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Universality: new criterion for non-existence

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Abstract
We find new "reasons" for a class of models for not having a universal model in a cardinal λ. This work, though has consequences in model theory, is really in combinatorial (set theory). We concentrate on a prototypical class which is a simply defined class of models, of combinatorial character—models of T_{ceq} (essentially another representation of T_{feq} which was already considered but the proof with T_{ceq} is more transparent). Models of T_{ceq} consist essentially of an equivalence relation on one set and a family of choice functions for it. This class is not simple (in the model theoretic sense) but seems to be very low among the non-simple (first order complete countable) ones. We give sufficient conditions for the non-existence of a universal model for it in λ. This work may be continued in Shelah et al. (Tba, In preparation. Preliminary number: Sh:F2150).

Keywords Classification theory · Non-simple theories · Universality · Combinatorial set theory

Mathematics Subject Classification Primary 03E05 · 03C45; Secondary 03C50 · 03C55

0 Introduction

On a recent survey on the universality spectrum see [24], an earlier survey is [6]; there have been several advances meanwhile (and this is one of the advances after [24]). See also [23], noting the example there works also for $\mu^+ < 2^\mu$ whenever μ is a strong limit singular. The problem for general first order theories is a model theoretic one, but specific examples are combinatorial set theoretic ones (and serve as proto-types for suitable families of theories); so

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combinatorialists may ignore model theoretic notions like "T is simple, has the tree property, is TP_2", and consider only the concrete universal theories considered; so ignore 1.4 (1),(2) and their proof. Here we concentrate on the theory T_{ceq}, which we considered as a prototypical ''minimal" non-simple T, so are expecting it (under $\leq_{\mathrm{univ}}$) to be low, so it is (like T_{feq}, see below), NSOP_1, see [2, 4, 7, 8, 19, 26]). True, there were non-existence results near a strong limit singular cardinal (see on the T_{feq} in [19], generalizing it the oak property [5], [22,Sect. 3]), but there were weak consistency results on existence (see [3, 19]). We had considered T_{feq}, a prototypical example of such theories, now T_{ceq} is essentially equivalent to it for our aims, see 1.4(3),(4) but T_{ceq} seem more transparent; we intend to deal with "to what family of T's versions of our proof apply, in particular, NTP_2 and non-simple"elsewhere.

We have hoped/expected that for the $\lambda > \mu = \mu^{<\mu}$ but $\lambda = \mu^+ < 2^\mu$ we shall have consistency results for theories like T_{feq} and the class of triangle free graphs, [9] and hopefull [13].

We first give a case with stronger set theoretic assumptions, but more transparent proof in Sect. 1. In Sect. 2 we give such proof under reasonable set theoretic assumptions, (close to the so called club guessing) but then have to consider finer points in combinatorial set theory on guessing clubs. Elsewhere we hope to have relevant complimentary consistency (see [9]) and families of theories.

A priory we think that T_{tfg}, the theory of triangle free graphs, is "more complicated"then $T_{\mathrm{feq}}, T_{\mathrm{ceq}}$, but now have doubts.

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Question 0.1 (1) *Does Sect. 1 apply to more theories than in Sect. 2?*
(2) *Can we characterize the dividing line? Simple/non-simple in our context.*
(3) *Does it help to have:*

$(*)$ *for some μ, $\mu < \lambda < 2^\mu$ there is no $\mathscr{A} \subseteq [\lambda]^\lambda$ which is $\mu - \mathrm{AD}$ of cardinality $> \lambda$?*

This would justify the use of $\mu - \mathrm{AD}$ family $\mathscr{A} \subseteq [\lambda]^\lambda$ in some consistency results, see [16], [9], see below.

Discussion 0.2 *Note that:*

$\boxplus$ *if $2 < n \leq \omega, \theta \leq \mu \leq \lambda < 2^\theta, \lambda \nrightarrow [\mu]^{<n}_\theta$ and we let T_n be the theory "$\{P_k$ is an irreflexive asymmetric k-place relation": $k < n, k \geq 2\}$ and T_n has a universal model M_* in λ <u>then</u> there is a μ-disjoint $\mathscr{A} \subseteq [\lambda]^\lambda$ of cardinality 2^θ.*

[Why? Without loss of generality the universe of M_* is λ. Let $\mathbf{c} : [\lambda]^{<n} \to \theta$ witness $\lambda \nrightarrow [\mu]^{<n}_\theta$ and for $u \subseteq \theta$ let $M_u = (\lambda, \ldots P_k^{M_u}, \ldots)_{k \in [2,n)}$ where $P_k^{M_u} = \{\eta \in {}^k\lambda : \eta$ is with no repetitions and $\mathbf{c}(\mathrm{Rang}(\eta)) \in u\}$. So there is an embedding f_u of M_u into M_*; now $\langle A_u = \{\mathrm{pr}(\alpha, f_u(\alpha)) : \alpha < \lambda\} : u \subseteq \theta\rangle$ is a family as promised when pr is a pairing function on λ. Why? If $A_{u_1} \cap A_{u_2}$ has cardinality $\geq \mu$ and $u_1 \neq u_2$ then (letting $B = \{\alpha < \lambda : f_{u_1}(\alpha) = f_{u_2}(\alpha)\}$) without loss of generality $u_1 \nsubseteq u_2$ and $\mathbf{c}{\restriction}B$ omits any member of $u_1 \backslash u_2$. The rest is left to the reader.]

0.1 Preliminaries

Notation 0.3 (1) *T is a theory with vocabulary $\tau_T = \tau(T)$ and is a first order, if not said otherwise.*
(2) (a) $\mathrm{EC}_T = \{M : M$ *a model of* $T\}$,
(b) $\mathrm{EC}_T(\lambda) = \{M \in \mathrm{EC}_T : M$ *of cardinality* $\lambda\}$,

(c) $\mathrm{EC}_T(\lambda!) = \{M \in \mathrm{EC}_T : M$ *has universe* $\lambda\}$,
(d) *for a set* A *of ordinals and ordinal* α *let* $\mathrm{suc}_A(\alpha)$ *be* $\min\{\beta \in A : \beta > \alpha\}$.
(3) *Let* pr *be be an (easily computable) pairing function on ordinals such that for* α, β *we have* $\mathrm{pr}(\alpha, \beta) < \max\{\omega, \alpha + |\alpha|, \beta + |\beta|\}$.

Convention 0.4 (1) (A) *If* T *is a f.o. theory not complete (like* $T^0_{\mathrm{ceq}}, T^0_{\mathrm{feq}}$, *usually universal),* *then* *embedding are the usual ones, (on* EC_T*) and* $\subseteq_T$ *(on* EC_T*) means* $\subseteq$ *and we assume* EC_T *has amalgamation and* JEP.
(B) *If* T *is complete,* *then* *embeddings are elementary (on* EC_T*) and* $\subseteq_T$ *means* $\prec$ *on* EC_T.
(C) *We say* f *is a* T*-embedding of* M *into* N *or* $f : M \longrightarrow_T N$ *when* M, N *are models of* T, f *embed* M *into* N *and* $f(M) \subseteq_T N$.
(1A) *In any case we always assume* T *has* JEP *(for* $\subseteq_T$ *of course).*
(2) *If* $\Delta \subseteq \mathbb{L}(\tau_T)$ *(such that* T *has* JEP *under* $\Delta-$*embedding) then* $\mathrm{univ}_{T,\Delta}(\lambda)$ *is the minimal* χ *such that there is a sequence* $\bar{M}$ *which is a* (λ, T, Δ)*-universal sequence which means:*

(a) $\bar{M} = \langle M_\alpha : \alpha < \chi\rangle$ *is a sequence of models of* T,
(b) *each* M_α *is of cardinality* λ,
(c) *for every model* M *of* T *of cardinality* λ *there is a* Δ*-embedding of* M *into some* M_α, *see below.*

(3) *For given* T, Δ *as above and models* M, N *of* T, *we say* f *is a* Δ*-embedding of* M *into* N *when:*

(a) f *is a function from* M *into* N,
(b) *if* $\varphi(x_0, \ldots, x_{n-1}) \in \Delta$ *and* $a_0, \ldots, a_{n-1} \in M$ *and* $M \models \varphi[a_0, \ldots, a_{n-1}]$ *then* $N \models \varphi[f(a_0), \ldots, f(a_{n-1})]$,
(c) *so* f *is one-to-one when* $(x \neq y) \in \Delta$.

(4) *For* T, Δ *as above in part (2) we may omit* Δ *when:*

(a) T *is complete,* $\Delta = \mathbb{L}(\tau_T)$, *all first order formulas,*
(b) T *not complete,* Δ *the set of quantifier free formulas in* $\mathbb{L}(\tau_T)$.

(5) *We may write* at, ep *for* $\Delta_{\mathrm{at}(T)} = \{\varphi \in \mathbb{L}(\tau_T) : \varphi$ *is atomic*$\}$, $\Delta_{\mathrm{ep}(T)} = \{\varphi \in \mathbb{L}(\tau_T) : \varphi$ *existential positive*$\}$ *respectively. We may write* τ *instead of* T. *We may write* φ *instead* $\Delta = \{\varphi\}$ *and* $\pm\varphi$ *instead* $\Delta = \{\varphi, \neg\varphi\}$.

Notation 0.5 (1) *Let* $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \xi, i, j$ *denote ordinals.*
(2) *Let* $\kappa, \lambda, \mu, \chi, \partial, \theta, \Upsilon$ *denote cardinals, infinite if not said otherwise.*
(3) *Let* k, ℓ, m, n *denote natural numbers.*
(4) *Let* φ, ψ, ϑ *denote formulas, f.o. if not said otherwise.*
(5) *For* $\lambda > \kappa$ *regular cardinals let* $S^\lambda_\kappa = \{\delta < \lambda : \mathrm{cf}(\delta) = \mathrm{cf}(\kappa)\}$ *and* $S^\lambda_{\leq\kappa} = \{\delta < \lambda : \mathrm{cf}(\delta) \leq \kappa\}$.

Definition 0.6 (1) $J^{\mathrm{bd}}_\theta = \{A \subseteq \theta : \sup(A) < \theta\}$, bd stands for bounding, for θ a regular cardinal or just a limit ordinal.
(1A) For θ regular uncountable let:

- $D^{\mathrm{club}}_\theta = \{A \subseteq \theta$: there is a club (= closed unbounded subset) E of θ such that $E \subseteq A\}$.
- NS_θ is the non-stationary ideal on θ.

(2) For a regular θ let:

(a) $\mathfrak{d}_\theta = \mathrm{Min}\{|\mathscr{F}| : \mathscr{F} \subseteq {}^\theta\theta$ is $<_{J^{\mathrm{bd}}_\theta}$-cofinal in $\mu T \mu\}$
(b) $\mathfrak{b}_\theta = \mathrm{Min}\{|\mathscr{F}| : \mathscr{F} \subseteq {}^\theta\theta$ has no $<_{J^{\mathrm{bd}}_\theta}$-upper bound$\}$.

(3) Let $\mathfrak{d}^{\mathrm{club}}_\theta$ be defined similarly using $<_{\mathrm{NS}_\theta}$ when θ is regular uncountable.
(4) For a model M and a set $u \subseteq M$ let $M{\restriction}u$ is defined naturally, allowing a function symbol to be interpreted as a partial function (and so an individual constant to be not defined) but $M{\restriction}u \subseteq M$ means $u = c\ell_M(u)$, see below.
(5) For a model M and $A \subseteq M$ let $\mathrm{c}`_M(A) = \mathrm{c}`(A, M)$ be the minimal subset B of M including A and closed under the functions of M; so $M{\restriction}\mathrm{c}`_M(A) \subseteq M$ and if $M$ has Skolem functions then $M{\restriction}\mathrm{c}`_M(A) \prec M$.

Recall

Definition 0.7 (1) For a regular uncountable cardinal λ let $\check{I}[\lambda] = \{S \subseteq \lambda$: some pair $(E, \bar{a})$ witnesses $S \in \check{I}(\lambda)$, see below$\}$.
(2) We say that (E, u) is a witness for $S \in \check{I}[\lambda]$ if:

(a) E is a club of the regular cardinal λ,
(b) $u = \langle u_\alpha : \alpha < \lambda\rangle$, $u_\alpha \subseteq \alpha$ and $\beta \in u_\alpha \Rightarrow u_\beta = \beta \cap u_\alpha$,
(c) for every $\delta \in E \cap S$, u_δ is an unbounded subset of δ of order-type $< \delta$ (and δ is a limit ordinal, necessarily δ is not a regular cardinal).

