ANALYTICAL GUIDE AND UPDATES FOR CARDINAL ARITHMETIC
E-12

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ABSTRACT. Part A: A revised version of the guide in [She94f], with corrections and expanded to include later works.
Part B: Corrections to [She94f].
Part C: Contains some revised proof and improved theorems.
Part D: Contains a list of relevant references.

Recent (July 2022) additions

- §14 = 14.1 on: no choice
- On inner models 13.8, see [GSS06]
- On Black Boxes and abelian groups 16.14 - 16, see [She11] and [She13b]
- On somewhat free scales 2.13, see [She13a],
- On n-dimensional Black Boxes, quite free abelian groups such that $\mathrm{Hom}(G, \mathbb{Z}) = \{0\}$, 16.18, see [She20],
- Survey on the existence of universal models; in particular, abelian groups 16.10, see [She21b].

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Notation -1.1. : $=^+$ appears in the following context: $\mu =^+ \sup\{\ldots\}$ means "both sides are equal, and if in the right side the sup is not obtained, then it is singular."

For a set C of ordinals, $\mathrm{acc}(C) = \{\alpha \in C : \alpha = \sup(\alpha \cap C)\}$, $\mathrm{nacc}(C) = C \setminus \mathrm{acc}(C)$.

The aim of this guide is to help the reader find out what is said in [She94f] and related works of the author, what are the theorems and definitions or where to look for them.

Let $[A]^\kappa = \{a \subseteq A : |a| = \kappa\}$, similarly $[A]^{<\kappa}$ and $[A]^{\leq\kappa}$. We denote $[A]^{\leq\kappa}$ also as $\mathcal{P}_{\leq\kappa}[A]$.

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Part A - Analytic guide

§ 0. $I[\lambda]$ and partial squares

See [She79], [She85a], [She94c, 2.3(5)], equivalent forms [She93a, 1.2], preservation of stationary subsets by μ-complete forcing [She79, 21], [She85a, 10].

Definition 0.1. Let $\lambda = \mathrm{cf}(\lambda) > \aleph_0$. For $S \subseteq \lambda$ we have: $S \in \check{I}[\lambda]$ iff for some club E of λ and $\langle C_\alpha : \alpha < \lambda \rangle$ we have: C_α is a closed subset of α, $\mathrm{otp}(C_\alpha) < \alpha$,

$$[\beta \in \mathrm{nacc}(C_\alpha) \Rightarrow C_\beta = \beta \cap C_\alpha] \text{ and } [\alpha \in E \cap S \Rightarrow \alpha = \sup(C_\alpha)]$$

(and every $\beta \in \mathrm{nacc}(C_\alpha)$ is a successor ordinal); note $E \cap S$ has no inaccessible cardinal as a member. Note that [She93a, 1.2] says that the definition just given is equivalent to those used in [She79], [She85a].

We can demand further $\alpha \in E \cap S \Rightarrow \mathrm{otp}(C_\alpha) = \mathrm{cf}(\alpha)$. But we can demand less: for each α we are given $< \lambda$ candidates for C_α, and for C a candidate for α and $\beta < \alpha, C \cap (\beta + 1)$ is a candidate for some $\gamma < \alpha$. $\check{I}[\lambda]$ is a normal ideal, and in many cases of the form "non-stationary ideal $+S$" (see [She79]; [She85a]).

Remark 0.2. $\check{I}[\lambda]$ is a normal ideal but many times it has the form $\{A \subseteq \lambda : A \cap S$ non-stationary$\}$ and then S is the "bad" set of λ. This holds for $\check{I}[\lambda] \restriction \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ if $\lambda = \lambda^{<\kappa}$ or less (see [She79], [She85a]).

Claim 0.3. *1) If λ is regular, then $S = S^{\lambda^+}_{<\lambda} = \{\delta < \lambda^+ : \mathrm{cf}(\delta) < \lambda\}$ is the union of λ sets on each of which we have a square (see below) hence belongs to $\check{I}[\lambda]$, see* [She91a, 4.1].
2) If $\lambda = \lambda^{<\kappa}$, then $\{\delta < \lambda^+ : \mathrm{cf}(\delta) < \kappa\}$ is the union of λ sets on each of which we have a square (see [She86b]*), hence the set belongs to $\check{I}[\lambda]$.*
3) Moreover, if $\lambda > \aleph_0$ is regular and $\alpha < \lambda \Rightarrow |\alpha|^{<\kappa} < \lambda$ or just $\alpha < \lambda \Rightarrow \mathrm{cov}(|\alpha|, \kappa, \kappa, 2) < \lambda$ then $\{\delta < \lambda : \mathrm{cf}(\delta) < \kappa\} \in \check{I}[\lambda]$ (see [She93a, 2.8]*, the case $\kappa = \aleph_0$ is trivial). By Dzamonja, Shelah* [DS95] *the same assumption gives $\{\delta < \lambda^+ : \mathrm{cf}(\delta) < \mathrm{cf}(\lambda)\}$ is the union of $\leq \lambda$ sets on each of which we have square. Also in* [DS95] *there are results on getting squares with λ singular and results with an inaccessible instead of λ^+.*

Definition 0.4. $S \subseteq \mu$ has a square if we have $S^+, S \subseteq S^+ \subseteq \mu$ and $\langle C_\alpha : \alpha \in S^+ \rangle$ such that: C_α is a closed subset of α of order type $< \alpha$, and $\alpha \in C_\beta \Rightarrow C_\alpha = \alpha \cap C_\beta$ and [α is a limit ordinal iff $\alpha = \sup(C_\alpha)$] for $\alpha \in S$; also if $\alpha \in S \Rightarrow \mathrm{cf}(\alpha) \leq \kappa (< \kappa)$, we can add "$\mathrm{otp}(C_\alpha) \leq \kappa (< \kappa)$".

Remark 0.5. Related ideals [She94c, 2.3,2.4] [She94a, 2.3,2.4,2.5,5.1,5.1A,5.2].

Claim 0.6. *If $\kappa^+ < \lambda = \mathrm{cf}(\lambda)$, then we can find a stationary*

$$S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \mathrm{cf}(\kappa)\},\ S \in \check{I}[\lambda]$$

[She93a, 1.5] *(somewhat more* [She93a, 1.4]*).*

Note 0.7. Negative consistency results: [She79], ("GCH + the bad set for $\aleph_{\omega+1}$ is stationary") Magidor, Shelah [MS94], Hajnal, Juhasz, Shelah [HJS86], consistency of $\check{I}[\lambda]$ large but stationary sets reflect [She91a].

Note 0.8. On killing stationary sets by forcing [She79], [She85a, 18,19], [She94a, 2.4].

Note 0.9. On consequences of pcf structure ([She79], [She94f, Ch.VIII,§5?], [She00a, 5.17,5.18]), e.g. (GCH) the bad stationary subsets of $\aleph_{\omega+1}$ do not reflect ([She79] or [She85a]).

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§ 1. Guessing clubs

Definition 1.1. Definition of ideals [She94l, 1.3,1.5,3.1]: definition of $g\ell$ [She94l, 2.1]: also [She94k, 1.8].

For example

Definition 1.2. For $\bar{C} = \langle C_\delta : \delta \in S\rangle$, $S \subseteq \lambda = \mathrm{cf}(\lambda) > \aleph_0$, C_δ a club of δ:

$$\mathrm{id}^b(\bar{C}) = \{A \subseteq \lambda : \text{ for some club } E \text{ of } \lambda, \text{ for no } \delta \in S \cap A \cap E, \text{ is } C_\delta \subseteq E\}$$

$$\begin{aligned}\mathrm{id}^a(\bar{C}) = \{A \subseteq \lambda : \quad & \text{for some club } E \text{ of } \lambda, \text{ for no } \delta \in S \cap A \cap E, \\ & \text{is } \sup(C_\delta \setminus E) < \sup C_\delta\}\end{aligned}$$

$$\begin{aligned}\mathrm{id}_p(\bar{C}) = \big\{A \subseteq \lambda : \quad & \text{for some club } E \text{ of } \lambda, \text{ for no } \delta \in S \cap A \cap E, \\ & \text{is } \delta = \sup\big(E \cap \text{ nacc}(C_\delta)\big)\big\}.\end{aligned}$$

Note 1.3. 1) Easy facts [She94l, 1.4,1.6].
2) For $\lambda, S \subseteq \lambda$ stationary, concerning the existence of $\bar{C} = \langle C_\delta : \delta \in S\rangle$ "guessing clubs of λ" [She94l, §2] (and [Shear, Ch.III,7.8A-G]).

Claim 1.4. *The following items give sufficient conditions for the properness of the above ideals for λ regular uncountable:*

- (a) *If $\delta \in S \Rightarrow \mathrm{cf}(\delta) < \mu$ for some $\mu < \lambda$, then we can find clubs C_δ for $\delta \in S$ such that $\mathrm{id}^b(\langle C_\delta : \delta \in S\rangle)$ is a proper ideal (i.e. for every club E of λ for some $\delta, C_\delta \subseteq E$) by* [She94l, 2.3(2)].
- (b) *If $\lambda = \mu^+, \mu$ regular, then*
  - (α) *There is a sequence $\langle (S_\varepsilon, \bar{C}_\varepsilon) : \varepsilon < \lambda\rangle$ such that:*
    - $\bullet_1$ *each S_ε is a subset of λ and*
      $$\bigcup\{S_\varepsilon : \varepsilon < \lambda^+\} = \{\delta < \lambda^+ : \mathrm{cf}(\delta) < \lambda\}$$
    - $\bullet_2$ *for each $\varepsilon < \lambda$, $\bar{C}_\varepsilon$ has the form $\langle C_{\varepsilon,\alpha} : \alpha \in S_\varepsilon\rangle$ and is a partial square, which means that: $C_{\varepsilon,\alpha}$ is a closed subset of α, is unbounded if α is a limit ordinal, is included in S_ε and*
      $$\beta \in C_{\varepsilon,\alpha} \ \Rightarrow\ C_{\varepsilon,\beta} = C_{\varepsilon,\alpha} \cap \beta$$
    - $\bullet_3$ *also, $C_{\varepsilon,\alpha}$ is of cardinality $< \lambda$*
  - (β) *For every limit ordinal $\delta(*) < \lambda$, for some partial square $\bar{C} = \langle C_\alpha : \alpha \in S_*\rangle$ of λ we have $\alpha \in S_* \Rightarrow \mathrm{otp}(C_\alpha) \leq \delta(*)$, and letting*
    $$S^* = \{\alpha \in S_* : \mathrm{otp}(\alpha) = \delta(*)\}$$
    we have that $\bar{C} \restriction S^*$ guesses clubs; that is, $\varnothing \notin \mathrm{id}^a(\bar{C})$.
  - (γ) *For every limit ordinal $\delta(*) < \lambda^+$, for some club E of λ^+, an ordinal $\varepsilon < \lambda$ and and limit ordinal $\Upsilon < \lambda$ divisible by $\delta(*)$ with the same cofinality, the sequence $\bar{C}_{\varepsilon,E}$ is as required in (β) replacing $\delta(*)$ by Υ where (of course $\bar{C}_\varepsilon$ is from sub-clause (α))*
    $$\bar{C}_{\varepsilon,E} = \langle C_{\varepsilon,\alpha} \cap E : \alpha \in S_\varepsilon \cap \alpha\rangle$$

    Why? clause (β) follows from clause (γ), clause (γ) follows from clause (α) as usual (trying enough times).
    For clause (α) see [She94l, 2.14(2)] *but the proof there is inaccurate, see* [She91a, 4.4,pg.47] *or see part (B), 44, see part B here.*

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(c) *If $\lambda = \mu^+$, μ regular, $S \subseteq S^\lambda_\mu$ is stationary, then we can find $\bar C = \langle C_\delta : \delta \in S\rangle$, C_δ a club of δ, $\mathrm{otp}(C_\delta) = \mu$, $[\alpha \in \mathrm{nacc}(C_\delta) \Rightarrow \mathrm{cf}(\alpha) = \mu]$ and $\mathrm{id}_p(\bar C)$ a proper ideal (i.e. for every club E of λ for some δ, $\delta = \sup(E \cap \mathrm{nacc}(C_\delta))$),* [She94l, 2.3(1)], [She03], [She97a, §3].

(d) *If $[\lambda = \mu^+$, μ singular, and $\delta \in S \Rightarrow \mathrm{cf}(\delta) = \mathrm{cf}(\mu) > \aleph_0]$ or $[\lambda$ inaccessible and $\delta \in S \Rightarrow \mathrm{cf}(\delta) \in (\aleph_0, \delta)]$, then for some $\bar C = \langle C_\alpha : \alpha \in S\rangle$ we have: $\mathrm{id}^a(\bar C)$ is proper and for each $\delta \in S$ we have: $\langle \mathrm{cf}(\alpha) : \alpha \in \mathrm{nacc}(C_\delta)\rangle$ converges to $|\delta|$ (and is strictly increasing)* [She94l, 2.6,2.7].

(e) *If $S^* \subseteq \lambda$ is stationary and does not reflect outside itself and $S \subseteq \lambda$ is stationary, then for some $\bar C = \langle C_\delta : \delta \in S\rangle$ we have $\mathrm{nacc}(C_\delta) \subseteq S^*$, and $\mathrm{id}_p(\bar C)$ is a proper ideal,* [She94l, 2.13].

(f) *Similar theorems with ideals* [She94k, 1.7,2.4], [She03, 1.11,1.12] *other related ideals* [She94k, 1.10].

(g) *More in the places above and* [She03, 2.6,2.8,2.9] *and* [KS93].

(h) *Assume $\lambda = \mathrm{cf}(\lambda)$, $S \subseteq \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \lambda\}$ is stationary and χ satisfies one of the following: $\lambda = \chi^+$ or $\chi = \min\{\tau < \lambda : (\exists \theta \le \tau)\ \tau^\theta \ge \lambda\}$ or λ strongly inaccessible not Mahlo. then we can find $\langle C_\delta, h_\delta : \delta \in S\rangle$ such that: $C_\delta = \{\alpha_{\delta,\zeta} : \zeta < \lambda\}$ is a club of $\delta, \alpha_{\delta,\zeta}$ increasing with ζ, $h_\delta : C_\delta \to \chi$ and for every club E of λ^+, for stationarily many $\delta \in S$, for each $i < \chi$,*

$$\{\zeta < \lambda : \alpha_{\delta,\zeta} \in E,\ \alpha_{\delta,\zeta+1} \in E,\ \text{and } h_\delta(\alpha_{\delta,\zeta}) = i\}$$

is a stationary subset of λ (see [She03, §3], [She97a, §3]*). If λ is a limit of inaccessibles, we can demand cf$(\alpha_{\delta,\zeta+1}) > \zeta$.*

(i) *If $\lambda, \bar C = \langle C_\delta : \delta \in S^+\rangle$ is as in 0.1, $S \subseteq S^+, \sup\{|C_\alpha|^+ : \alpha \in S\} < \lambda$ then for some club E of λ, $\bar C' = \langle g\ell(C_\delta, E) : \delta \in S^+ \cap \mathrm{acc}(E)\rangle$ is as in 0.1 and for every club $E_1 \subseteq E$ of λ, for stationarily many $\delta \in S$, we have $\alpha \in C'_\delta \Rightarrow \sup(C'_\delta \cap \alpha) \le \sup(E \cap \alpha)$.*

(j) *Assume $\lambda = \mathrm{cf}(\lambda)$ and $S \subseteq S^{\lambda^+}_\lambda = \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \lambda\}$ is stationary. Then we can find an S-club system $\bar C = \langle C_\delta : \delta \in S\rangle$ and $h : S \to \lambda$ such that for any club E of λ^+, for stationarily many $\delta \in S$, for every $i < \lambda$, the set $\mathrm{nacc}(C_\delta) \cap h^{-1}(\{i\})$ is unbounded in δ (under reasonable assumption $\big|\{C_\delta \cap \alpha : \alpha \in \mathrm{nacc}(C_\delta)\}\big| \le \lambda$), see* [She03, 3.3].

Note 1.5. On $\otimes_{\bar C}$, $\otimes^\kappa_{\bar C}$ for some S-club system [She94l, 2.12,2.12A,4.10] and a colouring theorem [She94l, 4.9] (see earlier [She88b]). Where λ is a Mahlo cardinal,

$\otimes_{\bar C}$ $\bar C$ has the form $\langle C_\delta : \delta \in S\rangle$, $S \subseteq \lambda$ a set of inaccessibles, C_δ a club of δ such that: for every club E of λ for stationary many $\delta \in S$, $E \cap \delta \setminus C_\delta$ is unbounded in δ

and for $\kappa < \lambda$:

$\otimes^\kappa_{\bar C}$ $\bar C$ has the form $\langle C_\delta : \delta \in S^\lambda_\in\rangle$, $S^\lambda_\in = \{\mu < \lambda : \mu \text{ inaccessible}\}$, such that: for every club E of λ, for stationarily many $\delta \in S^\lambda_\in \cap \mathrm{acc}(E)$, for no $\zeta < \kappa$ and $\alpha_\varepsilon \in S^\lambda_\in (\varepsilon < \zeta)$ is $\mathrm{nacc}(E) \cap \delta \setminus \bigcup_{\varepsilon<\zeta} C_{\alpha_\varepsilon}$ bounded in δ.

By [She94l, 4.9] if κ is a Mahlo cardinal and $\otimes^\kappa_{\bar C}$, then for some 2-place function c from κ to ω, for every pairwise disjoint $w_i \subseteq \kappa$, $|w_i| < \kappa$ for $i < \kappa$, and n, for some $i < j$, $\mathrm{Rang}(c \restriction w_i \times w_j) \subseteq (n, \omega)$. By [She94l, 4.10B], $\otimes^2_{\bar C} \Leftrightarrow \otimes^{\aleph_0}_{\bar C}$, also $\otimes^2_{\bar C}$ is a strengthened form of "κ not weakly compact", which fails under mild conditions ([She94l, 4.10A]). See more in [She94l, 4.13].

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Note 1.6. : $\mathrm{id}_p(\bar{C}, \bar{I})$ is decomposable [She94l, 3.2,3.3].

Note 1.7. If $\kappa^+ < \lambda$, we can find $\langle \mathscr{P}_\alpha : \alpha < \lambda \rangle$ such that:

(s) $\mathscr{P}_\alpha$ is a family of $< \lambda$ closed subsets of α,
(b) $\beta \in \mathrm{nacc}(C)$ and $C \in \mathscr{P}_\alpha \Rightarrow C \cap \beta \in \mathscr{P}_\beta$
(c) for every club E of λ for stationarily many $\alpha < \lambda$, there is $C \in \mathscr{P}_\alpha$, $\kappa = \mathrm{otp}(C)$, $\alpha = \sup(C)$ and $C \subseteq E$ [She93a, 1.3] (we can replace κ by $\delta(*)$, $|\delta(*)| = \kappa$).

Note 1.8. More on 1.4(c) in [She03, §3] and better in [She97a, §3].

Note 1.9. If we want to preserve

$$\alpha \in \mathrm{nacc}(C_\alpha) \cap \mathrm{nacc}(C_\beta) \cap \mathrm{nacc}(C_\gamma) \Rightarrow C_\beta \cap \alpha = C_\gamma \cap \alpha$$

we can weaken the guessing to: $\forall$ clubs E, $\exists^{\mathrm{stat}}\delta$ such that E is not disjoint to any interval of C_α. See the proof of [She96a, 6.2], [DS95].

Note 1.10. On ideals related to Jónsson algebras and guessing clubs: [She94k], [She03, §1] (used in §8 here).

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§ 2. Existence of lub

We discuss here lub of $\bar{f} = \langle f_\alpha : \alpha < \delta \rangle \mod I$, where $f_\alpha \in {}^\kappa\mathrm{Ord}$, I an ideal on κ, $\kappa^+ < \mathrm{cf}(\delta)$. See [She78], [She86a], [She88a, §14] and better [She94b, §1].

Definition 2.1. We say "f is a lub of $\langle f_\alpha : \alpha < \delta \rangle \mod I$" where I is an ideal on $\mathrm{Dom}(I)$, $f_\alpha : \mathrm{Dom}(I) \to$ ordinals, if $\bigwedge_{\alpha<\delta} f_\alpha \leq_I f$, and

$$\bigwedge_{\alpha<\delta} f_\alpha \leq f' \Rightarrow f \leq f' \mod I.$$

We say "f is an eub (exact upper bound) of $\langle f_\alpha : \alpha < \delta \rangle \mod I$" where I is an ideal on $\mathrm{Dom}(I)$, $f_\alpha : \mathrm{Dom}(I) \to$ ordinals, if $\bigwedge_{\alpha<\delta} f_\alpha \leq_I f$ and if $g <_I \max\{f, 1\}$ <u>then</u> for some $\alpha < \delta$ we have $g \leq_I f_\alpha$ (see [She94c, 1.4(4)]); usually $\alpha < \beta \Rightarrow f_\alpha \leq_I f_\beta$; "$f$ is an eub of $\langle f_\alpha : \alpha < \delta \rangle \mod I$" says more than "$f$ is a lub of $\langle f_\alpha : \alpha < \delta \rangle \mod I$".

Claim 2.2. *The trichotomy theorem on the existence of eub* [She94b, 1.2,1.6] *(slightly more* [She96a, 6.1]*, on eub $\neq$ lub, see example* [She96a, 6.1A]*).*

For example for I a maximal ideal on κ, $f_\alpha \in {}^\kappa\mathrm{Ord}$ for $\alpha < \delta$, $\mathrm{cf}(\delta) > \kappa^+$, $\bar{f} = \langle f_\alpha/I : \alpha < \delta \rangle$ increasing, <u>either</u> $\bar{f}$ has a $<_I$-eub, <u>or</u> for some sequence $\bar{w} = \langle w_i : i < \kappa \rangle$ of sets of ordinals, $|w_i| \leq \kappa$ we have:

$$\bigwedge_{\alpha<\delta} \bigvee_{\beta<\delta} \Big(\exists g \in \prod_{i<\kappa} w_i\Big)\big[f_\alpha/I < g/I < f_\beta/I\big].$$

The $\mathrm{cf}(\delta) > \kappa^+$ *is necessary by* [KS00].

Definition 2.3. [She94c, 2.6]. We define:

$$\begin{aligned}\mathrm{gd}_I(\bar{f}) =: \big\{\alpha < \delta : \ & \mathrm{cf}(\alpha) > \kappa, \text{ and there is an unbounded } A \subseteq \alpha \\ & \text{and members } s_i \text{ of } I \text{ for } i \in A \text{ such that:} \\ & i \in A \text{ and } j \in A \text{ and } i < j \text{ and} \\ & \zeta \in \kappa \setminus (s_i \cup s_j) \Rightarrow f_i(\zeta) \leq f_j(\zeta)\big\}.\end{aligned}$$

Sufficient conditions for the existence of eub [She94b, 1.7] is that $\mathrm{gd}_I(\bar{f})$ is a stationary subset of δ.

Definition 2.4. Let $\mathfrak{a}$ be a set of regular cardinals and $N \prec (\mathcal{H}(\lambda), \in)$: we define $\mathrm{Ch}^{\mathfrak{a}}_N(\theta) = \sup(N \cap \theta)$ for $\theta \in \mathfrak{a}$ [She94c, 3.5], [She94b, 3.4(stationary)], [She94a, 1.2,1.3,1.4] more [She94g, 3.3A,5.1A] and [She96a, §6].

Claim 2.5. *On the good/bad/chaotic division. For $\bar{f}$ a $<_I$-increasing sequence of functions from κ to ordinals, we have a natural division of $\{\delta < \ell g(\bar{f}), \mathrm{cf}(\delta) > \kappa^+\}$ to there:*

- $\bullet_1$ *to* $\mathrm{gd}_I(\bar{f})$ *(see 2.3 above),*
- $\bullet_2$ $\mathrm{ch}(\bar{f}) = \{\delta < \ell g(\bar{f})$: *for some ultrafilter D on $\ell g(\bar{f})$ disjoint to I and $w_i \subseteq$ ordinals for $i \in \mathrm{Dom}(I)$, $|w_i| \leq |\mathrm{Dom}(I)|$ and* $\bigwedge_{i<\delta} \bigvee_{j<\delta} (\exists g \in \prod w_i)[f_i \leq_D g \leq_D f_j]\}$ *and*
- $\bullet_3$ $\mathrm{bd}_I(\bar{f}) = \ell g(\bar{f}) \setminus (\mathrm{gd}_I(\bar{f}) \cup \mathrm{ch}_I(\bar{f}))$.

Note 2.6. In 2.5:

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(a) for every $\delta < \ell g(\bar{f})$ of uncountable cofinality there is a club C of δ such that $\delta \in \mathrm{gd}_I(\bar{f}) \wedge \alpha \in C \wedge \mathrm{cf}(\alpha) > \kappa \Rightarrow \alpha \in \mathrm{gd}_I(\bar{f})$ and $\delta \in \mathrm{ch}_I(\bar{f}) \Rightarrow C \subseteq \mathrm{ch}_I(\bar{f})$;

(b) for $\mathrm{bd}_I(\bar{f})$ to be non-trivial, $\ell g(\bar{f})$ should not be so small among the alephs.

There are connections to NPT (see §12) and $\check{I}[\mathrm{cf}(\ell g(\bar{f}))]$ (see §1) (and consistency of the existence of counterexamples; see [She79], [MS94], [She94b, 1.6], [She97b]).

Problem 2.7. Is the following consistent: $\{\delta < \aleph_{\omega+1} : \mathrm{cf}(\delta) = \aleph_2\} \notin \check{I}[\aleph_{\omega+1}]$ or $2^{\aleph_0} < \aleph_\omega$ and $\{\delta < \aleph_{\omega+1} : \mathrm{cf}(\delta) = (2^{\aleph_0})^+\} \notin \check{I}[\aleph_{\omega+1}]$ (also for inaccessibles) or $\bar{f} = \langle f_\alpha : \alpha < \aleph_{\omega+1}\rangle$, $f_\alpha \in \prod\limits_{n<\omega} \aleph_n$, $\mathrm{ch}_{J^{\mathrm{bd}}_\omega}(\bar{f}) \cap \{\delta < \aleph_{\omega+1} : \mathrm{cf}(\delta) = \aleph_2\}$ stationary or $(\forall S)[S \in \check{I}[\aleph_2]$ and $\bigwedge\limits_{\delta \in S} \mathrm{cf}(\delta) = \aleph_1 \Rightarrow S$ not stationary]?

Note 2.8. More on §2, see in §12 (in universes without full choice).

Note 2.9. See more in [She97f] for generalization to the case $\mathrm{cf}(\delta) \le |\mathrm{Dom} I|$. On existence of eub see [She97f, 3.10] and [She00a, 6.4].

Claim 2.10. *Assume $\lambda = \mathrm{cf}(\lambda) \ge \mu > 2^\kappa, f_\alpha \in {}^\kappa\mathrm{Ord}$ for $\alpha < \kappa$. then for some β^*_i $(i < \kappa)$ and $w \subseteq \kappa$ we have: $i \in w \Rightarrow \mathrm{cf}(\beta^*_i) > 2^\kappa$ and for every $f \in \prod\limits_{i \in w} \beta^*_i$ for unboundedly many $\alpha < \lambda$ we have $i \in w \Rightarrow f(i) < f_\alpha(i) < \beta^*_i$ and $i \in \kappa \setminus w \Rightarrow f_\alpha(i) = \beta^*_i$;* [She96a, 6.6D] *(slightly more general); more detailed proof* [She02, 6.1], *more variants* [She99, §7].

Note 2.11. On decreasing sequences see [She00a, 6.1,6.2]. See also [She06].

Claim 2.12. *The restriction in 2.5 to $\mathrm{cf}(\delta) > \kappa$ is necessary (by Kojman-Shelah* [KS00].

Claim 2.13. *See* [She13a] *for more; e.g. even if $\mathrm{pp}(\aleph_\omega) = \aleph_{\omega+1}$ then there is a $<_{J^{\mathrm{bd}}_\omega}$-increasing sequence $\langle f_\alpha : \alpha < \aleph_{\omega+1}\rangle$ of members of $\prod\limits_n \omega_n$ which is $(\aleph_\omega, \aleph_4)$-free; i.e. for every $\alpha < \aleph^+_\omega$ there is an equivalence relation E_α on each class of cardinality $< \aleph_4$ and $h : \alpha \to \omega$ such that if $\beta, \gamma < \alpha$ are not E_α-equivalent then*

$$\{f_\beta(n) : n \ge h(\beta)\} \cap \{f_\gamma(n) : n \ge h(\gamma)\} = \varnothing$$

(See more in [She13a], [She20].*)*

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§ 3. Uncountable cofinality and $\aleph_1$-complete filters and products: [She80a], [She86a], [She87]

Note 3.1. Assume $\langle \lambda_i : i \leq \kappa \rangle$ is an increasing continuous sequence of singulars, $\aleph_0 < \kappa = \mathrm{cf}(\kappa) < \lambda_0$. Let $\lambda = \lambda_\kappa$. If $\{i < \kappa : \mathrm{pp}(\lambda_i) = \lambda_i^+\}$ is a stationary subset of κ, <u>then</u> $\mathrm{pp}(\lambda) = \lambda^+$, [She94b, 2.4(1)].

Moreover, $\mathrm{pp}(\lambda_\kappa)$ is bounded by $\lambda_\kappa^{+\|h\|}$ where $\mathrm{pp}(\lambda_i) = \lambda_i^{+h(i)}$ hence we have a bound on $\mathrm{pp}(\lambda)$ in many cases [She94b, 2.4], [She94a, 1.10].

Note 3.2. Definition of various ranks and niceness of filters in [She94d, 1.1,1.2,1.4,3.12] (more generally on pair (t, D) or for $D \in \mathrm{Fil}(e, y)$ see [She93b, §5] and [She93a, §3,§4,§5]). For $\kappa = \mathrm{cf}(\kappa) > \aleph_0$, D a normal filter on κ and $f \in {}^\kappa\mathrm{Ord}$ let $\mathrm{rk}^2(f, D)$ be $\leq \alpha$ iff for every $A \in D^+$ and $g <_{D+A} f$ for some normal filter, $D_1 \supseteq D + A$ we have $\mathrm{rk}^2(g, D_1) \leq \beta$ for some $\beta < \alpha$. D is nice if $f \in {}^{\mathrm{Dom}(D)}\mathrm{Ord} \Rightarrow \mathrm{rk}^2(f, D) < \infty$.

Note 3.3. If for any $A \subseteq 2^{\aleph_1}$ in $K[A]$, there are Ramsey cardinals (or suitable Erdös cardinals which occurs if cardinal arithmetic is not trivial, essentially by Dodd and Jensen [DJ81]), <u>then</u> every normal filter on ω_1 is nice [She94d, 1.7,1.13]; more in [She94d, §1], [She93a, §3,§4,§5].

Note 3.4. [She94d, 2.2,2.2A,2.4,2.7], [She93a, §4] $A_e(f)$ [She94d, 3.3].

Note 3.5. Rank, basic properties:
[She94d, 2.3,2.4,2.8,2.9,2.10,2.11,2.12,2.14,2.21,3.4,3.8].

Note 3.6. Rank, connection to forcing:
[She94d, Definition 2.6 (E^t_p),2.6A,2.7A],[She93a, §3].

Note 3.7. Rank, relation with T_D
[She94d, 2.15,2.16,2.17,2.18,2.19,2.20,2.22].

Note 3.8. Ranks-going down: ranks when we divide ω_1, [She94d, 3.2] each f successor [She94d, 3.6], each f limit [She94d, 3.7].

Note 3.9. Rank, getting κ-like reduced products [She94d, 3.10,3.11,3.11A].

Note 3.10. Generic ultrapower with all $\kappa > \beth_2(\aleph_1)$ represented: [She94e, 1.3], just for one [She94e, 1.4] (earlier [She86a]).

Note 3.11. Ranks are $< \infty$ [She94d, 3.13-18].

Note 3.12. Preservative pairs (see 3.15), definition and basic properties [She94d, 4.15].

Note 3.13. Specific functions are preservative:
[She94d, 4.6] (H_s = successor), [She94d, 5.8] (H^{ia} = next inaccessible), [She94d, 5.9] ($H^{\epsilon-m}$ = next ϵ-Mahlo).

Note 3.14. The class of preservative pairs is closed under:

(A) $H^*(i)$ iterating H i times [She94d, 4.7,4.8,4.9]
(B) composition [She94d, 4.10]
(C) $\sup_{n<\omega} H^n$ [She94d, 4.11]
(D) iterating α times, $\alpha < \omega_1$ [She94d, 4.12]
(E) more [She94d, 4.13]
(F) induction [She94e, §2].

Note 3.15. Preservative pairs are bounds on cardinal exponentiation [She94d, 5.1,5.2,5.3].

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Note 3.16. If $\mathrm{rk}^2_E(f) = \mathrm{rk}^3_E(f) = \lambda$ inaccessible, then modulo (fil E) almost every $f(i)$ is inaccessible [She94d, 5.7].

Note 3.17. Generalizing normal filters and then ranks [She93b, §5], [She93a, §3,§4,§5].

Note 3.18. Combinatorial theorem using ranks, [She09b], if $\lambda > \mathrm{cf}(\lambda) > \aleph_0$ and $2^{\mathrm{cf}(\lambda)} < \lambda$ then $\lambda \rightarrow (\lambda, \omega + 1)^2$.

Note 3.19. For set theory with weak choice much remains (see [She97d], here §12).

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§ 4. Products, $T_D(f), \mathbf{U}$

We deal with computing $T_D(f), \mathbf{U}_D(f)$ and reduced products $\prod_{i<\kappa} f(i)/D$ from pcf, mainly when $(\forall i)[f(i) > 2^\kappa]$ see [She97f, §3], [She00a, §1], [She00a, §4] on T_D earlier, Galvin Hajnal [GH75].

Definition 4.1. 1) Define

$$T_D(f) =: \min\Big\{|\mathscr{F}| : \mathscr{F} \subseteq \prod_i (f(i)+1) \text{ and } f \neq g \in \mathscr{F} \Rightarrow f \neq_D g\Big\}$$

(i.e. $\{i : f(i) \neq g(i)\} \in D$) and $\mathscr{F}$ is maximal with respect to those properties.

$T_\Gamma(f) = \sup\{T_D(f) : D \in \Gamma\}$ for Γ set of filters on $\mathrm{Dom}(f)$, similarly for Γ set of ideals and naturally $T_\Gamma(\lambda)$.

2)

$$\begin{aligned}\mathbf{U}_D(f, <\theta) = \min\Big\{|\mathscr{A}| \ : \mathscr{A} \subseteq \prod [f(i)]^{<\theta},\ A \in \mathscr{A} \Rightarrow |A| < \theta,\\ \text{such that for every } g \in {}^\kappa\mathrm{Ord} \text{ with } g <_D f,\\ \text{for some } \bar{A} \in \mathscr{A} \text{ we have } \{i < \kappa : g(i) \in A_i\} \neq \varnothing \mod D\Big\}\end{aligned}$$

If $\theta = \kappa^+$ we may omit it, (note: if $\mathrm{cf}(\theta) > \kappa$ we can replace $\bar{A}$ by $\bigcup_{i<\kappa} A_i$.

