

We use small cancellation theory, but quote exactly what we use; we assume only knowledge of naive set theory and little group theory (except the theorem we quote).

(B70) Modest theory of short chains, II, JSL 44(1979), 491-502 (with Yuri Gurevich).

A chain (i.e. a linearly ordered set) is *short* if it embeds neither ω_1 nor ω_1^* . Shortness is easily expressible in the monadic language of order. Given a chain

C and a positive integer p we define (in the monadic language of order) p -modest subsets in C . A subset of C is *modest* in C if it is p -modest in C for every p . A chain is *modest* if it is modest in itself, it is *absolutely modest* if it is modest in its Dedekind completion.

Theorem 2.23. *The monadic theory of modest short chains coincides with that of countable chains.*

Decidability of the latter theory is due to Rabin; it is reproved here.

Theorem 2.24. *The monadic theory of any non-modest short chain is undecidable under the Continuum Hypothesis. (This use of the Continuum Hypothesis is eliminated in a later paper “Monadic theory of order and topology in ZFC” by the same authors.)*

The *modest theory* of a chain C is the theory of C in the monadic language of orders when the set variables range over absolutely modest subsets of C .

Theorem 2.25. *The modest theory of the real line R is decidable; it coincides with the theory of R with quantification over countable sets, with the theory of R with quantification over sets of cardinality $< 2^{\aleph_0}$, and with the modest theory of any complete short chain without jumps and ends having an everywhere dense absolutely modest subset.*

It is proved also that Theorem 3 is in some sense best possible.

(B71) A note on cardinal exponentiation, JSL 45(1980), 56-66.

For an \aleph_1 -complete filter D on I , for $f : I \rightarrow \text{Ord}$, let: $\|f\|_D$ (the rank of f) be $\sup\{\|g\|_D : g <_D f\}$ and

$$T_D(f) = \sup\{|G| : g \subseteq I(\text{Ord}), (\forall g \in G)g <_D f, (\forall g_1 \neq g_2 \in G)[g_1 \neq_D g_2]\}.$$

Our results are:

- improvements of the bounds on $\|f\|_D$, e.g. if $|\alpha| = \beth_\omega$ and $|I| < \beth_\omega$ then $\|\alpha\|_D < \beth_\omega^+$, hence
- corresponding bound on $2^{\aleph_\alpha}, \aleph_\alpha$ strong limit, $\text{cf}(\alpha) \geq \aleph_1$
- bounded on 2^{\aleph_α} in new cases: if $\aleph_\lambda(\aleph_0)$ is the first $\aleph_\alpha = \alpha$ of cofinality λ and $\aleph_{\omega_1}(\aleph_0)$ is strong limit then $2^{\aleph_{\omega_1}(\aleph_0)} < \aleph_{\omega_2}(\aleph_0)$ provided that the Chang conjecture holds.

The method is that in the proof rather than fixing a filter (say the club filter) we look at all possible filters, say normal filters on ω_1 (doing induction together) and that we get bounds on the rank by cases of $T_D(-)$ (a kind of converse to previous results).

(B72) Models with second order properties I. Boolean Algebras with no definable automorphisms, AML 14(1978), 57-72.

This is (the first) part of a series of papers, in which we try to build models with various second order properties.

Examples of such properties are:

- (a) let T be a (first order) theory, and we want a model M , such that $M \models T$ and every Boolean algebra definable in M , all its automorphisms are definable in M
- (b) we extend first order logic by allowing quantification over automorphism of Boolean algebras, and we want to show this logic is compact (see [She75d, Th. 2.6, pg. 353], [She72b, Th. 25]) for a definition of a second order quantification, and this amounts to assuming T satisfies various schemes, e.g., T has a model expanding $(\mathcal{H}(\kappa^+), \in)$
- (c) we have a model M of T in which a (definable) Boolean algebra is rigid, and we want to build other such models.

We want more concretely to build those models in specific cardinals, and replace “automorphisms of Boolean algebras” by automorphisms of other structures (e.g. ordered fields) branches of trees, etc.

Note that (a) is stronger than (b) which is stronger than (c); and in (b), (c) we can expand the language, so without loss of generality the theory has Skolem functions, and has enough build in set theory. But in (a) this is forbidden. Note that if we e.g. want to get a (λ^+, λ) -model in which a tree of height λ^+ has only definable branches, Chang’s original proof of his two-cardinal theorem is not appropriate, as he extends the language (so to encode finite sets). Note also that (b) is good for giving examples of compact logic. For example, allowing quantification over automorphisms of Boolean algebras, gives us a logic stronger than first order even on finite models (we can say a Boolean algebra is atomic, and has an automorphism of order two, which moves every atom. This distinguishes among the finite Boolean algebras between those with an even number of atoms. We shall prove this logic is compact (assuming GCH).

Let us return to this paper.

In [RS80] we prove (assuming \diamond_{\aleph_1}) that for a Boolean algebra B , the (two-sorted) model $(B, \text{Aut}(B))$ (with the Boolean operations, and the operations of applying an automorphism (where $\text{Aut}(B)$ is the group of automorphisms of (B)) is elementarily equivalent to $(B', \text{Aut}(B'))$ for some B' of cardinality \aleph_1 . We can add more relations and make other strengthening. Our first aim was to generalize this to uncountable cardinalities. This is done in §1, for an appropriate λ our results are stronger; because we prove a theory T has a model in λ^+ in which every automorphism of $(P(M) \cup Q(M), R^M)$ is inner (i.e. definable in the model) and we do not expand the language to have, essentially, some replacement axioms. Here the requirements on λ are severe; but this paper was written in order to exemplify the technique developed for solving the problem.

In §2 we use our technique to prove (a slightly stronger version of) the Chang two-cardinal theorem without expanding the language.

In some sense our problem is to omit types of power λ in models of power λ^+ .

(B73) Models with second order properties. II. Trees with no undefined branches, AML 14(1978), 73-87.

We prove several theorems of the form: a first order theory T has a model M (sometimes with additional conditions) such that (some) trees defined in M , have no branches except those defined in it. For this we omit types of cardinality λ for constructing models in λ^+ using “strong splitting”. We have some applications, e.g. an example for compact logic $\mathbb{L}(Q)$, where in $\mathbb{L}_{\omega_1, \omega}(Q)$ well-ordering is definable,

its compactness is proved in ZFC, and it is stronger than first order logic even for countable models; and eliminate extra set-theoretic axioms (like \diamond_{\aleph_1}) from some theorems. The paper also deals with constructing models of cardinality \aleph_1 .

Also we prove the absoluteness of the answer to “does $\psi \in \mathbb{L}_{\omega_1, \omega}(\exists^{\geq \aleph_1})$ have a model M such that every tree with sets of levels of cofinality \aleph_1 has no (full) branch which is not (first order) not definable (with parameters)?”. So if we can find such a model, assuming \diamond_{\aleph_1} , then we can really construct one² (e.g. a model of ZF or PA: the “classes” of it are branches of such a tree).

(B74) Appendix, Vaught two cardinal theorem revisited.

We gave an alternative proof to Vaught’s theorem $(\lambda^+, \lambda) \rightarrow (\aleph_1, \aleph_0)$ which reveal the corresponding partition theorems from [She71c], i.e. we characterize them.

(B75) A Banach space with few operators, IJM 30(1978), 181-191.

Assuming the axiom (of set theory) $\mathbf{V} = \mathbf{L}$ (explained below), we construct a Banach space with density character \aleph_1 such that every (linear bounded) operator T from B to B has the form $aI + T_1$, where I is the identity and T_1 has a separable range. The axiom $\mathbf{V} = \mathbf{L}$ means that all the sets in the universe are in the class \mathbf{L} of sets constructible from ordinals; in a sense this is the minimal universe. In fact, we make use of just one consequence of this axiom, \diamond_{\aleph_1} proved by Jensen, which is widely used by mathematical logicians.

(B76) Independence of strong partition relations for small cardinals and the free subset problem, J.45(1980), 505-509.

We prove the consistency of ZFC + GCH with the following (assuming $\text{con}(\text{ZFC})$ + “there are ω measurable cardinals”):

- (*) for any function f from finite subsets of \aleph_ω to ω there are pairwise disjoint $S_n \subseteq \omega_{2n}$ ($1 \leq n < \omega$) satisfying $|S_n| = \aleph_{2n}$ such that: if u, v are finite subsets of \aleph_ω and $(\forall n)[|S_n \cap u| = |S_n \cap v| \leq 1]$ then $f(u) = f(v)$.

On the free subset problem we prove the following: (*) implies that any algebra with countably many functions and power \aleph_ω has a free subset A of power \aleph_0 , i.e. $(\forall a \in A)$ (a is not in the subalgebra generated by $A \setminus \{a\}$).

We get similar independence results for $\aleph_\alpha > |\alpha|$, i.e. a free subset of power $|\alpha|$, and prove we cannot get a free set of power $|\alpha|^+$ in \aleph_α . Continued in [She81a].

(B77) Existentially-closed groups in \aleph_1 with special properties, Bull. Greek Math. Soc. 18(1977), 17-27.

We prove: for any countable e.c. group (= existentially closed group) G^* , there is an e.c. group $M \equiv_{\infty, \omega} G^*$ of cardinality \aleph_1 such that:

- (i) M is complete; i.e. every automorphism is inner
- (ii) (CH) M has no uncountable abelian subgroup.

