THE KEISLER–SHELAH ISOMORPHISM THEOREM AND THE CONTINUUM HYPOTHESIS II

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Abstract. We continue the investigation started in [2] about the relation between the Keilser–Shelah isomorphism theorem and the continuum hypothesis. In particular, we show it is consistent that the continuum hypothesis fails and for any given sequence $\mathbf{m} = \langle (\mathbb{M}^1_n, \mathbb{M}^2_n) : n < \omega \rangle$ of models of size at most $\aleph_1$ in a countable language, if the sequence satisfies a mild extra property, then for every non-principal ultrafilter $\mathcal{D}$ on ω, if the ultraproducts $\prod\limits_{\mathcal{D}} \mathbb{M}^1_n$ and $\prod\limits_{\mathcal{D}} \mathbb{M}^2_n$ are elementarily equivalent, then they are isomorphic.

1. Introduction

Ultraproducts arise naturally in model theory and many other areas of mathematics, see [6, Chapter VI]. An ultraproduct is a way to connect the notions of elementary equivalence and isomorphism. By a result of Keisler [4], the continuum hypothesis, CH, implies that in a countable language $\mathcal{L}$, two $\mathcal{L}$-models $\mathbf{M}, \mathbf{N}$ of size $\leq 2^{\aleph_0}$, are elementarily equivalent if and only if they have isomorphic ultrapowers with respect to an ultrafilter on ω. Recently the authors of this paper [2] have shown that Keisler's theorem is indeed equivalent to the CH, by showing that there are two elementary equivalent dense linear orders M and N of size $\leq \aleph_2$ which do not have isomorphic ultrapowers with respect

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to any ultrafilter on ω. Much earlier but after Keisler, Shelah [5] removed the CH from Keisler's theorem by weakening the conclusion and showed that if $\mathcal{L}$ is a countable language and $\mathbf{M}, \mathbf{N}$ are countable $\mathcal{L}$-models, then $\mathbf{M} \equiv \mathbf{N}$ if and only if they have isomorphic ultrapowers with respect to an ultrafilter on 2^ω. Shelah [7] has shown that the CH is an essential assumption for Keisler's theorem, even for countable models, by constructing a model of ZFC in which $2^{\aleph_0} = \aleph_2$ and there are countable graphs $\mathbf{\Delta} \equiv \mathbf{\Gamma}$ such that for no ultrafilter $\mathcal{U}$ on ω, $\mathbf{\Delta}^\omega/\mathcal{U} \simeq \mathbf{\Gamma}^\omega/\mathcal{U}$. See also [3], where some further connections between several variants of Keilser's theorem and cardinal invariants are found.

In this paper, we continue the investigations started in [2]. In Section 2, we consider some further extensions of Keisler's isomorphism theorem. To this end, we define the notion of an ultraproduct problem $\mathbf{m} = \langle(\mathbb{M}_n^{\mathbf{m},1}, \mathbb{M}_n^{\mathbf{m},2}, \tau_{\mathbf{m},n}) : n < \omega\rangle$ and show the consistency of the failure of the CH with the assertion that for any non-principal ultrafilter $\mathcal{D}$ on ω, if the ultraproducts $\prod\limits_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},1}$ and $\prod\limits_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},2}$ are elementarily equivalent, then they are indeed isomorphic. In Section 3 we show that the above conclusion fails if $\mathfrak{b} > \aleph_1$, where $\mathfrak{b}$ denotes the bounding number.

2. Extending the Keisler isomorphism theorem

Recall from [2] that there exists a dense linear order N of size $\aleph_2$ which is elementary equivalent to $M = (\mathbb{Q}, <)$, but for no ultrafilter $\mathcal{U}$ on $\omega, {}^\omega M/\mathcal{U} \simeq {}^\omega N/\mathcal{U}$. We also proved several consistent extensions of the Keisler's theorem for models of size at most $\aleph_1$ in the absence of the continuum hypothesis. In this section, we continue the work started in [2] and give a further extension of Keisler's isomorphism theorem. To this end, we start by making some definitions.

Definition 2.1. A *pseudo ultraproduct problem* is a sequence

$$\mathbf{m} = \langle(\mathbb{M}_n^{\mathbf{m},1}, \mathbb{M}_n^{\mathbf{m},2}, \tau_{\mathbf{m},n}) : n < \omega\rangle$$

where

(1) $\langle \tau_{\mathbf{m},n} : n < \omega \rangle$ is a $\subseteq$-increasing sequence of finite vocabularies, with $\tau_{\mathbf{m},0} = \emptyset$. Set $\tau_{\mathbf{m}} = \bigcup_n \tau_{\mathbf{m},n}$,[1]

(2) each $\mathbb{M}_n^{\mathbf{m},\ell}$ is a $\tau_{\mathbf{m}}$-model,

(3) $\kappa(\mathbf{m}) \leq \aleph_1$, where $\kappa(\mathbf{m}) = \sup\{||\mathbb{M}_n^{\mathbf{m},\ell}|| : \ell = 1, 2, \text{ and } n < \omega\}$ and $||\mathbb{M}_n^{\mathbf{m},\ell}||$ denotes the size of the universe of the model $\mathbb{M}_n^{\mathbf{m},\ell}$.

Definition 2.2. Suppose $\mathbf{m}$ is a pseudo ultraproduct problem and $k \leq n < \omega$.

(1) The Ehrenfeucht–Fraïssé game $\Game_k^n(\mathbf{m}) = \Game_{\tau_{\mathbf{m},k},k}(\mathbb{M}_n^{\mathbf{m},1}, \mathbb{M}_n^{\mathbf{m},2})$ is defined as a game between two players protagonist and antagonist where
  (a) it has $2(k+1)$ moves,
  (b) protagonist plays at even stages and antagonist plays at odd stages,
  (c) in the $(2l+1)$-th move, the antagonist chooses $A_l \subseteq \mathbb{M}_n^{\mathbf{m},1}$, $B_l \subseteq \mathbb{M}_n^{\mathbf{m},2}$ such that $|A_l| + |B_l| \leq k$,
  (d) in the $(2l+2)$-th move, the protagonist chooses f_l, a partial one-to-one function from $\mathbb{M}_n^{\mathbf{m},1} \restriction \tau_{\mathbf{m},k}$ into $\mathbb{M}_n^{\mathbf{m},2}$, which preserves ϕ and $\neg\phi$, for ϕ a strictly atomic formula (i.e., ϕ is of the form $x = y$ or $P(x_0, \cdots, x_{m-1})$ or $F(x_0, \cdots, x_{m-1}) = y$, where P is a predicate symbol and F is a function symbol or an individual constant) from $\tau_{\mathbf{m},k}$,
  (e) the protagonist has to satisfy $A_l \subseteq \mathrm{dom}(f_l)$, $B_l \subseteq \mathrm{range}(f_l)$ and $f_l \supseteq f_{l-1}$,

(2) We say that the protagonist looses the game $\Game_k^n(\mathbf{m})$, when there is no legal move for him to do.