(3) For $\kappa = \mathrm{cf}(\kappa) < \lambda = \mathrm{cf}(\lambda)$ let $\check{I}_{\leq\kappa}[\lambda]$ be the ideal $\{S \subseteq \lambda : S \subseteq S^\lambda_{\leq\kappa}, S \in \check{I}[\lambda]\}$

By ([14, 15] and better) [11, 18] we have:

Claim 0.8 *Let λ be regular uncountable.*

(1) *If $S \in \check{I}[\lambda]$ <u>then</u> we can find a witness $(E, \bar{a})$ for $S \in \check{I}[\lambda]$ such that (clauses (a), (b), (c) from 0.7(2) and):*

(d) $\delta \in S \cap E \Rightarrow \mathrm{otp}(a_\delta) = \mathrm{cf}(\delta)$,
(e) *if $\alpha \notin S$ then* $\mathrm{otp}(a_\alpha) < \mathrm{cf}(\delta)$ *for some $\delta \in S \cap E$.*

(2) *$S \in \check{I}[\lambda]$ <u>iff</u> there is a pair $(E, \bar{\mathcal{P}})$ such that:*

(a) *E is a club of the regular uncountable λ,*
(b) *$\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \lambda\rangle$, where $\mathcal{P}_\alpha \subseteq \{u : u \subseteq \alpha\}$ has cardinality $< \lambda$,*
(c) *if $\alpha < \beta < \lambda$ and $\alpha \in u \in \mathcal{P}_\beta$ then $u \cap \alpha \in \mathcal{P}_\alpha$,*
(d) *if $\delta \in E \cap S$ then some $u \in \mathcal{P}_\delta$ is an unbounded subset of δ of order type $< \delta$ (and δ is a limit ordinal).*

(3) *We say a stationary subset S has club guessing when some $\langle C_\delta : \delta \in S\rangle$ witnesses it, which means: C_δ is a club of S and for every club E of λ for some $\delta \in S$ we have $C_\delta \subseteq \lambda$.*

1 On T_{ceq} for Mahlo cardinals

As here we consider T_{ceq} the simplest, non-simple theory, we may consider how much does it behave like the class of graphs (equivalently random graph)? We prove that not by a non-existence result, but with quite specific set theoretic assumptions.

T_{ceq} is very close to (and equivalent for our purposes to) the older T_{feq} which is a prime example for a theory with the tree order property, equivalently non-simple (even TP_2 but having neither the strict order property nor even just the SOP_1). For it we get here parallel and better results than [19] where it is proved that there are limitations on the universality spectrum for T_{feq} and in [5], which generalize the results for any T with the so called oak property, see somewhat more in [22,Sect. 3]. The results in those papers are meaningful when SCH fails, that is, consider a cardinal λ such that: for some strong limit singular $\mu, \mu^+ < \lambda < 2^\mu$ if λ is regular then "usually"T_{feq}, has no universal in λ.

But what about $\lambda \in (\mu, 2^\mu)$ when for transparency we assume $\mu = \mu^{<\mu}$?. Here (in Sect. 1) we get further such non-existence results for (weakly inaccessible) Mahlo cardinals. In Sect. 2, we do better but the Mahlo case may cover more classes, comes first and the proofs are more transparent. The proof here (in Sect. 1) can be axiomatized as in Sect. 2 using:

$\boxplus$ $\mathrm{PGC}(\lambda, S)$ where S is a stationary set of regular cardinals $< \lambda$ means that some $\mathbf{U}$ witness $\mathrm{PGC}(\lambda, S)$ where $\mathbf{U} = \{\langle \omega(1+\varepsilon) : \varepsilon < \theta\rangle : \theta \in S\}$ (so $\mathbf{U} = S$).
Recall that "$\mathbf{U}$ witness $\mathrm{PGC}(\lambda, \theta)$" means $\mathrm{PGC}(\lambda, \theta) = \min\{|\mathbf{U}| : \mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ and $\mathbf{U}$ does P-guess clubs$\}$. See 2.1(5), Definition 2.1(3c) and 2.1(4).

First, recall (the reader can concentrate on the universal versions, $T^0_{\text{feq}}, T^0_{\text{ceq}}$, on T_{feq} see [19,2.1=Lb3,3.1=Lc3]):

Definition 1.1 $T_{\text{feq}} = T^1_{\text{feq}}$ is the model completion of the following (universal first order) theory, T^0_{feq} which is defined by:

(A) $\tau = \tau(T^0_{\text{feq}})$ consists of:

- (a) predicates P, Q (unary),
- (b) E (three place predicate written as xE_zy instead $E(x, y, z)$),

(B) a τ-model M is a model of T^0_{feq} <u>iff</u>:

- (a) the universe of M is the disjoint union of P^M and Q^M,
- (b) $xE_zy \to P(z) \wedge Q^M(x) \wedge Q^M(y)$,
- (c) for any fixed $z \in P^M$, E^M_z is an equivalence relation on Q^M.

Observation 1.2 *0) T_{feq} is well defined and* $\mathrm{univ}(\lambda, T_{\text{feq}}) = \mathrm{univ}(\lambda, T^0_{\text{feq}})$

(1) *So if $M \models T_{\text{feq}}$ <u>then</u>:*

- (a) *in (B)(c) of Def. 1.1, for each $x \in P^M$, E^M_x is with infinitely many equivalence classes,*
- (b) *if $n < \omega$, $x_1, \ldots, x_n \in P^M$ with no repetition and $y_1, \ldots, y_n \in Q^M$ then for some $y \in Q^M$, $\bigwedge_{\ell=1}^{n} yE^M_{x_\ell}y_\ell$,*
- (c) *if $n < \omega$ and $y_1, \ldots, y_n \in Q^M$ and e is an equivalence relation on $\{1, \ldots, n\}$ <u>then</u> for some $x \in P^M$ we have $y_\ell E^M_x y_k \Leftrightarrow \ell e k$,*
- (d) *P^M, Q^M are infinite.*

(2) *Hence T_{feq} has elimination of quantifiers and* $\mathrm{univ}_{T_{\text{feq}}}(\lambda) = \mathrm{univ}_{T^0_{\text{feq}}}(\lambda)$.

We present a close relative, the main one we consider here (and, as proved below, equivalent to T_{feq} for our purpose).

Definition 1.3 $T_{\text{ceq}} = T^1_{\text{feq}}$ is the model completion of the following (universal first order) theory, T^0_{ceq} which is defined by:

(A) $\tau = \tau(T^0_{\rm ceq}) = \tau(T_{\rm ceq})$ consists of: P, Q unary predicates, E a binary predicate and F a binary function symbol,
(B) a τ-model M is a model of the universal theory. $T^0_{\rm ceq}$ <u>iff</u>:

  (a) P^M, Q^M is a partition of M,
  (b) E^M is an equivalence relation on Q^M,
  (c) F^M is a function from $Q^M \times P^M$ into Q^M such that for every $c \in P^M$, $a \mapsto F^M(a, c)$ is choosing a representative for the a/E^M-equivalence class, that is, we have:

(α) $a \in Q^M \Rightarrow F^M(a, c) \in a/E^M$,
(β) if $a, b \in Q^M$ are E^M-equivalent then $F^M(a, c) = F^M(b, c)$.
(γ) if $c \notin P^M \vee a \notin Q^M$ then $F^M(a, c)$ is not defined (or, if you prefer, is equal to c).

Concerning λ in the neighborhood of a strong limit singular we shall not give details as we can just quote.

Claim 1.4 (0) *Concerning $T^0_{\rm ceq}$*

  (a) *For a model M of $T^0_{\rm ceq}$ and $A \subseteq M$ with n elements, the closure of A inside M has at most $n + n^2$ elements, (even at most $n + (n/2)^2$ elements),*
  (b) *$T^0_{\rm ceq}$ has amalgamation and JEP,*
  (c) *$T^0_{\rm ceq}$ has a model completion, that is $T_{\rm ceq}$ is well defined,*
  (d) $\mathrm{univ}(\lambda, T_{\rm ceq}) = \mathrm{univ}(\lambda, T_{\rm feq}) = \mathrm{univ}_{T^0_{\rm ceq}}(\lambda)$.

(1) *$T_{\rm ceq}$ is not simple, is* NSOP$_2$ *and even* NSOP$_1$ *and has the oak property, in fact, by* qf *(quantifier free) and even atomic formulas.*
(2) *We have $(A) \Rightarrow (B)$ <u>where</u>:*

  (A) (a) $\theta < \mu < \lambda < \chi$,
    (b) $\mathrm{cf}(\lambda) = \lambda, \theta = \mathrm{cf}(\theta) = \mathrm{cf}(\mu), \mu^+ < \lambda$,
    (c) $\chi := \mathrm{pp}_{\Gamma(\theta)}(\mu) > \lambda + |i^*|$,
    (d) *there is a set $\{(a_i, b_i) : i < i^*\}$ with $a_i \in [\lambda]^{<\mu}, b_i \in [\lambda]^\theta$ and $|\{b_i : i < i^*\}| \leq \lambda$ such that: for every $f : \lambda \to \lambda$ for some i, $f(b_i) \subseteq a_i$,*
  (B) (a) *$T_{\rm ceq}$ equivalently $T^0_{\rm ceq}$ has no universal model in λ,*
    (b) *Moreover,* $\mathrm{univ}(\lambda, T_{\rm ceq}) \geq \chi = \mathrm{pp}_{\Gamma(\theta)}(\mu)$.

(3) *$T_{\rm feq}$ can be interpreted in $T_{\rm ceq}$ hence* $\mathrm{univ}_{T_{\rm feq}}(\lambda) \leq \mathrm{univ}_{T_{\rm ceq}}(\lambda)$.
(4) *Also the inverse of part (3) holds.*

Proof (0) Easy as clause (d) follows by parts (3), (4).
(1) By part (3), (4) quoting [5] where the oak property was introduced.
(2) Follows from parts (3), (4) and [19,Claim 2.2].
(3) For a model M of $T^0_{\rm ceq}$ we define a model $N = N[M]$ of $T^0_{\rm feq}$ as follows:
$(*)_{N,M}$

  (a) $Q^N = P^M, P^N = Q^M/E^M$,
  (b) $E^N = \{(a, b, C) : C \in P^N = Q^M/E^M$ and $a, b \in Q^N$ and $(\forall c \in C)[F^M(c, a) = F^M(c, b)]$ equivalently, $(\exists c \in C)[F^M(c, a) = F^M(c, b)]\}$. Now check that $N \models T^0_{\rm feq}$ and $M \models T_{ceq} \Leftrightarrow N \models T_{feq}$.

(4) For a model N of $T^0_{\rm feq}$ we define a model $M = M[N]$ of $T^0_{\rm ceq}$ as follows:
$(*)_{M,N}$

(a) $P^M = Q^N$ and $Q^M = \{(c, A) : c \in P^N$ and $A \in Q^N/E^N_c\}$
(b) $E^M = \{((c_1, A_1), (c_2, A_2)) : c_1 = c_2 \in P^M$ and $A_1, A_2 \in Q^N/E^N_{c_2}\}$
(c) $F^M : Q^M \times P^M \to Q^M$ is defined by: If $d \in Q^M, b \in P^M$ hence for some $c \in P^N, A \in Q^N/E^N_c$ we have $d = (c, A)$ then we let $F^M(d, b) = (c, b/E^M_c)$.

Now check that $N \models T^0_{\rm feq}$ and $M \models T_{\rm ceq} \Leftrightarrow N \models T_{\rm feq}$.

$\square$

We now point out a new reason involved "large $\mathfrak{d}_\theta$'s" for not having a universal model in λ, even for many non-simple T's. In this section we deal with a case where the proof is simpler using $T_{\rm ceq}$ and λ a Mahlo cardinal.

Claim 1.5 (1) *Assume λ is a (weakly inaccessible) Mahlo cardinal and $S = \{\theta < \lambda : \theta$ regular (weakly inaccessible) and $\mathfrak{d}_\theta > \lambda\}$ is stationary in λ and S has club guessing. Then*

(a) $\text{univ}(\lambda, T_{\rm ceq})$ *is* $> \lambda$,
(b) *even,* $\geq \sup\{\chi^+$*: the set* $\{\theta \in S : \mathfrak{d}_\theta > \chi\}$ *is stationary and has club guessing*$\}$.

(2) *We have* $\chi < \text{univ}(\lambda, T_{\rm ceq})$ *when:*

(a) λ *is a Mahlo weakly inaccessible cardinal,*
(b) $\lambda \leq \chi$,
(c) $S \subseteq \{\theta < \lambda : \theta$ *is weakly inaccessible cardinal*$\}$ *is stationary.*
(d) $\bar{\mathscr{P}} = \langle \mathscr{P}_\theta : \theta \in S \rangle$,
(e) *if* $\theta \in S$ *then* $\mathscr{P}_\theta$ *is a set of* $\leq \lambda$ *clubs of* θ,
(f) $\bar{\mathscr{P}}$ *guess clubs of* λ*, that is, for every club* E *of* λ *for some* $C \in \mathscr{P}_\theta, \theta \in S$ *we have* $C \subseteq E$,
(g) $\mathfrak{d}_\theta > \chi$ *for every* $\theta \in S$.

Proof (1) Clearly
$(*)_0$ it suffices to:

(a) fix $\chi \geq \lambda$ such that $S_\chi = \{\theta \in S : \mathfrak{d}_\theta > \chi\}$ is stationary and has club guessing,
(b) prove $\text{univ}_T(\lambda) > \chi$.

Let $T = T_{\rm ceq}$, without less of generality assume $S = S_\chi$ and let:

$(*)_1$ $\langle C_\delta : \delta \in S \rangle$ witness "S has club-guessing";
$(*)_2$ if (A) below holds, then we define some objects in (B) where:

(A) (a) $M \in \text{EC}_T(\lambda!)$,
(b) $|P^M| = \lambda$,
(c) θ is regular and $\theta \in S$,
(d) E a club of θ.