[See more: [She96a], [She97c].

Note 4.2. If $\lambda > 2^{<\theta}, \theta \geq \sigma = \mathrm{cf}(\sigma) > \aleph_0$ and $\Gamma = \Gamma(\theta, \sigma)$ (the set of σ-complete ideals on a cardinal $< \theta$) we have

$$T_\Gamma(\lambda) = \mathrm{cov}(\lambda, \theta, \theta, \sigma)$$

(the latter can be computed from case of pp_Γ); [She94b, 5.9,p.94]. If $\theta^\kappa < \mathbf{U}_D(\lambda)$, <u>then</u> $T_D(f) = \mathbf{U}_D(f)$.

Note 4.3. A pcf characterization when $\lambda \leq T_D(f)$ holds, under $2^{\mathrm{Dom}(D)} < \min_i f(i)$ and $(\forall \alpha)(\alpha < \lambda \Rightarrow |\alpha|^{\aleph_0} < \lambda)$; see [She97f, 3.15]. (Note if $A_n \in D$, $\bigcap_{n<\omega} A_n = \varnothing$, then $T_D(f) = T_D(f)^{\aleph_0}$.)

See more in [She97f, §3].

Note 4.4. On sufficient conditions for $T_J(\bar{\lambda}) \geq \lambda$ and $T_J(\bar{\lambda}) = \lambda$, see [She06].

Note 4.5. Assume D is a filter on $\kappa, \mu = \mathrm{cf}(\mu) > 2^\kappa, f \in {}^\kappa\mathrm{Ord}$ and: D is $\aleph_1$-complete or $(\forall \sigma < \mu)(\sigma^{\aleph_0} < \mu)$. Then $(\exists A \in D^+) T_{D+A}(f) \geq \mu$ <u>iff</u> for some $A \in D^+$ and $\langle \lambda_i : i < \kappa \rangle = \bar{\lambda} \leq_{D+A} f$ we have $\prod_{i<\kappa} \lambda_i/(D+A)$ has true cofinality μ (for approximations see [She97f, §3], proof [She00a, 1.1], note $\Leftarrow$ is trivial). This is connected to the problem of the depth of products (e.g. ultraproducts) of Boolean Algebra.

Note 4.6. If $2^{2^\kappa} \leq \mu < T_D(\bar{\lambda})$ and $\mu^{<\theta} = \mu$, <u>then</u> for some θ-complete ideal $E \subseteq D$ we have $\mu < T_E(\bar{\lambda})$, [She97f, 3.20].

Note 4.7. On $\prod_{i<\kappa} \lambda_i/D$ see [She97f, 3.1-3.9B], essentially this gives full pcf characterization when it is $> 2^\kappa$. In particular for an ultrafilter D on κ with regularity θ (i.e. not θ-regular but σ-regular for $\sigma < \lambda$) and $\lambda_i > 2^\kappa$, we have

$$\prod_{i<\kappa} \lambda_i/D = \sup\Big(\mathrm{tcf} \prod_{i<\kappa} \{\lambda'_i/D : 2^\kappa < \lambda'_i = \mathrm{cf}(\lambda'_i) \leq \lambda_i\}\Big)^{<\mathrm{reg}(D)}$$

(see mainly [She97f, 3.9]).

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Note 4.8. Assume $\langle \lambda_i : i < \kappa \rangle$ tends to λ. A full characterization of $\prod\limits_{i<\kappa} \lambda_i/D = \lambda$ (via weak normal ultrafilters) appears in [GS12b].

Note 4.9. Assume D is an ultrafilter on κ and θ is the regularity of D (i.e. minimal θ such that D is not θ-regular). Then every $\lambda = \lambda^\theta > 2^\kappa$ can be represented as $\prod\limits_{i<\kappa} \lambda_i/D$. (Note $\lambda = \lambda^{<\theta}$ is necessary) (see [She00a, §6]).

Note 4.10. Assume $\theta < \kappa$, $J_* = [\kappa]^{<\theta}$, and $\lambda > \kappa^\theta$. Then

$$T_{J_*}(\lambda) = \sup\Big\{\text{tcf}\big(\prod_{\substack{n<n_i\\ i<\kappa}} \lambda_{i,n}/J\big) \ : \ n_i < \omega,\ \lambda_{i,n} \text{ regular} \in [\kappa^\theta, \lambda), \\ J \text{ is an ideal on } \bigcup_{i<\kappa} \{i\} \times n_i, \\ A \subseteq \kappa,\ |A| \geq \theta \Rightarrow \bigcup_{i \in A} \{i\} \times n_i \in J^+ \text{ and} \\ \prod_{\substack{n<n_i\\ i<\kappa}} \lambda_{i,n}/J \text{ has true cofinality}\Big\}.$$

This is just a case of the "θ-almost disjoint family $\subseteq [\lambda]^\kappa$" problem as clearly $T_J(\lambda) = \sup\big\{\mathscr{A} : \mathscr{A} \subseteq [\lambda]^\kappa$ is θ-almost disjoint; i.e. $A \neq B \in \mathscr{A} \Rightarrow |A \cap B| < \theta\big\}$.
See [She93b, §6].

Note 4.11. If $\lambda \geq \kappa > \beth_\omega(\theta)$ then in 4.9, $T_J(\lambda) = \lambda$.
(See [She00d]).

Note 4.12. ([She96a, 1.2]). Assume $\lambda > \mu = \text{cf}(\mu) > \theta > \aleph_0$ and $\text{cov}(\theta, \aleph_1, \aleph_1, 2) < \mu$. Then the following are equal

$$\lambda(0) = \min\big\{\kappa : \text{ if } \mathfrak{a} \subseteq \text{Reg} \cap \lambda \setminus \mu, |\mathfrak{a}| \leq \theta \text{ then we can partition } \mathfrak{a} \text{ to } \langle \mathfrak{a}_n : n < \omega \rangle \\ \text{such that } \mathfrak{b} \subseteq \mathfrak{a}_n \text{ and } |\mathfrak{b}| \leq \aleph_0 \Rightarrow \max \text{pcf}(\mathfrak{b}) \leq \kappa \\ \text{and } [\mathfrak{a}_n]^{\leq \aleph_0} \text{ is included in the ideal generated by} \\ \{\mathfrak{b}_\theta[\mathfrak{a}_n] : \theta \in \mathfrak{d}_n\} \text{ for some } \mathfrak{d}_n \subseteq \kappa^+ \cap \text{pcf}(\mathfrak{a}_n) \text{ of cardinality } < \mu\big\}$$

$$\lambda(1) = \min\big\{|\mathscr{P}| : \ \mathscr{P} \subseteq [\lambda]^{<\mu} \text{ and for every } A \in [\lambda]^{\leq\theta} \\ \text{for some partition } \langle A_n : n < \omega \rangle \text{ of } A \text{ we have:} \\ \langle \mathscr{P}_n : n < \omega \rangle,\ \mathscr{P}_n \subseteq \mathscr{P},\ |\mathscr{P}_n| < \mu,\ \mu > \underset{B \in \mathscr{P}_n}{\rightarrow} \sup(B) \\ \text{and } n < \omega \text{ and } a \in [A_n]^{\aleph_0} \Rightarrow (\exists A \in \mathscr{P}_n)[a \subseteq A]\big\}.$$

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§ 5. PCF THEORY: [She78], [She82, Ch.XIII,§5,§6], [She88a], [She90a]

$\mathfrak{a}$ denotes a set of regulars, $\min(\mathfrak{a}) > |\mathfrak{a}|$ (except for a generalization in [She94a, §3]).

For a partial order P let $\mathrm{cf}(P) = \min\{|A| : A \subseteq P, \bigwedge_{p\in P} \bigvee_{q\in A} p \le q\}$. We say that P has true cofinality if it has a well ordered cofinal subset whose cofinality is called $\mathrm{tcf}(P)$ (equivalently - a linearly ordered cofinal subset).

Note 5.1. $J_{<\lambda}[\mathfrak{a}], J_{\le\lambda}[\mathfrak{a}]$ [She94c, 1.2(2),(3)], also [She94c, 3.1], [She94a, 3.1]. For example we define:

$$J_{<\lambda}[\mathfrak{a}] =: \{\mathfrak{b} \subseteq \mathfrak{a} : \text{ for every ultrafilter } D \text{ on } \mathfrak{b}, \mathrm{cf}(\prod \mathfrak{b}/D) < \lambda\}.$$

$$J_{\lambda}[\mathfrak{a}] =: \{\mathfrak{b} \subseteq \mathfrak{a} : \text{ for every ultrafilter } D \text{ on } \mathfrak{b}, \mathrm{tcf}(\prod \mathfrak{b}/D) \ne \lambda\}.$$

Note 5.2. Definition of variants of pcf [She94c, 1.2(1),(2)], [She94a, 3.1] for example

$$\mathrm{pcf}(\mathfrak{a}) = \{\mathrm{cf}(\prod \mathfrak{a}/D) : D \text{ an ultrafilter on } \mathfrak{a}\}.$$

For given cardinals $\theta > \sigma$ let

$$\mathrm{pcf}_{\Gamma(\theta,\sigma)}(\mathfrak{a}) = \{\mathrm{tcf}(\prod \mathfrak{b}/J) : \begin{array}{l}\mathfrak{b} \subseteq \mathfrak{a}, |\mathfrak{b}| < \theta, J \text{ is a } \sigma\text{-complete ideal} \\ \text{on } \mathfrak{b} \text{ and } \prod \mathfrak{b}/J \text{ has true cofinality}\},\end{array}$$

$\Gamma(\theta)$ means $\Gamma(\theta^+, \theta)$.

Note 5.3. Trivial properties [She94c, 1.3,1.4].

Note 5.4. Basic properties [She94c, 1.5,1.8,2.6,2.8,2.10,2.12].

Note 5.5. $|\mathrm{pcf}(\mathfrak{a})| \le 2^{|\mathfrak{a}|}$ [She94c, 1.8(5)], $\mathrm{pcf}(\mathfrak{a})$ has a last member. [She94c, 1.9] Also, if $|\mathfrak{a} \cup \mathfrak{b}| < \min(\mathfrak{a} \cup \mathfrak{b})$ then $(\mathrm{pcf}(\mathfrak{a})) \cap (\mathrm{pcf}(\mathfrak{b}))$ has a last member; actually, $|\mathfrak{a}| < \min(\mathfrak{a})$, $|\mathfrak{b}| < \min(\mathfrak{b})$ suffices (by [She96a, 6.4A], we can take intersections of many $\mathfrak{a}_i$).

Note 5.6. If D, D_i $(i < \kappa < \min(\mathfrak{a}))$ are filters on $\mathfrak{a}$, E a filter on κ,

$$D = \{\mathfrak{b} \subseteq \mathfrak{a} : \{i : \mathfrak{b} \in D_i\} \in E\} \text{ and } \lambda_i = \mathrm{tcf}(\prod \mathfrak{a}/D_i)$$

(well defined) <u>then</u> $\mathrm{tcf}(\prod \mathfrak{a}/D_i)$, and $\mathrm{tcf}(\prod_{i<\kappa} \lambda_i/E)$ are equal [She94c, 1.10]. Moreover, $\bigwedge_i \kappa < \lambda_i$ is enough [She94c, 1.11].

(And see more in [She93b, 3.3,3.6], generalization [She97f, 1.10]).

Note 5.7. (Repeating 2.3) What is $\mathrm{Ch}^{\mathfrak{a}}_N$ (where $N \prec (\mathcal{H}(\chi), \in)$), $\mathrm{Ch}^{\mathfrak{a}}_N(\theta) = \sup(N \cap \theta)$ for $\theta \in \mathfrak{a}$) [She94c, 3.4], [She94b, 3.4], "stationary $F \subseteq \prod \mathfrak{a}$" [She94a, 1.2,1.3,1.4], more in [She96a, §6].

Note 5.8. $\mathrm{cf}(\prod \mathfrak{a}) = \max \mathrm{pcf}(\mathfrak{a})$ [She94b, 3.1,more 3.2], [She94c, 3.4], other representation [She97f].

Note 5.9. There is a generating sequence $\langle \mathfrak{b}_\theta[\mathfrak{a}] : \theta \in \mathrm{pcf}(\mathfrak{a})\rangle$; i.e. $J_{\le\theta}[\mathfrak{a}] = J_{<\theta}[\mathfrak{a}] + \mathfrak{b}_\theta[\mathfrak{a}]$, so $J_{<\lambda}[\mathfrak{a}]$ is the ideal on $\mathfrak{a}$ generated by $\{\mathfrak{b}_\theta[\mathfrak{a}] : \theta < \lambda\}$ and $\prod \mathfrak{b}_\theta[\mathfrak{a}]/J_{<\theta}[\mathfrak{a}]$ has true cofinality θ and $J_\lambda[\mathfrak{a}]$ is the ideal on $\mathfrak{a}$ generated by $\{\mathfrak{b}_\theta[\mathfrak{a}] : \theta < \lambda\} \cup \{\mathfrak{a} \setminus \mathfrak{b}_\lambda[\mathfrak{a}]\}$; [She94a, 2.6] also [She94c, 3.1] + [She93a, §1], more in [She94g, 4.1A]; nice good cofinal $\bar{f}$: [She94c, §3], [She94b, 3.4A], [She94a, 1.2,1.3,1.4], [She94g, 4.1A(2)]. Another representation is included in [She97f] (see 2.8 on the framework and 5.23 below); it uses 0.6 from [She93a].

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Note 5.10. If $\mathfrak{b} \subseteq \mathfrak{a}$ and $\mathfrak{c} = \mathrm{pcf}(\mathfrak{b})$, then for some finite $\mathfrak{d} \subseteq \mathfrak{c}, \mathfrak{b} \subseteq \bigcup_{\theta \in \mathfrak{d}} \mathfrak{b}_\theta[\mathfrak{a}]$ see [She94c, 3.2(5)].

Note 5.11. Cofinality sequence [She94c, 3.3], [She94a, 2.1], more in the proof of [She94g, 4.1].

Note 5.12. $\bar{f}$ is x-continuous (nice) [She94c, 3.3,3.5,3.8(1),(2)].

Note 5.13. For a discussion of when $\mathfrak{a}$ has a generating sequence which is smooth and/or closed [She94c, 3.6,3.8(3)], [She94g, 4.1A(4)]; *smooth* means $\mu \in \mathfrak{b}_\lambda[\mathfrak{a}] \Rightarrow \mathfrak{b}_\mu[\mathfrak{a}] \subseteq \mathfrak{b}_\lambda[\mathfrak{a}]$, *closed* means $\mathrm{pcf}(\mathfrak{b}_\lambda[\mathfrak{a}]) = \mathfrak{b}_\lambda[\mathfrak{a}]$. If for example $|\mathrm{pcf}(\mathfrak{a})| < \min(\mathfrak{a})$ we can have both [She94c, 3.8] and more, then we can use the "pcf calculus" style of proof. Proofs in this style can generally be carried further but become a little complicated, as done in [She96a, 6.7-6.7E] (particularly [She96a, 6.7C(3)].) On a generalization, see [She97f].

Note 5.14. If $\lambda = \max \mathrm{pcf}(\mathfrak{a})$, and $\mu =: \sup(\lambda \cap \mathrm{pcf}(\mathfrak{a}))$ is singular, then for $\mathfrak{c} \subseteq \mathrm{pcf}(\mathfrak{a})$ unbounded in μ, $\mathrm{tcf}(\prod \mathfrak{c}/J^{\mathrm{bd}}_\mu) = \lambda$ [She94c, 3.7], [She94a, 2.10(2)], where for A a set of ordinals, $J^{\mathrm{bd}}_A = \{B \subseteq A : \sup(B) < \sup(A)\}$.

Note 5.15. If $\lambda \in \mathrm{pcf}(\mathfrak{a})$, then for some $\mathfrak{b} \subseteq \mathfrak{a}$ we have: $\lambda = \max \mathrm{pcf}(\mathfrak{b})$ and $\lambda \cap \mathrm{pcf}(\mathfrak{b})$ has no last element and $\lambda \notin \mathrm{pcf}(\mathfrak{a} \setminus \mathfrak{b})$; see [She94a, ,2.10(1)].

Note 5.16. If $(\forall \mu)[\mu < \lambda \Rightarrow \mu^{<\kappa} < \lambda]$, then $J_{<\lambda}[\mathfrak{a}]$ is κ-complete [She94a, 1.6(1)].

Note 5.17. Localization: if $\lambda \in \mathrm{pcf}(\mathfrak{b})$, $\mathfrak{b} \subseteq \mathrm{pcf}(\mathfrak{a})$ (and we assume just $|\mathfrak{b}| < \min(\mathfrak{b})$), then, for some $\mathfrak{c} \subseteq \mathfrak{b}$ we have $|\mathfrak{c}| \leq |\mathfrak{a}|$ and $\lambda \in \mathrm{pcf}(\mathfrak{c})$, [She94a, 3.4].

Also if $\lambda \in \mathrm{pcf}_{\sigma\text{-complete}}(\mathfrak{b})$, $\mathfrak{b} \subseteq \mathrm{pcf}(\mathfrak{a})$ then for some $\mathfrak{c} \subseteq \mathfrak{b}$, we have $|\mathfrak{c}| \leq |\mathfrak{a}|$ and $\lambda \in \mathrm{pcf}_{\sigma\text{-complete}}(\mathfrak{c})$; see [She96a, 6.7F(4),(5)].

Note 5.18.

(*a*) $\mathrm{pcf}(\mathfrak{a})$ cannot contain an interval of Reg (= the class of regulars) of cardinality $|\mathfrak{a}|^{+4}$.

In fact:

(*b*) for no $\mathfrak{a}$ and χ is $\{i < |\mathfrak{a}|^{+4} : \chi^{+i+1} \in \mathrm{pcf}(\mathfrak{a})\}$ unbounded in $|\mathfrak{a}|^{+4}$.

[Why? If so, there is $\lambda \in \mathrm{pcf}((\chi, \chi^{+|\mathfrak{a}|^{+4}}) \cap \mathrm{pcf}(\mathfrak{a}))$ such that $\lambda > \chi^{|\mathfrak{a}|^{+4}}$, hence by localization for some $\mathfrak{c} \subseteq (\chi, \chi^{+|\mathfrak{a}|^{+4}}) \cap \mathrm{pcf}(\mathfrak{a})$ of cardinality $\leq |\mathfrak{a}|$ we have $\lambda \in \mathrm{pcf}(\mathfrak{c})$, hence for some limit ordinal $\delta < |\mathfrak{a}|^{+4}$, $\mathrm{pp}_{|\mathfrak{a}|}(\chi^{+\delta}) \geq \lambda \geq \chi^{+|\mathfrak{a}|^{+4}}$ and we get a contradiction by [She94g, §4].]

Note 5.19. Defining (μ, θ, σ)-inaccessibility [She93b, 3.1,3.2].

Note 5.20. On $\mathrm{pcf}(\mathfrak{b})$ for $\mathfrak{b} \subseteq \mathrm{pcf}(\mathfrak{a}), |\mathfrak{b}| < \min(\mathfrak{b})$ or even with no inaccessible accumulation points, see [She94c, 1.12], [She94a, §3], mainly: having $\mathfrak{b}^*_\lambda[\mathfrak{a}] \subseteq \mathrm{pcf}[\mathfrak{a}]$.

Note 5.21. Uniqueness of $\bar{f}$ ($<_J$-increasing cofinal) [She94c, 2.7,2.10].

Note 5.22. If $J = J_{<\lambda}[\mathfrak{a}]$, $\lambda = \mathrm{tcf}(\prod \mathfrak{a}/J)$, $\mathfrak{a} = \bigcup_{i<\alpha} \mathfrak{a}_i$, then for some finite $\mathfrak{b}_i \subseteq \mathrm{pcf}(\mathfrak{a}_i)$ (with $i < \alpha$) we have $\lambda = \max \mathrm{pcf}(\bigcup_{i<\alpha} \mathfrak{b}_i)$, and for $w \subseteq \alpha$ we have

$$\max \mathrm{pcf}\Big(\bigcup_{i \in w} \mathfrak{b}_i\Big) < \lambda \Leftrightarrow \Big(\bigcup_{i \in w} \mathfrak{a}_i\Big) \in J$$

[She94a, §1], more in [She96a, §6].

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Note 5.23. If $\bar\lambda = \langle \lambda_i : i < \kappa\rangle, I^*$ a weakly θ-saturated ideal on κ (see below) $\theta = \mathrm{cf}(\theta) < \lambda_i$, <u>then</u> the pcf analysis, e.g. from 5.9 holds for $\bar\lambda$ when we restrict ourselves to ideals on κ extending I^* (see [She97f, §1,§2]).

E.g. θ can play the role of $\kappa = \mathrm{Dom}(I)$ if I is weakly θ-saturated, i.e.

$(*)_{I,\theta}$ there is no division of κ to θ sets none of which is in I.

Note 5.24. If $|\mathfrak{a}| < \min(\mathfrak{a}), \aleph_0 \le \sigma = \mathrm{cf}(\sigma)$, then for some $\alpha < \sigma$ and $\lambda_\beta, \mathfrak{a}_\beta\ (\beta < \alpha)$ we have

(i) $\mathfrak{a} = \bigcup_{\beta<\alpha} \mathfrak{a}_\beta$,
(ii) $\lambda_\beta = \max\mathrm{pcf}(\mathfrak{a}_\beta)$
(iii) $\lambda_\beta \notin \mathrm{pcf}(\mathfrak{a} \setminus \mathfrak{a}_\beta)$ and
(iv) $\lambda_\beta \in \mathrm{pcf}_{\sigma-\mathrm{com}}(\mathfrak{a}_\beta)$.

[Why? We prove this by induction on $\max\mathrm{pcf}(\mathfrak{a})$, hence by the induction hypothesis we can ignore (iii) as we can regain it. Now let

$$J = \Big\{\mathfrak{b} \subseteq \mathfrak{a} : \begin{array}{l}\text{we can find } \alpha < \sigma,\ \langle \mathfrak{a}_\beta : \beta < \alpha\rangle \\ \text{such that } \mathfrak{b} = \bigcup_{\beta<\alpha} \mathfrak{a}_\beta \text{ and (ii), (iv) above}\end{array}\Big\}.$$

Clearly J is a family of subsets of $\mathfrak{a}$, includes the singletons, and is closed under subsets and under unions of $< \sigma$ members. If $\mathfrak{a} \in J$ we are done. If not, choose $\mathfrak{c} \subseteq \mathfrak{a}$ such that $\mathfrak{c} \notin J$ and (under these restrictions) $\lambda_\mathfrak{c} =: \max\mathrm{pcf}(\mathfrak{c})$ is minimal. Now by the minimality of $\lambda_\mathfrak{c}$, $J_{<\lambda_\mathfrak{c}}[\mathfrak{a}] \subseteq J$, so $\mathfrak{b}_{\lambda_\mathfrak{c}}[\mathfrak{a}]$ satisfies the requirement for $\mathfrak{b} \in J$ (with $\alpha = 1$). Contradiction].

Note 5.25. See more in 7.18 and [She97d] and particularly [She02].

Note 5.26. If $\lambda = \max\mathrm{pcf}(\mathfrak{b})$ and $\lambda \cap \mathrm{pcf}(\mathfrak{b})$ has no last element (see 5.15) and $\mu < \sup(\lambda \cap \mathrm{pcf}(\mathfrak{b}))$, <u>then</u> for some $\mathfrak{c} \subseteq \mathrm{pcf}(\mathfrak{b}) \setminus \mu$ of cardinality $\le |\mathfrak{b}|$ we have $\lambda = \max\mathrm{pcf}(\mathfrak{c})$ and $\theta \in \mathfrak{c} \Rightarrow \max\mathrm{pcf}(\theta \cap \mathfrak{c}) < \theta$ (see [She03, 2.4A,2.4(2)], an ex.).

Note 5.27. Above, the demand $|\mathfrak{a}| < \min(\mathfrak{a})$ was essential, but:

- If $\mathfrak{a} \subseteq \mathrm{pcf}(\mathfrak{b})$, $|\mathfrak{b}| < \min(\mathfrak{b})$ <u>then</u>, almost always, the pcf theorem holds for $\mathfrak{a}$, fully by [Shea].
- In the other direction, see 5.23.

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§ 6. Representation and pp

Note 6.1. Definition of pp and variants [She94b, 1.1]. For λ singular

$$\begin{aligned}\mathrm{pp}_\theta(\lambda) = \sup\{\mathrm{tcf}(\textstyle\prod \mathfrak{a}/J) : \ & \mathfrak{a} \text{ is a set of } \leq \theta \text{ regular cardinals,}\\ & \text{unbounded in } \lambda,\ J \text{ an ideal on } \mathfrak{a} \text{ including } J^{\mathrm{bd}}_{\mathfrak{a}}\\ & \text{and } \textstyle\prod \mathfrak{a}/J \text{ has true cofinality}\},\end{aligned}$$

$$\mathrm{pp}(\lambda) = \mathrm{pp}_{\mathrm{cf}(\lambda)}(\lambda),$$

$$\mathrm{pp}^+(\lambda) \text{ is the first regular without such a representation}$$

$$\mathrm{pp}_\Gamma(\lambda) \text{ means that we restrict ourselves to } J \text{ satisfying } \Gamma,$$

$$\mathrm{pp}^*_I(\lambda) = \mathrm{pp}_{\{I\}}(\lambda)$$

and

$$\mathrm{pp}_I(\lambda) = \sup\{\mathrm{pp}^*_J(\lambda) : J \text{ an ideal extending } I\},$$

$\lambda =^+$ $\mathrm{pp}(\lambda)$ means more than equality; the supremum in the right hand side is obtained if it is regular.

Note 6.2. Downward closure:
If $\lambda = \mathrm{tcf}(\prod_{i<\kappa} \lambda_i/I)$, $\lambda_i = \mathrm{cf}(\lambda_i) > \kappa$, and $\kappa < \lambda' = \mathrm{cf}(\lambda') < \lambda$, <u>then</u> for some λ'_i we have $\kappa \leq \lambda'_i = \mathrm{cf}(\lambda'_i) < \lambda_i$ and $\lambda' = \mathrm{tcf}(\prod_{i<\kappa} \lambda'_i/I)$. Moreover,

$$\mathrm{tlim}_I\, \lambda_i = \mu < \lambda' < \lambda \Rightarrow \mathrm{tlim}_I\, \lambda'_i = \mu$$

$\lambda' = \mathrm{tcf}(\prod \lambda'_i, \leq_I)$ is exemplified by μ^+-free $\bar{f}$, which means: if $w \subseteq \lambda'$ and $|w| \leq \mu$, then for some $\langle s_\alpha : \alpha \in w\rangle$, $s_\alpha \in I$ and for each $i < \kappa$, $\langle f_\alpha(i) : \alpha \in w, i \notin s_\alpha\rangle$ is without repetition; in fact, we get "strictly increasing." [She94b, 1.3,1.4,2.3] more [She94g, 4.1] a generalization [She97f, 3.12].

Note 6.3. If $\lambda > \kappa \geq \mathrm{cf}(\lambda)$, I an ideal on κ, κ is an increasing union of $\mathrm{cf}(\lambda)$ members of I, <u>then</u> $\{\mathrm{tcf}(\prod_{i<\kappa} \lambda_i/I) : \mathrm{tlim}_I\, \lambda_i = \lambda \text{ and } \lambda_i = \mathrm{cf}(\lambda_i)\}$ is an initial segment of $\mathrm{Reg} \setminus \lambda$, so the first member is λ^+, [She94b, 1.5,2.3].

Note 6.4. If $\lambda > \mathrm{cf}(\lambda) > \aleph_0$, <u>then</u> for some increasing continuous $\langle \lambda_i : i < \mathrm{cf}(\lambda)\rangle$ with limit λ, $\prod_{i<\mathrm{cf}(\lambda)} \lambda_i^+/J^{\mathrm{bd}}_{\mathrm{cf}(\lambda)}$ has true cofinality λ^+, [She94b, 2.1].

Note 6.5. $\mathrm{pp}(\lambda) > \lambda^+$ contradicts "large cardinal" type assumptions, for example "every μ-free abelian group is free" [She94b, 2.2,2.2B], for the parallel fact on cov see [She94b, 6.6].

Note 6.6.

(*a*) (inverse monotonicity) If $\mu > \lambda > \kappa \geq \mathrm{cf}(\lambda) + \mathrm{cf}(\mu)$ and $\mathrm{pp}^+_\kappa(\lambda) > \mu$, <u>then</u> $\mathrm{pp}^+_\kappa(\lambda) \geq \mathrm{pp}^+_\kappa(\mu)$

(*b*) so given $\kappa_0 \leq \kappa_1 < \mu$ if λ is minimal such that $\lambda > \kappa_1 \geq \kappa_0 \geq \mathrm{cf}(\lambda)$, $\mathrm{pp}(\lambda) \geq \mu$, <u>then</u>: $\mathfrak{a} \subseteq \mathrm{Reg} \cap [\kappa_1, \lambda)$, $|\mathfrak{a}| \leq \kappa_0$, $\sup(\mathfrak{a}) < \lambda$ implies $\max \mathrm{pcf}(\mathfrak{a}) < \lambda$; equivalently, $\lambda' \in (\kappa_1, \lambda)$ and $\mathrm{cf}(\lambda') \leq \kappa_0 \Rightarrow \mathrm{pp}_{\kappa_1}(\lambda') < \lambda$ [She94b, 2.3] (with more)

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(c) assume $\kappa \le \chi < \mu$, and

$$(\forall \lambda)[\lambda \in (\chi, \mu) \text{ and } \mathrm{cf}(\lambda) \le \kappa \Rightarrow \mathrm{pp}(\lambda) < \mu]$$

then for every $\mathfrak{a} \subseteq (\chi, \mu)$ of cardinality $\le \kappa$, $\sup(\mathfrak{a}) < \mu$ we have $\max \mathrm{pcf}(\mathfrak{a}) < \mu$ [by (d) below and 6.9 below]

(d) $\max \mathrm{pcf}(\mathfrak{a}) \le \sup\{\mathrm{pp}_{|\mathfrak{a} \cap \mu|}(\mu) : \mu \notin \mathfrak{a},\ \mu = \sup(\mathfrak{a} \cap \mu)\}$ [by the definition].

Similar assertion holds for $\mathrm{pp}_\Gamma, \Gamma$ is "nice" enough.

Note 6.7.

(A) If λ is singular, $\mu < \lambda$, then for some $\delta \le \mathrm{cf}(\lambda)$ and increasing sequence $\langle \lambda_i : i < \delta \rangle$ of regular cardinals in (μ, λ) and $\delta = \mathrm{cf}(\delta) \vee \delta < \omega_1$ we have: $\lambda_i > \max \mathrm{pcf}\{\lambda_j : j < i\}$ and $\lambda^+ = \mathrm{tcf}(\prod \lambda_i / J^{\mathrm{bd}}_\delta)$, [She94b, 3.3]

(B) If λ is singular and $\aleph_0 < \mathrm{cf}(\lambda) = \kappa$ and $\bigwedge_{\mu < \lambda} \mu^\kappa < \lambda$ and $\lambda < \theta = \mathrm{cf}(\theta) \le \lambda^\kappa$, then for some increasing sequence $\langle \lambda_i : i < \kappa \rangle$ of regulars $< \lambda, \lambda = \sum_{i<\kappa} \lambda_i$ and $\prod_{i<\kappa} \lambda_i / J^{\mathrm{bd}}_\kappa$ has true cofinality θ (see [She94a, 1.6(2)]). Moreover, we can demand $i < \kappa \Rightarrow \max \mathrm{pcf}\{\lambda_j : j < i\} < \lambda_i$. We can weaken the hypothesis to $\aleph_0 < \kappa = \mathrm{cf}(\lambda) < \lambda_0 < \lambda$ and $(\forall \mu)[\lambda_0 < \mu < \lambda$ and $\mathrm{cf}(\mu) \le \kappa \Rightarrow \mathrm{pp}(\mu) < \lambda]$ (see [She94a, 1.6(2)]. If we allow $\mathrm{cf}(\lambda) = \kappa = \aleph_0$ we still get this, but for possibly larger J, see [She96a, 6.5].

Note 6.8. $\mathrm{pp}_{\Gamma(\theta,\sigma)}$ can be reduced to finitely many $\mathrm{pp}_{\Gamma(\theta)}$, see [She94b, 5.8].

Note 6.9. If $\mu > \theta \ge \mathrm{cf}(\mu)$ and for every large enough $\mu' < \mu$:

$$[\mathrm{cf}(\mu') \le \theta \Rightarrow \mathrm{pp}_\theta(\mu') < \mu]$$

then

$$\mathrm{pp}(\mu) =^+ \mathrm{pp}_\theta(\mu) =^+ \mathrm{pp}_{\Gamma(\mathrm{cf}(\mu))}(\mu)$$

[She94a, 1.6(3)(5),1.6(2)(4)(6),1.6A].

Note 6.10. If $\langle \mathfrak{b}_\zeta : \zeta < \kappa \rangle$ is increasing, $\lambda \in \mathrm{pcf}(\bigcup_{\zeta<\kappa} \mathfrak{b}_\zeta) \setminus \bigcup_{\zeta<\kappa} \mathrm{pcf}(\mathfrak{b}_\zeta)$, then :

(A) for some $\mathfrak{c} \subseteq \bigcup_\zeta \mathrm{pcf}(\mathfrak{b}_\zeta), |\mathfrak{c}| \le \kappa$, we have $\lambda \in \mathrm{pcf}(\mathfrak{c})$

(B) if $\kappa = \mathrm{cf}(\kappa) > \aleph_0$, then for some club $C \subseteq \kappa$ and $\lambda_\zeta \in \mathrm{pcf}(\bigcup_{\xi<\zeta} \mathfrak{b}_\xi)$ for $\zeta \in C$, we have $\lambda = \prod_{\zeta \in C} \lambda_\zeta / J^{\mathrm{bd}}_\kappa$, λ_ζ $(\zeta \in C)$ is increasing and

$$\zeta \in C \Rightarrow \lambda_{\zeta+1} > \max \mathrm{pcf}\{\lambda_\xi : \xi \in C \text{ and } \xi \le \zeta\}$$

[She94a, 1.5].

Note 6.11. If $\mu > \theta \ge \mathrm{cf}(\mu) \ge \sigma = \mathrm{cf}(\sigma)$, and for every large enough $\mu' < \mu$:

$$[\sigma \le \mathrm{cf}(\mu') \le \theta \Rightarrow \mathrm{pp}_{\Gamma(\theta^+,\sigma)}(\mu') < \mu]$$

then

$$\mathrm{pp}_{\Gamma(\mathrm{cf}(\mu)^+,\sigma)}(\mu) = \mathrm{pp}_{\Gamma(\mathrm{cf}(\mu))}(\mu)$$

[She93b, 1.2] and more there.

