THEORIES WITH MINIMAL UNIVERSALITY SPECTRUM, E103

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Abstract. For a countable first order complete theory T, let the universality spectrum of $T, \mathrm{univ(T)}$ be the class of cardinals λ in which T has a universal model.

An old questions ask: is there a T which provably in ZFC has a minimal universality spectrum. We sort out the cases left open and eliminate some.

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Annotated Content

§ 0. Introduction

§ 0(A). **Background and aims.**

The following question was asked in [She80, §4], so is by now quite old. It is enough to decide countable many cases (see Fact 1.1).

Question 0.1. Is there a countable complete first order T which has a universal model in a cardinal λ <u>iff</u> $\lambda = 2^{<\lambda} > \aleph_0$?

This essentially says that the existence results of Jonsson [Jón56], [Jón60] (for universal theories with JEP and amalgamation under embeddings) and Morely-Vaught [MV62] (for complete first order T) are best possible. Which prove the "if" part. The parallel problem for universal-homogeneous and saturated was answered long ago.

By Kojman-Shelah [KS92] for many cardinals λ (such that $\lambda < 2^{<\lambda}$) the answer is no, even for the theory of dense linear order.

On background see the survey [Dža05] and recently [She21].

The questions was raised in [She80, §4]

§ 0(B). **Preliminaries.**

Convention 0.2. T will be a countable first order complete theory.

The following are used in §2 (see [She79a], [DFOS89, §4], [She93a], [She85] and more in [Shea]).

Definition 0.3.

1) For a regular uncountable cardinal λ let $\check{I}[\lambda] = \{S \subseteq \lambda\colon$ some pair $(E, \bar{a})$ witnesses $S \in \check{I}(\lambda)$, see below$\}$.

2) We say that (E, u) is a witness for $S \in \check{I}[\lambda]$ <u>if</u>:

(a) E is a club of the regular cardinal λ,
(b) $u = \langle u_\alpha : \alpha < \lambda\rangle, a_\alpha \subseteq \alpha$ and $\beta \in a_\alpha \Rightarrow a_\beta = \beta \cap a_\alpha$,
(c) for every $\delta \in E \cap S, u_\delta$ is an unbounded subset of δ of order-type $< \delta$ (and δ is a limit ordinal).

3) For regular $\lambda < \kappa$, let $\check{I}_\kappa[\lambda] = \check{I}[\lambda] \restriction \{\delta < \lambda\colon \mathrm{cf}(\delta) = \kappa\}$.

Claim 0.4. *Let λ be regular uncountable cardinal.*

1) If $S \in \check{I}[\lambda]$ <u>then</u> we can find a witness $(E, \bar{a})$ for $S \in \check{I}[\lambda]$ such that:

(a) $\delta \in S \cap E \Rightarrow \mathrm{otp}(a_\delta) = \mathrm{cf}(\delta)$,
(b) if $\alpha \notin S$ then $\mathrm{otp}(a_\alpha) < \mathrm{cf}(\delta)$ for some $\delta \in S \cap E$.

2) $S \in \check{I}[\lambda]$ <u>iff</u> there is a pair $(E, \bar{\mathcal{P}})$ such that:

(a) E is a club of the regular uncountable λ,
(b) $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \lambda\rangle$, where $\mathcal{P}_\alpha \subseteq \{u : u \subseteq \alpha\}$ has cardinality $< \lambda$,
(c) if $\alpha < \beta < \lambda$ and $\alpha \in u \in \mathcal{P}_\beta$ then $u \cap \alpha \in \mathcal{P}_\alpha$,
(d) if $\delta \in E \cap S$ then some $u \in \mathcal{P}_\delta$ is an unbounded subset of δ (and δ is a limit ordinal).

3) (a) If λ is regular, then $\{\delta < \lambda^+ : \mathrm{cf}(\delta) < \lambda \in \check{I}[\lambda]\}$.
(b) if $\lambda^{<\chi} = \lambda$ and λ, χ are regular cardinals then $\{\delta < \lambda : \mathrm{cf}(\delta) < \kappa\}$ belongs to $\check{I}[\lambda]$.
(c) If $\lambda > \kappa^+$ and λ, κ are regular then there is stationary set $S \subseteq \{\delta < \kappa\colon \mathrm{cf}(\delta) = \kappa\}$.

§ 1. Background and theorem

Note (and we shall use freely):

Fact 1.1. Given T_ℓ (for $\ell \le \ell_* \le \omega$) there is T such that $\mathrm{Univ}(T) = \bigcap\{\mathrm{Univ}(T_\ell) : \ell < \ell_*\}$.

Proof. Without loss of generality,

(∗) (a) τ_{T_ℓ}, the vocabulary of T_ℓ consists of predicates only,
(b) $\langle \tau(T_\ell) : \ell < \ell_*\rangle)$ are pairwise disjoint.

Next, let $T = \sum_{\ell<\ell_*} T_\ell$; that is, $\tau = \bigcup\{\tau(T_\ell) : \ell < \ell_*\} \cup \{P_\ell : \ell < \ell_*\}, P_\ell$ unary not in $\bigcup\{\tau(T_i) : i < \ell_*\} \cup \{P_i : i < \ell_* \wedge i \neq \ell\}$ and a τ-model M is a model of T iff:

(∗) (a) $\langle P^M_\ell : \ell < \ell_*\rangle$ is a sequence of pairwise disjoint sets,
(b) $M \restriction P^M_\ell \restriction \tau(T_\ell)$ is a model of T_ℓ for $\ell < \ell_*$,
(c) there are no more cases of the relations,
(d) if ℓ_* is finite, then $M \setminus \bigcup\{P^M_\ell : \ell < \ell_*\}$ is infinite.

It suffices to show:

⊞ T is as required.

This is easy. $\square_{1.1}$

This essentially says that the existence results of Jonsson [Jón56], [Jón60] (for universal theories with JEP and amalgamation under embeddings) and Morely-Vaught [MV62] (for complete first order T) are best possible. Which prove the "if" part. The parallel problem for universal-homogeneous and saturated was answered long ago.

By Kojman-Shelah [KS92] for many cardinals λ (such that $\lambda < 2^{<\lambda}$) the answer is no, even for the theory of dense linear order.

Theorem 1.2. *There is T such that: if $\lambda \notin \mathrm{Univ}(T)$, then for some cardinal μ:*

(∗) $\lambda = \mu^+$ and (a) or (b), where:
(a) μ is singular strong limit such that $\mu = \aleph_\delta, \delta < \mu$ and $2^\mu > \lambda$,
(b) μ is regular, $2^{<\mu} \le \lambda < 2^\mu$ and $\mathfrak{b}_\mu = \lambda_1 < \mathfrak{d}_\mu$.

Remark 1.3.

1) Recall 1.1,

2) More is done in [She96b] and [She16].

3) The proof of 1.2 is divided to cases some by quoting, some are proved below. Many of the cases prove this for $T = T_{\mathrm{lin}}$, the theory of linear orders (or any T which the strict order property), but not in all cases. In each case we get a more specific lower bound to $\mathrm{Univ}_T(\lambda)$. As there are finitely many case, this is enough by Fact 1.1.

Proof. Recalling Fact 1.1, the proof split to finitely many cases:

Case 1: λ singular.

Holds by [She17b] for the so called $T = T_{\mathrm{elo}}$.

Case 2: λ regular and for some regular κ we have $2^\kappa > \lambda > \kappa^+$.

By Kojman-Shelah [KS92], for T_{lin} (the theory of linear orders); also many singular cardinals were covered but irrelevant here.

Case 3: $\lambda = \aleph_0$ or just $\lambda < 2^{\aleph_0}$.

Let T_0 be $\mathrm{Th}(M_0)$ where $M_0 := ({}^\omega 2, P^{M_0}_n)_{n<\omega}$ and $P^{M_\gamma}_n := \{\eta \in {}^\omega 2 : \eta(n) = 1\}$.

(∗) We can assume cases 1, 2 and 3 fail, hence $\lambda = \mu^+, 2^{<\mu} \leq \lambda$.

[Why? Clearly $\lambda > \aleph_0$ is regular (if λ is a limit cardinal and $2^{<\lambda} > \lambda$, then for some $\kappa < \lambda$, $2^\kappa > \lambda$, hence $\kappa^+ < \lambda$, so this is covered by case 2). So without loss of generality $\lambda = \mu^+$. If $2^{<\mu} > \lambda$, then for some $\theta < \mu, 2^\theta > \lambda$ by case 1, therefore we are done, so we can assume $2^{<\mu} \leq \lambda$ indeed.]

Case 4: $\lambda = \mu^+$ with μ is singular.

We can assume not case 2, hence $\kappa < \mu \Rightarrow 2^\kappa \leq \lambda$. If $2^{<\mu} > \mu$, then by cardinal arithmetic $2^\mu = \lambda$, so we are done and so we can assume $2^{<\mu} \leq \mu$. Since by hypothesis μ is singular we have that $\theta < \mu \Rightarrow 2^\theta < \mu$ and so μ is strong limit.