²if we use a countable fragment \mathcal{L} of $\mathbb{L}_{\omega_1, \omega}(\exists^{\geq \aleph_1})$ to which ψ belongs and consider \mathcal{L} -definability with parameters, straightforwardly there is such a model if \diamond_{\aleph_1} hence provably in ZFC

(B78) Hanf number of omitting types for simple first-order theories, JSL 44(1979), 319-324.

Let T be a complete countable first-order theory such that every ultrapower of a model of T is saturated. If T has a model omitting a type p in every cardinality $< \beth_\omega$, then T has a model omitting p in every cardinality. There is also a related theorem, and an example showing the \beth_ω cannot be improved. Continued in [Sh:334].

(B79) On uniqueness of prime models, JSL 44(1979), 215-220.

We prove there are theories (stable or countable) for which over every A there is a prime model but it is not necessarily unique. We also give a simplified proof of the uniqueness theorem for countable stable theories.

(B80) A weak generalization of MA to higher cardinals, IJM 30(1978), 297-306.

We generalize MA to \aleph_1 -complete forcing notions satisfying a strengthening of the \aleph_2 -c.c. described in (*) below, and prove its consistency with ZFC + CH. (The number of dense sets is $< 2^{\aleph_1}$, of course). Where

- (*) for any $p_i \in P$ (for $i < \aleph_2$) there are regressive functions $f_n : \omega_2 \rightarrow \omega_2$, such that if $\alpha, \beta < \aleph_1$, $\bigwedge_{n < \omega} f_n(\alpha) = f_n(\beta)$ then p_α, p_β have a least upper bound.

We get similar consistency results (with ZFC + CH + " $2^{\aleph_1} = 2^{\aleph_2}$ ") like a uniformization property for suitable $\langle \eta_\delta : \delta \in S_1^2 \rangle$.

As a result we get the consistency of the decidability of the monadic theory of ω_2 .

(B81) The consistency with CH of some consequences of Martin Axiom plus $2^{\aleph_0} < 2^{\aleph_1}$, 31(1978), 19-33 (with U. Abraham and K. Devlin).

The main conclusion is the consistency with GCH of:

- (*) (1) for any ladder system $\langle \eta_\delta : \delta < \omega_1 \rangle$ (δ limit, η_δ an ω -sequence converging to δ) and $c_\delta \in {}^\omega \omega$, for some $h : \omega_1 \rightarrow \omega$ we have $(\forall \delta)(\exists^\infty k)[h(\eta_\delta(k)) = c_\delta(k)]$
 (*) (2) if G is a graph, ω_1 its set of models,

$$(\forall \alpha, \beta)[\alpha + \omega \leq \beta \rightarrow (\exists^{< \aleph_0} \gamma < \alpha)\langle \gamma, \beta \rangle \in G]$$

then G has chromatic number $\leq \aleph_0$.

The method the known iteration of forcing by Suslin trees following a work of Jensen; and a general principle SAD implying (*) is phrased, and its consistency proved.

(B82) Models with second order properties, III: omitting types for $\mathbb{L}(Q)$, Archive f. Math Logic, 21(1981), 1-11.

A known technique for building model a power \aleph_1 is by building an increasing continuous chain M_α ($\alpha < \omega_1$) of countable models, and in stage α we are allowing to promise to omit countably many types with parameters from M_α "strongly omitted" by M_α . We prove the parallel theorem with \aleph_0, \aleph_1 replaced by λ, λ^+ provided that \diamond_λ holds (or λ is strongly inaccessible). However, we prove \diamond_λ holds for any successor $\lambda \neq \aleph_1$ if GCH holds.

The obstacle for straightforward generalization is preserving the induction hypothesis in limit stages of cofinality $< \lambda$.

The main idea is to use \diamond_λ , while constructing $M_{\alpha+1}$ by approximation of power $< \lambda$, to “guess” obstacles in the future.

(B83) Existentially closed structures in the power of the continuum, Ann. Pure Appl. Logic 26(1984), 123-148 (with Giorgetta)

We deal with three cases:

- G is a countable universal locally finite group
- G is a countable existentially closed group,
- G is a countable existentially closed division ring.

In each case $K = \{H : H \text{ is } \mathbb{L}_{\infty, \omega}\text{-equivalent to } G\}$ which here means: every embedding of a finitely generated subalgebra of G into H can be extended to an embedding of G into H , and every countable subalgebra of H can be embedded into G .

We prove that K has members of power 2^{\aleph_0} ; in fact, many quite complicated members which e.g. do not have \aleph_1 pairwise commuting elements. The method is general: we build the model of power 2^{\aleph_0} as, essentially, an inverse limit of $\langle M_u^n : u \in \mathcal{P}(n) \rangle$, M_u^n finitely generated, so the problems become finitary: for $N \subseteq N_0$, N_1 finitely generated, amalgamate N_0, N_1 over N with some extra requirements. For those we have some criteria, but the main work is specific to each case.

(B84) On elementary equivalence of automorphism groups of Boolean Algebras, downward Skolem-Löwenheim theorems and compactness of related quantifiers, JSL 45(1980), 265-283 (with M. Rubin).

(B85) A note on the normal Moore space conjecture Can. J. Math., Vol.XXXI, 2(1979), 241-251 (with Devlin).

In this paper we show that if CH be assumed, then Jones’ spaces are not normal and that the GCH does not lead to a positive solution to the Jones conjecture. A brief survey of the progress on the problem to date is also included. Jones’ spaces are closely related to Aronszajn trees.

(B86) Souslin properties and Tree Topologies Bull. London Math Soc., (3), 39(1979), 537-552.

(B87) Classification theory for non-elementary classes. I. The number of uncountable models of $\psi \in \mathbb{L}_{\omega_1, \omega}$, Israel J. Math., 46(1983), 212-273.

The Main Theorem 2.26. *Assuming $\forall n < \omega$, $2^{\aleph_n} < 2^{\aleph_{n+1}}$; if $\psi \in \mathbb{L}_{\omega_1, \omega}$ (τ_ψ countable) has an uncountable model then:*

- *there exists n positive such that $I(\aleph_n, \psi) = 2^{\aleph_n}$ or*
- *ψ has models in every uncountable cardinality, and iff $(\exists \lambda > \aleph_0) I(\lambda, \psi) = 1$ then $(\forall \lambda > \aleph_0) I(\lambda, \psi) = 1$.*

As in [She75c] we consider a class of atomic models K of some first order complete theory.

It is proved that it is enough to prove the Main Theorem for K . Assuming $2^{\aleph_0} < 2^{\aleph_1}$ and $I(\aleph_1, \psi) < 2^{\aleph_1}$ basic stability machinery is introduced, i.e. a substitute to non-forking; and a substitute to saturated models in elementary classes.

The key to classify the complexity of the class K instead by stability, to do it by generalized amalgamation properties (remember the role of \aleph_0 - A.P. in [She75c]). The problem to find A.P.-s such that the failure of those A.P.-s will imply existence of many non-isomorphic models and if those generalized A.P.-s holds then to conclude existence of models in higher cardinalities.

The paper logically is divided to three parts: first developing basic stability theory. Second, introducing various amalgamation properties proving some interrelations among them and proving that if all the A.P.-s holds them K has models in every cardinality and the Morley categoricity theorem for K . The third part is by combinatorial work to prove that the failure of A.P. implies the existence of many non-isomorphic models.

We try to minimize the use of set theoretical assumptions: the main result of [She75c] is proved from $2^{\aleph_0} < 2^{\aleph_1}$ only. Most parts of the last part of the paper can be proved from the assumption $(\forall n < \omega)$ that $2^{\aleph_n} < 2^{\aleph_{n+1}}$.

It is an open problem to prove the Main Theorem in ZFC only, see also [She87a]. Continued in [She09].

(B88) Classification theory for non-elementary classes. II. Abstract elementary classes, Lecture Notes in Math. 1292(1987) 419-497.

We deal with a class of models K all of the same similarity type, and a partial order $<$ on K (which stands e.g. for elementary submodels) the only demand except the obvious is existence of $\lambda(K)$ which is a downward Skolem-Lowenheim number, and $(K, <)$ satisfies Tarski-Vaught elementary chain theorems. It is proved from ZFC only that $I(\aleph_1, K) = 1 \Rightarrow I(\aleph_2, K) \geq 1$ (this improves [She75c], [She83a], [She83b]) and assuming $2^{\aleph_0} < 2^{\aleph_1}$ we show that $1 \leq I(\aleph_1, K) < 2^{\aleph_1}$ implies $I(\aleph_2, K) \geq 1$, the methods are classifying by the amalgamation properties (as in [She75c] and [She83a], [She83b]). Most of the work is preparation to generalize the Main Theorem from [She83a], [She83b] to this context. Continued in [She01], [She09].

(B89) Boolean algebras with few endomorphisms, Proc. AMS 14(1979), 135-142.

Any Boolean algebra has some trivial endomorphism - the identity, an endomorphism into $\{0, 1\}$ corresponding to an ultrafilter, and some combinations of them. We build (assuming \diamond_{\aleph_1} and then CH only) Boolean algebra of power \aleph_1 , such that every endomorphism is definable from finitely many elements and ultrafilters. We also generalize a construction of Rubin of some special Boolean Algebras in \aleph_1 to higher cardinals. Continued in [Shear, Ch.I].

(B91) The structure of $\text{Ext}(G, \mathbb{Z})$ and $\mathbf{V} = \mathbf{L}$, Math Zeitschrift 162(1978), 39-50 (with H.L. Hiller and M.K. Huber).