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Note 6.12. : If $\mu > \kappa = \mathrm{cf}(\mu) > \aleph_0$ and for every large enough $\mu' < \mu$

$$(\mu')^\kappa < \mu \text{ or just } [\mathrm{cf}(\mu') \le \mathrm{cf}(\mu) \Rightarrow \mathrm{pp}_\kappa(\mu') < \mu],$$

then $\mathrm{pp}^+(\mu) = \mathrm{pp}^+_{J^{\mathrm{bd}}_\kappa}(\mu)$ and we can get the conclusion in 6.7(B) above [She94a, 1.8]. Generalization for $\Gamma(\theta,\sigma)$ in [She93b, 1.2].

Note 6.13. : If $\lambda > \kappa = \mathrm{cf}(\lambda) > \aleph_0, \lambda > \theta$ then for some increasing continuous sequence $\langle \lambda_i : i < \kappa \rangle$ with limit λ:

(*a*) for every $i < \kappa, \lambda_i < \mu < \lambda_{i+1}$ and $\mathrm{cf}(\mu) \le \theta \Rightarrow \mathrm{pp}_\theta(\mu) < \lambda_{i+1}$

or

(*b*) for every $i < \kappa, \mathrm{pp}_{\theta+\mathrm{cf}(i)}(\lambda_i) \ge \lambda$ [She94a, 1.9; more 1.9A].

Note 6.14. If $\sigma \le \mathrm{cf}(\mu) \le \theta < \kappa < \mu$ then:

$$\mathrm{pp}_\theta(\mu) < \mu^{+\theta^+} \Rightarrow \mathrm{pp}_\kappa(\mu) = \mathrm{pp}_\theta(\mu);$$

and

$$\mathrm{pp}_{\Gamma(\theta^+,\sigma)}(\mu) < \mu^{+\theta^+} \Rightarrow \mathrm{pp}_{\Gamma(\kappa^+,\sigma)}(\mu) = \mathrm{pp}_\theta(\mu)$$

[She94a, 3.6; more 3.7,3.8].

Note 6.15. If $\langle \mu_i : i \le \kappa \rangle$ is increasing continuous, $\mu_0 > \kappa^{\aleph_0} > \kappa = \mathrm{cf}(\kappa) > \aleph_0$ and $\mathrm{cov}(\mu_i, \mu_i, \kappa^+, 2) < \mu_{i+1}$, then for some club E of κ we have

$$\delta \in E \cup \{\kappa\} \Rightarrow \mathrm{pp}_{J^{\mathrm{bd}}_{\mathrm{cf}(\delta)}}(\mu_\delta) = \mathrm{cov}(\mu_\delta, \mu_\delta, \kappa^+, 2)$$

so e.g. for most limit $\delta < \omega_1$, $\mathrm{pp}_{J^{\mathrm{bd}}_\omega}(\beth_\delta) =^+ \beth_{\delta+1}$ (see [Sheb, part C, remark to X, §5, p.412].

Note 6.16. If $\mathrm{pp}^+_\sigma(\mu) > \lambda = \mathrm{cf}(\lambda)$, (so $\mathrm{cf}(\mu) \le \sigma$) then

(*a*) for some $\mathfrak{a}$ an unbounded subset of μ, $|\mathfrak{a}| \le \sigma$, $\lambda = \mathrm{tcf}(\prod \mathfrak{a}/J^{\mathrm{bd}}_{\mathfrak{a}}) = \max \mathrm{pcf}(\mathfrak{a})$ or

(*b*) for some $\mathfrak{a} \subseteq (\mu, \lambda)$ of cardinality $\le \sigma$, $\lambda = \max\mathrm{pcf}(\mathfrak{a})$ and $\theta \in \mathfrak{a} \Rightarrow \max\mathrm{pcf}(\theta \cap \mathfrak{a}) < \theta$ (see [She03, 2.4A,2.4(2)]).

Note 6.17. For singular μ we like to have, for $\lambda \in (\mu, \mathrm{pp}^+(\mu)) \cap \mathrm{Reg}$, an increasing sequence $\bar\lambda$ of regulars of length $\mathrm{cf}(\mu)$ converging to μ such that $\lambda = \mathrm{tcf}(\prod \bar\lambda, <_{J^{\mathrm{bd}}_{\mathrm{cf}\mu}})$. We had such results for $\lambda = \mu^+$ and when $\alpha < \mu \Rightarrow |\alpha|^{\mathrm{cf}(\mu)} < \mu$; see [She94a, §1] (and more in [She94a]).

$\otimes_1$ If $\lambda > \kappa \ge \mathrm{cf}(\lambda) > \aleph_0$, and $\mu < \lambda$ of cofinality $\le \kappa$ (satisfying $\mathrm{pp}_\kappa(\mu) < \lambda$) is large enough, then
  (a) for every regular $\theta \in (\lambda, \mathrm{pp}^+_\kappa(\mu))$ there is an increasing sequence $\langle \lambda_i : i < \mathrm{cf}(\lambda) \rangle$ of regular cardinals $< \lambda$, with limit λ, such that θ is the true cofinality of $\prod \lambda_i / J^{\mathrm{bd}}_{\mathrm{cf}\lambda}$. This means that
  (b) $\mathrm{pp}_\kappa(\lambda) = \mathrm{pp}(\lambda) = \mathrm{pp}^*_{J^{\mathrm{bd}}_{\mathrm{cf}\lambda}}(\lambda)$

Note 6.18. Now from a paper by Gitik and the author [GS13], we quote directly:

(A) Assume that $\kappa > \aleph_0$ is a weakly compact cardinal. Let $\mu > 2^\kappa$ be a singular cardinal of cofinality κ. Then for every regular $\lambda < \mathrm{pp}^+_{\Gamma(\kappa)}(\mu)$ there is an increasing sequence $\langle \lambda_i : i < \kappa \rangle$ of regular cardinals converging to μ such that $\lambda = \mathrm{tcf}(\prod_{i<\kappa} \lambda_i, <_{J^{\mathrm{bd}}_\kappa})$.

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(B) Let μ be a strong limit cardinal and θ a cardinal above μ. Suppose that at least one of them has an uncountable cofinality. Then there is $\sigma_* < \mu$ such that for every $\chi < \theta$ the following holds:

$$\theta > \sup\left\{ \sup \mathrm{pcf}_{\sigma_*\text{-complete}}(\mathfrak{a}) : \mathfrak{a} \subseteq \mathrm{Reg} \cap (\mu^+, \chi),\ |\mathfrak{a}| < \mu \right\}.$$

As an application we show that:

(C) if κ is a measurable cardinal and $\mathbf{j} : \mathbf{V} \to \mathbf{M}$ is the elementary embedding by a κ-complete ultrafilter over κ, then for every τ the following holds:
  (a) if $\mathbf{j}(\tau)$ is a cardinal then $\mathbf{j}(\tau) = \tau$;
  (b) $|\mathbf{j}(\tau)| = |\mathbf{j}(\mathbf{j}(\tau))|$;
  (c) for any κ-complete ultrafilter W on κ, $|\mathbf{j}(\tau)| = |\mathbf{j}_W(\tau)|$.

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§ 7. Covering number

Definition 7.1. [She94b, 5.1]

$$\mathrm{cov}(\lambda,\mu,\theta,\sigma)=\min\big\{|\mathscr{P}| \;:\; \mathscr{P}\subseteq[\lambda]^{<\mu} \text{ such that } \forall a\subseteq\lambda \text{ with } |a|<\theta, \; \exists\alpha<\sigma \text{ and } \exists A_i\in\mathscr{P}\ (i<\alpha) \text{ such that } a\subseteq\bigcup_{i<\alpha}A_i\big\}.$$

So $\mathrm{cov}(\lambda,\kappa^+,\kappa^+,2)=\mathrm{cf}([\lambda]^{\le\kappa},\subseteq)$.

Note 7.2. Basic properties [She94b, 5.2,5.3] see also [She94b, 3.6]; for example if $\lambda>\theta>\mathrm{cf}(\lambda)\ge\sigma$, then for some $\mu<\lambda$ we have

$$\mathrm{cov}(\lambda,\lambda,\theta,\sigma)=\mathrm{cov}(\lambda,\mu,\theta,\sigma).$$

Note 7.3. cov and cardinal arithmetic and $T_\Gamma(\lambda)$ see e.g. [She94b, 5.10], [She94b, 5.6,5.7,5.9,5.10, Definition of T_Γ]. For example,

$$\lambda^\kappa=\mathrm{cov}(\lambda,\kappa^+,\kappa^+,2)+2^\kappa.$$

By this and 7.4, 7.5 below we shall use assumptions on cases of pp rather than conventional cardinal arithmetic.

Note 7.4. On cov = pp: if $\lambda\ge\mu\ge\theta>\sigma=\mathrm{cf}(\sigma)>\aleph_0$, $\lambda>\mu\vee\mathrm{cf}(\mu)\in[\sigma,\theta)$, then $\mathrm{cov}(\lambda,\mu,\theta,\sigma)=\sup\{\mathrm{pp}_{\Gamma(\theta,\sigma)}(\chi):\chi\in[\mu,\lambda],\mathrm{cf}(\chi)\in[\sigma,\theta)\}$, we have $=^+$ if $\mu=\theta$; [She94b, 5.4].

Assuming for simplicity $\lambda=\mu$, if $=^+$ fails, then for some $\mathfrak{a}\subseteq\mathrm{Reg}\cap\mu$ we have $|\mathfrak{a}|<\mu,\sup(\mathfrak{a})=\mu$ and

$$\mathrm{cov}(\lambda,\mu,\theta,\sigma)=\sup\{\mathrm{tcf}(\textstyle\prod\mathfrak{b}/J): \mathfrak{b}\subseteq\mathfrak{a},|\mathfrak{b}|<\theta,\mu=\sup(\mathfrak{b}), J \text{ is an ideal on } \mathfrak{b} \text{ extending } J^{\mathrm{bd}}_{\mathfrak{b}}\};$$

see [She02, 6.12].

Note 7.5. The parallel of 7.4 for $\sigma=\aleph_0$ "usually holds", i.e.:

(*a*) for λ singular, $\mathrm{cov}(\lambda,\lambda,\mathrm{cf}(\lambda)^+,2)=\mathrm{pp}(\lambda)$ if for every singular $\chi<\lambda$, $\mathrm{pp}(\chi)=\chi^+$; [She94g, §1] (and weaker assumptions and intermediate stages there)

(*b*) if $\mathrm{cf}(\lambda)=\aleph_0$, $\bigwedge_{\mu<\lambda}\mu^{\aleph_0}<\lambda$ and $\mathrm{pp}(\lambda)<\mathrm{cov}(\lambda,\lambda,\aleph_1,2)$, then

$$\{\mu:\lambda<\mu=\aleph_\mu<\mathrm{pp}(\lambda)\}$$

is uncountable [She94g, 5.9], more in [She96a, 6.4]. If λ is a strong limit, then the set has cardinality $>\lambda$;

(*c*) few exceptions: if $\langle\lambda_i:i\le\kappa\rangle$ is increasing continuous and $\kappa=\mathrm{cf}(\kappa)>\aleph_0$, $\bigwedge_{i<\kappa}\mathrm{cov}(\lambda_i,\lambda_i,\kappa^+,2)<\lambda_\kappa$, then for some club C of κ, $\delta\in C\cup\{\kappa\}$ implies equality, i.e. $\mathrm{cov}(\lambda_\delta,\lambda_\delta,\aleph_1,2)=\mathrm{pp}(\lambda_\delta)$. [She94g, 5.10]

(*d*) For example, for a club of $\delta<\omega_1$, $2^{\beth_\delta}$ suffices for the parallel of 6.4, [She94g, 5.13]

(*e*) if on λ there is a $\aleph_1$-saturated λ-complete ideal extending J^{bd}_λ (for example, λ a real valued measurable) then $\mathrm{cov}(\lambda,\aleph_1,\aleph_1,2)\le\lambda$ [She96a, §3] and more there

(*f*) in clause (c), if $\kappa^{\aleph_0}<\lambda_0$ we can add $\mathrm{pp}(\lambda_\delta)=^+\mathrm{pp}_{J^{\mathrm{bd}}_\omega}(\lambda_\delta)$; of course, there $\mathrm{cov}(\lambda_\delta,\lambda_\delta,\aleph_1,2)=\mathrm{cov}(\lambda_\delta,\lambda_\delta,\kappa^+,2)$.

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Note 7.6. cov = minimal cardinality of a stationary S [She94b, 3.6,5.12], [She94g, 3.6,3.8,3.8A,5.11,5.2A], [She94f, Ch.VII,§1,§4], [She93b, 2.6(using 2.2),3.7], finally [She93a, 3.6]; for example

$$\mathrm{cf}(\mathscr{S}_{\leq\kappa}(\lambda),\subseteq) = \min\{|S| : S \subseteq [\lambda]^{\leq\kappa} \text{ is stationary}\}.$$

Moreover, we got a measure one set of this cardinality for an appropriate filter; for another filter see [She00c].

Note 7.7. Covering by normal filters (prc), [She94a, §4], [She93b, §1], generalization [She93b, §5], essentially [She96a, proof of 4.2 second case]. To quote [She93b, §1].

Note 7.8. On $\mathrm{cf}_J(\prod \mathfrak{a}, <_I)$, a generalization, see [She94g, 3.1].

Note 7.9. Computing $\mathrm{cf}^\sigma_{<\theta}(\prod \mathfrak{a})$ [She94g, 3.2]; computing from it $\mathrm{pp}(\lambda)$ for non-fixed point λ by it [She94g, 3.3].

Note 7.10. cov is $\mathrm{cf}^\sigma_{<\theta}(\prod(\mathrm{Reg}\cap\lambda), <_{J^{\mathrm{bd}}_\lambda})$ is $\mathrm{pp}_{\Gamma(\theta,\sigma)}$, when $\mathrm{cf}(\sigma) > \aleph_0$ [She94g, 3.3,3.4,3.5].

Note 7.11. $\mathrm{cov}(\lambda,\lambda,\mathrm{cf}(\lambda)^+,2) =^+ \mathrm{pp}(\lambda)$ when λ is singular non-fixed point [She94g, 3.7(1), and more 3.7(1)-(5),3.8].

Note 7.12. Computing $\mathrm{cov}(\lambda,\theta,\theta,2)$ by using $\mathrm{cf}_{<\theta}$ when $\theta > \mathrm{cf}(\lambda) = \aleph_0$, see [She94g, 5.1,5.2,5.3,5.4,5.4A,5.5],restriction to subset of $\lambda \cap \mathrm{Reg}$ is [She94g, 5.5A]. See more in [She94g, 6.5]; here, in 5.7, 5.8.

Note 7.13. Finding a family $\mathscr{P}$ of subsets of λ covering many of the countable subsets of λ, for example, if $a \in [\lambda]^{\aleph_1}$ we can find $H : a \to \omega$ such that each countable subset of $H^{-1}(\{0,\dots,n\})$ is included in a member of $\mathscr{P}$. I.e. we characterize the minimal cardinality of such $\mathscr{P}$ by pcf [She93b, 2.1-2.4], [She96a, 1.2] more in [She02].

Note 7.14. Characterizing the existence of $\mathscr{P} \subseteq [\lambda]^{\aleph_1}$, $|\mathscr{P}| > \lambda$ with pairwise finite intersection [She93b, §6] more in [She96a, 1.2], [She02].

Note 7.15. If $\lambda \geq \mu > \sigma = \mathrm{cf}(\sigma) > \aleph_0$, <u>then</u> $\{\mathrm{cov}(\lambda,\mu,\theta,\sigma) : \mu \geq \theta > \sigma\}$ is finite [GS93].

Note 7.16. Let $\lambda > \kappa > \aleph_0$ be regular, then: $\bigwedge_{\mu<\lambda} \mathrm{cov}(\mu,\kappa,\kappa,2) < \lambda$ <u>iff</u> for every $\mu < \lambda$ and $\langle a_\alpha : \alpha < \lambda\rangle$, $a_\alpha \subseteq \mu$, $|a_\alpha| < \kappa$ for some unbounded $s \subseteq \lambda$, $|\bigcup_{a\in s} a_\alpha| < \kappa$ (a problem of Rubin-Shelah [RS87], see [She94a, 6.1], [She96a, 3.1]). For λ successor of regular, a stronger theorem: see [She94a, §6]; more [She02, 6.13,6.14].

Note 7.17. If $\mu > \lambda \geq \kappa$, $\theta = \mathrm{cov}(\mu,\lambda^+,\lambda^+,\kappa)$ and $\mathrm{cov}(\lambda,\kappa,\kappa,2) \leq \mu$ (or at least $\leq \theta$), <u>then</u> $\mathrm{cov}(\mu,\lambda^+,\lambda^+,2) = \mathrm{cov}(\theta,\kappa,\kappa,2)$, [She96a, 2.1].

Note 7.18. 1) If $\lambda \geq \beth_\omega$, <u>then</u> for some $\kappa < \beth_\omega$, $\mathrm{cov}(\lambda,\beth^+_\omega,\beth^+_\omega,\kappa) = \lambda$, [She00d, 1.1]; any strong limit singular can serve instead of $\beth_\omega$.
2) For a singular limit cardinal μ (for example $\mu = \aleph_\omega$) sufficient conditions (for replacing $\beth_\omega$ by μ) are given in [She00d, 2.1,4.1]. For example such a condition is

$$(*)_{\kappa,\mu}\ \ \mathfrak{a} \subseteq \mathrm{Reg}\setminus\mu \wedge |\mathfrak{a}| < \mu \Rightarrow |\mathrm{pcf}_{\kappa-\mathrm{complete}}(\mathfrak{a})| < \mu.$$

3) So for every $\lambda \geq \beth_\omega$ for some n and $\mathscr{P} \subseteq [\lambda]^{<\beth_\omega}$ of cardinality λ, every $X \in [\lambda]^{\leq\beth_\omega}$ is the union of $\leq \beth_n$ sets from $\mathscr{P}$; ([She00d, 2.5]) and the inverse [She00d, 4.2] (see [She02]).

Also if the statement above holds for e.g. $\aleph_\omega$ then $(*)_{\aleph_n,\aleph_{\omega+1}}$ holds (by [She00d, 2.6]).

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§ 8. Bounds in cardinal arithmetic

Note 8.1. If $\langle \lambda_i : i \le \kappa \rangle$ is increasing continuous, J a normal ideal on κ and $\mathrm{pp}_J(\lambda_i) \le \lambda_i^{+h(i)}$, then $\mathrm{pp}_J(\lambda_\kappa) \le \lambda_\kappa^{\|h\|}$ [She94b, 2.4], [She94a, 1.10,1.11] where $\|h\|$ is Galvin Hajnal rank, i.e.

$$\|h\| = \sup\big\{\|f\| + 1 : f <_{D_\kappa} h\big\},$$

D_κ the club filter on κ.

Note 8.2. Let C_0 be the class of infinite cardinals and define by induction:

$$C_\zeta =: \{\lambda : \text{ for every } \xi < \zeta,\ \lambda \text{ is a fixed point of } C_\zeta\}$$

(i.e. $\lambda = \mathrm{otp}(C_\zeta \cap \lambda)$), then for example

$$\mathrm{pp}\big(\omega_1\text{-th member of } C_1 \setminus \beth_2(\aleph_1)\big) < \beth_2(\aleph_1)^+\text{-th member of } C_1 \setminus \beth_2(\aleph_1)$$

[She94d, 5.6].

Note 8.3. For $\zeta < \omega_1$ we have

$$\mathrm{pp}_{\mathrm{nor}}(\aleph^\zeta_\omega(\beth_2(\aleph_1))) < \aleph^\zeta_{(\beth_2(\aleph_1))^+}(\beth_2(\aleph_1))$$

and more on $\aleph^\zeta_\delta$, see [She94d, 5.4,5.5], where

$$\aleph^0_\alpha(\lambda) = \lambda^{+\alpha},\ \aleph^{\zeta+1}_0(\lambda) = \lambda,\ \aleph^{\zeta+1}_{\alpha+1}(\lambda) = \aleph^i_\zeta(\aleph_0)$$

where

$$\zeta = \aleph^{i+1}_\alpha(\lambda) + 1 \text{ and } \aleph^{\zeta+1}_\delta(\lambda) = \bigcup_{\alpha<\delta} \aleph^{i+1}_\alpha(\lambda),$$

and for i limit,

$$\aleph^i_0(\lambda) = \lambda,\ \aleph^i_{\alpha+1}(\lambda) = \bigcup_{j<i} \aleph^j_{\alpha+1}(\aleph^i_\alpha(\lambda)) \text{ and } \aleph^i_\delta(\lambda) = \bigcup_{\alpha<\delta} \aleph^i_\alpha(\lambda).$$

Note 8.4. If there are no [there are $\le \aleph_1$] inaccessibles below λ, $\lambda > 2^{\aleph_1}$, $\mathrm{cf}(\lambda) = \aleph_1$, then there are no [there are $\le 2^{\aleph_1}$] inaccessibles below $\mathrm{pp}(\lambda)$ [She94d, 5.10], similarly for Mahlo, ϵ-Mahlo.

Note 8.5. If $\bigwedge_{\delta<\omega_1} \mathrm{pp}(\aleph_\delta) < \aleph_{\omega_1}, \mathrm{pp}(\aleph_{\omega_1}) = \aleph_{\alpha^*}$, then there are $|\alpha^*|$ subsets of ω_1 with pairwise countable intersection [She94a, 1.7(1), more(2)] getting Kurepa trees [She94a, 2.8.2.9].

Note 8.6. The minimal counterexample to Tarski statement is simple, Jech-Shelah [JS91].

In [Tar25] Tarski showed that for every limit ordinal β, $\prod_{\xi<\beta} \aleph_\xi = \aleph_\beta^{|\beta|}$, and conjectured that

$$\prod_{\xi<\beta} \aleph_{\sigma_\xi} = \aleph_\alpha^{|\beta|}$$

holds for every ordinal β and every increasing sequence $\{\sigma_\xi\}_{\xi<\beta}$ such that $\lim_{\xi<\beta} \sigma_\xi = \aleph_\alpha$.

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Now: if a counterexample exists, then there exists one of length $\omega_1 + \omega$ (Jech and Shelah [JS91]).

Note 8.7. $\mathrm{pp}(\aleph_{\alpha+\delta}) < \aleph_{\alpha+|\delta|^{+4}}$ [She94g, 2.1,2.2, more 2.3-2.8].

Note 8.8. If $\delta < \aleph_4, \mathrm{cf}(\delta) = \aleph_0$ then $\mathrm{pp}(\aleph_\delta) < \aleph_{\omega_4}$. If $|\delta| + \mathrm{cf}(\delta)^{+3} < \kappa$, then $\mathrm{pp}(\aleph_{\alpha+\delta}) < \aleph_{\alpha+\kappa}$ [She94g, 4.2,4.3,4.4], more [She93b, 3.3-3.6].

Note 8.9. More on the number of inaccessibles: [She96a, §4].

E.g. [She96a, 4.4]. For transparency, assume that for no core model $\mathbf{K}[A]$, with A a set of ordinals, do we have covering (here the SCH holds). Then

(A) Assume $\mu > \mathrm{cf}(\mu) = \aleph_1$, $\mu_0 < \mu$,
$$\sigma \geq \big|\{\lambda \in (\mu_0, \mu) : \lambda \text{ inaccessible}\}\big| < \mu.$$
Then
$$\sigma^{+4} > \big|\{\lambda : \mu < \lambda < \mathrm{pp}_{\Gamma(\sigma,\aleph_1)}(\mu),\ \lambda \text{ inaccessible}\}\big|$$

(B) The parallel of [She94g, 4.3].

Note 8.10. By Gitik and Shelah [GS93]:

(*a*) If μ is a Jónsson limit cardinal not strong limit, then $\langle 2^\sigma : \sigma < \mu\rangle$ is eventually constant.

(*b*) If μ is a limit cardinal, $\mu_0 < \mu$ and $\bigwedge_{\theta \in (\mu_0,\mu)} \mu \to [\theta]^{<\omega}_{\theta,\mu_0}$, then $\langle 2^\theta : \mu_0 < \theta < \mu\rangle$ has finitely many values.

(*c*) If on μ there is a μ_0^+-saturated, uniform μ-complete ideal for example μ a real value measurable $\leq 2^{\aleph_0}$, then the assumption of (b) holds, hence its conclusion.

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§ 9. Jónsson algebras

Note 9.1. Definition and previously known results: [She94b, 4.3,4.4]. A Jónsson algebra is one with no proper subalgebra with the same cardinality. A Jónsson cardinal is λ such that there is no Jónsson algebra with countable vocabulary and cardinality λ.

Note 9.2. Definition of $\mathrm{id}_j(\bar{C})$, $\mathrm{id}^j_\theta(\bar{C})$ see [She94k, 1.8], $\mathrm{id}^j_J(\bar{C})$ see [She94k, 1.16] (also with k instead of j).

Note 9.3. Jónsson games: Definition [She94k, 2.1], connection to [She94k, 2.3] (for example $\lambda = \aleph_{\omega+1}$).

Note 9.4. λ^+ (for a singular λ) is not a Jónsson cardinal when:

- (a) λ is not an accumulation point of inaccessible Jónsson cardinals [She94b, 4.5 more 4.6]
- (b) weaker hypothesis (for $\lambda^+ \rightarrow [\lambda^+]^{<\omega}_\kappa$) [She03, 2.5]
- (c) $\lambda = \beth^+_\omega$ (see [She03], [ES05] more there)
- (d) on every large enough regular $\mu < \lambda$, there is an algebra M on μ which has no proper subalgebra with set of elements a stationary subset of μ, see [She97a, 3.3].

Note 9.5. Sufficient condition for "λ not Jónsson" [She94l, 1.8,1.9] for $\lambda \nrightarrow [\lambda]^{<\omega}_\sigma$ [She94l, 1.10,3.5,3.6,3.7].

Note 9.6. λ inaccessible is not Jónsson when: λ not Mahlo [She94l, 3.8], λ has a stationary subset S not reflecting in inaccessibles [She94l, 3.9], λ not λ-Mahlo [She94k], λ not $\lambda \times \omega$-Mahlo [She03, 1.14], there is a set S of singulars satisfying, $\mathrm{rk}_\lambda(S) > \mathrm{rk}_\lambda(S^+)$ where $S^+ = \{\kappa < \lambda : \kappa \text{ inaccessible}, S \cap \kappa \text{ stationary}\}$, [She03, 1.15].

Note 9.7. If μ^+ is a Jónsson cardinal, $\mu > \mathrm{cf}(\mu) > \aleph_0$, <u>then</u> $\mathrm{cf}(\mu)$ is "almost" μ^+-supercompact [She03, 2.8] other [She03, 2.10].

Note 9.8. If λ is regular, and for every regular large enough $\mu < \lambda$, for some $f : \mu \to \lambda$ we have $\|f\|_{J^{\mathrm{bd}}_\mu} \geq \lambda$ (or at least this holds for "enough" μ's), <u>then</u> on λ there is a Jónsson algebra, [She94k, 2.12+2.12A]. More sufficient conditions there.

Note 9.9. See more [She03], [ES05].

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§ 10. Colouring = negative partition relations: (see [She88a], [She90b], [She91b])

Note 10.1. Definition of Pr_ℓ: Pr_0, see [She94f, AP,1.1], $\mathrm{Pr}_1^{(-)}$, see [She94f, AP,1.2], $\mathrm{Pr}_2^{(y)}$, see [She94f, AP,1.3], $\mathrm{Pr}_3^{(y)}$, see [She94f, AP,1.4], Pr_4, see [She94l, 4.3].

For example: $\mathrm{Pr}_1(\lambda,\mu,\theta,\kappa)$ means: there is a 2-coloring of λ by θ colours (= symmetric 2-place function from λ to θ) such that: if $\langle w_i : i<\mu\rangle$ is a sequence of pairwise disjoint subsets of $\lambda, \bigwedge_i |w_i| < \kappa$ and $\zeta<\theta$, <u>then</u> for some $i<j$, on $w_i \times w_j$ the coloring c is constant. In $\mathrm{Pr}_0(\lambda,\mu,\theta,\kappa)$ we replace ζ by $h:\kappa\times\kappa\to\theta$ and demand $\alpha\in w_i$ and $\beta\in w_j \Rightarrow c(\alpha,\beta) = h(\mathrm{otp}(w_i\cap\alpha),\mathrm{otp}(w_j\cap\beta))$. If $\mu=\lambda$ we may omit it, if $\kappa=\aleph_0$ we may omit it. (See [She94f, AP,1.2]).

Note 10.2. Trivial implications [She94f, AP,1.6,1.6A,1.7] and $\mathrm{Pr}_1 \Rightarrow \mathrm{Pr}_0$ by [She94l, 4.5(3)], $\mathrm{Pr}_4 \Rightarrow \mathrm{Pr}_1 \Rightarrow \mathrm{Pr}_0$ by [She94l, 4.5(1)]. For example if $\mathrm{Pr}_1(\lambda,\mu,\theta,\sigma), \chi = \chi^{<\sigma} + 2^\theta \le \mu \le \lambda < 2^\chi$ then $\mathrm{Pr}_0(\lambda,\mu,\chi,\sigma)$. Other such Pr and implications [She97a, §2,§4].

Note 10.3. Colouring for successor of singular: [She94b, 4.1,4.7], [She03, §2] for example $\mathrm{Pr}_1(\lambda^+,\lambda^+,(\mathrm{cf}(\lambda))^+,2)$ for λ singular.

Note 10.4. Combining Pr_ℓ's [She94b, 4.8,4.8A].

Note 10.5. Using pcf:

- (a) if $\lambda = \mathrm{tcf}(\prod \mathfrak{c}/J^{\mathrm{bd}}_{\mathfrak{c}})$ and $\left[\theta\in\mathfrak{c} \Rightarrow |\mathfrak{c}\setminus\theta| = |\mathfrak{c}|\right]$, <u>then</u> $\mathrm{Pr}_1(\lambda,\lambda,2^{|\mathfrak{c}|},\mathrm{cf}(\mathfrak{c}))$; see [She94b, 4.1B], [She88a].
- (b) getting colouring on $\lambda\in\mathrm{pcf}(\mathfrak{a})$ from colourings on every $\theta\in\mathfrak{a}$, see [She94b, 4.1D].

Note 10.6. Using guessing of clubs: Definition and basic properties of for example $(Dx)^\lambda_{\kappa,\sigma,\theta,\tau}$ [She94l, 4.1].

Note 10.7. Proof of such properties [She94l, 4.2], [She03, 2.6]

- (a) if λ is a regular $\lambda>\sigma>\kappa$ <u>then</u> $\mathrm{Pr}_1(\lambda^+,\lambda^+,\sigma,\kappa)$, [She94l, §4]
- (b) if λ is inaccessible with a stationary subset S not reflecting in inaccessibles and $\sigma \le \min_{\delta\in S}\mathrm{cf}(\delta)$ and $\kappa<\lambda$ then $\mathrm{Pr}_1(\lambda,\lambda,\kappa,\sigma)$, [She94l, 4.1+4.7]
- (c) if $\lambda=\mu^+, \mu > 2^{\mathrm{cf}(\mu)}, \kappa<\mu$, <u>then</u> $\mathrm{Pr}_1(\lambda,\lambda,\mathrm{cf}(\mu),\kappa)$, [She03, 2.7]
- (d) if $\lambda=\mu^+, \mu>\mathrm{cf}(\mu)$ <u>then</u> $\mathrm{Pr}_1(\lambda,\lambda,\mathrm{cf}(\mu),\mathrm{cf}(\mu))$, [She94b, 4.1(1)]
- (e) by [ES05] we get such properties for e.g. $\lambda = \beth^+_\omega$
- (f) if $\lambda=\aleph_2$ and $\mu=\aleph_0$ or if $\lambda=\mu^{++}$, μ regular then $\mathrm{Pr}_1(\lambda,\lambda,\lambda,\mu)$ ([She97a, §1]).

Note 10.8. $(E2)$ implies Pr_4 [She94l, 4.4].

Note 10.9. $(D2) \Rightarrow \mathrm{Pr}_1$ [She94l, 4.7].

Note 10.10. Concerning the results in [She81a] on partition relations restriction of the kind appearing there are necessary (we use FILL) see, some day [S$^+$a].

Note 10.11. Galvin conjecture:

- (a) $\aleph_n \nrightarrow [\aleph_1]^{n+1}_{\aleph_0}$ ([She92, 5.8(1)], more there), but
- (b) for the naturally defined $h:\omega\to\omega$ if CON(ZFC + $\lambda\to(\aleph_1)^\omega_2$) <u>then</u> it is consistent with ZFC that: $2^{\aleph_0} = \lambda \to [\aleph_1]^n_{h(n)}$, (we can even get $X\in[\lambda]^{\aleph_1}$ which exhibits the conclusion simultaneously for all n, $\lambda\to[\aleph_1]^n_{h_1(n)}$, if $h_1(n)\ge n$, $h_1(n)/h(n)\to\infty$), [She92, 3.1]

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(c) if κ is measurable indestructible by adding (even many) Cohen subsets to κ, then a generalization of Halpern Lauchli theorem holds to ${}^{\kappa>}2$ (but using some $\langle <_\alpha : \alpha < \kappa \rangle$, $<_\alpha$ a well order of ${}^\alpha 2$) ([She92, 4.1,4.2 + §2]). See more in [She96e], [She00e], [RS00], and [Shed].

Note 10.12. More on colouring (improving results on Jónssonness from [She03] to colouring) see [ES05], e.g. for $\lambda = \beth_\omega^+$ we have $\mathrm{Pr}_1(\lambda, \lambda, \lambda, \aleph_0)$.

Note 10.13. More on Pr_i's in [She06, §3]. E.g., if $\mu > \aleph_0$ is strong limit, $\chi \geq \mu$, $\lambda = 2^\chi$ is singular, then $\chi \in \mu \cap \mathrm{Reg} \setminus \{\aleph_0\} \Rightarrow \mathrm{Ps}_1(\mathrm{cf}(\lambda), \lambda, \kappa)$.

See more in [She21a] and from there, on [Rin14], [She19].

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§ 11. Trees and linear orders

Note 11.1. Let $\mathfrak{a}_\delta = \{\lambda_i : i < \delta\}$ and $\mathfrak{a}_i = \{\lambda_j : j < i\}$ for every $i < \delta$. If $\lambda = \max \mathrm{pcf}(\mathfrak{a}_\delta)$ and $\lambda_i > \max \mathrm{pcf}(\mathfrak{a}_i)$, then we can find in $\prod(\mathfrak{a}_\delta)$ a $<_{J_{<\lambda}[(\mathfrak{a}_\delta)]}$-increasing cofinal sequence $\langle f_\alpha : \alpha < \lambda\rangle$ such that $\big\{\{f_\alpha \restriction \mathfrak{a}_j : j < i\} : i < \delta, \alpha < \lambda\big\}$ forms a tree with δ levels, level i of cardinality $\max \mathrm{pcf}(\mathfrak{a}_j) < \lambda_i$ and $\geq \lambda$ many δ-branches [She94b, 3.5].