The case $2^\mu = \lambda = \mu^+$ is covered, so assume $2^\mu > \lambda = \mu^+$. If $\mu = \aleph_\mu$, then we can apply 2.1, so we can assume that for some limit ordinal $\delta < \mu, \mu = \aleph_\delta$. So we fall under clause (a) of the theorem.

Case 5: $\lambda = \mu^+$ and μ regular $> \aleph_0$.

So $\lambda \geq 2^{\aleph_0}$ (otherwise use case 3) and $2^{<\mu} \in \{\mu, \lambda\}$ otherwise use case 2. Therefore by 0.4(3)(b) there is a stationary set $S \subseteq S^\lambda_\mu$ which is from $\check{I}[\lambda]$, hence by 3.4 without loss of generality, $\mathfrak{b}_\mu = \lambda$ and by 4.2 without loss of generality, $\mathfrak{d}_\mu > 2^\mu$. So we fall under clause (b) of the theorem. $\square_{1.3}$

§ 2. Successor of singular fix point

Theorem 2.1. *If* $\lambda = \mu^+ < 2^\mu, \mu = \aleph_\mu$ *and*[1] $\kappa = \mathrm{cf}(\mu) < \mu$ *then* $\mathrm{univ}_{\mathrm{T_{lin}}}(\lambda) \ge 2^\mu$, *equivalently* $\mathrm{univ}_{T_{\mathrm{dlo}}}(\lambda) \ge 2^\mu$.

Proof.

Case 1: $\kappa > \aleph_0$:

By 2.2, below.

$(*)_1$ choose $\langle \bar\alpha_\delta : \delta \in S\rangle$ such that:
- (a) $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ is stationary,
- (b) $\bar\alpha_\delta = \langle \alpha_{\delta,i} : i < \kappa\rangle$ is increasing with limit δ such that $\mu | \alpha_{\delta,i}$,
- (c) club guessing: for every club E of λ, for stationarily many δ many $\delta \in S$ we have $\{\alpha_{\delta,i} : i < \kappa\} \subseteq E$,
- (d) $|\{\bar\alpha_\delta \restriction i : \alpha_{\delta,i} = \alpha\}| \le \mu$ for every $i < \kappa$ and $\alpha < \lambda$ (recall $\bar\alpha_\delta$ is increasing but not necessarily continuous).

$(*)_2$
- (a) for $M \in \mathrm{EC}_{T_{\mathrm{lin}}}(\lambda!), a \in M$ and $\alpha < \lambda$ let $\mathrm{cf}(a,\alpha,M)$ be the cofinality of $M \restriction \{\beta < \alpha : \beta <_M a\}$,
- (b) we say $\bar\beta$ witness $\mathrm{cf}(a,\alpha,M) = \theta$ when $\bar\beta = \langle b_i : i < \theta\rangle$ is $<_M$-increasing and cofinal in $M \restriction \{\beta < \alpha : \beta <_M a\}$,
- (c) for $a \in M \in \mathrm{EC}_{T_{\mathrm{lin}}}(\lambda!)$ and $\delta \in S$, let $\mathrm{inv}(a, \bar\alpha_\delta, M) = \langle \mathrm{cf}(a, \alpha_{\delta,i+1}, M) : i < \kappa\rangle$.

Now,

$(*)_{2.1}$ for every sequence $\bar\theta = \langle \theta_i : i < \kappa\rangle$ of regular cardinals $< \mu$ and for transparency $\ne \kappa$, there are $M = M_{\bar\theta}$ and $\bar F_{\bar\theta}$ such that:
- (a) $M \in \mathrm{EC}_{\mathrm{T_{lin}}}(\lambda!)$, $\bar F_{\bar\theta} = \langle F_{\bar\theta,\varepsilon} : \varepsilon < \mu\rangle$, $F_{\bar\theta,\varepsilon} \in {}^\lambda\lambda$,
- (b) if $\alpha < \lambda$ is divisible by μ, then:
  - (α) if $\beta_2 <_M \beta$ are $< \alpha$ then for some $\gamma \in (\alpha, \alpha+\mu)$ we have $\beta_2 <_M \gamma <_M \beta$,
  - (β) moreover, if $\beta \in [\alpha, \alpha+\mu)$, (so α is definable from β) then in $M \restriction \{\gamma : \alpha \le \gamma < \alpha+\mu$ and $(\forall \xi < \alpha)(\xi <_M \gamma \equiv \xi < \beta)\}$ there is an $<_M$-increasing sequence of length μ; moreover such sequence is $\bar F_{\bar\theta}(\beta) = \langle F_{\bar\theta,\varepsilon}(\beta) : \varepsilon < \mu\rangle$.
- (c) if $\sigma = \mathrm{cf}(\sigma) < \mu$ is $\ne \kappa$ and $\langle \alpha_i : i < \sigma\rangle$ is an $<_M$-increasing sequence of ordinals $< \alpha$ then for some $\beta \in [\alpha, \alpha+\mu)$ we have $(\forall \xi < \alpha)(\xi < \beta \equiv \bigvee_{i<\sigma} \xi < \alpha_i)$,
- (d) if $\delta \in S$ and $i < \kappa$ then $\mathrm{cf}(\delta, \alpha_{\delta,i+1}, M) = \theta_i$ as witnessed by $\langle F_{\bar\theta,\varepsilon}(\beta) : \varepsilon < \theta_i\rangle$, for some $\beta \in [\alpha_{\delta,i}, \alpha_{\delta,i} + \mu]$ fix such $\beta = \beta^0_{\delta,i}$.

[Why? Straightforward as in [KS92].]

$(*)_3$ if $\bar\theta, M_{\bar\theta}, \bar F_{\bar\theta}$ are as above, $N \in \mathrm{EC}_{\mathrm{T_{lin}}}(\lambda!)$ and f embeds $M_{\bar\theta}$ into M, then for some club E of λ, we have $\delta \in S \cap \mathrm{acc(E)}$, $\{\alpha_{\delta,i} : i < \kappa\} \subseteq E$ moreover $\mathrm{inv}(\delta, \bar\alpha_\delta, M_\delta) = \mathrm{inv}(f(\delta), \delta, N)$.

[Why? Similarly to [KS92], but we elaborate. Let $\mathscr{B}$ be a model $(\lambda, <^{\mathscr{B}}_1, <^{\mathscr{B}}_2, <^{\mathscr{B}}_3, F^{\mathscr{B}})$ where:

$(*)_{3.1}$
- (a) $<^{\mathscr{B}}_1 = <_{M_{\bar\theta}}$,
- (b) $<^{\mathscr{B}}_1 = <_N$,
- (c) $<^{\mathscr{B}}_2$ is the order of the ordinals (up to λ),
- (d) $G^N = f$,

[1] Earlier we demand "μ is strong limit" but quoting [KS92] this is not necessary.

(e) F^N satisfies $F_{\bar\theta,\varepsilon}(\beta^\bullet_{\delta,i}) = F^N(\varepsilon,\beta^\bullet_{\delta,i})$.

Let,

$(*)_{3.2}$ $E = \{\delta < \lambda : \delta < \lambda$ is divisible by μ, closed under f and $\mathscr{B} \restriction \delta \prec \mathscr{B}\}$.

Clearly E is a club of λ and it suffices to prove:

$(*)_4$ it suffices to prove that for any assume $\delta \in S \cap E$ and $i < \kappa \Rightarrow \alpha_{\delta,i} \in E$ and prove that:
- if $i < \kappa$ then $\mathrm{cf}(f(\delta), \alpha_{\delta,i+1}, N) = \theta_i$.

It is actually clear but we elaborate.

Let $\bar\beta_i = \langle \beta_{i,\varepsilon} : \varepsilon < \theta_i\rangle$ be increasing in $M_{\bar\theta}$ such that $\beta_\varepsilon \in [\alpha_{\delta,i}, \alpha_{\delta,i} + \mu)$ and $\bar\beta_i$ witness $\mathrm{cf}(\delta, \alpha_{\delta,i+1}, M_{\bar\theta}) = \theta_i$, which means $(\forall\gamma < \alpha_\delta, i+1)(\gamma < \delta \equiv \bigvee_{\varepsilon<\theta_i} \gamma <_{M_{\bar\theta}} \beta_{i,\varepsilon})$ exist by $(*)_3$.

Hence,

$(*)_{4.1}$ it suffices to prove: $\bar\beta^f_{\delta,i} = \langle f(\beta_{i,\varepsilon}) : \varepsilon < \theta_1\rangle$ witness $\mathrm{cf}(f(\delta), \alpha_{\delta,i+1}, N) = \theta_i$.

[Why? Should be clear.]

Now clearly $\varepsilon < \theta_i \Rightarrow \beta_{i,\varepsilon} <_{M_{\bar\theta}} \delta \Rightarrow f(\beta_{i,\varepsilon}) <_N f(\delta)$ recalling f embedded into $M_{\bar\theta}$ into N, so too finite as:

$(*)_{4.2}$ assume toward contradiction $\gamma_\bullet < \beta_{\delta,i+1}, N \models$"$f(\beta_{i,\varepsilon}) < \gamma_i < f(\delta)$" for $\varepsilon < \theta_i$.