(B) we define:

(a) for $a \in P^M$ hence $a < \lambda$ let $g_a = g_{M,E,a}$ be the following function from θ to θ:

- for $\alpha < \theta$, $g_a(\alpha)$ is the minimal $\beta \in E$ such that: $\beta \in E \backslash (\alpha + 1)$ and $(\beta_1 \in Q^M \cap \beta) \wedge (F^M(\beta_1, a) < \theta) \Rightarrow F^M(\beta_1, a) < \beta$,

(b) $\mathscr{G}^0_{M,E} = \{g_{M,E,a} : a \in P^M\}$, note that E determine θ,
(c) for $\theta \in S$ let $\mathscr{G}^*_{M,\theta} = \{g_{M,C_\theta,a} : a \in P^M\}$.

Now easily

$(*)_3$ for a, M, θ, E as above, $g_{M,E,a}$ is a well defined non-decreasing function from θ into θ, in fact, into $E \subseteq \theta$,

$(*)_4$ if $M, N \in \mathrm{EC}_T(\lambda!)$ and f embeds M into N then for some club E^* of λ: if $\theta \in S, \theta = \sup(E^* \cap \theta)$, $E \subseteq E^* \cap \theta$ is a club of θ and $a \in P^M$ then $g_{M,E,a} \leq g_{N,E,f(a)}$ (so, the only way E^* influences is the demand "$E \subseteq E^*$").

Recall that, $\theta \in S \Rightarrow \eth_\theta > \chi \geq \lambda$ and we shall prove that $\mathrm{univ}(\lambda, T) > \chi$; this suffices. So assume $\langle M_\alpha : \alpha < \chi \rangle$ is a sequence of members of $\mathrm{EC}_T(\lambda!)$.

So for each $\theta \in S$ the set $\mathscr{G}_\theta = \cup\{\mathscr{G}^*_{M_\alpha,\theta} : \alpha < \chi$ satisfies $|Q^{M_\alpha} \cap \theta| = \theta\}$ has cardinality $\leq \chi$ recalling $\lambda \leq \chi$.

For $\theta \in S$, as $|\mathscr{G}_\theta| < \eth_\theta$, necessarily there is an increasing $g_\theta \in {}^\theta\theta$ such that $g \in \mathscr{G}_\theta \Rightarrow g_\theta \nleq g \mod J^{\mathrm{bd}}_\theta$ and without loss of generality, $g_\theta \in {}^\theta(C_\theta)$. Now we define a model $N \in \mathrm{EC}_{T^0_{\mathrm{ceq}}}(\lambda!)$ with $\tau_N = \tau(T_{\mathrm{ceq}}) = \tau(T^0_{\mathrm{ceq}})$ as follows:

(A) universe is λ,
(B) (a) Q^N is the set of odd ordinals $< \lambda$,
  (b) E^N is an equivalence relation on Q^N such that for every $\alpha < \beta < \lambda$ satisfying β is divisible by $|\alpha|, \alpha \in Q^N$ we have $|\alpha/E^N \cap \beta| = |\beta|$,
  (c) if $\alpha = 4\beta + 1 < \lambda$ then α/E^N is disjoint to α,
  (d) if $\theta \in S$ and $\alpha < \theta$ then $\theta > F^N(4\alpha + 1, \theta) > g_\theta(4\alpha + 1)$.

This is easy to do.

To show that $\bar{M}$ does not witness $\mathrm{univ}(\lambda, T) \leq \chi$ is suffice to show that N cannot be embedded in M_α for any $\alpha < \chi$. Toward contradiction assume that $\alpha < \chi$ and f is an embedding of N into M_α. Let $E = \{\delta < \lambda : \delta$ a limit ordinal such that $((M_\alpha \restriction \delta, N \restriction \delta, f \restriction \delta, < \restriction \delta) \prec (M_\alpha, N, f, < \restriction \lambda))\}$. Clearly E is a club of λ hence for some $\theta \in S_\chi$ we have $C_\theta \subseteq E$. Let $h \in {}^\theta\theta$ be $g_{M_\alpha, C_\theta, f(\theta)}$, so is well defined and belongs to $\mathscr{G}^0_{M_\alpha, C_\theta}$ hence to $\mathscr{G}^*_{M_\alpha,\theta}$ hence to $\mathscr{G}_\theta$ hence $g_\theta \nless h \mod J^{\mathrm{bd}}_\theta$. Now,

$\bullet_1$ choose $\alpha < \theta$ such that $h(\alpha) < g_\theta(\alpha)$,
$\bullet_2$ let $\gamma = 4\alpha + 1$,
$\bullet_3$ $F^N(\gamma, \theta) \in (g_\theta(\alpha), \theta)$ by the choice of N i.e. (B)(d) above,
$\bullet_4$ $g_\theta(\alpha) \in C_\theta$ by the choice of g_θ,
$\bullet_5$ $C_\theta \subseteq E$ by the choice of θ,
$\bullet_6$ $g_\theta(\alpha) \in E$ by $\bullet_4$ and $\bullet_5$,
$\bullet_7$ every member of E is closed under f and f^{-1},
[Why? By the choice of E it is closed under f, as f is one-to-one similarly for f^{-1}.]
$\bullet_8$ $f(F^N(\gamma, \theta)) \in [g_\theta(\alpha), \theta)$,
[Why? By $\bullet_3$, $\bullet_6$ and $\bullet_7$.]
$\bullet_9$ $f(F^N(\gamma, \theta)) = F^{M_\alpha}(f(\gamma), f(\theta))$,
[Why? As f embed N into M_α.]
$\bullet_{10}$ $F^{M_\alpha}(f(\gamma), f(\theta)) \in [g_\theta(\alpha), \theta)$,
[Why? by $\bullet_8$ and $\bullet_9$.]
$\bullet_{11}$ $h(\alpha)$ is a member of C_θ, hence a limit ordinal and in E,
[Why? By the choice of h.]
$\bullet_{12}$ $\alpha < h(\alpha)$, $\gamma < h(\alpha)$ and $f(\gamma) < h(\alpha)$,
[Why? First, $\alpha < h(\alpha)$ by the choice of h. Second, $\gamma < h(\alpha)$: as $h(\alpha)$ is limit $> \alpha$ by $\bullet_{11}$ and $\gamma = 4\alpha + 1$ by $\bullet_1$. Third $f(\gamma) < h(\alpha)$: as $\gamma < h(\alpha)$ and $h(\alpha) \in E$ by $\bullet_{11}$ and so $h(\alpha)$ is closed under f by $\bullet_7$]

$\bullet_{13}$ $F^{M_\alpha}(f(\gamma), f(\theta)) < h(\alpha)$,
[Why? By the choice of h as $g_{M_\alpha, C_\theta, f(\theta)}$ and $\bullet_{12}$].
Now by the inequalities $\bullet_1$, $\bullet_8$, $\bullet_9$ and $\bullet_{13}$ we get $h(\alpha) < g_\theta(\alpha) \le f(F^N(\gamma, \theta)) = F^{M_\alpha}(f(\gamma), f(\theta)) < h(\alpha)$, contradiction.

(2) Similarly.

$\square$

Remark 1.6 Under the assumption of 1.5(2), we can similarly prove that: for every sequence $\langle (E_\xi, \mathscr{G}_{\xi,\theta}) : \xi < \chi, \theta \in S \rangle$ satisfying clause (A) below, there is a sequence $\langle g_{\theta,\alpha} : \theta \in S, \alpha < \lambda \rangle$ with $g_{\theta,\alpha} \in {}^\theta\theta$ satisfying clause (B) below, where:

(A) E_ξ is a club of λ for $\xi < \chi$ and $\mathscr{G}_{\xi,\theta} \subseteq {}^\theta\theta$ has cardinality $\le \lambda$ for $\xi < \lambda, \theta \in S$.
(B) for every $\xi < \chi$ and club E of λ there are $\theta \in \mathrm{acc}(E_\xi) \cap E \cap S$ and $\alpha < \lambda$ such that $\theta = \sup(E_\xi \cap E \cap \theta)$ and $g \in \mathscr{G}_{\xi,\theta} \Rightarrow g_{\theta,\alpha} \not\le g \mod J^{\mathrm{bd}}_{E_\xi \cap E \cap \theta}$.

2 On successor Cardinals and club guessing

We first introduce the relevant notions (in 2.1); (we could add clause 2.1(2)(b) into the definition of $\mathbf{U}_{\lambda,\theta}$ in 2.1(1), but so far it does not matter[1]). We then investigate it and use it for sufficient conditions for"no universal".

Definition 2.1 Assume $\lambda > \theta$ are regular and $D \subseteq \mathscr{P}(\theta)$ is upward closed non-empty satisfying $D \subseteq [\theta]^\theta$, omitting D means $D = \{\theta\}$; and $\mathfrak{B}$ is a model with universe λ and countable vocabulary but $\mathfrak{B}$ is locally finite when $\theta = \aleph_0$. Saying "for D-most $\varepsilon < \theta$" will mean "for some $X \in D$ for every $\varepsilon \in X$". The main case[2] is $\theta > \aleph_0$, this is necessary for the "full" cases (see parts (2)), but not for the others; we may forget to assume $\theta > \aleph_0$.

(1) Let $\mathbf{U}_{\lambda,\theta} = \{\bar{u} : \bar{u} = \langle u_i : i < \theta \rangle$ is $\subseteq$-increasing continuous, and $i < \theta \Rightarrow i \subseteq u_i \in [\lambda]^{<\theta}$ (hence $\theta \subseteq \cup\{u_i : i < \theta\} \in [\lambda]^\theta$) and $\bigwedge_{i<\theta} u_i \cap \theta \in \theta\}$.

(1A) We shall say that $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ obeys $\mathfrak{B}$ when every $\bar{u} \in \mathbf{U}$ does, which means that for every $\varepsilon < \theta$ we have $\mathfrak{B} \restriction u_\varepsilon \subseteq \mathfrak{B}$, (if $\mathfrak{B}$ has Skolem functions this is equivalent to $\mathfrak{B} \restriction u_\varepsilon \prec \mathfrak{B}$ which implies $\theta > \aleph_0$).

(2) We say $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ fully D-guesses clubs <u>when</u> $\theta > \aleph_0$ and for every model M with universe λ and countable vocabulary there is $\bar{u} \in \mathbf{U}$ which fully D-guesses M meaning[3]
(a) (α) if[4] $\varepsilon < \theta$ then $c\ell(u_\varepsilon, M) \subseteq \sup(u_\varepsilon)$, moreover (actually follows using an expansion of M) $M \restriction \sup(u_\varepsilon) \prec M$,

(β) $(\exists \mathscr{X} \in D)(\forall \varepsilon)[\varepsilon \in \mathscr{X} \Rightarrow c\ell_M(u_\varepsilon) = u_\varepsilon \subseteq M]$, i.e. for D-most $\varepsilon < \theta$ the set u_ε is closed under the functions of M, (in an equivalent definition $M_\varepsilon \restriction u_\varepsilon \prec M$ as we can expand M by Skolem functions).

(b) the sequence $\mathrm{ord}(\bar{u}) = \langle \sup(u_\varepsilon) : \varepsilon < \theta \rangle$ is strictly increasing.

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[1] Note that it is relevant to "fully D-guess clubs"implies "almost guess clubs", see 2.15

[2] We may omit clause (b) from the definition 2.1(3) of "fully D-guess clubs", the only problem this cause is for it implying the other versions, (see 2.15).

[3] We may omit in 2.1(2) the clauses $(a)(\alpha)$, (b) but then we have problems with "FGC $\Rightarrow$ AGC and the gain is doubtful.

[4] This implies a case of club guessing.

(3) We say $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ almost D-guesses clubs <u>when</u>:

(a) for every model M with universe λ and countable vocabulary and $A \in [\lambda]^\lambda$ for some $\bar{u} \in \mathbf{U}$ we have:

(α) if $\varepsilon < \theta$ then $c\ell(u_\varepsilon, M) \subseteq \sup(u_\varepsilon)$; as in $(a)(\alpha)$ of part (2) without the moreover,
(β) for D-most $\varepsilon < \theta$ we have $A \cap u_{\varepsilon+1} \nsubseteq \sup(u_\varepsilon)$,
(γ) $c\ell(\bigcup\limits_{\varepsilon<\theta} u_\varepsilon, M) = \bigcup\limits_{\varepsilon<\theta} u_\varepsilon$, that is, $M{\restriction}(\bigcup\limits_{\varepsilon<\theta} u_\varepsilon) \subseteq M$,

(b) if $\bar{u} \in \mathbf{U}$ then $\mathrm{ord}(\bar{u}) = \langle \sup(u_\varepsilon) : \varepsilon < \theta \rangle$ is strictly increasing.

(3A) We say $\mathbf{U}$ medium D-guesses clubs <u>when</u> as in part (3) omitting clause $(a)(\gamma)$.
(3B) We say $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ semi-D-guesses clubs <u>when</u>:

$(a)'$ as (a) in part (3) but replacing (β) by:

$(\beta)'$ for D-most $\varepsilon < \theta$ for some $\zeta \in [\varepsilon, \theta)$ and $\alpha \in A$ we[5] have $\alpha \in (u_{\zeta+1} \backslash u_\zeta) \cap (\sup(u_{\varepsilon+1}) \backslash \sup(u_\varepsilon))$,

(b) as in part (3).