Let G be a torsion free abelian group, then it is well known that $\text{Ext}(G, \mathbb{Z})$ is divisible, hence determined by its p -ranks (p zero or prime) $\nu_p(G)$.

We prove that for non-free G (assuming $\mathbf{V} = \mathbf{L}$): $\nu_0(G)$ is $\min\{2^{|H|} : H \text{ a subgroup of } G, G/H \text{ free}\}$, hence $\nu_0(G)$ is a successor cardinal and $\nu_p(G) \leq \nu_0(G)$.

We determine the possible $\nu_p(G)$ when $\text{Hom}(G, \mathbb{Z}) = 0$, and we characterize the abelian group to which there exists co-Moore spaces; and assert that instead $\mathbf{V} = \mathbf{L}$ it is enough to assume no stationary subset of a regular cardinal is small. Continued in [MRS99].

(B92) Remarks on Boolean Algebras, Algebra Universalis, 11(198), 77-89.

In §4, we show that if B satisfies the κ -chain condition, $|B| > \lambda = \lambda^{<\kappa}$ then B has large independent subsets, e.g. of power $\geq \lambda^+$. We can replace λ^+ by $\chi = \text{cf}(\chi)$ if $(\forall \alpha < \chi)(|\alpha|^{<\kappa} < \lambda)$, and find an independent subset A' of $A \in [B]^\chi$ of cardinality χ . If χ is singular, $(\forall \alpha < \chi)(|\alpha|^{<\kappa} < \lambda)$ and $A \in [B]^{\chi^+}$ we can find an independent $A' \subseteq A$ of cardinality χ .

In fact, our results are almost complete.

We prove some theorems on Boolean Algebras. In §1, we prove that if S is a subset of a (Boolean Algebra) B , $|S|$ singular strong limit, then $S^- = \{a-b : a, b \in S\}$ has a subset of power $|S|$ which is a pie subset (a subset no two elements of which are comparable).

In §2 we prove the consistency of $ZFC + 2^{\aleph_0} > \aleph_1$ with “there is a Boolean Algebra B of cardinality 2^{\aleph_0} , with no uncountable pie subset, nor an uncountable chain.”

In §3, we prove that if $|B| = \lambda^+$, $\lambda = \lambda^{<\kappa}$, B satisfies the κ -chain condition, then $B \setminus \{0\}$ is the union of λ ultrafilters (proper, of course). We shall show in [She81b] that this is not necessarily true when $|B| = \lambda^{++}$, and other connected results.

(B93) Simple unstable theories, Ann. Math. Logic, 19(1980), 177-203.

Being simple is a property of first order complete theories weaker than stability. We start their investigation, using a test question asking if $|T| < \kappa$, $\mu = \mu^\kappa < \lambda = \lambda^{|T|} \leq 2^\mu$, $M \models T$, $\|M\| = \lambda$ implies M has a κ -saturated elementary extension of cardinality λ . For non-simple the answer is no, for simple theories the answer is consistently yes. We start to develop a parallel of [She78, Ch.II,III].

(B94) Weakly compact cardinals: a combinatorial proof, JSL 44(1979), 559.

We give here a direct purely combinatorial proof that weak compactness is equivalent to a combinatorial property (2). This property (2) is apparently stronger, and from it, all other usual equivalent definitions and usual properties of weakly compact cardinals can be deduced. So this proof may be useful for books which want to present weakly compact cardinals, but not logic.

Remark 2.27. For μ the first inaccessible we can define a μ -tree with no μ -branch by:

$$T = \{h : \text{Dom } h \text{ an ordinal } \alpha < \mu, h(i) < 1 + i, \text{ and for strong limit } i, j \in \text{Dom } h, h(i) \neq h(j)\}.$$

Theorem 2.28. *For μ strongly inaccessible the following are equivalent:*

- μ is weakly compact, i.e., every μ -tree has a μ -branch
- for every family of functions $f_\alpha : \alpha \rightarrow \alpha$ ($\alpha < \mu$) there is a function $f : \mu \rightarrow \mu$ such that: $(\forall \alpha < \mu)(\exists \beta) \alpha \leq \beta < \mu$ and $[f_\beta \upharpoonright \alpha = f \upharpoonright \alpha]$.

(B95) Canonization theorems and applications, JSL, 45(1981), 345.

We prove canonization theorems generalizing Erdős-Rado partition theorems. We introduce a very general partition relation which generalizes $\lambda \rightarrow (\kappa)_\mu^\chi$ in many aspects. We prove a corresponding partition theorem (which works simultaneously for sequences of cardinals). E.g. assume $\langle \lambda_i : i < \alpha \rangle$ a sequence of cardinals, $|\alpha| + \kappa < \lambda_i$, $\langle \mu_i = \beth_n(\lambda_i)^+ : i < \alpha \rangle$ is strictly increasing, $F : [\bigcup_i \mu_i]^{<\omega} \rightarrow \theta$, then we can find $\langle A_i : i < \alpha \rangle$, $A_i \in [\mu_i \setminus \bigcup_{j < i} \mu_j]^{\lambda_i^+}$ and F is restricted to increasing sequences from $\bigcup_{i < \alpha} A_i$ does not depend on the last n arguments. Moreover, in the inductive proof if we restrict how the sequences are distributed between the A_i 's, e.g. to those of \bar{n} -kind, where \bar{a} is of the \bar{n} -kind if

$$n_\ell = \left| \left\{ k < \ell g(\bar{a}) : \ell g(\bar{n}) - k + 1 = \{ i < \alpha : \emptyset \neq \bar{a} \cap \mu_i \setminus \bigcup_{j < i} \mu_j \} \right\} \right|.$$

If in the last one there is one element (e.g. $n_{\ell g(\bar{n})-1} = 1$, we can gain 2 places provided we increase the function $\beth_n(\lambda_i)^+$ which mainly consists of n exponentiation, by some plusses. If $\alpha < \omega$, we can gain more.

As an application we complete the answer to the following question:

Given a Hausdorff space X of cardinality λ , characterize the pairs of cardinals (λ, χ) such that X has a subspace Y , of cardinality χ , which is discrete. Also, generalize Hajnal's free subset theorem for tuples instead of one-place functions.

(B96) Algebraically closed groups of large cardinality, JSL 44(1979), 232-552 (with Martin Ziegler).

Let M be a countable algebraically closed group, κ an uncountable cardinal. We will prove in this paper the following theorems.

Theorem 2.29. *There is an algebraically closed group N of cardinality κ which is $\mathbb{L}_{\infty, \omega}$ -equivalent to M .*

Theorem 2.30. *There is an algebraically closed group N of cardinality κ which is $\mathbb{L}_{\infty, \omega}$ -equivalent to M , and contains a free abelian group of cardinality κ .*

Theorem 2.31. *There are 2^κ non-isomorphic algebraically closed groups of cardinality κ which are $\mathbb{L}_{\infty, \omega}$ -equivalent to M .*

Theorem 2.32. *There is an algebraically closed group N of cardinality κ which is $\mathbb{L}_{\infty, \omega}$ -equivalent to M and satisfies: every subgroup of N of uncountable regular cardinality contains a free subgroup of the same cardinality.*

The method is to construct things parallel to Ehrenfeucht-Mostowski models, using things like centralizers of elements.

(B97) Unordered types of ultrafilters, Topology Proc. 3(1979), 199-204 (with M.E. Rudin).

It is proved in ZFC that there are the maximal number of incomparable ultrafilters on ω , in the Rudin-Kaisler ordering (and similarly for other cardinals).

(B98) Whitehead groups may not be free even assuming CH,II

(A) We prove the consistency of $ZFC + 2^{\aleph_0} = 2^{\aleph_1} + \text{club}$ (i.e. \clubsuit), where club is the following weakening of diamond:

- (*) there is $\langle A_\delta : \delta < \omega_1 \rangle$, A_δ an unbounded subset of δ such that for any unbounded $A \subseteq \omega_1$, $\{\delta : A_\delta \subseteq A\}$ is stationary. (Under CH, club and diamond were known to be equivalent).

The method is to start with a model of $ZFC + GCH$, blow up 2^{\aleph_1} and then collapse \aleph_1 .

(B) For simplicity let Φ be a family of ω -sequences from a set called $\text{Dom}(\Phi)$; (Φ, h) has the uniformization property if for any functions $f_\eta : \text{Rang}(\eta) \rightarrow \text{ordinals}$ satisfying $f_\eta(i) < h(i)$, there is $f : \text{Dom}(\Phi) \rightarrow \text{ordinals}$, such that for every $\eta \in \Phi$ for all but finitely many n , $f(\eta(n)) = f_\eta(n)$.

We investigate such properties.

(B1) We prove that for a stationary $S \subseteq \omega_1$ and η_δ converging to δ , then property “ $(\langle \eta_\delta : \delta \in S \rangle, 2)$ has the uniformization property” does not depend on S only: in fact, seemingly trivial changes in η_δ can change its truth value.

(B2) We prove that if in the weak diamond (see [DS78]), restricting ourselves to a stationary co-stationary S , replacing 2 by 3 give a non-equivalent statement.