Now,

$(*)_{4.4}$ $\mathfrak{B} \restriction\models \psi[\gamma_\bullet, \beta_{\delta,i}, \theta_i]$ which means $(\forall x)[\gamma_\bullet \le_2 \theta(x)](\forall x)[\gamma_\bullet \le_2 G(x) \equiv (\forall\varepsilon < \theta_i)(F(\beta_{\delta,i},\varepsilon) \le_1 x)]$.

[Why? Just think.]

But,

$(*)_{4.5}$ $\mathscr{B} \models \neg\psi[\gamma_\bullet, \beta_{\delta,i}, \theta]$ because $x \mapsto \delta$ is a counter-example.

Now,

$(*)_5$ $\mathrm{univ}_{\mathrm{T_{lin}}}(\lambda) \ge 2^\mu$.

[Why? Toward contradiction assume $\chi < 2^\mu$ and $\bar M = \langle M_\xi : \xi < \chi\rangle$ witness $\mathrm{univ}_{\mathrm{T_{lin}}} \le \chi$, so $M_\xi \in \mathrm{EC}_{T_{\mathrm{lin}}}(\lambda!)$. So $\Lambda^2_\mu = \{\mathrm{inv}(a, \bar\alpha_\delta, M) : \delta \in S$ and $a \in M\} \cap \Lambda^1_\mu$ is of cardinality $\le \chi + \lambda$ where $\Lambda^1_\mu = \{\bar\theta : \bar\theta$ a sequence of length κ of regular cardinals $< \mu$ but $\ne \kappa\}$. Recalling $\mu = \aleph_\mu$ by the case assumption clearly $|\lambda^1_\mu| = \mu^\kappa = 2^\mu$, there is $\bar\theta \in \Lambda^1_\mu \setminus \Lambda^1_\mu$. Easily $M_{\bar\theta}$ is not embeddable into M_ξ for $\xi < \chi$.]

$(*)_6$ $\mathrm{univ}_{\mathrm{T_{lin}}}(\lambda) = 2^\lambda$.

[Why? Let $\Lambda^*_\mu = \{\mathbf{s} : \mathbf{s}$ has the form $\langle\bar\theta_\delta : \delta \in S\rangle = \langle\bar\theta_{\mathbf{s},\delta} : \delta \in S\rangle$ where for each $\delta \in S, \bar\theta_{\mathbf{s},\delta} = \langle\theta_{\mathbf{s},\delta,i} : i < \kappa\rangle \in \Lambda^1_\mu\}$.] We choose $M_{\mathbf{s}} \in \mathrm{EC}_{T_{\mathrm{lin}}}(\lambda!)$ parallely to $(*)_3$; that is, replacing clause (d) there by:

d') if $\delta \in S$ then $\mathrm{cf}(\delta, \alpha_{\delta,i}, M) = \theta_{\mathbf{s},\delta,i}$.

As $|\Lambda^1_\mu| = 2^\mu$ clearly there is a subset $\Lambda_* \subseteq \Lambda^*_{mu}$ of cardinality 2^μ such that $\mathbf{s} \ne \mathbf{r} \in \Lambda_* \Rightarrow \lambda > \sup\{\delta \in S : \bar\theta_{\mathbf{s},\delta} = \bar\theta_{\mathbf{r},\delta}\}$. We continue as in the proof of $(*)_5$].

<u>Case 2</u>: $\kappa = \mathrm{cf}(\mu) = \aleph_0$ and $\mu = \aleph_\mu$:

$\odot_1$ (a) choose $\theta = \mathrm{cf}(\delta) \in (\kappa, \mu)$,
(b) choose $\langle\bar\alpha_\delta : \delta \in S\rangle$ such that:
(α) $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$,

(β) $\bar{\alpha}_\delta = \langle \alpha_{\delta,i} : i < \theta \rangle$ is increasing continuous with limit δ (not necessarily continuous),

(γ) for every $i < \theta$ and $\alpha < \lambda$ the set $\Omega_{\alpha,i} = \{\bar{\alpha}_\delta \restriction i : \delta \in S$ satisfies $\alpha_{\delta,i} = \alpha\}$ has cardinality $\leq \mu$,

(δ) $\langle \bar{\alpha}_\delta : \delta \in S \rangle$ guess clubs.

[Why? See 2.2 below.]

We continue as in the proof of $(*)_5$ in Case 1, using θ instead of κ, now using $\mathrm{Inv}(a, \delta, S) = \langle \mathrm{inv}(a, \alpha_{\delta,i+1}, M) : i < \theta \rangle$. $\square_{2.1}$

As in [She93a] but proved to be more self-contained.

Fact 2.2. Assume $\lambda = \mu^+, \mu > \theta = \mathrm{cf}(\theta)$. Assume further $S^\bullet \in \check{I}_\theta[\lambda]$ (see [She79a], here 0.3) is stationary such that $S^\bullet \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$, where such $S^\bullet$ exists by [She93a], here 0.4(3)(c).

Then we can find $\mathbf{s} = \langle \bar{\alpha}_\delta : \delta \in S \rangle$ such that:

(a) $S \subseteq S^\bullet$ is stationary,

(b) for $\delta \in S, \bar{\alpha}_\delta = \langle \alpha_{\delta,i} : i < \theta \rangle$ is increasing continuous with limit δ,

(c) for every successor $i < \theta$ and $\alpha < \lambda$ the set $\Omega_{\alpha,i} = \{\bar{\alpha}_\delta \restriction i : \delta \in S$ satisfies $\alpha_{\delta,i} = \alpha\}$ has cardinality $\leq \mu$, moreover is a singleton.

Proof. Recalling $S \in \check{I}_\theta[\lambda]$ we can find $\mathbf{s}_\theta = \langle \bar{\alpha}^0_\delta : \delta \in S_\theta \rangle$ as in (a), (b) but not necessarily continuous such that $\alpha = \alpha^0_{\delta,i} \Rightarrow \bar{\alpha}^0_\delta \restriction i = \bar{\beta}_\alpha$ (exists by the definition of $S \in \check{I}_\theta[\lambda]$). Next, for any club E of λ letting $\mathbf{s}_E = \langle \bar{\alpha}^1_{\delta,E} : \delta \in S_E \rangle$ where $S_E = \{\delta \in S : (\delta \cap E)$ has order type $\delta\}$ and $\alpha_{\delta,E,i} = \sup(E \cap \alpha^0_{\delta,i})$, (so called glueing), so $\alpha^1_{E,\delta}$ is $\leq$-increasing, not necessarily $<$-increasing, just $\leq$-increasing and $\delta \in S_E = \delta = \bigcup\{\alpha^1_{\delta,E,i} : i < \theta\}$.

As there or [She94, Ch. III], (trying θ^+ times), for some club E_* of λ:

$(*)^1_{E_*}$ for every club E of λ for stationarily many $\delta \in S$ we have $\delta \in S_E$ and $(\forall i < \theta)(\alpha_{\delta,E_*,i} \in E)$ equivalent to $(\forall i < \theta)(\alpha_{\delta,E,i} = \alpha_{\delta,E_*,i})$ i.e., $\bar{\alpha}_{\delta,E} = \bar{\alpha}_{\delta,E_*}$.

Now,

$(*)_2$ if $\alpha < \lambda$ and $i < \theta$ then $\Omega_{\alpha,i}, \Omega_\alpha$ have cardinality $\leq \mu$ where:
- $\Omega_{\alpha,i} = \{\bar{\alpha}_{\delta,E_*} \restriction i : \delta \in S$ satisfies $\alpha_{E,\delta,i} = \alpha$ and $\delta = \sup(E_* \cap \delta)\}$ and,
- $\Omega_\alpha = \bigcup\{\Omega_{\alpha,i} : i < \kappa\}$.

[Why? Because for each $\alpha < \lambda, i < \theta$, the $\Omega_{\alpha,i}$ is included in $\{\bar{\alpha}^1_\delta \restriction i : \alpha_{\delta,i} \in [\alpha, \min(E_*) \setminus (\alpha + 1))\}$.]

Clearly we are done.

$\square_{2.2}$

Discussion 2.3.

1) Why in [She20, §2] when $\mu < \aleph_\mu$ we do not use T_{lin} (= theory of linear order)? The point is that (recall is strong limit singular):

$(*)$ if I_1 is a linear order of cardinality μ then for some linear order I_2 of cardinality μ extending I_1 we have: if $\theta < \mu$ and $\theta = \mathrm{cf}(\theta) \neq \mathrm{cf}(\mu)$ and $\langle a_\alpha : \alpha < \theta \rangle$ is $<_{I_1}$-increasing then:

(a) there is $t \in I_2 \setminus I_1$ such that $(\forall s \in I_1)(s <_{I_2} \Leftrightarrow \bigvee_{\alpha<\theta} a_\alpha <_{I_2} b)$,

(b) if $\theta > \aleph_0$ then there is a club E of θ and $\bar{t} = \langle t_\delta : \delta \in E \rangle$ such that: if $\delta \in E$ then:

$$(\forall s \in I_1) \left(s <_{I_2} b \Leftrightarrow \bigvee_{\alpha<\theta} a_\alpha <_{I_2} b \right).$$

2) Still we can prove $\text{univ}_{\text{T}_{\text{dlo}}}(\lambda) \geq 2^\mu$ (every $= 2^\alpha$) when:

(a) some $\bar{C} = \langle C_\delta : \delta \in S \rangle$ guess clubs, $\text{otp}(C_\delta) = \mu$,
(b) we may allow $\sup(C_\delta) < \delta$,
(c) enough to use $\langle (C_\delta, D_\delta) : \delta \in S \rangle, D_\delta$ a filter on δ such that $\mathscr{P}(C_{\delta \setminus D_\delta}) > \lambda$ and the guessing of clubs means:
$\odot_1$ for every club E of λ for some $\delta \in S$ we have $E \cap \text{nacc}(C_\delta) \in D_\delta$.
(d) We can return to [She20, §6] deciding, e.g. $\lambda = \kappa^{+^\kappa} = \aleph_{\kappa+\kappa}$.
(e) Again going back to ZFC: maybe it is enough when:
(∗) there is a linear order I of cardinality $< \lambda$ with $\leq \lambda$ cuts (no $> \lambda$).