The following easy lemma will be useful later.

Lemma 2.3. *Suppose* $\mathbf{m}$ *is a pseudo ultraproduct problem and* $k \leq n < \omega$. *Let* f *be the last move of protagonist in the game* $\Game_k^n(\mathbf{m})$. *If* $\phi(\nu_0, \cdots, \nu_{l-1})$ *is a* $\tau_{\mathbf{m},k}$*-formula and* $x_0, \cdots, x_{l-1} \in \mathrm{dom}(f)$, *then*

$$\mathbb{M}_n^{\mathbf{m},1} \models \phi_{\mathrm{dom}(f)}(x_0, \cdots, x_{l-1}) \Leftrightarrow \mathbb{M}_n^{\mathbf{m},2} \models \phi_{\mathrm{range}(f)}(f(x_0), \cdots, f(x_{l-1})),$$

[1]Thus we allow that $\tau_{\mathbf{m}}$ to be finite.

where for any set D, ϕ_D is obtained from ϕ by replacing all quantifiers $\exists x$ and $\forall x$ by the restricted quantifiers $\exists x \in D$ and $\forall x \in D$ respectively.

Proof. By induction on the complexity of the formula ϕ. This is true for strictly atomic formulas by the assumption and it is easy to see that if it holds for ϕ, ϕ_0 and ϕ_1, then it also holds for $\neg\phi$ and $\phi_0 \wedge \phi_1$. Now suppose that $\phi(\nu_0, \cdots, \nu_{l-1}) = \exists \nu \psi(\nu, \nu_0, \cdots, \nu_{l-1})$ and let $x_0, \cdots, x_{l-1} \in \mathrm{dom}(f)$. If $x \in \mathrm{dom}(f)$ is such that $\mathbb{M}_n^{\mathbf{m},1} \models \psi_{\mathrm{dom}(f)}(x, x_0, \cdots, x_{l-1})$, then by the induction hypothesis $\mathbb{M}_n^{\mathbf{m},2} \models \psi_{\mathrm{range}(f)}(f(x), f(x_0), \cdots, f(x_{l-1}))$ and hence $\mathbb{M}_n^{\mathbf{m},2} \models \exists \nu \in \mathrm{range}(f) \psi_{\mathrm{range}(f)}(\nu, f(x_0), \cdots, f(x_{l-1}))$. It then follows that $\mathbb{M}_n^{\mathbf{m},2} \models \phi_{\mathrm{range}(f)}(f(x_0), \cdots, f(x_{l-1}))$. Conversely suppose that for some $y \in \mathrm{range}(f)$, $\mathbb{M}_n^{\mathbf{m},2} \models \psi_{\mathrm{range}(f)}(y, f(x_0), \cdots, f(x_{l-1}))$. Let $x \in \mathrm{dom}(f)$ be such that $y = f(x)$. Then by the induction hypothesis $\mathbb{M}_n^{\mathbf{m},1} \models \psi_{\mathrm{dom}(f)}(x, x_0, \cdots, x_{l-1})$ and hence $\mathbb{M}_n^{\mathbf{m},1} \models \exists \nu \in \mathrm{dom}(f) \psi_{\mathrm{dom}(f)}(\nu, x_0, \cdots, x_{l-1})$. Thus $\mathbb{M}_n^{\mathbf{m},1} \models \phi_{\mathrm{dom}(f)}(x_0, \cdots, x_{l-1})$. $\square$

Definition 2.4. Suppose $\mathbf{m}$ is a pseudo ultraproduct problem and $n < \omega$. Then $\mathbf{k}_{\mathbf{m},n}$ is the maximal $k \leq n$ such that the protagonist has a winning strategy in the Ehrenfeucht–Fraïssé game $\eth_k^n(\mathbf{m}) = \eth_{\tau_{\mathbf{m},k},k}(\mathbb{M}_n^{\mathbf{m},1}, \mathbb{M}_n^{\mathbf{m},2})$. Set also $\mathbf{k}_{\mathbf{m}} = \langle \mathbf{k}_{\mathbf{m},n} : n < \omega \rangle$.

We now define the notion of an ultraproduct problem.

Definition 2.5. An *ultraproduct problem* is a pseudo ultraproduct problem $\mathbf{m}$ such that $\limsup_{n<\omega} \mathbf{k}_{\mathbf{m},n} = \infty$.

To each pseudo ultraproduct problem we assign a natural countably generated filter on ω and a cardinal invariant, which play an important role for the rest of the paper. We start by defining such notions in a more general context. Let us first fix some notation.

Notation 2.6. *(1) For a filter $\mathcal{D}$ on ω, let $\forall_{\mathcal{D}} x \phi(x)$ mean "$\exists A \in \mathcal{D}\ \forall n \in A\ \phi(n)$".*
(2) The notation $\forall^* x \phi(x)$ means "for all but finitely many $x, \phi(x)$ holds".

Definition 2.7. (1) Given a sequence $\mathbf{k} = \langle \mathbf{k}_n : n < \omega \rangle \in {}^{\omega}\omega$, let $\mathcal{D}_{\mathbf{k}}$ be the filter on ω generated by co-bounded subsets of ω and the sets

$$\{n < \omega : \mathbf{k}_n > k\},$$

where $k < \omega$.

(2) Suppose $\mathbf{k} = \langle \mathbf{k}_n : n < \omega \rangle \in {}^\omega\omega$ is such that $\limsup_{n<\omega} \mathbf{k}_n = \infty$, then set

$$\mathfrak{d}_{\mathbf{k}} = \min\left\{ |\mathcal{F}| : \mathcal{F} \subseteq \prod_{n<\omega} [\omega]^{\mathbf{k}_n} \text{ and } (\forall \eta \in {}^\omega\omega)\,(\exists f \in \mathcal{F})\,\big(\forall_{\mathcal{D}_{\mathbf{k}}} n\,(\eta(n) \in f(n))\big)\right\}.$$

Remark 2.8. Suppose $\mathbf{k} = \langle \mathbf{k}_n : n < \omega \rangle \in {}^\omega\omega$. If $\limsup_{n<\omega} \mathbf{k}_n = \infty$, then $\mathcal{D}_{\mathbf{k}}$ is a countably generated non-principal proper filter on ω. Otherwise, $\mathcal{D}_{\mathbf{k}} = \mathcal{P}(\omega)$.