(B3) We prove that $\text{Dom}(\Phi)$ may be ω , Φ uncountable but still $(\Phi, 2)$ has the uniformization property, i.e. this is consistent with $ZFC + 2^{\aleph_0} = 2^{\aleph_1}$. Though this looks like a consequence of $MA + 2^{\aleph_0} > \aleph_1$, it in fact contradicts MA. So the result says that there are infinite $A_\alpha \subseteq \omega$ for $\alpha < \omega_1$ almost disjoint (i.e. $\alpha < \beta \Rightarrow |A_\alpha \cap A_\beta| < \aleph_0$) and for any $h_\alpha \in {}^{(A_\alpha)}2$ for $\alpha < \omega_1$ there is $h \in {}^\omega 2$ such that $\alpha < \omega_1 \Rightarrow h_\alpha \subseteq^* h$. In fact we have a condition on $\langle A_\alpha : \alpha < \alpha^* \rangle$ preserved by finite support iteration such that it guarantees that the natural forcing of adding h for any given $\bar{h} = \langle h_\alpha : \alpha < \alpha^* \rangle$ satisfies it.

Group Theory:

(C) We prove that occurrence of non-free Whitehead groups, and $(\Phi, 2)$ with the uniformization property Φ “non-trivial” are closely related (for power \aleph_1 -equivalent). Hence by (B1), the property of being Whitehead is quite delicate, maybe depending on seemingly trivial changes.

We prove a set theoretic result which shows that for many non-free G , $\text{Ext}(G, \mathbb{Z})$ is large (compare with [HHS78]).

(B99) Equi-consistency results, Notre Dame J. Formal Logic 26(1985), 178-188 (with L. Harrington).

There are two topics:

MA and measurability and the Baire property: The following are equi-consistent:

- $ZFC +$ there is a weakly compact cardinal
- $ZFC + MA$ for every real r , $\aleph_1^{L[r]} < \aleph_1$
- $ZFC + MA +$ every Δ_3^1 -set of reals is measurable
- $ZFC + MA +$ every Δ_3^1 -set of reals has the Baire property.

We can replace Δ_3^1 by first order definable with a real and ordinal parameter.

Reflection of stationary sets: The following are equi-consistent:

- ZFC + there is a Mahlo cardinal
- ZFC + GCH + every stationary $S \subseteq S_0^2$ has an initial segment stationary.

(B100) Independence results, JSL 45(1980), 563-573.

THE METHOD: We introduce the oracle chain condition which (for \aleph_1) helps us to use iterated forcing of length \aleph_2 , each step increasing the set of reals, by enabling us to omit types (i.e. to promise no real satisfying \aleph_1 conditions will be added later on).

We then introduce proper forcing, which is a very general condition guaranteeing \aleph_1 is not collapsed and is preserved by countable support iteration.

In fact we also introduce a blending of the two.

THE RESULTS: We prove the consistency of: $2^{\aleph_0} = \aleph_2$ and:

- there is a universal linear order of power \aleph_1
- there are countable complete $T \subseteq T_1$, T is not \aleph_0 -stable, not superstable, but still there is a unique $L(T)$ -reduct of a model of T_1 of power \aleph_1 .

The results are complimentary to [She78, Ch.VII]. More see [She89].

(B101) The theorems of Beth and Craig in abstract model theory. II. Compact Logics, Archive Math Logik 21(1980), 13-36 (with J.A. Makowski).

Various compact logics such as stationary logic, positive logic, logics with various cardinality quantifiers and cofinality quantifiers are studied. Counterexamples to the theorems of Beth and Craig are given. Back and forth arguments are studied for the first two logics, transfer theorems presented for positive logic and a new compactness proof for the cofinality quantifier is given.

(B102) Forcing with stable posets, JSL 47(1982), 37-42 (with U. Abraham).

(B103) On partitions of the real line, IJM 32(1979), 299-304 (with D.H. Fremlin).

We proved that the real line is not necessarily the disjoint union of \aleph_1 non-empty G_δ -sets. This follows from known results and: if the real line can be partitioned into κ sets which are G_δ -sets (κ uncountable) then the real line can be covered by κ nowhere dense closed sets.

(B104) The \aleph_2 -Souslin hypothesis, Trans. AMS 264(1981), 411-419 (with R. Laver).

We prove, assuming $\text{con}(\text{ZFC} + \text{there is a weakly compact cardinal})$ the consistency of “ZFC + CH + $2^{\aleph_1} = 2^{\aleph_2} = \aleph_3$ + there is no \aleph_2 -Souslin tree.”

(B105) On uncountable abelian groups, IJM 32(1979), 311.

In §1 we characterize the Whitehead groups of power $< 2^{\aleph_0}$, assuming Martin Axiom: they are the \aleph_1 -free groups satisfying possibility II or III from [She74]; and, equivalently, they are \aleph_1 -coseparable or equivalently $\text{Ext}(-, \mathbb{Z}_\omega) = 0$.

In §2 we construct an \aleph_1 -free group satisfying possibility II which is not strongly \aleph_1 -free. Hence $MA + 2^{\aleph_0} > \aleph_1$ implies there is a Whitehead group which is not strongly \aleph_1 -free.

We also prove (assuming $\mathbf{V} = \mathbf{L}$ or even $2^{\aleph_0} < 2^{\aleph_1}$) that there is a strongly \aleph_1 -free, separable, not \aleph_1 -separable group of cardinality \aleph_1 . At last we construct an \aleph_2 -free (hence strongly \aleph_1 -free) non-separable, non-Whitehead group of cardinality 2^{\aleph_1} . The method is using a tree which imitates \diamond_{\aleph_1} in a sense.

In §3 we deal with hereditarily separable groups. If $\mathbf{V} = \mathbf{L}$ (or every stationary subset of a regular cardinal is not small they are just the free groups). (This strengthens the theorem: if $\mathbf{V} = \mathbf{L}$ every Whitehead group is free.) But $MA + 2^{\aleph_0} > \aleph_1$ implies there are non-Whitehead, hereditarily separable groups of cardinality \aleph_1 . We also prove, assuming $2^{\aleph_0} < 2^{\aleph_1}$, that any hereditarily separable group is strongly \aleph_1 -free (a little more, in fact).

CORRECTIONS: Omit 3.3,(iii) and 3.6.

(B106) Martin axiom does not imply that every two \aleph_1 -dense sets of reals are isomorphic, IJM 38(1981), 161-176 (with U. Abraham).

RESULTS: We prove the consistency of the following with ZFC + “ $2^{\aleph_0} = \aleph_2$ ” (assuming $\text{con}(\text{ZFC})$, of course).

- 1) There are two \aleph_1 -dense sets of reals which cannot be made isomorphic by any c.c.c. forcing notion (so without loss of generality MA holds).
- 2) There is an \aleph_1 -dense set of reals such that any $f : A \rightarrow A$ is order preserving on some uncountable $A' \subseteq A$.
- 3) For every $A \subseteq R$ of power \aleph_1 , any one-to-one function $f : A \rightarrow R$ is the union of countably many monotonic functions.
- 4) MA + there are entangled sets of reals (defined there).

METHOD: A basic fact is that in order to make two \aleph_1 -dense subsets of R isomorphic, there is a reasonably canonical forcing notion doing it, if we are given a small enough closed unbounded subset as a parameter.

(B108) On successor of singular cardinals, study in logic and the foundation of Math, Vol. 97(ed. boffa, Van dalen and McAllon) No. Holland, 1979, 357-386.

Let us for simplicity assume $\lambda = \aleph_{\omega+1}$, \aleph_ω strong limit. We find a set $S^* \subseteq \lambda$, determine modulo the closed unbounded filter, prove the equivalence of some definitions and:

- if P is \aleph_n -complete forcing notion, $S_1 \subseteq \{\delta < \lambda, \text{cf}(\delta) < \aleph_n, \delta \in S^*\}$ is stationary, then S_1 remains stationary after forcing with P . However, there is an \aleph_n -complete forcing notion which makes S^* non-stationary.
- S^* has no initial segment δ stationary if GCH holds
- $S^* = \emptyset$ (i.e. not stationary) iff a weak version of squares holds
- it is consistent (with ZFC + GCH) that S^* is stationary (if supercompact cardinals are)(the large cardinals are necessary).

Using (2) and (3) we get a proof, using different methods according to whether S^* is stationary or not, of

COROLLARY: (\aleph_ω strong limit) there is an $\aleph_{\omega+1}$ -free non-free group of power $\aleph_{\omega+1}$ (similarly for transversal). For higher λ we get less information.

Donder noted that as if there is a special Aronszajn on $\aleph_{\omega+1}$ then the weak squares (from 3) holds we can (from (4), (3)) deduce: ZFC + GCH is consistent

with “there is no special Aronszajn tree on $\aleph_{\omega+1}$ ” (if a supercompact is consistent).

(B109) Infinite games and reduced powers, AML 20(1981), 77-108 (with W. Hodges).

By [She71a], if A, B are elementarily equivalent structures then there are ultrafilters D, E such that $A^I/D \cong B^J/E$. The paper proves analogues of this theorem where the language can be infinitary and D, E are κ -complete for κ a regular cardinal.

1) Let PL_κ be the language of prenex formulas of $L_{\kappa, \kappa}$ with arbitrary well-ordered game quantifiers of length $< \kappa$. If A, B have mutually consistent PL_κ -theories and κ is strongly compact then A, B have isomorphic limit ultrapowers where the filter involved are κ -complete.

2) Let PH_κ be the language of Horn formulas in PL_κ . If for every regular κ , A and B have mutually consistent PH_κ -theories then A, B have isomorphic limit reduced powers with κ -complete filters.

3) If there is a proper class of measurable cardinals, then it is consistent that (1), (2) hold with “limit” deleted; this uses game-theoretic arguments of Galvin and Laver.

4) The natural analogue of (2) for classes of structures and limit reduced products also holds and gives interpolation and preservation theorems for PH_κ which had been derived proof-theoretically by Hodges.