(3C) We say $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ pseudo-D-guess clubs <u>when</u>:

$(a)''$ if M is as above and $A \in [\lambda]^\lambda$ then for some $\bar{u} \in \mathbf{U}$ we have:
(α) as is part (3) clause $(a)(\alpha)$,
(β) for D-most $\varepsilon < \theta$ for some $\zeta \in [\varepsilon, \theta)$ and $\alpha \in A$ we have $\alpha \in (u_{\zeta+1} \backslash u_\zeta) \cap (\sup(u_{\varepsilon+1}) \backslash \sup(u_\varepsilon))$,

(b) as above.

(3D) We say $\mathbf{U}$ is (λ, θ)-reasonable (or just reasonable when (λ, θ) are clear from the context) <u>when</u> $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ satisfies clause (3)(b).
(4) We say $\mathbf{U}$ does $X - D$-guess clubs <u>when</u>:

- $\mathbf{U}$ does fully D-guess clubs and $X = F$,
- $\mathbf{U}$ does almost D-guess clubs and $X = A$,
- $\mathbf{U}$ does semi-guess clubs and $X = S$,
- $\mathbf{U}$ does medium D-guess clubs and $X = M$,
- $\mathbf{U}$ does pseudo guess clubs and $X = P$.

(5) Let $\mathrm{XGC}_D(\lambda, \theta) = \min\{|\mathbf{U}| : \mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ and $\mathbf{U}$ does $X - D$-guess clubs$\}$.
(5A) Similarly $\mathrm{XGC}_D(\lambda, \theta, \mathfrak{B})$ when we restrict ourselves to $\mathbf{U}$ obeying $\mathfrak{B}$.
(6) We say $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ is bounded <u>when</u> there is an F witnessing it which means: F is a function from $\{\bar{u}{\restriction}(\zeta+1) : \bar{u} \in \mathbf{U}, \zeta < \theta\}$ into λ such that $F(\bar{u}_1{\restriction}(\zeta_1+1)) = F(\bar{u}_2{\restriction}(\zeta_2+1)) \Rightarrow \bar{u}_1{\restriction}(\zeta_1+1) = \bar{u}_2{\restriction}(\zeta_2+1)$ and $F(\bar{u}{\restriction}(\zeta+1)) < \sup(u_{\zeta+1})$.
(7) We say "strongly bounded"when in addition $F(\bar{u}{\restriction}(\zeta+1)) \in u_{\zeta+1}$ for every $\zeta < \theta$.
(8) We say $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ is weakly bounded, <u>when</u> there is a function F witnessing it which means:

(a) $\mathrm{Dom}(F) = \{\mathrm{ord}(\bar{u}{\restriction}(\zeta+1)) : \bar{u} \in \mathbf{U}$ and $\zeta < \theta\}$ recalling $\mathrm{ord}(\bar{u}) = \langle \sup(u_\varepsilon) : \varepsilon < \theta \rangle$,
(b) $\mathrm{Rang}(F) \subseteq \lambda$ and $F(\mathrm{ord}(\bar{u}){\restriction}(\zeta+1)) < \sup(u_{\zeta+1})$ for $\bar{u} \in \mathbf{U}$ and $\zeta < \theta$,
(c) if $\zeta_1, \zeta_2 < \theta$ are successor of successor ordinals and $\bar{u}_1, \bar{u}_2 \in \mathbf{U}$ and $F(\mathrm{ord}(\bar{u}_1){\restriction}\zeta_1) = F(\mathrm{ord}(\bar{u}_2){\restriction}\zeta_2)$ then $\mathrm{ord}(\bar{u}_1){\restriction}\zeta_1 = \mathrm{ord}(\bar{u}_2){\restriction}\zeta_2$.

[5] The"$\alpha \notin u_\zeta$" follows, and "D -most" can be replaced by "all".

(9) Let

(a) if $\bar{u} \in \mathbf{U}_{\lambda,\theta}$ and $f : \theta \to \theta$ is $\le$-increasing continuous with limit θ then $\bar{u}^{[f]} = \bar{u}[f]{:=}\langle u_{f(\varepsilon)} : \varepsilon < \theta\rangle]$,
(b) if $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ and $f : \theta \to \theta$ is $\le$-increasing continuous with limit θ then $\mathbf{U}^{[f]}{:=}\mathbf{U}[f] = \{\bar{u}[f] : \bar{u} \in \mathbf{U}\}$,
(c) if $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ and $\mathscr{F}$ is a set of $\le$-increasing continuous function from θ into θ with limit θ then $\mathbf{U}[\mathscr{F}] = \{\bar{u}[f] : \bar{u} \in \mathscr{F}, f \in \mathscr{F}\}$,
(d) if $w \in [\theta]^\theta$ then $f_w = f[w]$ is the $g : \theta \to \theta$ such that $\langle g(\varepsilon) : \varepsilon < \theta$ but the closure of w in order. so is $\le$-increasing continuous with limit θ.

(10) In (a),(b) of part (9) above we may write $\bar{u}[w]$, $\mathbf{U}[w]$ for $w \in [\theta]^\theta$ meaning $\bar{u}[f]$, $\mathbf{U}[f]$ where $f = f_w$, writing $\mathbf{U}[W]$, $W \subseteq [\theta]^\theta$ mean $\cup\{\mathbf{U}[w] : w \in W\}$.

(11) Now for $X \in \{F, A, S, M, P\}$ we let (naturally and we can add $\mathfrak{B}$ as in part (5A)):

(a) $\mathrm{AXGC}_D(\lambda, \theta) = \mathrm{Min}\{|\mathbf{U}| : \mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ does $X - D$-guess clubs and is strongly bounded$\}$,
(b) $\mathrm{CXGC}_D(\lambda, \theta)$ is defined as in (a) but $\mathbf{U}$ is just bounded,
(c) $\mathrm{WXGC}_D(\lambda, \theta)$ is defined as in clause (a) but $\mathbf{U}$ is weakly bounded.

Some of the obvious implications are:

Observation 2.2 *1) If* $\mathbf{U}$ *fully D-guesses clubs, <u>then</u>* $\mathbf{U}$ *almost D-guesses clubs,*
2) If $\mathbf{U}$ *almost D-guesses clubs <u>then</u>* $\mathbf{U}$ *semi-guess-club and medium D-guesses clubs.*
3) If $\mathbf{U}$ *semi-D-guesses-clubs or medium D-guesses clubs <u>then</u>* $\mathbf{U}$ *does pseudo D-guesses clubs.*
4) If $D_1 \subseteq D_2 \subseteq [\theta]^\theta$ *then "*$\mathbf{U}$ *does* $X - D_1$*-guess clubs" implies "*$\mathbf{U}$ *does* $X - D_2$*-guess clubs" for* $X \in \{F, A, M, S, P\}$*, we may write* {full, almost, medium, semi, pseudo}.
5) Assume $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ *and* $\mathfrak{B}$ *is as in 2.2. Then there is* $\mathbf{U}'$ *such that:*

(a) $\mathbf{U}' \subseteq \mathbf{U}_{\lambda,\theta}$
(b) $|\mathbf{U}'| \le |\mathbf{U}|$
(c) if $\mathbf{U}$ *does* $X - D$*-guess clubs, for* $X \in \{F, A, M, S, P\}$ *as in part (4) then so does* $\mathbf{U}'$*,*
(d) $\mathbf{U}$ *obeys* $\mathfrak{B}$*, (see 2.1(1)).*

6) In 2.1(11) the number is $\ge \lambda$*.*
7) We may replace " countable vocabulary" by "vocabulary of cardinality $< \theta$*".*

Proof 2.2 E.g.
5) Let $\mathbf{U}' = \{\bar{u}' \in \mathbf{U} :$ for some $\bar{u} \in \mathbf{U}$ for every $\varepsilon < \theta$ we have $\mathrm{c}`_{\mathfrak{B}}(u_\varepsilon) = u'_\varepsilon \subseteq \sup(u_\varepsilon)\}$, it suffice to prove that $\mathbf{U}'$ is as required. The main point is to verify the appropriate version of clause (a) in Def 2.1. So let M be a model with universe λ and countable vocabulary, we have to find a suitable meber of $\mathbf{U}'$. By renaming, without loss of generality the vocabulary of M is disjoint to the one of $\mathfrak{B}$ and let M' be a common expansion of M and $\mathfrak{B}$ with $\tau(M') = \tau(M) \cup \tau(\mathfrak{B})$. Let $E = \{\delta : M{\restriction}\delta \prec M\}$. So $(M', E, < {\restriction}\lambda)$ is as required in clause (a) for $\mathbf{U}$ hence there is a suitable $\bar{u} \in \mathbf{U}$. We can check that in all cases $\bar{u}' = \langle \mathrm{c}`_{\mathfrak{B}}(u_\varepsilon) : \varepsilon < \theta\rangle \in \mathbf{U}$ is as required here, so we are done.
7) Recall 2.1(1), the statement "$u_\epsilon \cap \theta \in \theta$".

Definition 2.3 1) For the model theory: for a model $M \in \mathrm{EC}_T(\lambda!)$, $\Delta \subseteq \mathbb{L}(\tau_T)$ and $u \subseteq \lambda$, $A \subseteq M$ let $M^{[A]}{\restriction}_\Delta u$ be the model $M{\restriction}u$ expanded by all the restriction to u of all relations definable by a Δ-formula with parameters from A.
1A) If $\Delta = \mathbb{L}_{\mathrm{qf}}(\tau_M)$ <u>then</u> we may omit Δ; writing $\bar{a}$ instead A means $\mathrm{Rang}(\bar{a})$.
2) For $M \in \mathrm{EC}_T(\lambda!)$, $\bar{u} \in \mathbf{U}_{\lambda,\theta}$ and $\bar{a} \in {}^{\omega>}M$ let $g_{\bar{a},\bar{u},M}$ be the function from θ to θ such that for $\zeta < \theta$, $g_{\bar{a},\bar{u},M}(\zeta)$ is the minimal $\varepsilon \in (\zeta, \theta)$ such that $M^{[\bar{a}]}{\restriction}u_\varepsilon \prec M^{[\bar{a}]}{\restriction} \bigcup\limits_{\xi<\theta} u_\xi$.

Claim 2.4 *We assume that $\mathfrak{B}$ is a model with universe λ and countable vocabulary, (for the case of full club guessing, we add locally finite when $\theta = \aleph_0$.)*

(1) We have

(A) $\mathrm{CSGC}(\lambda, \theta) = \lambda$, *moreover* $\lambda = \mathrm{CSGC}(\lambda, \theta, \mathfrak{B})$ *provided that:*

- $\lambda = \mathrm{cf}(\lambda) = \theta^{++}$ *and* $\theta = \mathrm{cf}(\theta)$

(B) $\mathrm{ASGC}(\lambda, \theta) = \lambda$ *provided*[6] *that*

- $\lambda = \theta^{++}, \theta = \mathrm{cf}(\theta)$,
- *there is a stationary set* $S \subseteq S^{\theta^+}_{\theta}$ *from* $\check{I}_\theta[\theta^+]$,

(C) $\mathrm{AFGC}(\lambda, \theta) = \lambda$ *even with a reasonable witness. provided that:*

- $\lambda = \lambda^\theta$ *and* $\theta = \mathrm{cf}(\theta) > \aleph_0$,

(D) $\mathrm{MGC}_D(\lambda, \theta) = \lambda$ *<u>when</u>:*

$(*)$ $\theta = \mathrm{cf}(\theta) < \lambda$ *and there is* $\mathscr{S}$ *such that:*

(a) $\mathscr{S} \subseteq \{w : w \subseteq \lambda, \mathrm{otp}(w) = \theta\}$,
(b) $\mathscr{S}$ *has cardinality* λ,
(c) if $A \in [\lambda]^\lambda$ *then for some* $w \in \mathscr{S}$ *the set* $w \cap A$ *has cardinality* θ,
(d) $D = [\theta]^\theta$.

(E) $\mathrm{AGC}(\lambda, \theta) = \lambda$ *<u>when</u>*

$(*)$ *we have:*

(a),(b),(c),(d) as in (D) above,
(e) the cofinality of $([\lambda]^\theta, \subseteq)$ *is equal to* λ.

(2) For regular $\lambda > \theta = \mathrm{cf}(\theta)$ *we have:*

(A) if $\mathrm{SGC}_D(\lambda, \theta) = \lambda$ *and* $\mathfrak{b}_\theta \leq \lambda$ *<u>then</u>* $\mathrm{AGC}_D(\lambda, \theta) = \lambda$ *when* $D = [\theta]^\theta$,
(B) if $\mathrm{SGC}(\lambda, \theta)) = \lambda$ *and* $\mathfrak{d}_\theta \leq \lambda$ *<u>then</u>* $\mathrm{AGC}(\lambda, \theta) = \lambda$ *recalling that the default value of* D *is* $\{\theta\}$.