Note:

(*a*) The lexicographic order on $\mathscr{F} = \{f_\alpha : \alpha < \lambda\}$ has density $\sum_{i<\delta} \lambda_i$.

(*b*) If $\prod \lambda_i / I$ is as in [She94b, 1.4(1)(see 1.3)] then F is $(\Sigma\lambda_i)^+$-free (see 6.3). Hence any set of cardinality $\leq \Sigma\lambda_i$ is the union of $\leq \mathrm{gen}(I)$ many sets F' each satisfying "for some $s \in I$ we have $\langle f_\alpha \restriction (\delta \setminus s) : f_\alpha \in F'\rangle$ is increasing": i.e.

$$\big[\alpha < \beta,\ f_\alpha, f_\beta \in F',\ i \in \delta \setminus s\big] \Rightarrow f_\alpha(i) < f_\beta(i).$$

Here $\mathrm{gen}(I) = \min\{|\mathscr{P}| : \mathscr{P} \subseteq I$ generates the ideal $I\}$. [She94b, 1.4(3)]

(*c*) if $\lambda > 2^{|\delta|}$, then we can have such trees with exactly λ branches [She88b]; somewhat more: [She96a, 6.6B].

See more in part (C).

Note 11.2. There are quite many $\langle \lambda_i : i < \delta\rangle$, λ as in 11.1: for example, if

$$\aleph_0 < \kappa = \mathrm{cf}(\mu) < \mu_0 < \mu < \lambda = \mathrm{cf}(\lambda) < \mathrm{pp}_\kappa(\mu)$$

then we can find such $\langle \lambda_i : i < \kappa\rangle$ with limit μ with $\mu_0 < \lambda_i < \mu$, if $\bigwedge_{\alpha<\mu} |\alpha|^\kappa < \mu$ or at least $(\forall \mu' < \mu)[\mathrm{pp}_\kappa(\mu') < \mu]$, see [She94a, 1.6(2),(4)]. Also $\mathrm{pp}(\aleph_{\alpha+\delta}) < \alpha_{\alpha+|\delta|^+}$ helps to get such examples, see [She97e, §5], [RS98].

Note 11.3. For $\lambda > \kappa = \mathrm{cf}(\kappa)$ the following cardinals are equal:

$$\sup\{\mu : \text{some tree with } \lambda \text{ nodes has } \geq \mu \text{ many } \kappa\text{-branches}\}$$

and

$$\sup\{\mathrm{pcf}(\mathfrak{a}) : \begin{array}{l} |\mathfrak{a}| < \min(\mathfrak{a}),\ \mathrm{cf}(\mathrm{otp}(\mathfrak{a})) = \kappa,\ \mathfrak{a} \subseteq \mathrm{Reg} \cap \lambda^+ \setminus \kappa \text{ and} \\ \theta \in \mathfrak{a} \Rightarrow \max \mathrm{pcf}(\mathfrak{a} \cap \theta) < \theta\} \end{array}$$

see [She00a, 2.2].

Note 11.4. Definition of Ens, entangled linear order and basic facts. (Ens stands for *entangled sequence.*) See for example [She94f, AP,2.1,2.2 more 2.3].

A linear order $\mathcal{I}$ is λ-entangled if given any $n < \omega$ and pairwise distinct $x^e_\zeta \in \mathcal{I}$ $(e < n, \zeta < \lambda)$ and $w \subseteq \{0, 1, \ldots, n-1\}$ there are $\zeta < \xi$ such that for $e < n$ we have: $x^e_\zeta < x^e_\xi \Leftrightarrow e \in w$. We say $\mathcal{I}$ is entangled if it is $|\mathcal{I}|$-entangled; $\mathrm{Ens}(\lambda, \mu)$ means there are μ linear orders $\mathcal{I}_\zeta$ $(\zeta < \mu)$ each of cardinality λ and if $n < \omega$, $\zeta_e < \mu$ distinct $(e < n)$ and $w \subseteq n$ and if $x^e_\zeta \in \mathcal{I}_\zeta$ are distinct then for some $\alpha < \beta < \mu$ we have $\mathcal{I}_{\zeta_e} \models x^e_\alpha < x^e_\beta \Leftrightarrow e \in w$.

For more on σ-entangled linear orders see [She97e]; first

Claim 11.5. $\mathrm{Ens}(\mathrm{cf}(2^{\aleph_0}))$, *by Bonnet-Shelah* [BS85]. *The proof gives* $\mathrm{Ens}(\mathrm{cf}(2^\lambda), \aleph_0)$ *when there is a linear order of cardinality* 2^λ *and density* λ.

Note 11.6. $\mathrm{Ens}(\lambda^+, \mathrm{cf}(\lambda))$ for λ singular [She94b, 4.9 more 4.11,4.14] more [She94a, 5.3].

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Note 11.7. For μ regular uncountable and a linear order $\mathcal{I}$ of power μ, $\mathcal{I}$ is entangled iff the interval Boolean algebra of $\mathcal{I}$ is λ-narrow (see Bonnet-Shelah [BS85] (using different names), later [She94j, 2.3] or [She97e, §1]).

Note 11.8. A sufficient condition for existence of entangled linear order of cardinality λ is: $\lambda = \max \mathrm{pcf}(\mathfrak{a})$, $\kappa = |\mathfrak{a}|$, $[\theta \in \mathfrak{a} \Rightarrow \theta > \max \mathrm{pcf}(\theta \cap \mathfrak{a})]$, $2^\kappa \geq \sup(\mathfrak{a})$, $\mathfrak{a}$ divisible to κ sets not in $J_{<\lambda}[\mathfrak{a}]$, [She94b, 4.12]. If we omit "$2^\kappa \geq \sup(\mathfrak{a})$" we can still prove $\mathrm{Ens}(\lambda, \kappa)$; [She94b, 4.10A] more in [She94b, 4.10F,4.10G], [She94a, 5.4,5.5,5.5A].

Note 11.9. If $\mathrm{cf}(\lambda) < \lambda \leq 2^{\aleph_0}$, <u>then</u> there is an entangled linear order in λ^+, [She94b, 4.13].

Note 11.10. If $\lambda \in \mathrm{pcf}(\mathfrak{a})$ and $\left[\theta \in \mathfrak{a} \Rightarrow \theta > \max \mathrm{pcf}(\theta \cap \mathfrak{a})\right]$ and for each $\theta \in \mathfrak{a}$ there is an entangled linear order or just $\mathrm{Ens}(\theta, \max \mathrm{pcf}(\theta \cap \mathfrak{a}))$, <u>then</u> there is one on λ, [She94b, 4.10C].

Note 11.11.

(*a*) If $\kappa^{+4} \leq \mathrm{cf}(\lambda) < \lambda < 2^\kappa$, <u>then</u> there is an entangled linear order in λ^+, [She93b, 4.1 more 4.2,4.3].

(*b*) There is a class of cardinals λ for which there is an entangled linear order of cardinality λ^+, [She94a, §5]. It is not clear if we can demand e.g. $\lambda = \lambda^{\aleph_0}$, but if this fails, then for κ large enough, $\kappa^{\aleph_0} = \kappa \Rightarrow 2^\kappa < \aleph_{\kappa^{+4}}$ (see (a), more in [She97e]).

(*c*) There is a class of cardinals λ for which there is a Boolean algebra B of cardinality λ^+ with neither chain nor antichain of cardinality λ^+; i.e. if $Y \subseteq B$, $|Y| = |B|$ then $(\exists x, y \in Y)[x < y]$ and $(\exists x, y \in Y)[x \not\leq y \wedge y \not\leq x]$. In fact, for any sequence $\langle x_\alpha : \alpha < \lambda^+ \rangle$ of distinct members of B:
   - (*i*) $(\exists \alpha < \beta)(x_\alpha < x_\beta)$,
   - (*ii*) $(\exists \alpha < \beta)(x_\alpha > x_\beta)$ and
   - (*iii*) $(\exists \alpha < \beta)[x_\alpha \not\leq x_\beta \wedge x_\beta \not\leq x_\alpha]$; see [She97e, 4.3].

(*d*) Moreover, in part (c), for any given λ_0, letting μ be the minimal $\mu = \aleph_\mu > \lambda_0$ then we can find B as there with density μ (everywhere); similarly in (b).

(*e*) Moreover in (c) (and (b)) if the density character is μ, $\ell \in \{0,1,2\}$, $\theta = \mathrm{cf}(\theta) < \mu$ and $x_\alpha \in B$ (for $\alpha < \lambda$) are distinct then for some $w \subseteq \lambda$, $|w| = \theta$ we have for any $\alpha, \beta \in w$, $\alpha < \beta$:

$$\ell = 0 \Rightarrow x_\alpha < x_\beta$$

$$\ell = 1 \Rightarrow x_\alpha > x_\beta$$

$$\ell = 2 \Rightarrow x_\alpha \not\leq x_\beta \wedge x_\beta \not\leq x_\alpha$$

Similarly in part (b).

(*f*) If 2^λ is singular, then there is an entangled linear order of cardinality $(2^\lambda)^+$ (the assumption implies $(\forall \mu)[\lambda < \mathrm{cf}(\mu) < \mu \leq 2^\lambda < \mathrm{pp}(\mu)]$ (i.e. $\mu = 2^\lambda$), this suffices as we can use 6.6(b), 6.12, 11.8; see [She97e, 5.5, pg.65]).

Note 11.12. Universal linear orders: see Section 13, Model Theory.

Note 11.13. For every λ there is $\mu, \lambda \leq \mu < 2^\lambda$ such that (A) or (B):

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(A) $\mu = \lambda$ and for every regular $\chi \leq 2^\lambda$ there is a tree T of cardinality λ with $\geq \chi$ branches (so a linear order of cardinality $\geq \chi$ and density $\leq \lambda$)

(B) $\mu > \lambda$, and:

(α) $\mathrm{pp}(\mu) = 2^\lambda$, $\mathrm{cf}(\mu) \leq \lambda$, $(\forall \theta)[\mathrm{cf}(\theta) \leq \lambda < \theta < \mu \Rightarrow \mathrm{pp}_\lambda(\theta) < \mu]$. Hence, by [She94a, §1] for every regular $\chi \leq 2^\lambda$ there is a tree from [She94b, 3.5]: $\mathrm{cf}(\mu)$ levels, every level of cardinality $< \mu$ and χ $(\mathrm{cf}(\mu))$-branches

(β) for every $\chi \in (\lambda, \mu)$, there is a tree T of cardinality λ with $\geq \chi$-branches of the same height

(γ) $\mathrm{cf}(\mu) = \mathrm{cf}(\lambda_0)$ for $\lambda_0 = \min\{\theta : 2^\theta = 2^\lambda\}$ and even $\mathrm{pp}_{\Gamma(\mathrm{cf}(\mu))}(\mu) = 2^\lambda$ see [She94b, 5.11], [She93b, 4.3] and [She96a, 3.3]; see more in [She09a, 2.10].

Note 11.14. If $\theta_{n+1} = \min\{\theta : 2^\theta > 2^{\theta_n}\}$ for $n < \omega$, $\sum_{n<\omega} \theta_n < 2^{\theta_0}$, <u>then</u> for some $n > 0$ and regular $\mu \in [\theta_n, \theta_{n+1})$ for every regular $\chi \leq 2^{\theta_n}$, there is a tree with μ nodes and $\geq \chi$ μ-branches [She96a, 3.4].

Note 11.15. Kurepa trees: there are two contexts that arise

(a) we can get Kurepa trees of singular cardinality: if $\bar{\lambda} = \langle \lambda_i : i < \delta \rangle$ and $\delta < \lambda_i = \mathrm{cf}(\lambda_i)$, $\lambda_i > \max \mathrm{pcf}\{\lambda_j : j < i\}$ <u>then</u> there is a tree with δ levels, the i-th level of cardinality $< \lambda_i$, and at least $\max \mathrm{pcf}\{\lambda_i : i < \delta\}$ δ-branches, see [She94b, 3.5], hence can derive consequences from conventional cardinal arithmetic assumptions

(b) if for example $\mathrm{pp}(\aleph_{\omega_1}) > \aleph_{\omega_2}$ and for a club of $\delta < \omega_1, \mathrm{pp}(\aleph_\delta) < \aleph_{\omega_1}$, <u>then</u> there is an $(\aleph_1)$-Kurepa tree (see [She94a, 2.8] for more). We get a large family of sets with small intersection in more general circumstances [She94a, 1.7].

(c) If $\bigwedge_{\alpha<\mu} |\alpha|^\kappa < \mu$, $\mathrm{cf}(\mu) = \kappa > \aleph_0$ and $\mu \leq \lambda < \mu^\kappa$, <u>then</u> there is a tree with μ nodes, κ levels and exactly λ branches, λ of them of height κ. We can derive results on linear orders (really they are the same problems). If we speak on the number of κ-branches (or for linear order number of Dedekind cuts of cofinality from at least one side κ), instead of "$\bigwedge_{\alpha<\mu} |\alpha|^\kappa < \mu$" it suffices that

(∗) (a) $2^\kappa < \mu_0 < \mu$
(b) if $\mu_0 < \chi < \mu$ and $\mathrm{cf}(\chi) \leq \kappa$ then $\mathrm{pp}(\kappa) < \mu$.

See [She89] or [She96a, 6.6(1)]. (Similarly, other results can be translated between trees and linear orders).

§ 12. Boolean Algebras and General Topology

Note 12.1. Concerning Boolean algebras and topology. λ-c.c. is not productive and $\lambda - L$-spaces exist and $\lambda - S$-spaces exist and more follows from $\mathrm{Pr}_1^-(\lambda, 2)$ (or appropriate colouring) see [She94i, 1.6A] so [She94b, 4.2] is a conclusion of this. This is translated to results on cellularity of topological spaces (cellularity $\leq \lambda \Leftrightarrow \lambda^+$-c.c.).

We have

(a) if $\lambda \geq \aleph_1$, for some λ^+-c.c. Boolean algebras B_1, B_2 we have: $B_1 \times B_2$ is not λ^+-c.c. (why? now $\mathrm{Pr}_1(\lambda^+, \lambda^+, 2, \aleph_0)$ suffice [She94f, 1.6A] and it holds by [She91b] or [She94l, 4.8(1),p.177] if λ regular $> \aleph_1$, [She94b, 4.1,p.67] if λ is singular and lastly by [She97a, §1] if $\lambda = \aleph_1$)

(b) if λ is inaccessible and has a stationary subset not reflecting in any accessible, then for some λ-c.c. Boolean algebras B_1, B_2 we have: $B_1 \times B_2$ is not λ-c.c. (see [She94l, 4.8])

(c) if λ is Mahlo, $\otimes_\lambda^{\aleph_0}$ (see 1.5) then for some λ-c.c. Boolean algebras B_n, for any proper filter I on ω extending J_ω^{bd} we have $\prod B_n/I$ fails the λ-c.c. [She94l, 4.11].

Note 12.2. Concerning Topology: characterizing by pp when there are $f_\alpha \in {}^\kappa\sigma$ for $\alpha < \theta$ such that $\alpha < \beta \Rightarrow \bigvee_{i<\kappa} f_\alpha(i) < f_\beta(i)$, see [She93b, 3.7], is needed for Gerlits, Hajnal and Szentmiklossy [GHS92]. The condition is (when θ is regular for simplicity)

$$2^\kappa \geq \theta \text{ or } (\exists \mu)\big[\mathrm{cf}(\mu) \leq \kappa < \mu \text{ and } \mu \leq \sigma \text{ and } \mathrm{pp}^+(\mu) > \theta\big]$$

(for θ singular, just ask if it holds for every regular $\theta_1 < \theta$).

(Why not just $\theta \leq \sigma^\kappa$? Because if e.g. $\kappa = \beth_\omega$, $\theta = \beth_{\omega+1}$, $\sigma = \aleph_0$ we do not know whether $\mathrm{pp}^+(\kappa) = \theta^+$).

Note 12.3. Concerning Topology: let X be a topological space, B a basis of the topology (not assuming the space to be Hausdorff or even T_0). If $\lambda = \aleph_0$ or λ is strong limit of cofinality $\aleph_0$, and the number of open sets is $> |B| + \lambda$, then it is $\geq \lambda^{\aleph_0}$; see for $\lambda = \aleph_0$ [She93c], for $\lambda > \aleph_0$, [She94h] relaying on [She00d].

Note 12.4. Concerning Topology: densities of box products: for example if μ is strong limit singular, $\mu = \sum_{i<\mathrm{cf}(\mu)} \lambda_i$, $\mathrm{cf}(\lambda_i) = \aleph_0$, $2^{\lambda_i} = \lambda_i^+$, λ_i are strong limit cardinals, $\max \mathrm{pcf}\{\lambda_i : i < \mathrm{cf}(\mu)\} < 2^\mu$, $\mathrm{cf}(\mu) < \theta < \mu$, then the density of the $\mathrm{cf}(\mu)^+$-box product ${}^\theta\mu$ is 2^μ [She96a, §5].

Gitik Shelah [GS98] prove consistency results.

Note 12.5.

(a) the results in 12.3 come from starting to analyze the following: given a μ_i-complete filter D_i on λ_i, for $i < \kappa$, what is

$$\min\Big\{|A| : \; A \subseteq \prod_{i<\kappa} \lambda_i \text{ such that for every } \langle A_i : i < \kappa\rangle \in \prod_{i<\kappa} D_i \text{ we have } A \cap \prod_{i<\kappa} A_i \neq \varnothing\Big\}$$

continued in [She00b] and then [She99].

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(b) This is applied also to the problem of λ-Gross spaces. These are vector spaces V over a field F with an inner product such that for any subspace $U \subseteq V$ of dimension λ,

$$\dim \Big\{ x \in V : \bigwedge_{y \in U} (x,y) = 0 \Big\} < \dim V.$$

See Shelah Spinas [SS96].

Note 12.6. A well known problem in general topology is whether every Hausdorff space can be divided to two sets each not containing a homeomorphic copy of Cantor's discontinuum. In [She00d] we have a sufficient condition for this (e.g. $|\mathfrak{a}| \leq \aleph_0 \Rightarrow |\mathrm{pcf}(\mathfrak{a})| \leq \aleph_0$ and $2^{\aleph_0} \geq \aleph_\omega$, by [She00d, 3.6(2)], the $(*)_1$ version relying on [She00d, Th.2.6]). But we can prove: if $c\ell$ is a closure operation on $\mathcal{P}(X)$ (i.e. $a \subseteq c\ell(a) = c\ell(c\ell(a))$, $a \subseteq b \Rightarrow c\ell(a) \subseteq c\ell(b)$) and $|a| \geq \aleph_0 \Rightarrow |c\ell(a)| > \beth_\omega$, then we can partition X to two sets, each not containing any infinite $a = c\ell(a)$. (Can prove more).

Related weaker problem is to find large $A \subseteq {}^\omega\lambda$ containing no large closed subsets, λ strong limit of cofinality $\aleph_0$, if $\mathrm{pp}(\lambda) = 2^\lambda$ is easy (hence for higher cofinalities this holds and e.g. for many $\beth_\delta$, $\delta < \omega_1$). See [She94b, 6.9], more in [She96a, 3.3,3.4]. See more in [She00d], [She04].

Note 12.7. If $\mathrm{pp}(\lambda) > \lambda^+$, $\mathrm{cf}(\lambda) = \aleph_0$ (or just a consequence from [She94b, §1], see 6.2 here), then there is first countable λ-collectionwise Hausdorff (and even λ-metrizable), not λ^+-collectionwise Hausdorff space (see [She96b]; when we assume just $\mathrm{cov}(\lambda, \lambda, \aleph_1, 2) > \lambda^+$ use [She94b, §6]).

Note 12.8. If $\lambda < \lambda^{<\lambda}$, then there is a regular $\kappa < \lambda$ and tree T with κ levels, for each $\alpha < \kappa$, T has $< \lambda$ members of level $\leq \alpha$, and T has $> \lambda$ many κ-branches. If $\lambda < \lambda^{<\lambda}$ and $\neg(\exists \mu)\big[\mu$ strong limit and $\mu \leq \lambda < 2^\mu\big]$, then this is above $2^\kappa > \lambda$; see [She96a, 6.3].

Note 12.9. Depth of homomorphic images of ultraproducts of Boolean algebras, [She97f, §3] and resolved for $\lambda_i > 2^{|\mathrm{Dom}(D)|}$ in [She00a, §3].

Note 12.10. If λ is strong limit singular, $\kappa = \mathrm{cf}(\lambda)$ and e.g. $2^\lambda = \lambda^+$, then for some Boolean algebras B_1, B_2 we have: B_1 is λ^+-c.c., B_2 is $(2^\kappa)^+$-c.c. but $B_1 \times B_2$ is not λ^+-c.c. (see for more [She00b]). More constructions in [She99].

Note 12.11. If $\lambda = \lambda^{\beth_\omega}$, B a $\beth_\omega$-c.c. Boolean algebra of cardinality $\leq 2^\lambda$ then B is λ-linked (that is $B \setminus \{0\}$ is the union of $\leq \lambda$ sets of pairwise non-disjoint elements), see [She00b, §8].

Note 12.12. On the measure algebra, [She99].

Note 12.13. On independent sets in Boolean Algebra, [She99].

Note 12.14. On ultraproducts of Boolean Algebra: $s(B)$, spread, i.e. constructing examples of $\mathrm{inv}(\prod_{i<\kappa} B_i/D) > \prod_{i<\kappa} \mathrm{inv}(B_i)/D$, see:

(a) for inv being s, (spread), Roslanowski Shelah [RS98], [She99]
(b) similarly hd (hereditarily density)
(c) similarly hL (hereditarily Lindelof)
(d) for inv being Depth, [She01a], [She05b], [GS08], [GS12b]
(e) for inv being Length, [She01a].

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§ 13. Strong covering, forcing, and partition calculus

Note 13.1. Preservation under forcing: essentially pcf and pp are preserved except for forcing notion involving large cardinals. Specifically if (the pair of universes) (V, W) satisfies κ-covering [i.e. $V \subseteq W$ and if $a \subseteq \mathrm{Ord}, W \models |a| < \kappa$ then for some $b \in V, a \subseteq b \subseteq \mathrm{Ord}$ and $W \models |b| < \kappa$] and $\mathfrak{a} \subseteq \mathrm{Ord} \setminus \kappa$ is a set from W of cardinality $< \kappa$ of regulars of W <u>then</u>

$$\mathrm{pcf}^V \left\{ \mathrm{cf}^V(\theta) : \theta \in \mathrm{pcf}^V(\mathfrak{a}) \right\} = \left\{ \mathrm{cf}^V(\lambda) : \lambda \in \mathrm{pcf}^W(\{\mathrm{cf}^W(\theta) : \theta \in \mathfrak{a}\}) \right\}$$

(this applies for example to (K, V) if there is no inner model with measurable by Dodd and Jensen [DJ81]).

Note 13.2. The strong covering lemma: see [She98b, Ch.XIII,§1,§2] or better [She94f, Ch.VII,§1,§2]; see more in [She93b, 2.6,p.407] and [She00c], each can be read independently.

Suppose $W \subseteq V$ is a transitive class of V including all the ordinals and is a model of ZFC, let $\lambda > \kappa$ be cardinals of V.

We say (W, V) satisfies the strong (λ, κ)-covering property if for every model $M \in V$ with universe λ and predicates and function symbols there is $N \prec M$ of cardinality $< \kappa$, $N \cap \kappa \in \kappa$, $N \in V$ but the universe of N belongs to W; we also use stronger versions (like the set of such N's is positive or even equal to $[\lambda]^{\leq \kappa}$ modulo some ideal, <u>or</u> weaker versions like union of few sets from W).

Those papers do this without using fine structure assumptions, just that (W, V) satisfies (λ, κ)-covering and related properties.

Note 13.3. Application of ranks (see 3.2) to partition calculus: Shelah Stanley [SS00]. If there is a nice filter of κ (see 3.2) and $\lambda, \mathrm{cf}(\lambda) > \kappa = \mathrm{cf}(\kappa), (\forall \mu < \lambda)\, \mu^\kappa < \lambda$ <u>then</u> $\lambda \to (\lambda, \omega + 1)^2$.

Note 13.4. Also, with ranks. If $\lambda > \mathrm{cf}(\lambda) > \aleph_0$, <u>then</u> $\lambda \to (\lambda, \omega + 1)^2$ in ZFC (see [She09b]).

Note 13.5. <u>Polarized Partition Relations</u>

If λ is strong limit singular and $2^\lambda > \lambda^+$, <u>then</u> $\binom{\lambda^+}{\lambda} \to \binom{\lambda}{\lambda}^{1,1}_2$, see [She98a].

Note 13.6. If $\lambda > \mathrm{cf}(\lambda)$ is a limit of measurables and some pcf assumptions are forced, then even $\binom{\lambda^+}{\lambda} \to \binom{\lambda^+}{\lambda}^{1,1}_2$, see [GS12a].

Note 13.7. See [She97d].

Note 13.8. On inner models, see [GSS06].

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§ 14. Axiom of Choice, weak versions

Note 14.1. Set theory with weak choice: [She97d], [She12], [She14].

Note 14.2. The intermediate axiom of choice: $(\mathrm{Ax})_4$ (see [Shee]).

1) We suggest considering $(\mathrm{Ax})_4$ in addition to ZF + DC, which tells us each $[\alpha]^{\aleph_0}$ is well ordered. This is orthogonal to $\mathbf{V} = \mathbf{L}[\mathbb{R}]$.
2) In particular, it gives: given a set I, for every ordinal α, ${}^I\alpha$ is covered by a sequence of few well ordered sets (depending only on I), even uniformly.
3) There is a class of successor cardinals which are regular (and even so called "explicitly successor").

Note 14.3. (See [She16b]) We have a quite strong pcf theorem which the minimal cofinality is big enough compared to the index set.

Note 14.4. (See [She16b]) Black Boxes and constructing abelian groups.

Note 14.5. On splitting stationary sets, see Larson and Shelah [LS09].

Note 14.6. A ZFC conclusion — we quote from [Shee, §3]:
We prove that if $\mu > \kappa = \mathrm{cf}(\mu) > \aleph_0$, then from a well-ordering of $\mathcal{P}(\mathcal{P}(\kappa)) \cup {}^{\kappa>}\mu$ we can define a well-ordering of ${}^\kappa\mu$. If, e.g., μ is a strong limit singular cardinal of uncountable cofinality, then by using a well order of $\mathcal{H}(\mu)$ we can define a well-ordering of $\mathcal{P}(\mu)$, hence of $\mathcal{H}(\mu^+)$. Lastly, we give sufficient conditions (in ZF + DC) on singular μ for μ^+ to be regular. Actually, if $\mu = \mu^{\aleph_0} + 2^{2^\kappa}$, $\kappa = \kappa^{\aleph_0}$, and $X \subseteq \mu$ codes $\mathcal{P}(\mathcal{P}(\kappa))$ and ${}^\omega\mu$, then by using X as a parameter we can define a well-ordering of ${}^\kappa\mu$.

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§ 15. Transversals and (λ, I, J)-sequences

See [She85b] (and [She75]), a transversal is a one to one choice function.

Note 15.1. If I is an ideal on $\kappa, \lambda > \mathrm{cf}(\lambda)$ and $\mathrm{pp}_I(\lambda) > \mu$, <u>then</u> we can find a family of functions $f_\alpha(\alpha < \mu)$ from κ to λ, which is λ^+-free for I i.e. any λ of them are strictly increasing on each $x \in \mathrm{Dom}(I)$ if for each α we ignore a set $s_\alpha \in I$ such that $i \in \kappa \setminus (s_\alpha \cup s_\beta) \Rightarrow f_\alpha(i) < f_\beta(i)$ (so $\{\mathrm{Rang}(f_\alpha) : \alpha \in u\}$ has a transversal when $u \subseteq \mu, |u| \leq \lambda$) [She94b, 1.5A] (the case μ singular changes nothing for this purpose). So $\mathrm{NPT}(\lambda^+, \kappa)$ (see Definition below). On weakening "$\mathrm{pp}_I(\lambda) > \mu$" to "$\mathrm{pp}^+_I(\lambda) > \mu$" for μ successor of regular see [She94a, §6] (μ singular-easy). On weakening $\mathrm{pp}_I(\lambda) > \mu$ to $\mathrm{cov}(\lambda, \lambda, \kappa^+, 2) > \mu$, see [She94b, §6] for some variants; in particular $\mathrm{NPT}_{J^{\mathrm{bd}}_\omega}(\lambda^+, \aleph_0)$ when $\mathrm{cov}(\lambda, \lambda, \kappa^+, 2) > \lambda$ by [She94b, 6.3,p.99].

Note 15.2. Definitions of variants of NPT, [She94b, 6.1], [She94a, 6.3] for example $\mathrm{NPT}(\lambda, \kappa)$ means that there is a family $\{A_i : i < \lambda\}$ of sets each of cardinality $\leq \kappa$, and $< \lambda$ of them have a transversal, but not all. Similarly for $\mathrm{NPT}_J(\lambda, \kappa)$ we have $f_\alpha : \mathrm{Dom}(J) \to$ ordinals as in 15.1.

Note 15.3. Trivial and easy facts [She94b, 6.2,6.7], why concentrating on "$\mathrm{NPT}(\lambda^+, \aleph_1)$, $\mathrm{cf}(\lambda) = \aleph_0$" [She94b, 6.4].

Note 15.4. If $\lambda > \mathrm{cf}(\lambda) = \aleph_0$ and $\mathrm{cov}(\lambda, \lambda, \aleph_1, 2) > \lambda^+$, <u>then</u> $\mathrm{NPT}_{J^{\mathrm{bd}}_\omega}(\lambda^+, \aleph_1)$, [She94b, 6.3] more in [She94b, 6.5,6.8], [She94a, 6.1], [She94a, 6.2] application to [RS87], [She94a, 6.4,6.5].

Note 15.5. When λ is a strong limit of cofinality $\aleph_0$, there is $T \subseteq {}^\omega\lambda, |T| = 2^\lambda$ with no large dense subset, [She94b, 6.9] (there is a subclaim with more information).

Note 15.6. If I is an ideal on κ, $\mu > \kappa \geq \mathrm{cf}(\mu)$, $\mathrm{pp}^+_I(\lambda) > \lambda = \mathrm{cf}(\lambda) > \mu$, $\lambda_i = \mathrm{cf}(\lambda_i) > \kappa$ (for $i < \kappa$), $\mathrm{tlim}_I \lambda_i = \mu$, $\langle f_i : i < \lambda\rangle$ is $<_I$-increasing cofinal in $\prod_{i<\kappa} \lambda_i / I$, <u>then</u> for some $A \subseteq \lambda$, $|A| = \lambda$ for every $B \subseteq A$ of cardinality λ and $\delta < \mu^+$ there is $B' \subseteq A$ of order type δ and $\langle s_\alpha : \alpha \in B'\rangle$ such that: $s_\alpha \in I$, $\alpha < \beta$ and $\zeta \in \kappa \setminus (s_\alpha \cup s_\beta)$ and $\alpha \in B'$ and $\beta \in B' \Rightarrow f_\alpha(\zeta) < f_\beta(\zeta)$ so $\langle \mathrm{Rang}(f_\alpha \restriction (\kappa \setminus s_\alpha) : \alpha \in B'\rangle$ is a sequence of pairwise disjoint sets. (For somewhat more, see [She96a, 6.2,6.2A(3)]).

Note 15.7. (κ-MAD families)

Let $\kappa = \mathrm{cf}(\kappa) > \aleph_0$. For any $\mu \geq 2^\kappa$, letting

$$\chi = \chi^\kappa_\mu = \sup\{\mathrm{pp}_{J^{\mathrm{bd}}_\kappa}(\mu') : 2^\lambda \leq \mu' \subseteq \mu,\ \mathrm{cf}(\mu') = \kappa\}$$

we have:

(a) every κ-almost disjoint subfamily of $[\mu]^\kappa$ (i.e. intersection of two has cardinality $< \kappa$) has cardinality $\leq \chi$; also $\chi^\kappa_\mu = T_{J^{\mathrm{bd}}_\kappa}(\mu)$

(b) trivially there is maximal κ-almost disjoint family $\subseteq [\mu]^\kappa$ and all such families have the same cardinality which is in χ

(c) if $\chi_0 = \chi, \chi_{n+1} = \chi^\kappa_{(\chi_n)}, \chi_\omega = \sum_{n<\omega} \chi_n$ <u>then</u>

(α) $\chi_n = \sup\{\mathrm{pp}_{J_n}(\mu') : 2^\lambda \leq \mu' \leq \mu, \mathrm{cf}(\mu') = \kappa\}$ where

$$J_n = \{A \subseteq \kappa^n : (\exists^{<\kappa}\alpha_0)(\exists^{<\kappa}\alpha_1)\dots(\exists^{<\kappa}\alpha_{n-1})\langle\alpha_0, \dots, \alpha_{n-1}\rangle \in A\}$$

(β) $\chi^\kappa_{(\chi_\omega)} = \chi_\omega$ (hence is doubtful if it is consistent to have $\chi_n \neq \chi_{n+1}$).

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Note 15.8. $\bar{\eta} = \langle \eta_\alpha : \alpha < \lambda \rangle$ is a (λ, I, J)-sequence for $\bar{I} = \langle I_i : i < \delta \rangle$ iff each $\eta_\alpha \in \prod_{i<\delta} \mathrm{Dom}(I_i), J$ is an ideal on δ, I is an ideal on λ, each I_i is an ideal on $\mathrm{Dom}(I_i)$, and

$$X \in I^+ \Rightarrow \big\{ i < \delta : \{\eta_\alpha(i) : \alpha \in X\} \in I_i \big\} \in J.$$

The definition was introduced in [She00b] and considered again in [She99]. In [She99] first the case of the Erdös-Rado ideal defined there was considered. For the case $\bar{I} = \langle J^{\mathrm{bd}}_{\lambda_i} : i < \delta \rangle$ and $\lambda = \mathrm{tcf}(\prod_{i \le \delta} \lambda_i / J^{\mathrm{bd}}_\delta)$ and $J = J^{\mathrm{bd}}_\lambda, I = J^{\mathrm{bd}}_\delta$, the existence of a (λ, I, J)-sequence comes from pcf theory. Also the case $I_i = \prod_{\ell < n_i} J^{\mathrm{bd}}_{\lambda_{i,\ell}}$ for $\langle \lambda_{i,\ell} : \ell < n_i \rangle$ increasing a sufficient pcf condition for the existence of a (λ, I, J)-sequence was given in [She99] which holds sometimes (for any given $\langle n_i : i < \delta \rangle$. Also in [She99] the case $I_i = J^{\mathrm{bd}}_{\langle \lambda_{i,\ell} : \ell < n \rangle}$, for $\langle \lambda_{i,\ell} : \ell < n \rangle$ a decreasing sequence of regulars was considered, giving a sufficient condition which requires pcf to be reasonably complicated. A most case $I_i = \prod_{\ell < n} J^{\mathrm{nst},\theta}_{\lambda_{i,\ell}}, \lambda_{i,\ell}$ regular decreasing, $J^{\mathrm{nst},\theta}_\lambda$ is the ideal of non-stationary sets $+\{\delta < \lambda : \mathrm{cf}(\delta) \ne \theta\}$, when e.g. $\delta < \kappa < \lambda_{i,\ell}$ and we prove existence for some $\langle \lambda_{i,\ell} : \ell < n, i < \delta \rangle$. Many applications for Boolean algebras can be found in [She99].