5) Necessary and sufficient conditions are given for a structure A to have a limit reduced power with κ -complete filters in which a given sentence is true; as a corollary, when $\kappa > \omega$. The amalgamation property fails for PH_κ -elementary extensions.

6) Under various conditions, examples are given of sentences which are preserved in reduced products over κ -complete filters but are not logically equivalent to sentences in PH_κ , also answering a related question of Kueker.

7) A non-standard infinitary language is described which has good interpolation and Feferman-Vaught properties.

(B110) Better quasi-order for uncountable cardinals, IJM 42(1982), 177-226.

We generalize Nash-Williams theory on better quasi-order to uncountable cardinals.

Let the well-ordering number of a quasi-order Q be the minimal λ such that for any $q_i \in Q$ ($i < \lambda$) for some $i < j$ we have $q_i \leq q_j$, and then we say Q is λ -well ordered. We define some variants of λ -better quasi-order, and prove for them preservation theorems under the known infinitary operations, (like Seq, \mathcal{P}). We then compute the well ordering number of some classes and of the result of the infinitary operations on arbitrary quasi-order.

The main results are:

- the well-ordering number of the class of graphs ordered by embeddings (taking edges to edges rather than to paths (or disjoint paths)) is the first beautiful cardinal κ_0 (it is a mild large cardinal, somewhat bigger than weakly compact, and still compatible with L)
- also the well-ordering number of the class of models with one-place predicates, under elementary embeddings, is the same

- the well-ordering number of $\{M : M \text{ a linear order and is the union of } \leq \lambda \text{ scattered ordered}\}$ ($\lambda > \aleph_0$), ordered by embeddability, is the first beautiful cardinal $> \lambda$.

(B112) S^- Forcing, a black box theorem for morasses with applications: super-Souslin trees and generalized MA, Israel J. Math 43(1982), 185-224 (with L. Stanley).

(A) Quite a few consistency results can be gotten in the following way: starting with $\mathbf{V} \models GCH$, we force an example of power \aleph_2 by countable approximations. So a countable approximation depends on a countable set α of ordinals, so it is $\tau(\alpha)$ for some term τ , and the number of τ -s is $2^{\aleph_0} = \aleph_1$. We state a black box principle, which says that many such proofs can be carried in the universe \mathbf{V} (if it satisfies the principle), moreover the proof that the black box principle applies is the same as that the forcing works.

Of course, \aleph_2 can be replaced by higher cardinals.

(B) We prove that the principle follows from $\mathbf{V} = \mathbf{L}$, really much less is needed: the morass

(C) We got some consequences of the principle, among them the morass, so they are equivalent.

(D) Another application is the existence of super-Souslin trees. An \aleph_2 -super-Souslin tree T is an \aleph_2 -tree T , such that for any $\alpha < \omega_2$, $\bar{\alpha} = \langle \alpha_n : \alpha < \omega \rangle$, and ω distinct members of T_α , we have a function

$$F_{\bar{\alpha}} : \left\{ \langle \beta_b : n < \omega \rangle : \alpha_n \leq \beta_n \wedge (\exists \beta) \bigwedge_n \beta_n \in T_\beta \right\} \rightarrow \omega_1$$

with $F_{\bar{\alpha}}(\beta) = F_{\bar{\alpha}}(\gamma)$ implying β_n, γ_n are comparable for infinitely many n 's. (So it is essentially Souslin as a special Aron tree is an Aron tree).

Now an \aleph_2 -super-Souslin tree is \aleph_2 -Souslin (ignoring possibly finitely many branches), and it remains so in any extension not adding reals.

Hence: if $CH + SH_{\aleph_2}$ then \aleph_2 is inaccessible in \mathbf{L} .

(B114) Isomorphism types of Aronszajn trees, IJM 50(1985), 75-113 (with U. Abraham).

We investigate Aronszajn trees, under embeddability and isomorphism when a restriction to a club (closed unbounded set of levels) C is allowed, get consequences of $2^{\aleph_0} < 2^{\aleph_1}$, and consistency results with CH and with $2^{\aleph_0} = 2^{\aleph_1}$. We also formulate a combinatorial principle that follows from the weak diamond (which is equivalent to $2^{\aleph_0} < 2^{\aleph_1}$) and hopefully will serve to obtain results like those in (A) below.

The results have variants speaking on Specker orders.

(A) **Consequences** of $2^{\aleph_0} < 2^{\aleph_1}$ (§1).

If CH or even if $2^{\aleph_0} < 2^{\aleph_1}$ holds, then:

- there are 2^{\aleph_1} pairwise really different Aronszajn trees, i.e. not one of these trees is embeddable on a club into the other
- there is an Aronszajn tree T such that for every closed unbounded $C \subseteq \omega_1$, $T \upharpoonright C$ is rigid, (i.e. the only embedding of $T \upharpoonright C$ into $T \upharpoonright C$ is the identity, so T is a *really rigid* tree).

We can combine (b) with (a):

- (c) for every Aronszajn tree T^1 there is an Aronszajn tree T^2 such that T^1 is not embeddable into T^2 on a closed unbounded set, i.e., there is no prime Aronszajn tree.

The method is the combinatorial principle mentioned above.

(B) **Consistency results with CH** (§2).

- (a) GCH + the following is consistent:
 - (a₁) there is a universal Aronszajn tree T i.e., a tree T such that for every Aronszajn tree T^* , $T^* \upharpoonright C$ is order embeddable into $T \upharpoonright C$ for some closed unbounded $C \subseteq \omega_1$. (The universal tree T is a special Aronszajn tree), so in this model there are no Souslin trees. Thus (a₁) is a strengthening of Jensen's "CH is consistent with Souslin's hypothesis," whose method we use.
 - (a₂) Every two Aronszajn trees contain subtrees which are isomorphic on a closed unbounded set.
- (b) We have some consistency result with CH concerning Souslin trees. For example, CH + "there is a Souslin tree" do not imply that there are 2^{\aleph_1} really different Souslin trees. Using this construction, Shelah proved that the Malitz-Magidor logic is not necessarily compact.

Problem: Does the existence of two really different Souslin trees follow from the existence of a Souslin tree?

(C) **Consistency results with $2^{\aleph_0} = 2^{\aleph_1}$.**

- (a) Martin's axiom + $2^{\aleph_0} = \aleph_2$ + every two Aronszajn trees are isomorphic on a closed unbounded set is consistent.
- (b) Martin's axiom + $2^{\aleph_0} > \aleph_1$ does not imply that every two Aronszajn trees are isomorphic on a closed unbounded set.
- (c) Martin's axiom + $2^{\aleph_0} = \kappa$ + every two Aronszajn trees are isomorphic on a closed unbounded set is consistent. κ here is "any" regular cardinal such that $\kappa^{<\kappa} = \kappa$.

(The difference between this item and (a) is that in (a) we get $2^{\aleph_0} = \aleph_2$ and here 2^{\aleph_0} is as big as we want; moreover, the proofs are different: in (a) we use proper forcing and in (c), a technique of using generic reals).

(B115) Superstable fields and groups, AML 18(1980), 227-270 (with G. Cherlin).

The main object is to prove: for any infinite field F , $\text{Th}(F)$ is superstable iff F is algebraically closed. This is done by investigating connected groups (G connected if it has no proper definable subgroup of finite index) and indecomposable subgroups (same rank and multiplicity one). Also extend some known results for ω -stable groups with Morley rank ≤ 3 to superstable groups (i.e. the theory of the group is superstable).

An important ingredient is looking at definable groups in models of a stable theory and deal with local ranks $\text{rk}(-, \Delta, \kappa)$, $\kappa \in \{2, \aleph_0\}$ for Δ -s which are translation closed.

(B116) Positive results in abstract model theory: a theory of compact logics, Ann. Pure Appl. Logic 25(1983), 263-299 (with J.A. Makowski).

We prove that compactness is equivalent to the amalgamation property provided the occurrence number of the logic is smaller than the first measurable cardinal. We also relate compactness to the existence of certain regular ultrafilters associated with the logic and develop a general theory of compactness and its consequences. Continued in [She85b].

(B117) Combinatorial problems on trees: Partitions, Δ -systems and large free subsets, Ann. Pure Appl. Logic 33(1987), 43-81 (with M. Rubin).

We prove partition theorems on trees and generalize to a setting of trees the theorems of Erdős and Rado on Δ -systems and the theorems of Fodor and Hajnal on free sets. For simplicity we ignore some of the results, e.g. on trees with a non-fix splitting, or with $> \omega$ levels. Let μ be an infinite cardinal and T_μ be the tree of finite sequences of ordinals $< \mu$, with the partial ordering of being an initial segment: $\eta < \nu$ denotes that η is an initial segment of ν . A subtree of T_μ is a non-empty subset of T_μ closed under initial segments. $T \leq T_\mu$ means that T is a subtree of T_μ and $\langle T, \leq \rangle \cong T_\mu$.