Definition 2.9. If $\mathbf{m}$ is a pseudo ultraproduct problem, then set $\mathcal{D}_{\mathbf{m}} = \mathcal{D}_{\mathbf{k}_{\mathbf{m}}}$. Furthermore, if it is an ultraproduct problem, then set $\mathfrak{d}_{\mathbf{m}} = \mathfrak{d}_{\mathbf{k}_{\mathbf{m}}}$.

The next simple lemma will be very useful.

Lemma 2.10. *Suppose* $\mathbf{m}$ *in an ultraproduct problem.*

(1) *If* $\mathcal{D} \supseteq \mathcal{D}_{\mathbf{m}}$ *is a non-principal ultrafiler on* ω. *Then the ultraproducts* $\prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},1}$ *and* $\prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},2}$ *are elementary equivalent.*

(2) *If* $\mathcal{D}$ *is a non-principal ultrafiler on* ω *and* $\prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},1} \equiv \prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},2}$, *then* $D \supseteq \mathcal{D}_{\mathbf{m}}$.

Proof. (1) Suppose $\mathbf{m}$ is an ultraproduct problem and $D \supseteq \mathcal{D}_{\mathbf{m}}$ is a non-principal ultrafiler on ω. Let ϕ be a $\tau_{\mathbf{m}}$-statement. Then, for some $k < \omega$, it is a $\tau_{\mathbf{m},k}$-statement. We prove by induction on the complexity of ϕ that

$$(*)_\phi : \qquad \prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},1} \models \phi \iff \prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},2} \models \phi.$$

If ϕ is a strictly atomic formula, then $E = \{n < \omega : \mathbf{k}_{\mathbf{m},n} > k\} \in \mathcal{D}_{\mathbf{m}} \subseteq \mathcal{D}$, and for each $n \in E$ we have $\mathbb{M}_n^{\mathbf{m},1} \models \phi$ if and only if $\mathbb{M}_n^{\mathbf{m},2} \models \phi$, from which the result follows. It is also clear that if $(*)_\phi$ holds then $(*)_{\neg\phi}$ holds and that if $(*)_{\phi_0}$ and $(*)_{\phi_1}$ hold, then $(*)_{\phi_0 \wedge \phi_1}$ holds. Now suppose that $\phi = \exists x \psi(x)$ and $(*)_\psi$ is true. Suppose $\prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},1} \models \phi$. Then for some $\bar{x} = \langle x_n : n < \omega \rangle \in \prod_{n<\omega} \mathbb{M}_n^{\mathbf{m},1}$, $\prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},1} \models \psi([\bar{x}]_{\mathcal{D}})$, and hence

$$A = \{n < \omega : \mathbb{M}_n^{\mathbf{m},1} \models \psi(x_n)\} \in \mathcal{D}.$$

We may further suppose that for any $n \in A, \mathbf{k}_{\mathbf{m},n} > k$. For $n \in A$ let g_n be the last move of the protagonist in the game $\eth^n_{\mathbf{k}_{\mathbf{m},n}}(\mathbf{m})$, in which antagonist always chooses $A_\ell = \{x_n\}$

and $B_\ell = \emptyset$. Thus for any such n, by the induction hypothesis and Lemma 2.3 we have $\mathbb{M}^{\mathbf{m},2}_n \models \psi(g_n(x_n)))$. Let $\bar{y} = \langle g_n(x_n) : n < \omega \rangle$. Then $\prod\limits_{\mathcal{D}} \mathbb{M}^{\mathbf{m},2}_n \models \psi([\bar{y}]_{\mathcal{D}})$ and hence $\prod\limits_{\mathcal{D}} \mathbb{M}^{\mathbf{m},2}_n \models \phi$. Conversely, suppose that $\prod\limits_{\mathcal{D}} \mathbb{M}^{\mathbf{m},2}_n \models \phi$. Then for some $\bar{y} = \langle y_n : n < \omega \rangle \in \prod_{n<\omega} \mathbb{M}^{\mathbf{m},2}_n$, $\prod\limits_{\mathcal{D}} \mathbb{M}^{\mathbf{m},2}_n \models \psi([\bar{y}]_{\mathcal{D}})$, and hence

$$B = \{n < \omega : \mathbb{M}^{\mathbf{m},2}_n \models \psi(y_n)\} \in \mathcal{D}.$$

We again assume that $\mathbf{k}_{\mathbf{m},n} > k$ for every $n \in B$. For $n \in B$, let h_n be the last move of the protagonist in the game $\Game^n_{\mathbf{k}_{\mathbf{m},n}}(\mathbf{m})$, in which antagonist always chooses $A_\ell = \emptyset$ and $B_\ell = \{y_n\}$. Thus for any such n, $y_n \in \text{range}(h_n)$ and hence for some $x_n, y_n = h_n(x_n)$. By the induction hypothesis and Lemma 2.3 we have $\mathbb{M}^{\mathbf{m},1}_n \models \psi(x_n)$. Let $\bar{x} = \langle x_n : n < \omega \rangle$. Then $\prod\limits_{\mathcal{D}} \mathbb{M}^{\mathbf{m},1}_n \models \psi([\bar{x}]_{\mathcal{D}})$ and hence $\prod\limits_{\mathcal{D}} \mathbb{M}^{\mathbf{m},1}_n \models \phi$. We are done.