Note 15.9. The family $\{\kappa : \mathrm{NPT}(\kappa, \aleph_1)\}$ is not too small, see [She79], Magidor Shelah [MS94], [She97b].

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§ 16. Model Theory, algebra, and Black Boxes

Note 16.1. $L_{\infty,\lambda}$-equivalent non-isomorphic models in λ: if $\lambda > \mathrm{cf}(\lambda) > \aleph_1$ there are such models of cardinality λ (if $\mathrm{cf}(\lambda) = \aleph_1$, it suffices to have: $\langle \lambda_i : i < \mathrm{cf}(\lambda) \rangle$ is an increasing sequence of regulars with limit λ and that

$$\{\delta < \mathrm{cf}(\lambda) : \text{there is an unbounded } a \subseteq \delta \text{ with } \lambda > \max \mathrm{pcf}\{\lambda_i : i \in a\}\}$$

is stationary; not known if this fails in some universe of set theory, see [She94b, §7].

Note 16.2. Universal Models: for example, the class of linear orders.

If λ is regular and $\exists \mu\ [\mu^+ < \lambda < 2^\mu]$, then there is in λ no universal linear order, not even a universal model (for elementary embeddings) for T in λ where T is a first order theory with the strict order property. For almost all singular λ we have those results, more specifically if λ is not a fixed point of the second order the result holds; and if it fails for λ the consequences for pp are not known to be consistent, see [KS92a] which rely on guessing clubs.

Note 16.3. A much weaker demand on the first order T suffices in 16.2: NSOP_4, see [She96c, §2]. On the remaining cardinals see some information in [She93d, §3]; on complimentary consistency (only for $\lambda = \aleph_1$) see [She80b, §4].

Note 16.4. Universal models for $(\omega + 1)$-trees with $(\omega + 1)$-levels and or stable unsuperstable T:

Similar results: if λ regular $(\exists \mu)[\mu^+ < \lambda < \mu^{\aleph_0}]$ then there is no universal member; also for most singular [KS92b].

Similarly if $\kappa = \mathrm{cf}(\kappa) < \kappa(T)$, $(\exists \mu)[\mu^+ < \lambda < \mu^\kappa]$.

Note 16.5. Universal abelian groups have similar results for pure embedding (under reasonable restrictions (mainly the groups are reduced, because there are divisible universal abelian groups the interesting cardinals are $\lambda^{\aleph_0} > \lambda > 2^{\aleph_0}$). For torsion free reduced abelian groups, $\mathfrak{K}^{\mathrm{rtf}}$, or reduced separable p-groups, $\mathfrak{K}^{\mathrm{rs}(p)}$ if $2^{\aleph_0} + \mu^+ < \lambda = \mathrm{cf}(\lambda) < \mu^{\aleph_0}$, then there is no universal. For "most" λ, λ regular can be omitted. (This and more [KS95]).

Note 16.6. We can use the usual embedding but restrict the class of abelian groups. The natural classes: $\mathfrak{K}^{\mathrm{rtf}}$ (torsion free, reduced i.e. has no divisible subgroups) and $\mathfrak{K}^{\mathrm{rs}}(p)$ (reduced separable p-groups). But in addition we restrict ourselves to the abelian groups which are $(< \lambda)$-stable (see [She96d]; club guessing is used).

Note 16.7. For classes $\mathfrak{K}^{\mathrm{rtf}}, \mathfrak{K}^{\mathrm{rs}(p)}$ from 15.7 of abelian groups under embeddings see [She97c]: mainly if $\lambda^{\aleph_0} > \lambda > 2^{\aleph_0}$ there are negative results except when some pcf phenomena not known to be consistent (also club guessing is used). Below the continuum there are independence results. More on the existence of universals see [She93d] on metric spaces see [She97c] and on normed spaces [DS04].

Note 16.8. For cardinals $\geq \beth_\omega$, for the classes $\mathfrak{K}^{\mathrm{rtf}}, \mathfrak{K}^{\mathrm{rs}(p)}$ the results in 16.7 are improved to have demands on the cardinals like 16.4, see [She01b].

Note 16.9. There exists a reflexive abelian group, whose cardinality is the first measurable cardinal, [She10]. We do not succeed in proving the existence of arbitrarily large reflexive groups, but show that it would follow from relatively weak assumptions. Furthermore, we show that any condition which would preclude their existence must be quite stringent.

Note 16.10. For a survey on the existence of universal abelian groups under various embedding relations, see [She21b, §10] with references and table (see [She21b, pg. 302].)

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Note 16.11. 1) Existence of a λ-free but not free abelian group of cardinality λ for arbitrarily large $\lambda <$ [first fixed point]. (See Magidor and the author, [MS94].)

2) Consistency of "for the first $\lambda = \aleph_\lambda$, every λ-free algebra is free" for any variety; also [MS94].
3) See more in 16.14, [She20].

Note 16.12. Diamonds and Omitting Types: In the omitting type theorem for $\mathbb{L}(Q)$ in the λ^+ interpretation, not only $\lambda = \lambda^{<\lambda}$ (needed even for the completeness) was used in [She81b] but $(D\ell)_\lambda$ [for λ successor this is $\diamondsuit_\lambda$, generally it means: there is $\langle \mathscr{P}_\alpha : \alpha < \lambda \rangle$, $\mathscr{P}_\alpha$ a family of $< \lambda$ subsets of λ such that for every $A \subseteq \lambda$ for stationarily many $\delta < \lambda$, $A \cap \delta \in \mathscr{P}_\delta$]. Now by [She00d]: if $\lambda > \beth_\omega$ then $\lambda = \lambda^{<\lambda} \Leftrightarrow (D\ell)_\lambda$. In fact: if $\lambda = \lambda^{<\lambda}$ and $(\forall \mu < \lambda)(\mu^{<\kappa>_{\rm tr}} < \lambda) \Rightarrow (D\ell)_{S^\lambda_\kappa}$ (where $(D\ell)_{S^\lambda_\kappa}$ is defined as above but for $\alpha \in S^\lambda_\kappa =: \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$, and $\mu^{<\kappa>_{\rm tr}} = \sup\{\lambda :$ there is a tree with μ-nodes and λ many κ-branches$\}$).

Note 16.13. There are uses for proving Black Boxes (see [Shear, Ch.III,§6]), those are construction principles provable in ZFC, and have quite many applications, see there for references.

Note 16.14. 1) In [She13b] we prove[1] that for strong limit singular μ we can find quite large and quite free sets $\subseteq {}^{\mathrm{cf}(\mu)}\mu$ and quite strong Black Boxes.

2) We define the following objects as in [She13b, §1]:
   (a) Let $\mathbf{C} = \{$strong limit singular $\mu : \mathrm{pp}(\mu) =^+ 2^\mu\}$, with $=^+$ as on pg.4
   (b) $\mathbf{C}_\kappa = \{\mu \in \mathbf{C} : \mathrm{cf}(\mu) = \kappa\}$
   (c) The set $\mathcal{F} \subseteq {}^\kappa\mu$ is called (θ, σ, J)-free, where J is an ideal on κ, when

$$f_1 \neq f_2 \in \mathcal{F} \Rightarrow \{i < \kappa : f_1(i) = f_2(i)\} \in J$$

   and every $\mathcal{F}' \subseteq \mathcal{F}$ of cardinality $< \theta$ is $[J, \sigma]$-free, which means that:
   - there is a sequence $\langle u_f : f \in \mathcal{F}' \rangle$ of members of J such that for every pair $(\gamma, i) \in \mu \times \kappa$, the set $\{f \in \mathcal{F}' : f(i) = \gamma \wedge i \notin u_f\}$ has cardinality $< 1 + \sigma$.

   (d) We may replace "$\mathcal{F} \subseteq {}^\kappa\mu$" by a sequence $\bar{C} = \langle C_\delta : \delta \in S \rangle$ with C_δ a set of order type κ, or even a by a set $\{C_\delta : \delta \in S\}$. This means that the definition applies to $\{f_\delta : \delta \in S\}$, where f_δ is an increasing function $\kappa \to C_\delta$ for each δ; similarly for the other parts.

Definition 16.15. [See [She13b, 0.5=LOp.14]]

Assume we are given a quadruple $(\lambda, \mu, \theta, \kappa)$ of cardinals.[2] Let $\mathrm{BB}^-(\lambda, \mu, \theta, \kappa)$ mean that some pair $(\bar{C}, \bar{\mathbf{c}})$ satisfies clauses (A) and (B) below; we call the pair $(\bar{C}, \bar{\mathbf{c}})$ a *witness* for $\mathrm{BB}^-(\lambda, \mu, \theta, \kappa)$. Let $\mathrm{BB}(\lambda, \mu, \theta, \kappa)$ mean that some witness $(\bar{C}, \bar{\mathbf{c}})$ satisfies clause (A) below and for some sequence $\langle S_i : i < \lambda \rangle$ of pairwise disjoint subsets of λ (or of S), each $(\bar{C} \restriction S_i, \bar{\mathbf{c}} \restriction S_i)$ satisfies clause (B) below,[3] where:

(A) (a) $\bar{C} = \langle C_\alpha : \alpha \in S \rangle$ and $S = S(\bar{C}) \subseteq \lambda = \sup(S)$
   (b) $C_\alpha \subseteq \alpha$ has order type κ
   (c) $\bar{C}$ is μ-free (see 0.6) [but when we replace κ by J then we say "$\bar{C}$ is (μ, J)-free".]

(B) (a) $\bar{\mathbf{c}} = \langle \mathbf{c}_\alpha : \alpha \in S \rangle$

[1] The proof of [She13b, 2.6, Case 2] appears incomplete, but the claim is proved in [She05a, 1.11].

[2] but we may replace λ by an ideal I on $S \subseteq \lambda = \sup(S)$, so writing λ would mean that $S = \lambda$; also, we may replace κ by an ideal J on κ, so writing κ would mean that $J = J^{\rm bd}_\kappa$.

[3] thus replacing S and $\bar{\mathbf{c}}$ by S_i and $\bar{\mathbf{c}} \restriction S_i$

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(b) $\mathbf{c}_\alpha$ is a function from C_α to θ

(c) if $\mathbf{c} : \bigcup_{\alpha \in S} C_\alpha$, then $\mathbf{c}_\alpha = \mathbf{c} \restriction C_\alpha$ for each $\alpha \in S$
[but when we replace λ by an ideal I on S, then we demand that the set $\{\alpha \in S : \mathbf{c}_\alpha = \mathbf{c} \restriction C_\alpha\}$ is not in I].

Remark 16.16. The reader may recall that if S is a stationary subset of

$$\{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$$

for a regular cardinal λ, S is non-reflecting, and $\bar{C} = \langle C_\alpha : \alpha \in S\rangle$ satisfies $C_\delta \subseteq \delta - \sup(C_\delta)$ and $\mathrm{otp}(C_\delta) = \kappa$, then $\diamondsuit_S$ implies $\mathrm{BB}(\lambda, \lambda, \lambda, \kappa)$. Therefore, if $\mathbf{V} = \mathbf{L}$ then for every regular $\kappa < \lambda$ with λ a non-weakly compact cardinal we have $\mathrm{BB}(\lambda, \lambda, \lambda, \kappa)$.

Theorem 16.17. *(From* [She11, 1.18-LOp.15]*.) We have* $\mathrm{BB}(\lambda, \bar{C}, (\lambda, \theta), {<}\mu)$ *when:*

(A) $\mu \in \mathbf{C}_\kappa$, $\lambda = \mathrm{cf}(2^\mu)$ *and* $\theta < \mu$, $\sigma = \mathrm{cf}(\sigma) < \mu$*;*
(B) $S \subseteq S^\lambda_\sigma$ *is stationary;*
(C) $\bar{C} = \langle C_\alpha : \alpha \in S\rangle$, $C_\delta \subseteq \delta$, $|C_\delta| \leq \mu$*;*
(D) $\chi < 2^\mu \Rightarrow \chi^{\langle \sigma \rangle_{\mathrm{tr}}} < 2^\mu$*;*
(E) $\bar{C}$ *is* shallow*: that is,* $\big|\{C_\delta \cap \alpha : \alpha \in C_\delta\}\big| < \lambda$ *for* $\alpha < \lambda$*.*

The BB Trichotomy Theorem 16.15. If $\mu \in \mathbf{C}_\kappa$ and $\kappa > \sigma = \mathrm{cf}(\sigma)$, then at least one of the following holds:

(A) There is a μ^+-free $\mathcal{F} \subseteq {}^\kappa\mu$ of cardinality 2^μ
(B) (a) $\lambda := 2^\mu = \lambda^{<\lambda}$ (so λ is regular) and $\chi < \lambda \Rightarrow \chi^\sigma < \lambda$
 (b) if $S \subseteq S^\lambda_\sigma$ is stationary and $\bar{C} = \langle C_\alpha : \alpha \in S\rangle$ is a weak ladder system (i.e., $C_\delta \subseteq \delta$)[4] then
 (c) letting $J^{\mathrm{nst}}_S = \{A \subseteq \lambda : A \cap S$ is not stationary in $\lambda\}$ we have[5]
  $\bullet_1$ $\mathrm{BB}(J^{\mathrm{nst}}_S, \bar{C}, \theta, {\leq}\mu)$ for every $\theta < \mu$ provided that $\delta \in S \Rightarrow |C_\delta| < \mu$
  $\bullet_2$ $\mathrm{BB}(J^{\mathrm{nst}}_S, \bar{C}, (2^\mu, \theta), {<}\lambda)$ for any $\theta < \mu$
(C) (a) $\lambda_2 = 2^\mu$ is regular, $\chi < \lambda_2 \Rightarrow \chi^\sigma < \lambda_2$, and $\lambda_1 = \min\{\partial : 2^\partial > 2^\mu\}$ is both regular and strictly less than 2^μ
 (b) like (B)(b) for $\lambda = \lambda_2$ but $|C_\delta| < \lambda_1$ for $\delta \in S$ (so $C_\delta = \delta$ would not work)
 (c) $\mathrm{BB}(J^{\mathrm{nst}}_S, \mu^+, \theta, \kappa)$ for every $\theta < \mu$ and any stationary subset S of λ_1
 (c)′ like (B)(b), but for $\lambda = \lambda_1$, S a club or simply not in the weak diamond ideal [DS78].

Note 16.18. In [She20]

1) We get scales with some two-cardinal freeness properties. This is used to get somewhat free n-dimensional scales.
2) Hence for every $n < \omega$ there is an $\aleph_{\omega_1 n+1}$-free (but not $\aleph_{\omega_1 n+2}$-free) abelian group G such that $\mathrm{Hom}(G, \mathbb{Z}) = \{0\}$.
3) Also, a complementary consistency result.

Note 16.19. On tiny models: On tiny models see Laskovski, Pillay and Rothmaler [LPR92], M is tiny if $\mu = \|M\| < |T|, T$ categorical in $|T|^+$, where $|T|$ is the number of formulas up to equivalence. Assume further that for T not every regular type is trivial, then existence of such T for given μ is equivalent to the existence

[4] Bear in mind that a choice of $C_\delta = \delta$ would satisfy this
[5] What about freeness? We may get it by the choice of $\bar{C}$; also, if $\bar{C}$ is a ladder system (particularly if *strictly*), we will get a weak form (e.g. stability).

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of $A_i \in [\mu]^\mu$ for $i < \mu^+$ such that $\bigwedge_{i<j} |A_i \cap A_j| < \aleph_0$, hence necessarily $\mu < \beth_\omega$. (Proved in the appendix of [She00d]).

Note 16.20. On cofinalities of the symmetric group: Let Sp be the family of regular λ such that the permutation group of ω is the union of a strictly increasing chain of subgroups. Now Sp has closure properties under pcf, say if $n < \omega \Rightarrow \lambda_n \in \mathrm{Sp}$ then $\mathrm{pcf}\{\lambda_n : n < \omega\} \subseteq \mathrm{Sp}$ (Shelah and Thomas [ST97]).

Note 16.21. Hanf number

On application to Hanf numbers see Grossberg Shelah [GS86].

Note 16.22. On the number of non-isomorphic models: see [She09a, §2]. See more in [She16a], e.g. on groups. A survey on existence of universal, see [Dža05], more recently [S$^+$b].

Note 16.23. In addition, if $\mu = \aleph_\mu > \mathrm{cf}(\mu)$ then there is no universal linear order of cardinality $\lambda = \mu^+$. See [S$^+$c].

Note 16.24. We can sum: (see [S$^+$c, 6.1])

there is T such that if $\lambda \notin \mathrm{Univ}(T)$, then for some cardinal μ

(∗) $\lambda = \mu^+$ and (a) or (b), where:

(a) μ is singular strong limit such that $\mu = \aleph_\delta$, $\delta < \mu$ and $2^\mu > \lambda$,

(b) μ is regular, $2^{<\mu} \leq \lambda < 2^\mu$ and $\mathfrak{b}_\mu = \lambda_1 < \mathfrak{d}_\mu$.

(see [S$^+$c, 6.1])

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§ 17. Discussion

Thesis 17.1. Artificially/naturality thesis.

Probably you will agree that, for a polyhedron, v (number of vertices) e (number of edges) and f (number of faces) are natural measures, whereas $e+v+f$ is not, but from a deeper point of view $v-e+f$ runs deeper than all. In this vein we claim: for λ regular, 2^λ is the right measure of $\mathcal{P}(\lambda)$, and λ^κ is a good measure of $[\lambda]^{\le\kappa}$. However, the various cofinalities are better measures. λ^κ is an artificial combination of more basic things of two kinds: the function $\lambda \mapsto 2^\lambda$ (λ regular which is easily manipulated) and the various cofinalities we discuss (which are not). For example $\mathrm{pp}(\aleph_\omega) < \aleph_{\omega_4}$ is the right theorem, not $\aleph_\omega^{\aleph_0} < \aleph_{\omega_4} + (2^{\aleph_0})^+$ (not to say: $2^{\aleph_\omega} < \aleph_{\omega_4}$ when $\aleph_\omega$ is strong limit). Also the equivalence of the different definitions which give apparently weak and strong measures, show naturality:

- (*a*) $\mathrm{cf}([\aleph_\omega]^{\aleph_0}) = \mathrm{pp}(\aleph_\omega)$
- (*b*) $\min\{|S| : S \subseteq [\lambda]^{\le\kappa} \text{ stationary}\} = \mathrm{cf}([\lambda]^{\le\kappa}, \subseteq)$ for $\kappa < \lambda$
- (*c*) if $\lambda \ge \mu > \theta > \sigma = \mathrm{cf}(\sigma) > \aleph_0$ <u>then</u>

$$\mathrm{cov}(\lambda,\mu,\theta,\sigma) = \sup\{\mathrm{pp}_{\Gamma(\mathrm{cf}(\chi),\theta,\sigma)}(\chi) : \mu \le \chi \le \lambda, \sigma \le \mathrm{cf}(\chi) < \theta\}.$$

Note, $\chi \le \mathrm{pp}_\theta(\lambda)$ says $[\lambda]^{\le\kappa}$ is at least as large as χ in a strong sense, whereas $\chi \ge \min\{|S| : S \subseteq [\lambda]^{\le\kappa} \text{ stationary}\}$ says that $[\lambda]^{\le\kappa}$ can be exhausted very well by χ "points" (for the right filters: measure 1).

We tend to think the pp's are enough, but there is a gap is our understanding concerning cofinality $\aleph_0$, mainly: is it true that

$$(*)\quad \lambda > \mathrm{cf}(\lambda) = \aleph_0 \Rightarrow \mathrm{cf}([\lambda]^{\le\kappa}, \subseteq) = \mathrm{pp}_{\aleph_0}(\lambda).$$

We have many approximations saying that this holds in many cases (see 7.5).

More generally, we should replace power by products, and cardinality by cofinality, and therefore deal with $\mathrm{pcf}(\mathfrak{a})$.

Note 17.2. <u>The Cardinal Arithmetic below the continuum thesis</u>:

We should better investigate our various cofinalities without assuming anything on powers (for example, the difference between the old result $\mathrm{pp}(\aleph_\omega) < \aleph_{(2^{\aleph_0})^+}$ and the latter result $\mathrm{pp}(\aleph_\omega) < \aleph_{\omega_4}$ is substantial); as

- (*a*) you should try to get the most general result (when it has substance of course)
- (*b*) if we add many Cohen reals, all non-trivial products are $\ge 2^{\aleph_0}$, but our various cofinalities do not change, so we should not ignore this phenomenon
- (*c*) even if we want to bound 2^λ for λ strong limit singular, we need to investigate what occurs in the interval $[\lambda, 2^\lambda]$ which is a problem of the form indicated above; this is central concerning the problem (see [She96a]): if λ is the ω_1-fixed point then 2^λ is $<$ the ω_4-th fixed point
- (*d*) looking at cardinal arithmetic without assumptions on the function $\lambda \mapsto 2^\lambda$, makes induction on cardinality more useful.

Thesis 17.3.

(*A*) $\mathrm{pp}(\lambda)$ is the right power set operation.

$\lambda \mapsto 2^\lambda$ (λ regular) is very elastic, you can easily manipulate it, but $\mathrm{pp}(\lambda)$ (λ singular) and $\mathrm{cov}(\lambda,\mu,\theta,\sigma)$ are not; it is hard to manipulate them, and we can prove theorems about them in ZFC.

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(B) pcf, cov are basic operations, with non-trivial ZFC results.

1) Consider $[\lambda]^{\leq\kappa}$, the family of subsets of λ of cardinality $\leq \kappa$, when $\lambda > \kappa$ (see 17.1).
2) λ^κ is the crude measure of $[\lambda]^{\leq\kappa}$.

It is very interesting to measure it, and cardinality is generally a very crude measure; $\mathrm{pp}_\kappa(\lambda)$ is a fine measure; and we have intermediate ones: $\mathrm{cf}([\lambda]^{\leq\kappa}, \subseteq)$, $\min\{|S| : S \subseteq [\lambda]^{\leq\kappa}$ stationary$\}$ and more. The best is when we can compute cruder numbers from finer ones; particularly when they are equal, so we could use different definitions for the same cardinal depending on what we want to prove. So we want to show that the $\mathrm{pp}_{\Gamma(\mathrm{cf}(\lambda))}(\lambda)$ for λ singular is enough.

Note 17.4. $\mathrm{pp}_{\Gamma(\theta,\sigma)}(\lambda)$ is the finest we have for what we want; they are like the skeleton of set theory; you can easily change your dress and even can manage to change how much flesh you have; but changing your bones is harder. You may take hypermeasurable λ, blow up 2^λ and make it singular; this does not affect for example $\mathrm{pp}_{\Gamma(\aleph_1)}(\lambda^*)$ when $\lambda^* > \lambda, \mathrm{cf}(\lambda^*) = \aleph_1$ (even if $\lambda^* <$ new 2^λ), nor $\mathrm{cov}(\lambda^*, \lambda^*, \aleph_1, \sigma)(\sigma = 2, \aleph_1)$; they measure really how many subsets of λ^* of cardinality $\aleph_1$ there are - not through some $\lambda' < \lambda^*$ having many subsets of cardinality $\leq \aleph_1$.

Note 17.5. Subconscious remnants of GCH have continued to influence the research: concentration on strong limit cardinals; but from our point of view, even if $2^{\aleph_0}$ is large and $\mu < 2^{\aleph_0} \Rightarrow 2^\mu = 2^{\aleph_0}$, the cardinal arithmetic below $2^{\aleph_0}$ does not become simpler.

Also GCH was used as an additional assumption (or semi-axiom), but rarely was the negation of CH used like this: simply because one didn't know to prove interesting theorems from $\neg$CH. But now we know that violations of GCH have interesting consequences (see below).

Note 17.6. Up to now we have many consequences of GCH (or instances of it) and few of the negations of such statements. We now begin to have consequences of the negation, for example see here 11.10; so we can hope to have proofs by division to cases. For example, let λ be a strong limit singular; if $\mathrm{pp}(\lambda) > \lambda^+$ then $\mathrm{NPT}(\lambda^+, \mathrm{cf}(\lambda))$ and if $\mathrm{pp}(\lambda) \leq \lambda^+$ then $2^\lambda = \lambda^+$ (and $\diamondsuit^*_{\{\delta<\lambda^+:\mathrm{cf}(\delta)\neq\mathrm{cf}(\lambda)\}}$) and so various constructions are possible (see here 11.10(b) and [She97e] on more, also [She96b], [RS98]).

Note 17.7. The right problems.

An outside viewer may say that the main problem,

$$(\aleph_\omega = \beth_\omega \Rightarrow 2^{\aleph_\omega} < \aleph_{\omega_1})$$

was not solved. As an argument we may accuse others: maybe $\aleph_{\omega_4}$ is the right bound. But more to the point is our feeling that this is not the right problem. The right problems are:

(α) Does $\mathrm{pcf}(\mathfrak{a})$ always have cardinality $\leq |\mathfrak{a}|$?
(β) Is $\mathrm{cov}(\lambda, \lambda, \aleph_1, 2) =^+ \mathrm{pp}(\lambda)$ when $\mathrm{cf}(\lambda) = \aleph_0$?

Now (α) is just a member of a family of problems quite linearly ordered by implication discussed in [She93a, §6], [She00d], which seem unattackable both by the forcing methods and ZFC methods. The borderline between chaos and order seems

$(\alpha)^-$ Can $\mathrm{pcf}(\mathfrak{a})$ have an accumulation point which is an inaccessible cardinal? (Hopefully not.)

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Similarly (β) is the remnant of the conjecture that all $\mathrm{cov}(\lambda, \mu, \theta, \sigma)$ can be expressed by the values of $\mathrm{pp}_{\Gamma(\theta,\sigma)}(\lambda')$ and even $\mathrm{pp}_{\Gamma(\mathrm{cf}(\lambda'))}(\lambda')$; this has been proved in many cases (see 7.5). On an advance see [She00d].

Also though $(\alpha), (\beta)$ have not been solved, much of what we want to derive from them has been proved.

Another problem on which no light was shed is:

(γ) if λ is the first fixed point, find a bound on $\mathrm{pp}(\lambda)$ (or better $\mathrm{cov}(\lambda, \lambda, \aleph_1, 2)$).

We can hope for the ω_4-th fixed point, to serve as a bound but will be glad to have the first inaccessible as a bound. Even getting a bound assuming GCH below λ would open our eyes. This becomes a problem after [She86a], [She82, Ch.XII,§5,§6].

(δ) Generalize [She94b, §1] to deal with what occurs above $\mathrm{tlim}_I\ \lambda_i$ (for example 4.1, (λ, σ)-entangled linear order).

More accurately, assume $\prod\limits_{i<\delta(*)} \lambda_i/J$ has true cofinality $\lambda, \mu = \mathrm{tlim}_I(\lambda_i) = \sup(\lambda_i), \lambda_i$ regular $> \delta(*)$, and $\sup_{i<\delta(*)} \lambda_i < \theta = \mathrm{cf}(\theta) < \lambda$. We can find regular $\lambda'_i < \lambda_i$ such that $\mathrm{tcf}(\prod \lambda'_i/J) = \theta)$ as exemplified by $\bar{f}$, which is μ^+-free (hence $\mathrm{tlim}(\lambda'_i) = \lambda_i$) in addition: if $\delta < \theta, \mathrm{cf}(\delta) < \theta$ and $\mathrm{cf}(\delta) > 2^{|\delta(*)|}$ (or just $\bar{f} \restriction \delta$ has a $<_J$-lub) then without loss of generality f_δ/J is the $<_J$-lub of $\bar{f} \restriction \delta$, we want to know something on $\langle \mathrm{cf}(f_\delta(\alpha)) : \alpha < \delta \rangle$. For more information see [She94g, 4.1,4.1A].

Note that we also do not know, for example

(ε) if $\mathrm{cf}(\lambda) \le \kappa < \lambda$, is $\mathrm{cf}(\mathrm{pp}_\kappa(\lambda)) > \lambda$? (we know that it is $> \kappa$)

(ζ) we believe pcf considerations will eventually have impact on cardinal invariants of the continuum, but this has not materialized so far.

Note 17.8. The perspective here led to phrasing some hypotheses, akin to GCH or SCH.

The "strong hypothesis" says $\mathrm{pp}(\lambda) = \lambda^+$ for (every) singular λ. Note it is like GCH but is not affected by, say, c.c.c. forcing; it follows from $\neg 0^\#$ and from GCH; its negation is known to be consistent and I feel it is a natural axiom. Other hypotheses may still follow from ZFC: for example, the medium hypothesis says $|\mathrm{pcf}(\mathfrak{a})| \le |\mathfrak{a}|$, and the weak says $\{\mu : \mathrm{pp}(\mu) \ge \lambda,\ \mu < \lambda,\ \mathrm{cf}(\mu) = \aleph_0 [> \aleph_0]\}$ is countable [finite]. There are intermediate ones; such hypotheses and consequences are dealt with in [She93a, §6], see more in [She00d], [She02]. Particularly concerning the connection of the medium and weak ones, (see 13.3, 7.18), $\mathrm{ZF} + \mathrm{DC} + [\alpha]^{\aleph_0}$ well ordered suffice, see [Shee].

Note 17.9. In a major advance, Gitik has proven in [Git20] that the following version of the weak hypothesis fails: $\{\mu : \mathrm{cf}(\mu) = \kappa < \mu,\ \mathrm{pp}_{\kappa\text{-complete}}(\mu) \ge \lambda\}$ may be large.

Still open is whether, in the RGCH (see [She00d]), we can replace $\beth_\omega$ by $\aleph_\omega$.

* * *

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§ 18. Part B - Corrections to the book [She94f]

page 50,line 22: see more in Part C.

page 51,line 12: replace λ^+ by μ.

page 51,line 13: see more in Part C.

page 66,Theorem 3.6: second line of theorem:
replace $\lambda^{\beta+1}$ by $\lambda_0^{\beta+1}$
add after the second line of Remark 3.6A:
2) This is essentially the proof from [She82, Ch.XIII,§6] and more appears in Ch.IX

first line of the proof:
replace $\lambda > \aleph_0$ by "$\lambda_0 > |\alpha|^+$ (why? as we can replace λ_0 by λ_0^+ and deduce the result on the original λ_0 from the result on λ_0^+)"

replace fifth line of the proof:

$$N_h = \bigcap \big\{\text{Skolem Hull}_M \big(\lambda_0 \cup \bigcup_{\beta<\alpha} C_\beta\big) : C_\beta \text{ a club of } f(\lambda^{+\beta+1}) \text{ for } \beta < \alpha\big\}$$

add in the end of the proof:
Clearly this family is a family of subsets of λ each of cardinality at most λ_0 of the right cardinality. So we have to prove just that it is cofinal. So let X be a subset of λ of cardinality at most λ_0, and we shall find a member of the family which includes it. Let χ be large enough. By 3.4 we can find an elementary submodel N_i of $(\mathcal{H}(\chi), \in, <^*_\chi)$, for $i \leq \delta =: |\alpha|^+$ each of cardinality such that $\{F, \lambda_0, \alpha^*, \lambda, X, f, g\} \in N_i$ and $i < j \to N_i \in N_j$ increasing continuous with i and condition (b) form 3.4 holds for $f \in F$.

It is enough to prove that

(∗) N_f includes $N_\delta \cap \lambda$

for this it is enough to prove

(∗∗) if C_β is a club of $\lambda_0^{\beta+1}$ for each $\beta < \alpha^*$ and M' is the Skolem Hull in M of $\lambda_0 \cup \bigcup\{C_\beta : \beta < \alpha^*\}$ then M' include $N_\delta \cap \lambda$.

For this we prove by induction on $\gamma \leq \alpha$ that

$(**)_\gamma$ M' includes $\lambda \cap \lambda_0^{+\gamma}$.

Case 1: $\gamma = 0$.
In this case as M includes λ_0 this is trivial.

Case 2: γ a limit cardinal ordinal.
In this case the induction hypothesis implies the conclusion trivially.