The following are extracts:

Theorem 2.33. (1) *A partition theorem. Suppose $\text{cf}(\lambda) \neq \text{cf}(\mu)$, $F : T_\mu \rightarrow \lambda$ and for every branch b of T_μ we have $\sup(\{F(\alpha) : \alpha \in b\}) < \lambda$. Then there is $T \leq T_\mu$ such that $\sup(\{F(\alpha) : \alpha \in T\}) < \lambda$.*

Theorem 2.34. (2) *A theorem on large free subtrees. Let $\lambda^+ \leq \mu$, $F : T_\mu \rightarrow P(T_\mu)$, for every branch b of T_μ : $|\bigcup\{F(\alpha) : \alpha \in b\}| < \lambda$, and for every $\alpha \in T_\mu$ and $\beta \in f(\alpha)$, $\beta \not\prec \alpha$. Then there is $T \leq T_\mu$ such that for every $\alpha, \beta \in T$, $\beta \notin F(\alpha)$.*

Let $P_\lambda(C)$ denote the ideal in $P(C)$ of all subsets of C whose power is less than λ . Let $\text{cov}(\mu, \lambda)$ mean that μ is regular, $\lambda < \mu$, and for every $\kappa < \mu$ there is $D \subseteq P_\lambda(\kappa)$ such that $|D| < \mu$ and D generates the ideal $P_\lambda(\kappa)$ of $P(\kappa)$. Note that if for every $\kappa < \kappa^{<\lambda} < \text{cf}(\mu) = \mu$, then $\text{cov}(\mu, \lambda)$ holds. Let $\alpha \wedge \beta$ denote the maximal common initial segment of α and β .

Theorem 2.35. (3) *A theorem on Δ -systems. Suppose $\text{cov}(\mu, \lambda)$ holds, $F : T_\mu \rightarrow P(C)$ and for every branch b of T_μ we have $|\bigcup\{F(\alpha) : \alpha \in b\}| < \lambda$, then there is $T \leq T_\mu$ and a function $K : T \rightarrow P_\lambda(C)$ such that for every incomparable $\alpha, \beta \in T$, $F(\alpha) \cap F(\beta) \subseteq K(\alpha \wedge \beta)$.*

(B118) On the expressibility hierarchy of Magidor-Malitz quantifiers, JSL 48(1983), 542-557 (with M. Rubin).

We prove that the logics of Magidor Malitz and their generalization by Rubin are distinct even for PC classes; during this we prove the existence of forcing notions satisfying one variant of the countable chain condition but not another and use the preservation of the satisfaction of sentences in \mathbb{L}^n (see below) under appropriate chain condition.

Let $M \models Q^n x_1 \dots x_n \varphi(x_1 \dots x_n)$ mean that there is an uncountable subset A of $|M|$ such that for every $a_1, \dots, a_n \in A$, $M \models \varphi[a_1, \dots, a_n]$.

Theorem 2.36. (1) 1) (\diamond_{\aleph_1}). For every $n \in \omega$ the class $K_{n+1} = \{\langle A, R \rangle : \langle A, R \rangle \models Q^{n+1}x_1 \dots x_{n+1}R(x_1, \dots, x_{n+1})\}$ is not an \aleph_0 -PC-class in the logic L^n , obtained by closing first order logic under Q^1, \dots, Q^n , i.e., for no countable L^n -theory T , K_{n+1} is the class of reducts of the models of T .
 2) (CH) For every $n \in \omega$ there is a forcing notion (P, \leq) satisfying $(Q^i x_1 \dots x_i)(x_1, \dots, x_i)$ have no upper bound) iff $i > n$.

Theorem 2.37. (2) (\diamond_{\aleph_1}). Let $M \models (Q^E x, y)\varphi(x, y)$ mean that there is $A \subseteq |M|$ such that $E_{A, \varphi} =: \{\langle a, b \rangle : a, b \in A \text{ and } M \models \varphi[a, b]\}$ is an equivalence relation on A with uncountably many equivalence classes, and such that each equivalence class is uncountable. Let $K^E = \{\langle A, R \rangle : \langle A, R \rangle \models (Q^E x, y)R(x, y)\}$, then K^E is not an \aleph_0 -PC-class in the logic gotten by closing first order logic under the set of quantifiers $\{Q^n : n \in \omega\}$ which were defined in Theorem 1.

A more general version of the second theorem is proved in the paper.

(B119) Iterated forcing and changing cofinalities, IJM 40(1981), 1-32.

(B120) Free limits and Aronszajn trees, IJM 38(1981) 33-38.

Our main result is the consistency of the “ZFC + Souslin hypothesis + not every Aronszajn tree is special.”

For this end we introduce some variant of being special, investigate the connections between them, and for one “being S -special” ($S \subseteq \omega_1$ stationary co-stationary) we prove that if we S -specialize all Aronszajn trees (by iterated forcing), some “bad” tree we specify does not become special; as some related property of the forcing is preserved.

We use (as clearer though CS is O.K.) an iteration in which we take a kind of limit called free (it is even bigger than inverse limit).

(B121) On the standard part of non-standard models of set theory, JSL 48(1983), 33-38 (with M. Magidor and J. Stavi).

We characterize the ordinals α of uncountable cofinality such that α is the standard part of a non-standard model of ZFC iff α is weakly compact in some sense; more exactly:

- (*) there is $\gamma > \alpha$ such that:
 - (a) for arbitrarily large $\beta < \gamma$, \mathbf{L}_β is a model of ZFC
 - (b) in L_γ , α has the tree property, i.e., if whenever $T \in L_\gamma$, is a tree $\subseteq \alpha$, of height α and each level has A -cardinality $< \alpha$, and the function $\beta \mapsto$ level of β in T , is in A then T has a branch of length α in A .

Remark 2.38. We can replace $\mathbf{L}_\beta \models$ ZFC by $\mathbf{L}_\beta \models$ KP.

(B122) On Fleissner diamond, NDJFL, 22(1981), 29-35.

We prove that $ZFC + GCH + \diamond_\lambda^+$ is consistent with (where $1 < \kappa < \lambda$, λ regular $> \aleph_0$): there are $S_{\alpha, \beta} \subseteq \alpha$ (for $\alpha < \lambda$, $\beta < \kappa$) such that:

- $\{S_{\alpha, \beta} : \beta < \kappa\} : \alpha < \lambda\}$ is a diamond sequence, i.e., for every $A \subseteq X$ for stationarily many $\alpha < \lambda$, $A \cap \alpha \in \{S_{\alpha, \beta} : \beta < \kappa\}$

- for no function $f : \lambda \rightarrow \kappa$ is $\langle S_{\alpha, f(\alpha)} : \alpha < \lambda \rangle$ a diamond sequence; moreover, even $\langle \{S_{\alpha, \beta} : \beta < \kappa\} \setminus \{S_{\alpha, f(\alpha)} : \alpha < \kappa\} \rangle$ is not a diamond sequence.

(B123) Monadic theory of order and topology in ZFC, AML 23(1982), 179-198 (with Yuri Gurevich).

True first-order arithmetic is interpreted in the monadic theory of the real line, in the monadic theory of any non-modest short chain, in the monadic theory of Cantor Discontinuum, in the monadic theory of any non-modest metrizable space. It was known that existence of such interpretations is consistent with ZFC.

(B124) \aleph_ω may have a strong partition relation, IJM 38(1981), 283-288.

We prove the consistency of ZFC + GCH with the following (assuming $\text{con}(\text{ZFC})$ + there are ω measurable cardinals):

- (*) for any function f from finite subsets of \aleph_ω to ω , there are pairwise disjoint $S_n \subseteq \aleph_\omega$, $|S_n| = \aleph_n$ such that:
if u, v are finite subsets of \aleph_ω and $(\forall n)[|S_n \cap u| = |S_n \cap v| \leq n]$ then
 $f(u) = f(v)$.

Note that (*) implies: every Banach space of power $\geq \aleph_\omega$ contains an unconditional (infinite) basis.

(B125) The consistency of $\text{Ext}(G, \mathbb{Z}) = \mathbb{Q}$, IJM 39(1981), 82.

For abelian groups, if $\mathbf{V} = \mathbf{L}$, $\text{Ext}(G, \mathbb{Z})$ cannot have cardinality \aleph_0 . We show that GCH does not imply this (hence some complete $\mathbb{L}_{\infty, \aleph_1}$ sentence has exactly \aleph_0 non-isomorphic models of power \aleph_1 , but not proved here). (Compare [Sh:91]).

The proof continues in [She77].

(127) On Boolean Algebras with no uncountably many pairwise comparable or uncomparable elements, NDJFL 22(1981), 301-308.

We shall prove, assuming CH, the existency of a Boolean algebra of power \aleph_1 , having neither uncountable set of pairwise comparable elements nor uncountable set of pairwise incomparable elements.

We finish commenting on generalizations to higher cardinals.

(B128) Uncountable constructions, IJM 51(1985), 273-297.

We suggest a principle which unites many proofs from \diamond_{\aleph_1} , and is similar to forcing and deduce from it several new results. The paper is supposed to help non-set theorists to build counterexamples by reducing the problems to questions on finitely generated objects: (assuming \diamond_{\aleph_1}).

Boolean Algebras:

- 1) For pedagogical reasons we build a Rubin Boolean Algebra.
- 2) There is a Boolean Algebra which is not 1-Rubin but among any \aleph_1 elements there are two comparable ones (even a, b, c with $a \cap b = c$).

E.C. Groups: For any countable e.c.g. and uncountable K , there is an uncountable $H \equiv_{\infty, \omega} G$, such that K cannot be embeddable into H .

Banach spaces: There is a non-separable Banach space B so that for any non-separable subspace B_1 , B/B_1 is separable.

(B129) The number of non-isomorphic models of cardinality λ , $\mathbb{L}_{\infty,\lambda}$ equivalent to a fixed model, NDJFL, 22(1981), 5-10.

We show that if $\mathbf{V} = \mathbf{L}$ and λ is regular but not weakly compact, then if M is a model of cardinality λ then $\{N/\cong : N \equiv_{\infty,\lambda} M, \|N\| = \lambda\}$ has either one number or 2^λ members.