(2) Suppose by the way of contradiction that $D \not\supseteq \mathcal{D}_{\mathbf{m}}$. Let $k < \omega$ be such that $\{n < \omega : \mathbf{k}_{\mathbf{m},n} > k\} \in \mathcal{D}_{\mathbf{m}} \setminus D$. It then follows that

$$A = \{n < \omega : \mathbf{k}_{\mathbf{m},n} \leq k\} \in D.$$

Thus for any $n \in A$ with $n > k$, as $\mathbf{k}_{\mathbf{m},n} < k + 1$, the protagonist loses the game $\Game^n_{k+1}(\mathbf{m})$, and hence antagonist has a winning strategy. In particular, by enlarging k if necessary, we can find some formula $\phi_n(\nu_0, \cdots, \nu_{l_n-1})$, which is a boolean combination of strict atomic formulas of $\tau_{\mathbf{m},k+1}$, and some $a_0, \cdots, a_{l_n-1} \in \mathbb{M}^{\mathbf{m},1}_n$ such that $\mathbb{M}^{\mathbf{m},1}_n \models \phi_n(a_0, \cdots, a_{l_n-1})$, but for no $b_0, \cdots, b_{l_n-1} \in \mathbb{M}^{\mathbf{m},2}_n$ we have $\mathbb{M}^{\mathbf{m},2}_n \models \phi_n(b_0, \cdots, b_{l_n-1})$. It thus follows that $\mathbb{M}^{\mathbf{m},1}_n \models \exists x_0 \cdots x_{l_n-1} \phi_n$ but $\mathbb{M}^{\mathbf{m},2}_n \models \neg\exists x_0 \cdots x_{l_n-1} \phi_n$. As $\tau_{\mathbf{m},k+1}$ is finite and $\mathcal{D}$ is an ultrafiler, for some set $B \subseteq A$ in $\mathcal{D}$ we have $\phi_n = \phi$ for some fixed formula ϕ. But then $\prod\limits_{\mathcal{D}} \mathbb{M}^{\mathbf{m},1}_n \models \exists x_0 \cdots x_{l_n-1} \phi$ while $\prod\limits_{\mathcal{D}} \mathbb{M}^{\mathbf{m},1}_n \models \neg\exists x_0 \cdots x_{l_n-1} \phi$, which contradicts our assumption $\prod\limits_{\mathcal{D}} \mathbb{M}^{\mathbf{m},1}_n \equiv \prod\limits_{\mathcal{D}} \mathbb{M}^{\mathbf{m},2}_n$. $\square$

We would like to construct a model of ZFC in which the continuum is large and for every pseudo ultraproduct problem $\mathbf{m}$, if $\mathcal{D}$ is a non-principal ultrafilter on ω and if the structures $\prod\limits_{\mathcal{D}} \mathbb{M}^{\mathbf{m},1}_n$ and $\prod\limits_{\mathcal{D}} \mathbb{M}^{\mathbf{m},2}_n$ are elementarily equivalent, then $\prod\limits_{\mathcal{D}} \mathbb{M}^{\mathbf{m},1}_n \cong \prod\limits_{\mathcal{D}} \mathbb{M}^{\mathbf{m},2}_n$.

We may assume that $\mathbf{m}$ is an ultraproduct problem, as otherwise $\mathcal{D}_{\mathbf{m}} = \mathcal{P}(\omega)$, and everything becomes trivial.

The next lemma reduces the construction of such a model to controlling the size of $\mathfrak{d}_{\mathbf{m}}$'s.

Proposition 2.11. *Suppose $\mathbf{m}$ is an ultraproduct problem, $\mathfrak{d}_{\mathbf{m}} = \aleph_1$ and $\mathcal{D}$ is a non-principal ultrafilter on ω such that $\prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},1}$ and $\prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},2}$ are elementarily equivalent. Then these ultraproducts are isomorphic, i.e., $\prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},1} \cong \prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},2}$.*

Remark 2.12. By [7], it is consistent that $\mathfrak{d} = \aleph_1 < \mathfrak{d}_{\mathbf{m}}$, where $\mathfrak{d}$ is the dominating number. It also follows from [7] that in Lemma 2.11, we can not replace $\mathfrak{d}_{\mathbf{m}} = \aleph_1$ by $\mathfrak{d} = \aleph_1$.

The following definition plays a key role in the proof of Lemma 2.11.

Definition 2.13. Suppose $\mathbf{m}$ is an ultraproduct problem.

(1) We say that $\mathbf{s} \in \mathrm{AP}_{\mathbf{m}}$ iff
   (a) $\mathbf{s} = \langle \mathbf{g}_{\mathbf{s},n} : n < \omega \rangle$,
   (b) $\mathbf{g}_{\mathbf{s},n}$ is an initial segment of a play of $\partial^n_{\mathbf{k}_{\mathbf{m},n}}(\mathbf{m})$ of length $l_{\mathbf{s},n}$ with the last function $f_{\mathbf{s},n}$, where the protagonist plays with a winning strategy,
   (c) $\lim_{\mathcal{D}_{\mathbf{m}}} \langle \mathbf{k}_{\mathbf{m},n} - l_{\mathbf{s},n} : n < \omega \rangle = \infty$.

(2) Define the partial order $\leq_{\mathrm{AP}_{\mathbf{m}}}$ on $\mathrm{AP}_{\mathbf{m}}$ by $\mathbf{s} \leq_{\mathrm{AP}_{\mathbf{m}}} \mathbf{t}$ iff

$$\{n : \mathbf{g}_{\mathbf{s},n} \text{ is an initial segment of } \mathbf{g}_{\mathbf{t},n}\} \in \mathcal{D}_{\mathbf{m}}.$$

We are now ready to complete the proof of Lemma 2.11.

Proof of Lemma 2.11. Assume the hypotheses in the lemma hold. The proof of the next claim is evident.

Claim 2.14. (1) $\mathrm{AP}_{\mathbf{m}} \neq \emptyset$.

(2) If $\langle \mathbf{s}_\ell : \ell < \omega \rangle$ is a $\leq_{\mathrm{AP}_{\mathbf{m}}}$-increasing sequence from $\mathrm{AP}_{\mathbf{m}}$, then it has a $\leq_{\mathrm{AP}_{\mathbf{m}}}$-upper bound.

(3) Suppose $\mathbf{s} \in \mathrm{AP}_{\mathbf{m}}$, $\ell \in \{1,2\}$, and $\bar{w} = \langle w_n : n < \omega \rangle \in \prod\limits_n [\mathbb{M}_n^{\mathbf{m},\ell}]^{\mathbf{k}_{\mathbf{m},n}}$. Then for some $\mathbf{t} \in \mathrm{AP}_{\mathbf{m}}$, we have

(a) $\mathbf{s} \leq_{\mathrm{AP}_{\mathbf{m}}} \mathbf{t}$,

(b) if $l_{\mathbf{s},n} < \mathbf{k}_{\mathbf{m},n}$, then

(i) $\ell = 1 \implies w_n \subseteq \mathrm{dom}(f_{\mathbf{t},n})$,

(ii) $\ell = 2 \implies w_n \subseteq \mathrm{range}(f_{\mathbf{t},n})$.