Case 3: $\gamma = \beta + 1$.
Use the induction hypothesis and the choice of the functions f and g. (See more Ch.IX, 3.3)

page 136,lines 21,22,23:
replace by:
No problem to define. We define B_i^α (for $i < \lambda, \alpha \in S$) by induction on α:

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$$B_i^\alpha = \begin{cases} \{\beta : \mathrm{cf}(\beta) \neq \lambda \text{ and } \beta \in A_i^\alpha \vee \beta = \sup(\beta \cap A_i^\alpha)\} & \underline{\text{if}}\ \mathrm{cf}(\alpha) \neq \aleph_1 \\ \bigcap \{ \bigcup_{\beta \in C} B_i^\beta : C \text{ a club of } \alpha \text{ such that } \bigwedge_{\beta \in C} \mathrm{cf}(\beta) = \aleph_0 \} & \underline{\text{if}}\ \mathrm{cf}(\alpha) = \aleph_1 \end{cases}$$

(or see [She91a, 4.1]).

page 210, line 15:
add: or λ is not Mahlo and we can use Ch.III.

page 222,line 24: replace by:
Definition 1.4. 1) We say D is strongly nice if it is strongly nice to every

page 224,line 8: replace by:

$$\sup \Big\{ \prod_{i<\omega_1} f(i)/D : D \text{ is a normal filter extending } D^* \Big\}$$

page 228,line 1: replace D^* by $D^* \in V^*$.

pages 334-337: see a rewriting in [Shea]

page 334,line -4 replace by:
(2) The first phrase follows from part 1 and check the second

page 335,line 4: replace "$f \restriction \mathfrak{b}\mu[\mathfrak{a}] \leq f_\alpha^\mu$" by "$f \restriction \mathfrak{b}_\mu[\mathfrak{a}] \leq f_\alpha^\mu$"

page 335,line 18: space after $\varnothing$; replace $\bigcap_{\ell=1}^n \mathfrak{b}_{\sigma_\ell}[\mathfrak{a}]$ by $\bigcup_{\ell=1}^n \mathfrak{b}_{\sigma_\ell}[\mathfrak{a}]$

page 336,line 3: replace $\mathfrak{b}$ by $\mathfrak{c}$

page 336,line -7: replace $\square_{3.3}$ by $\square_{3.2}$

page 381,lemma3.5 and page 383,line 21: No! But see [She94g, 5.12] and [She02, §6]

page 410,line -1: replace by: $\{\delta < \sigma : \mathrm{cov}(\lambda_\delta, \lambda_\delta, \theta^+, 2) < \mu_\delta\}$ contains a club of σ, where

$(*)(i)$ let μ_δ be $\mathrm{pp}_\theta^{\mathrm{cr}}(\lambda_\delta)$ the first regular $\mu > \lambda_\delta$ such that:
if $\mathfrak{a} \subseteq \mathrm{Reg} \cap \lambda_\delta \setminus |\mathfrak{a}|^+$, then $\sup\{\max \mathrm{pcf}(\mathfrak{b}) : \mathfrak{b} \subseteq \mathfrak{a}, |\mathfrak{b}| \leq \theta$ and $(\forall \chi < \lambda_\delta) \max \mathrm{pcf}(\mathfrak{b} \cap \chi) < \lambda_\delta\}$ (so normally this means $\mathrm{cov}(\lambda_\delta, \lambda_\delta, \theta^+, 2) =^+ \mathrm{pp}_\theta(\lambda_\delta)$).

page 411,line 1: replace by:

(ii) $\mathrm{cov}(\lambda, \lambda, \theta^+, 2) < \mathrm{pp}_\theta^{\mathrm{cr}}(\lambda)$ which normally means $\mathrm{cov}(\lambda, \lambda, \theta^+, 2) =^+ \mathrm{pp}_\theta(\lambda)$, e.g. if $\mathrm{cov}(\lambda_i, \theta^+, \theta^+, 2) < \lambda$ for a club of $i < \sigma$

(iii) if e.g. $\sigma^{\aleph_0} < \lambda$, then we can add $\{\delta < \sigma$: if $\mathrm{cf}(\delta) = \aleph_0$ then $\mathrm{pp}_{J_\omega^{\mathrm{bd}}}^{\mathrm{cr}}(\lambda_\delta) > \mathrm{cov}(\lambda_\delta, \lambda_\delta, \theta, 2)\}$ contains a club (for the changes needed for the proof see below, Part C).

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page 417,line 11: add:

Here examples are constructed for λ singular and in [She97a] for $\lambda = \aleph_1$ which was the last case.

page 418,line 20: sequence of not sequence of ...

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§ 19. Part C - Expansions for [She94f]

§17 Short Expansions

page 50,line 22: add: [this is the proof of II,1.4(3)].

Case 1: otp(A) is zero.
Trivial.

Case 2: otp(A) is a successor ordinal.
Let α be the last member of A and let A' by $A \setminus \{\alpha\}$. Clearly the order type of A' is (strictly smaller than that of A) hence by the induction hypothesis we can find $s'_\beta \in i$ for $\beta \in A'$ as required. Define s_β for $\beta \in A$ as follows:

if $\beta = \alpha$, then $s_\beta = \varnothing$ and if $\beta \in A'$ then $s_\beta =: \{i < \kappa : i \in s'_\beta \text{ or } f_\alpha(i) \leq f_\beta(i)\}$. Now s_β is a subset of κ and if $\beta = \alpha$ is the union of two sets: s'_β and $\{i < \kappa : f_\alpha(i) \leq f_\beta(i)\}$, now the first belongs to I by its choice and the second as we know $f_\beta <_I f_\alpha$ (because $\beta < \alpha$). So S_β, their union is in I, too.

This holds also in the case $\beta = \alpha$. So $s_\beta \in I$ for $\beta \in A$, and it is easy to check the requirements.

Case 3: otp(A) is a limit ordinal.
Let δ be sup(A), so is a limit ordinal. So by 1.4(ii)(δ) there is a closed unbounded subset C of δ and sets $\tau_\alpha \in I$ for $\alpha \in C$ such that $i \in \kappa \setminus \tau_\alpha \setminus s_\beta$ and $\alpha < \beta$ implies $f(i) < f_\beta(i)$.

Without loss of generality $0 \in C$ (let $t_0 =: \{i < \kappa : f_0(i) \geq f_{\min(A)}(i)\}$).

Now for every $\alpha \in C$ let $A_\alpha =: A \cap [\alpha, \min(A \setminus (\alpha+1))$. Clearly $\text{otp}(A_\alpha) < \text{otp}(A)$, let $A'_\alpha =: A_\alpha \cup \{\alpha\}$. So $\text{otp}(A'_\alpha) = 1 + \text{otp}(A_\alpha) < \text{otp}(A)$ (as the latter is a limit ordinal). So we can apply the induction hypothesis, getting s'_β for $\beta \in A'_\alpha$ as guaranteed there.

Now we define s_β for $\beta \in A$ as follows: let $\alpha_\beta =: \sup(C \cap \beta)$ and $\gamma_\beta =: \min(A \setminus (\alpha + 1))$. So $\beta \in A_{\alpha_\beta}$, hence s'_β is well defined, and let

$$s_\beta =: s'_\beta \cap \{i < \kappa : \text{ it is not true that } f_{\alpha_\beta}(i) \leq f_\beta(i)\}.$$

Now check.

* * *

page 51, line 13: add to the end of line (this is line 7 of the proof of II,1.5A).

Of course, we do not have knowledge on the relation between $f_\alpha(i)$ and $f_\beta(j)$, so we just e.g. use f'_α defined by $f'_\alpha(i) =: \kappa f_\alpha(i) + i$ (so f'_α is a function from κ to λ, as $\kappa < \lambda$). Now $\langle f'_\alpha : \alpha < \mu \rangle$ is as required (note that $\big\langle \{f_\alpha(i) : i < \mu\} : i < \kappa \big\rangle$ is a sequence of pairwise disjoint subsets of λ).

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§ 20. More on II,3.5

This refinement is used in [Shef].

Claim 20.1. *Assume*

(a) $\mathfrak{a} = \{\lambda_i : i < \delta\}$ *is an increasing sequence of regular cardinals* $> \delta$
(b) $\lambda = \mathrm{tcf}_\pi(\mathfrak{a}, <_{J^{\mathrm{bd}}_\delta})$
(c) $\lambda_0 > 2^{|i|}$ *for* $i < \delta$ *or just* $i < \delta \Rightarrow \lambda_0 > |\mathrm{pcf}(\mathfrak{a})\lambda_i|$
(d) $\mathrm{cf}(\delta) > \aleph_0$
(e) $S =: \Big\{i < \delta :$ *for some* $i_0 < i_1,\ \mathrm{pcf}\{\lambda_j : i_0 < j < i\} \setminus \sum\limits_{j<i} \lambda_j$ *is a singleton cardinal* $< \underset{j<\delta}{\to \sup} \lambda_j\Big\}$ *is stationary.*

then we can find $\langle f_\alpha : \alpha < \lambda\rangle$ *such that*

(a) $f_\alpha \in \prod\limits_{i<\delta} \lambda_i$ *is* $<_{J^{\mathrm{bd}}_\delta}$*-increasing and cofinal*
(b) *if* $f \in \prod\limits_{i<\delta} \lambda_i$ *and* $(\forall i < \delta)(\exists \alpha < \lambda)[f \restriction i = f_\alpha \restriction i]$ *then* $f \in \{f_\alpha : \alpha < \lambda\}$.

Remark 20.2. This is just the proof of [She94f, Ch.II,3.5], just we use more of it.

Proof. Let $\mu = \sum\limits_{i<\delta} \lambda_i$ and $\mu_j = \sum\limits_{i<j} \lambda_i$ for $j < \delta$.

Recall $\mathfrak{a} = \{\lambda_i : i < \delta\}$, so $\min(\mathfrak{a}) > |\mu \cap \mathrm{pcf}(\mathfrak{a})|$. Let $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\theta : \theta \in \mathrm{pcf}(\mathfrak{a})\rangle$ be a generating sequence for $\mathrm{pcf}(\mathfrak{a})$. Choose $\langle \bar{f}^\theta : \theta \in \mathrm{pcf}(\mathfrak{a})\rangle$ as in claim 20.4 below. Now we let

$$\mathscr{F} = \Big\{f \in \prod_{i<\delta} \lambda_i : \text{for every } \theta \in \mathrm{pcf}(\mathfrak{a}), \text{ for some } n < \omega \text{ and } \theta_0 < \ldots < \theta_{n-1} \text{ from } \mathrm{pcf}(\mathfrak{b}_\theta) \text{ and } \alpha_0 < \theta_0, \ldots, \alpha_{n-1} < \theta_{n-1} \text{ we have } f \restriction \mathfrak{b}_\theta = \max\{f^{\theta_\ell}_{\alpha_\ell} : \ell < n\}\Big\}.$$

Let $f_\alpha := f^\lambda_\alpha$ for $\alpha < \lambda$.
First clearly

$(*)_1$ $\alpha < \lambda \Rightarrow f_\alpha \in \mathscr{F}$.

Secondly, the main point is

$(*)_2$ if $f', f'' \in \mathscr{F}$ then $f' <_{J^{\mathrm{bd}}_{\mathfrak{a}\delta}} f''$ or $f' =_{J^{\mathrm{bd}}_\delta} f''$ or $f'' <_{J^{\mathrm{bd}}_{\mathfrak{a}}} f'$.

Why does $(*)_2$ hold? Given $f', f'' \in \mathscr{F}$, let

$$\mathfrak{c}_1 = \{\theta \in \mathfrak{a} : f'(\theta) < f''(\theta)\},$$
$$\mathfrak{c}_2 = \{\theta \in \mathfrak{a} : f'(\theta) = f''(\theta)\},$$
$$\mathfrak{c}_3 = \{\theta \in \mathfrak{a} : f'(\theta) > f''(\theta)\},$$

so $\langle \mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3\rangle$ is a partition of $\mathfrak{a}$. Let $E = \{i < \delta :$ for $\ell = 1, 2, 3$ if $\sup(\mathfrak{c}_\ell) = \sup(\mathfrak{a})$ then $\sup(\mathfrak{c}_\ell \cap \lambda_i) = \sup(\mathfrak{a} \cap \lambda_i)$ and if $\sup(\mathfrak{c}_\ell) < \sup(\mathfrak{a})$ then $\sup(\mathfrak{c}_\ell) < \lambda_j$ for some $j < i\}$.

Clearly E is a club of δ; by clause (c) of the assumption, $S \subseteq \delta$ is stationary hence $S \cap E \neq \varnothing$, so let $i \in S \cap E$ and let θ_i be the single member of $\mathrm{pcf}(\mathfrak{a} \cap \lambda_i) \setminus \mu_i = \mathrm{pcf}(\{\lambda_j : j < i\}) \setminus \mu_i$ (recall the definition of S). So $\mathfrak{b}_{\theta_i}$ contains an end-segment of $\mathfrak{a} \cap \lambda_i$ — say $\mathfrak{b}'$. By the choice of $\mathscr{F}$ and the assumption $f', f'' \in \mathscr{F}$ and the choice

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of θ_i, we know that for some end segment $\mathfrak{b}''$ of $\mathfrak{b}', f' \restriction \mathfrak{b}'' \in \{f^{\theta_i}_\alpha \restriction \mathfrak{b}'' : \alpha < \theta_i\}$ and without loss of generality also $f'' \restriction \mathfrak{b}'' \in \{f^{\theta_i}_\alpha \restriction \mathfrak{b}'' : \alpha < \theta_i\}$. So for some β', $\beta'' < \theta_i$ we have $f' \restriction \mathfrak{b}'' = f^{\theta_i}_{\beta'} \restriction \mathfrak{b}''$ and $f'' \restriction \mathfrak{b}'' = f^{\theta_i}_{\beta''} \restriction \mathfrak{b}''$.

Now $\beta' < \beta'' \vee \beta' = \beta'' \vee \beta' > \beta''$ and accordingly we get one of the three possibilities in $(*)_2$.

Now clearly we are done. $\square_{20.1}$

Claim 20.3. *1) In 20.1 we can weaken assumption (e) to*

- $(e)^-_{\mathfrak{a}}$ *letting $\langle \mu_i : i < \sigma\rangle$ be increasing continuous with limit* $\sup(\mathfrak{a})$ *so* $\sigma = \mathrm{cf}(\sup(\mathfrak{a}))$ *for some normal filter D on* $\mathrm{cf}(\sup(\mathfrak{a})) = \mathrm{cf}(\delta)$ *we have:*
- $(e)^-_D$ *if* $\mathfrak{a}' \subseteq \mathfrak{a}(=: \{\lambda_i : i < \delta\})$ *and* $\sup(\mathfrak{a}') = \sup(\mathfrak{a})$ *then*
$$\{i < \sigma : \max(\mathrm{pcf}(\mathfrak{a}' \cap \mu_i)) = \max(\mathrm{pcf}(\mathfrak{a} \cap \mu_i))\} \in D.$$

2) Assume $\mathfrak{a}$ has no last element, $\mathrm{cf}(\sup(\mathfrak{a})) > \aleph_0$, *and* $\lambda = \mathrm{tcf}(\pi\mathfrak{a}/J^{\mathrm{bd}}_{\mathfrak{a}})$ *and* $\mu < \sup(\mathfrak{a}) \Rightarrow \max\mathrm{pcf}(\mathfrak{a} \cap \mu) < \sup(\mathfrak{a})$, *(e.g.* $\mathfrak{a} = \{\lambda_i : i < \delta\}$ *from 20.1 assuming clauses (a)-(d) of 20.1.*

<u>then</u> for some unbounded $\mathfrak{a}^* \subseteq \mathfrak{a}$, we have (clause (a), (b), (c) of 20.1 and) clause $(e)^-_{\mathfrak{a}^*}$ of part (1) holds (hence the conclusion of 20.1).

Proof. 1) Let $\mathfrak{a} = \{\lambda_i : i < \delta\}$ such that λ_i is regular increasing with i.

We repeat the proof of 20.1. So our problem is that in proving $(*)_2$, so we have $f', f'' \in \mathscr{F}$ and having defined the partition $\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3$ of $\mathfrak{a}$, at least two parts are unbounded in $\mathfrak{a}$ say $\mathfrak{c}_{\ell_1}, \mathfrak{c}_{\ell_2}$

$\boxtimes$ if $\theta \in \mathrm{pcf}(\mathfrak{c}_\ell) \setminus \{\lambda\}$ then $\mathfrak{b}_\theta \setminus \mathfrak{c}_\ell \in J_\theta[\mathfrak{a}]$.

[Why $\boxtimes$? As in the proof of 20.1, we know that $f', f'' \in \mathscr{F}$ hence for some $\mathfrak{c} \in \mathbf{J}_{<\theta}[\mathfrak{a}]$ we have $f' \restriction (\mathfrak{b}_\theta \setminus \mathfrak{c}), f'' \restriction (\mathfrak{b}_\theta \setminus \mathfrak{c})$ belongs to $\{f^\theta_\alpha \restriction (\mathfrak{b}_\theta \setminus \theta) : \alpha < \theta\}$ and we continue as there.] Now for $\ell = 1, 2, 3$ we have $\sup(\mathfrak{c}_\ell) = \sup(\mathfrak{a}) \Rightarrow S_\ell := \{j < \sigma : \max\mathrm{pcf}(\mathfrak{c}_\ell \cap \mu_\ell) = \max\mathrm{pcf}(\mathfrak{a} \cap \mu_\ell)\} = \mathrm{cf}(\sup(\mathfrak{a}))\} \in D$ also $E := \{i : \sup(\mathfrak{a} \cap \mu_i) = \mu_i\}$ is a club of σ. Hence $S = E \cap S_1 \cap S_2 \cap S_3 \in D$.

So for the D-majority of $j < \sigma$ we have $\sup(\mathfrak{c}_{\ell_1} \cap \mu_j) = \mu_j = \sup(\mathfrak{c}_{\ell_2} \cap \mu)$ and $\max\mathrm{pcf}(\mathfrak{c}_{\ell_j} \cap \mu_j) = \max\mathrm{pcf}(\mathfrak{a} \cap \mu_j) = \max\mathrm{pcf}(\mathfrak{c}_{\ell_2} \cap \mu_j)$ and we get contradiction by $\boxtimes$.

2) We try to choose $\langle \mathfrak{a}_\eta : \eta \in {}^n\sigma\rangle$ by induction on $< \omega$ such that

- (i) $\mathfrak{a}_{\langle\ \rangle} = \mathfrak{a}$
- (ii) $\mathfrak{a}_\eta \subseteq \mathfrak{a}_{\eta \restriction n}$ for $\eta \in {}^{n+1}\sigma$
- (iii) $\sup(\mathfrak{a}_\eta) = \sup(\mathfrak{a})$
- (iv) for every $\eta \in {}^n\sigma$ for some club E_η of σ we have: for every $j \in E_\eta$ there is $i < j$ such that $\max\mathrm{pcf}(\mathfrak{a}_{\eta\frown\langle i\rangle} \cap \mu_j) < \max\mathrm{pcf}(\mathfrak{a}_n \cap \mu)$.

Now for $n = 0$ there is no problem and if $\mathfrak{a}_n, \langle \mathfrak{a}_\eta : \eta \in {}^n\sigma\rangle$ has been chosen but there is no suitable $\langle \mathfrak{a}_\eta : \eta \in {}^{n+1}\sigma\rangle$ then for some $\eta \in {}^n\sigma$ letting

$$\mathscr{P}_\eta = \big\{\{i < \sigma : \max\mathrm{pcf}(\mathfrak{b} \cap i) < \max\mathrm{pcf}(\mathfrak{a}_\eta \cap i)\} : \mathfrak{b} \subseteq \mathfrak{a}_\eta, \sup(\mathfrak{b}) = \sup(\mathfrak{a}_\eta)\big\},$$

the normal ideal D_η (on σ) which $\mathscr{P}_\eta$ generates satisfies $\varnothing \notin D_\eta$ so $\mathfrak{a}_\eta, D_\eta$ are as required. Lastly, not all the $\mathfrak{a}_\eta$'s are defined as then we let

$$E = \{i < \sigma : i \text{ a limit ordinal such that } \eta \in {}^{\omega>}i \Rightarrow i \in E_\eta\}$$

Clearly E is a club of σ. Now for any $i \in E$, we choose, by induction on $n < \omega$, a sequence $\eta_n \in {}^n i$ such that $\eta_n \triangleleft \eta_{n+1}$ and $\max\mathrm{pcf}(\mathfrak{a}_{\eta_n}) > \max\mathrm{pcf}(\mathfrak{a}_{\eta_{n+1}})$. We let $\eta_0 = \langle\ \rangle$, and η_{n+1} will exist by clause (iv). So $\langle \max\mathrm{pcf}(\mathfrak{a}_{\eta_n}) : n < \omega\rangle$ is a strictly decreasing sequence of cardinals, a contradiction. So we are done. $\square_{20.3}$

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Claim 20.4. *Assume*

(a) $|\mathrm{pcf}(\mathfrak{a})| < \min(\mathfrak{a})$, $\mathfrak{a}$ *as usual a set of regular cardinals*

(b) $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\theta : \theta \in [\mathfrak{a}] \rangle$ *a generating sequence for* $\mathrm{pcf}(\mathfrak{a})$ *(exists by x.x - FILL) which is closed (i.e.* $\mu \in \mathfrak{b}_\theta \Rightarrow \mathfrak{b}_\mu \subseteq b_\theta$*) and smooth (i.e.* $\mathrm{pcf}(\mathfrak{b}_\mu) \cap \mathfrak{a} = \mathfrak{b}_\mu$*). We can choose by induction on* $\theta \in \mathrm{pcf}(\mathfrak{a})$, $\bar{f}^\theta = \langle f^\theta_\alpha : \alpha < \lambda \rangle$ *such that*

(α) $f^\theta_\alpha \in \prod \mathfrak{b}_\theta$ *is* $<_{J_{<\theta}[\mathfrak{b}_\theta]}$*-increasing and cofinal*

(β) *if* $\theta \in \mathrm{pcf}(\mathfrak{a})$, $\alpha < \theta$ *and* $\mu \in \mathfrak{b}_\theta$, *then for some* $n < \omega$, $\mu_0, \ldots, \mu_{n-1} \in \mathrm{pcf}(\mathfrak{b}_\mu)$, $\beta_0 < \mu_0, \ldots, \beta_{n-1} < \mu_{n-1}$, *and* $\beta < \mu$ *we have*

$$f \restriction \mathfrak{b}_\mu = \max[\{f^{\mu_\ell}_{\beta_\ell} : \ell < n\}].$$

Proof. This is a restatement of [She94f, Ch.VII,§1]. $\square$

Claim 20.5. *Assume* κ *is regular and* $\bar{\theta} = \langle \theta_i : i < \kappa \rangle$ *is a sequence of regular cardinals* $> \kappa^+$. *then for some* $u, E, \bar{\lambda}, \lambda$ *and* D *we have*

(a) $u \subseteq \kappa$ *is unbounded*

(b) $\lambda = \mathrm{tcf}(\prod_{i \in u} \theta_i, <_{J^{\mathrm{bd}}_u} \rangle$

(c) $E := \{\delta < \kappa : \delta$ *a limit ordinal and* $\delta = \sup(u \cap \delta)$ $\}$

(d) $\bar{\lambda} = \langle \lambda_\delta : \delta \in E \rangle$

(e) $\lambda_\delta = \max \mathrm{pcf}\{\theta_i : i \in u \cap \delta \setminus j\}$ *for every* $j \in [j_\delta, \delta)$

(f) $\lambda = \mathrm{tcf}(\prod_{i \in u} \theta_i, <_{J^{\mathrm{bd}}_u})$

(g) $\mathscr{D}$ *is a normal filter on* κ *extending* $\mathscr{D}_\kappa$

(h) *if* $A \in \mathscr{D}^+$, $v_\delta \subseteq u \cap \delta$, $j_\delta < \delta$, $\lambda_\delta > \max \mathrm{pcf}\{\theta_i : i \in u \cap \delta \setminus v_\delta \setminus j_\delta\}$ *for* $\delta \in A$ *then* $\bigcup\{v_\delta : \delta \in A\}$ *is a co-bounded subset of* u.

Remark 20.6. We can add:

(i) if v is an unbounded subset of u then the set

$$\Big\{i < \kappa : \max \mathrm{pcf}\big(\{\theta_j : j \in i \cap v\}\big) = \max \mathrm{pcf}\big(\{\theta_j : j \in i \cap u\}\big)\Big\}$$

belongs to $\mathscr{D}$.

Proof. By the pcf theorem there is $u_0 \in [\kappa]^\kappa$ such that

(∗) $\lambda = \mathrm{tcf}\big(\prod_{i \in u_0} \theta_i, <_{J^{\mathrm{bd}}_{u_0}}\big)$ is well defined.

Now for every $u \in [u_0]^\kappa$ we define E_u and $\langle \lambda^u_\delta : \delta \in E_u \rangle$ as in clauses (c),(e) and stipulate $\lambda^u_i = 0$ for $i \in \kappa \setminus E_u$ and let $\bar{\lambda}^u = \langle \lambda^u_i : i < \kappa \rangle$. So $\gamma_u = \mathrm{rk}_{\mathscr{D}_\kappa}(\langle \lambda^u_i : i < \kappa \rangle)$ is a well defined ordinal and we can choose $u_1 \in [u]^\kappa$ such that γ_{u_1} is minimal. Let

$$\mathscr{D}^*_u = \big\{A \subseteq \kappa : A \in \mathscr{D}_\kappa \text{ or } A \in \mathscr{D}^+_\kappa \setminus \mathscr{D}_\kappa \text{ and } \gamma_{u_1} < \mathrm{rk}_{D + (\kappa \setminus A)}(\langle \lambda^u_i : i < \kappa \rangle)\big\}.$$

As for clause (h), but [She00a], $\mathscr{D}_u$ is a normal filter on κ (extending $\mathscr{D}_\kappa$). For proving [?] assume that $A \in \mathscr{D}^+_u$, $\bar{v} = \langle v_\delta : \delta \in A \rangle$, $\bar{j} = \langle j_\delta : \delta \in A \rangle$ and $v_\delta \subseteq u_1 \cap \delta$, $j_\delta \cap \delta$, $j_\delta < \delta$ and $\lambda^{u_1}_\delta > \max \mathrm{pcf}\{\theta_i : i \subseteq \delta \cap u_1 \setminus v_\delta \setminus j_\delta\}$.

We should prove that $v := u_1 \setminus \bigcup\{v_\delta : \delta \in A\}$ is bounded in κ. Toward contradiction assume $\kappa = \sup(v)$ and we shall prove that $\gamma_v < \gamma_u$, thus deriving the desired contradiction

$(**)_1$ $\gamma_{u_1} = \mathrm{rk}_{\mathscr{D}_\kappa}(\bar{\lambda}^{u_1})$.

But by the choice of $\mathscr{D}_u$

$(**)_2$ $\mathrm{rk}_{\mathscr{D}_\kappa}(\bar{\lambda}^{u_1}) = \mathrm{rk}_{\mathscr{D}_\kappa + A}(\bar{\lambda}^{u_1})$.

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Now clearly by our assumption

$(**)_3$ $\delta \in A \Rightarrow \lambda^v_\delta < \lambda^{u_1}_\delta$

hence

$(**)_4$ $\bar\lambda^v < \bar\lambda^{u_1} \mod (\mathscr{D}_\kappa + A)$ hence

$(**)_5$ $\mathrm{rk}_{\mathscr{D}_\kappa + A}(\bar\lambda^{u_1}) > \mathrm{rk}_{\mathscr{D}_\kappa + A}(\bar\lambda^v)$.

Now by a monotonicity property of $\mathrm{rk}_D(\bar\lambda^v)$ in D

$(**)_6$ $\mathrm{rk}_{\mathscr{D}_\kappa + A}(\bar\lambda^v) \geq \mathrm{rk}_{\mathscr{D}_\kappa}(\bar\lambda^v)$.

But

$(**)_7$ $\mathrm{rk}_{\mathscr{D}_\kappa}(\bar\lambda^v) = \gamma_v$.

Together $(**)_1 - (**)_7$ gives $\gamma_{u_1} > \gamma_v$, contradicting the choice of u_1. The contradiction comes from assuming that v is unbounded in κ, so $\sup(v) < \kappa$, thus finishing the proof of clause (h) and of the claim. $\square_{20.5}$

Remark 20.7. We can replace $(J^{\mathrm{bd}}_\kappa, \mathscr{D}_\kappa)$ by other such pairs (on κ or on $[\mu]^{<\kappa}$).

Observation 20.8. *Assume $\theta = \mathrm{cf}(\theta)$ and $\bar\lambda = \langle \theta_i : i < \kappa\rangle$ is an increasing sequence of regular cardinals $> \kappa^{++}$ and $\lambda = \mathrm{tcf}(\prod \theta_i, <_{J^{\mathrm{bd}}_\sigma})$. Then we can find an $u, \mathscr{F}, \bar f$ such that*

(a) *$u \subseteq \kappa$ is unbounded*
(b) *$\bar f = \langle f_\alpha : \alpha < \lambda\rangle$ such that*
(c) *$f_\alpha \in \prod\limits_{i \in u} \theta_i$*
(d) *$\langle f_\alpha : \alpha < \lambda\rangle$ is $<_{J^{\mathrm{bd}}_u}$-increasing cofinal in $(\prod\limits_{i\in u} \theta_i, <_{J^{\mathrm{bd}}_u})$*
(e) *$\mathscr{F} \subseteq \prod\limits_{i\in u}\theta_i$ includes $\{f_\alpha : \alpha < \lambda\}$ and $\big|\{f \restriction \delta : f \in \mathscr{F}\}\big| \leq \lambda_\delta$ for $\delta \in E$*
(f) *if $f \in \prod\limits_{i\in u}\theta_i$ for every $j < \theta$ for some $g \in \mathscr{F}$ we have $f_\alpha \restriction (j \cap u) = g \restriction (j \cap u)$ then $f \in \mathscr{F}$*
(g) *$\mathscr{F}$ is linearly ordered by $<_{J^{\mathrm{bd}}_u}$.*

Let $u, E, \langle \lambda_i : i < \kappa\rangle, \mathscr{D}$ be as in the previous claim. As we can ...?

For $j \in E_i$ let $J_j = \{v \subseteq u \cap j : \max \mathrm{pcf}(\lambda_i : i \in [j', j) \cap u) < \lambda_i$ for some $j' < j\}$. For each $\delta \in E_u$ choose $\langle f^\delta_\alpha : \alpha < \lambda^u_j\rangle$ such that

$\circledast_\delta$ (a) $f^\delta_\alpha \in \prod\limits_{i \in u\cap\delta} \theta_i$
(b) *$\bar f^\delta = \langle f^\delta_\alpha : \alpha < \lambda_j\rangle$ is $<_{J_\delta}$-increasing and cofinal in $(\prod\limits_{j \in u\cap\delta}\theta_i, <_{J_i})$*
(c) *if $\bar f^j \restriction \delta$ has a $<_{J_j}$-l.u.b. then f^j_δ is an increasing $<_{J_j}$-l.u.b.*

Let

$$\circledast\ \mathscr{F}^* = \{f \in \prod_{i\in u} \lambda_i : (\forall \delta \in E)(\exists \alpha < \lambda_\delta)\ f \restriction (u \cap \delta) = f^\delta_\alpha \mod J_\delta\}.$$

For $f \in \mathscr{F}$ let g^+_f be the function with domain E, $g^+_f(\delta) = \alpha$, α as above (clearly it is unique). By [Shear, xxx] *- FILL*

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⊛ *we can find* $f_\alpha \in \mathscr{F}$ *for* $\alpha < \lambda$ *such that* $\bar f = \langle f_\alpha : \alpha < \lambda\rangle$ *is* $<_{J^{\rm bd}_u}$*-increasing cofinal in* $(\prod\limits_{i \in u} \lambda_i, <_{J^{\rm bd}_u})$ *(and if* $\alpha > \kappa, \bar f \restriction \alpha$ *has a* $<_{J^{\rm bd}_u}$*-e.u.b. then* f_α *is such* $<_{J^{\rm bd}_u}$*-e.u.b..*

Assume toward contradiction

⊠ $f_1, f_2 \in \mathscr{F}$ *and* $u_1 = \{i \in u : f_1(i) < f_2(i)\}$ *is unbounded in* κ *and also* $u_2 := u \setminus u_1$ *is unbounded* $\theta\kappa$*.*

Now E *is partitioned to*

$$A_1 = \{\delta \in E : g_{f_1}(\delta) < g_{f_2}(\delta)\} \text{ and}$$
$$A_2 = \{\delta \in E : g_{f_1}(\delta) \geq g_{f_2}(\delta)\}.$$

Hence for some $\ell \in \{1,2\}$ *we have* $A_\ell \in D^+$*. So for each* $\delta \in A_\delta$ *we can find* v_δ *such that*

(a) $u \setminus v_\delta \in J_\delta$ *and* $v_\delta \subseteq u \cap \delta$
(b) $f_k \restriction v_\delta = f^\delta_{g_{f_k}(\delta)} \restriction v_\delta$ *for* $k = 1,2$
(c) *if* $\ell = 1$ *then* $g_{f_1}(\delta) < g_{f_2}(\delta)$ *and* $f^\delta_{g_{f_1}(\delta)} \restriction v_\delta < f^\delta_{g_{f_2}(\delta)} < f^\delta_{g_{f_2}(\delta)} \restriction v_\delta$
(d) *if* $\ell = 2$ *then* $g_{f_1}(\delta) \geq g_{f_2}(\delta)$ *and* $f^\delta_{g_{f_1}(\delta)} \restriction v_\delta \geq f^\delta_{g_{f_2}(\delta)} \restriction v_\delta$.

This is clearly possible.

Now if $i \in v := \bigcup\{v_\delta : \delta \in A_i\}$ *then* $[f_1(i) < f_2(i) \Leftrightarrow \ell = 1]$ *but by the previous claim (clause (b)) and clause (a),* v *is a co-bounded subset of* $u, f_1 < f_2$ *mod* $J^{\rm bd}_u$ *or* $f_2 \leq f_1 \mod J^{\rm bd}_u$ *so we are done.*

Conclusion 20.9. *Assume* $\mu > \kappa = {\rm cf}(\mu) > \aleph_0, \langle \mu_i : i < \kappa\rangle$ *is increasing continuous sequence with limit* $i, {\rm cf}(\mu_i) \leq \kappa$ *and* ${\rm pp}(\mu_i) < \mu_{i+1}$ *for* $i < \kappa$*. then we can find* $\mathscr{F}$ *as in 20.8 of cardinality (and cofinality)* ${\rm pp}(\mu)$*.*

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§ 21. More on III,4.10: Densely running away from Colours

Question 21.1. [Hajnal]: Let $\lambda = (2^{\aleph_0})^+$. Is there $c : [\lambda]^2 \to \omega$ such that

$$(\forall A \in [\lambda]^\lambda)(\forall n < \omega)(\exists B \in [A]^\lambda)\big[n \notin \mathrm{Rang}(c \restriction [B]^2)\big]?$$

Answer: yes.

Clearly it is equivalent to the property $P_7(\lambda, \aleph_0, 2)$ defined below for $\lambda = (2^{\aleph_0})^+$. Now Claim 21.3 covers the case $\lambda = (2^{\aleph_0})^+$ and then we have more. We look again at [Shear, Ch.III,4.9-4.10C,pp.177-181].

Definition 21.2. $\mathrm{Pr}_7(\lambda, \sigma, \theta)$ where $\lambda \geq \theta \geq 1, \lambda \geq \sigma = \mathrm{cf}(\sigma)$ means that there is $c : [\lambda]^2 \to \sigma$ such that

$$(\forall A \in [\lambda]^\lambda)(\forall \alpha < \sigma)(\exists B \in [A]^\lambda)\big[\min \mathrm{Rang}(c \restriction [B]^2) > \alpha\big]$$

(So far, θ is redundant). Moreover, if $w_\alpha \in [\lambda]^{<1+\theta}$ for $\alpha < \lambda$ are pairwise disjoint and $\zeta < \sigma$ <u>then</u> for some $X \in [\lambda]^\lambda$ we have

- $(*)$ if $\alpha < \beta$ are from X then $(\forall i \in w_\alpha)(\forall j \in w_\beta)(c\{i,j\} \geq \zeta)$.

Claim 21.3. *Assume λ is a regular uncountable cardinal, $2 \leq \kappa < \lambda$ and $\otimes^\kappa_\lambda$ holds or just $\oplus^\kappa_\lambda$ (see below).*

<u>then</u> there is a symmetric 2-place function c from λ to $\aleph_0$ such that:

- $(*)$ *if $\langle w_i : i < \lambda\rangle$ is a sequence of pairwise disjoint non-empty subsets of $\lambda, |w_i| < \kappa$ and $n < \omega$, <u>then</u> for $Y \in [\lambda]^\lambda$ for every $i < j$ from Y we have:*

$$\max(w_i) < \min(w_j)$$

$$\bigwedge_{\alpha \in w_i} \bigwedge_{\beta \in w_j} c(\alpha, \beta) > n.$$

(i.e. $\mathrm{Pr}_7(\lambda, \aleph_0, \kappa)$*).*

Note that Definition 21.4(1) is from [Shear, Ch.III,4.10,p.178].