Check what [KS92] demands - no.

Claim 2.4. *Assume λ is a singular (or just limit) cardinal which is not strong limit. Then some linear order I of cardinality $< \lambda$ has $\geq$ cuts.*

Proof. By [CS16] which relies on [She96a] and prove more but we elaborate.
Let $\kappa = \min\{\kappa : 2^\kappa \geq \lambda\}$. By our assumption:

$(*)_0$ (a) $\kappa < \lambda$,
(b) $2^{<\kappa} \leq \lambda$.

If $2^{<\kappa} < \lambda$ the conclusion is obvious; use $I = ({}^{\kappa>}2, <_{\text{lex}})$.
So assume:

$(*)_1$ $2^{<\kappa} < \lambda$.

If κ is a successor cardinal, letting $\kappa = \theta^+$ we have $2^\theta < \lambda$, hence $2^{<\kappa} = 2^\theta$, so contradicting $(*)_1$.
If $\kappa = \aleph_0$ then $I = (\mathbb{Q}, <)$ is an example so we are left with the case:

$(*)_2$ $\kappa > \aleph_0$ is a limit cardinal.

So we can find $\bar{\kappa}$ such that:

$(*)_3$ (a) $\bar{\kappa} = \langle \kappa_i : i < \text{cf}(\kappa) \rangle$ is increasing with limit κ,
(b) $\langle 2^\kappa : i < \text{cf}(\kappa) \rangle$ is increasing with limit λ,
(c) $2^\kappa \neq \lambda$ because $\text{cf}(2^\kappa) > \kappa = \text{cf}(\lambda)$,
(d) $2^\kappa > \lambda$.

Now we apply [She96a, Lemma 3.3, pg 76-77] with κ here standing for λ there; check the three cases there. $\square$

§ 3. No universal for some T

Question 3.1. Maybe $\mathfrak{d}_\mu > \lambda = \mu^+$ or $\mathfrak{b}_\mu > \lambda = \mu^+$ suffices?

In §6B we deal with the class of linear orders. Here we use a less natural class, the models of $T_{\rm elo}$ from [She20, 2.1 = Lh5], which is a countable complete first order theory with elimination of quantifiers.

For it by [She20, 2.6 = Lh11] or older we have:

Theorem 3.2. *If μ is a singular cardinal satisfying $\mu < 2^{<\mu}$, <u>then</u> $T = T_{\rm elo}$ (and equivalently $T^0_{\rm elo}$) has no universal model of cardinality λ; moreover,* $\text{univ}_{\rm T_{elo}}(\mu, \rm T) \geq 2^{<\mu}$.

But here we like to deduce a result for $\lambda = \mu^+$, and need $\text{univ}(\lambda, \rm T_{elo}) > \mu^+$ so the problematic case is $2^{<\mu} = \mu^+$.

However as μ is singular then implies $2^{<\mu} = \mu^+$. So,

Theorem 3.3. *If $\mu > \text{cf}(\mu)$ and $2^{<\mu} > \mu^+$ <u>then</u>* $\text{univ}(\mu^+, \rm T_{elo}) \geq 2^{<\mu}$.

Proof. By cardinal arithmetic and 3.2. $\square_{3.3}$

Claim 3.4. *Assume $\lambda = \mu^+, \mu$ regular $> \aleph_0, \mathfrak{b}_\mu > \lambda$ and $(2^{<\mu} = \lambda)$ hence there is a set $S \in \check{I}_\mu[\lambda]$. <u>Then</u> $\text{univ}_{T_{\rm ceq}}(\lambda) \geq \mathfrak{b}_\mu$, where $T_{\rm ceq}$ is as in* [Shear].

Proof. Let

- $(*)_1$ $\langle \eta_\varepsilon : \varepsilon < \lambda \rangle$ lists $\bigcup_{i<\mu} {}^i\lambda$ which has cardinality λ.

Now for every club E of λ we can choose $\alpha_\varepsilon \in E$ by induction on $\varepsilon < \mu$ such that $\alpha_\varepsilon > \bigcup_{\zeta<\varepsilon} \alpha_\zeta$ and $(\exists \xi < \alpha_\varepsilon)(\langle \alpha_\zeta : \zeta < \varepsilon \rangle = \eta_\zeta)$.

This implies that there is such S. Hence

- $(*)_2$ we can find $\bar{C}$ such that:
  - (a) $\bar{C} = \langle C_\delta : \delta \in S \rangle$,
  - (b) C_δ is a club of δ of order type μ disjoint to S,
  - (c) if $\alpha < \lambda$, then $\Omega_\alpha = \{C_\delta \cap \alpha : \alpha \in S \text{ satisfies } \alpha \in \text{nacc}(C_\delta)\}$ has cardinality $\leq \mu$.

Let $\bar{\alpha}_\delta = \langle \alpha_{\delta,i} : i < \mu \rangle$ list C_δ in increasing order.

- $(*)_3$ Without loss of generality for every club E of λ for stationarily many $\delta \in S$ for stationarily many $i < \mu$, we have $\text{suc}_E(\alpha_{\delta,i}) \geq \alpha_{\delta,i}$ (even equal).

For every club E of we can choose M_E from $\text{EC}_{T_{\rm ceq}}(\lambda!)$ such that for every $\delta \in S$ and $\varepsilon \in E$ we have $F^{M_E}(\alpha_{\bar{\gamma},i}, \delta) \in \alpha_{\delta,i+2} \setminus \alpha_{\delta,i+1}$, as in [Shear].

As in [Shear] we are done. $\square_{3.4}$

Definition 3.5. Assume $\lambda = \mu^+ > \aleph_1, \mu$ regular, $\mathfrak{b}_\mu = \lambda$ and let

- (a) for stationarily $A \subseteq S$, let $J^{\rm club}_A = \{B \subseteq \mu : B \cap A \text{ is not stationarily}\}$,
- (b) for stationarily $A \subseteq S$, let $\mathfrak{b}[J^{\rm club}_A] := \min\{|\mathscr{F}| : \mathscr{F} \subseteq {}^\mu\mu$ has no $<_{J^{\rm club}_A}$-upper bound in ${}^\mu\mu$,
- (c) $\text{id}_\mu = \{A \subseteq \mu : A \text{ is not stationarily or } A \text{ is stationarily and } \mathfrak{b}[J^{\rm club}_A] \leq \lambda\}$.

Observation 3.6. *For λ, μ as in 3.6, id_μ is a normal ideal on μ.*

Question 3.7. Assume $2^\mu > \lambda = \mu^+ > \aleph_1, \mu$ regular and the ideal id_μ is not μ-saturated. <u>Then</u> $\text{univ}_{T_{\rm ceq}}(\lambda) \geq 2^\mu$.

Claim 3.8. *Assume $2^\mu > \lambda = \mu^+ > \aleph_1$ and id_μ is μ-saturated. Then we can find $\bar{C}$ such that:*

(a) $\bar{C} = \langle C_\delta : \delta \in S \rangle$ is as in the proof of 3.4,
(b) if E is a club of λ, then for stationarily many $\delta \in S$, we have $\{i < \mu : \alpha_{\delta,i} \in E$ and $\alpha_{\delta,i} \in E\} = \mu \mod \mathrm{id}_\mu$.

Proof. Choose $\bar{C}$ as in $(*)_2, (*)_3$ in the proof of 3.4. Try μ times. $\square_{3.8}$

Claim 3.9. *Let λ, μ be as in 4.4 and $\bar{C}$ as there. Then* $\mathrm{univ}_{T_{\mathrm{ceq}}}(\lambda) \geq (\mu^\mu/\mathrm{id}_\mu) = 2^\mu$.

Proof. The tree μ^λ has $2^\mu > \lambda$ μ-branches and ${}^{\mu>}\lambda$ has cardinality λ. Hence for some subtree $\mathscr{T} \subseteq {}^{\mu>}\lambda$ of cardinality $\mu, \lim_\mu(\mathscr{T})$ has cardinality $> \lambda$ (and more). This implies $|\mu^\mu/\mathrm{id}_\mu| = 2^\mu$. $\square_{3.9}$

§ 4. Non-existence of universal

Question 4.1. For $\bar{C} = \langle C_\delta : \delta \in S \subseteq S^\lambda_\mu \rangle, \chi \subseteq \mathrm{cf}(\lambda) > \lambda$ can we demand:

⊙ E_ξ a club of λ for $\xi < \lambda$ <u>then</u> for some λ, and $X \in [\chi]^\chi$ and $w \in (D^{\mathrm{club}}_\mu)^+$:
- $\xi \in X \Rightarrow \{\alpha \in C_\delta : \bigcap_{\xi \in X} \mathrm{suc}_{C_\delta(\alpha) \in E_\varepsilon}\} = w \mod D^{\mathrm{club}}$.
- Less ...