Proof. Clause (1) is clear. Clause (2) follows by an easy diagonalization argument, but let us elaborate a proof. For each $\ell < \omega$ set

$$\eta_\ell = \langle \mathbf{k}_{\mathbf{m},n} - l_{\mathbf{s}_\ell,n} : n < \omega \rangle \in \omega^\omega.$$

Then $\lim_{\mathcal{D}_{\mathbf{m}}} \eta_\ell(n) = \infty$, and for all $i < \omega$,

$$(\omega, <)^\omega / \mathcal{D}_{\mathbf{m}} \models \mathrm{id}_i / \mathcal{D}_{\mathbf{m}} < \eta_\ell / \mathcal{D}_{\mathbf{m}},$$

where id_i is the constant sequence i on ω. As the structure $(\omega, <)^\omega / \mathcal{D}_{\mathbf{m}}$ is $\aleph_1$-saturated, we can find some $\eta \in \omega^\omega$ such that for all $\ell, i < \omega$,

$$(\omega, <)^\omega / \mathcal{D}_{\mathbf{m}} \models \mathrm{id}_i / \mathcal{D}_{\mathbf{m}} < \eta / \mathcal{D}_{\mathbf{m}} < \eta_\ell / \mathcal{D}_{\mathbf{m}}.$$

For $n < \omega$ let ℓ_n be the maximal natural number ℓ such that:

(a) $\ell \leq \eta(n)$,

(b) $\langle g_{\mathbf{s}_i,n} : i \leq \ell + 1 \rangle$ is $\trianglelefteq$-increasing,

(c) $\mathbf{k}_{\mathbf{m},n} - l_{\mathbf{s}_i,n} \geq \eta(n)$ for every $i \leq \ell$.

Note that each ℓ_n is well-defined as on the one hand, $\ell_n \leq \eta(n)$, and on the other hand, $\ell = 0$ satisfies the above requirements.

Let $\ell_* < \omega$. Then

$$D_{\ell_*} = \{ n < \omega : \ell_n \geq \ell_*, \mathbf{k}_{\mathbf{m},n} - l_{\mathbf{s}_{\ell(n)},n} \geq \eta(n) \} \in \mathcal{D}_{\mathbf{m}}.$$

To see this, note that the set

$$\{n<\omega : \forall j \leq \ell_*\big(\eta(n) > \ell_*, \mathbf{g}_{\mathbf{s}_j,n} \trianglelefteq \mathbf{g}_{\mathbf{s}_{j+1},n} \text{ and } k_{\mathbf{m},n} - l_{\mathbf{s}_j,n} \geq \eta(n)\big)\}$$

belongs to $\mathcal{D}_{\mathbf{m}}$ and is included in D_{ℓ_*}, so $D_{\ell_*} \in \mathcal{D}_{\mathbf{m}}$ as well.

Now set $\mathbf{s} = \langle \mathbf{g}_{\mathbf{s},n} : n<\omega\rangle$, where $\mathbf{g}_{\mathbf{s},n} = \mathbf{g}_{\mathbf{s}_{\ell_n},n}$ for each $n<\omega$. We show that $\mathbf{s}$ is as required.

In order to show that $\mathbf{s} \in \mathrm{AP}_{\mathbf{m}}$, it only suffices to show that

$$\lim_{\mathcal{D}_{\mathbf{m}}} \mathbf{k}_{\mathbf{m},n} - l_{\mathbf{s},n} = \infty.$$

This follows easily from the following inequalities

$$\mathbf{k}_{\mathbf{m},n} - l_{\mathbf{s},n} = \mathbf{k}_{\mathbf{m},n} - l_{\mathbf{s}_{\ell_n},n} \geq \eta_{\ell_n}(n) \geq \eta(n),$$

and the fact that $\lim_{\mathcal{D}_{\mathbf{m}}} \eta(n) = \infty$.

Now fix $\ell_* < \omega$. Then for all n with $\ell_n > \ell_*$ we have

$$\mathbf{g}_{\mathbf{s}_{\ell_*},n} \trianglelefteq \mathbf{g}_{\mathbf{s}_{\ell_n},n} = \mathbf{g}_{\mathbf{s},n},$$

hence

$$D_{\ell_*} \subseteq \{n<\omega : \mathbf{g}_{\mathbf{s}_\ell,n} \trianglelefteq \mathbf{g}_{\mathbf{s},n}\},$$

where D_{ℓ_*} is as defined above. As $D_{\ell_*} \in \mathcal{D}_{\mathbf{m}}$, we have $\{n<\omega : \mathbf{g}_{\mathbf{s}_\ell,n} \trianglelefteq \mathbf{g}_{\mathbf{s},n}\} \in \mathcal{D}_{\mathbf{m}}$, and hence $\mathbf{s}_\ell \leq_{\mathrm{AP}} \mathbf{s}$.

Finally (3) follows from the way we defined the game $\Game^n_k(\mathbf{m})$, noting that protagonist plays with a winning strategy. $\square$

Definition 2.15. For each $\mathbf{s} \in \mathrm{AP}_{\mathbf{m}}$, we define $H_{\mathbf{s}}$ as the set

$$H_{\mathbf{s}} = \left\{(h_1,h_2) : h_1 \in \prod_n \mathbb{M}^{\mathbf{m},1}_n,\ h_2 \in \prod_n \mathbb{M}^{\mathbf{m},2}_n \text{ and } \{n<\omega : f_{\mathbf{s},n}(h_1(n)) = h_2(n)\} \in \mathcal{D}_{\mathbf{m}}\right\}.$$

The proof of the next claim follows from the way that we defined the game $\Game^n_k(\mathbf{m})$ 2.4 and the fact that by Lemma 2.10 we have $\mathcal{D} \supseteq \mathcal{D}_{\mathbf{m}}$ (see also the proof of lemmas 2.3 and 2.10).

Claim 2.16. Let $\mathbf{s} \in \mathrm{AP}_{\mathbf{m}}$. Then

$$h_{\mathcal{D},\mathbf{s}} = \{(h_1/\mathcal{D}, h_2/\mathcal{D}) : (h_1, h_2) \in H_{\mathbf{s}}\}$$

defines a partial elementary mapping from $\prod\limits_{\mathcal{D}} \mathbb{M}^{\mathbf{m},1}_n$ into $\prod\limits_{\mathcal{D}} \mathbb{M}^{\mathbf{m},2}_n$.