We do not rely on the full power of $\mathbf{V} = \mathbf{L}$ but use a combinatorial consequence for regular non-weakly compact. The use of some set theoretical assumption stronger than GCH is necessary as in [She81d] we construct a model of ZFC + GCH where a sentence $\psi \in \mathbb{L}_{\infty,\omega_1}$ has \aleph_0 non-isomorphic models of cardinality \aleph_1 . Also the restriction on λ to be not weakly compact is not a coincidence; see [She82].

(B130) Stability over a predicate, NDJFL, 16(1985), 361-376 (with A. Pillay).

This is a beginning of a theory suppose to measure how much $M \upharpoonright P$ determines M , for this we look at $\{M : M \upharpoonright P = N\}$ and count the number of isomorphism types (over N). In the cases the number is not big, we have parallels to some of the theorems on stable theories.

(B133) On the number of non-isomorphic models in $\mathbb{L}_{\infty,\kappa}$ when κ is weakly compact, NDJFL, 23(1982) 21-26.

We construct for κ weakly compact a model M of cardinality κ which has exactly θ non-isomorphic models $M \equiv_{\infty,\kappa} M$ of cardinality κ , for any $\theta \leq \kappa$ we want. This together with [She81c] completes the answer of this problem when $\mathbf{V} = \mathbf{L}$.

(B135) Rigid homogeneous chains, Math. Proc. of Cambridge Philo. Soc. 87(1981), 7-17 (with A.M.W. Glass, Yuri Gurevich and W.C. Holland).

A chain C is *rigidly homogeneous* if for any two points a, b of C there is a unique automorphism f of C such that $f(a) = b$. The group of automorphisms of a rigidly homogeneous chain C is isomorphic to a subgroup of the additive group of reals. We classify rigidly homogeneous chains by elementary properties of their automorphism groups.

The chain of integers is the only countable rigidly homogeneous chain. The cardinality of every rigidly homogeneous chain is $\leq 2^{\aleph_0}$. Ohkuma constructed rigidly homogeneous chains of cardinality 2^{\aleph_0} . We prove consistency of existence as well as consistency of non-existence of rigidly homogeneous chains of uncountable cardinality $< 2^{\aleph_0}$.

(B136) Construction of many complicated uncountable structures and Boolean Algebras, IJM, 45(1983), 100-146.

This paper has three aims:

1) To make the results of [She78, Ch.VIII] on constructing models more available for applications, by separating the combinatorial parts: for this we have to set up a suitable framework. We believe it is useful for proving things like: there are many non-isomorphic structures, there are many structures no one embeddable into the other, there are rigid structures or undecomposable ones or structures with few endomorphisms, etc.

There are few kinds of structures (mainly trees) which served as index sets, and we have many complicated such structures. In an application we have in some sense to interpret those structures in the one we are interested in.

2) A second aim is to strengthen the results of [She78, Ch.VIII]. In particular we mainly are interested there in showing that there are many non-isomorphic models of an unsuperstable theory. We got many times results on the number of models not elementarily embeddable in each other, being a side benefit. Here we consider the later case in more detail, and in a somewhat stronger combinatorial results needed later.

We also consider some more kinds of index structures (mainly a tree of pairs).

3) We apply the previous results to problems on Boolean Algebras from the list of van Douwen, Monk, and Rubin.

In particular

- (a) if $\lambda = \aleph_0$ there is a complete Boolean Algebra of power λ satisfying the c.c.c. with no non-trivial one-to-one endomorphism
- (b) if we waive “complete” then is such B of any power $\lambda > \aleph_0$
- (c) in any $\lambda > \aleph_0$ there is a Bonnet-rigid Boolean Algebra.

4) To solve some problems from that list, unconnected to 1) and 2):

- (a) if a Boolean Algebra B has no dense subset of power $< \lambda$ then it has a λ pairwise incomparable element
- (b) (\diamond_{\aleph_1}) there is a Boolean Algebra of power \aleph_1 satisfying many rigidity conditions: Bonnet rigid endo-rigid indecomposable.

The idea of this proof is to generalize Keisler criterion for omitting types (specific to atomless Boolean Algebras) to the case an ultrafilter is given in each countable stage of the construction. 1 Continued in [Shear].

(B137) The Singular cardinals problem: independence results, London Math. Soc. Lecture Note Ser. 87(1983) 116-134.

Assuming the consistency of a supercompact cardinal, we prove the consistency of:

- \aleph_ω strong limit $2^{\aleph_\alpha} = \aleph_{\alpha+1}$, $\alpha < \omega_1$ arbitrary;
- \aleph_{ω_1} strong limit, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$, $\alpha < \omega_2$ arbitrary;
- \aleph_δ strong limit, $\text{cf}(\delta) = \aleph_0$, 2^{\aleph_δ} arbitrarily large before the first inaccessible cardinal; for \aleph_δ “large enough”.

We start with κ supercompact, e.g. an ω -sequence $\langle \kappa_n : n < \omega \rangle$ through it and collapse to make κ to \aleph_ω . The new point is that we change our mind on what is \mathbf{V} — it becomes larger when we determine κ_n for bigger n .

(B138) On the structure of $\text{Ext}(A, \mathbb{Z})$ in ZFC, JSL, 50(1985), 302-315 (with G. Sageev).

We prove:

- (GCH) for any $\aleph_0 < \kappa < \aleph_\omega$, and cardinals $\nu_p \leq \kappa^+$ (for $0 < p < \omega$ prime) there is a κ -free abelian group of power κ , $\nu_0(G) = \kappa^+$, $\nu_p(G) = \nu_p$ (see [HHS78]).

(B139) On the number of non-conjugate subgroups, Algebra Universalis, 16(1983), 131-146.

Let G a given group of cardinality λ ($\lambda > \aleph_0$) and $\text{nc}(G)$ the number of non-conjugate subgroups of G , so we are interested in lower bound of the number of equivalence relations.

We prove for many uncountable cardinals λ that if $|G| = \lambda$ then $\text{nc}(G) \geq \lambda$, (e.g. when GCH holds).

This is proved for $\aleph_0 < \lambda < 2^{\aleph_0}$ (for $\lambda = \aleph_1$ assuming $2^{\aleph_0} = \aleph_1$ by [She80] this is best possible); for λ such that $\forall \mu[\lambda \neq 2^\mu]$. For $\lambda = 2^\mu$ and μ not regular non-strong limit and $\text{Ded}(\mu) > \lambda$. It is an open problem if the theorem is true for other cardinals. Continued in [She87b].

(B140) On endo-rigid \aleph_1 -strongly free abelian groups of power \aleph_1 , IJM, 40(1981), 291-295.

We prove that if $2^{\aleph_0} < 2^{\aleph_1}$ then there is an abelian group of power \aleph_1 , which is strongly \aleph_1 -free but endo-rigid where:

- G is endo-rigid iff every homomorphism $h : G \rightarrow G$ has the form $h(x) = n(x)$ for every x ; (for some integer n).

(B141) The monadic theory of ω_2 , JSL 48(1983), 387-398 (with Y. Gurevich and M. Magidor).

Assume consistency of ZFC plus existence of a weakly compact cardinal. Then

- (i) for every $S \subseteq \omega$, ZFC plus “ S and the monadic theory of ω_2 are recursive in each other” is consistent, and
- (ii) ZFC plus “the full second-order theory of ω_2 is interpretable in the monadic theory of ω_2 ” is consistent.

(B142) The structure of saturated free algebras, Algebra Universalis, 17(1983), 191-199 (with J. Baldwin).

Let V be a variety with countable similarity type. Suppose M is an uncountable algebra in V which is both free and saturated.

Lemma 2.39. $\text{Th}(M)$ is \aleph_0 -stable.

Theorem 142.1. There exists a finite set $q_1, q_2, \dots, q_\kappa$ of types (over finite sets) each with weight one such that there exist infinite sets of indiscernibles Y_1, \dots, Y_n each Y_i based on some q_i such that M is generated as an algebra by $Y \cup \dots \cup Y_n$.

(B143) The monadic theory and the “next world”, IJM 49(1984), 55-68 (with Yuri Gurevich).

Let R be the real line in a model \mathbf{V} of ZFC and B be the Boolean Algebra of regular open subsets of R in \mathbf{V} . Let V^B be the corresponding Boolean valued model of ZFC.

Theorem 143.1. There is an algorithm interpreting the full second-order \mathbf{V}^B -theory of \aleph_0 in the monadic \mathbf{V} -theory of R .

Theorem 143.2. There is an algorithm interpreting the full second-order \mathbf{V}^B -theory of 2^{\aleph_0} in the monadic \mathbf{V} -theory of R if \mathbf{V} satisfies the Continuum Hypothesis.

Theorems 1 and 2 are results of applications of a general interpretation theorem.

Continued in [She88].

(B217) There are Noetherian domains in every cardinality with free additive groups, Abstracts Amer. Math. Soc. 7(1986), 369 (with G. Sageev).
(Based on Notices AMS 86T-03-269)

Theorem 2.40. *There are Noetherian rings (in fact domains) with a free additive group, in every infinite cardinality.*

Remark 2.41. 1) For \aleph_1 this was proved by O’Neill.
2) The work was done in Sept., ’83.
3) We thank Fuchs for suggesting to us the problem.