Theorem 4.2. *If μ is regular, $\lambda = \mu^+$ and $2^\mu > \mathfrak{d}_\mu, S \in \check{I}_\mu[\lambda]$ and $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta)\}$ is stationary <u>then</u>,* $\mathrm{univ}_{\mathrm{T_{elo}}}(\lambda) > \mathfrak{d}_\mu$.

Remark 4.3.

1) The existence of S is not a heavy assumption as $2^{<\mu} \leq \lambda$ implies it.

2) In 4.2 we can add: if $\lambda > \theta$ are regular, $\mathfrak{d}_\theta < 2^\theta$, $\mathrm{MGC_{ub}}(\lambda)$ and $\theta = \lambda$, <u>then</u> $\mathrm{univ}(\lambda, T_{\mathrm{ceq}}) \geq \mathfrak{d}_\theta$ (where on MGC is from [Shear]).

Proof. So assume:

$(*)_1$ $\bar{M} = \langle M_\xi : \xi < \xi_* \rangle$ where $M_\xi \in \mathrm{EC}_{T_{\mathrm{dno}}}(\lambda!)$ and $\xi_* < \mathfrak{d}_\mu$.

We shall find $N \in \mathrm{EC}_{T_{\mathrm{lno}}}(\lambda)$ not embeddable into M_ξ for every $\xi < \xi_*$. thus proving the theorem.

$(*)_2$ Let $\bar{C}$ be such that:
- (a) $\bar{C} = \langle C_\delta : \delta \in S \rangle$,
- (b) C_δ is a club of δ of order type μ,
- (c) $\alpha \in \mathrm{nacc}(C_{\delta_1}) \cap \mathrm{nacc}(C_{\delta_2})$ implies $C_{\delta_2} \cap \alpha = C_{\delta_2} \cap \alpha$ (check, or just $\mu \geq |\{C_\delta \cap \alpha : \alpha \in \mathrm{nacc}(C_\delta)\}|$),
- (d) let $\langle \alpha_{\delta,i} = \alpha(\delta, i) : i < \mu \rangle$ list C_δ in increasing order.

[Why? By the assumption of S, that is, by 2.2].

$(*)_3$
- (a) Let $\chi = \mathfrak{d}_\mu$ and $\bar{f} = \langle f_\alpha : \alpha < \chi \rangle$ is cofinal in $({}^\mu\mu, <_{J^{\mathrm{bd}}_\mu})$ without loss of generality each f_α is increasing,
- (b) let $E_\alpha = \{\delta < \mu : \delta$ is a limit ordinal such that f_α maps δ into $\delta\}$, clearly a club of δ,
- (c) let $f'_\alpha \in {}^\mu\mu$ be $f'_\alpha(i) =$ the i-th member of E_α.

$(*)_4$ for $M \in \mathrm{EC}_{T_{\mathrm{lim}}}(\lambda!), \delta \in S, \alpha < \chi$ and $\zeta < \lambda$ let $u_{\delta,\alpha,\zeta}[M] = \{i < \mu :$ if ζ and some $\varepsilon \in [\alpha(\delta, f'_\alpha(i), \alpha(\delta, f'_\alpha(i+1)))]$ realizes the same quantifier type in M over $\{\gamma : \gamma < f'_\alpha(i)\}$, <u>then</u> $\varepsilon <_M \zeta\}$.

$(*)_5$ $\mathscr{P} = \{u_{\delta,\alpha,\zeta}[M] : \xi < \xi_*, \delta \in S, \alpha < \chi$ and $\zeta < \lambda\}$ is a subset of $\mathscr{P}(\mu)$ of cardinality $\leq |\xi_*| + |S| + \chi + \lambda < 2^\mu$ hence,

$(*)_6$ there is $u_* \in [\mu]^\mu$ such that $u \notin \mathscr{P}$ (or more),

$(*)_7$ let N be the linear order of cardinality λ constructed from $(\bar{C}, u_*)$, see [S$^+$, k8] (used in [S$^+$, h20]).

It suffices to prove N is not embeddable in M_ξ for $xi < \xi_*$, so toward contradiction assume f embeds N into M_ξ.

Clearly $E_* = \{\delta < \lambda : \delta$ is a limit ordinal $< \lambda$ such that f maps δ to $\delta\}$, clearly E_* is a club of λ hence there is $\delta \in S$ such that $\delta = \sup(E \cap \delta)$, so clearly $E = \{i < \mu : f$ maps $\alpha_{\delta_*,i}$ into itself$\}$ is a club of μ. Hence there is $\gamma < \mathfrak{d}_\mu = \chi$ such that $E_\gamma \subseteq E$ and let $\beta = f(\delta) \in M_\alpha$.

Now we shall get a contradiction to $u_* \neq u_{\delta,\alpha,\gamma}[M_\xi]$ by [Shear, 2.6 = Lh11]. $\square_{4.2}$

Remark 4.4. What about 4.3(2)?

Similarly because by [She79b, Ch. III], without loss of generality $\bar{C}$ satisfies:

(∗) for every club E of λ for stationarily many $\delta \in S, \delta = \sup\{\alpha \in S : \mathrm{nacc}(C_\delta)\}$ is stationary subset of δ.

Of course, we choose u_* such that:

(∗) $u_* \neq u_{\delta,\alpha,\zeta}[M_\xi]$ modulo D^{club}_M for all relevant $(\xi, \delta, \alpha, \zeta)$.

Claim 4.5.

1) If $\mu = \mathrm{cf}(\mu) > \aleph_0$ and $\mathfrak{d}_\mu = 2^\mu$ <u>then</u> there is D such that:

(a) D is a uniform ultrafilter on μ,
(b) μ^μ/D has cofinality $\mathrm{cf}(2^\mu)$,
(c) if $\langle \alpha_i : i < \mu\rangle \in {}^{\mu>}\mu$ in increasing continuous <u>then</u> for some increasing continuous sequence $\langle i(\varepsilon) : \varepsilon < \mu\rangle \in {}^\mu\mu$ the set $\bigcup\{[\alpha_{i(2\zeta+1)}, \alpha_{i(2\zeta+2)}) : \zeta < \mu\}$ belongs to D.

2) We can add in part (1):

(d) $(\mu, <)^M/D$ has cofinality $\mathrm{cf}(2^\mu)$.

Remark 4.6.

1) Can we add. If $\mu = \mathrm{cf}(\mu) > \aleph_0$ and $\mathfrak{d}^{\mathrm{club}}_\mu = 2^\mu$ <u>then</u> there is a filter D on μ on μ extending the club filter such that $\mathrm{cf}[(\mu, <)^M/D] = \mathrm{cf}(2^\mu)$.

2) But not clear if this helps.

Proof. Let $\langle \bar\alpha_\xi : \xi < 2^\mu\rangle$ list the increasing continuous sequences $\bar\alpha \in {}^\mu\mu$; so $\bar\alpha_\xi = \langle \alpha_{\xi,i} : i < \mu\rangle$. Let $\langle g^*_\xi : \xi < 2^\mu\rangle$ list ${}^\mu\mu$. Now we choose $h_\xi \in {}^\mu\mu, \langle A_{\xi,u} : u \in [\xi]^{<\aleph_0}\rangle$ by induction on $\xi < 2^\mu$ such that:

⊕ (a) $A_{\xi,u} \subseteq \mu$,
(b) if $n < \omega$ and $u \in [\xi+1]^{<\aleph_0}$ <u>then</u> $B_u = \bigcap\{A_{\zeta,v} : \zeta \in u \text{ and } v \subseteq \zeta\cap u\} \subseteq u$ has cardinality μ,
(c) for some increasing continuous sequence $\eta = \eta_\xi \in {}^\mu\mu$ we have $A_{\xi,u} \subseteq \bigcup\{[\alpha_{\xi,\eta(2i+1)}, \alpha_{\xi,\eta(2i+2)}) : i < \mu\}$,
(d) if $\zeta \in u \in [\xi]^{<\aleph_0}$ then $h^*_\zeta \restriction A_{\xi_u}, g^*_\zeta \restriction A_{\xi,u} < h^\bullet_\xi \restriction A_{\xi,u}$.

If we succeed to carry the induction then $\{B_u : u \in [2^\mu]^{<\aleph_0}\} \subseteq [\mu]^{\aleph_0}$ is closed under intersection of two hence there is a uniform ultrafilter D on μ which extends $\{B : u \subseteq 2^\mu \text{ is finite}\}$ hence $\{A_\xi : \xi < 2^\mu\}$ so we shall be done; note $\{g^*_\xi : \xi < 2^\mu\}$ is increasing cofinal in $(\mu, <)^\mu/D$.