Definition 21.4. 1) For a Mahlo (inaccessible) cardinal λ and $\kappa < \lambda$ let

- $\otimes^\kappa_\lambda$ there is $\bar{C} = \langle C_\delta : \delta \in S^\lambda_\in\rangle$, where $S^\lambda_\in =: \{\delta < \lambda : \delta \text{ is inaccessible}\}, C_\delta$ a club of δ, such that: for every club E of λ for some $\delta \in \mathrm{acc}(E) \cap S^\lambda_\in$ of cofinality $\geq \kappa$, for <u>no</u> $\zeta < \kappa$ and $\alpha_\varepsilon \in S^\lambda_\in$ (for $\varepsilon < \zeta$) do we have
  - $(*)$ $\mathrm{nacc}(E) \cap \delta \setminus \bigcup_{\varepsilon < \zeta} C_{\alpha_\varepsilon}$ is bounded in δ.

2) For λ regular $> \kappa = \mathrm{cf}(\kappa) \geq \aleph_0$, let

- $\oplus^\kappa_\lambda$ there is $\bar{C} = \langle C_\delta : \delta \in S\rangle, S = \{\delta < \lambda : \delta \text{limit}\}, C_\delta$ a club of δ such that: for every club E of λ for some $\delta \in \mathrm{acc}(E)$ of cofinality $\geq \kappa$, for <u>no</u> $\zeta < \kappa$ and $\alpha_\varepsilon \in S$ (for $\varepsilon < \zeta$) do we have
  - $(*)'$ $S^\lambda_{\geq\kappa} \cap E \setminus \bigcup_{\varepsilon<\zeta} C_{\alpha_\varepsilon}$ is bounded in δ where $S^\lambda_{\geq\kappa} = \{\delta < \lambda : \mathrm{cf}(\delta) \geq \kappa\}$.

Remark 21.5. 1) For λ Mahlo, the property $\otimes^2_\lambda$ holds <u>if</u> there are stationary subsets S_i of λ for $i < \lambda$ such that for no $\delta < \lambda$, $\bigwedge_{i<\delta} [S_i \cap \delta$ a stationary in $\delta]$ (we can consider only δ inaccessible).

[Why? Choose C_δ a club of δ disjoint to S_i for some $i(\delta) < \delta$, such that $\min(C_\delta) > i(\delta)$].

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2) This is close to [She88b, §3], see [She94f, Ch.III,2.12]. As in [She88b, §3], the proof is done such that from appropriate failures of Chang conjectures or existence of colourings we can get stronger colourings here. For the result as stated also $c(\beta,\alpha) = \ell g[\rho(\beta,\alpha)]$ is O.K., but the proof as stated is good for utilizing failure of Chang conjecture (as in [She88b, §3]).
3) Note that $\otimes^2_\lambda$ is closely related to $\otimes_{\bar C}$ from [She94f, Ch.III,2.12]. Also if $\kappa \le \aleph_0$, then in $\otimes^\kappa_\lambda$ we can replace $\text{nacc}(E)$ by E.
4) Note that λ weakly compact fails even $\otimes^2_\lambda$ and forcing notion P which is θ-c.c. for some $\theta < \lambda$ preserves this.

Observation 21.6. *In Definition 21.4 in* $(*)$ *and* $(*)'$ *if* $\kappa \le \aleph_0$ *it does not matter whether we write* E *or* $\text{nacc}(E)$.

Observation 21.7. *1)* $\otimes^2_\lambda$ *implies* $\otimes^{\aleph_0}_\lambda$.
2) $\oplus^2_\lambda$ *implies* $\oplus^{\aleph_0}_\lambda$.
3) If $\kappa_1 < \kappa_2 < \lambda$ *then* $\otimes^{\kappa_2}_\lambda \Rightarrow \otimes^{\kappa_1}_\lambda$ *and* $\oplus^{\kappa_2}_\lambda \Rightarrow \oplus^{\kappa_1}_\lambda$.
4) $\otimes^\kappa_\lambda \Rightarrow \oplus^\kappa_\lambda$ *if* λ *is inaccessible* $> \aleph_0$.

Proof. 1) Let $\bar C$ exemplify $\otimes^2_\lambda$ and we shall show that it exemplifies $\otimes^{\aleph_0}_\lambda$, assume not and let E be a club of λ which exemplifies this. We choose by induction on $k < \omega$ a club E_k of $\lambda : E_0 = E$, if E_k is defined let

$$A_k =: \{\delta < \lambda : \quad \delta \in \text{acc}(E_k) \cap S^\lambda_\in \text{ and for no } \alpha \in S^\lambda_\in \text{ is } E_k \cap \delta \setminus C_\alpha \text{ bounded in } \delta\}.$$

As $\bar C$ exemplifies $\otimes^2_\lambda$, clearly A_k is a stationary subset of λ and let

$$E_{k+1} = \{\delta \in E_k : \delta = \sup(A_k \cap \delta)\}.$$

Let $\delta(*) \in \bigcap_{k<\omega} E_k$ which necessarily belong $\subseteq E$. By the choice of E we can find $n < \omega = \kappa$ and $\alpha_\ell \in S^\lambda_\in$ for $\ell < n$ such that $\text{nacc}(E) \cap \delta(*) \setminus \bigcup_{\ell<n} C_{\alpha_\ell}$ is bounded in $\delta(*)$. Now we choose by induction on $k \le n, \delta_k \in \text{acc}(E_{n+1-k})$ such that $\delta_k < \delta(*)$ and $\text{nacc}(E_{n+1-k}) \cap \delta_k \setminus \bigcup_{\ell<n-k} C_{\alpha_\ell}$ is bounded in δ_k. For $k = 0$ any large enough $\delta \in \delta(*) \cap E_{n+1}$ is O.K. For $k+1$ use the definition of E_{n+1-k}. For $k = n, \delta_n$ gives a contradiction to the choice of E.
2) Same proof replacing $S^\lambda_\in$ by $S^\lambda_{\ge\kappa}$.
3) The same $\bar C$ witnesses it.
4) Here λ is inaccessible. That is, we have to show that:
the version with $(*) \Rightarrow$ the version with $(*)'$

Let $\bar C' = \langle C'_\delta : \delta \in S^\lambda_\in\rangle$ exemplifies $\otimes^\kappa_\delta$. We define $S = \{\delta < \lambda : \delta \text{ limit}\}$ and $\bar C = \langle C_\delta : \delta \in S\rangle$ as follows: if $\delta \in S^\lambda_\in (\subseteq S)$ we let $C_\delta = C'_\delta$ and if $\delta \in S \setminus S^\lambda_\in$ let C_δ be a club of δ of order type $\text{cf}(\delta)$ with $\text{cf}(\delta) < \delta \Rightarrow \min(C_\delta) > \text{cf}(\delta)$ and if δ is a successor cardinal, say θ^+ then $\min(C_\delta) > \theta$ (possible as $\delta \notin S^\lambda_\in \Rightarrow \text{cf}(\delta) < \delta \vee (\exists\theta < \delta)(\delta = \theta^+)$). We shall show that $\langle C_\delta : \delta \in S\rangle$ exemplify $\oplus^\kappa_\lambda$.

Given a club E of λ, let

$$E_0 = \{\delta \in E : \delta \text{ a limit cardinal, } \text{otp}(\delta \cap E) = \delta \text{ and } \delta > \kappa\}$$

$$\text{and } E_1 = \{\delta \in E_0 : \text{otp}(\delta \cap E_0) \text{ is divisible by } \kappa^+\},$$

so E_1 is a club of λ so by the version with $(*)$ there is $\delta \in \text{acc}(E_1) \cap S^\lambda_\in$ hence $\text{cf}(\delta) > \kappa$ satisfying $(*)$, i.e. the requirement in 21.4(1); we shall show that it satisfies the requirement in 21.4(2) thus finishing.

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So let $\zeta < \kappa$ and $\alpha_\varepsilon \in S$ for $\varepsilon < \zeta$ and we should prove that $Y =: S^\lambda_{\geq\kappa} \cap E \setminus \bigcup_{\varepsilon<\zeta} C_{\alpha_\varepsilon}$ is unbounded in δ, so fix $\beta^* < \delta$ and we shall prove that $Y \cap (\beta^*, \delta) \neq \varnothing$ thus finishing.

Let ζ be the disjoint union of u_0, u_1, u_2, where

$$\begin{aligned} u_0 =& \{\varepsilon < \zeta : \alpha_\varepsilon < \delta\} \\ u_1 =& \{\varepsilon < \zeta : \alpha_\varepsilon \geq \delta \text{ and } \alpha_\varepsilon \in S \setminus S^\lambda_\in\} \\ u_2 =& \{\varepsilon < \zeta : \alpha_\varepsilon \geq \delta \text{ and } \alpha_\varepsilon \in S^\lambda_\in\}. \end{aligned}$$

By the choice of δ we know that $Y_2 = \text{nacc}(E_1) \cap \delta \setminus \bigcup_{\varepsilon \in u_2} C_{\alpha_\varepsilon}$ is unbounded in δ. As $\text{cf}(\delta) \geq \kappa$ (see its choice, i.e. $\delta \in \text{acc}(E) \cap S^\lambda_\in \wedge \min(E) > \kappa$), we can find $\beta \in Y_2$ such that $\beta < \delta$, $\beta > \beta^*$ and $\beta > \alpha_\varepsilon$ for $\varepsilon \in u_0$.

Now $\text{cf}(\beta) = \kappa^+$ as $\beta \in Y_2 \subseteq \text{nacc}(E_1)$ and by the choice of E_1. Also

$$\varepsilon \in u_0 \Rightarrow \sup(C_{\alpha_\varepsilon}) < \beta \quad \text{and} \quad \varepsilon \in u_2 \Rightarrow \sup(C_{\alpha_\varepsilon} \cap \beta) < \beta$$

(as otherwise $\beta \in C_{\alpha_\varepsilon}$, contradicting $\beta \in Y_2$), so we can find $\beta_0 < \beta$ such that $\varepsilon \in u_0 \cup u_2 \Rightarrow \sup(C_{\alpha_\varepsilon} \cap \beta) < \beta_0$. Now for $\varepsilon < \zeta$, if $C_{\alpha_\varepsilon} \cap (\beta_0, \beta) \neq \varnothing$ then $\varepsilon \in u_1$, so by the choice of C_{α_ε} we know $|C_{\alpha_\varepsilon}| = \text{cf}(\alpha_\varepsilon) < \min(C_{\alpha_\varepsilon}) < \beta$, noting that β is a cardinal as E_0 is a set of cardinals. By the definition of E_0, E_1 we know that $E \cap S^\lambda_{\geq\kappa} \cap \beta$ has cardinality β hence $E \cap S^\lambda_{\geq\kappa} \setminus \beta_0$ has cardinality β, so we finish. $\square_{21.6}$

Proof. By 21.11(4) without loss of generality $\oplus^\kappa_\lambda$, so let $\bar{C}$ be as required in $\oplus^\kappa_\lambda$.

We define e_α for every ordinal $\alpha < \lambda$ as follows:

- (a) if $\alpha = 0$, $e_\alpha = \varnothing$
- (b) if $\alpha = \beta + 1$, $e_\alpha = \{0, \beta\}$
- (c) if α is a limit ordinal, then we let $e_\delta = C_\delta \cup \{0\}$.

Let S be the set of limit ordinals $< \lambda$. For $\alpha < \beta$ we define by induction on $\ell < \omega$ the ordinals $\gamma^+_\ell(\beta, \alpha)$, $\gamma^-_\ell(\beta, \alpha)$.

$\underline{\ell = 0}$: $\gamma^+_\ell(\beta, \alpha) = \beta$, $\gamma^-_\ell(\beta, \alpha) = 0$

$\underline{\ell = k + 1}$: $\gamma^+_\ell(\beta, \alpha) = \min(e_{\gamma^+_k(\beta,\alpha)} \setminus \alpha)$ if $\alpha < \gamma^+_k(\beta, \alpha)$ and $\gamma^-_\ell(\beta, \alpha) = \sup(e_{\gamma^+_k(\beta,\alpha)} \cap \alpha)$ $\underline{\text{if}}$ $\alpha < \gamma^+_k(\beta, \alpha)$ and $\alpha \notin \text{acc}(e_{\gamma^+_k(\beta,\alpha)})$.

Note that $\gamma^-_\ell(\beta, \alpha) < \alpha \leq \gamma^+_\ell(\beta, \alpha)$ if they are defined and then $\ell > 0 \Rightarrow \gamma^+_\ell(\beta, \alpha) < \gamma^+_{\ell-1}(\beta, \alpha)$ (prove by induction). So if $\alpha < \beta < \lambda$ for some $k = k(\beta, \alpha) < \omega$ we have: $\gamma^+_\ell(\beta, \alpha)$ is defined iff $\ell \leq k$ and: $\gamma^-_\ell(\beta, \alpha)$ is defined iff $\ell < k \vee [\ell = k$ and $\gamma^+_k(\beta, \alpha) = \alpha]$ and: $\gamma^+_k(\beta, \alpha) = \alpha$ or $\alpha \in \text{acc}(e_{(\gamma^+_k(\beta,\alpha))})$. Let $\rho(\beta, \alpha) = \langle \gamma^+_\ell(\beta, \alpha) : \ell \leq k(\beta, \alpha) \rangle$. Note (we shall use it freely):

$\otimes_1$ if $\gamma < \alpha < \beta, k \leq k(\beta, \alpha)$ and $\gamma^-_k(\beta, \alpha)$ is defined and $\bigwedge_{\ell \leq k} \gamma^-_\ell(\beta, \alpha) < \gamma$ $\underline{\text{then}}$

- (α) $\ell \leq k \Rightarrow \gamma^+_\ell(\beta, \alpha) = \gamma^+_\ell(\beta, \gamma)$
- (β) $\ell \leq k \Rightarrow \gamma^-_\ell(\beta, \alpha) = \gamma^-_\ell(\beta, \gamma)$
- (γ) $k(\beta, \gamma) \geq k(\beta, \alpha)$ and $\rho(\beta, \alpha) \trianglelefteq \rho(\beta, \gamma)$.

Now we define $c\{\alpha, \beta\} = c(\beta, \alpha) = c(\alpha, \beta)$ for $\alpha < \beta < \lambda$ as follows:

$$c(\beta, \alpha) = k(\beta, \alpha) + 1.$$

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So assume $\bar{w} = \langle w_i : i < \lambda \rangle$ is a sequence of pairwise disjoint subsets of λ, $|w_i| < \kappa$ and $n(*) < \omega$. Without loss of generality for some $\kappa^* < 1 + \kappa$, $\bigwedge_{i<\lambda} |w_i| = \kappa^*$ and $i < \min(w_i)$ and $[i < j \Rightarrow \sup(w_i) < \min(w_j)]$. Let $w_i = \{\alpha^i_\varepsilon : \varepsilon < \kappa^*\}$. Let $\chi \geq (2^\lambda)^+$, and we choose by induction on $n < \omega$ and for each n by induction on $i < \lambda$, $N^n_i \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ such that $\|N^n_i\| < \lambda$, $\{\langle N^n_\varepsilon : \varepsilon \leq j \rangle : j < i\} \subseteq N^n_i$, $\bar{w} \in N^0_i$, N^n_i increasing continuous in i and $\langle N^m_i : i < \lambda \rangle \in N^n_0$ for $m < n$.

Let us define for $\ell < \omega$

$$E^\ell = \{\delta < \lambda : N^\ell_\delta \cap \lambda = \delta\}$$

$$S^\ell = \{\delta \in S^\lambda_{\geq\kappa} \cap \mathrm{acc}(E^\ell) : \begin{array}{l} \text{for } \underline{\text{no}}\ \zeta < \kappa \text{ and } \alpha_\varepsilon < \lambda \\ \text{for } \varepsilon < \zeta \text{ do we have} \\ \delta > \sup[S^\lambda_{\geq\kappa} \cap E^\ell \cap \delta \setminus \bigcup_{\varepsilon<\zeta} C_{\alpha_\varepsilon}]\}. \end{array}$$

Note that $\alpha < \lambda \Rightarrow (E^\ell, S^\ell) \in N^{\ell+1}_\alpha$ hence $\delta \in E^{\ell+1} \Rightarrow \delta = \sup(\delta \cap S^\ell)$.

We know that S^ℓ is a stationary subset of λ as E^ℓ is a club of λ because $\oplus^\lambda_\kappa$ is exemplified by $\bar{C}$.

Choose $\delta_{n(*)} \in E^{2(n(*)+1)} \cap S^{2(n(*)+1)}$ and then choose $\alpha(*) < \lambda$ such that $\alpha(*) > \delta_{n(*)}$. We now choose by downward induction on $m < n(*)$ ordinals δ_m, ζ^*_m such that:

$(*)(i)$ $\delta_m < \delta_{m+1}$

(ii) $\delta_m \in E^{2m} \cap S^{2m}$

(iii) $\delta_m > \sup\{\gamma^-_\ell(\beta, \delta_{m+1}) : \beta \in w_{\alpha(*)},\ \ell \leq k(\beta, \delta_{m+1}),\ \gamma^-_\ell(\beta, \delta_{m+1}) \text{ well defined}\}$

(iv) $\delta_m \notin \bigcup\{C_\gamma : \gamma = \gamma^+_{k(\beta,\delta_{m+1})}(\beta, \delta_{m+1}) \text{ for some } \beta \in w_{\alpha(*)}\}$

(v) $\zeta^*_m < \delta_m, \zeta^*_m < \zeta^*_{m+1}$ if $m + 1 < n(*)$

(vi) if $\alpha \in [\zeta^*_m, \delta_m)$ then $(\forall \beta' \in w_\alpha)(\forall \beta'' \in w_{\alpha(*)})\big[\rho(\beta'', \delta_m) \triangleleft \rho(\beta'', \beta')\big]$

[Why can we do it? Assume $\delta_{m+1} \in S^{2(m+1)}$ has already been defined and we shall find δ_m, ζ_m as required. Let

$$Y_m = \{\gamma^-_\ell(\beta, \delta_{m+1}) : \beta \in w_{\alpha(*)},\ \ell \leq k(\beta, \delta_{m+1}),\ \gamma^-_\ell(\beta, \delta_{m+1}) \text{ well defined}\}$$

so Y_m is a subset of δ_{m+1} of cardinality $< \kappa$, but $\delta_{m+1} \in S^{2(m+1)}$ (if $m = n(*) - 1$ by the choice of δ_n, if $m < n - 1$ by the induction hypothesis). But $S^{2(m+1)} \subseteq S^\lambda_{\geq\kappa}$, hence $(\forall \delta \in S^{2(m+1)})[\mathrm{cf}(\delta) \geq \kappa]$, hence $\sup(Y_m) < \delta_{m+1}$. Also as $\delta_{m+1} \in E^{2(m+1)} \cap S^{2(m+1)}$ by the definition of $S^{2(m+1)}$, there is

$$\xi^*_m \in S^\lambda_{\geq\kappa} \cap E^{2(m+1)} \cap \delta \setminus \bigcup \big\{\leq_\gamma : (\exists \beta \in w_{\alpha(*)})[\gamma = \gamma^+_{k(\beta,\delta_{m+1})}(\beta, \delta_{m+1})]\big\} \setminus \sup(Y_m)$$

As each e_γ is closed and there are $< \kappa$ of them,$\zeta^*_m < \xi^*_m$. where

$$\zeta^*_m = \sup\Big\{\{\sup Y_m\} \cup \big\{\sup(e_\gamma \cap \xi^*_m) : (\exists \beta \in w_{\alpha(*)})[\gamma = \gamma^+_{k(\beta,\delta_{m+1})}(\beta, \delta_{m+1})]\big\}\Big\}$$

So we can find $\delta_m \in (\zeta^*_m, \xi^*_m) \cap S^\lambda_{\geq\kappa} \cap E^{2m} \cap S^{2m}$ as required and choose $\zeta_m < \delta_m$ large enough.]

$(**)$ For every $\alpha \in [\zeta^*_0, \delta_0)$ we have

$$(\forall \beta' \in w_\alpha)(\forall \beta'' \in w_\alpha)[c\{\beta', \beta''\} \geq n].$$

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[Why? By clause (vi) above.]

Let

$$W = \Big\{\delta < \lambda : \quad \delta > \zeta^*_0 \text{ and for some } \alpha'' \geq \delta \text{ we have for every } \alpha' \in (\zeta^*_0, \delta) \text{ we have } (\forall \beta' \in w_{\alpha'})(\forall \beta'' \in w_{\alpha''})\big[c\{\beta', \beta''\} \geq n\big]\Big\}.$$

As $\delta_0 \in E_0$ (see $(*)(ii)$) so by E_0's definition, $\delta_0 = N^0_{\delta_0} \cap \lambda$ hence $\zeta_0 \in N^0_{\delta_0}$. Now $\bar{w} \in N^0_{\delta_0}$ (read definition) hence $W \in N^0_{\delta_0}$ and by $(*) + (**)$ and W's definition $\delta_0 \in W$, hence W is a stationary subset of λ. For $\delta \in W$, let $\alpha''(\delta)$ be as in the definition of W. So $E = \big\{\delta^* : (\forall \delta \in W \cap \delta^*)[\alpha''(\delta) < \delta^*]\big\}$, it is a club of λ hence $W' = W \cap E$ is a stationary subset of λ and $\{\alpha''(\delta) : \delta \in W'\}$ is as required. $\square_{21.3}$

Conclusion 21.8. *If* $\lambda = \mathrm{cf}(\lambda) > \aleph_0$ *is not Mahlo (or is Mahlo as in 21.4(1) or 21.4(2)),* κ *then* $\mathrm{Pr}_7(\lambda, \aleph_0, \aleph_0)$.

Proof. By 21.3 it suffices to prove $\oplus^{\aleph_0}_\lambda$. This holds by 21.9, 21.11 and 21.12 below. $\square_{21.8}$

Claim 21.9. *1) If* $\lambda = \mu^+$ *then* $\oplus^{\mathrm{cf}(\mu)}_\lambda$.
2) If λ *is (weakly) inaccessible, not Mahlo or Mahlo as in 21.4(1), e.g. as in 21.5(1), and* $\aleph_0 \leq \kappa < \lambda$ *then* $\oplus^\kappa_\lambda$.

Proof. 1) Choose C_δ a club of δ of order type $\mathrm{cf}(\delta)$.

Repeat the proof of 21.6(2), using

$$E_0 = \{\delta < \lambda : \delta > \mu \text{ and } \mathrm{otp}(E \cap \delta) = \delta \text{ is divisible by } \mu^2\}.$$

The only point slightly different is $|C_{\alpha_\varepsilon} \cap (\beta_0, \beta)| < |\beta|$ (now β is not a cardinal). For μ singular, $|C_{\alpha_\varepsilon}| < \mu = |\beta| = |\beta \cap S^\lambda_{\geq \kappa} \cap E \setminus \beta_0|$, and for μ regular we choose δ of cofinality μ and everything is easy.
2) Now $\oplus^\lambda_\kappa$ holds trivially (choose a club E^*_0 of λ with no inaccessible member and choose C_δ a club of δ of order type $\mathrm{cf}(\delta)$ such that $\mathrm{cf}(\delta) < \delta \Rightarrow \min(C_\delta) > \mathrm{cf}(\delta)$ and $\delta \notin E^* \Rightarrow \min(C_\delta) > \sup(E^* \cap \delta)$, now for any club E choose $\delta \in \mathrm{acc}(E \cap E^*)$) So we can apply 21.6(2). $\square_{21.9}$

Definition 21.10. $\mathrm{Pr}_8(\lambda, \mu, \sigma, \theta)$ means:

there is $c : [\lambda]^2 \to [\sigma]^{<\aleph_0} \setminus \{\varnothing\}$ such that <u>iff</u> $w_\alpha \in [\lambda]^{<\theta}$ for $\alpha < \lambda$ are pairwise disjoint and $\zeta < \sigma$ <u>then</u> for some $Y \in [\lambda]^\mu$ we have $\alpha', \alpha'' \in Y$ and

$$\alpha' < \alpha'' \Rightarrow (\forall \beta' \in w_{\alpha'},\ \forall \beta'' \in w_{\alpha''})\big[\zeta \in c\{\beta', \beta''\}\big].$$

Observation 21.11. *Note that* $\mathrm{Pr}_8(\lambda, \lambda, \sigma, \theta) \Rightarrow \mathrm{Pr}_7(\lambda, \sigma, \theta)$ *because we can use* $c'\{\alpha, \beta\} = \max[c\{\alpha, \beta\}]$.

Claim 21.12. *1) If* λ *is regular and* $\aleph_0 \leq \sigma \leq \lambda$ *then* $\mathrm{Pr}_8(\lambda^+, \lambda^+, \sigma, \lambda)$.
2) If μ *is singular,* $\lambda = \mu^+$ *and* $\aleph_0 \leq \sigma \leq \mathrm{cf}(\mu)$ *then* $\mathrm{Pr}_8(\lambda, \lambda, \sigma, \mathrm{cf}(\mu))$.
3) If λ *is inaccessible* $> \aleph_0$*,* $S \subseteq \lambda$ *stationary not reflecting in inaccessibles and* $\sigma < \lambda$*,* $\theta = \min\{\mathrm{cf}(\delta) : \delta \in S\}$ *then* $\mathrm{Pr}_8(\lambda, \lambda, \sigma, \theta)$.

Proof. The proofs in [She94f, Ch.III,§4] gives this - in fact this is easier. E.g.
1) Follows by Claim 21.3 (and [She94f, Ch.III,4.2(2),p.162]) but let us give some details.

Let $\bar{e}$ be as there (i.e. $\bar{e} = \langle e_\alpha : \alpha < \lambda^+ \rangle$, $e_0 = \varnothing$, $e_{\alpha+1} = \{\alpha\}$, e_δ a club of δ of order type $\mathrm{cf}(\delta)$). Let $h : \lambda^+ \to \sigma$ be such that $(\forall \zeta < \sigma)(\exists^{\mathrm{stat}} \delta < \lambda^+)(\mathrm{cf}(\delta) = \lambda \text{ and } h(\delta) = \zeta)$, $h_\alpha = h \restriction e_\alpha$, $\bar{h} = \langle h_\alpha : \alpha < \lambda^+ \rangle$.

Let $\gamma(\beta, \alpha), \gamma_e(\beta, \alpha), \rho_{\bar{h}}$ be as there (Stage A,p.164) and also the colouring d: for $\alpha < \beta < \lambda^+$

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$$d(\beta,\alpha) = \max\{h(\gamma_{\ell+1}(\beta,\alpha) : \gamma_{\ell+1}(\beta,\alpha) \text{ well defined}\}.$$

By Stage B there the result should be clear. $\square_{21.12}$

Hajnal has shown the following:

Theorem 21.13. *Assume $\lambda = (2^{<\kappa})^+$, $\kappa = \mathrm{cf}(\kappa) > \omega$, I is a normal ideal concentrating on $S_{\kappa,\lambda} = \{\alpha < \lambda : \mathrm{cf}(\alpha) = \kappa\}$, $\Lambda \subset [\lambda]^2$ is such that $\Lambda \cap [B]^2 \neq \varnothing$ for all $B \in I^+$ and $\Lambda = \bigcup_{\eta<\xi} \Lambda_\eta$ for some $\xi < \kappa$.*

Then there exist I and T such that $I \subset J$, $T \subset \xi$, J is a normal ideal and for all $\eta \in T$ and $B \in J^+$ we have

$$[B]^2 \cap \Lambda_\eta \neq \varnothing \textit{ and } G \cap [B]^2 \subset \bigcup\{\Lambda_\eta : \eta \in T\}.$$

This comes from the following:

Lemma 21.14. *Assume $\lambda = (2^{<\kappa})^+, \kappa = \mathrm{cf}(\kappa) > \omega$.*

I is a normal ideal concentrating on $S_{\kappa,\lambda}, P$ is a partial order not containing decreasing sets of type κ.

Assume further that

$$p : \mathcal{P}(\lambda) \to P \textit{ and}$$

$$p(A) \leq_p p(B) \textit{ for } A \subset B.$$

Then there is an $A \in I^+$ and a normal ideal $J \supset I$ satisfying $B \in J$ iff $B \in I$ or $p(B) \prec_P p(A)$ for $B \subset A$.

<u>page 412</u>: add at the end

The following improves [She94f, Ch.IX,5.12,p.410].

Claim 21.15. *1) Assume*

- *(a)* $\sigma = \mathrm{cf}(\sigma) > \aleph_0$
- *(b)* $\langle \lambda_i : i < \sigma\rangle$ *increasing continuous,* $\lambda = \sup\{\lambda_i : i < \sigma\}$
- *(c)* $\sigma \leq \theta < \lambda$ *and* $\sigma^{\aleph_0} < \lambda$
- *(d)* $\mathrm{cov}(\lambda_i, \lambda_i, \theta^+, 2) < \lambda$ *for* $i < \sigma$.

<u>*then*</u>

- *(α)* $\mathrm{pp}_\theta(\lambda) =^+ \mathrm{cov}(\lambda,\lambda,\theta^+,2)$ *and* $\mathrm{pp}^{\mathrm{cr}}_{J^{\mathrm{bd}}_\sigma}(\lambda) = \mathrm{cov}(\lambda,\lambda,\theta^+,2)^+$ *(on* $\mathrm{pp}^{\mathrm{cr}}$ *see below).*
- *(β)* $S^* = \{\delta < \sigma : \mathrm{cov}(\lambda_\delta,\lambda_\delta,\theta^+,2)^+ = \mathrm{pp}^{\mathrm{cr}}_{J^{\mathrm{bd}}_{\mathrm{cf}(\delta)}}(\lambda_\delta)\}$ *contains a club of* σ.

2) Instead of "$\sigma^{\aleph_0} < \lambda$" it suffices

- $\otimes$ *for some club C of σ, <u>iff</u> $i < \delta \in C, \delta$ of cofinality $\aleph_0$ and set $\mathfrak{a} \subseteq \lambda_\delta$ of cardinality $\leq \lambda_i$ and $\mathfrak{a}$ is a set of regular cardinals, <u>then</u>*

$$\lambda > \left|\left\{\mathrm{tcf}(\prod \mathfrak{b}/J^{\mathrm{bd}}_{\mathfrak{b}}) : \mathfrak{b} \subseteq \mathfrak{a},\ \sup(\mathfrak{b}) = \lambda_\delta,\ \mathrm{otp}(\mathfrak{b}) = \omega,\ \prod \mathfrak{b}/J^{\mathrm{bd}}_{\mathfrak{b}} \textit{ has true cofinality}\right\}\right|.$$

(So without loss of generality $\lambda_{\delta+1}$ is above this cardinality.)

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Definition 21.16. Let J be an ideal on some ordinal $\mathrm{Dom}(J)$. We let

$$\begin{aligned}\mathrm{pp}^{\mathrm{cr}}_J(\lambda) = \min\big\{\mu : \mu \text{ regular } > \lambda, \text{ and}\\ \sup\{\mathrm{tcf}\prod_t \lambda_t/J : \bar\lambda = \langle \lambda_t : t \in \mathrm{Dom}(J)\rangle\\ \text{is strictly increasing with limit } \lambda\} < \mu\big\}.\end{aligned}$$

Proof. Proof of 21.16 1) Similar to the proof of [She94f, Ch.IX,5.12]. We assume toward contradiction that the desired conclusion fails.

Without loss of generality

$(*)_0(a)$ each λ_i is singular of cofinality $< \sigma$

(b) $\theta^{+3} < \lambda_0$ and $\sigma^{\aleph_0} < \lambda_0$

(c) $\mathrm{cov}(\lambda_i, \lambda_i, \theta^+, 2) < \lambda_{i+1}$

(d) $\mu \in (\lambda_0, \lambda_{i+1}) \Rightarrow \mathrm{pp}_\theta(\mu) < \lambda_{i+1}$.

[Why? Clearly we can replace $\langle \lambda_i : i < \sigma\rangle$ by $\bar\lambda \restriction C = \langle \lambda_i : i \in C\rangle$ for any club C of σ, hence it is enough to show that each of the demands holds for $\bar\lambda \restriction C$ for any small enough club C of σ. Now (a) holds whenever $C \subseteq \{i < \lambda : i \text{limit}\}$, clause (b) holds for $C \subseteq [i_0, \sigma)$ when $\theta^{+3} < \lambda_{i_0}$ and clause (c) holds as $\mathrm{cov}(\lambda_i, \lambda_i, \theta^+, 2) < \lambda$ and use [She94f, Ch.II,5.3,10] + Fodor's lemma and monotonicity of cov.

Lastly, clause (d) holds as if $\{\mu < \lambda : \mathrm{pp}_\theta(\mu) \ge \lambda \mathrm{cf}(\mu) \le \theta\}$ is unbounded in λ, we get a contradiction by [She94f, Ch.II,2.3(4)].]

Let $\lambda_\sigma =: \lambda$. By [She94f, Ch.VIII,1.6(3)] we have (but shall not use)

$(*)_1$ if $\delta \le \sigma$ and $\mathrm{cf}(\delta) > \aleph_0$ <u>then</u> $\mathrm{pp}^+_\theta(\lambda_\delta) = \mathrm{pp}^{\mathrm{cr}}_{J^{\mathrm{bd}}_{\mathrm{cf}\lambda_\delta}}(\lambda_\delta)$ (and $\mathrm{cf}(\lambda_\delta) = \mathrm{cf}(\delta)$).

Now by clause (d)

$(*)_2$ $\mathfrak{a} \subseteq \mathrm{Reg} \cap \lambda_i \setminus \lambda_0$, $|\mathfrak{a}| \le \theta$ and $\sup(\mathfrak{a}) \le \lambda_i$ <u>implies</u> $\max \mathrm{pcf}(\mathfrak{a}) < \mathrm{pp}^+_\theta(\lambda_i)$.

Let

$$S =: \big\{i \le \sigma : \mathrm{cov}(\lambda_i, \lambda_i, \theta^+, 2) \ge \mathrm{pp}^{\mathrm{cr}}_{J^{\mathrm{bd}}_{\mathrm{cf}\lambda_i}}(\lambda_i)\big\}.$$

So it is enough to prove that S is not stationary.