Proof. Sketch of Proof

Let \mathbb{Z} be the ring of integers, X a set of distinct variables, $\mathbb{Z}[X]$ the ring of polynomials over \mathbb{Z} , $\mathbb{Z}(X)$ its field of quotients, and R_X the additive subgroup of $\mathbb{Z}(X)$ generated by

$$\{p/q : p, q \in \mathbb{Z}[X], p \text{ not divisible (non-trivially) by any integer}\} \subseteq \mathbb{Z}(X).$$

It is known that R_X is a Noetherian domain. Let, for a ring R , R^+ be its additive group. For $Y \subseteq X$ we can define $\mathbb{Z}[Y]$, $\mathbb{Z}(Y)$, R_Y similarly. \square

Lemma 2.42. 1) R_X^+ is a free abelian group.

2) If $n \geq 0$, $Y \subseteq X$, $x(1), \dots, x(n) \in X \setminus Y$ pairwise distinct, $W = \{x(1), \dots, x(n)\}$, $W(\ell) = W - \{x(\ell)\}$ then $R_{W \cup Y}^+ / \sum_{\ell=1}^n R_{W(\ell) \cup Y}^+$ is a free abelian group.

Proof. 1) Follows by 2) for $n = 0, Y = X$.

2) This is phrased because it is the natural way to prove 1) by induction on $|Y|$, for all n simultaneously (a degenerated case of [She83a]). If $|Y| > \aleph_0$, let $Y = \{y(\alpha) : \alpha < \lambda\}$ with no repetitions, so $\lambda = |Y|$, $Y_\alpha = \{y(i) : i < \alpha\}$. It suffices for each $\alpha < \lambda$ to prove that $G_\alpha =: (R_{Y_{\alpha+1}}^+ + \sum_{\ell=1}^n R_{Y \cup W(\ell)}^+) / (\sum_{\ell=1}^n R_{Y_{\alpha+1} \cup W(\ell)}^+ + R_{Y_\alpha \cup W}^+)$ is free.

We now show that G_α is isomorphic to $G'_\alpha = R_{Y_{\alpha+1}}^+ / (\sum_{\ell=1}^n R_{Y_{\alpha+1} \cup W(\ell)}^+ + R_{Y_\alpha \cup W}^+)$.

For this it is enough to show $(\sum_{\ell=1}^n R_{Y_\alpha \cup W(\ell)}^+) \cap R_{Y_{\alpha+1}}^+ = \sum_{\ell=1}^n R_{Y_{\alpha+1} \cup W(\ell)}^+$, as the right side is included in the left side trivially we have to show

$$\sum_{\ell=1}^n \frac{p_\ell}{q_\ell} \in \sum_{\ell=1}^n R_{Y_{\alpha+1} \cup W(\ell)}^+$$

if $\frac{p_\ell}{q_\ell} \in R_{Y_\alpha \cup W(\ell)}^+$ and $\sum_{\ell=1}^n \frac{p_\ell}{q_\ell} \in R_{Y_{\alpha+1} \cup W}^+$ which is easy by projections). But G'_α is free by induction hypothesis. \square

The next claim completes the case “ y countable”.

Claim 2.43. *If $Y \cup \{x(1), \dots, x(n)\} \subseteq X$, $x(\ell) \in X \setminus Y$ distinct, $G = R_{W \cup Y}^+$, $I = \sum_i I_i$, $I_i = R_{W(i) \cup Y}^+$ then G/I is free, when Y is countable.*

Proof. It suffices to prove:

- (a) G/I is torsion free
 (b) if $a_1, \dots, a_k \in G/I$ are independent, then

$$\left\{ m \in \mathbb{Z}^+ : \text{there are } \langle q_1, \dots, q_k \rangle \in L \text{ such that } m \text{ divides } \sum_{i=1}^k q_i a_i \text{ in } G/I \right\}$$

is finite, where

$$L = \{ \langle q_1, \dots, q_k \rangle : q_i \in \mathbb{Z}, \text{ not all zero with no common divisor} \}.$$

Let $x_1(q) \in X$ for $q = 1, \dots, n$ be new distinct variable and let $V = \{x_1(1), \dots, x_1(n)\}$. For $u \subseteq \{1, \dots, n\}$ let us define $h_u : R_{V \cup W \cup Y} \rightarrow R_{V \cup Y}$ an isomorphism $h_u(y) = y$ for $y \in Y$, $h_u(x(q)) = x_1(q)$ if $q \in u$, $h_u(x(q)) = x(q)$ if $q \notin u$. So let $a_1 + I, \dots, a_k + I$ be independent.

Suppose $\langle q_1, \dots, q_k \rangle \in L$, $m_0 m_1 \in \mathbb{Z} \setminus \{0\}$, $m_0 m_1$ divides $\sum_i m_0 q_i a_i + I$. So for some $s \in R_{W \cup Y}$ and $p_\ell \in I_\ell$ for $\ell = 1, \dots, n$ we have: $\sum_i m_0 q_i a_i = m_0 m_1 s + \sum_{\ell=1, \dots, n} p_\ell$.

Let u vary on subsets of $\{1, \dots, n\}$, $b_u = \sum_u (-1)^{|u|} h_u(a_\ell) \in R_{V \cup W \cup Y}$, so

$$\sum_i m_0 q_i b_i = \sum_u \sum_i m_0 q_i (h_u(a_i)) = m_0 m_1 \sum_u h_u(s) + \sum_{\ell=1, \dots, n} \sum_u h_u(p_\ell).$$

However for each $\ell = 1, \dots, n$ we have $\sum_u h_u(p_\ell)$ is zero (as $x(\ell)$ does not appear in it).

So $\sum_i m_0 q_i b_i$ is divisible by $m_0 m_1$ in $R_{V \cup W \cup Y}^+$. As $R_{V \cup W \cup Y}^+$ is free, it suffices to prove $\{b_i : i = 1, \dots, k\}$ is independent, equivalently they are linearly independent (over the rationals) in $Z(Y \cup W \cup V)$. But, if not, we can substitute suitable numbers for $x_1(1), \dots, x_1(n)$ and get contradiction to “ $\{a_i + I : i = 1, \dots, n\}$ is independent.”

That is let R' be a subring of $R_{V \cup W \cup Y}$ generated by $\mathbb{Z}[X] \cup \{\frac{1}{q_1}, \dots, \frac{1}{q_m}\}$ for some $m, q, \dots, q_\ell \in \mathbb{Z}[X]$ such that $h_u(a_i) \in R'$. Let g be a homomorphism from R' to $R_{W \cup Y}$ which is the identity on $R_{W \cup Y}$ and maps each $x_1(q)$ to an integer (so we require from $\langle g(x_i(q)) : q = 1, \dots, n \rangle$ to make some finitely many polynomials over the integers nonzero which is possible). Now $\ell \in u \subseteq \{1, \dots, n\} \Rightarrow h_u(a_i) \in I_\ell$. So it is enough to show that $\langle g(b_i) : i = 1, \dots, k \rangle$ is linearly independent. But $g(b_i) = \sum_u g(h_u(a_i)) \in g h_\emptyset(a_i) + I = g(a_i) + I = a_i + I$. \square

(C1) On Shelah compactness of cardinal, IJM, 31(1978), pgs.34-56 and pg.394 (with S. Ben-David)

We deal with the compactness property of cardinals presented by Shelah, who proved a compactness theorem for singular cardinals. We improve that result in eliminating axiom I there and show a new application of that theorem together with a straightforward proof of it for the special case discussed. We discuss compactness for regular cardinals and show some independence results: one of them, a part of which is due to A. Litman, is the independence from ZFC + GCH of the gap—one two cardinal problem for singular cardinals.

(C2) The Shelah P -point independence results, IJM, 43(1982), 28-48. [C2], (E. Wimmers).

Present the proof of some forcing extension there is no P -point. We start with $\mathbf{V} \models "2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} = \aleph_2^+."$ Consider a list $\langle D_\alpha : \alpha < \aleph_2 \rangle$ of the filter D on \mathbb{N} such that $\mathcal{P}(\mathbb{N})/D$ satisfies the c.c.c. (and every co-finite subset of \mathbb{N} belongs). We then force a close enough subset of the CS product of $\prod_{\alpha} \mathbb{Q}_{D_\alpha}$ for appropriate \mathbb{Q}_{D_α} .

(C3) The uniformization property for \aleph_2 , IJM, 36(1980), 248-256. [C3], (C. Steinhorn and J. King).

We deal with the following problem: let $S_1^2 = \{\delta < \aleph_2 : \text{cf}(\delta) = \aleph_1\}$, η_δ an ω_1 -sequence converging to δ , $\Phi = \langle \eta_\delta : \delta \in S_1^2 \rangle$. Assuming GCH, can (Φ, \aleph_1) have the uniformization property, at least for some Φ (see [She77])? The answer is positive, so not only $\text{GCH} \not\rightarrow \diamond_{S_1^2}$, but also GCH is consistent with S_1^2 being small.

We start with $V \models \text{GCH}$ and use iteration of length ω_3 , with support of power $\leq \aleph_1$, of the obvious forcings introducing a uniformizing functions each condition being an initial segment. The proof is like [She77] taking $N \prec (\mathcal{H}(\aleph_1), E)$, closed under countable sequences. But trying to build trees as there we run into two problems:

- (a) even if we get a tree of candidates for being a conditions, we cannot find \aleph_1 -closed N_1 with $N_1 \cap \omega_2 \notin S_1^2$ (if we replace S_1^2 by $S \subseteq S_1^2$, $S_1^2 \setminus S$ stationary, our life would be much easier).

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