Let $\mathcal{F} \subseteq \prod\limits_n [\omega]^{\mathbf{k}_{\mathbf{m},n}}$ witness $\mathfrak{d}_{\mathbf{m}} = \aleph_1$.

For each $\ell \in \{1,2\}$ let $\langle \mathbb{M}^{\mathbf{m},\ell}_{n,i} : i < \omega_1 \rangle$ be a $\prec$-increasing and continuous chain of countable elementary submodels of $\mathbb{M}^{\mathbf{m},\ell}_n$ whose union is $\mathbb{M}^{\mathbf{m},\ell}_n$. For each $i < \omega_1$ and $\ell \in \{1,2\}$, as $||\mathbb{M}^{\mathbf{m},\ell}_{n,i}|| \leq \aleph_0$, there is some $\mathcal{F}^\ell_i \subseteq \prod\limits_n [\mathbb{M}^{\mathbf{m},\ell}_{n,i}]^{\mathbf{k}_{\mathbf{m},n}}$ of cardinality $\aleph_1$ such that

$$\left(\forall \eta \in \prod_n \mathbb{M}^{\mathbf{m},\ell}_{n,i}\right) \left(\exists f \in \mathcal{F}^\ell_i\right) \forall_{\mathcal{D}_{\mathbf{m}}} n \left(\eta(n) \in f(n)\right).$$

Let $\langle f^\ell_j : j < \omega_1 \rangle$ enumerate $\bigcup\limits_{i<\omega_1} \mathcal{F}^\ell_i$. By induction on $i < \omega_1$, and using Claim 2.14, we choose $\mathbf{s}_i$ such that:

(1) $\mathbf{s}_i \in \mathrm{AP}_{\mathbf{m}}$,

(2) $i < j \implies \mathbf{s}_i \leq_{\mathrm{AP}_{\mathbf{m}}} \mathbf{s}_j$,

(3) if $i = 2j+1$, then

$$n < \omega \text{ and } l_{\mathbf{s}_{2j}} < \mathbf{k}_{\mathbf{m},n} \implies f^1_j(n) \subseteq \mathrm{dom}(f_{\mathbf{s}_i,n}),$$

(4) if $i = 2j+2$, then

$$n < \omega \text{ and } l_{\mathbf{s}_{2j+1}} < \mathbf{k}_{\mathbf{m},n} \implies f^2_j(n) \subseteq \mathrm{range}(f_{\mathbf{s}_i,n}).$$

Now let $\mathcal{D}$ be a non-principal ultrafilter on ω which extends $\mathcal{D}_{\mathbf{m}}$. Let

$$h = \bigcup \{h_{\mathcal{D},\mathbf{s}_i} : i < \omega_1\}.$$

Claim 2.17. h is an isomorphism from $\prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},1}$ onto $\prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},2}$.

Proof. By Claim 2.16, each $h_{\mathcal{D},\mathbf{s}_i}$ is a partial elementary embedding from $\prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},1}$ into $\prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},2}$, and furthermore, for $i < j < \omega_1$ we have $h_{\mathcal{D},\mathbf{s}_i} \subseteq h_{\mathcal{D},\mathbf{s}_j}$. By the construction of the sequence $\langle \mathbf{s}_i : i < \omega_1 \rangle$, we clearly have $\mathrm{dom}(h) = \prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},1}$ and $\mathrm{range}(h) = \prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},2}$. So we are done. □

Lemma 2.11 follows. □

The following is an immediate corollary of Lemma 2.11.

Corollary 2.18. *Suppose* $\mathbf{m}$ *is an ultraproduct problem,* $\mathfrak{d}_{\mathbf{m}} = \aleph_1$ *and for each* $n < \omega$, $\mathbb{M}_n^{\mathbf{m},1} \equiv \mathbb{M}_n^{\mathbf{m},2}$. *If* $\mathcal{D}$ *is a non-principal ultrafilter on* ω*, then* $\prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},1} \cong \prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},2}$.

We now turn to the problem of controlling the size of $\mathfrak{d}_{\mathbf{m}}$'s. We again prove a slightly stronger result. We first start with the following simple lemma.

Lemma 2.19. *Assume* MA_κ*, and let* $\mathbf{k} = \langle \mathbf{k}_n : n < \omega \rangle \in {}^{\omega}\omega$ *be such that* $\lim_{\mathcal{D}_{\mathbf{k}}} \mathbf{k}_n = \infty$. *Let* $A \subseteq {}^{\omega}\omega$ *be of size* $\leq \kappa$. *Then there exists* $f \in \prod_{n<\omega} [\omega]^{\leq \mathbf{k}_n}$ *such that for each* $\eta \in A$, *we have* $\forall_{\mathcal{D}_{\mathbf{k}}} n\, (\eta(n) \in f(n))$.

Proof. Let $\mathbb{P}$ be the following covering forcing notion. The conditions are pairs $p = (k_p, f_p)$, where

(α) $k_p < \omega$,
(β) $f_p \in \prod_{n<\omega} [\omega]^{\leq \mathbf{k}_n}$ and $\{|f_p(n)| : n < \omega\}$ is bounded.

Given conditions $p, q \in \mathbb{P}$, let $p \leq q$ (q is stronger than p) if

(γ) $k_q \geq k_p$,
(δ) $f_q \restriction k_p = f_p \restriction k_p$,
(ϵ) $\forall_{\mathcal{D}_{\mathbf{k}}} n, f_q(n) \supseteq f_p(n)$.

It is easily seen that the forcing notion $\mathbb{P}$ is c.c.c. Indeed let $A \subseteq \mathbb{P}$ be of size $\aleph_1$. By shrinking A if necessary, we can assume that $k_p = k_*$ for some fixed $k_* < \omega$ and all $p \in A$.