(3) For $\lambda > \theta = \mathrm{cf}(\theta)$ *such that*[7] $\lambda > \theta^+$ *we have* $\mathrm{SGC}(\lambda, \theta) = \lambda$ *provided that (e.g.* $\lambda = \theta^{+n}$ *for some* $n > 0$*):*

$\boxplus^3_{\lambda,\theta}$ $\mathrm{cf}([\lambda]^\theta, \subseteq) = \lambda$.

(4) If $\mathbf{U}_1 \subseteq \mathbf{U}_{\lambda,\theta}$ *medium guesses clubs, <u>then</u> there is* $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ *which medium guesses clubs of cardinality* $\leq |\mathbf{U}_1|$ *and for* $\bar{u} \in \mathbf{U}$ *we have:*

(a) if $u = \cup\{u_i : i < \theta\}$ *then* $u \subseteq \delta = \sup(u)$ *for some* $\delta < \lambda$ *of cofinality* θ*; (this actually follows by 2.1(3)(b)),*
(b) if $\mathfrak{b}_\theta \leq \lambda$ *then* $u = \cup\{u_i : i < \theta\}$ *and* $\bar{u} \in \mathbf{U}_{\lambda,\theta}$ *then* $\mathrm{otp}(u \setminus \theta) = \theta$,

(5) If $\lambda \geq \theta^+$ *and* $\theta = \mathrm{cf}(\theta) > \aleph_0$ *and* $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$ *is stationary and some* $\bar{C} = \langle C_\delta : \delta \in S\rangle$ *guesses clubs, <u>then</u>* $\mathrm{PGC}(\lambda, \theta) = \lambda$.

(6) If $\mathrm{cf}([\lambda]^\theta, \subseteq) = \lambda, \theta > \aleph_0$ *and* $\mathfrak{d}_\theta \leq \lambda$ *then* $\mathrm{FGC}(\lambda, \theta) = \lambda$, *moreover* $\mathrm{BFGC}(\lambda, \theta) = \lambda$, *(in fact, looking at [18] we get strongly bounding).*

6 We can weaken the demand: if we weaken the demand in Definition 2.1(5) to "for stationary many $\varepsilon < \theta$" and $\theta \geq \aleph_2$.

7 see footnote to part (2)

Discussion 2.5 (1) *In 2.4 we have* ZFC *results, we may get stronger results (on the full and almost versions) in some forcing extensions see 2.14 and* [10].

(2) *We can look at the cases of Definition 2.1 for singular λ, replacing $(u_\zeta \setminus \sup(u_\varepsilon))$ by $u_\zeta \setminus u_\varepsilon$, but we have not arrive to it.*

(3) *When we have clause $(a)(\gamma)$ of the Definition 2.1(3) there is less need of clause $(a)(\alpha)$. E.g. in 2.4(1)(C) we do not need "λ regular".*

(4) *In clauses (D), (E) of 2.1(1) we may add bounded/weakly bounded under natural assumption.*

Proof Without loss of generality $\mathfrak{B}$ has a pairing function $\mathrm{pr}^{\mathfrak{B}}$ and its inverses as well as $\alpha+1, \alpha+\beta$ and $\alpha\beta$.

(1) Clause (A): First, choose $S, S^+, \bar{C}$ such that (partial square guessing clubs):

$(*)_1$

(a) $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta$ and $\delta > \theta^+\}$ is stationary,

(b) $S \subseteq S^+ \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) \leq \theta$ and $\delta > \theta^+\}$, moreover if $\delta \in S$ then $\delta = \sup(S^+ \cap \delta)$,

(c) $\bar{C} = \langle C_\alpha : \alpha \in S^+\rangle$,

(d) C_α is a closed subset of α of order type $\leq \theta$, and $\mathrm{otp}(C_\alpha)$ is a limit ordinal iff $\alpha = \sup(C_\alpha)$,

(e) for $\alpha \in S^+$ we have $\alpha \in S \Leftrightarrow \mathrm{otp}(C_\alpha) = \theta$,

(f) if $\alpha \in C_\beta$ then $\alpha \in S^+$ and $C_\alpha = C_\beta \cap \alpha$,

(g) $\bar{C} \restriction S$ guess clubs, i.e.: if E is a club of λ then for stationarily many $\delta \in S$ we have $C_\delta \subseteq E$,

(h) if $\alpha \in S^+$ then $\alpha > \theta^+$ and α is closed under $\mathfrak{B}$, that is $\mathfrak{B}{\restriction}\alpha \subseteq \mathfrak{B}$,

(i) if $\alpha \in S^+$ then θ^2 divides δ.

Why do they exist (provably in ZFC)? see [11,1.3=L1.3(b)], but we elaborate (for the case $\theta > \aleph_0$); by [17,4.4(1), pg. 47] (with θ^+, λ here standing for λ, λ^+ there):

$(*)_{1.1}$ there are $W, \overline{S}, \overline{C}_i$ $(i < \theta^+)$ such that:

(A) $W = \{\delta < (\theta^+)^+ : \mathrm{cf}(\delta) < \theta^+\}$ hence is in $\check{I}[(\theta^+)^+]$.

(B) W is the union of λ sets which have the square property, i.e., there are sequences $\overline{S} = \langle S_i : i < \lambda\rangle$ and $\overline{C}_i = \langle C^i_\delta : \delta \in S_i\rangle$ for $i < \lambda$ such that:

- (a) $W \subseteq \bigcup_{i<\lambda} S_i$,
- (b) For $\delta \in S_i$, C^i_δ is a subset of $\delta \cap S_i$ of cardinality $< \lambda$ closed in δ, and if δ is a limit ordinal then C^i_δ is unbounded in δ,
- (c) For all δ_1, δ_2 if $\delta_2 \in S_i$ and $\delta_1 \in C^i_{\delta_2}$ then $\delta_1 \in S_i$ and $C^i_{\delta_1} = C^i_{\delta_2} \cap \delta_1$. (Notice that δ_1 may also be a successor ordinal.)

Easily (and as in [25,a. III] making θ^+ tries):

$(*)_{1.2}$ there are $\zeta < \theta^+$, $i < \lambda$ and a club E_* of λ such that: for every club $E \subseteq E_*$ of λ for some δ we have $\delta \in S_i$, $\mathrm{cf}(\delta) = \theta$, $\delta = \sup(C^i_\delta \cap E_*)$, $C^i_\zeta \cap E = C_\delta \cap E_*$ and $\mathrm{otp}(C^i_\delta \cap E_*) = \zeta$.

$(*)_{1.3}$ without loss of generality $\alpha \in E_* \Rightarrow (\mathrm{c`}_{\mathfrak{B}}(\alpha) = \alpha) \wedge (\theta^+)^2 | \alpha \wedge \alpha \geq \theta^+$,

$(*)_{1.4}$ let:

(a) $e \subseteq \zeta$ is unbounded in ζ and $\mathrm{otp}(e) = \theta$

(b) $S = \{\delta \in S_i : \mathrm{cf}(\delta) = \theta,\ \mathrm{otp}(C^i_\delta \cap E_*) = \zeta\}$,

(c) $S^+ = \{\alpha : \alpha \in S$ or for some $\delta \in S$ we have $\alpha \in C^i_\delta$ and $\mathrm{otp}(\alpha \cap C^i_\delta) \in e\}$.

(d) $C_\delta = \{\alpha \in C^i_\delta : \mathrm{otp}(\alpha \cap C^i_\delta) \in e\}$ for $\delta \in S$.

Now $S, \delta^+, \langle C_\delta : \delta \in S^+\rangle$ satisfies all the demands, proving $(*)_1$.

$(*)_2$ For $\delta \in S$ let $\langle \gamma^\bullet_{\delta,\varepsilon} : \varepsilon < \theta \rangle$ list C_δ in increasing order.

Second, fix $\bar{f}, \bar{g}$ such that:

$(*)_3$

(a) $\bar{f} = \langle f_\alpha : \alpha \in [\theta^+, \lambda) \rangle$,
(b) f_α is a one-to-one function from θ^+ onto α,
(c) $\bar{g} = \langle g_\xi : \xi \in [\theta, \theta^+) \rangle$,
(d) g_ξ is a one-to-one function from θ onto ξ.

$(*)_4$

(a) for $\delta \in S$ let $e_\delta = \{\xi < \theta^+$: if $\alpha \in C_\delta$ then $\mathrm{Rang}(f_\alpha \restriction \xi) = \alpha \cap \mathrm{Rang}(f_\delta \restriction \xi)$ and this set includes $C_\delta \cap \alpha$ and has cardinality $\theta\}$
(b) e_δ is a club of θ^+.

[Why clause (b) holds? As $\mathrm{otp}(C_\delta) = \theta$ and $\alpha \in C_\delta \cup \{\delta\} \Rightarrow |\alpha| = \theta^+$, this should be clear.]

$(*)_5$ for $\delta \in S$ and $\xi \in e_\delta$ let:

(a) $u_{\delta,\xi} = \mathrm{Rang}(f_\delta \restriction \xi)$, it belongs to $[\delta]^\theta$ and it includes C_δ,
(b) we choose $\bar{u}_{\delta,\xi} = \langle u_{\delta,\xi,\varepsilon} : \varepsilon < \theta \rangle$ by $u_{\delta,\xi,\varepsilon} = c\ell_{\mathfrak{B}}(\{f_{\gamma^\bullet_{\delta,\upsilon}}(g_\xi(\zeta)) : \upsilon < \omega(1+\varepsilon)$ and $\zeta < \omega(1+\varepsilon)\} \cup \{\gamma^\bullet_{\delta,\upsilon} : \upsilon < \omega(1+\varepsilon)\})$,
(c) for $w \in [\theta]^\theta$ let $\bar{u}^{[w]}_{\delta,\xi}$ be $\langle u^{[w]}_{\delta,\varepsilon}; \varepsilon < \theta \rangle$ where $u^{[w]}_{\delta,\varepsilon} = u_{\delta,\iota}$ where: $\iota \in w$ is the minimal ι that satisfies $\mathrm{otp}(w \cap \iota) = \varepsilon$, this fits 2.1(9)(d).

Note that (recalling $(*)_2$):

$(*)_6$ For $\delta \in S, \xi \in e_\delta$
we have:

(a) $\bar{u}_{\delta,\xi}$ is a $\subseteq$-increasing continuous sequence of subsets of $u_{\delta,\varepsilon}$,
(b) each $u_{\delta,\xi,\varepsilon}$ include $C_{\gamma^\bullet_{\delta,\omega(1+\varepsilon)}}$ and is an unbounded subset of $\gamma^\bullet_{\delta,\omega(1+\varepsilon)}$ and it is of cardinality $< \theta$,
(c) $\cup\{u_{\delta,\xi,\varepsilon} : \varepsilon < \theta\}$ is equal to $u_{\delta,\xi}$,
(d) $u_{\delta,\xi,\varepsilon}$ is computable from $\mathrm{pr}^{\mathfrak{B}}(\gamma^\bullet_{\delta,\varepsilon}, \xi)$ recalling that $\mathrm{pr}^{\mathfrak{B}}$ is a pairing function, using as parameters $\bar{f}, \bar{g}$ which were fixed in $(*)_2$.

[Why? should be clear.]

Lastly, $(*)_7$ let:

(a) $\mathbf{U} = \{\bar{u}_{\delta,\xi} : \delta \in S, \xi \in C_\delta\}$
(b) $\mathbf{U}_w = \{\bar{u}^{[w]}_{\delta,\xi} : \delta \in S$ and $\xi \in e_\delta\}$ for $w \in [\theta]^\theta$.

We shall prove that (why the w? for the use in the proof of part (4) of the claim):

$(*)_8$ if $w \in [\theta]^\theta$ then $\mathbf{U}_w$ witnesses $\mathrm{WSGC}(\lambda, \theta) \leq \lambda$.

Fix w now and we shall deal with all the demands:

$(*)_{8.1}$ $\mathbf{U}_w$ has cardinality $\leq \lambda$; in fact is equal to λ.

[Why? As $|\mathbf{U}_w| \leq |\{(\delta, \xi) : \delta \in S, \xi \in e_\delta \subseteq \theta^+\}| \leq \lambda + \theta^+ = \lambda$. The other inequality is also easy as $\cup\{u_{\delta,\xi} : \delta \in S, \xi \in e_\delta\} = \lambda$ and each $u_{\delta,\xi}$ has cardinality $\theta < \lambda$.]

$(*)_{8.2}$ $\mathbf{U}_w \subseteq \mathbf{U}_{\lambda,\theta}$ is reasonable.

[Why? By the choices above.]

$(*)_{8.3}$ $\mathbf{U}_w$ semi-guess clubs.

[Why? Let M and A be as in Definition 2.1(3B)(a)$'$; without loss of generality M expand $\mathfrak{B}$ and let M^+ be the expansion of M by the relation $<^{M^+}$, the order of the ordinals $< \lambda$ and

$P^{M^+} = A$, and let $E := \{\delta < \lambda : M^+ \restriction \delta \prec M^+\}$, clearly E is a club of λ. By the choice of $\bar{C}$ there is $\delta \in S$ such that $C_\delta \subseteq E$ (hence $\delta \in E$). Note that if $\alpha \in C_\delta$ then $A \cap \alpha$ is unbounded in α.