Let for $i \le \sigma, \mu_i =: \mathrm{pp}^{\mathrm{cr}}_{J^{\mathrm{bd}}_{\mathrm{cf}(\lambda_i)}}(\lambda_i)$, so $\lambda_{i+1} > \mu_i > \lambda_i, \mu_i$ is regular. Note that $\mu_\sigma = pp_\theta(\lambda_\sigma) = \mathrm{pp}^{\mathrm{cr}}_{J^{\mathrm{bd}}_\sigma}(\lambda_i)$ by [She94f, Ch.VIII,1.6(3)].

Clearly

$(*)_3$ $\lambda_i < \mu_i = \mathrm{cf}(\mu_i) \le \mathrm{cov}(\lambda_i, \lambda_i, \theta^+, 2)^+$.

We can find $\bar A = \langle A_\zeta : \zeta < \lambda\rangle$ such that:

$(*)_4(a)$ $\zeta < \lambda_0 \Rightarrow A_\zeta = \varnothing$

(b) $\lambda_i \le \zeta < \lambda_{i+1} \Rightarrow A_\zeta \subseteq \lambda_i$ and $|A_\zeta| < \lambda_i$

(c) for every $A \subseteq \lambda_i$ of cardinality $\le \theta$, for some ζ, $\lambda_i < \zeta < \mathrm{cov}(\lambda_i, \lambda_i, \theta^+, 2)$ (which is $< \lambda_{i+1}$) we have $A \subseteq A_\zeta$.

Choose χ regular large enough, now choose by induction on $i \le \sigma$ an elementary submodel M^*_i of $(\mathcal{H}(\chi), \in, <^*_\chi)$, $\|M^*_i\| < \mu_i$, $M^*_i \cap \mu_i$ is an ordinal such that

$(*)_5$ if $i \le \sigma$, <u>then</u>

$$\bigcup_{j<i} M^*_j \cup \{\zeta : \zeta \le \lambda_i\} \cup \{\langle \lambda_i : i < \sigma\rangle,\ \bar A,\ \langle M^*_j : j < i\rangle\} \subseteq M^*_i.$$

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Let $\mathscr{P}_i = M^*_i \cap [\lambda_i]^{<\lambda_i}$. It is enough to show that

$$S_1 = \{i \le \sigma : \begin{array}{l} \text{for some } Y \subseteq \lambda_i,\ |Y| \le \theta \text{ and} \\ Y \text{ is not a subset of any member of } \mathscr{P}_i\} \end{array}$$

is not stationary and $\sigma \notin S_1$ (in fact S, S_1 are equal).
[Why? As clearly $S \subseteq S_1$.]

We assume S_1 is a stationary subset of σ or $\sigma \in S_1$ and eventually will finish by getting a contradiction.

For each $i \in S_1$ choose $Y_i \subseteq \lambda_i$ of cardinality $\le \theta$ which is not a subset of any member of $\mathscr{P}_i$. Let $Y = \bigcup\limits_{i \in S_1} Y_i$, so $Y \subseteq \lambda$, $|Y| \le \theta$; and for each $i < \sigma$ we can find an ordinal $\zeta(i)$ such that $\lambda_i \le \zeta(i) < \mathrm{cov}(\lambda_i, \lambda_i, \theta, 2)$ (which is $< \lambda_{i+1}$) and $Y \cap \lambda_i \subseteq A_{\zeta(i)}$. Now $|A_{\zeta(i)}| < \lambda_i$, hence by Fodor's Lemma there is $i(*) < \sigma$ such that

$$S_2 =: \{i < \sigma : |A_{\zeta(i)}| < \lambda_{i(*)}\}.$$

is a stationary subset of σ. Let $Z =: \{\zeta(i) : i \in S_2\}$. Now if $\sigma \in S_1$, then by [She94f, Ch.IX,II,5.4] and [She94f, Ch.II,§1] we have

$$\mathrm{pp}^{\mathrm{cr}}_{J^{\mathrm{bd}}_\sigma}(\lambda) = \mathrm{cov}(\lambda, \lambda, \sigma^+, \sigma) =^+ \mathrm{pp}_{\Gamma(\sigma^+, \sigma)}(\lambda)$$

so there are $j^* < \sigma$ and $B_j \in \mathscr{P}_\sigma = M_\sigma \cap [\lambda]^{<\lambda}$ for $j < j^*$ such that $Z \subseteq \bigcup\limits_{j<j^*} B_j$. So for some $j < j^*$ we have $|Z \cap B_j| = \sigma$. Now the set

$$A^* = \bigcup \{A_\gamma : \gamma \in B_j, |A_j| \le \lambda_{i(*)}\}$$

belongs to M_σ, has cardinality $\le \lambda_{i(*)} \times |B_j| < \lambda$ and

$$\begin{array}{rl} Y = & \bigcup\{Y \cap \lambda_i : i \in S_2 \text{ and } \zeta(i) \in B_j\} \subseteq \\ & \bigcup\{A_{\zeta(i)} : i \in S_2 \text{ and } \zeta(i) \in B_j\} \subseteq A^* \in \mathscr{P}_\sigma \end{array}$$

contradiction. So we have finished the case $\sigma \in S_1$ and from now on we shall deal with the case $\sigma \notin S_1$ hence S_1 is a stationary subset of σ, hence without loss of generality $S_2 \subseteq S_1$. Note that if $\delta < \sigma$ and $\mathrm{cf}(\delta) > \aleph_0$, we can apply this proof to $\lambda_\delta, \langle \lambda_i : i < \delta \rangle$ (for $\sigma' = \mathrm{cf}(\delta)$) hence

$(*)_6$ $i \in S_2 \Rightarrow \mathrm{cf}(i) = \aleph_0$.

Clearly

$(*)_7$ for no $i \in S_2$ and $Z' \subseteq Z \cap \lambda_i$ is Z' unbounded in λ_i and is contained in a member of M^*_i of cardinality $< \lambda_i$.

Now we want to work as in the proof of [She94f, CH.IX,3.5], but for σ places at once with "nice" behavior on a club of σ, in the end the model is the Skolem Hull of the union of $\aleph_0$ sets, so one "catches" an unbounded subsets of Z. Let $\bar\lambda = \langle \lambda_i : i \le \sigma \rangle$.

We shall choose by induction on $k < \omega$,

$$N^a_k, N^b_k, g_k, \langle A^k_\ell : \ell < \omega \rangle, \langle \langle A^k_{\ell,i} : i \le \sigma \rangle : \ell < \omega \rangle$$

such that:

(a) for $x \in \{a, b\}, N^x_k$ is an elementary submodel of $(\mathcal{H}(\chi), \in, <^*_\chi, \sigma, \bar\lambda)$ of cardinality $\le \sigma$ and N^x_k is the Skolem Hull of $N^x_k \cap \lambda$ and $N^a_k \prec N^b_k$

(b) $N^a_0[N^b_0]$ is the Skolem Hull of $\{i : i \le \sigma\}$ [of $Z \cup \{i : i \le \sigma\}$] in $(\mathcal{H}(\chi), \in, <^*_\chi, \sigma, \bar\lambda)$

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(c) $g_k \in \prod(\mathrm{Reg} \cap N^a_k \cap \lambda \setminus \lambda_0^+)$

(d) for $x \in \{a, b\} : N^x_{k+1}$ is the Skolem Hull of

$$N^x_k \cup \{g_k(\kappa) : \kappa \in \mathrm{Dom}(g_k)\} \cup (N^b_k \cap \lambda_0)$$

(e) $N^a_k \cap \lambda = \bigcup_{\ell<\omega} A^k_\ell$

(f) $A^k_\ell = \bigcup_{i<\sigma} A^k_{\ell,i}$ and $\langle A^k_{\ell,i} : i < \sigma\rangle$ is continuous increasing (in i) and $A^k_{\ell,i} \subseteq \lambda_i$ and $|A^k_{\ell,i}| < \sigma$

(g) if $\kappa \in \mathrm{Reg} \cap \lambda \cap N^a_k \setminus \lambda_0^+$ then $\sup(N^b_k \cap \kappa) < g_k(\kappa) < \kappa$

(h) if $\mathfrak{a} \subseteq A^k_\ell$ has order type ω and $\sup(\mathfrak{a}) = \lambda_i$ and $\mathfrak{a}$ is a subset of some $\mathfrak{b} \in M^*_i$ of cardinality $\leq \lambda_0$, <u>then</u> for some infinite $\mathfrak{b} \subseteq \mathfrak{a}, g_k \restriction \mathfrak{b}$ is included in some function $h^k_{\mathfrak{a}} \in M^*_i$ such that $|\mathrm{Dom}(h^k_{\mathfrak{a}})| \leq \lambda_0$.

For $X \in \mathcal{H}(\chi)$ and a function F we let

$$A(X, F) =: \{F(x_1, \ldots, x_n) : x_1, \ldots, x_n \in X\}.$$

Let us carry the induction for $k = 0$; we define N^a_0, N^b_0 by clause (b) and define $\{A^0_\ell : \ell < \omega\}$ as

$$\big\{A(\sigma + 1, F) : F \text{ a definable function in } (\mathcal{H}(\chi), \in, <^*_\chi, \sigma, \bar\lambda)\big\}.$$

For $k+1$, let $g'_k \in \prod(\mathrm{Reg} \cap \lambda \cap N^a_k \setminus \lambda_0^+)$ be defined by $g'_k(\kappa) = \sup(N^b_k \cap \kappa)$ (note: the domain of g'_k is determined by N^a_κ, the values — by N^b_k).

We now shall find g_k satisfying:

(α) $\mathrm{Dom}(g_k) = \mathrm{Dom}(g'_k), g_k \in \prod(\mathrm{Dom}(g'_k))$

(β) $g'_k < g_k$

(γ) if $i < \sigma, \ell < \omega$ and $\mathfrak{a} \subseteq \mathrm{Reg} \cap A^k_\ell \setminus \lambda_0^+$ is unbounded in λ_i and is a subset of some $\mathfrak{b} \in M^*_i$ of cardinality $\leq \lambda_0$ and is of order type ω, <u>then</u> for some infinite $\mathfrak{b} \subseteq \mathfrak{a}$ we have $g_k \restriction \mathfrak{b}$ is included in some $h_{\mathfrak{b}} \in M^*_i$ such that $|\mathrm{Dom}(h_{\mathfrak{b}})| \leq \lambda_0$

(δ) if $\mathfrak{a} \subseteq \lambda_i \cap \mathrm{Reg} \cap A^k_{\ell,i} \setminus \lambda_0^+$ and $\mathfrak{a} \in M^*_{i+1}$ <u>then</u> $g_k \restriction \mathfrak{a} \subseteq h$ for some function from M^*_{i+1}.

Note: a function choosing $\langle \bar f^{\mathfrak{a},\mu} : \mu \in \mathrm{pcf}(\mathfrak{a})\rangle$ satisfying $(*)_{\mathfrak{a}}$ below for each $\mathfrak{a} \subseteq \mathrm{Reg} \cap \lambda \setminus \theta^+, |\mathfrak{a}| \leq \theta$ is definable in $(\mathcal{H}(\chi), \in, <^*_\chi)$, so each M^*_i is closed under it where

$(*)_{\mathfrak{a}}$ $\bar f^{\mathfrak{a},\mu} = \langle f^{\mathfrak{a},\mu}_\alpha : \alpha < \mu\rangle$ satisfies

(α) $f^{\mathfrak{a},\mu}_\alpha \in \prod \mathfrak{a}$,

(β) $\alpha < \beta \Rightarrow f^{\mathfrak{a},\mu}_\alpha <_{J_{<\mu}[\mathfrak{a}]} f^{\mathfrak{a},\mu}_\beta$

(γ) if $\theta < \mathrm{cf}(\alpha) < \min(\mathfrak{a})$ <u>then</u> $f^{\mathfrak{a},\mu}_\alpha(\kappa) = \min\{\bigcup_{\beta\in C} f^{\mathfrak{a},\mu}_\beta(\kappa) : C \text{ a club of } \alpha\}$

(δ) if $f \in \prod \mathfrak{a}$ then for some $\alpha < \mu$ we have $f < f^{\mathfrak{a},\mu}_\alpha \mod J_\mu[\mathfrak{a}]$.

Let $\langle \mathfrak{a}_{i,\zeta} : \zeta < \zeta_i \leq \sigma^{\aleph_0}\rangle$ list the $\mathfrak{a}$ such that $\mathrm{tcf}(\prod \mathfrak{a}/J^{\mathrm{bd}}_{\mathfrak{a}})$ is well defined and for some $n < \omega$, $\mathfrak{a} \subseteq A^k_n$, $\mathfrak{a} \subseteq \mathrm{Reg} \cap \lambda_i \setminus \lambda_0^+$, $\mathrm{otp}(\mathfrak{a}) = \omega$, $\lambda_i = \sup(\mathfrak{a})$ and there is $\mathfrak{b} \subseteq \mathrm{Reg} \cap \lambda_i \setminus \lambda_0^+$, $\mathfrak{b} \in M^*_i$, $|\mathfrak{b}| \leq \lambda_0$ such that $\mathfrak{a} \subseteq \mathfrak{b}$: note that the number of such $\mathfrak{a}$-s is $\leq \sigma^{\aleph_0}$. Let $\{\mathfrak{b}_{i,\zeta} : \zeta < \zeta_i \leq \sigma^{\aleph_0}\}$ be such that $\mathfrak{b}_{i,\zeta} \subseteq \mathrm{Reg} \cap \lambda_i \setminus \lambda_0^+$, $\mathfrak{b}_{i,\zeta} \in M^*_i$, $|\mathfrak{b}_{i,\zeta}| \leq \lambda_0$ and $\mathfrak{a}_{i,\zeta} \subseteq \mathfrak{b}_{i,\zeta}$.

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So apply [She94f, CH.VIII,§1]; i.e. let $\theta_1 = \theta + \sigma^{\aleph_0}$ choose $\langle M^k_\zeta : \zeta < \theta_1^{++}\rangle$ increasing continuous, $M^k_\zeta \prec (\mathcal{H}(\chi), \in, <^*_\chi)$, $\langle M^k_\zeta : \zeta \le \xi\rangle \in M^k_{\xi+1}$, $\|M^k_\zeta\| \le \lambda_0$ and g'_k, $\langle A^k_{\ell,i} : i < \sigma, \ell < \omega\rangle$, Z, $\langle \mathfrak{b}_{i,\zeta} : i \in S_2, \zeta < \zeta_i\rangle$ belong to M^k_0; and the function $g_k(\kappa) =: \sup(\kappa \cap \bigcup\limits_{\zeta<\theta_1^{++}} M^k_\zeta)$ satisfies clauses $(\alpha), (\beta), (\gamma), (\delta)$ above. Now N^a_{k+1}, N^b_{k+1} are defined by clause (d). Note that by the definition of μ'_i we have: for every $i < \sigma$, $\zeta < \zeta_i$, for some infinite $\mathfrak{a} = \mathfrak{a}^*_{i,\zeta} \subseteq \mathfrak{a}_{i,\zeta}$ we have $\mu_{i,\zeta}$, $k = \max \mathrm{pcf}(\mathfrak{a}) < \mu_i$. Moreover $\prod \mathfrak{a}/J^{\mathrm{bd}}_{\mathfrak{a}}$ has true cofinality. So our main demand on g_k is: $g_{k+1} \restriction \mathfrak{a}^*_{i,\zeta} = f^{\mathfrak{b}_{i,\zeta},\mu_{i,\zeta,k}}_\delta \mod J^{\mathrm{bd}}_{\mathfrak{a}^*_{i,\zeta}}$ for a suitable δ, so $\delta = \sup(\mu_{i,\zeta,k} \cap M^k_{\theta^{++}})$ is O.K. (For clause (γ) use (b) + (c) above.)

Now let $\{A^{k+1}_\ell : \ell < \omega\}$ be a list of:

$$\Big\{\lambda \cap A\big(\bigcup_{m\le n} A^k_m \cup \mathrm{Rang}[g_k \restriction \bigcup_{m<n} A^k_m], F\big) : \quad n<\omega \text{ and } F \text{ a definable function in } \big(\mathcal{H}(\chi), \in, <^*_\chi, \theta, \bar\lambda\big)\Big\}$$

and if

$$A^{k+1}_\ell = \lambda \cap A\big(\bigcup_{m<n} A^k_m \cup \mathrm{Rang}[g_k \restriction \bigcup_{m<n} A^k_m], F^{k+1}_\ell\big) \text{ and } i < \sigma$$

$g_k \restriction (\bigcup\limits_{m<n} A^k_{m,i} \cap \lambda_i \setminus \lambda_0^+)$ is included in some function: $\mathrm{Dom}(h^k_{\ell,i}) = \mathfrak{b}^k_{\ell,i}$, $h^k_{\ell,i}(\kappa) = \sup(\kappa \cap M^k_{\theta^{++}})$.

Having finished the inductive definition note that:

$(*)_8$ $\bigcup\limits_k N^a_k \prec \bigcup\limits_k N^b_k \prec (\mathcal{H}(\chi), \in, <^*_\chi, \theta, \bar\lambda)$.

[Why? As $N^a_k \prec N^b_k \prec (\mathcal{H}(\chi), \in, <^*_\chi, \theta, \bar\lambda)$ by clause (a) and clause (d).]

$(*)_9$ $\bigcup\limits_k N^a_k \cap \lambda_0 = \bigcup\limits_k N^b_k \cap \lambda_0$.

[Why? $N^b_k \cap \lambda_0 \subseteq N^a_{k+1} \cap \lambda_0$ (see clause (d)).]

$(*)_{10}$ if $\mu \in \mathrm{Reg} \cap \lambda^+ \setminus \lambda_0^+$ and $\mu \in \bigcup\limits_k N^a_k$ then $\bigcup\limits_{k<\omega} N^a_k$ contains an unbounded subset of $\mu \cap \bigcup\limits_{k<\omega} N^b_k$.

[Why? By clauses (d) + (g).]

So clearly (as usual)

$$\bigcup_k N^a_k \cap \lambda = \bigcup_k N^b_k \cap \lambda.$$

but $Z \subseteq N^b_0 \subseteq \bigcup\limits_{k<\omega} N^b_k$ and $Z \subseteq \lambda$ hence $Z \subseteq \bigcup\limits_{k<\omega} N^a_k \cap \lambda$. So for each $i \in S_2$, we can find $\langle(\bar a^{i,k}, w^{i,k}, u^{i,k}, \bar F^{i,k}) : k \le k(i)\rangle$ such that:

(a) $\bar a^{i,k(i)} = \langle \zeta(i)\rangle$
(b) $\bar a^{i,k} = \langle a^{i,k}_n : n < n^{i,k}\rangle$
(c) each $a^{i,k}_n$ belongs to $N^a_k \cup (\lambda_0 \cap N^b_{k+1})$
(d) $w^{i,k} = \{n < n^{i,k} : a^{i,k}_n \in \lambda_0 \cap N^b_{k+1}\}$

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(*e*) $u^{i,k} = \{n < n^{i,k} : a^{i,k}_n \in N^a_k \cap \mathrm{Reg} \cap \lambda \setminus \lambda^+_0\}$

(*f*) $\bar F^{i,k} = \langle F^{i,k}_n : n \in n^{i,k} \setminus w^{i,k}\rangle$, and $F^{i,k}_n$ is a definable function in $(\mathcal{H}(\chi), \in, <^*_\chi, \theta, \bar\lambda)$

(*g*) if $k > 0$, then $a^{i,k}_n = F^{i,k}_n(\dots, a^{i,k-1}_m, \dots, g_{k-1}(a^{i,k-1}_{m'}), \dots)_{m<n^{i,k-1}, m' \in u^{i,k-1}}$.

Let $a^{i,k}_n \in A^k_{\ell(i,k,n)}$. Note $(*)$ We can find stationary $S_3 \subseteq S_2$ such that:

$(*)$ if $i \in S_3$ then $k(i) = k(*)$ and for $k \le k(*)$ we have $n^{i,k} = n^k$, $w^{i,k} = w^k$, $u^{i,k} = u^k$, $\bar F^{i,k} = \bar F^k$, $\ell(i,k,n) = \ell(k,n)$.

We can also find a stationary $S_4 \subseteq S_3$ such that:

$(*)$ if $i_1 < i_2$ are in S_4 then $a^{i_1,k}_n \in A^k_{\ell(k,n),i_2}$

$(**)$ if $k < k(*), n \in u^k$ then $\langle a^{i,k}_n : i \in S_4\rangle$ is constant or strictly increasing and if it is strictly increasing and its limit is $\neq \lambda$ (hence is $< \lambda$) then it is $< \lambda_{\min(S_4)}$.

Let $E = \{\delta < \sigma : \delta = \sup(\delta \cap S_4)$ and if $n \in u^k$, and $\langle a^{i,k}_n : i \in S_4\rangle$ is strictly increasing with limit λ then $\langle a^{i,k}_n : i \in S_4 \cap \delta\rangle$ is strictly increasing with limit $\lambda_\delta\}$.

Now choose $\delta(*) \in E \cap S_1$, and choose b, a subset of $\delta(*) \cap S_4$ of order type ω with limit $\delta(*)$. We can choose $b^{k,n} \in [b]^{\aleph_0}$ for $k \le k(*)$, $n \le n^k$ such that: $b^{0,0} = b$, $b^{k,n+1} \subseteq b^{k,n}$, $b^{k+1,0} = b^{k,n^k}$, and if $n \in u^k$, $\langle a^{i,k}_n : i \in S_4\rangle$ strictly increasing with limit λ then $\prod\{a^{i,k}_n : i \in b^{k,n+1}\}/J^{\mathrm{bd}}_{b^{k,n+1}}$ has true cofinality which necessarily is $< \mu_{\delta(*)}$.

So (recall $n^{k(*)} = 1$) $b^* = b^{k(*),1}$ is a subset of $S_4 \cap \delta(*)$ of order type ω with limit $\delta(*)$ and $b^* \subseteq b^{k,n}$ for $k \le k(*)$, $n \le n^k$ and $b^* \subseteq b^{k,n+1}$ hence $n \in u^k$ and

$$\langle a^{i,k}_n : i \in S_4\rangle \text{ strictly increasing } \Rightarrow \mu_{\delta(*)} > \max \mathrm{pcf}\{a^{i,k}_n : i \in b^*\}.$$

Now we prove by induction on $k \le k(*)$ that for each $n < n^k$ for some $\mathfrak{B}_{k,n} \in M^*_{\delta(*)}$ with $\|\mathfrak{B}_{k,n}\| \le \lambda_0$ we have $\{a^{i,k}_n : i \in b^*\} \subseteq \mathfrak{B}_{k,n}$. For $k = 0$ clearly $A^0_{\ell(k,n)} \in M^*_{\delta(*)}$ has cardinality $\le \sigma$. For $k > 0$, for each $n < n^k$ we use the "$b^{k,n+1} \subseteq b$ and the choice of g_{k-1} and clause $(*)_0(c)$. So we get a contradiction to $(*)_7$ so we are done.

2) A variant of the proof of part (1). First, it is enough to prove, for each $i(*) < \sigma$ restrict ourselves to $S^* \cup \{\delta < \sigma$: the cardinal appearing in $\otimes$ is $\ge \lambda_{i(*)}\}$, then without loss of generality $i(*) = 0$ and see that $\zeta_i \le \lambda_0$ is O.K. $\square_{21.16}$

Remark 21.17. 1) Note that if we just omit "$\sigma^{\aleph_0} < \lambda$" we still get that for a club of $\delta < \sigma$, $\mathrm{cf}(\delta) > \aleph_0$ or $\mathrm{cf}(\delta) = \aleph_0$ and $\mathrm{pp}^{\mathrm{cr}}_{J^{\mathrm{bd}}_\omega}(\lambda_\delta)$; if $< \mathrm{cov}(\lambda_i, \lambda_i, \theta^+, 2)$ is still $\ge \lambda^{+\lambda_\delta}_\delta$.

Conclusion 21.18. *If μ is strong limit singular of uncountable cofinality* <u>*then*</u> *for a club of $\mu' < \mu$ we have $(2^{\mu'})^+ =^+ \mathrm{pp}^{\mathrm{cr}}_{J^{\mathrm{bd}}_{\mathrm{cf}(\mu')}}(\mu')$.*

Conclusion 21.19. *If $\beth_\delta$ is a singular cardinal of uncountable cofinality,* <u>*then*</u> *for a club of $\alpha < \delta$, if $\mathrm{cf}(\alpha) = \aleph_0$ then*

$(*)_1$ $2^{\beth_\alpha} =^+ \mathrm{pp}(\lambda)$

$(*)_2$ *there is $S \subseteq {}^\omega(i_\alpha)$ of cardinality $2^{\beth_\alpha}$ containing no perfect subset (and more — see* [She94b, §6]*).*

§ 22. Guessing clubs by countable C-s

Recently[6] Zapletal [Zap01] proved a beautiful theorem

Theorem 22.1. *If I is a "nice" (definition) of a σ-complete ideal on $\mathcal{P}(\mathbb{R})$ for suitable* LC *if* $\mathrm{ZFC} + \mathrm{LC} \vdash \mathrm{cov}(I) = 2^{\aleph_0}$ *then* $\mathrm{ZFC} + \mathrm{LC} \models \mathrm{Unif}(I) < \aleph_4$.

He also showed that $\aleph_4$ cannot be replaced by $\aleph_2$. The $< \aleph_4$ comes from quoting guessing clubs. The following shows we can replace $\aleph_4$ by $\aleph_3$ (other continuation see [SZ97], [SZ99]).

Claim 22.2. *Assume $\delta^* < \omega_1$ is a limit ordinal and $S \subseteq S^{\aleph_2}_{\aleph_0}$ is stationary. then we can find $\bar{C} = \langle C_\alpha : \alpha \in S\rangle$ such that*

- (a) $C_\alpha \subseteq S$
- (b) $C_\alpha \subseteq \alpha$
- (c) $\beta \in C_\alpha \Rightarrow \beta \in S$ *and* $C_\beta = C_\alpha \cap \beta$
- (d) $\mathrm{otp}(C_\alpha) \leq \delta^*$
- (e) *for every club E of ω_2 the set*
$$\{\delta \in S : \delta = \sup(C_\delta),\ \delta^* = \mathrm{otp}(C_\delta), \textit{and } C_\delta \subseteq E\}$$
is a stationary subset of ω_2.

Proof. For each $\alpha < \omega_2$ choose $\langle a^\alpha_i : i < \omega_1\rangle$; it will be an increasing continuous sequence of countable subsets of α with union α. For each $\alpha < \omega_2$ let

$$\begin{aligned} C^0_\alpha = \big\{ i < \omega_1 : {} & i \text{ is a limit ordinal such that} \\ & \forall \beta \in a^\alpha_i\ [a^\beta_i = a^\alpha_i \cap \beta], \\ & \alpha < \omega_1 \Rightarrow \alpha \subseteq a^\alpha_i, \text{and} \\ & j < i \Rightarrow \text{the closure of } a^\alpha_j \text{ is } \subseteq a^\alpha_i \cup \{\alpha\}\big\}. \end{aligned}$$

Clearly

$(*)_1(a)$ each C^0_α is a club of ω_1

(b) if $i \in C^0_\alpha$ and $\beta \in a^\alpha_i$ then $i \in C^0_\beta$.

Now

$(*)_2$ for some $\zeta = \zeta^* < \omega_1$, for every club E of ω_1 the following set $F_{\zeta^*}(E) \cap S$ is non empty where

⊠ $F_\zeta(E)$ is the set of $\delta < \omega_2$ such that:
- (a) $\delta = \mathrm{otp}(\delta \cap E \cap S)$
- (b) $\delta = \sup(E \cap \delta \cap S) = \sup(a^\delta_\zeta \cap E \cap S)$
- (c) $\mathrm{otp}(a^\delta_\zeta \cap E \cap S)$ is divisible by δ^*
- (d) $\zeta \in C^0_\delta$ hence $\beta \in a^\delta_\zeta \Rightarrow \zeta \in C^0_\beta$.

[Why does $(*)_2$ hold? Otherwise for each $\zeta < \omega_1$ there is a club E_ζ of ω_2 such that $F_\zeta(E_\zeta) = \varnothing$. Let $E^* = \bigcap\{E_\zeta : \zeta < \omega_1\} \setminus \omega_1$, clearly E^* is a club of ω_2, and so is $E' = \{\delta < \omega_2 : \delta = \mathrm{otp}(E^* \cap \delta \cap S)\}$ and choose $\delta \in E' \cap S$, exists as E' is a club of ω_2 and $S \subseteq S^2_0$ is stationary. Easily the set

$$C^* = \{\zeta < \omega_1 : \zeta \text{ limit}, \mathrm{otp}(a^\delta_\zeta \cap E \cap S) \text{ is divisible by } \delta^*\}$$

[6] added Fall 2002

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is a club of ω_1. So there is $\zeta^* \in C^* \cap C^0_\delta$, clearly $\delta \in F_{\zeta^*}(E^*)$ hence $\delta \in F_{\zeta^*}(E_\zeta)$: contradiction.]

$(*)_3$ if $E_1 \subseteq E_0$ are clubs of ω_2 then $F_{\zeta^*}(E_1) \subseteq F_{\zeta^*}(E_0)$.

[Why? Note that $a \subseteq b \subseteq \delta = \sup(a)$ and $\delta^*|\mathrm{otp}(a)| \Rightarrow \delta^*\mathrm{otp}(b)$.]

$(*)_4$ for some club E_0 of ω_2 for every club E_1 of ω_2 the set $F_{\zeta^*}(E_1, E_0) \neq \varnothing$ where $F_{\zeta^*}(E_1, E_0) = \{\delta : \delta \in F_{\zeta^*}(E_0)$ and $a^\delta_{\zeta^*} \cap E_0 \cap E_1 = a^\delta_{\zeta^*} \cap E_0\}$.

[Why? If not we choose by induction on $\varepsilon < \omega_1$ a club E_ε of ω_2 such that $i < \varepsilon \Rightarrow E_\varepsilon \subseteq E_i$ and $F_{\zeta^*}(E_{\varepsilon+1}, E_\varepsilon) = \varnothing$. So $E^* = \cap\{E_\varepsilon : \varepsilon < \omega_1\}$ is a club of ω_2 so we can find $\delta \in F_{\zeta^*}(E^*)$, hence $\delta \in \cap\{F_{\zeta^*}(E_\varepsilon) : \varepsilon < \omega_1\}$ by $(*)_3$. Now trivially $\langle a^\delta_{\zeta^*} \cap E_\varepsilon : \varepsilon < \omega_1\rangle$ is a decreasing sequence of subsets of $a^\delta_{\zeta^*}$ which is countable and $\varepsilon < \omega_1 \Rightarrow a^\delta_{\zeta^*} \cap E_\varepsilon \neq a^\delta_{\zeta^*} \cap E_{\varepsilon+1}$ as $F_{\zeta^*}(E_{\varepsilon+1}, E_\varepsilon) = \varnothing$, contradiction.]

We fix E_0 as in $(*)_4$,

$(*)_5$ for some $\xi_\zeta < \omega_1$ we have: for every club E_1 of ω_2 for some $\delta \in F_{\zeta^*}(E_1, E_0)$ we have $\mathrm{otp}(a^\delta_{\zeta^*} \cap S \cap E_0) = \xi$.

[Why? As in the proof of $(*)_4$.]

So necessarily ξ is divisible by δ^*. Choose $b \subseteq \xi = \sup(b)$, $\mathrm{otp}(b) = \delta^*$. Let

$$S' = \{\alpha < \omega_1 : \mathrm{otp}(a^\alpha_{\zeta^*} \cap S \cap E_0) \in b \cup \{\xi\}\}.$$

Now we define $\bar{C} = \langle C_\alpha : \alpha \in S\rangle$ as follows: if $\alpha \in S \setminus S'$ we let $C_\alpha = \varnothing$ and if $\alpha \in S'$ we let $C_\alpha = \{\beta : \beta \in a^\alpha_{\zeta^*} \cap S \cap E_0$ and $\mathrm{otp}(\beta \cap a^\alpha_{\zeta^*} \cap S \cap E_0) \in b\}$.

Now you can check that $\bar{C} = \langle C_\alpha : \alpha \in S\rangle$ is as required. (Noting that is clause (e), "stationarily many" "at least one" are equivalent demands.) $\square_{22.2}$

Remark 22.3. Can we demand above that if C_α has no last element then C_α is a closed subset of α?

Not clear to me, but we can find

⊛ there is $\langle \mathscr{C}_\alpha : \alpha \in S\rangle$ such that
- (a) $\mathscr{C}_\alpha$ is a countable family of countable subsets of $\alpha \cap S$, each of order type $\leq \delta(*)$
- (b) if $C \in \mathscr{C}_\alpha$ then C is closed as a subset of α
- (c) if $\beta \in C \in \mathscr{C}_\alpha$ then $C \cap \beta \in \mathscr{C}_\beta$
- (d) if E is a club of ω_2 then for stationarily many $\alpha \in S$ for some $C \in \mathscr{C}_\alpha$ we have $\delta(*) = \mathrm{otp}(C)$ and $C \subseteq E$.

In some cases Zapletal [Zap01] uses $\mathfrak{d} \leq \mathfrak{b}^{+n}$ we can replace this by $\mathrm{cf}([\mathfrak{d}]^{\aleph_0}, \subseteq) = \mathfrak{d}$ because

Claim 22.4. *Assume κ is regular uncountable.*

If $\lambda > \kappa$ and $\mathrm{cf}([\lambda]^{<\kappa}, \subseteq) = \lambda$, <u>then</u> for any $\alpha < \kappa$ there is $Y \subseteq {}^\alpha([\lambda]^{<\kappa})$ which is $E_{\kappa,\alpha}(\lambda)$-positive, i.e.,

⊛ *if $\chi > \lambda$ and $\xi \in \mathcal{H}(\chi)$ then there is $\bar{N} = \langle N_i : i \leq \alpha\rangle$ such that $x \in N_i \prec (\mathcal{H}(\chi) \in <^*_\chi)$*

$$\|N_i\| < \kappa$$

$$N_i \cap \kappa \in \kappa$$

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$$N_i \text{ increasing continuous}$$

$$\bar{N} \restriction (i+1) \in N_{i+1}.$$

Proof. As in [She93a, §2] (fill!)

Let $W_0 = \{\alpha \in E_0 : \{\xi^*, \zeta^*\} \subseteq C^0_\alpha$ and $\text{otp}(a^\alpha_{\xi^*} \cap E_0) \leq \zeta\}$ and for $\alpha \in W_0$ let $b_\alpha = a^\alpha_{\xi^*} \cap E_0 \cap S$. Clearly

⊛) (a) $\alpha \in W_0 \Rightarrow b_\alpha \subseteq W_0$ and $\text{otp}(b_\alpha) \leq \zeta$

(b) $\alpha \in b_\beta \cap \beta \in W \Rightarrow b^*_\alpha = b_\beta \cap \alpha$

(c) if E_1 is a club of ω_2 then for stationarily many $\alpha \in W_0 \cap S$ we have $b_\alpha \subseteq E_1$ and $\text{otp}(b_\alpha) = \xi$.

$\square_{22.4}$

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