Still why can we carry the induction? Arriving to ξ, for any $u \subseteq \zeta + 1$ we define $f_{\xi,u} \in {}^\mu\mu$ by:

$(*)^1_{\zeta,u}$ for $i < \mu$ let $f_{\xi,u}(i) = \min\{j : j \in [i+1, \mu) \text{ such that } (\alpha_{\xi,i+1}, \alpha_{\xi,j}) \cap B_u \neq \emptyset\}$.

$(*)^2_{\zeta,u}$

(a) let $E_{\xi,u} = \{\delta < \mu : \delta$ is a limit ordinal $< \mu$ such that $i < \delta \Rightarrow f_{\xi,u}(i) < \delta$ and $i < \delta \Rightarrow \alpha_{\xi,i} < \delta\}$,
(b) $E_{\xi,u}$ is a club of λ,
(c) let $\langle i(\zeta, u, \xi) : \zeta < \mu\rangle$ list $E_{\xi,u}$ in increasing order,
(d) $\alpha_{\xi,i(\zeta,u,\xi)} = i(\zeta, \xi)$.

As $\{u \subseteq \xi : u \text{ finite has cardinality } < 2^\mu\}$.

$(*)^3_{\zeta,u}$ there is $\eta_\xi \in {}^\mu\mu$ increasing such that $u \in [\xi]^{<\aleph_0} \Rightarrow \{\zeta < \lambda : f_{\xi,u}(\zeta) < \eta_\xi(i)\} \in [\mu]^M$.

$(*)^4_{\zeta,u}$ let $E_\bullet = \{\delta < \mu : \text{ if} \zeta < \delta \text{ then } \eta_\xi(\zeta) < \delta\}$, a club of λ.

Lastly,

$(*)^5_{\zeta,u}$ let $i_\varepsilon(\varepsilon)$ be:

- if $\varepsilon = 2\zeta$ then $i(\varepsilon) = i_\xi(\varepsilon)$ is the ε-th member of E,
- if $\varepsilon = 2\zeta + 1$ then $i_\xi(\varepsilon) = i(2\zeta) + 1$.

So we let:

$(*)^6_{\zeta,u}$ $A_\varepsilon = \bigcup\{[\alpha_{\xi,i_\xi(2\varepsilon+1)}, \alpha_{\xi,i_\xi(2\varepsilon+2)}) : \varepsilon < \mu\} = \bigcup\{[\alpha_{\xi,i(\varepsilon)}, \alpha_{\varepsilon,i(\varepsilon)+1}) : i \in E_\bullet\}$.

It is as required so we are done proving. $\square_{4.5}$

Claim 4.7. *Assume $\lambda = \mu^+, \mu$ an uncountable regular cardinal, $\mathfrak{d}_\mu = \lambda$ and let $S \in \check{I}_\mu[\lambda]$ be stationarily.* *<u>Then</u>* $\text{univ}_{T_{\text{elo}}}(\lambda) \geq 2^\mu$ *and* $\text{univ}_{T_{\text{lin}}}(\lambda) \geq 2^\mu$.

Question 4.8. Does this already exists?

Proof. For transparency assume $\mu > \aleph_0$ and we deal with T_{elo}.

$(*)_1$ Let $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \mu\}$ be stationarily. Let ....

$(*)_2$ Let $\bar{c}$ be such that:
  (a) $\bar{c} = \langle C_\delta : \delta \in S\rangle$,
  (b) $C_\alpha = \{\alpha_{\delta,i} : i < \mu\}, \alpha_{\delta,i}$ is increasing and with limit δ.

$(*)_3$ Let $\langle E_\varepsilon : \varepsilon < \lambda\rangle$ be a sequence of clubs of μ which is cofinal in (clubs of $\mu, \subseteq$).

$(*)_4$ Let $\bar{E}$ be such that:
  (a) $\bar{E} = \langle E_\varepsilon : \varepsilon < \lambda\rangle$,
  (b) E_ε a club of μ,
  (c) if E is a club of μ <u>then</u> for some $\varepsilon < \lambda$ we have $E_\varepsilon \subseteq E$.

$(*)_5$ For $N \in \text{EC}_{T_{\text{ceq}}}(\lambda!), \varepsilon < \lambda, \delta \in S$ and $a \in P^{M,}$ let $u_{N,\varepsilon,\delta,a} := \{i \in E_\varepsilon$: for some $\beta < \alpha_{\delta,\text{suc}(i,E_\varepsilon)}$ we have $\alpha_{\delta,i} < \beta$ and $(\forall\alpha)(\beta \leq_\mu \alpha \in u_\delta \Rightarrow R_1(a,\alpha))$

$(*)_6$ For every $u \subseteq \mu$ there is $M_n \in \text{EC}_{T_{\text{elo}}}(\lambda)$ such that if $\delta \in S, \varepsilon < \lambda$ <u>then</u> for some $a \in M_u$ we have:
  $(*)$ if $i \in E_\varepsilon, j = \text{suc}_{E_\varepsilon}(i)$ and $\alpha_{\delta,i} \leq \beta < \alpha_{\delta,j}$ <u>then</u> $R^M_1(\beta, a) \Leftrightarrow \text{otp}(E_\varepsilon \cap i) \in u \Leftrightarrow \neg R^M_2(\beta, a)$.

[Why? as in [She20].]

$(*)_7$ If $u \subseteq \mu, M_u$ is as in $(*)_7$ and $N \in \text{EC}_{T_{\text{elo}}}(\lambda!)$ and $M_u$is embeddable into N, <u>then</u> $n \in \{u_{N,\varepsilon,\delta,a} : \varepsilon < \lambda, \delta \in S$ and $a \in N\}$.

[Why? As in [She20].]

$(*)_8$ $\text{univ}_{T_{\text{elo}}}(\lambda) \geq 2^\mu$.

[Why? Easy by $(*)_5 + (*)_6$.] $\square_{4.7}$

§ 5. Linear orders in singulars revisited

The original aim was to prove the consistency of the existence of a universal linear order in $\mu < 2^{<\mu}$ with $\mu > \aleph_1$. Presently not achieved.

§ 5(A). Background.

Discussion 5.1. Among those who notice Kojman-Shelah [KS92], most probably conclude that there is a universal one of cardinality λ essentially iff GCH holds near λ, i.e. $2^{<\lambda} = \lambda$ well except $\lambda = \aleph_1$. Now concentrating on λ regular, it was clear that some cases are left open, mainly λ successor of $\mu = \mu^{<\mu}$, (in particular, μ inaccessible). Still it seems that theory of linear order is essentially maximal.

However, we can look at it differently. It seems reasonable that [She78] can be generalized to such μ. We have made some tries but not too strongly. However, this necessitates a strong failure of guessing clubs on S^λ_μ. Concerning μ singular in [KS92] it was clear that we have to wait for advances in cardinal arithmetic; that is, pcf calculus, as only in extreme cases (of pcf) the non-existence remains open.

In [She96b, §3], the property T is the theory of linear order or T having the strict order property is replaced by "$T \in$ SOP", a weaker model theoretic demand, but the set theoretic side is similar. In [She16] again the stress was of the model theory side finding T without SOP_4. This gives a side benefit, as said there: no need for the club guessing sequence to be lean.

This work started by working on the open cases $\lambda = \mu^+$, first for μ regular considering the value of $\mathfrak{d}_\lambda$ and then the singular μ case.

Considering the singular λ case motivates us looking again at trying to improve [She00], see also [She02], [She06], but eventually this is not necessary. But this was not done for the class of linear orders of trees. Trying to look for an exact reference, I see (as I recall) that while the results in [KS92] for regular $\lambda > \aleph_1$, use linear order with many cuts in the proof but not as an assumption in the theorem; however I discover, for singular λ, there is such a demand in the theorem. So it says less than what I attributed to it (so my memory improves the result, but alas, not the proof). For singular λ there are not only pcf demands but:

$\boxplus_\lambda$ there is a linear order I of cardinality $< \lambda$ with $> \lambda$ Dedekind cuts,

And moreover,

$\boxplus^+_\lambda$ for arbitrarily large regular $\kappa < \lambda$, there is a tree with κ nodes and $> \lambda$ κ branches.

Why this does not occur for regular? If $\kappa = \min\{\kappa : 2^\kappa > \lambda\}$ and $\lambda < 2^{<\lambda}$ then $\kappa < \lambda$ hence 2^λ is the sum of $< \lambda$ cardinals $< \lambda$ hence $2^{<\kappa} < \lambda$ and $({}^{\kappa>}2, \triangleleft)$ exemplify $\boxplus_\lambda$. But for λ singular, this is not the case, so [KS92] was correct to distinguish but does not explicate the natural problem below; and [She16] (as not said there) makes a real advance on the singular case: for T with (any variant of) the olive property we have only pcf obstacles. So it is natural to ask:

Question 5.2. Is the use of $\boxplus^+_\lambda$ for singular λ, in proving $\lambda \neq \mathrm{univ}_{T_{\mathrm{dlo}}}(\lambda)$, necessary? Our answer is mixed: on the one hand this set theoretic case arises and, on the other hand, except when some very problematic case of pcf, in this case there is no universal linear order. Moreover, a canonical linear order similarly to the so-called (λ, κ)-limit models, for linear orders such canonical (non-universal linear order exists; see [She14], [She12], [She10], and [Sheb]) ((2022-07-25) Not sure.