Furthermore, we can assume that $f_p(n) = f_q(n)$ for all $p, q \in A$ and all $n < k_*$. We show that any two conditions in A are compatible. Thus let $p, q \in A$. Let $l < \omega$ be such that

$$\forall n < \omega \big(|f_p(n)|, |f_q(n)| < l\big).$$

Let $E = \{n < \omega : \mathbf{k}_n > 2l\}$. Then $E \in \mathcal{D}_{\mathbf{k}}$. Define $r \in \mathbb{P}$ as $r = (k_r, f_r)$, where:

(1) $k_r = k_*$,
(2) for all $n < k_*$, $f_r(n) = f_p(n) = f_q(n)$,
(3) for all $n \in E \setminus k_*$, $f_r(n) = f_p(n) \cup f_q(n)$.
(4) for all $n \in \omega \setminus (E \cup k_*)$, $f_r(n) = f_p(n)$.

The condition r is easily seen to be an extension of both of p and q.

For each $\eta \in A$, the set

$$D_\eta = \{p \in \mathbb{P} : \forall_{\mathcal{D}_{\mathbf{k}}} n, \eta(n) \in f_p(n)\}$$

is clearly dense in $\mathbb{P}$. Thus by MA_κ we can find a filter G such that $G \cap D_\eta \neq \emptyset$ for all $\eta \in A$. Then the function f, defined as $f(n) = \bigcup_{p \in G} f_p(n)$, has the required properties. $\square$

Proposition 2.20. *Suppose the GCH holds and $\lambda > \mathrm{cf}(\lambda) = \aleph_1$. Then there exists a c.c.c. forcing notion $\mathbb{P}$ of size λ such that in $V^{\mathbb{P}}$, we have:*

(1) $2^{\aleph_0} = \lambda$,
(2) *if* $\mathbf{k} \in {}^\omega\omega$ *is such that* $\limsup_{n<\omega} \mathbf{k}_n = \infty$, *then* $\mathfrak{d}_{\mathbf{k}} = \aleph_1$.

Proof. Let $\langle \lambda_i : i < \omega_1 \rangle$ be an increasing sequence of regular cardinals cofinal in λ. Let $\mathbb{P} = \langle \langle \mathbb{P}_i : i \leq \omega_1 \rangle, \langle \dot{\mathbb{Q}}_i : i < \omega_1 \rangle \rangle$ be a finite support iteration of c.c.c. forcing notions such that:

- for each $i < \omega_1$, $|\mathbb{P}_i| = \lambda_i$,
- for each $i < \omega_1, V[G_{\mathbb{P}_i}] \models$"Martin's axiom",
- $(2^{\aleph_0})^{V[G_{\mathbb{P}_i}]} = \lambda_i$.

It is easily seen, using Lemma 2.19, that $\mathbb{P} = \mathbb{P}_{\omega_1}$ is as required. Indeed, suppose that $\mathbf{k}$ is as in clause (2). Pick $i_* < \omega_1$ such that $\mathbf{k} \in V[G_{\mathbb{P}_{i_*}}]$. For each $i_* \leq i < \omega_1$, pick, using

Lemma 2.19, some $f_i \in \prod_{n<\omega} [\omega]^{\mathbf{k}_n} \cap V[G_{\mathbb{P}_{i+1}}]$ such that:

$$(\forall \eta \in {}^{\omega}\omega \cap V[G_{\mathbb{P}_i}])\, (\forall_{\mathcal{D}_{\mathbf{k}}} n)\, (\eta(n) \in f_i(n))\,.$$

Then the family $\mathcal{F} = \{f_i : i_* \leq i < \omega_1\}$ witnesses $\mathfrak{d}_{\mathbf{k}} = \aleph_1$. $\square$

Remark 2.21. By [8], the conclusion of Lemma 2.20 also holds in the Sacks model.

Putting the above results together we get the following theorem, which extends [2, Theorem 1.2].

Theorem 2.22. *Assume the GCH holds and* $\lambda > \mathrm{cf}(\lambda) = \aleph_1$*. Then there exists a c.c.c. forcing notion* $\mathbb{P}$ *of size* λ *such that in* $V^{\mathbb{P}}$*:*

(1) $2^{\aleph_0} = \lambda$,
(2) *for every ultraproduct problem* $\mathbf{m}$*, if* $\mathcal{D}$ *is a non-principal ultrafilter on* ω *and if* $\prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},1} \equiv \prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},2}$*, then* $\prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},1} \cong \prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},2}$.

Proof. By Lemmas 2.10, 2.18 and 2.20. $\square$

The models appearing in a pseudo ultraproduct problem that we have considered so far were of size $\aleph_0$ or $\aleph_1$. We now consider the same situation where the models can be finite as well.

Definition 2.23. (see [1] and [8, Chapter V]) Suppose $f, g \in {}^{\omega}(\omega+1 \setminus \{0,1\})$ and $g \leq f$. Then

$$\mathfrak{d}_{f,g} = \min\left\{ |\mathcal{F}| : \mathcal{F} \subseteq \prod_{n<\omega} [f(n)]^{<1+g(n)} \text{ and } (\forall \eta \in \prod_{n<\omega} f(n))(\exists a \in \mathcal{F})(\forall n, \eta(n) \in a(n)) \right\}.$$

Given $f \in {}^{\omega}(\omega+1 \setminus \{0,1\})$, we define the notion of a (pseudo) f-ultraproduct problem as follows.

Definition 2.24. Suppose $f \in {}^{\omega}(\omega+1 \setminus \{0,1\})$. Then a (pseudo) f-ultraproduct problem $\mathbf{m} = \langle (\mathbb{M}_n^{\mathbf{m},1}, \mathbb{M}_n^{\mathbf{m},2}, \tau_{\mathbf{m},n}) : n < \omega \rangle$ is defined as in the notion of a (pseudo) ultraproduct problem (see definitions 2.1 and 2.5), but we require for each $n < \omega$ and $\ell = 1, 2$, $||\mathbb{M}_n^{\mathbf{m},\ell}|| \leq f(n)$.

Given an f-ultraproduct problem $\mathbf{m}$, the sequence $\mathbf{k_m}$ and the filter $\mathcal{D}_\mathbf{m}$ are defined as before. The proof of the next lemma is essentially the same as in the proof of Lemma 2.11.

Lemma 2.25. *Suppose $f \in {}^\omega(\omega+1 \setminus \{0,1\})$, $\mathbf{m}$ is an f-ultraproduct problem, $f \geq \mathbf{k_m}$, $\mathfrak{d}_{f,\mathbf{k_m}} = \aleph_1$ and $\mathcal{D} \supseteq \mathcal{D}_\mathbf{m}$ is a non-principal ultrafilter on ω. Then $\prod\limits_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},1} \cong \prod\limits_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},2}$.*

The following is analogous to Theorem 2.22 for f-ultraproduct problems.