Now recall that $M \restriction \delta \prec M$, $\langle u_{\delta,\xi} : \xi \in e_\delta \rangle$ is $\subseteq$-increasing continuous with union δ, each $u_{\delta,\xi}$ is of cardinality $\leq \theta$ and e_δ is a club of θ^+ hence $e = \{\xi \in e_\delta : M^+ \restriction u_{\delta,\xi} \prec M^+\}$ is a club of θ^+. So if $\xi \in e$ then $A \cap u_{\delta,\xi}$ is unbounded in $u_{\delta,\xi}$. Now choose $\xi \in e$, so $\bar{u} = \bar{u}_{\delta,\xi}$ is as required.]

$(*)_{8.4}$ $\mathbf{U}$ is weakly bounded.

[Why? Just think, recalling $(*)_1$ and Definition 2.1(8), that is, note that $\langle C_\delta \cap \alpha : \delta \in S^+ \rangle$ has cardinality $\leq \theta^+$ for each $\alpha < \lambda$ because $\beta \in C_{\delta_1} \cap C_{\delta_2} \Rightarrow C_{\delta_1} \cap \beta = C_{\delta_2} \cap \beta$ and $\delta > \alpha \Rightarrow \sup(C_\delta \cap \alpha) \in C_\delta$, anyhow below we shall get more.]

$(*)_9$ $\mathbf{U}$ is bounded hence $\mathrm{CSGC}(\lambda, \theta)$ holds, in fact:

(a) if $u_1 = u_{\delta_1,\xi_1,\varepsilon_1}$, $u_2 = u_{\delta_2,\xi_2,\varepsilon_2}$ and $\mathrm{pr}(\gamma^\bullet_{\delta_1,\varepsilon_1}, \xi_1) = \mathrm{pr}(\gamma^\bullet_{\delta_2,\varepsilon_2}, \xi_2)$ then:

(α) $\langle \gamma^\bullet_{\delta_1,\varepsilon} : \varepsilon \leq \varepsilon_1 \rangle = \langle \gamma^\bullet_{\delta_2,\varepsilon} : \varepsilon \leq \varepsilon_2 \rangle$
(β) $u_1 = u_2$

(b) $\mathrm{pr}(\gamma^\bullet_{\delta,\varepsilon}, \xi_1) < \gamma^\bullet_{\delta,\varepsilon+1}$.

[Why? Clause (a) holds by $(*)_6(d)$ and clause (b) by $(*)_1(h)$.]

We have finished proving $\lambda = \mathrm{CSGC}(\lambda, \theta)$, and even $\mathrm{CSGC}(\lambda, \theta, \mathfrak{B})$, that is clause (A) of part (1).

Clause (B): Fix a stationary $S \subseteq S^{\theta^+}_\theta$ which belongs to $\check{I}_\theta[\theta^+]$, see Def 0.7. By 0.8 we can choose $\langle \zeta_{\xi,\varepsilon} : \varepsilon < \theta \rangle$ for $\xi \in S$ such that for any such $\xi \in S$, $\langle \zeta_{\xi,\varepsilon} : \varepsilon < \theta \rangle$ is increasing continuous with limit ξ and $(\zeta_{\xi_1,\varepsilon+1} = \zeta_{\xi_2,\varepsilon+1}) \wedge (\upsilon \leq \varepsilon) \Rightarrow \zeta_{\xi_1,\upsilon} = \zeta_{\xi_2,\upsilon}$ (and more). Without loss of generality this is $\overline{C} = \langle C_\alpha : \alpha < \lambda$ such that if $\xi \in S$, $\varepsilon < \theta$ then for some $\alpha < \zeta_{\xi,\varepsilon}$ we have $\{\zeta_{\xi,i} : i \leq \varepsilon\} = C_\alpha$ and for $\overline{u} \in \mathbf{U}$ such that $\zeta = \sup(\bigcup_\varepsilon u_\varepsilon)$ we require that $\alpha \in u_{\varepsilon+1}$ (and $\overline{C}$ definable in $\mathfrak{B}$).

Now in the proof of clause (A) of part (1) we choose $\bar{f}, \bar{g}$ as in $(*)_3$ but in addition $\bar{g} = \langle g_\alpha : \alpha \in [\theta, \theta^+) \rangle$ satisfies that: if $\alpha = \zeta_{\xi,\varepsilon}$, ε a limit ordinal then g_α is computable from $\langle g_\beta : \beta \in \{\zeta_{\xi,\iota} : \iota < \varepsilon\} \rangle$ e.g. as follows: for $\iota < \theta$ let $W^\iota_\alpha = \{\gamma < \alpha :$ for some $\beta \in \{\zeta_{\xi,e} : e < \varepsilon\}$ we have $g_\beta(\gamma) < \iota\}$ and then define g^ι_α by induction on $\iota < \theta$ such that:
$(*)$

(a) g^ι_α is a function from W^ι_α onto some ordinal $< \theta$.
(b) g^ι_α is increasing with ι.
(c) $g^\iota_\alpha \restriction (W^\iota_\alpha \setminus \bigcup \{W^j_\alpha : j < \iota\})$ is order preserving.

Also we can restrict ourselves to $\xi \in S$ such that $u_{\delta,\xi}$ is closed under pr. Then we can restrict ourselves to (w, δ, ξ) such that $\varepsilon_1 < \varepsilon_2 \in w \Rightarrow \mathrm{pr}(\gamma^\bullet_{\delta,\varepsilon_1}, \zeta_{\xi,\varepsilon_1}) \in u_{\delta,\xi,\varepsilon_2}$.

Clause (C): Easy but we elaborate.

We are assuming $\lambda = \lambda^\theta, \theta = \mathrm{cf}(\theta)$; so $\mathbf{U} = \mathbf{U}_{\lambda,\theta}$ is trivially a subset of $\mathbf{U}_{\lambda,\theta}$ of cardinality λ and let F be a one-to-one function from $\{\overline{u} \restriction \epsilon : \overline{u} \in \mathbf{U}$ and $\epsilon < \theta\}$, clearly exist. Let M be a model with universe λ and we have to find $\bar{u}$ as promised. Toward this we choose u_ε by induction on ε as follow:

(a) u_ε is a subset of λ of cardinality $< \theta$,
(b) $u_\varepsilon = \mathrm{c}`(u_\varepsilon, M)$ and has no last member,
(c) if $\varepsilon = \zeta + 1$ then some $\alpha \in A \setminus \sup(u_\zeta)$ belongs to u_ε,
(d) if ε is a limit ordinal then $u_\varepsilon = \cup \{u_\zeta : \zeta < \varepsilon\}$

(e) If $\epsilon = \zeta + 1$ then $F(\bar{u}\restriction\epsilon) \in u_\epsilon$.

There is no problem to carry the induction and $\langle u_\varepsilon : \varepsilon < \theta\rangle$ is as required.

Clause (D) Recall that $\mathscr{S} \subseteq \{w \subseteq \lambda : \mathrm{otp}(w) = \theta\}$ and more by our assumption. For each $w \in \mathscr{S}$ let $\bar{u}_w$ be $\langle u_{w,\varepsilon} : \varepsilon < \theta\rangle$ were $u_{w,\varepsilon} = \{\alpha \in W : \mathrm{otp}(w \cap \alpha) < \omega(1+\varepsilon)\} \cup \varepsilon$. Now let $\mathbf{U} = \{\bar{u}_w : w \in \mathscr{S}\}$, it suffice to prove that $\mathbf{U}$ witness $\mathrm{MGC}(\lambda, \theta) = \lambda$.

Clearly most demands hold: $\mathbf{U}$ is a subset of $\mathbf{U}_{\lambda,\theta}$ of cardinality λ, and for each $\bar{u} \in \mathbf{U}$ the sequence $\langle sup(u_\varepsilon) : \varepsilon < \theta\rangle$ is increasing. The main point is, to be given M, A as in clause (a) of Def 2.1(3) and to prove that sub-clauses (α), (β) there hold.

Let M^+ be an expansion of M by the order $<^{M^+}$ of the ordinals $< \lambda$, $R^{M^+} = A$, pr and let $E = \{\delta < \lambda : M^+\restriction\delta \prec M^+\}$, clearly it is a club of λ. Now there is no harm in replacing A by a smaller sub-set so let $A' = \{\alpha \in A : \alpha = \min(A\backslash\beta)$ for some $\beta \in E\}$. Clearly $A' \in [\lambda]^\lambda$ so by the choice of $\mathscr{S}$ there is $w \in \mathscr{S}$ such that $w \cap A$ has cardinality θ.

Now $\bar{u}_w \in \mathbf{U}$ is as required.

Clause E:

By [18] there is a stationary $\mathscr{A} \subseteq [\lambda]^\theta$ of cardinality λ, see details in the proof of part (3). Now for each $w \in \mathscr{S}$ let $\mathscr{A}_w = \{v \in \mathscr{A} : w \subseteq v\}$, so it is non-empty. Now for each $w \in \mathscr{S}$ let $\langle \alpha_{w,\varepsilon} : \varepsilon < \theta\rangle$ list the members of w in increasing order. Also for each such pair (w, v) let $\bar{u}_{w,v} = \langle u_{w,v,\varepsilon} : \varepsilon < \theta\rangle$ be such that:

(a) $u_{w,v,\varepsilon}$ is a subset of v of cardinality $< \theta$,
(b) $u_{w,v,\varepsilon}$ is increasing continuous with ε,
(c) $u_{w,v,\varepsilon}$ includes $\{\alpha_{w,\zeta} : \zeta < \omega(1+\varepsilon)\}$ if $\theta > \aleph_0$ and is $\{\alpha_{w,\zeta} : \zeta < 1+\varepsilon\}$,
(d) $u_{w,v,\varepsilon}$ is included in $\cup\{\alpha_{w,\zeta} : \zeta < \omega(1+\varepsilon)\}$
(e) $\cup\{u_{w,v,\varepsilon} : \sigma < \theta\} = v \cap \cup\{\alpha_{w,\varepsilon} : \varepsilon < \theta\}$

Lastly we define $\mathbf{U}$ as the set $\{\bar{u}_{w,v} : w \in \mathscr{S}, v \in \mathscr{A}_w\}$; so it suffice to prove that $\mathbf{U}$ witnesses $\mathrm{MGC}_D(\lambda, \theta) = \lambda$; this is as in previous cases.

(2) As in [20,Ch.III], and anyhow not used .

(3) By [21] there is $\mathscr{S}$ such that:

$(*)_1$

(a) $\mathscr{S} \subseteq [\lambda]^\theta$ has cardinality λ
(b) $\mathscr{S}$ is stationary, i.e. for every model M_* with universe λ and vocabulary $\leq \theta$ there is $w \in \mathscr{S}$ such that $M_*\restriction w \prec M_*$.

Now as we can increase $\mathscr{S}$, without loss of generality:

$(*)_2$ $\mathscr{S} \cap [\alpha]^\theta$ is a stationary subset of $[\alpha]^\theta$ for every $\alpha \leq \lambda$.

We continue as in the proof of part (1), maybe details will be given in [10] and anyhow this will not be used here.

(4) Let $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ medium guess clubs,

Now clause (a) follows by 2.1(3)(b).

For clause (b), for $\bar{u} \in \mathbf{U}_1$, let $\langle \alpha_{u,\varepsilon} : \varepsilon < \theta\rangle$ list $\bigcup_\varepsilon u_\varepsilon$ and for each $\bar{u}$ and increasing $g \in {}^\theta\theta$ we define $w^g_{\bar{u},\varepsilon}$ by induction on $\varepsilon < \theta$ as follows. For $\varepsilon = 0$ let $w^g_{\bar{u},\varepsilon} = \mathrm{c}`_{\mathfrak{B}}(\emptyset)$, for ϵ limit let $w_{\overline{u},\varepsilon} = \bigcup\{w_{\overline{u},\zeta} : \zeta < \epsilon\}$, so let $\epsilon = \zeta + 1$. Let $\iota < \theta$ be minimal $\geq g(\zeta), \epsilon$, such that $\sup(w^g_{\overline{u},\zeta}) < \sup\{\alpha_{\overline{u},\iota(1)} : \iota(1) < i\}$ and let $w^g_{\overline{u},\varepsilon} = \mathrm{c}`_{\mathfrak{B}}(\{\alpha_{\overline{u},\iota(1)} : i(1) < \iota\})$. Lastly let $\mathscr{F} \subseteq \{g \in {}^\theta\theta$:g is increasing$\}$ be $<_{J^{\mathrm{bd}}_\theta}$ $-$unbounded of cardinality $\mathfrak{b}_\theta$ and let $\mathbf{U} = \{\langle \overline{w}^g_{\overline{u},\varepsilon} : \epsilon < \theta\rangle : g \in \mathscr{F}$ and $\overline{u} \in \mathbf{U}_1\}$. Now check.