Claim 5.3. *We have* $\mathrm{univ}(\mu, \mathrm{T}) \geq 2^\kappa$ *when:*

$\boxplus$ *for some* χ, λ, κ *we have:*

(a) $2^\kappa \geq \chi \geq \mu \geq \lambda = \mathrm{cf}(\lambda) > \kappa$,
(b) $\mathbf{U}_{J^{\rm bd}_\lambda}(\mu) < \chi$, *if* $\lambda = \mu$,
(c) *we have* α) *or* β), *where:*
  (α) $\kappa^+ < \lambda$.
  (β) $\lambda = \kappa^+$ *and there is a lean* $\bar{C} = \langle C_\delta : \delta \in S_{\mathrm{cf}(\kappa)^\lambda}\rangle$ *guessing clubs of* $\mathrm{otp}(C_\delta) = \kappa, \sup(C_\delta) \subseteq \delta, C_\delta$ *closed in* $\sup(C_\delta)$.
(d) T *has* SOP_4 *(so if* T *is no complete this is witnessed by a quantifier free formula, or pair of contradictory existential formulas).*

Remark 5.4.
1) We use the "no ded-inaccessible" (see [CS16]).
2) Unlike [KS92], we replace $(\exists \chi)(\chi < \lambda < \mathrm{ded}(\chi))$ by $(\exists \chi)(\chi < \lambda \leq^+ \mathrm{ded}(\chi))$. Check:

- $\bullet_1$ do we need $\langle (C_\delta, \{g_{\delta,\gamma} : \gamma < \chi\}) : \delta \in S$ lean?
- $\bullet_2$ does [KS92] use it See §3.
- $\bullet_3$ put it in §3.
- $\bullet_4$ check for the olive properties [easier].
- $\bullet_5$ remember $\mathrm{SOP}_{\leq n}$ equivalent.
- $\bullet_6$ phrase properties sufficient for $\chi_1 \notin \mathrm{Sp}_{\mathrm{univ}}$, in ZFC.
- $\bullet_7$ try §2 for T with linear order.

Proof. Stage A:

$(*)_1$ Let $\mathscr{A} \subseteq [\mu]^\lambda$ has cardinality $\mathbf{U}_{J^{\rm bd}_\lambda}$ and witness it,
$(*)_2$ Let $\bar{C} = \langle C_\delta : \delta \in S\rangle$ be a lean sequence of guessing clubs of λ with $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \mathrm{cf}(\kappa), \mathrm{otp}(C_\delta) = \kappa$ closed in $\sup(C_\delta) \subseteq \delta\}$.

[Why exists? See clause (c) of the assumption if α) uses [She93b], if β) obvious.] $\square_{5.4}$

Let $\mathscr{F}_\xi = \{(C_\delta \cap (\xi + 1)), g_{\alpha,\delta} \restriction (C_\delta \cap C(\xi + 2)) : \delta \in S$ and $\alpha < \chi\}$. (Debt: is not enough that $|\mathscr{F}_{\alpha,\xi}| < \lambda$ for $\alpha < \chi, \xi < \lambda$? so, $\bar{C}$ lean is enough?) where $g_{\alpha,\delta}(\zeta) = (\mathrm{otp}(\zeta \cap C_\delta), \mathrm{otp}(\mathrm{suc}_{C_\delta}(\zeta) \cap C_\delta), \delta)$ for $\zeta \in C_\delta, \alpha < \chi$.
For each $\alpha < \chi$.

§ 5(B). Peculiar models.

It seem we get a canonical model but not a universal one.

Claim 5.5. *If (A) + (C) then (C),* when:

(A) *(a)* κ *is a Mahlo cardinal,*
  (b) $\lambda > \kappa$ *is strong limit singular of cofinality* κ,
  (c) $\kappa_i = \kappa_i^{<\kappa_i}$ *is increasing for* $i < \kappa$ *with limit* κ,
  (d) $\lambda_i = \lambda_i^{\kappa_i} \in (\kappa, \lambda)$ *is increasing with limit* λ,
  (e) $\lambda_i > \Pi_{j<i} 2^{\lambda_j}$,
  (f) *for limit* $\delta < \kappa$ *we have* $2^{\sup\{\kappa_i : i < \delta\}} = (\sum\{\kappa_i : i < \delta\})^+$ *and* $\Pi_{i<\delta}\lambda_i = (\Sigma\{\lambda_i : i < \lambda\})^+$,
(B) *(a)* *let* $\mathbb{Q}_i = \mathrm{Cohen}_{\kappa_i}(\lambda_i)$, *the forcing of adding* λ_i *many* κ_i*-Cohens so* $\mathbb{Q}_i$ *satisfies the* κ_i^+*-cc and is* κ_i*-complete of cardinality* λ_i,
  (b) $\mathbb{P} = \Pi_{i<\kappa}\mathbb{Q}$ *is the product with the Easton support.*
(C) *in* $\mathbf{V}^{\mathbb{P}}$ *we have:*
  (a) *cardinals and cofinalities are the same as in* $\mathbf{V}$,
  (b) $\kappa_i = \mathrm{cf}(\kappa_i)$ *increasing with* $i < \kappa = \mathrm{cf}(\kappa)$ *and* $\lambda_i > \kappa_i$,
  (c) $2^{\kappa_i} = \lambda_i, 2^{<\kappa} = \sum_{i<\kappa} \lambda_i = \lambda$ *and* $\theta < \kappa_0 \Rightarrow 2^\theta = (2^\theta)^{\mathbf{V}} < \kappa$,

(d) λ *is singular,* $\lambda = \Sigma_{i<\kappa} 2^{\kappa_i}$ *hence* $\kappa = \mathrm{cf}(\lambda)$ *and* $\lambda < 2^\kappa$,
(e) if I *is a linear order of cardinality* $< \lambda$, *then* I *has* $\le \lambda$ *cuts,*
(f) if $\theta = \mathrm{cf}(\theta) < \lambda$ *and* $\mu \in [\theta, \lambda)$, *then* $\mathrm{trp}_\theta(\mu) \le (\mu^\theta)^{\mathbf{V}}$.

Proof. We are assuming κ is a Mahlo cardinal so let:

$(*)_1$ (A) $p \in \mathbb{Q}_i$ iff:
- a) p is a function,
- b) $\mathrm{dom}(p) \in [\lambda_i]^{<\kappa_i}$,
- c) if $\alpha \in \mathrm{dom}(p)$ then $p(\alpha) = {}^{\kappa_i}2$.

(B) order natural,
(C) the generic of $\mathbb{Q}_i$ is $\underset{\sim}{\bar\eta} = \langle \underset{\sim}{\eta}_{i,\alpha} : \alpha < \lambda_i \rangle, \underset{\sim}{\eta}_{i,\alpha} \in {}^{(\kappa_i)}2$.

$(*)_2$ (a) for $u \subseteq \lambda_i$ let $\mathbb{Q}_{i,u} = \{p \in \mathbb{Q}_i : \mathrm{dom}(p) \subseteq u\}$,
(b) for $v \subseteq \kappa$ and $\bar u = \langle u_i : i \in v\rangle$ with $u_i \subseteq \lambda_i$ let $\mathbb{P}_i = \Pi_{i \in v} \mathbb{Q}_{i,u_i}$, product with Easton support,
(c) for u, v and $\bar u$ as above $\mathbb{P}_{\bar u} \lessdot \mathbb{P}$.

The cardinal arithmetic is well know so should be clear. Clause (e) of (C) of the Claim follows from clause (f). To prove it, i.e. concerning $\mathrm{trp}_\theta(\mu)$ where $\theta = \mathrm{cf}(\theta) \le \mu < \lambda$, toward contradiction assume $\mu < \lambda, \theta = \mathrm{cf}(\theta) \le \mu$ and $p \Vdash_{\mathbb{P}}$ "$\underset{\sim}{cF}$ is a subtree of ${}^{\theta>}\mu$ with $\le \mu$ nodes and $> \lambda$ θ-branches", without loss of generality $\mu > \kappa$ and as $\Vdash_{\mathbb{P}} \lambda = \Sigma\{2^{\kappa_i} : i < \kappa\}, \lambda_i = 2^{\kappa_i}$ increasing with i, necessarily $\theta \ge \kappa$.

As $\mathbb{P}$ satisfies the κ^+-cc, we can find $\bar u = \langle u_i : i < \kappa\rangle, u_i \in [\lambda_i]^{\le \mu}$ such that $p \in \mathbb{P}_{\bar u}$ and $\underset{\sim}{\mathscr{F}}$ is a $\mathbb{P}_{\bar u}$-name. Now $\mathbb{P}_{\bar u}$ has cardinality $\le \mu \times \kappa$ which is $< \lambda$ because λ is a strong limit cardinal. As $\theta > \kappa$ or $\theta = \kappa$ recalling κ is Mahlo clearly $\mathbb{P}$ satisfies the θ-cc so the forcing notion $\mathbb{P}/\mathbb{P}_{\bar u}$ satisfies the θ-cc hence it does not add θ-branches to $\underset{\sim}{\mathscr{F}}$, contradiction. $\square_{5.5}$

Definition 5.6. Assume:

$\boxplus_{\lambda,\kappa}$ (a) $\lambda > \kappa = \mathrm{cf}(\lambda)$,
(b) $\theta = \mathrm{cf}(\theta) \le \mu < \lambda \Rightarrow \mathrm{trp}_\theta(\mu) < \lambda$; [hence κ is weakly inaccessible]
(d) $\lambda_i \in (\kappa, \lambda)$ is increasing with i with limit λ,
(d) if $\theta = \mathrm{cf}(\theta) < \lambda$ then for every $i < \kappa$ large enough $\mathrm{trp}_\theta(\lambda_\mathrm{i}) = \lambda_\mathrm{i}$; this is more than clause (b.)