Theorem 2.26. *Assume the GCH holds. Then there exists a cardinal and cofinality preserving forcing notion $\mathbb{P}$ such that in $V^\mathbb{P}$:*

(1) $2^{\aleph_0} \geq \aleph_2$,
(2) *for every $f \in {}^\omega(\omega+1 \setminus \{0,1\})$ and every f-ultraproduct problem $\mathbf{m}$ with $\mathbf{k_m} \leq f$, if $\mathcal{D}$ is a non-principal ultrafilter on ω and if $\prod\limits_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},1} \equiv \prod\limits_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},2}$, then $\prod\limits_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},1} \cong \prod\limits_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},2}$.*

Proof. By [1], there exists a cardinal and cofinality preserving forcing notion $\mathbb{P}$ which forces the failure of the continuum and such that in $V^\mathbb{P}$, for each $f \in {}^\omega(\omega+1 \setminus \{0,1\})$ and each f-ultraproduct problem $\mathbf{m}$ with $\mathbf{k_m} \leq f$, we have $\mathfrak{d}_{f,\mathbf{k_m}} = \aleph_1$. Now the result follows from Lemmas 2.10 and 2.25. □

3. An impossibility result

In this section, we show that we cannot extend Theorem 2.22 to get a model of MA + $2^{\aleph_0} > \aleph_1$ in which for every ultraproduct problem $\mathbf{m}$ and every non-principal ultrafilter $\mathcal{D}$ on ω if $\prod\limits_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},1} \equiv \prod\limits_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},2}$, then $\prod\limits_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},1} \cong \prod\limits_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},2}$. Indeed we prove the following stronger result. Recall that the bounding number is defined as

$$\mathfrak{b} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq {}^\omega\omega \text{ and } \forall f \in {}^\omega\omega \exists g \in \mathcal{F}, g \not\leq^* f\},$$

where $\leq^*$ is the eventual domination order.

Theorem 3.1. *Assume $\mathfrak{b} > \aleph_1$. Then there exists an ultraproduct problem $\mathbf{m}$ such that:*

(1) $\mathbb{M}_n^{\mathbf{m},1} \equiv \mathbb{M}_n^{\mathbf{m},2}$ *for every* $n < \omega$,

(2) *For every non-principal ultrafilter* $\mathcal{D}$ *on* ω, $\prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},1} \not\cong \prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},2}$.

Proof. We follow [2]. For every $n < \omega$, let $\mathbb{M}_n^{\mathbf{m},1} = (\mathbb{Q}, <)$ be the dense linear order of rational numbers and $\mathbb{M}_n^{\mathbf{m},2} = (N, <_N)$ be a dense linear order of size $\aleph_1$ which has a point a with $\mathrm{cf}(N_a, <_N) = \aleph_1$, where $N_a = \{d \in N : d <_N a\}$.

Suppose by the way of contradiction that $\prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},1} \cong \prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},2}$, for some non-principal ultrafilter $\mathcal{D}$ on ω and let $\pi : \prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},1} \cong \prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},2}$ witness such an isomorphism.

For notational simplicity set $M_* = \prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},1} = (\mathbb{Q}, <)^\omega/\mathcal{D}$ and $N_* = \prod_{\mathcal{D}} \mathbb{M}_n^{\mathbf{m},2} = N^\omega/\mathcal{D}$. Let $a_* = [\langle a : n < \omega \rangle]_{\mathcal{D}} \in N_*$. By [2, Claim 2.2], $\mathrm{cf}((N_*)_{a_*}) = \aleph_1$, and hence $\mathrm{cf}((M_*)_{a_\dagger}) = \aleph_1$ where $a_\dagger \in M_*$ is such that $\pi(a_\dagger) = a_*$. By [2, Claim 2.4], $\mathrm{cf}((M_*)_{b_\dagger}) = \aleph_1$ for every $b_\dagger \in M_*$, in particular $\mathrm{cf}((M_*)_{0_\dagger}) = \aleph_1$ where $0_\dagger = [\langle 0 : n < \omega \rangle]_{\mathcal{D}} \in M_*$.

Claim 3.2. $\mathrm{cf}((M_*)_{0_\dagger}) \geq \mathfrak{b}$.

Proof. Suppose $\kappa < \mathfrak{b}$ and let $\langle [f_i]_{\mathcal{D}} : i < \kappa \rangle$ be an increasing sequence in M_* where for each $i < \kappa$, $[f_i]_{\mathcal{D}} <_{M_*} 0_\dagger$. We may assume that $-1 < f_i(n) < 0$ for every $n < \omega$. For each $i < \kappa$ set

$$g_i(n) = \min\{k < \omega : f_i(n) < -\frac{1}{k}\}$$

Then $\mathcal{G} = \{g_i : i < \kappa\} \subseteq {}^\omega\omega$ and $|\mathcal{G}| \leq \kappa$. Thus we can find some $g : \omega \to \omega$ such that for all $i < \kappa$, $g_i \leq^* g$. Define $f : \omega \to \mathbb{Q}$ by

$$f(n) = -\frac{1}{g(n)}.$$

For any $i < \kappa$ we can find some $\ell_i < \omega$ such that $g(n) > g_i(n)$ for all $n > \ell_i$ and hence for any such n,

$$f(n) = -\frac{1}{g(n)} > -\frac{1}{g_i(n)} > f_i(n).$$

It follows that the sequence $\langle [f_i]_{\mathcal{D}} : i < \kappa \rangle$ is bounded from above by $[f]_{\mathcal{D}} < 0_\dagger$. Thus $\mathrm{cf}((M_*)_{0_\dagger}) > \kappa$. As $\kappa < \mathfrak{b}$ was arbitrary, $\mathrm{cf}((M_*)_{0_\dagger}) \geq \mathfrak{b}$ and we are done. $\square$

But this implies that $\mathrm{cf}((M_*)_{0_\dagger}) \geq \mathfrak{b} > \aleph_1$ which contradicts $\mathrm{cf}((M_*)_{0_\dagger}) = \aleph_1$. The theorem follows. $\square$

Remark 3.3. Martin's axiom implies $\mathfrak{b} = 2^{\aleph_0}$, and hence the conclusion of Theorem 3.1 holds under $\mathrm{MA} + 2^{\aleph_0} > \aleph_1$.

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