(5), (6) Combine things above. □

Discussion 2.6 *Assume $\lambda > \theta \geq \sigma = \mathrm{cf}(\sigma)$, ($2^\sigma > \lambda$ in the interesting case). Let $\mathbf{U}_{\lambda,\theta,\sigma} = \{\bar{u} : \bar{u} = \langle u_\varepsilon : \varepsilon < \sigma\rangle$ is $\subseteq$-increasing and $u_\varepsilon \in [\lambda]^\theta\}$ and repeat the definition. Of doubtful help, otherwise $(\theta^{++}, \theta^+, \theta)$ would have helped.*

Theorem 2.7 *(1) Assume* $\lambda = \mathrm{cf}(\lambda) > \theta = \mathrm{cf}(\theta)$, $D = [\theta]^\theta$, $\mathrm{AGC}_D(\lambda,\theta) = \lambda$ *and* $\mathfrak{b}_\theta > \lambda$. *Then* $\lambda \notin \mathrm{Univ}(T_{\mathrm{ceq}})$*; moreover,* $\mathrm{univ}_{T_{\mathrm{ceq}}}(\lambda) \geq \mathfrak{b}_\theta$.

(1A) In part (1) we can replace $\mathrm{AGC}_D(\lambda,\theta)$ *by* $\mathrm{MGC}(\lambda,\theta)$.

(2) If $\lambda = \mathrm{cf}(\lambda) > \theta = \mathrm{cf}(\theta)$ *and*[8] $\mathrm{FGC}(\lambda,\theta) = \lambda$ *and* $\chi = \mathfrak{d}_\theta > \lambda$ *or just* $\mathrm{cf}({}^\theta\theta, \leq_D)) \geq \chi > \lambda$, *then* $\mathrm{univ}_{T_{\mathrm{ceq}}}(\lambda) \geq \chi$.

(3) If D is a uniform filter on θ, $({}^\theta\theta, <_D)$ *is* $(<\chi)$*-directed and* $\chi > \lambda$, $\mathrm{FGC}_D(\lambda,\theta) = \lambda$ *then* $\mathrm{univ}_{T_{\mathrm{ceq}}}(\lambda) \geq \chi$.

Remark 2.8 (1) The Claim 2.11 below shows that we cannot weaken the assumption on T too much.

(2) Note that the above works also for $\theta = \aleph_0$.

(3) See more in [10].

Proof (1) So let ($T = T_{\mathrm{ceq}}$ and):

$(*)_0$ $\mathfrak{B}$ is as in Definition 2.1, such that:

(a) $\mathfrak{B}$ has universe λ,
(b) $\tau_{\mathfrak{B}}$, the vocabulary of $\mathfrak{B}$, is countable,
(c) $\mathfrak{B}$ has a pairing function $\mathrm{pr} : \lambda \times \lambda \to \lambda$ i.e., a one-to-one 2-*place* function from λ into λ and pr_1 and pr_2 its inverses.

$(*)_1$ assume $\alpha_* < \mathfrak{b}_\theta$ and $M^*_\alpha \in \mathrm{EC}_T(\lambda!)$ for $\alpha < \alpha_*$; it suffices to find $N \in \mathrm{EC}_{T_{\mathrm{ceq}}}(\lambda!)$ not embeddable into M^*_α for every $\alpha < \alpha_*$

$(*)_2$ let $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ witness $\mathrm{AGC}_D(\lambda,\theta,\mathfrak{B}) = \lambda$ or just $\mathrm{MGC}_D(\lambda,\theta,\mathfrak{B}) = \lambda$.
[Why? Is there such $\mathbf{U}$? by the assumption of the theorem and apply 2.2(5).]

$(*)_3$ for $\bar{u} \in \mathbf{U}$, $\alpha < \alpha_*$ and $d \in P^{M^*_\alpha}$ we define the set $E_{\bar{u},d,\alpha}$; clearly is a club of θ, as follows:

- $E_{\bar{u},d,\alpha} = \{\varepsilon < \theta : \varepsilon$ is a limit ordinal such that u_ε is closed inside $\cup\{u_\zeta : \zeta < \theta\}$ under the functions of M^*_α and the function $F^{M^*_\alpha}(-,d)$, or just if $a \in u_\varepsilon$, $b \in \bigcup_{\zeta<\theta} u_\zeta$ and $M^*_\alpha \models$ "$F^{M^*_\alpha}(a,d) = b$" then $b \in \sup(u_\varepsilon)\}$.

So $\mathscr{E} = \{E_{\bar{u},d,\alpha}, : \bar{u} \in \mathbf{U}, \alpha < \alpha_*$ and $d \in P^{M^*_\alpha}\}$ is a set of clubs of θ of cardinality $\leq |\mathbf{U}| + |\alpha_*| + |P^{M_\bullet}| < \mathfrak{b}_\theta$. Hence,

$(*)_{3.1}$ there is an increasing function $g : \theta \to \theta$ such that $(\forall E \in \mathscr{E})(\forall^\infty \varepsilon < \theta)(g(\varepsilon) > \mathrm{suc}_E(\varepsilon))$.

Now we can construct $N = N_g \in \mathrm{EC}_T(\lambda!)$ such that:

$(*)_4$

(a) $P^N = \{3\beta : \beta < \lambda\}$ hence $Q^N = \{3\beta+1, 3\beta+2 : \beta < \lambda\}$
(b) if $\alpha = 3\beta + 1 < \lambda$ (hence $\alpha \in Q^N$) then $\alpha = \min(\alpha/E^N)$
(c) if $\bar{u} \in \mathbf{U}$ then for some $\alpha(\bar{u}) = \alpha_{\bar{u},g} \in P^N$ we have: if $\beta \in (u_{\varepsilon+1}\backslash u_\varepsilon) \cap Q^N$ and $(\beta/E^N) \cap u_\varepsilon = \emptyset$ but $\upsilon < \theta \Rightarrow (\beta/E^N) \cap \sup(\bigcup_{\zeta<\theta} u_\zeta) \not\subseteq \sup(u_\upsilon)$ then $F^N(\beta, \alpha(\bar{u})) \in \sup(\bigcup_{\zeta<\theta} u_\zeta)\backslash \sup(u_{g(\varepsilon+1)})$.

Now toward contradiction assume that:

$(*)_5$ f embeds N_g into M^*_α and $\alpha < \alpha_*$.

8 Recall that this means that $D = \{\theta\}$

Let $N_g^+ = (N_g, <^{N_g^+})$ where $<^{N_g^+} = \{(\alpha, \beta) : \alpha < \beta < \lambda\}$ and let $M_\bullet$ be a model with universe λ expanding M_α^* and (a renaming of) N_g^+; (that is $\tau(M_\bullet)$ contains also a disjoint copy τ' of $\tau(N_g^+)$ such that the restriction of $M_\bullet$ to τ' is the suitable copy of N_g^+).
Also we have $f = G^{M_*}$ for some unary function symbol $G \in \tau(M_\bullet)$ and $M_\bullet$ has Skolem functions and $\tau(M_\bullet)$ is countable and expand $\mathfrak{B}$.
$(*)_6$

(a) Let $E = \{\delta < \lambda : M_\bullet \restriction \delta \prec M_\bullet\}$, so a club of λ,
(b) Let $H : \lambda \to \lambda$ be $H(\alpha) = \mathrm{suc}_E(\alpha)$,
(c) Let $M_* = (M_\bullet, H)$.

$(*)_7$ Let $A = \{\mathrm{pr}^{\mathfrak{B}}(3\beta + 1, f(3\beta + 1)) : \beta < \lambda\}$.
As f is a one-to-one function from λ to λ necessarily $A \in [\lambda]^\lambda$. By the choice of $\mathbf{U}$ and of D as $[\theta]^\theta$ there is $\bar{u}$ such that:
$(*)_8$

(a) $\bar{u} \in \mathbf{U}$,
(b) if $\varepsilon < \theta$ then $u_\delta = \mathrm{c}\ell(u_\varepsilon, \mathfrak{B})$,
(c) $\mathrm{c}\ell(u_\varepsilon, M_*) \subseteq \sup(u_\varepsilon)$, and (essentially follows) $M_\bullet \restriction \sup(u_\varepsilon) \prec M_\bullet$,
(d) the set $v = \{\varepsilon < \theta : A \cap u_{\varepsilon+1} \setminus \sup(u_\varepsilon) \neq \emptyset\}$ has cardinality θ,
(e) $M_* \restriction \bigcup_{\varepsilon<\theta} u_\varepsilon \prec M_*$, (not used when we assume only $\mathrm{MGC}(\lambda, \theta) = \lambda$),

Now let $d = f(\alpha_{\bar{u},g})$, so it is a member of $P^{M_\alpha^*}$, and
$(*)_9$ if $\varepsilon \in v$ then for some $a = a_\varepsilon \in u_{\varepsilon+1} \setminus \sup(u_\varepsilon)$ we have $a_\varepsilon \in A$,
$(*)_{10}$

(a) for $\varepsilon \in v$ let $a_\varepsilon = \mathrm{pr}^{\mathfrak{B}}(\beta_\varepsilon, f(\beta_\varepsilon))$, so $\beta_\varepsilon \in \{3\gamma + 1 : \gamma < \lambda\}$,
(b) Let $v_1 = \{\varepsilon \in v : g(\varepsilon) > \mathrm{suc}_{E(\bar{u},d,\alpha)}(\varepsilon)\}$,
(c) For $\varepsilon < \theta$ let $\zeta_\varepsilon = \mathrm{suc}_{E(\bar{u},d,\alpha)}(\varepsilon)$,

$(*)_{11}$ $v_1 \in [v]^\theta$,
[Why? Because $v \in [\theta]^\theta$ and the choice of f (here the use of $\mathfrak{b}_\theta$ matters.)]
Now,

$\boxplus$ Assume $\epsilon \in v_1$,

$\bullet_1$ $a_\varepsilon \in u_{\varepsilon+1} \setminus u_\varepsilon$,
[Why? by the choice of a_ε.]
$\bullet_2$ if $\zeta < \theta$ then $\sup(u_\zeta)$ is closed under $x + 1$ so a limit ordinal and f^{-1}; that is, $\forall \beta < \lambda[\beta < \sup(u_\varepsilon) \iff f(\beta) < \sup(e_\varepsilon)]$,
[Why? As $\mathrm{c}\ell(\sup(u_\varepsilon), M_*) \subseteq \sup(u_\varepsilon)$ by $(*)_8(c)$.]
$\bullet_3$ $\beta_\varepsilon \in u_{\varepsilon+1} \setminus \sup(u_\varepsilon)$ and $f(\beta_\varepsilon) \in u_{\varepsilon+1} \setminus \sup(u_\varepsilon)$,
[Why? As $\beta_\varepsilon = \mathrm{pr}_1^{\mathfrak{B}}(a_\varepsilon)$, $f(\beta_\varepsilon) = \mathrm{pr}_2^{\mathfrak{B}}(a_\varepsilon)$ and $a_\varepsilon \in u_{\varepsilon+1}$ by $\bullet_1$ (and $u_{\varepsilon+1} = \mathrm{c}\ell_{\mathfrak{B}}(u_{\varepsilon+1})$ by $(*)_8(b)$) we have $\beta_\varepsilon \in u_{\varepsilon+1}$, $f(\beta_\varepsilon) \in u_{\varepsilon+1}$. Now, $\beta_\varepsilon < \sup(u_\varepsilon) \iff f(\beta_\varepsilon) < \sup(u_\varepsilon)$ by $\bullet_2$ and $(\beta_\varepsilon, f(\beta_\varepsilon) < \sup(u_\varepsilon)) \Rightarrow a_\varepsilon = \mathrm{pr}^{\mathfrak{B}}(\beta_\varepsilon, f(\beta_\varepsilon)) \subseteq \sup(u_{\varepsilon+1})$. But, $a_\varepsilon \notin \sup(u_\varepsilon)$, together $\beta_\varepsilon, f(\beta_\varepsilon) \notin \sup(u_\varepsilon)$ and we are done proving $\bullet_3$.]
$\bullet_4$ $F^{M_\alpha^*}(f(\beta_\varepsilon), d) \notin [\zeta_\varepsilon, \sup\left(\bigcup_\varepsilon u_\varepsilon\right))$,
[Why? by the definition of $E_{\bar{u},d,\alpha}$ in $(*)_3$.]
$\bullet_5$ $F^{M_\alpha^*}(f(\beta_\varepsilon), d) = f(F^{N_g}(\beta_\varepsilon, \alpha_{\bar{u},g}))$,
[Why? As f embed N_g into M_α^* and the choice of d as $f(\alpha_{\bar{u},g}.)$]

$\bullet_6$ $F^{N_g}(\beta_\varepsilon, \alpha_{\bar{u},g}) \in \left[\sup(u_{g(\varepsilon)}), \sup(\bigcup_\zeta u_\zeta)\right)$,
[Why? By the choice of N_g, $\alpha_{\bar{u},g}$ and $\bullet_3$.]
$\bullet_7$ $f(F^{N_g}(\beta_\varepsilon, \alpha_{\bar{u},g})) \in \left[\sup(u_{g(\varepsilon)}), \sup(\bigcup_\zeta u_\zeta)\right)$,
[Why? By $\bullet_2$ and $\bullet_6$.]
$\bullet_8$ $f(F^{N_g}(\beta_\varepsilon, \alpha_{\bar{u},g})) \in \left[\sup(u_{\zeta(\varepsilon)}), \sup(\bigcup_\zeta u_\zeta)\right)$ recalling $\zeta(\varepsilon) = J_\varepsilon = \mathrm{suc}_{E(\bar{u},d,\alpha)}(\varepsilon)$,
[Why? By $\bullet_7$, the definition of $E_{\bar{u},d,\alpha}$ and the assumption on $\varepsilon \in V_1$.]
$\bullet_9$ $F^{M^*_\alpha}(f(\beta_*), d) \in \left[\zeta_\varepsilon, (\sup(\bigcup_\zeta u_\zeta)\right)$,
[Why? By $\bullet_8$ and $\bullet_5$.]
But $\bullet_4$ and $\bullet_9$ are contradictory, so we are done proving part (1).
(1A), (2), (3) Similarly to the proof of part (1) with some changes will be done in [10]. □

Conjecture 2.9 (1) *Assume T (is countable complete first order) with the* TP_2. *If $\lambda > \theta > \aleph_0$ are regular, $\mathfrak{d}_\theta > \lambda$ and $\mathfrak{d}_\kappa >$* $\mathrm{FGC}(\lambda, \theta)$ *(maybe θ inaccessible), then* $\mathrm{univ}_T(\lambda) \geq \mathfrak{d}_\kappa$.
(2) *Assume T (is countable complete first order) non-simple. If $\lambda > \theta > \aleph_0$ are regular, $\mathfrak{d}_\theta > \lambda$ and $\mathfrak{d}_\kappa >$* $\mathrm{CFGC}(\lambda, \theta)$, *then* $\mathrm{univ}_T(\lambda) \geq \mathfrak{d}_\kappa$.