We say a linear order M is λ-peculiar when:

$(*)^1_{\bar M}$ some pair $(\bar M, \bar{\mathbf{I}})$ strongly witness "M is λ-peculiar" which means that:
$(*)^1_{\bar M, \bar{\mathbf{I}}}$ (a) $\bar M = \langle M_\alpha : \alpha \le \lambda\rangle$ is an increasing continuous sequence of linear orders,
(b) M_α has cardinality $\le |\alpha| + \aleph_0$,
(c) if $\alpha < \lambda$ and (I_1, I_2) is a cut of M_α of cofinality (θ_1, θ_2) then:
- if $\theta_1 \ne \theta_2$ or $\theta_1 = \theta_2 \wedge \mathrm{trp}_{\theta_1}(|\alpha| + \aleph_0) = |\aleph| + \aleph_0$, then there is one and only one $s \in I_{\alpha+1} \setminus I_\alpha$ realizing the cut (I_1, I_2),
- if not, then the cut is not realized.

Claim 5.7.
1) In 5.5, the cardinals λ, κ *are as assumed in 5.6.*
2) If λ, κ *are as assumed in 5.6, then:*

(a) there is a λ*-peculiar linear order,*
(b) up to isomorphism there is a unique λ*-peculiar linear order.*

Proof.

1) Should be clear.

2) Note that the clause (a) it is easy. Now for clause (b) we have to prove:

$(*)_3$ if M_1, M_2 are λ-peculiar, then they are isomorphic.

Why? There is no problem to carry the induction so in particular $f_{1,\lambda}, f_{2,\lambda}$ are well defined, so it suffices to prove $\mathrm{dom}(f_{\ell,\lambda}) = M_{\ell,\lambda}$ for $\ell = 1, 2$. Assume toward contradiction that this fails, and by symmetry without loss of generality there is $s \in M_{1,\lambda}(N_{1,\lambda})$ and s is minimal in this set.

Let $i_1 = i(1) < \kappa$ be such that $s < \lambda_i$, by the choice of the $N_{1,\alpha}$'s, necessarily $N_{1,\lambda_{i(1)}+\varepsilon}$ for $\varepsilon < \lambda_{i(1)+1}$, necessarily there is $\gamma \in N_{1,\lambda}$ which is $> s$, and choose such minimal γ; let $i_2 = i(2) \in (i_1, \kappa)$ be such that $\gamma < \lambda_{i(2)}$ and $\mathrm{cf}(\gamma) < \lambda_{i(2)}$.

Now by the inductive choice for $\alpha \in [\lambda_{i(2)}, \lambda^+_{i(2)})$ (see clause $(\mathrm{d})_2$) there are $\beta < \lambda^+_{i(2)}$ even $< \lambda_{i(2)} + \lambda_{i(2)}$ and $t \in N_{1,\beta+1} \cap \gamma \setminus s$.

Now t contradicts the choice of γ. $\square_{5.7}$

Discussion 5.8. Do we have:

$\odot$ any linear order of cardinality $\leq \lambda$ can be extend to a λ-peculiar linear order M.

A natural try is, that is, we have to prove:

$(*)_1$ if N is a linear order of cardinality $\leq \lambda$, then some λ-peculiar of M' extend it.

Why this holds? Without loss of generality the set of elements of N is λ. We choose $M_\alpha, f_\alpha, \mathbf{I}_\alpha, g_\alpha$ and bu induction on $\alpha \leq \lambda$ such that:

$(*)_2$ (a) M_α is a linear order with universe an ordinal $< (|\alpha| + \aleph_0)^+$,
- (b) (α) $\langle M_\beta : \beta \leq i \rangle$ satisfies the relevant demands of Definition 5.6, i.e. clauses (a), (b), (c) there,
  - (β) $\mathbf{I}_j$ is the set of cuts of M_j.
- (c) f_j is an order preserving function from a submodel N_α of $N \restriction \alpha$ into M_α,
- (d) f_α is increasing continuous with α,
- (e) if $\alpha = \beta + 1$ and $a \in M_\alpha \setminus M_\beta$ and there is $b \in N \setminus N_\beta$ such that $(\forall c \in N_\beta)[c <_N b \equiv f_\beta(c) < a]$ and $b_a \in N \setminus N_\beta$ is minimal (in λ) satisfying the demand above then $f_\alpha(b_\alpha) = a$.

Clearly we can carry the induction and then $M^+_\kappa = \bigcup_{i<\kappa} M_i$ is λ-peculiar and $f_\kappa = \bigcup_{i<\kappa} f_i$ embeds N_λ into M_κ.

If $N = N_\lambda$ we are done, so as $N_\lambda \subseteq N$ assume $N_\lambda \subsetneq N$ and let $\beta \in N$ be minimal (under the order type of μ).

Not clear.

Remark 5.9. Sort out strong witness / weak witness. For strong witnesses the uniqueness is obvious.

So assume:

$\oplus_1$ $M_1, M_2 \in \mathscr{K}_{\lambda!}$ are λ-peculiar and we shall prove that M_1, M_2 are isomorphic, this suffices,

$\oplus_2$ let $\langle M_{\ell,\alpha} : \alpha \leq \lambda \rangle$ witness M_ℓ is λ-peculiar.

Now, by induction on $\alpha \leq \lambda$ we choose $N_{1,\alpha}, N_{2,\alpha}, f_{1,\alpha}, f_{1,\alpha}$ such that:

$\oplus^3_\alpha$ (a) $N_{\ell,\alpha} \subseteq M_{\ell,\alpha}$ for $\ell = 1, 2$, (c)$_1$ if $\alpha = \beta + 1, s \in M_{1,\beta} \setminus N_{1,\beta}$ realizes the cut$(\mathrm{I}_1, \mathrm{I}_2)$ of $N_{1,\beta}$ and the cut $f_{1,\beta}(I_1, I_2)$ of $N_{2,\beta}$ is realized in $M_{2,\alpha}$ and there is no $t < s$ in $M_{1,\beta} \setminus N_{1,\beta}$ realizing (N_1, N_2), then $s \in \mathrm{dom}(f_\alpha)$,

(c)$_2$ like (c)$_1$ interchanging 1 and 2,

(d)$_1$ if $\alpha = \lambda_i, i < \kappa$ and $\gamma = \min(N_{1,\alpha} \cap \{\lambda_i\})$ has cofinality $\leq \lambda_i$ and $\beta = \sup(\gamma \cap N_{1,\alpha})$ is $< \gamma$, then we choose an increasing sequences $\langle \alpha_{\gamma,\varepsilon} : \varepsilon < \mathrm{cf}(\gamma) \rangle$ of ordinals from the interval (β, γ) with limit γ and demand $\alpha_{\gamma,\varepsilon} \in N_{1,\alpha+\varepsilon+1}$ (really one such α suffices)

(d)$_2$ like (d)$_1$ but for $M_2, N_{2,\alpha}, N_{2,\alpha+\varepsilon+1}$ instead of $M_1, N_{1,\alpha}, N_{1,\alpha+\varepsilon+1}$.

Remark 5.10.

1) In this Definition 5.6 assuming $\langle \lambda_i : i < \kappa \rangle$ is increasing with limit λ such that $(\forall \theta \in \mathrm{Reg} \cap \lambda)(\exists^\infty \mathrm{i} < \kappa)(\mathrm{trp}_\theta(\lambda_\mathrm{i}) = \lambda_\mathrm{i})$, we may fix $j(I_1, I_2)$ for $(I_1, I_2) \in \mathbf{I}$, e.g. by:

(∗) (a) if (I_1, I_2) has non-equal cofinalities, then $j(I_1, I_2) = i$,
(b) if (I_1, I_2) has cofinality (θ, θ), then $j(I_1, I_2) = \min\{j : j \geq i \text{ and } \mathrm{trp}_\theta(\lambda_\mathrm{i}, \theta) \leq \lambda_\mathrm{i}\}$.

2) By this we may make not only the λ-peculiar M unique (up to isomorphism), but (with more restrictions) also the sequence $\bar{M}$.

3) In 5.5 what about non-Mahlo κ? The problem is "$\mathrm{trp}_\kappa(\kappa) < \lambda$". Let $S \subseteq \mathrm{Card} \cap \kappa$ be stationary non-reflecting and revise the forcing in order to have this Debt.

4) Question: what about $\mathbf{K}_{\mathrm{lf}}$? (See [She17a], [Sheb]).

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