Remark 2.10 See hopefully [13], [10].

Claim 2.11 *Assume $\mu = \mu^{<\mu} \leq \theta = \mathrm{cf}(\theta) < \lambda = \mathrm{cf}(\lambda) < \chi = \chi^\lambda$, λ is strongly inaccessible Mahlo and for transparency* GCH *holds in the interval $[\mu, \chi)$. For some $\mathbb{P}$:*

(a) $\mathbb{P}$ is a $(< \mu)$-complete forcing of cardinality χ neither collapsing any cardinal, nor changing cofinalities
(b) $(2^\mu)^{\mathbf{V}[\mathbb{P}]} = \chi$
(c) in $\mathbf{V}^{\mathbb{P}}$ we have $\mathfrak{d}_\partial = \chi$ for every inaccessible $\partial \in [\mu, \lambda)$,
(d) in $\mathbf{V}^{\mathbb{P}}$ there is $\bar{\mathscr{P}}$ as in 1.5(2),
(e) T_{ceq} has no universal member in λ moreover $\mathrm{univ}(\lambda, T_{\mathrm{ceq}}) \geq \chi$
(f) the results of [16] holds, i.e. there is a universal random graph in λ, and see [24]

Remark 2.12 Note that here the case "$\cup_{\varepsilon<\theta} u_\varepsilon \cap \sup(u_\zeta)$ has cardinality θ" does not arrise.

Proof Our proof is based on the proof [16], but the quoted [1] has to be changed, see full proof in a work by Mark Poór and the author in preparation.

That is, we choose:
$(*)$ $\mathbb{P} = \mathbb{P}_3$, where $\langle \mathbb{P}_k, \mathbb{Q}_\ell : k \leq 3, \ell < 3\rangle$ is an iteration and:
(A) $\mathbb{Q}_0$ is adding χ λ-Cohen, so it satisfies:
$\bullet_1 \mathbb{Q}_0$ is a $(< \lambda)$-complete forcing notion of cardinality χ,
$\bullet_2 \mathbb{Q}_0$ neither collapse some cardinal nor change any cofinality (in fact is λ^+-cc),
$\bullet_3$ in $\mathbf{V}^{\mathbb{Q}_0}$ there is a family $\mathscr{A}_0$ of χ-many subsets of λ each of cardinality λ, the intersection of any two having cardinality $< \lambda$,
(B) in $\mathbf{V}^{\mathbb{Q}_0} = \mathbf{V}^{\mathbb{P}_1}$ the forcing notion $\mathbb{Q}_1$ satisfies:
$\bullet_1 \mathbb{Q}_1$ is a $(< \mu)$-complete λ-cc forcing notion of cardinality χ, (yes, λ-cc not λ^+-cc),
$\bullet_2 \mathbb{Q}_1$ does neither collapses some cardinal nor changes any cofinality,
$\bullet_3$ in $\mathbf{V}^{\mathbb{P}_2}$ there is a family $\mathscr{A}_1$ of χ-many subsets of λ, each of cardinality λ, the intersection of any two having cardinality $< \mu$,
$\bullet_4$ in $\mathbf{V}^{\mathbb{P}_2}$ we have $\mathfrak{d}_\partial = \chi$ for every weakly inaccessible $\partial \in (\mu, \lambda)$,
(C) in $\mathbf{V}^{\mathbb{P}_2}$ we have $\mathbb{Q}_2$ which is $(< \mu)$-complete μ^+-cc forcing notion, forcing that there is a universal graph of cardinality λ.

Now, why are there such $\mathbb{Q}_\ell$-s? For clause (A) use the forcing of adding χ λ-Cohens.

For clause (B) we use $\mathbb{Q}_1$ such that

$(*)\mathbb{Q} = \mathbb{Q}'_1 \times \mathbb{Q}'_1$: starting with $(\mu, \lambda, \chi, \mathscr{A}_0)$ as above:

(a) $\mathbb{Q}'_1$ is the product with Easton support of $\langle \mathbb{Q}_{1,\theta} : \theta \in S \rangle$ where $S = \{\theta : \theta \in (\mu, \lambda)$ is an inaccessible non-Mahlo$\}$ and for $\theta \in S$, $\mathbb{Q}_{1.\theta}$ is the forcing adding χ many θ-Cohens,

(b) $\mathbb{Q}''_2$ forces a refinement $\mathscr{A}_1$ of $\mathscr{A}_0$ to a family as required as in [1] but with Easton support; so each condition has cardinality $< \lambda$ and has Easton support.

Now why clause (B)$\bullet_1, \bullet_2, \bullet_3$ holds? As said above noting that [1,6.1] use full support while use Easton support

Lastly for clause (C) we apply [16].

Having constructed $\mathbb{P} = \mathbb{P}_3$ we have to check that is is as required.

Now being $(< \mu)$-complete, of cardinality χ, pedantically of density χ, is obvious by the properties of the $\underset{\sim}{\mathbb{Q}}_\ell$-s. Similarly concerning "no cardinal is collapsed and no cofinality changed", so clause (a) of Claim 2.11 holds. Also forcing the existence of a universal graph of cardinality λ holds by the choice of $\underset{\sim}{\mathbb{Q}}_2$, so clause (f) of 2.11 holds.

Next, clause (c) there saying $\mathfrak{d}_\partial = \chi$ holds because it obviously holds in $\mathbf{V}^{\mathbb{P}_2}$ by the choice of $\underset{\sim}{\mathbb{Q}}'_1$ and the later forcing preserve it because it satisfies the μ^+-cc. Now, lastly, why clause (d) of 2.11 holds? First, in $\mathbf{V}^{\mathbb{P}_1}$ we have GCH in the interval $[\mu, \lambda)$ so there is such $\bar{\mathscr{P}}$, and $\mathbb{P}_3/\mathbb{P}_1$ satisfies the λ-cc so the old clubs of λ are dense. and this continue to holds in $\mathbf{V}^{\mathbb{P}}$. Hence the non-existence of a universal model of T_{ceq} in λ, (holds by 1.5(2)). $\square$

Question 2.13 (1) *Can we for theories T satisfying* $\mathrm{NSOP}_1 + \mathrm{TP}_2$ *get similar results?*

(2) *Is T_{ceq} in some sense minimal non-simple in a suitable family of theories?*

Claim 2.14 *Assume $\lambda > \theta = \mathrm{cf}(\theta) > \aleph_0$ and $\lambda = \lambda^\theta$ and $\mathbb{P}$ is a θ-cc forcing notion.*

(1) In $\mathbf{V}^{\mathbb{P}}$ there is a reasonable strongly bounding $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ of cardinality λ witnessing $\lambda = \mathrm{FGC}(\lambda, \theta, \mathfrak{B})$

(2) Assume $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ fully guess clubs <u>then</u> in $\mathbf{V}^{\mathbb{P}}$, $\mathbf{U}$ still fully guess clubs.

(3) In part (2), if $\mathbf{U}$ is reasonable/bounding/strongly bounding/weakly bounding in $\mathbf{V}$, <u>then</u> so it is in $\mathbf{V}^{\mathbb{P}}$.

Remark 2.15 (1) This showed help in consistency results.

(2) Similarly for the other versions of guessing clubs from 2.1, but take care of what is D.

(3) In 2.14(2) we can replace"fully guess clubs" by a slightly stronger version of"almost guess clubs" specifically, instead of one set A we have $\sigma < \theta$ sets A.

(4) Also In 2.14(2) we can use versions with D-s.

Proof Part (1) follows by parts (2),(3) because in $\mathbf{V}$ there is such $\mathbf{U}$ by 2.4(1)(C). The point is:

$(*)$ $(A) \Rightarrow (B)$ where:

(A) if $\chi > \lambda$ and $\{\mathfrak{B}, \mathbb{P}, \lambda\} \cup \{\varepsilon : \varepsilon \leq \theta\} \subseteq N \prec (\mathscr{H}(\chi), \in)$ and $\|N\| < \lambda$ and $\mathbb{P}$ satisfies the θ^+-cc where $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathfrak{B}}$ a model with universe λ and vocabulary of cardinality $< \theta$"

(B) $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{N} \cap \lambda = c\ell(|N|, \underset{\sim}{\mathfrak{B}})$ and $\underset{\sim}{\mathfrak{B}} \restriction |N|$ is an elementary submodel of $\underset{\sim}{\mathfrak{B}}''$. $\square$

Definition 2.16 Assuming $\theta = \mathrm{cf}(\theta) \leq \lambda$ we let $\mathfrak{d}^\dagger_{\theta\lambda}$ be the cofinality of the partial order $(\mathscr{F}^\dagger_{\theta,\lambda}, \leq^\dagger_{\theta,\lambda})$ where:

$(*)_1$ $\mathscr{F}^\dagger_{\theta,\lambda}$ is the family of subsets of ${}^\theta\theta$ of cardinality $\leq \lambda$

$(*)_2$ let $\leq = \leq^\dagger_{\theta,\lambda}$ is the following partial order on $\mathscr{F}\dagger_{\theta,\lambda}$:

$F_1 \leq F_2$ if $(\forall f_1 \in F_1)(\exists f_2 \in F_2)[f_1 \leq f_2]$

The following was part of 2.7, maybe we shall return to it.

Claim 2.17 *(1) If* $\lambda > \theta > \aleph_0$ *are regular,* $\mathrm{FGC}(\lambda, \theta) = \lambda$ *and* $\mathfrak{d}_\theta > \lambda$ *then* $\mathrm{univ}(\lambda, T_{\mathrm{ceq}}) \geq \mathfrak{d}_\theta$.

(2) Above we can replace $\mathfrak{d}_\theta$ *by* $\mathfrak{d}^{\dagger}_{\theta,\lambda}$

Proof (1) Like the proof of part (1) of 2.7 so we mainly note the changes.

$(*)_1$ as above.
$(*)_2$ $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ witness $\mathrm{FGC}(\lambda, \theta) = \lambda$.
$(*)_3$ we let:

(a) M^+_α is an expansion of M^*_α by a pairing function and Skolem functions,
(b) $\mathscr{Z} = \{(\alpha, \bar{u}, d) : \alpha < \alpha_*, \cup_{\varepsilon<\theta} u_\varepsilon$ is closed under the function $F^{M^+_\alpha}$ and each u_ε is closed under the functions of $M^+_\alpha\}$,
(c) for $(\alpha, \bar{u}, d) \in \mathscr{Z}$ let $E_{\alpha,\bar{u},d} = \{\varepsilon < \theta : u_\varepsilon$ is closed under $F^{M^*_\alpha}(-, d)\}$,
(d) above let $g_{\alpha,\bar{u},d} \in {}^\theta\theta$ be such that $g_{\alpha,\bar{u},d}(\varepsilon) = \min(E_{\alpha,\bar{u},d} \setminus (\varepsilon + 1))$, $g \in {}^\theta\theta$

Next choose $g \in {}^\theta\theta$ not bounded by any well defined $g_{\alpha,\bar{u},d}$ Now we choose $N = N_g$ as follows:

$(*)_4$ we let
(a), (b) as above,
(c) for every $\bar{u} \in \mathbf{U}$ for some $\alpha(\bar{u}) = \alpha_{\bar{u},v} \in P^N$ for every $\varepsilon < \theta$ we have:
$F^N(-, d)$ maps $\cup_{\varepsilon<\theta} u_\varepsilon$ into itself
for $\varepsilon < \theta$ we have: $\varepsilon \in v$ iff there is $a \in u_{\varepsilon+1} \setminus u_\varepsilon$ such that $F^N(a, d) \notin u_{\varepsilon+1}$.
The rest is as in the proof of part (1) of 2.7.

□

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